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# Some properties of continuous Yao graph

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**Abstract.** Given a set  $S$  of points in the plane and an angle  $0 < \theta \leq 2\pi$ , the *continuous Yao graph*  $cY(\theta)$  with vertex set  $S$  and angle  $\theta$  defined as follows. For each  $p, q \in S$ , we add an edge from  $p$  to  $q$  in  $cY(\theta)$  if there exists a cone with apex  $p$  and angular diameter  $\theta$  such that  $q$  is the closest point to  $p$  inside this cone.

In this paper, we prove that for  $0 < \theta < \pi/3$  and  $t \geq \frac{1}{1-2\sin(\theta/2)}$ , the continuous Yao graph  $cY(\theta)$  is a  $\mathcal{C}$ -fault-tolerant geometric  $t$ -spanner where  $\mathcal{C}$  is the family of convex regions in the plane. Moreover, we show that for every  $\theta \leq \pi$  and every half-plane  $h$ ,  $cY(\theta) \ominus h$  is connected, where  $cY(\theta) \ominus h$  is the graph after removing all edges and points inside  $h$  from the graph  $cY(\theta)$ . Also, we show that there is a set of  $n$  points in the plane and a convex region  $C$  such that for every  $\theta \geq \frac{\pi}{3}$ ,  $cY(\theta) \ominus C$  is not connected.

Given a geometric network  $G$  and two vertices  $x$  and  $y$  of  $G$ , we call a path  $P$  from  $x$  to  $y$  a *self-approaching path*, if for any point  $q$  on  $P$ , when a point  $p$  moves continuously along the path from  $x$  to  $q$ , it always get closer to  $q$ . A geometric graph  $G$  is *self-approaching*, if for every pair of vertices  $x$  and  $y$  there exists a self-approaching path in  $G$  from  $x$  to  $y$ . In this paper, we show that there is a set  $P$  of  $n$  points in the plane such that for some angles  $\theta$ , Yao graph on  $P$  with parameter  $\theta$  is not a self-approaching graph. Instead, the corresponding continuous Yao graph on  $P$  is a self-approaching graph. Furthermore, in general, we show that for every  $\theta > 0$ ,  $cY(\theta)$  is not necessarily a self-approaching graph.

**Keywords:**  $t$ -spanner, Region-fault tolerant spanner, Continuous Yao graph, Self-approaching graph

## 1 Introduction

Let  $S$  be a set of  $n$  points in  $\mathbb{R}^d$  and let  $t \geq 1$  be a real number. A geometric graph is an edge-weighted graph on  $S \subseteq \mathbb{R}^d$  such that the weight of each edge is the Euclidean distance between its endpoints. A geometric graph  $G$  with vertex set  $S$  is called a  $t$ -spanner for  $S$ , if for each two points  $p$  and  $q$  in  $S$ , there exists a path  $Q$  in  $G$  between  $p$  and  $q$  whose length is at most  $t$  times  $|pq|$ , the Euclidean distance between  $p$  and  $q$ . The length of a path is defined to be the sum of the lengths, or weight, of all edges on the path. The path  $Q$  is called a  *$t$ -spanner path* (or  *$t$ -path*) between  $p$  and  $q$ . We denote the length of path  $Q$  by  $|Q|$ . The *stretch factor* (or *dilation*) of  $G$  is the smallest value of  $t$  for which  $G$  is a  $t$ -spanner. The  $t$ -spanners were introduced by Peleg and Schäffer [14] in the scope

of distributed computing and, then, by Chew [6] in the scope of computational geometry. The  $t$ -spanners are applicable in many scopes such as graph theory, network topology design, distributed systems, robotics. We refer the reader to [5,8,12,16] for reading about the  $t$ -spanners and their applications.

The problem of efficient construction of a  $t$ -spanner for a given point set and a constant  $t > 1$  has been studied extensively. One can see the major algorithms for building spanners in the book by Narasimhan and Smid [13].

The Yao graph used by Andrew Yao to construct Euclidean minimum spanning tree on high-dimensional Euclidean space [17]. For a set  $S$  of points in the plane, the Yao graph  $Y_k$ , for  $k \geq 2$ , is defined as follows. At each point  $u \in S$ , we draw  $k$  cones with apex at  $u$  and angle  $\frac{2\pi}{k}$ . For each point  $u \in S$  and cone  $C$ , we add an edge between  $u$  and the closest point to  $u$  in  $C$ . If one chooses  $k$  sufficiently large, the Yao graph becomes a  $t$ -spanner.

In 2014, Barba et al. [3] introduced the *continuous Yao graph* as a variation of Yao graph. The continuous Yao graph  $cY(\theta)$  with vertex set  $S$  and angle  $\theta$  defined as follows: For each  $p, q \in S$ , we add an edge from  $p$  to  $q$  in  $cY(\theta)$  if there exists a  $\theta$ -cone, a cone with aperture  $\theta$ , with apex at  $p$  such that  $q$  is the closest point to  $p$  inside this  $\theta$ -cone. In continuous Yao graph, for each  $u \in S$ , we rotate a  $\theta$ -cone with apex  $u$  around  $u$  continuously, and connect  $u$  to the closest point inside the  $\theta$ -cone during this rotation. They showed that  $cY(\theta)$  has stretch factor at most  $1/(1-2\sin(\theta/4))$  for  $0 < \theta < 2\pi/3$ . Unlike Yao graphs that always have a linear number of edges for any constant  $k$ , in the worst case, continuous Yao graphs may have a quadratic number of edges. Since  $cY(\theta) \subseteq cY(\gamma)$  for any  $\theta \geq \gamma$ , the continuous Yao graphs are useful in potential applications that require scalability. Moreover, when some rotations apply on the input point set the continuous Yao graphs are invariant [3].

One of the useful properties of a network is *fault tolerance* that is after one or more vertices or edges fail, the remaining graph is still a good network of alive vertices. In particular, a graph  $G = (S, E)$  is called  *$k$ -vertex fault-tolerant  $t$ -spanner* [10] for  $S$ , denoted by  $(k, t)$ -VF $T$ S for a given real number  $t \geq 1$  and non-negative positive integer  $k$ , if for each set  $S' \subseteq S$  with cardinality of at most  $k$ , the graph  $G \setminus S'$  is a  $t$ -spanner for  $S \setminus S'$ . Also,  $G$  is called a  *$k$ -edge fault-tolerant  $t$ -spanner* [10] for  $S$ , denoted by  $(k, t)$ -EF $T$ S, if for each set  $E' \subseteq E$  with cardinality at most  $k$  and for each pair of points  $p$  and  $q$  in  $S$ , the graph  $G \setminus E'$  contains a path  $P$  between  $p$  and  $q$  with  $|P| \leq t|P_S|$  where  $P_S$  is the shortest path between  $p$  and  $q$  in the graph  $K_S \setminus E'$  in which that  $K_S$  is Euclidean complete graph on  $S$ . Levcopoulos et al. [10] for the first time considered the problem of constructing fault-tolerant spanners in Euclidean spaces efficiently. They proposed three algorithms that construct  $k$ -vertex fault-tolerant spanners. Some other works on the fault tolerant spanners have been done [7,11].

In 2009, Abam et al. [1] introduced the concept of *region-fault tolerant spanner* for planar point sets. For a fault region  $F$  and a geometric graph  $G$  on a point set  $S$ , assume  $G \ominus F$  is the remaining graph after removing the vertices of  $G$  that lie inside  $F$  and all edges that intersect  $F$ . For a set  $\mathcal{F}$  of regions in the plane, an  $\mathcal{F}$ -fault tolerant  $t$ -spanner is a geometric graph  $G$  on  $S$  such that for

any region  $F \in \mathcal{F}$ , the graph  $G \ominus F$  is a  $t$ -spanner for  $\mathcal{G}_c(S) \ominus F$ , where  $\mathcal{G}_c(S)$  is the complete geometric graph on  $S$ . They showed for any set of  $n$  points in the plane and any family  $\mathcal{C}$  of convex regions, one can construct a  $\mathcal{C}$ -fault tolerant spanner of size  $O(n \log n)$  in  $O(n \log^2 n)$  time.

In 2013, Alamdari et al. [2] introduced the concept of *self-approaching* and *increasing-chord* graph drawings. The problem is that, we are given a graph and we need to check if the graph has an self-approaching or increasing-chord embedding in the Euclidean space.

A geometric graph is self-approaching, if for every pair  $s$  and  $t$  of vertices of the graph, there exists a self-approaching path from  $s$  to  $t$ , denoted by  $st$ -path. A path from  $s$  to  $t$  is a self-approaching path if for each  $q$  on the path, not only the vertices, but any place of the path, if a point  $p$  starts at  $s$  and moves toward  $q$ , it always get closer to  $q$  in its movement. Also, a graph  $G$  is called increasing-chord if, for each pair  $u$  and  $v$  of its vertices, there exists a path between  $u$  and  $v$  such that the path is self-approaching both from  $u$  to  $v$  and from  $v$  to  $u$ . Obviously, an increasing-chord graph is a self-approaching graph.

In the geometric context, the position of the vertices of the graph is fixed, so we just want to know whether a given geometric graph is self-approaching or increasing-chord. One of the interesting properties of these graph is the following. It is known that the stretch factor (or dilation) of any self-approaching graph is at most 5.3332 [9] and the stretch factor of any increasing-chord graph is at most 2.094 [15].

**Our results.** In this paper, we prove the following results. For the rest of the paper  $S$  denotes a set of  $n$  points in the plane.

1. The continuous Yao graph  $cY(\theta)$  on  $S$  is a  $\mathcal{C}$ -fault-tolerant geometric  $t$ -spanner of  $S$ , where  $0 < \theta < \pi/3$  and  $t \geq \frac{1}{1-2\sin(\theta/2)}$ .
2. For any  $\theta \leq \pi$  the graph  $cY(\theta) \ominus h$  on  $S$  is connected, where  $h$  is an arbitrary half-plane in the plane.
3. There is a set of  $n$  points in the plane and a convex region  $C$  such that for every  $\theta \geq \frac{\pi}{3}$ ,  $cY(\theta) \ominus C$  is not connected.
4. There is a set  $P$  of  $n$  points in the plane such that for some angles  $\theta$ , Yao graph on  $P$  with parameter  $\theta$  is not a self-approaching graph. Instead, the corresponding continuous Yao graph on  $P$  is a self-approaching graph.
5. For every  $\theta > 0$ ,  $cY(\theta)$  is not necessarily a self-approaching graph.

## 2 $cY(\theta)$ is fault-tolerant

In this section, we show that the continuous Yao graph  $cY(\theta)$  for  $0 < \theta < \pi/3$  and  $t \geq \frac{1}{1-2\sin(\theta/2)}$  is a  $\mathcal{C}$ -fault-tolerant geometric  $t$ -spanner where  $\mathcal{C}$  is the family of convex regions in the plane. Furthermore, we show that for every  $\theta \leq \pi$  and every half-plane  $h$ ,  $cY(\theta) \ominus h$  is connected. Moreover, we show that there is a set of  $n$  points such that for a some convex region  $C$ ,  $cY(\theta) \ominus C$  is not connected for every  $\theta \geq \frac{\pi}{3}$ . We need the following lemmas.

**Lemma 1** ([4]). *Let  $a, b$  and  $c$  be three points such that  $|ac| \leq |ab|$  and  $\angle bac \leq \alpha < \pi$ . Then*

$$|bc| \leq |ab| - (1 - 2\sin(\alpha/2))|ac|.$$

**Lemma 2** ([1]). *A geometric graph  $G$  on  $S$  is a  $\mathcal{C}$ -fault tolerant  $t$ -spanner if and only if it is an  $\mathcal{H}$ -fault-tolerant  $t$ -spanner, where  $\mathcal{H}$  is the family of all half-planes.*

Now, we prove the following theorem:

**Theorem 1.** *Let  $\theta$  be a real number with  $0 < \theta < \pi/3$  and let  $t$  be a real number with  $t \geq \frac{1}{1-2\sin(\theta/2)}$ . For any point set  $S$ , the continuous Yao graph  $cY(\theta)$  is a  $\mathcal{C}$ -fault-tolerant geometric  $t$ -spanner.*

*Proof.* By Lemma 2, it is sufficient to prove that  $cY(\theta)$  is an  $\mathcal{H}$ -fault-tolerant geometric  $t$ -spanner.

Let  $h$  be an arbitrary half-plane in  $\mathcal{H}$ . We must show that for each pair of points  $p, q \in S$  outside  $h$ , there is a  $t$ -path between  $p$  and  $q$  in  $cY(\theta) \ominus h$ . The proof is by induction on the rank of distance  $|pq|$ .

For the base step, suppose that the pair  $p$  and  $q$  is the closest pair in  $cY(\theta) \ominus h$ . Without loss of generality, assume the distance between  $p$  and  $h$  is less than or equal to the distance between  $q$  and  $h$ . Since  $p$  and  $q$  are outside  $h$  and  $\theta < \pi/2$ , there is a  $\theta$ -cone  $C_p$  with apex at  $p$  completely outside  $h$  such that  $q \in C_p$ . Since pair  $p$  and  $q$  is the closest pair, by the construction of  $cY(\theta)$ , the edge  $(p, q)$  must be in  $cY(\theta) \ominus h$ . Note that if both  $p$  and  $q$  have same distance to  $h$ , then we can choose both  $p$  and  $q$  as apex, but we have to choose the point such that the corresponding  $\theta$ -cone does not intersect  $h$ .

For the induction hypothesis step, suppose for each pair  $u, v \in S$  outside  $h$  with  $|uv| < |pq|$  there is a  $t$ -path between  $u$  and  $v$  connecting them in  $cY(\theta) \ominus h$ .

Now we consider the induction step. Since  $p$  and  $q$  are outside  $h$ , there is a  $\theta$ -cone  $C_p$  with apex at  $p$  such that  $q \in C_p$  (here we assumed that  $p$  is closer than  $q$  to  $h$ ).

Suppose that  $r$  is the closest point to  $p$  inside the cone  $C_p$  (see Fig. 1). Since  $\theta < \pi/3$ ,  $1 - 2\sin(\theta/2) > 0$ , and also since  $|pr| \leq |pq|$ , by Lemma 1 we have  $|rq| < |pq|$ . Therefore, by the induction hypothesis, there is a  $t$ -path  $Q$  between  $r$  and  $q$  in  $cY(\theta) \ominus h$ . Now consider the path  $P := \{(p, r)\} \cup Q$ . Clearly the path  $P$  connects  $p$  and  $q$ , and  $P$  is in  $cY(\theta) \ominus h$ . Now by Lemma 1, we have

$$\begin{aligned} |P| &= |pr| + |Q| \\ &\leq |pr| + t|rq| \\ &\leq |pr| + t(|pq| - (1 - 2\sin(\theta/2))|pr|) \\ &= t|pq| + (1 - t(1 - 2\sin(\theta/2)))|pr| \\ &\leq t|pq| \quad \left( \text{since } t \geq \frac{1}{1 - 2\sin(\theta/2)} \right). \end{aligned}$$

Thus  $P$  is a  $t$ -path in  $cY(\theta) \ominus h$  between  $p$  and  $q$ . This completes the proof.  $\square$

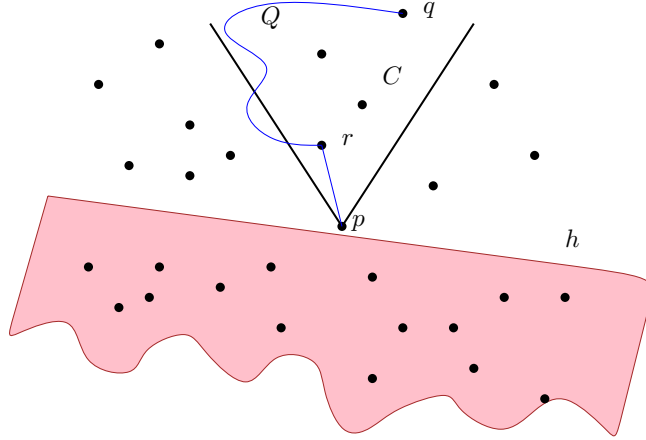


Fig. 1. Illustrating of the proof of Theorem 1.

Note that even though  $cY(\theta)$  is a  $\mathcal{C}$ -fault tolerant spanner, it is not necessarily connected. It is because in fault-tolerant spanners, we compare the graph with the complete graph. So if the complete graph after removing points in some regions becomes disconnected, then some of fault-tolerant spanners of the points set might be disconnected. In the following, we show that for every half-plane  $h$  in the plane, the graph  $cY(\theta) \ominus h$  is a connected graph for every  $\theta \leq \pi$ .

**Lemma 3.** *The continuous Yao graph  $cY(\pi)$  on each set  $S$  contains  $CH(S)$ , the convex hull of  $S$ .*

*Proof.* Proof is straightforward. □

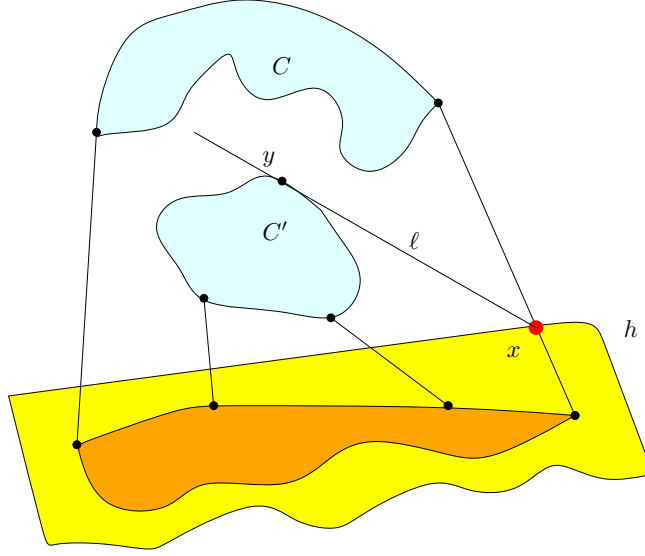
**Theorem 2.** *For any  $\theta \leq \pi$  the graph  $cY(\theta) \ominus h$  on  $S$  is connected, where  $h$  is a half-plane.*

*Proof.* It is easy to see that for every  $\alpha, \beta > 0$ , if  $\alpha \geq \beta$  then  $cY(\alpha) \subseteq cY(\beta)$ . So, to prove the theorem, it is sufficient to show that  $cY(\pi) \ominus h$  is connected for every half-plane  $h$ .

Let  $h$  be an arbitrary half-plane in the plane. Suppose that  $cY(\pi) \ominus h$  is not connected.

Since  $cY(\pi) \ominus h$  is disconnected, it has more than one connected component. At least one of the connected components of  $cY(\pi) \ominus h$  contains a point from  $CH(S)$ . Suppose  $C$  and  $C'$  are two (distinct) connected components of  $cY(\pi) \ominus h$  such that at least one of them, say  $C$ , contains a point from  $CH(S)$ . If both of  $C$  and  $C'$  contains a point from  $CH(S)$  then by Lemma 3, part of  $CH(S)$  which lies outside  $h$  is connected which is a contradiction because we assumed that  $C$  and  $C'$  are distinct connected components of  $cY(\pi) \ominus h$ .

Now, assume  $C'$  contains no vertices on  $CH(S)$ . Let  $x$  be the intersection of boundary of  $h$  and  $CH(S)$  and let  $\ell$  be a line through  $x$  and tangent to  $C'$  (see Fig. 2). We have two cases.



**Fig. 2.** Illustrating of the proof of Theorem 2: case 1.

**Case 1:**  $\ell \cap C'$  contains exactly one point denoted by  $y$ .

If  $x \notin S$  then clearly by the construction of graph  $cY(\pi)$ , the vertex  $y$  should be connected to a vertex of  $C$  that is contradiction, since we assumed that  $C$  and  $C'$  are distinct connected components of  $cY(\pi) \ominus h$ . Now, suppose that  $x \in S$ . Since  $\ell \cap C'$  contains exactly one point, we can with continuously rotating the line  $\ell$  around the point  $y$  (in Fig. 2, we rotate  $\ell$  counter-clockwise), find a line  $\ell'$  such that  $\ell' \cap C'$  contains exactly one point that is  $y$ , and also the point in the intersection of  $\ell'$  and boundary  $h$  does not belong to  $S$ . Hence, by the construction of graph  $cY(\pi)$ , point  $y$  should be connected to a vertex of  $C$ . That is a contradiction with that  $C$  and  $C'$  are distinct connected components of  $cY(\pi) \ominus h$ .

**Case 2:**  $\ell$  is tangent to  $C'$  in at least two points.

We claim that there is a line  $\ell'$  that is tangent to exactly one point of  $S$  on the convex hull of  $C'$  (denoted by  $s$ ). Suppose that  $s$  be the farthest point of  $S$  on  $\ell$  with respect to  $x$ , and suppose that  $r$  be the next point after  $s$  on the convex hull of  $C'$  in counterclockwise order (see Fig. 3). Let  $\ell''$  be a line that through of  $s$  and  $r$ , and let  $z$  be the intersection of  $\ell''$  and boundary of  $h$ . Now suppose that  $w$  be a point on the segment  $xz$  ( $w \neq x, z$ ). Let  $\ell'$  be a line that through of  $w$  and  $s$ . Clearly  $\ell'$  is tangent to exactly one point of  $S$  that is  $s$  on the convex hull of  $C'$ . Now by the construction of  $cY(\pi)$ , vertex  $s$  should be connected to a vertex of connected component of  $C$ . This contradicts with the assumption that  $C$  and  $C'$  are distinct connected components of  $cY(\pi) \ominus h$ .

According to the above mentioned cases,  $cY(\pi) \ominus h$  is connected. This completes the proof.  $\square$

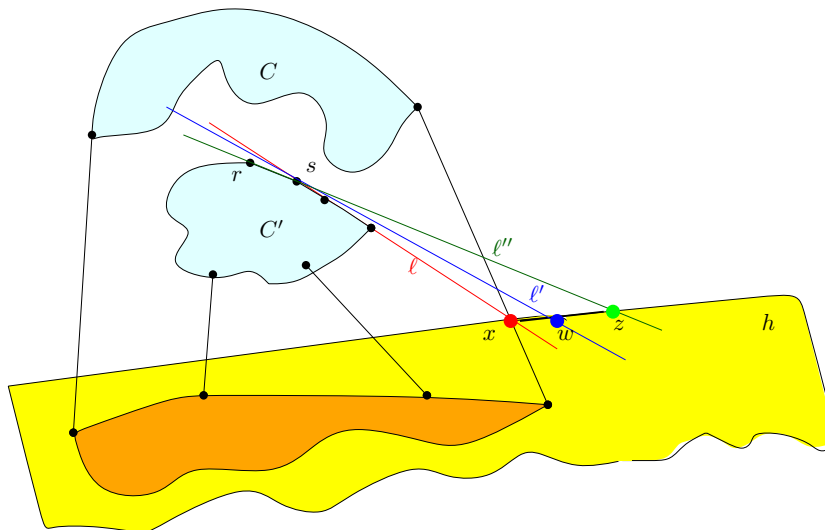


Fig. 3. Illustrating of the proof of Theorem 2: case 2.

In the following, we give an example such that for every  $\theta \geq \frac{\pi}{3}$ ,  $cY(\theta) \cap C$  for some convex region  $C$  is not necessarily connected.

Assume  $P_0 = \{(0, 0), (1, \sqrt{3}), (2, 0)\}$  which is the vertices of an equilateral triangle. Let  $P_i$  be the translation of  $P_0$  by value  $c \times i$  horizontally where  $c$  is a sufficiently large constant positive integer. Here, we consider  $c = 10$  (see Fig. 4). We choose the set  $P := \bigcup_{i=0}^{k-1} P_i$ .

Let  $\theta$  be an angle with  $\theta \geq \frac{\pi}{3}$ . Consider  $cY(\theta)$  on  $P$ . Note that for  $\theta \geq \frac{\pi}{3}$ ,  $cY(\theta)$  on the vertices of an equilateral triangle is exactly the complete graph of their vertices. Now, let  $C$  be a convex region such that  $C$  only contains the points  $(1, \sqrt{3}), (2, 0), (10, 0)$  and  $(11, \sqrt{3})$  from  $P$  (see Fig. 4). Since in the  $cY(\theta)$  the vertex  $(0, 0)$  is only connected to the vertices  $(1, \sqrt{3})$  and  $(2, 0)$ , clearly in  $cY(\theta) \cap C$  there is no path between  $(0, 0)$  and  $(12, 0)$ . Hence  $cY(\theta) \cap C$  is not connected.

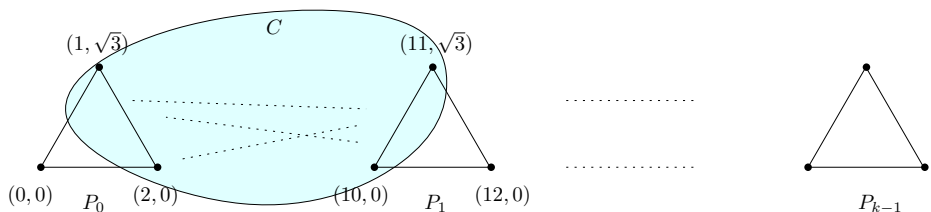


Fig. 4. An example that  $cY(\theta)$  is disconnected after convex region fault.



### 3 $cY(\theta)$ is not self-approaching

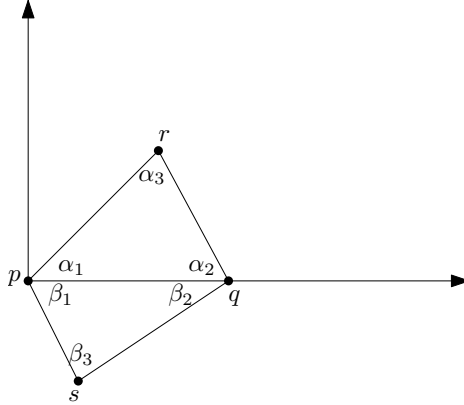
In this section, we show that there is a set  $P$  of  $n$  points in the plane such that for some angles  $\theta$ , Yao graph on  $P$  with parameter  $\theta$  is not a self-approaching graph. Instead, the corresponding continuous Yao graph on  $P$  is a self-approaching graph. Finally, we show that there is a set of  $n$  points in the plane such that for every  $\theta > 0$ , continuous Yao graph  $cY(\theta)$  is not a self-approaching graph.

In the following, we assume that the number of cones is even. With making some modifications, similar results holds for odd number of cones.

Now, let  $p = (0, 0)$ ,  $q = (1, 0)$ ,  $r = (r_1, r_2)$  and  $s = (s_1, s_2)$  be four points in the plane. For three points  $x, y$  and  $z$ , let  $\angle yxz$  be the angle between the segment  $xy$  and  $xz$ . Now, assume that  $\angle qpr = \alpha_1$ ,  $\angle pqr = \alpha_2$ ,  $\angle prq = \alpha_3$ ,  $\angle qps = \beta_1$ ,  $\angle pqs = \beta_2$ ,  $\angle psq = \beta_3$ , for some angles  $\alpha_i$  and  $\beta_i$  with  $0 < \alpha_i < \pi/2$  and  $0 < \beta_i < \pi/2$  for  $i = 1, 2, 3$  (see Fig. 5).

We call the quadrilateral on the four points  $p, q, r$  and  $s$  a *bad quadrilateral* if we have the following properties:

1.  $|pq| \cos \alpha_1 < |pr| < |pq|$
2.  $|pq| \cos \alpha_2 < |qr| < |pq|$
3.  $|pq| \cos \beta_1 < |ps| < |pq|$
4.  $|pq| \cos \beta_2 < |qs| < |pq|$



**Fig. 5.** A bad quadrilateral on the points  $p, q, r$  and  $s$ .

Now, we consider  $k$  cones, generated by the  $k$ -rays through origin, where the  $i$ th ray has angle  $(i - 1)\theta$  with the positive  $x$ -axis. The  $i$ th cones contains the points  $p$  such that the angle of line  $op$  with positive  $x$ -axis is bigger than or equal  $(i - 1)\theta$  and less than  $i\theta$ .

Let  $V$  be a set of four points  $p = (0, 0)$ ,  $q = (1, 0)$ ,  $r = (r_1, r_2)$  and  $s = (s_1, s_2)$  such that the quadrilateral on the vertices  $p, q, r$  and  $s$  is a bad quadrilateral. It

is easy to see that, for every  $\theta$  with  $\alpha_1 < \theta \leq \frac{\pi}{2}$  such that  $\frac{2\pi}{\theta}$  is even number and  $\beta_2 < \theta$ , the Yao graph  $Y_k$  on  $V$  where  $k = \frac{2\pi}{\theta}$  does not contain the edge  $\{p, q\}$  but it contains the edges  $\{p, r\}$ ,  $\{p, s\}$ ,  $\{q, r\}$  and  $\{q, s\}$ . So the Yao graph  $Y_k$  on the point set  $V$  is not a self-approaching graph, since every path from  $p$  to  $q$  is not a  $pq$ -path. Indeed, the following lemma shows that  $cY(\theta)$  on  $V$  is a self-approaching graph.

**Lemma 4.** *Let  $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$  or  $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$ . For every  $\theta$  such that*

- a)  $\alpha_1 < \theta \leq \frac{\pi}{2}$ , and
- b)  $\theta < \alpha_1 + \beta_1$  or  $\theta < \alpha_2 + \beta_2$ , and
- c)  $\frac{2\pi}{\theta}$  is even number, and
- d)  $\beta_2 < \theta$ ,

*continuous Yao graph  $cY(\theta)$  on  $V$  contains the edge  $\{p, q\}$ . Furthermore,  $cY(\theta)$  on  $V$  is a self-approaching graph.*

*Proof.* We prove the theorem for the case that  $\theta < \alpha_1 + \beta_1$ . Similar argument works for the case  $\theta < \alpha_2 + \beta_2$ .

Since  $\theta < \alpha_1 + \beta_1$ , by construction of  $cY(\theta)$ , clearly there is a  $\theta$ -cone  $C$  with apex  $p$  that only contains the point  $q$ . Hence the edge  $\{p, q\}$  is in  $cY(\theta)$ .

On the other hand, the Yao graph on  $V$  with angle parameter  $\theta$  is a subgraph of  $cY(\theta)$ , so the edges  $\{p, r\}$ ,  $\{p, s\}$ ,  $\{q, r\}$  and  $\{q, s\}$  are in  $cY(\theta)$ . So, we only need to show that there is a self-approaching path from  $r$  to  $s$  and a self-approaching path from  $s$  to  $r$ .

Since  $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$  or  $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$ , the path  $r \rightarrow q \rightarrow s$  or  $r \rightarrow p \rightarrow s$  is an  $rs$ -path and in the reverse direction is an  $sr$ -path. So there is an  $xy$ -path between all ordered pairs  $\{x, y\}$  for  $x, y \in V$ . Hence,  $cY(\theta)$  is a self-approaching graph on  $V$ . This completes the proof.  $\square$

Next, we will give another point set such that the Yao graph on the point set is not self-approaching graph, but the continuous Yao graph is a self-approaching graph.

Let  $P$  be a set of  $n$  points as follows:

$$P = V \cup \{x_1, x_2, \dots, x_{n-4}\}, \quad (1)$$

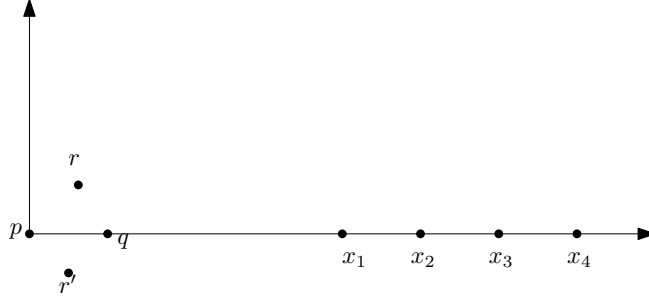
with  $x_j = (c + j, 0)$  for  $1 \leq j \leq n - 4$  where  $c$  is a sufficiently large constant positive integer. Here, we consider  $c = 19$  (see Fig. 6).

Now, using Lemma 4 we can easily find the following result.

**Lemma 5.** *Let  $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$  or  $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$ . For every  $\theta$  such that*

- a)  $\alpha_1 < \theta \leq \frac{\pi}{2}$ , and
- b)  $\theta < \alpha_1 + \beta_1$  or  $\theta < \alpha_2 + \beta_2$ , and
- c)  $\frac{2\pi}{\theta}$  is even number, and
- d)  $\beta_2 < \theta$ ,

*continuous Yao graph  $cY(\theta)$  on  $P$  is self-approaching and  $Y_k$  is not a self-approaching for  $k = \frac{2\pi}{\theta}$ .*



**Fig. 6.** The set  $P$  with  $n = 8$ .

Note that there are some point set that satisfy the conditions mentioned in above lemmas. For example, let  $V = \{p, q, r, s\}$  with  $p = (0, 0)$ ,  $q = (1, 0)$ ,  $r = (0.7839, 0.4422)$ ,  $s = (0.2161, -0.4422)$ , and suppose that  $k = 4$  and  $\theta = \frac{\pi}{2}$ . We can easily verify that the quadrilateral on the points of  $V$  is a bad quadrilateral. Moreover we have

$$\alpha_1 = 29.4271^\circ, \alpha_2 = 63.9535^\circ, \alpha_3 = 86.6198^\circ,$$

$$\beta_1 = 29.4271^\circ, \beta_2 = 63.9535^\circ, \beta_3 = 86.6198^\circ.$$

Now, suppose that  $P = V \cup \{x_1, x_2, \dots, x_{n-4}\}$  where  $x_i = (19 + i, 0)$ . This point set satisfies Lemma 5.

At the first view, it seems that the continuous Yao graph is a self-approaching graph, but this is not true in general. Next, we give a point set such that for each  $\theta$ , the continuous Yao graph  $cY(\theta)$  on the point set is not a self-approaching graph.

**Theorem 3.** *There is a set  $P$  of  $n$  points such that for every  $\theta > 0$ ,  $cY(\theta)$  on  $P$  is not a self-approaching graph.*

*Proof.* We prove the theorem for  $0 < \theta \leq \frac{2\pi}{3}$ . Since for every  $\alpha_1$  and  $\alpha_2$  with  $\alpha_1 \leq \alpha_2$  we have  $cY(\alpha_2) \subseteq cY(\alpha_1)$ , the theorem holds for every  $\theta > 0$ .

Consider two points  $p = (0, 0)$  and  $q = (1, 0)$ . Let  $C$  be a circle centered at the midpoint of the segment  $pq$ , with radius  $\frac{1}{2}$ . Let  $D_p$  and  $D_q$  be circles respectively centered at  $p$  and  $q$  with radius one. Let  $\ell$  be the perpendicular bisector of segment  $pq$  (see Fig. 7).

Consider  $x$  and  $y$  as points outside  $C$  and inside  $D_p \cap D_q$  such that  $\angle xpq < \frac{\theta}{2}$  and  $\angle ypq < \frac{\theta}{2}$ . Let  $x'$  and  $y'$  be the symmetries of  $x$  and  $y$  with respect to line  $\ell$ , respectively.

Now let  $V = \{p, q, x, y, x', y'\}$ , and consider  $cY(\theta)$  on  $V$ . Since  $\angle xpy = \angle x'qy' < \theta$ ,  $cY(\theta)$  does not contain the edge  $\{p, q\}$ . So, by our selection of points  $x, y, x'$  and  $y'$ , there is no self-approach path between  $p$  and  $q$ . Hence,  $cY(\theta)$  on  $V$  is not self-approaching.

One can extend this point set to a point set with arbitrary number of points such that  $cY(\theta)$  is not a self-approaching graph of the point set. To this end, one

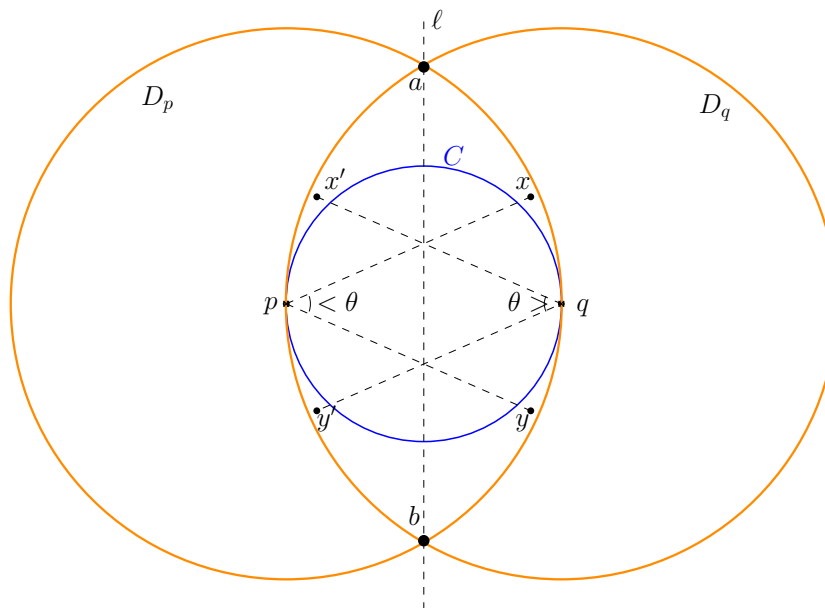


Fig. 7. Illustrating of the proof of Theorem 3.

can add any point to  $V$  which is sufficiently far from the points in the original set  $V$ , because adding points which are far can not help in making a self-approaching path from  $p$  to  $q$ .  $\square$

#### 4 Concluding remarks

We proved that for  $0 < \theta < \pi/3$  and  $t \geq \frac{1}{1-2\sin(\theta/2)}$ , the continuous Yao graph  $cY(\theta)$  is a  $\mathcal{C}$ -fault-tolerant geometric  $t$ -spanner. Furthermore, we showed that for every  $\theta \leq \pi$  and every half-plane  $h$ ,  $cY(\theta) \ominus h$  is connected. Finally, we showed that for every  $\theta > 0$ ,  $cY(\theta)$  is not necessarily a self-approaching graph. The question whether  $cY(\theta)$  for  $\frac{\pi}{3} \leq \theta \leq \pi$  is a  $\mathcal{C}$ -fault-tolerant geometric spanner with constant stretch factor remains open.

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