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# Optimal control theory, sub-Riemannian geometry and swimming of copepod

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## Abstract

In [17], based on the observation of copepods, Takagi proposed a model to analyze the swimming of microorganisms using sinusoidal paddling or sequential paddling followed by a recovery stroke in unison, and they are compared with the concept of efficiency. Our aim is to provide an interpretation in the frame of optimal control theory and sub-Riemannian geometry. The Maximum principle is used to select two types of periodic control candidates as minimizers: sinusoidal up to time reparameterization and the sequential paddling, interpreted as an abnormal stroke in sub-Riemannian geometry. Geometric analysis combined with numerical simulations are decisive tools to compute the optimal solutions, refining Takagi computations. A family of simple strokes with small amplitudes emanating from a center is characterized as an invariant of SR-geometry and allow to identify the metric used by the swimmer.

**keywords:** Stokes flow, Optimal control theory, sub-Riemannian geometry, Abnormal closed geodesics

**AMS classification numbers:** 70Q05, 93C10, 49K15

## 1 Introduction

This article is entirely devoted to analysis of combining optimal control theory and sub-Riemannian (SR-) geometry of the swimming process of the copepod – an abundant variety of zooplankton – observed and modeled in the frame of swimming at low Reynolds number by Takagi [17]. See Fig.1 for the pictures of the copepod (left) and the 2-link symmetric micro-robot to mimic the animal mechanism of the line locomotion.

Note that it is the simplest model of swimming based on *slender body theory for Stokes flow*: two symmetric links, the body being reduced to an infinitesimal sphere, and furthermore it can be easily observed and designed. Hence it is an excellent testbed to understand the swimming mechanism and useful for more complicated models: Purcell swimmers [16, 14] or to take into account more links, this in the same mathematical frame (SR-geometry).

In microrobot modeling, to produce the displacement along the line  $0x_0$ , we use a pair of two symmetric links with equal length normalized to  $l = 1$ ,  $\theta_1, \theta_2$  are the respective angles of the two links and they satisfy the triangle constraint  $\mathcal{T} : 0 \leq \theta_1 \leq \theta_2 \leq \pi$ .

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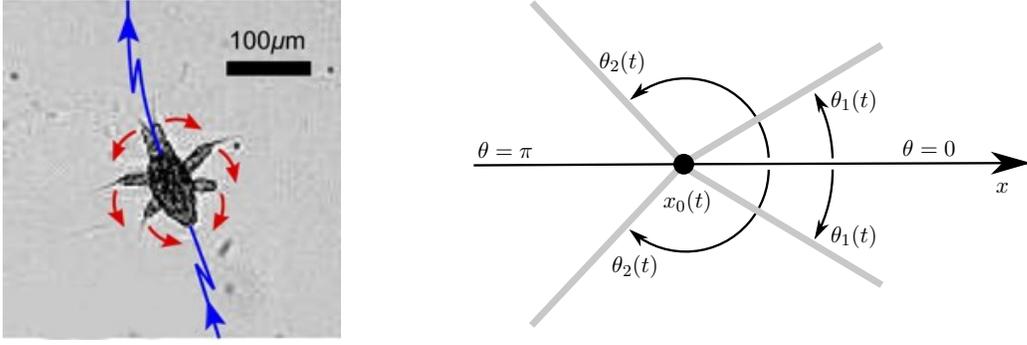


Figure 1: (left) Observation of a zooplankton. (right) Sketch of the 2-link symmetric swimmer.

Stokes theory relates the speed of the displacement variable  $x_0$  to the speed of the shape vector  $\theta = (\theta_1, \theta_2)$  by the equation

$$\dot{x}_0 = \sum_{i=1}^2 \frac{\dot{\theta}_i \sin(\theta_i)}{\Delta}, \quad \Delta = \sum_{i=1}^2 1 + \sin^2 \theta_i. \quad (1)$$

To parameterize the motion as a control system, one introduces the dynamics

$$\dot{\theta}_1 = u_1, \quad \dot{\theta}_2 = u_2.$$

It provides a control system written as

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \quad q = (x_0, \theta) \quad (2)$$

with state constraints  $\mathcal{T} : 0 \leq \theta_1 \leq \theta_2 \leq \pi$ . A *stroke*  $\gamma$  is a closed curve in the  $\theta$ -plane produced by a periodic control  $u(t)$ ,  $u = (u_1, u_2)$ .

To compare different strokes or different swimmers one introduced a metric in the shape variable. A choice of particular interest, in particular in relation with the concept of efficiency defined by [13], is the *mechanical energy dissipated by drag forces*:

$$E = \int_0^T \dot{q}^\top M(q) \dot{q} dt \quad (3)$$

where  $T$  is the period of the stroke and  $M$  is the matrix

$$M = \begin{pmatrix} 2 - 1/2(\cos^2 \theta_1 + \cos^2 \theta_2) & -1/2 \sin \theta_1 & -1/2 \sin \theta_2 \\ -1/2 \sin \theta_1 & 1/3 & 0 \\ -1/2 \sin \theta_2 & 0 & 1/3 \end{pmatrix} \quad (4)$$

Using (1) this amounts to minimize the quadratic form

$$E = \int_0^T (a(q)u_1^2 + 2b(q)u_1u_2 + c(q)u_2^2) dt$$

where

$$a(q) = \frac{1}{3} - \frac{\sin^2 \theta_1}{2\Delta(\theta)}, \quad b(q) = \frac{-\sin \theta_1 \sin \theta_2}{2\Delta(\theta)}, \quad c(q) = \frac{1}{3} - \frac{\sin^2 \theta_2}{2\Delta(\theta)}.$$

For geometric and numeric computations using continuation methods, a simplified metric is introduced that is the *euclidean (flat) metric* associated to quadratic cost

$$\int_0^T (u_1^2 + u_2^2) dt,$$

noting that in fine the qualitative analysis of the problem is rather robust with respect to the choice of the metric and the parameters of the metric can be fitted with the observations [12], this providing the solution to the *inverse optimal problem*.

Denoting by  $g$  such a metric, the length of a stroke  $\gamma = (\theta_1, \theta_2)$  is

$$L(\gamma) = \int_0^T g(\theta, \dot{\theta})^{1/2} dt$$

and our optimal control problem is rewritten, thanks to Maupertius principle, as a SR-problem:

$$\dot{q} = \sum_{i=1}^2 u_i F_i(q), \quad \min_{u(\cdot)} \int_0^T g(\theta, \dot{\theta}) dt$$

with appropriate boundary conditions associated to *periodic controls*;

$$\theta(0) = \theta(T),$$

with state constraints  $\mathcal{T} : 0 \leq \theta_1 \leq \theta_2 \leq \pi$  and  $x_0(T) - x_0(0)$  stands for the displacement of a stroke and we can set  $x_0(0) = 0$ . Using the energy minimization point of view, the period of a stroke can be fixed to  $T = 2\pi$ . Note also that the problem is equivalent to a time minimal control problem by parameterizing by arc-length (which experimentally will allow to compare different strokes or swimmers).

From the general point of view of SR-geometry, the problem amounts to compute for each  $q(0) = q_0$ , the SR-sphere  $S_{q_0}(r)$  with  $r$  radius  $r$ , formed by extremities of minimizers starting from  $q_0$  and fixed length and moreover requiring that the optimal control is periodic. This is equivalent to fix the displacement  $x_0(2\pi)$  ( $x_0(0) = 0$ ) and to compute strokes minimizing the length. Also in this geometric frame one can introduce the concept of *geometric efficiency* of a stroke  $\gamma$  that is the ratio

$$\mathcal{E} = x_0(2\pi)/L(\gamma) \tag{5}$$

which turns out to be equivalent to the concept of efficiency in fluid mechanics used in [17, 12] and defined by  $\mathcal{E}' = x_0(T)^2/E(\gamma)$ .

Note that more general concepts of efficiency can be used as in [9]  $\text{Eff} \sim x_0(T)^2 M_{11}(\gamma(0))/E(\gamma)$ , with  $M = (M_{ij})_{ij}$  defined in (4), which is depending upon  $\theta(0)$  and more generally any function  $h(\theta(0), x_0(T), E(\gamma))$ .

The main contribution of this article is that in the optimal control theory frame, one can construct a one parameter family of geodesic strokes for the metric defined by  $g$ , forming a foliation of the interior of the triangle  $\mathcal{T}$  which corresponds to a geodesic stroke, see Fig.2. Moreover an unique stroke of this family is the solution of maximizing the efficiency.

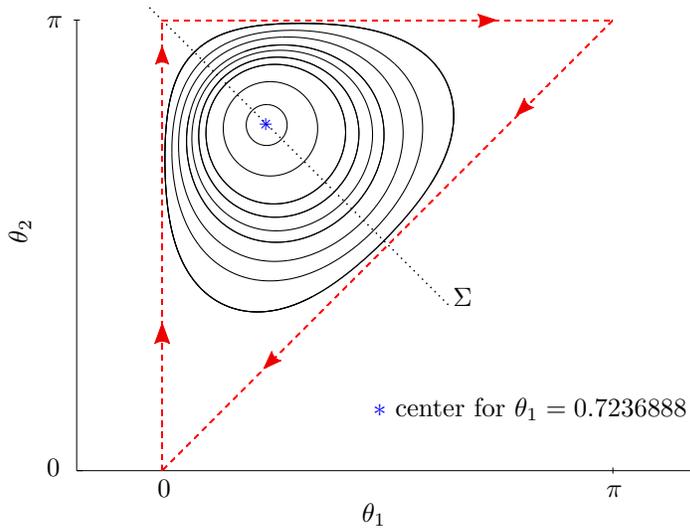


Figure 2: One parameter family of geodesic strokes for the euclidean cost.

This construction is based on the application of the Maximum Principle [15, 18] complemented by second order optimality conditions [4] and such curves are numerically evaluated using continuation and shooting methods implemented in the `HamPath` software [10].

Note such family started with strokes of *small amplitudes* emanating from a center  $C$  which can be evaluated by constructing graded normal form introduced in SR-geometry [3] and asymptotics of the corresponding displacement and length can be computed. Another asymptotics can be computed for larger amplitudes near the boundary of the triangle formed by the sequential paddling followed by the recovery stroke in unison. An interpretation of this triangle stroke is given in the frame of SR-optimality and non optimality is related to the corners, vs the physical interpretation in [17].

The article is organized in three sections. The first section contained a geometric analysis, in relation with SR-geometry and the important problem of computing geodesic strokes with small amplitudes, each providing an optimal solution the problem of minimizing the length, for a given solution (hence extremities belong to the SR-sphere), with small radius. The second section is a direct application of the maximum principle in the frame of periodic control, complemented by second order optimality conditions and numerical simulations to compute strokes in the problem of maximizing different concepts of efficiencies and gives an explanation to the problem of smoothness of the corresponding solution. A final section is devoted to the analysis of the optimality of the triangle abnormal stroke, to be related to the asymptotics of displacement and efficiencies in relation with the computation in [12].

## 2 A geometric analysis of the problem in the frame of SR-geometry and application to compute strokes with small amplitudes

### 2.1 Geodesic computations

They are presented in the case of the copepod swimmer but can be generalized to any swimmer in relation with the decomposition  $q = (\mathbf{q}_1, \mathbf{q}_2)$ :  $\mathbf{q}_1$  displacement variable,  $\mathbf{q}_2$ : shape variables, providing a SR-problem where the dimension of the nonholonomic constraints is given by:  $\dim \mathbf{q}_2 = \dim$  of number of controlled links (simplified copepod: 2 vs biological copepod: 3, Purcell: 2).

The optimal control problem related to energy minimization is written

$$\min \int_0^T L(q(t), u(t)) dt$$

over arcs  $q(\cdot)$  on  $[0, T]$  s.t.  $\dot{q}(t) = F(q(t), u(t)) = \sum_{i=1}^2 u_i F_i(q)$  a.e. and the class of admissible controls is the set  $\mathcal{U}$  of bounded measurable mapping valued in  $U = \mathbb{R}^2$ , and with the following boundary conditions: where  $T$  can be fixed to  $2\pi$  and the dynamics write:

$$F(q, u) = \left( u_1 \frac{\sin q_2}{2 + \sin^2 q_2 + \sin^2 q_3} + u_2 \frac{\sin q_3}{2 + \sin^2 q_2 + \sin^2 q_3}, u_1, u_2 \right).$$

Here, the state variables  $q = (x_0, \theta)$ ,  $\theta = (\theta_1, \theta_2)$  are the link angles and  $w_0$  stands for the displacement. The cost defined by a Riemannian metric on the shape variables:  $L(u_1, u_2) = a(\theta)u_1^2 + 2b(\theta)u_1u_2 + c(\theta)u_2^2$  which can be either the mechanical energy or the euclidean metric.

**Maximum principle in SR-geometry [15]** We introduce the pseudo-Hamiltonian  $p\dot{q} + p_0(a(q)u_1^2 + 2b(q)u_1u_2 + c(q)u_2^2)$ , where  $p_0$  is a constant and according to the Maximum principle, minimizers are found among *extremal curves*, which are solutions a.e. of the following equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = \frac{\partial H}{\partial q}, \quad \frac{\partial H}{\partial u} = 0.$$

This leads to consider the following *normal case*. Assume  $p_0 \neq 0$  and it can be normalized to  $p_0 = -1/2$ . Controls, solving  $\partial H/\partial u = 0$ , are given by

$$u_1 = \frac{cH_1 - bH_2}{ab - b^2}, \quad u_2 = \frac{aH_2 - bH_1}{ab - b^2}$$

where  $H_i = p \cdot F_i$  denotes  $F_i$ 's Hamiltonian lifts and plugging such controls in  $H$  yields the *normal Hamiltonian*

$$H_n = \frac{1}{2(ac - b^2)} (aH_2^2 - 2bH_1H_2 + cH_1^2). \quad (6)$$

The corresponding solutions  $z = (q, p)$  are called *normal extremals*.

*Abnormal case.* If  $p_0 = 0$ , additional extremals  $z = (q, p)$  appear and they are called *abnormal*. They are solutions of the implicit equations

$$H_1(z(t)) = H_2(z(t)) = 0 \quad (7)$$

and they can be computed by (time) differentiation.

Recall that if  $F, G$  are two smooth vector fields, the Lie bracket is computed as:

$$[F, G](q) = \frac{\partial F}{\partial q}(q)G(q) - \frac{\partial G}{\partial q}(q)F(q).$$

If  $H_F(z) = p \cdot F(q)$ ,  $H_G(z) = p \cdot G(q)$ , the Poisson bracket is

$$\{H_F, H_G\}(z) = dH_F(\vec{H}_G) = p \cdot [F, G](q)$$

with  $\vec{H}_G = \frac{\partial H_G}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H_G}{\partial q} \frac{\partial}{\partial p}$ .

Hence differentiating twice using  $\dot{z} = u_1 \vec{H}_F(z) + u_2 \vec{H}_G(z)$ , the abnormal controls are given by

$$\begin{aligned} H_1 = H_2 = \{H_1, H_2\} &= 0 \\ u_1 \{\{H_1, H_2\}, H_1\} + u_2 \{\{H_1, H_2\}, H_2\} &= 0 \end{aligned} \quad (8)$$

and they can be (generically) computed solving provided one Poisson bracket  $\{\{H_1, H_2\}, H_i\}$ ,  $i = 1, 2$ , is non zero.

**Definition 2.1.** *The exponential mapping is, for fixed  $q_0 = q(0)$  the map:  $\exp_{q_0}(t, p(0)) \mapsto \Pi(\exp(t\vec{H}_n(z(0))))$  where  $\Pi$  is the projection:  $(q, p) \rightarrow p$ . A projection of an extremal is called a geodesic. It is called strictly normal if it is the projection of a normal extremal but not an abnormal one. A time  $t_c$  is a conjugate time if the exponential mapping is not of full rank at  $t_c$  and  $t_{1c}$  denotes the first conjugate time and  $q(t_{1c})$  is called the first conjugate point along the reference geodesic  $t \mapsto q(t)$ .*

**Definition 2.2.** *Fixing  $q_0$ , the wave front  $W(q_0, r)$  is the set of extremities of geodesics (normal or abnormal) with length  $r$  and the sphere  $S(q_0, r)$  is the set of extremities of minimizing geodesics. The conjugate locus  $C(q_0)$  is the set of first conjugate points of normal geodesics starting from  $q_0$  and the cut locus  $C_{cut}(q_0)$  is the set of points where geodesics cease to be optimal.*

**Definition 2.3.** *According to the previous definitions, a stroke is called (strictly) normal if it is a (strictly) normal geodesic with periodic control while an abnormal stroke is a piecewise smooth abnormal geodesic with periodic control.*

## 2.2 Computation in the copepod case

One has

$$F_i = \frac{\sin(\theta_i)}{\Delta} \frac{\partial}{\partial x_0} + u_i \frac{\partial}{\partial \theta_i}, \quad i = 1, 2.$$

with  $\Delta = \sum_{i=1}^2 (1 + \sin^2(\theta_i))$ . We get

$$[F_1, F_2](q) = \tilde{f}(\theta_1, \theta_2) \frac{\partial}{\partial x_0}$$

with

$$\tilde{f}(\theta_1, \theta_2) = 2 \sin(\theta_1) \sin(\theta_2) (\cos(\theta_1) - \cos(\theta_2)) / \Delta^2(\theta)$$

Furthermore,

$$[[F_1, F_2], F_i] = \frac{\partial \tilde{f}}{\partial \theta_i} \frac{\partial}{\partial x}, \quad i = 1, 2.$$

and we have simple formulas to generate all Lie brackets.

**Definition 2.4.** *A point  $q_0$  is called a Darboux or a contact point if at  $q_0$   $F_1, F_2$  and  $[F_1, F_2]$  are linearly independent and a Martinet point if  $F_1, F_2$  are linearly independent,  $[F_1, F_2] \subset \text{span}\{F_1, F_2\}$  but at least for one  $i = 1, 2$ ,  $[[F_1, F_2], F_i] \notin \text{span}\{F_1, F_2\}$ .*

According to this terminology and Lie brackets computations, we have

**Proposition 2.5.** 1. All interior points of the triangle  $\mathcal{T} : 0 \leq \theta_1 \leq \theta_2 \leq \pi$  are contact points.

2. The sides of the triangle (vertices excluded) are Martinet points and the triangle  $\mathcal{T}$  is a (piecewise smooth) abnormal strokes.

**Geometry comment.** Hence the observed stroke by Takagi [17] of sequential paddling followed by a recovery stroke in unison corresponding to the policy:  $\theta_2 : 0 \rightarrow \pi$ ,  $\theta_1 : 0 \rightarrow \pi$ ,  $\theta_i \rightarrow 0$ ,  $i = 1, 2$  with  $\theta_1 = \theta_2$  where the copepod swimmer followed the triangle  $\mathcal{T}$  boundary of the physical domain is the unique abnormal stroke.

Moreover it has a nice geometric interpretation using Stokes' theorem and curvature control methods.

**Lemma 2.6.** One has:

$$1. \oint_{\partial D} \sum_{i=1}^2 \frac{\sin(\theta_i)}{\Delta} d\theta_i = \int_D \left[ \frac{\partial}{\partial \theta_2} \left( \frac{\sin(\theta_1)}{\Delta} \right) - \frac{\partial}{\partial \theta_1} \left( \frac{\sin(\theta_2)}{\Delta} \right) \right] d\theta_1 \wedge \theta_2 = \int_D d\omega.$$

2. The points where  $d\omega = 0$  are precisely the abnormal triangle, and  $d\omega < 0$  in the interior domain and  $d\omega > 0$  in the exterior.

## 2.3 SR-classification in dimension 3 and strokes with small amplitudes for the copepod swimmer

In the geometric approach of SR-geometry, the problem of computing strokes with small amplitudes, is precisely the problem of computing the points of the SR-sphere with small radius associated to periodic normal strokes. The key notion for computing such spheres is the concept of *privileged coordinates*, see [3, 11], forming near each point a set of graded coordinates. The concept of nilpotent or model of order  $-1$  allows to get estimates of the SR-balls with small radii and more precise approximations have to be used, especially to evaluate the conjugate locus. In particular, it was developed in [1, 2] and [6, 5] respectively in the contact and Martinet case. Those computations are recalled next, and are crucial in our analysis.

### 2.3.1 The contact case.

The SR-problem is written as  $(D, g)$ , where  $D$  is a two-dimensional distribution:  $D = \text{span}\{F_1, F_2\}$  and  $g$  is a SR-metric. If  $D$  is a contact distribution, near a point  $q_0 \in \mathbb{R}^3$  identified to 0, one has the following.

- **Heisenberg-Brockett nilpotent model.** The nilpotent model of order  $-1$  is the so-called Heisenberg-Brockett model introduced in [7] where the SR-problem is defined by the orthonormal frame  $D = \text{span}\{\hat{F}, \hat{G}\}$

$$\hat{F} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

with  $q = (x, y, z)$  and the weights of  $x, y$  are one and the weight of  $z$  is two, forming a set of privileged coordinates.

- **Generic model.** Using this gradation, the normal form of order 0 is similar and the generic model is given by the normal form of order 1

$$F = \hat{F} + yQ(w) \frac{\partial}{\partial z}, \quad G = \hat{G} - xQ(w) \frac{\partial}{\partial z},$$

$w = (x, y)$  and  $Q$  is quadratic in  $w$ ,  $Q = \alpha x^2 + 2\beta xy + \gamma y^2$ , where  $\alpha, \beta, \gamma$  are parameters.

We introduce  $\tilde{Q}(w) = (1 + Q(w))$  and one can write

$$F = \frac{\partial}{\partial x} + y\tilde{Q} \frac{\partial}{\partial z}, \quad G = \frac{\partial}{\partial y} - x\tilde{Q} \frac{\partial}{\partial z}.$$

Computing, one has

$$[F, G] = (2\tilde{Q} + x \frac{\partial Q}{\partial x} + y \frac{\partial Q}{\partial y}) \frac{\partial}{\partial z}$$

and using Euler formula

$$[F, G] = 2(1 + 2Q) \frac{\partial}{\partial z} = f(w) \frac{\partial}{\partial z}.$$

**Geodesics equations.** They are computed in this approximation, using Poincaré coordinates associated with the frame  $(F, G, \frac{\partial}{\partial z})$ :

$$H_1 = p \cdot F, \quad H_2 = p \cdot G, \quad H_3 = p \cdot \frac{\partial}{\partial z}.$$

The normal Hamiltonian is

$$H_n = 1/2(H_1^2 + H_2^2).$$

One has

$$\dot{H}_1 = dH_1(\vec{H}_n) = \{H_1, H_2\} H_2 = (p \cdot [F, G]) H_2 = p_z f(w) H_2$$

with  $p_z = H_3$  is a constant (isoperimetric situation).

$$\dot{H}_2 = dH_2(\vec{H}_n) = -\{H_1, H_2\} H_1 = -p_z f(w) H_1.$$

Since  $f(w) = 2 + O(|w|^2)$  and for  $p_z$  non zero, we introduce the parameterization

$$ds = p_z h(w) dt. \tag{9}$$

Denoting  $\phi'$  the derivative of the a function  $\phi$  with respect to  $s$ , we get

$$H'_1 = H_2, \quad H'_2 = -H_1$$

**Lemma 2.7.** *In the  $s$ -parameter, the normal controls are solutions of the linear pendulum equation  $H''_1 + H_1 = 0$  and are trigonometric functions.*

**Geometric comments.** This calculation confirms up to reparameterization the choice of sinusoidal paddling in [17].

The remaining equation to be integrated using (2.1) are

$$x' = \frac{H_1}{p_z h(w)}, \quad y' = \frac{H_2}{p_z h(w)}, \quad z' = \frac{(H_1 y - H_2 x) \tilde{Q}}{p_z h(w)}. \tag{10}$$

The solution can be estimated with *micro-local expansions*, see [1, 5] for the problem of computing conjugate points. *Note that the weights are similar to analyze periodicity.*

**Heisenberg-Brockett solution.** For the model of order  $-1$ , the geodesics equation can be computed, thanks to Liouville integrability and we recall the following [7].

**Lemma 2.8.** *The solution starting from  $q(0) = 0$  is parameterized by*

$$x(t) = \frac{A}{\lambda} (\sin(\lambda t + \varphi) - \sin(\varphi)) \tag{11}$$

$$y(t) = \frac{A}{\lambda} (\cos(\lambda t + \varphi) - \cos(\varphi)) \tag{12}$$

$$z(t) = \frac{A^2}{\lambda} t - \frac{A^2}{\lambda^2} \sin(\lambda t) \tag{13}$$

where  $A, \lambda, \varphi$  are parameters defined by  $p(0)$ .

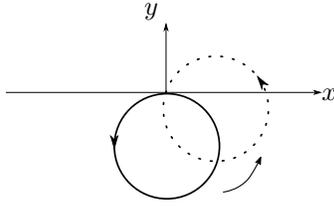


Figure 3: One parameter family of circles which are the geodesics of the Heisenberg-Brockett problem.

**Relation with the copepod problem.** In relation with the swimmer problem, some information can be obtained taking  $(x, y)$  as the shape variables. Indeed projecting the geodesics flows, we get a one parameter family of circles on each energy level associated to  $A$ , each of them deduced by a proper rotation  $R_\alpha$  along the  $z$ -axis, associated to a symmetry of the SR-model, see Fig.3.

The associated displacement is given by

$$z = \oint (ydx - xdy) = \int d\omega$$

where  $d\omega = 2(dy \wedge dx)$  is proportional to the standard  $\mathbb{R}^2$ -volume form.

The  $z$  variable cannot be identified to the displacement variable due to the following.

**Lemma 2.9.** *After a period  $t = 2\pi/\lambda$ ,  $z(2\pi/\lambda) = A^2 2\pi/\lambda^2$  but one has  $\dot{z} > 0$  and hence  $z$  is always increasing.*

Since a property of the problem is that for the observed motion the displacement variable is always such that for each stroke is divided into intervals for which the displacement variable is alternatively increasing or decreasing. To identify the displacement variable for strokes of small amplitude the response is given by the privileged coordinates. The analysis goes as follows.

We choose a point  $\theta(0) = (\theta_1(0), \theta_2(0))$  in the interior of the triangle identified to 0 using the translation

$$x = \theta_1 - \theta_1(0), \quad y = \theta_2 - \theta_2(0)$$

and using Taylor expansion one has the following expansion

$$\dot{z} = \frac{u_1 \sin \theta_1 + u_2 \sin \theta_2}{2 + \sin^2 \theta_1 + \sin^2 \theta_2} = c_1 u_1 + c_2 u_2 + o(1)$$

where  $c_1, c_2$  are constants. Using  $\dot{x} = u_1, \dot{y} = u_2$  we set

$$Z = z - c_1 x - c_2 y \tag{14}$$

to get an equation of the form

$$\dot{Z} = u_1 \alpha_1(w) + u_2 \alpha_2(w)$$

with  $w = (x, y)$ ,  $\alpha_i = o(1)$ . Such normalizations being for simplicity perform on the euclidean model, but they are similar for the mechanical energy.

Hence, we proved

**Lemma 2.10.**  *$q = (x, y, Z)$  will form near  $(0, \theta_1(0), \theta_2(0))$  a graded system of coordinates with respective weights  $(1, 1, 2)$ , thus providing a relation between the physical coordinates and the normal coordinates.*

Still, the nilpotent model is not suitable to make a neat analysis, due to the symmetry of revolution described in Fig.3, where every projection on the  $(x, y)$ -plane is periodic, and this property is not stable by perturbation. Hence to get further analysis one has to use higher order approximations. A natural approach is to use the construction of the normal form of order zero, noting that in this construction one mixed the shape and displacement variables and moreover one fixes the gauge of the rotation  $R_\alpha$ .

## 2.4 Computations of the normal form of order 1

We consider the following vector fields

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial x} + (a_{30}^0 x^3 + a_{21}^0 x^2 y + a_{12}^0 x y^2 + a_{03}^0 y^3 + a_{20}^0 x^2 + a_{11}^0 x y + a_{02}^0 y^2 + a_{10}^0 x \\
&\quad + a_{01}^0 y + a_{00}^0) \frac{\partial}{\partial z} \\
F_2 &= \frac{\partial}{\partial y} + (-a_{30}^0 y^3 - a_{21}^0 x y^2 - a_{12}^0 x^2 y - a_{03}^0 x^3 + a_{20}^0 y^2 + a_{11}^0 x y + a_{02}^0 x^2 - a_{10}^0 y \\
&\quad - a_{01}^0 x + a_{00}^0) \frac{\partial}{\partial z}.
\end{aligned} \tag{15}$$

We apply at each step, an action on the vector fields as described in the algorithm of [1]. We denote  $(x, y, z)$  the old variables and  $(X, Y, Z)$  the new variables and we rewrite the new vector fields as  $F_1(x, y, z), F_2(x, y, z)$ .

Step 1. We remove the terms  $x^i \partial/\partial z, i \geq 0$  in  $F_1$  and the terms  $y^i \partial/\partial z, i \geq 0$  in  $F_2$  by a change of coordinates  $\varphi_1$  such that

$$\begin{aligned}
\varphi_1(x, y, z) &= (x, y, z - 1/4 a_{30}^0 x^4 - 1/3 a_{20}^0 x^3 - 1/2 a_{10}^0 x^2 - a_{00}^0 x \\
&\quad + 1/4 a_{03}^0 y^4 + 1/3 a_{02}^0 y^3 + 1/2 a_{01}^0 y^2 - a_{00}^0 y)
\end{aligned}$$

and the resulting vector fields are written as

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial x} + (a_{21}^1 x^2 y + a_{12}^1 x y^2 + a_{03}^1 y^3 + a_{11}^1 x y + a_{02}^1 y^2 + y/2) \frac{\partial}{\partial z} \\
F_2 &= \frac{\partial}{\partial y} + (-a_{21}^1 x y^2 - a_{12}^1 x^2 y - a_{03}^1 x^3 + a_{11}^1 x y + a_{02}^1 x^2 - x/2) \frac{\partial}{\partial z}
\end{aligned}$$

Step 2. To remove the terms of order 0 in  $F_1, F_2$ , a change of coordinates  $\varphi_2$  is performed where

$$\begin{aligned}
\varphi_2(x, y, z) &= [x + c_{300} x^3 + c_{210} x^2 y + c_{120} x y^2 + c_{030} y^3 + c_{110} (x y + y^2) + c_{011} y z + c_{001} z, \\
&\quad y + c_{030} x^3 + c_{120} x^2 y + c_{210} x y^2 + c_{300} y^3 - c_{110} (x y + x^2) - c_{011} x z + c_{001} z, \\
&\quad z + d_{300} x^3 + d_{210} x^2 y + d_{120} x y^2 + d_{030} y^3 + d_{101} x z + d_{011} y z],
\end{aligned} \tag{16}$$

followed by the feedback

$$u_1 \leftarrow u_1 + u_2 (2c_{001} x + 2c_{001} y), \quad u_2 \leftarrow u_2 - u_1 (2c_{001} x + 2c_{001} y).$$

The resulting vector fields are written as

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial x} + (a_{30}^2 x^3 + a_{21}^2 x^2 y + a_{12}^2 x y^2 + a_{03}^2 y^3 + y/2) \frac{\partial}{\partial z} \\
F_2 &= \frac{\partial}{\partial y} + (-a_{30}^2 y^3 - a_{12}^2 x^2 y - a_{21}^2 x y^2 - a_{03}^2 x^3 - x/2) \frac{\partial}{\partial z}
\end{aligned}$$

Step 3. We remove the terms  $a_{30}^3 x^3 \partial/\partial z$  and  $a_{12}^3 x y^2 \partial/\partial z$  in  $F_1$  and the terms  $-a_{30}^3 y^3 \partial/\partial z$  and  $-a_{12}^3 x^2 y \partial/\partial z$  in  $F_2$  by the change of coordinates

$$\varphi_3(x, y, z) = (1/\sqrt{2} x - 1/\sqrt{2} y, 1/\sqrt{2} x + 1/\sqrt{2} y, z - a_{30}^3/4 x^4 + a_{30}^3/4 y^4)$$

followed by the feedback

$$u_1 \leftarrow 1/\sqrt{2} u_1 + 1/\sqrt{2} u_2, \quad u_2 \leftarrow -1/\sqrt{2} u_1 + 1/\sqrt{2} u_2.$$

Therefore, we obtain

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial x} + ((-a_{21}^2/2 + a_{12}^2/2 + 3/2 a_{03}^2) x^2 y + (a_{21}^2/2 - a_{12}^2/2 + a_{03}^2/2) y^3 + y/2) \frac{\partial}{\partial z} \\
F_2 &= \frac{\partial}{\partial y} + ((a_{21}^2/2 + a_{12}^2/2 - 3/2 a_{03}^2) x y^2 + (-a_{21}^2/2 - a_{12}^2/2 - a_{03}^2/2) x^3 - x/2) \frac{\partial}{\partial z}
\end{aligned}$$

Step 4. We normalize the terms of order 1 with a quadratic form  $Q$  such that  $\partial^2 Q / \partial(xy)^2 = 0$  by the diffeomorphism

$$\varphi_4(x, y, z) = (x, y, z - 1/4(a_{03}^2 - a_{21}^2) x^3 y - 1/4(a_{21}^2 - a_{03}^2) xy^3)$$

$$F_1 = \frac{\partial}{\partial x} + \frac{y}{2} (1 + Q(x, y)) \frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial y} - \frac{x}{2} (1 + Q(x, y)) \frac{\partial}{\partial z} \quad (17)$$

where

$$Q(x, y) = (1/2 a_{21}^2 + a_{12}^2 + 3/2 a_{03}^2) x^2 + (1/2 a_{21}^2 - a_{12}^2 + 3/2 a_{03}^2) y^2$$

**Application.** We apply this algorithm for the copepod model where we consider the initial point on the axis of symmetry  $\Sigma$  (see Fig.2)  $\theta_1(0) = \theta_{10}$ ,  $\theta_2(0) = \pi - \theta_{10}$  and the computation is for the euclidean model.

One sets:  $x = \theta_1 - \theta_{10}$ ,  $y = \theta_2 + \theta_{10} - \pi$  to compute the expression near  $(x, y) \sim (0, 0)$  and we denote  $z$  the displacement variable. Computing the Taylor expansion of  $F_1, F_2$  near 0 up to order 4, the resulting vector fields can be put in the form of (15).

In this step 2, the important point is to keep track of the mixing of the shape variables with the displacement variable. Up to order 2, we have the following.

**Lemma 2.11.** *After step 2, up to order 2 with respect to the gradation (1, 1, 2) the relation between the new coordinates and Heisenberg coordinates is given by*

$$X = \varphi_2^1(x, y, z), \quad Y = \varphi_2^2(x, y, z) \quad (18)$$

where  $\varphi_2^j$  is the  $j$ -th component of  $\varphi_2$ . In particular, this transformation mixed the shape and the displacement variable.

Lemma 2.11 is the key tool to solve the periodicity equations on the model of order 0.

$$\theta(0) = \theta(2\pi), \quad \dot{\theta}(0) = \dot{\theta}(2\pi).$$

**Definition 2.12.** *Let  $\mathcal{G}$  the subgroup of diffeomorphisms  $\varphi$  of  $\mathbb{R}^3$  defined by  $\varphi(\theta, z) = (\psi_1(\theta), \psi_2(\theta, z))$  where  $\psi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $\psi_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ .*

**Proposition 2.13.** *The center of swimmings is on  $\Sigma$ , obtained by restricting  $\varphi_2$  to  $\mathcal{G}$  i.e. taking  $c_{001} = c_{011} = 0$  where  $c_{001}, c_{011}$  are defined in (16). For the copepod, the computations yield*

$$c_{001}(\theta_{10}) = 1/2 \frac{(\cos(\theta_{10}))^4 + 3(\cos(\theta_{10}))^2 - 2}{((\cos(\theta_{10}))^2 - 2) \sin(\theta_{10}) \cos(\theta_{10})}$$

$$c_{011}(\theta_{10}) = 1/2 \frac{(\cos(\theta_{10}))^8 + 6(\cos(\theta_{10}))^6 + 5(\cos(\theta_{10}))^4 - 12(\cos(\theta_{10}))^2 + 4}{((\cos(\theta_{10}))^6 - 5(\cos(\theta_{10}))^4 + 8(\cos(\theta_{10}))^2 - 4)(\cos(\theta_{10}))^2}$$

and we check numerically that these two coefficients  $c_{001}$  and  $c_{011}$  are both zero for one value of  $\theta_{10} \in [-\pi, \pi]$  which is  $\theta_{10}^* \simeq 0.7236888$ . Moreover  $(\theta_{10}^*, \pi - \theta_{10}^*)$  corresponds to the center of the family of simple strokes presented in Fig.2.

Finally, for the copepod model, the quadratic form in the normal form (17) is

$$Q(x, y) = 1/8 \frac{(-11(\cos(\theta_{10}))^6 + 17(\cos(\theta_{10}))^4 - 8(\cos(\theta_{10}))^2 + 4)x^2}{((\cos(\theta_{10}))^6 - 5(\cos(\theta_{10}))^4 + 8(\cos(\theta_{10}))^2 - 4)(\cos(\theta_{10}))^2}$$

$$+ 1/8 \frac{(5(\cos(\theta_{10}))^8 - 33(\cos(\theta_{10}))^6 + 2(\cos(\theta_{10}))^4 + 28(\cos(\theta_{10}))^2 - 8)y^2}{((\cos(\theta_{10}))^6 - 5(\cos(\theta_{10}))^4 + 8(\cos(\theta_{10}))^2 - 4)(\cos(\theta_{10}))^2}$$

and at the center point  $(0.7236888, \pi - 0.7236888)$ , we have

$$Q(x, y) = -0.7165898586 x^2 - 0.7379854942 y^2.$$

**Proposition 2.14.** *The only stroke that can be computed with the vector fields  $F_1, F_2$  given by the normal form (17) is reduced to the point  $(\theta_{10}^*, \pi - \theta_{10}^*)$  with  $\theta_{10}^* \simeq 0.7236888$  which correspond to the center of the family of simple strokes obtained with the true system (2) presented in Fig.2.*

**Remark 2.15.** *The shape variables and the displacement variable are mixed only in step 2. Therefore, when this algorithm is applied at the center  $(\theta_{10}^*, \pi - \theta_{10}^*)$ , we use only diffeomorphisms of  $\mathcal{G}$ .*

#### 2.4.1 The Martinet case

The analysis at a Martinet point  $q_0$  not belonging to the vertices of the triangle  $\mathcal{T}$  is more intricate and as before relies to the construction of a generic model, which in the case of order 0.

**Martinet model of order 0.** The point  $q_0$  is identified to zero and there exists local coordinates  $q = (x, y, z)$  such that the SR-geometry is given by  $(D, g)$  where:

- $D = \text{span}\{F, G\}$ ,  $F = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}$  and  $G = \frac{\partial}{\partial z}$  with  $q = (x, y, z)$  are graded coordinates with respective weights  $(1, 1, 3)$ .
- The metric  $g$  is of the form  $a(q)dx^2 + b(q)dy^2$  where we have
  - Model of order  $-1$ . (Flat Martinet case):  $a = c = 1$
  - Generic model:

$$a = (1 + \alpha y)^2 \sim 1 + 2\alpha y \text{ (order zero)}$$

$$c = (1 + \beta x + \gamma y)^2 \sim 1 + 2\beta x + 2\gamma y \text{ (order zero)}$$

where  $\alpha, \beta, \gamma$  are parameters and the square in the formula is related to the crucial computation presented next.

**Geodesic equations.** We introduce the orthonormal frame

$$F_1 = \frac{F}{\sqrt{a}}, \quad F_2 = \frac{G}{\sqrt{c}}, \quad F_3 = \frac{\partial}{\partial z}$$

and denoting  $H_i = p \cdot F_i$ , the normal Hamiltonian is given by  $H_n := 1/2(H_1^2 + H_2^2)$ .

We parameterize by arc-length:  $H_1^2 + H_2^2 = 1$ ,  $H_1 = \cos \chi$ ,  $H_2 = \sin \chi$  and  $H_3 = p_z = \lambda$  constant, assuming  $\lambda \neq 0$ . Hence the geodesic equations become

$$\begin{aligned} \dot{x} &= \frac{\cos \chi}{\sqrt{a}}, \quad \dot{y} = \frac{\sin \chi}{\sqrt{b}}, \quad \dot{z} = \frac{y^2 \cos \chi}{2\sqrt{a}}, \\ \dot{\chi} &= \frac{1}{\sqrt{ac}}(y\lambda - \alpha \cos \chi - \beta \sin \chi) \end{aligned} \tag{19}$$

We introduce the new parameterization

$$\sqrt{ac} \frac{d}{dt} = \frac{d}{ds} \tag{20}$$

and denoting by  $\phi'$  the derivative of a function  $\phi$  with respect to  $s$ , we get the equations

$$y' = \sin \chi(1 + \alpha y), \quad \chi' = (y\lambda - \alpha \cos \chi - \beta \sin \chi)$$

and the second order differential equation

$$\chi'' + \lambda \sin \chi + \alpha^2 \sin \chi \cos \chi - \alpha\beta \sin^2 \chi + \beta\chi' \cos \chi = 0. \tag{21}$$

As a consequence, we obtain the following result.

**Proposition 2.16.** *The generic case projects, up to a time reparameterization, onto a two dimensional equation (21), associated with a generalized dissipative pendulum depending on the parameters  $\alpha, \beta$  only.*

**Geometric application** This pendulum equation describing the dynamics of the controls  $u_1 = H_1 = \sin \chi$ ,  $u_2 = H_2 = \cos \chi$  describes the main properties of the generic model of order 0.

- *Flat case:*  $\alpha = \beta = \gamma = 0$ , it corresponds to the standard pendulum which is integrable using elliptic functions.
- *Generic case:*
  - $\beta = 0$ : it corresponds to the integrable case with the existence of an additional first integral, linear in the momentum and identified to  $p_x$ . Moreover using (19) again, the equations can be integrated using elliptic functions.
  - $\beta \neq 0$ : due to dissipation, we are in the non-integrable case.

An additional property being:

**Lemma 2.17.** *The abnormal line identified to  $t \rightarrow (t, 0, 0)$  is strict if and only if  $\alpha \neq 0$ .*

**Application to the copepod.** For the copepod case, previous computations tell us that the sides of the triangle  $\mathcal{T}$  are strict and periodic strokes can appear for different values of the modulus  $k$  of the elliptic functions and they are:

- $k = 0$ : circles,
- $k \simeq 0.65$ : eight shapes called Bernoulli lemniscates.

Note that in this analysis  $\beta = 0$  is not a stable model and moreover the triangle constraint  $\mathcal{T} : 0 \leq \theta_1 \leq \theta_2 \leq \pi$ .

### 2.4.2 Numerical simulations.

Numerical simulations using the `HamPath` software are presented on Fig.4 based on the geometric analysis and to achieved the computations of strokes corresponding respectively to simple loops, limaçons and eight shapes curves. Note that they are performed using the true model.

- *Simple loops.* They are generated by perturbations of simple loops, by period doubling.
- *Eight-shapes.* They appear only on the sides of the triangle.

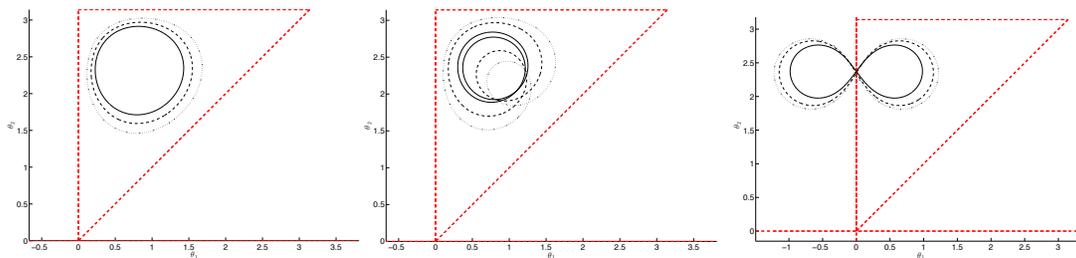


Figure 4: One parameter family of simple loops, limaçons and Bernoulli lemniscates normal strokes for the euclidean metric.

## 3 A powerful approach of the problem using optimal control theory and numerical simulations

In this section we use the general tools of optimal control: Maximum principle, necessary second order optimality conditions and numerical simulation with the `HamPath` software [10] to make a complete analysis of the problem.

### 3.1 The Maximum principle with periodic controls

First of all, one needs a neat statement of the *transversality conditions* suitable to analyze the maximizing different concepts of efficiencies with periodic control and we use [18]. The problem is written in the extended state space  $\tilde{q} = (q, q^0)$ .

$$\begin{cases} \min_{u(\cdot) \in \mathcal{U}} b(\tilde{q}(0), \tilde{q}(2\pi)) \\ \dot{q} = \sum_{i=1}^2 u_i F_i(q) = F(q, u), \quad u \in \mathbb{R}^2 \\ \dot{q}^0 = \sum_{i=1}^2 u_i^2, \quad q^0(0) = 0. \end{cases} \quad (22)$$

with an end-points conditions of the form  $(\tilde{q}(0), \tilde{q}(2\pi)) \in C$ , where  $C \subset \mathbb{R}^3 \times \mathbb{R}^3$  is a given closed set.

We denote  $\bar{p} = (p, p_0)$  the extended adjoint vector. The pseudo- Hamiltonian takes the form

$$H(\bar{q}, \bar{p}, u) = \sum_{i=1}^2 u_i H_i + p_0(u_1^2 + u_2^2)$$

and  $H_i = p \cdot F_i(q)$ .

From the maximum principle (cf. [18]), an optimal control pair  $(q, u)$  satisfies the following necessary conditions that we split into two distinct parts

**Standard conditions.**

$$\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}}, \quad \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}}, \quad \frac{\partial H}{\partial u} = 0.$$

and the transversality conditions

$$(\bar{p}(0), -\bar{p}(2\pi)) \in \lambda \nabla h(\bar{q}(0), \bar{q}(2\pi)) + N_C(\bar{q}(0), \bar{q}(2\pi)) \quad (23)$$

where  $N_C$  is the (limiting) normal cone to the (closed) set  $C$ ,  $(\bar{p}, \lambda) \neq 0$ ,  $\lambda \geq 0$ .

**Application.**  $q = (x_0, \theta_1, \theta_2)$

- Maximizing the geometric efficiency with periodic condition

$$\theta(0) = \theta(2\pi), \quad h = -\frac{x_0^2(2\pi)}{E}$$

where  $E$  is the mechanical energy

$$E = \int_0^{2\pi} (u_1^2 + u_2^2) dt.$$

In this case, from (23) we deduce

**Lemma 3.1.** *Denoting  $p_\theta = (p_{\theta_1}, p_{\theta_2})$  we have:  $p_\theta(0) = p_\theta(2\pi)$ , to produce a smooth stroke in the normal case  $p_0 \neq 0$ . Moreover at the final point  $(p_{x_0}, p_0)$  has to be collinear to the gradient to the level set  $h(x_0, x^0) = c$  where  $c$  is the minimum of the cost.*

- Maximizing an efficiency depending on  $\theta(0)$ , with periodic condition:

$$\theta(0) = \theta(2\pi), \quad h = -\frac{m(\theta(0)) x_0^2(2\pi)}{E}$$

where  $m$  is a chosen smooth function.

In this case (23) leads to

$$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial h}{\partial \theta(0)} \quad (24)$$

hence producing a *jump* of the adjoint vector.

### 3.2 Second-order necessary condition

It is the standard necessary optimality related to the non existence of conjugate point, see [6] or [20].

**Proposition 3.2.** *Let  $(x_0(t), \theta(t))$ ,  $t \in [0, 2\pi]$  be a strictly normal stroke. Then a necessary optimality condition is the non existence of conjugate time  $t_c \in ]0, 2\pi[$ .*

It can be checked numerically using the `HamPath` code.

### 3.3 Applications: numerical simulations

We present a sequence of simulations using the `HamPath` software. They concern both the euclidean cost  $\int_0^{2\pi} (u_1^2 + u_2^2) dt$  or the mechanical energy (3).

- Fig.5: Different kind of normal strokes for the mechanical cost: simple loop, limaçon, eight and computations of conjugate points. Only simple loops are candidates to be optimal strokes.
- One-parameter family of simple strokes, each of them associated with a different energy. The corresponding efficiency is represented in Table 1 for the euclidean cost and in Table 2 for the mechanical energy. They are compared with the efficiency of the abnormal stroke and we deduce the normal stroke with the maximum efficiency.

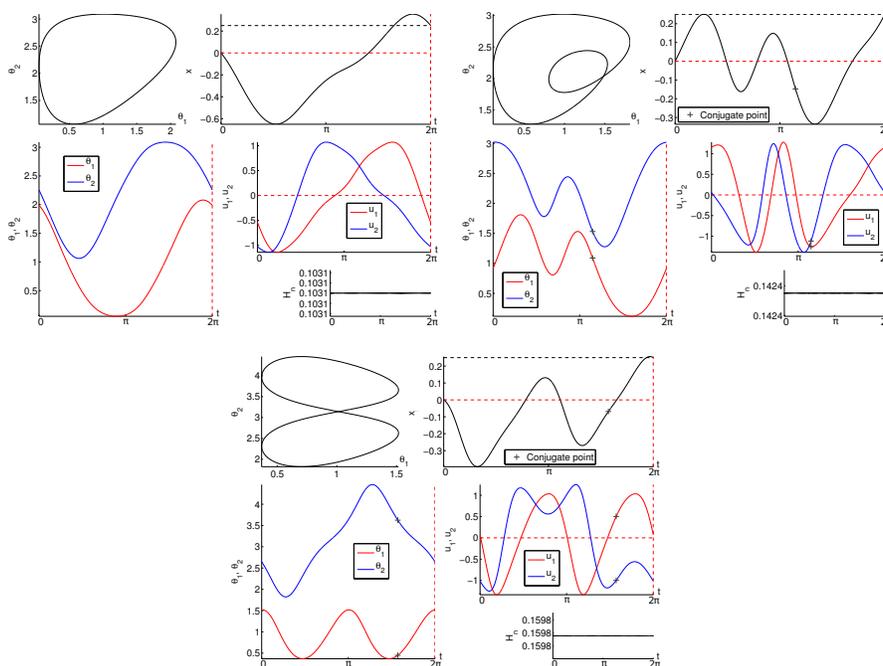


Figure 5: Normal strokes: simple loop (left), limaçon with inner loop (right) and eight case, and a two self-intersecting case (bottom). First conjugate points on  $[0, 2\pi]$  appear with a cross except for the simple loop stroke.

A remarkable property is the following

**Property 3.3.** *in the euclidean case, the one-parameter family of simple loops is symmetric with respect to the straight line  $\Sigma : \theta_2 = \pi - \theta_1$ , see Fig.2 and corresponds to an adjoint vector  $p(0)$  transverse to  $\Sigma$ .*

Moreover from this numeric computation, we observe that the *abnormal triangle stroke*, has a very low efficiency and in particular is *not optimal*.

*Indeed, we can produce the same displacement by doing twice a normal stroke, with similar energy.*

**Remarks about algorithms.** Two algorithms can be used for the problem of efficiency in the smooth case, both implementable in the `HamPath` code.

Method 1. The displacement is fixed to  $x_0(2\pi) = x_T$  and the one parameter family is computed solving the shooting equation

$$\theta(0) = \theta(2\pi), \quad p_\theta(0) = p_\theta(2\pi)$$

to generate the one parameter family  $\mathcal{F}_\lambda$  of periodic stroke initiated from  $\theta_0, p_0$  with  $\lambda = 0$  and the one parameter family is generated by continuation. Maximizing the efficiency is obtained by using the Jacobi equation (also called variational equation) to compute the derivative of the mapping  $\lambda \rightarrow h(\exp 2\pi H_n(\theta(\lambda), p_\theta(\lambda)))$ . Note that this algorithm is parent to the method of computing conjugate points. This amounts to use Table (2) or Table (1) for computing the maximum efficiency by derivation.

Method 2. A second method is to use the Maximum principle and the transversality condition (23) and to solve the associated shooting equation.

Both methods have been used in the smooth case, providing the same solutions.

**Non smooth case.** As in [8], we consider a cost depending upon  $\theta(0)$ , namely

$$h = -\frac{x_0^2(2\pi)m(\theta(0))}{E}$$

where  $m(\theta_1(0)) = 2 - \cos^2(\theta_1(0))$ . In Fig.7 is illustrated the corresponding optimal solution satisfying the transversality conditions (23). It is a non smooth stroke and it can be compared to the previous smooth solution associated to geometric efficiency.

Types of $\gamma$	$x_0(T)$	$L(\gamma)$	$x_0(T)/L(\gamma)$
Simple loops	$5.500 \times 10^{-2}$	1.984	$2.520 \times 10^{-2}$
	$1.400 \times 10^{-1}$	3.785	$3.698 \times 10^{-2}$
	$1.700 \times 10^{-1}$	4.340	$3.917 \times 10^{-2}$
	$2.000 \times 10^{-1}$	4.946	$4.043 \times 10^{-2}$
	$2.100 \times 10^{-1}$	5.109	$4.110 \times 10^{-2}$
<b>Optimal stroke</b> Fig.6 (left)	<b><math>2.169 \times 10^{-1}</math></b>	<b>5.180</b>	<b><math>4.187 \times 10^{-2}</math></b>
	$2.200 \times 10^{-1}$	5.354	$4.109 \times 10^{-2}$
	$2.300 \times 10^{-1}$	5.624	$4.089 \times 10^{-2}$
	$2.500 \times 10^{-1}$	6.305	$3.965 \times 10^{-2}$
	$2.740 \times 10^{-1}$	9.046	$3.028 \times 10^{-2}$
Abnormal	$2.742 \times 10^{-1}$	10.73	$2.555 \times 10^{-2}$
Limaçon	$2.000 \times 10^{-1}$	6.147	$3.253 \times 10^{-2}$

Table 1: Geometric efficiency for the anormal stroke and different normal strokes with the euclidean cost.

Types of $\gamma$	$x_0(T)$	$L(\gamma)$	$x_0(T)/L(\gamma)$
Simple loops	$0.500 \cdot 10^{-1}$	$9.935 \times 10^{-1}$	$5.03 \times 10^{-2}$
	$1.500 \cdot 10^{-1}$	1.860	$8.06 \times 10^{-2}$
	$1.700 \times 10^{-1}$	2.021	$8.41 \times 10^{-2}$
	$2.000 \times 10^{-1}$	2.280	$8.77 \times 10^{-2}$
	$2.100 \times 10^{-1}$	2.501	$8.84 \times 10^{-2}$
	$2.200 \times 10^{-1}$	2.472	$8.89 \times 10^{-2}$
<b>Optimal stroke</b> Fig.6 (right)	<b><math>2.228 \times 10^{-1}</math></b>	<b>2.561</b>	<b><math>8.902 \times 10^{-2}</math></b>
	$2.300 \times 10^{-1}$	2.586	$8.895 \times 10^{-2}$
	$2.500 \times 10^{-1}$	2.854	$8.76 \times 10^{-2}$
	$2.600 \times 10^{-1}$	3.044	$8.54 \times 10^{-2}$
Abnormal	$2.742 \times 10^{-1}$	4.933	$5.56 \times 10^{-2}$
Limaçon	$2.500 \times 10^{-1}$	3.353	$7.46 \times 10^{-2}$

Table 2: Geometric efficiency for the anormal stroke and different normal strokes with the mechanical cost.

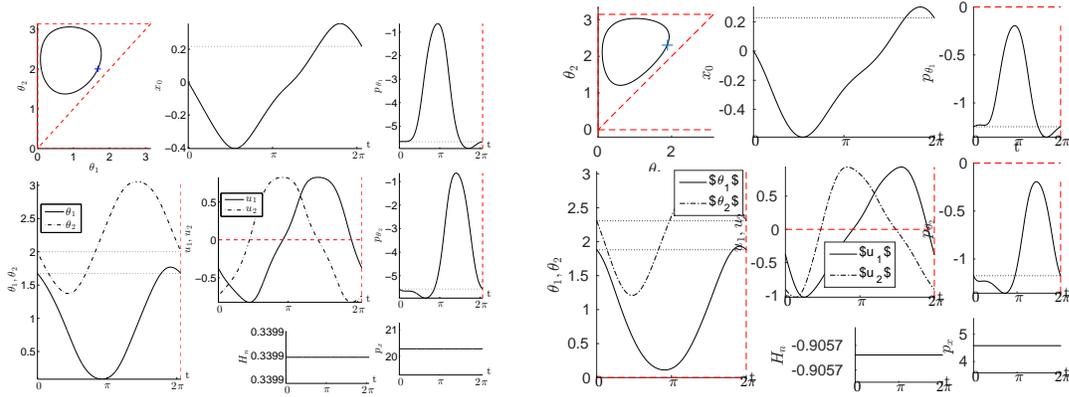


Figure 6: Normal stroke of the Copepod swimmer for the euclidean cost (left) and the mechanical energy (right), obtained by the transversality conditions of the maximum principle (23).

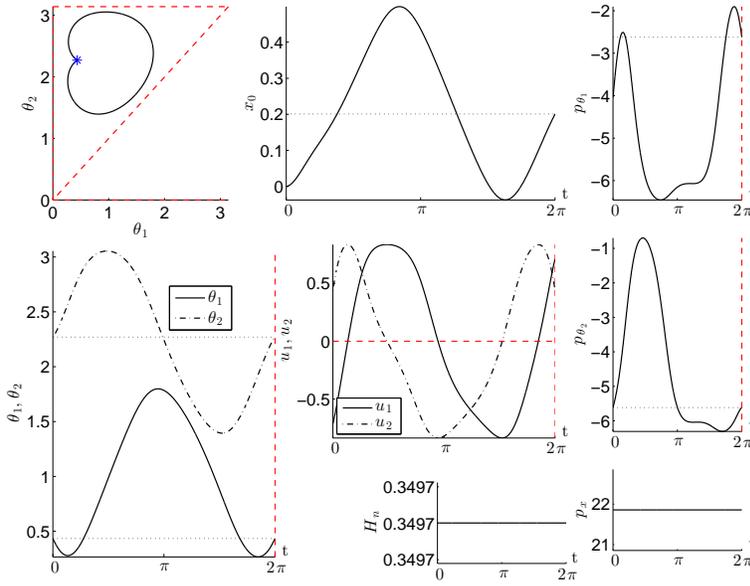


Figure 7: Non-smooth normal stroke of the Copepod swimmer for the cost depending upon the initial angle  $\theta(0)$ , obtained by the transversality conditions of the maximum principle (23)

## 4 Geometric efficiency and optimality of the abnormal stroke

In the previous section, based on numerical simulation the abnormal stroke is shown to be non optimal. In this section, we propose a complete mathematical analysis using a penalty method.

In this section we shall discuss how the geometric efficiency can be related to a reference optimal control problem for the Copepod model in which we minimize integral cost criteria. We shall consider two different right end-point conditions: the ‘exact displacement’ case ( $x_0(T) = x_T$ ) and the case in which we are given a lower bound for the Copepod displacement ( $x_0(T) \geq x_T$ ). For the latter we shall derive also normality conditions for minimizers.

### 4.1 Some comments on the efficiency term for the case of ‘exact displacement’ ( $x_0(T) = x_T$ )

We consider here the following optimal control problem for the Copepod model:

$$\left\{ \begin{array}{l} \text{Minimize } \int_0^T \ell(q(t), u(t)) dt \\ \text{over arcs } q(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^3) \text{ s.t.} \\ \dot{q}(t) = F(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ q_1(T) = x_T, \quad q_1(0) = 0 \\ q_2(0) = q_2(T), \quad q_3(0) = q_3(T), \end{array} \right. \quad (25)$$

in which  $x_T > 0$  is a given number which represents the Copepod desired displacement,  $T = 2\pi$  in the time period,  $U = \mathbb{R}^2$ ,  $\ell : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ , and  $F : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is the function

$$F(q, u) := \left( u_1 \frac{\sin(q_2)}{2 + \sin^2(q_2) + \sin^2(q_3)} + u_2 \frac{\sin(q_3)}{2 + \sin^2(q_2) + \sin^2(q_3)}, u_1, u_2 \right).$$

The state variable is  $q = (x, \theta = (\theta_1, \theta_2))$  in which  $x$  stands for the swimmer displacement, and  $\theta = (\theta_1, \theta_2)$  are the link amplitudes.

We provide first a simple relationship between minimizers of the optimal control problem (25) and minimizers of an optimal control problem in which we maximize the *geometric efficiency*  $\mathcal{E} = \frac{q_1(T)}{q_0(T)}$  where  $q_0(T) = \int_0^T \ell(q(t), u(t)) dt$ .

**Remark 4.1.** We observe first the fact that, for any integral cost term  $\ell = \ell(q, u)$  (Lipschitz w.r.t.  $q$  and Borel measurable w.r.t.  $u$ ), if  $(q^*, u^*)$  is a minimizer for (25), then  $(q^*, u^*)$  is a minimizer also for the optimal control problem in which we replace the integral term in (25) with  $J_{\mathcal{E}}(q(\cdot), u(\cdot)) := -\frac{q_1^2(T)}{q_0(T)}$  (i.e. a problem in which we want to maximize the geometric efficiency  $\mathcal{E}$ ).

Indeed if  $(q^*, u^*)$  is a minimizer for (25), then  $q_0^*(T) \leq q_0(T)$  for all admissible trajectories for problem (25). Taking the reciprocal, since the right end-point has to be the same ( $q_1(T) = q_1^*(T) = x_T$ ), we immediately obtain:

$$-\frac{(q_1^*(T))^*}{q_0^2(T)} \leq -\frac{q_1^2(T)}{q_0(T)},$$

for all admissible trajectories.

We study next the case in which we minimize a simplified cost which represent the energy of the system (this is the sub-Riemannian problem):  $\ell(q, u) = u_1^2 + u_2^2$ . Remark 4.1 suggests to investigate a class of optimal control problems in which we consider the geometric efficiency in the cost. If, in addition, we allow free right end-point conditions, then we can establish normality of the Maximum Principle. We consider:

$$\left\{ \begin{array}{l} \text{Minimize } J_{\mathcal{E}}((q_0(\cdot), q(\cdot)), u(\cdot)) := -\frac{q_1^2(T)}{q_0(T)} \\ \text{over arcs } (q_0(\cdot), q(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^4) \text{ s.t.} \\ \dot{q}(t) = F(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ \dot{q}_0(t) = \ell(q(t), u(t)) = u_1^2(t) + u_2^2(t), \quad \text{for a.e. } t \in [0, T] \\ u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ q_0(0) = 0, \quad q_1(0) = 0 \\ q_2(0) = q_2(T), \quad q_3(0) = q_3(T) , \end{array} \right. \quad (26)$$

Suppose that we have a minimizer  $((q_0^*, q^*), u^*)$  for (26). Necessarily we have  $q_0^*(T) \neq 0$  and  $q_1^*(T) \neq 0$ . Write  $E^* := q_0^*(T) = \int_0^T \ell(q^*(t), u^*(t)) dt$  the corresponding energy and consider the following optimal control problem (for some constant  $K > 0$ ):

$$\left\{ \begin{array}{l} \text{Minimize } J_{\mathcal{E}, K}((q_0(\cdot), q(\cdot)), u(\cdot)) := \left[ -\frac{q_1^2(T)}{q_0(T)} + K|q_0(T) - E^* \right] \\ \text{over arcs } (q_0(\cdot), q(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^4) \text{ s.t.} \\ \dot{q}(t) = F(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ \dot{q}_0(t) = \ell(q(t), u(t)) = u_1^2(t) + u_2^2(t), \quad \text{for a.e. } t \in [0, T] \\ u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ q_0(0) = 0, \quad q_1(0) = 0 \\ q_2(0) = q_2(T), \quad q_3(0) = q_3(T) , \end{array} \right. \quad (27)$$

**Proposition 4.2.** Let  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  be a local minimizer for (26), then we can find a positive constant  $K$  such that  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  is a local minimizer for (27).

**Proof.**

Assume that  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  is local a minimizer for (25). Then, there exists  $\varepsilon_0 \in (0, x_T/2)$  such that

$$J_{\mathcal{E}}((q_0^*, q^*)(\cdot), u^*(\cdot)) \leq J_{\mathcal{E}}((q_0, q)(\cdot), u(\cdot)) , \quad (28)$$

for all trajectory-control pair  $(q(\cdot), u(\cdot))$  satisfying all the conditions of the control system in (25), and with

$$\|q^*(\cdot) - q(\cdot)\|_{L^\infty} \leq \varepsilon_0, \quad \|u^*(\cdot) - u(\cdot)\|_{L^\infty} \leq \varepsilon_0 . \quad (29)$$

We claim that we can find  $\varepsilon_1 \in (0, \varepsilon_0)$  small enough such that

$$J_{\mathcal{E}, K}((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot)) \leq J_{\mathcal{E}, K}((q_0(\cdot), q(\cdot)), u(\cdot)) , \quad (30)$$

for all trajectory-control pair  $((q_0(\cdot), q(\cdot)), u(\cdot))$  satisfying all the conditions of the control system in (27), and with

$$\|(q_0^*(\cdot), q^*(\cdot)) - (q_0(\cdot), q(\cdot))\|_{L^\infty} \leq \varepsilon_1, \quad \|u^*(\cdot) - u(\cdot)\|_{L^\infty} \leq \varepsilon_1 . \quad (31)$$

Suppose by contradiction that for each  $\varepsilon \in (0, \varepsilon_0)$  we can find a trajectory/control pair  $((q_0^\varepsilon(\cdot), q^\varepsilon(\cdot)), u^\varepsilon(\cdot))$  satisfying all the requirements of the control system in (27), such that

$$J_{\mathcal{E},K}((q_0^\varepsilon(\cdot), q^\varepsilon(\cdot)), u^\varepsilon(\cdot)) < J_{\mathcal{E},K}((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot)) = -\frac{[q_1^*(T)]^2}{q_0^*(T)} \quad (32)$$

and

$$\|(q_0^*(\cdot), q^*(\cdot)) - (q_0^\varepsilon(\cdot), q^\varepsilon(\cdot))\|_{L^\infty} \leq \varepsilon, \quad \|u^*(\cdot) - u^\varepsilon(\cdot)\|_{L^\infty} \leq \varepsilon. \quad (33)$$

Then, we define  $\rho := \sqrt{\frac{q_0^*(T)}{q_0^\varepsilon(T)}}$  and take the trajectory/control pair  $((\tilde{q}_0(\cdot), \tilde{q}(\cdot)), \tilde{u}(\cdot))$  where

$$\tilde{u}(\cdot) := \rho u^\varepsilon(\cdot), \quad \tilde{q}_0(0) = 0, \quad \tilde{q}_1(0) = 0$$

and

$$\tilde{q}_2(\cdot) = \rho q_2^\varepsilon(\cdot), \quad \tilde{q}_3(\cdot) = \rho q_3^\varepsilon(\cdot).$$

It is straightforward to see that  $((\tilde{q}_0(\cdot), \tilde{q}(\cdot)), \tilde{u}(\cdot))$  is a solution of the control system in (27), and, since  $\tilde{q}_0(T) = q_0^*(T)$ ,

$$J_{\mathcal{E},K}((\tilde{q}_0(\cdot), \tilde{q}(\cdot))) = -\frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)}.$$

Observe that taking  $\varepsilon_1 \in (0, \varepsilon_0)$  small enough, then for all  $\varepsilon \in (0, \varepsilon_1)$ , we obtain that

$$\|q^*(\cdot) - \tilde{q}(\cdot)\|_{L^\infty} \leq \varepsilon_0, \quad \|u^*(\cdot) - \tilde{u}(\cdot)\|_{L^\infty} \leq \varepsilon_0. \quad (34)$$

And choosing  $K > 0$  big enough, we also have that

$$\left| \frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)} - \frac{[q_1^\varepsilon(T)]^2}{q_0^\varepsilon(T)} \right| \leq K |q_0^\varepsilon(T) - \tilde{q}_0(T)| = K |q_0^\varepsilon(T) - E^*|. \quad (35)$$

From (32) and (35) we would eventually deduce that

$$\begin{aligned} J_{\mathcal{E}}((\tilde{q}_0(\cdot), \tilde{q}(\cdot)), \tilde{u}) &= -\frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)} \\ &\leq -\frac{[q_1^\varepsilon(T)]^2}{q_0^\varepsilon(T)} + K |q_0^\varepsilon(T) - E^*| = J_{\mathcal{E},K}((q_0^\varepsilon(\cdot), q^\varepsilon(\cdot)), u^\varepsilon(\cdot)) \\ &< J_{\mathcal{E}}((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot)), \end{aligned}$$

which contradicts the (local) minimality properties of  $(q^*(\cdot), u^*(\cdot))$  (cf. (28)-(29)).

□

**Corollary 4.3.** *Let  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  be a local minimizer for (26), then the Maximum Principle applies in the normal form.*

**Proof.** If  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  is a local minimizer for (26), then from Proposition 4.2 we can find a positive constant  $K$  such that  $((q_0^*(\cdot), q^*(\cdot)), u^*(\cdot))$  is a local minimizer also for (27). Invoking a standard Maximum Principle for problem (27), we easily deduce that the Lagrange multiplier  $\lambda$  associated with the cost  $J_{\mathcal{E}}$  is necessarily non-zero.

□

We observe that ‘normal form’ in Corollary 4.3 means that the Lagrange multiplier  $\lambda$  associated with the cost  $J_{\mathcal{E}}$  is non-zero. This does not necessarily implies that also the Lagrange multiplier  $p_0$  associated with  $q_0$  is non-zero. (In the next section we will see that if we impose a lower bound for the Copepod displacement, then we will obtain that both  $\lambda \neq 0$  and  $p_0 \neq 0$ .)

## 4.2 Geometric efficiency and normality for the Copepod model

In this section we consider the following optimal control problem in which we are given a lower bound  $x_T > 0$  for the Copepod displacement:

$$\left\{ \begin{array}{l} \text{Minimize } J_{\mathcal{E}}((q_0(\cdot), q(\cdot)), u(\cdot)) := \left[ -\frac{q_1^2(T)}{q_0(T)} \right] \\ \text{over arcs } (q_0(\cdot), q_1(\cdot), q_2(\cdot), q_3(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^4) \text{ s.t.} \\ (\dot{q}_1, \dot{q}_2, \dot{q}_3)(t) = F(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ \dot{q}_0(t) = \ell(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ q_0(0) = 0, \\ q_1(0) = 0, \quad q_1(T) \geq x_T \\ q_2(0) = q_2(T), \quad q_3(0) = q_3(T). \end{array} \right. \quad (36)$$

We are interested in establishing normality properties of the Copepod model (36). More precisely we shall prove that the optimal control problem (36) does not allow *abnormal* minimizers. This is in particular valid for the case of interest of this report, in which we consider two different Lagrangians:

(I)  $\ell(q, u) = u_1^2 + u_2^2$  (this is the sub-Riemannian problem).

(II) the case in which the Lagrangian represents the mechanical energy of the system.

(In either case, the Lagrangian  $\ell$  does not depend of  $q_1$ , and can be expressed in terms of regular functions of  $\sin(q_2)$ ,  $\sin(q_3)$ ,  $\cos(q_2)$ ,  $\cos(q_3)$ .) Therefore, henceforth in this section, we shall assume that either (I) or (II) above are satisfied.

### The Maximum Principle

Consider the pseudo-Hamiltonian (also referred to as ‘unmaximized’ Hamiltonian)  $H : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$H(q, p, u) := \langle (p_1, p_2, p_3), F(q, u) \rangle - p_0 \ell(q, u).$$

The Maximum Principle for problem (36) takes the following form. Let  $(\bar{q}, \bar{u})$  be a (local) minimizer for (36). Then, there exist a vector-valued function  $p = (p_0, p_1, p_2, p_3) \in W^{1,1}([0, T]; \mathbb{R}^4)$ ,  $\alpha \geq 0$ ,  $\beta_1 \in \mathbb{R}$ ,  $\beta_2 \in \mathbb{R}$  and  $\lambda \geq 0$  such that

(i)  $\lambda + \|p\|_{L^\infty} + |\beta_1| + |\beta_2| + \alpha \neq 0$ ;

(ii)  $-\dot{p}(t) = \partial_q H(\bar{q}(t), p(t), \bar{u}(t)) \quad \text{a.e.}$ ;

(iii)  $H(\bar{q}(t), p(t), \bar{u}(t)) = \max_{u \in U} H(\bar{q}(t), p(t), u) = r \quad \text{for a.e. } t, \text{ for some } r \in \mathbb{R}$ ;

(iv)  $p_2(0) = p_2(T) = \beta_1$ ,  $p_3(0) = p_3(T) = \beta_2$ ,  $p_0(T) = \lambda \frac{q_1^2(T)}{q_0^2(T)}$ ,  
and  $p_1(T) = \alpha + 2\lambda \frac{q_1(T)}{q_0(T)}$ , with  $\alpha = 0$  if  $\bar{q}_1(T) > x_T$ .

Observe that, since  $F$  and  $\ell$  do not depend on  $q_0$  or  $q_1$ , condition (ii) provides the following relations:

$$\dot{p}_0(t) = 0, \quad \dot{p}_1(t) = 0 \quad \text{a.e.}$$

(which yield  $p_0(\cdot) \equiv \lambda \frac{q_1^2(T)}{q_0^2(T)}$  and  $p_1(\cdot) \equiv \alpha + 2\lambda \frac{q_1(T)}{q_0(T)}$ ) and

$$-\dot{p}_2(t) = p_1(\bar{u}_1(t) \partial_{q_2} \varphi_1(\bar{q}(t)) + \bar{u}_2(t) \partial_{q_2} \varphi_2(\bar{q}(t))) - p_0 \partial_{q_2} \ell(\bar{q}(t), \bar{u}(t)) \quad \text{a.e. } t \in [0, T] \quad (37)$$

$$-\dot{p}_3(t) = p_1(\bar{u}_1(t) \partial_{q_3} \varphi_1(\bar{q}(t)) + \bar{u}_2(t) \partial_{q_3} \varphi_2(\bar{q}(t))) - p_0 \partial_{q_3} \ell(\bar{q}(t), \bar{u}(t)) \quad \text{a.e. } t \in [0, T], \quad (38)$$

in which

$$\varphi_1(q) = \varphi_1(q_2, q_3) := \frac{\sin(q_2)}{2 + \sin^2(q_2) + \sin^2(q_3)}, \quad \varphi_2(q) = \varphi_2(q_2, q_3) := \frac{\sin(q_3)}{2 + \sin^2(q_2) + \sin^2(q_3)}.$$

**Definition 4.4.** We say that the (local) minimizer for (36)  $(\bar{q}, \bar{u})$  is normal if the Maximum Principle applies with  $\lambda \neq 0$ .

**Remark 4.5.** Notice that for problem (36) we shall prove that both  $\lambda \neq 0$  and  $p_0 \neq 0$  (the multiplier  $p_0$  is associated with the state variable  $q_0$ ). Indeed, since  $\lambda$  and  $p_0$  turn out to be proportional, the minimizer  $(\bar{q}, \bar{u})$  can be considered abnormal if the (first order) necessary conditions for optimality are applicable with  $\lambda = 0$  or, equivalently,  $p_0 = 0$ .

**Proposition 4.6.** All minimizers for (36) are necessarily normal.

**Proof.**

**Step 1.** Let  $(\bar{q}, \bar{u})$  be a minimizer for (36), and assume that the Maximum Principle applies with  $\lambda = 0$  (i.e.  $(\bar{q}, \bar{u})$  is abnormal). Then  $\bar{q}_1(T) = x_T$ . Indeed, supposing by contradiction that  $\bar{q}_1(T) > x_T$ , and bearing in mind the transversality condition for the adjoint variable  $p_1$ , we would obtain that  $\alpha = 0$  and, therefore  $p_1(\cdot) \equiv 0$ . On the other hand the Maximality condition (iii) yields:

$$p_2(t) = -p_1\varphi_1(\bar{q}(t)), \quad p_3(t) = -p_1\varphi_2(\bar{q}(t)) .$$

We deduce that  $p_2(\cdot), p_3(\cdot)$  are the smooth periodic functions:

$$p_2(t) = -p_1 \frac{\sin(\bar{q}_2(t))}{2 + \sin^2(\bar{q}_2(t)) + \sin^2(\bar{q}_3(t))}, \quad p_3(t) = -p_1 \frac{\sin(\bar{q}_3(t))}{2 + \sin^2(\bar{q}_2(t)) + \sin^2(\bar{q}_3(t))} . \quad (39)$$

Then from the formulae derived for  $p_2(\cdot), p_3(\cdot)$  we would also have  $p(\cdot) \equiv 0$ , obtaining that  $(p(\cdot), \lambda) = (0, 0)$ . But, this would contradict the non-triviality condition (i) of the Maximum Principle. Then, we deduce that  $\bar{q}_1(T) = x_T$ .

**Step 2.** We introduce a new optimal control problem in which we replace the constraint inequality  $q_1(T) \geq x_T$  by a penalty term in the cost to minimize.

$$\left\{ \begin{array}{l} \text{Minimize } \tilde{J}_{\mathcal{E}}((q_0(\cdot), q(\cdot)), u(\cdot)) := \left[ -\frac{q_1^2(T)}{q_0(T)} + K \max\{-(q_1(T) - x_T); 0\} \right] \\ \text{over arcs } (q_0, q(\cdot)) \in W^{1,1}([0, T]; \mathbb{R}^4) \text{ s.t.} \\ \dot{q}(t) = F(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ \dot{q}_0(t) = \ell(q(t), u(t)), \quad \text{for a.e. } t \in [0, T] \\ u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ q_0(0) = 0, \quad q_1(0) = 0, \\ q_2(0) = q_2(T), \quad q_3(0) = q_3(T) . \end{array} \right. \quad (40)$$

Here we take  $K > 0$  such that

$$K > \frac{2C}{x_T},$$

in which  $C > 0$  is a constant such that

$$\left| \frac{q_1^2(T)}{q_0(T)} \right| \leq C, \quad \text{for all trajectory-control pairs } (q(\cdot), u(\cdot)) .$$

(Observe that such a constant  $C$  always exists, take for instance  $C = \frac{\sqrt{2}\pi}{2}$ .)

We claim that if  $(\bar{q}, \bar{u})$  is a minimizer for (36), then it is a minimizer also for (40). Indeed, if  $(\bar{q}, \bar{u})$  is a minimizer for (36), then

$$J_{\mathcal{E}}(\bar{q}(\cdot), \bar{u}(\cdot)) \leq J_{\mathcal{E}}(q(\cdot), u(\cdot)) , \quad (41)$$

for all trajectory-control pair  $(q(\cdot), u(\cdot))$ .

Assume, by contradiction, that we can find a trajectory/control pair  $(\hat{q}(\cdot), \hat{u}(\cdot))$  satisfying all the requirements of the control system in (40), such that

$$\tilde{J}_{\mathcal{E}}(\hat{q}(\cdot), \hat{u}(\cdot)) < \tilde{J}_{\mathcal{E}}(\bar{q}(\cdot), \bar{u}(\cdot)) . \quad (42)$$

From the choice of  $K$  we necessarily have that  $\tilde{q}_1(T) \geq x_T/2$ . Consider the trajectory/control pair  $(\tilde{q}(\cdot), \tilde{u}(\cdot))$  satisfying the following properties:

$$\tilde{u}(s) := \begin{cases} 2\hat{u}(2s) & \text{if } s \in [0, T/2] \\ 2\hat{u}(2s - T) & \text{if } s \in (T/2, T] \end{cases} \quad (43)$$

$\tilde{q}_0(0) = 0$ ,  $\tilde{q}_1(0) = 0$ , and

$$(\tilde{q}_2, \tilde{q}_3)(s) = \begin{cases} (\hat{q}_2, \hat{q}_3)(2s) & \text{if } s \in [0, T/2] \\ 2\hat{u}(2s - T) & \text{if } s \in (T/2, T] \end{cases} . \quad (44)$$

(Roughly speaking, employing a 'bigger' control, we construct a trajectory which, in the  $(q_2, q_3)$ -variables, does twice the path of  $(\hat{q}(\cdot), \hat{u}(\cdot))$  on the same time interval.) As a consequence, we obtain  $\tilde{q}_1(T) \geq x_T$ , and a straightforward calculation provides

$$\frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)} = \frac{\hat{q}_1^2(T)}{\hat{q}_0(T)} .$$

We would deduce that

$$\begin{aligned} J_{\mathcal{E}}(\tilde{q}(\cdot), \tilde{u}(\cdot)) &= -\frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)} = -\frac{\hat{q}_1^2(T)}{\hat{q}_0(T)} \leq \tilde{J}_{\mathcal{E}}(\hat{q}(\cdot), \hat{u}(\cdot)) \\ &< \tilde{J}_{\mathcal{E}}(\bar{q}(\cdot), \bar{u}(\cdot)) = J_{\mathcal{E}}(\bar{q}(\cdot), \bar{u}(\cdot)) , \end{aligned}$$

which contradicts the minimality of  $(\bar{q}, \bar{u})$  for (36).

**Step 3.** We claim that the optimal control problem (40) has no abnormal minimizers. To see this consider the Maximum Principle for problem (40), which asserts that if  $(\bar{q}, \bar{u})$  is an abnormal minimizer for (40), then, there exist a vector-valued function  $p = (p_0, p_1, p_2, p_3) \in W^{1,1}([0, T]; \mathbb{R}^4)$ ,  $\beta_1 \in \mathbb{R}$ ,  $\beta_2 \in \mathbb{R}$  and  $\lambda \geq 0$  such that

- (i)'  $\lambda + \|p\|_{L^\infty} + |\beta_1| + |\beta_2| \neq 0$ ;
- (ii)  $-\dot{p}(t) = \partial_q H(\bar{q}(t), p(t), \bar{u}(t))$  a.e.;
- (iii)  $H(\bar{q}(t), p(t), \bar{u}(t)) = \max_{u \in U} H(\bar{q}(t), p(t), u) = r$  for a.e.  $t$ , for some  $r \in \mathbb{R}$ ;
- (iv)'  $p_2(0) = p_2(T) = \beta_1$ ,  $p_3(0) = p_3(T) = \beta_2$ ,  $p_0(T) = \lambda \frac{\tilde{q}_1^2(T)}{\tilde{q}_0(T)}$ ,  
and  $p_1(T) \in \lambda(-2\frac{\tilde{q}_1(T)}{\tilde{q}_0(T)} + [-K, 0])$ .

From condition (ii) we know that  $p_0(\cdot)$  and  $p_1(\cdot)$  are constants, and system (37)-(38) are valid. Then, from the relations (iv)' above we would deduce that  $p_0 \equiv 0$  and  $p_1 \equiv 0$ , and therefore (37)-(38) yield also  $p_2 \equiv 0$  and  $p_3 \equiv 0$ . This would mean that  $(p(\cdot), \lambda) = (0, 0)$ , which contradicts the non-triviality condition (i)' of the Maximum Principle.

In conclusion, from step 2 above, if  $(\bar{q}, \bar{u})$  is an abnormal minimizer for (36), then it would be an abnormal minimizer for (40) as well, but this is not admissible owing to step 3.

□

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