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Coprimeness of Fractional Representations

Catherine Bonnet[1] and Yutaka Yamamoto[2]

Abstract— **Coprimeness of a fractional representation plays various crucial roles in many different contexts, for example, stabilization of a given plant, minimality of a state space representation, etc. It should be noted however that coprimeness depends crucially on the choice of a ring (or algebra) where such a representation is taken, which reflects the choice of a plant, and particular problems that one studies. Such relationships are particularly delicate and interesting when dealing with infinite-dimensional systems. This paper discusses various coprimeness issues for different rings, typically for H^∞ and pseudorational transfer functions. The former is related to H^∞-stabilizability, and the latter to controllability of behaviors. We also give some intricate examples where a seemingly non-coprime factorization indeed turns out to be a coprime factorization over H^∞. Some future directions are also indicated.**

I. Introduction

Coprimeness is often a central issue for various systemic studies. For example, coprimeness over the ring of stable transfer functions [14] or H^∞ [4] gives the existence of a stabilizing controller constructed over the respective ring. Here the coprimeness of a pair (p,q) means the Bézout identity

$$px+qy=1 \tag{1}$$

for some elements x,y chosen in that ring.

If one changes the choice of a ring, for example, to the ring of polynomials, equation (1) translates to the minimality of a state space realization. For example, if one associates a controllable realization to the fractional representation p/q of a transfer function, the Bézout condition above is equivalent to the observability of such a realization, and hence minimality (canonicity) of the realization, for the coprimeness here means that there is no redundant mode generated by the denominator q.

This implicitly imports the fact that nonexistence of a common factor between p and q implies the Bézout condition (1). It is due to the fact that the ring of polynomials is a Euclidean domain (and hence a principal ideal domain), and we see that the choice of the ring crucially determines the system properties we deal with.

The dependence on the choice of a ring becomes particularly crucial when we study infinite-dimensional systems. There can be several non-equivalent notions of coprimeness, and also depending on the context and the choice of the ring, there are variety of results that give us deeper insights into infinite-dimensional systems.

This paper discusses such varied notions of coprimeness, and also give some examples and counterexamples, which lead us to a wider perspective for infinite-dimensional systems.

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II. Preliminaries and Pseudorationality

Let $\mathscr{E}'(\mathbb{R})$ denote the space of distributions having compact support contained in $\mathbb{R}$. $\mathscr{E}'(\mathbb{R}_-)$ is its subspace with support contained in the negative half line $(-\infty,0]$. Distributions such as Dirac's delta δ_a placed at $a\in\mathbb{R}$, its derivative δ_a' are examples of elements in $\mathscr{E}'(\mathbb{R})$. If $a\le 0$, then they belong to $\mathscr{E}'(\mathbb{R}_-)$. An impulse response function $p\times m$ matrix G ($\operatorname{supp} G\subset[0,\infty)$) is said to be *pseudorational* ([15]) if there exist matrices Q and P having entries in $\mathscr{E}'(\mathbb{R}_-)^{p\times p}$ and $\mathscr{E}'(\mathbb{R}_-)^{p\times m}$, respectively, such that

1) $G=Q^{-1}*P$ where the inverse is taken with respect to convolution;
2) $\operatorname{ord}\det Q^{-1}=-\operatorname{ord}\det Q$, where $\operatorname{ord} q$ denotes the order of a distribution q [11].

As an example, consider the delay-differential equation:

$$\begin{aligned}\dot{x}(t) &= x(t-1)+u(t)\\ y(t) &= x(t).\end{aligned}$$

This can be expressed as $x=(\delta'-\delta_1)^{-1}*u$, where δ_a is the Dirac delta distribution ($\delta=\delta_0$), and δ' its derivative. Shifting the time axis by 1, we obtain $x=(\delta'_{-1}-\delta)^{-1}*\delta_{-1}*u$, and this is pseudorational.

In what follows, we will deal only with the SISO case, i.e., single-input and single-output case, for simplicity. For definitions and pertinent properties, see related references in [20], [15], [17], etc.

According to the Paley-Wiener theorem given in the Appendix for a pseudorational pair (p,q), each component p and q are Laplace transformable, and have only discrete zeros. Hence for a pseudorational impulse response G, its Laplace transform, i.e., *transfer function*, $\hat{G}(s)$ is $\hat{p}(s)/\hat{q}(s)$, and hence is the ratio of entire functions satisfying the Paley-Wiener estimate (15).

Definition 2.1: Let R be $\mathscr{E}'(\mathbb{R})$, $\mathscr{E}'(\mathbb{R}_-)$ or H^∞. The pair (p,q), $p,q\in R$ is said to be *spectrally coprime* if $\hat{p}(s)$ and $\hat{q}(s)$ have no (finite) common zeros. It is *approximately coprime* if there exist sequences $\phi_n,\psi_n\in R$ such that $p*\phi_n+q*\psi_n\to\delta$ in R. It is said to satisfy the Bézout identity

(equation) or simply is *Bézout* over R where $R = \mathscr{E}'(\mathbb{R})$ or $\mathscr{E}'(\mathbb{R}_-)$, if there exist $x, y \in R$ such that

$$p * x + q * y = \delta. \tag{2}$$

When $R = H^\infty$, this translates in the Laplace domain to

$$px + qy = 1. \tag{3}$$

III. Coprimeness over H^∞

Let us start with a very simple case, which also indicates an interesting difference of coprimeness over $\mathscr{E}'(\mathbb{R}_-)$ and H^∞.

Example 3.1: Consider the transfer function

$$G(s) := \frac{1}{(s+1)(s(1-e^{-s})+1)}. \tag{4}$$

As Logemann [6] showed, this transfer function belongs to H^∞. Since $G \in H^\infty$, it admits the following trivial coprime factorization

$$p := G, q := 1.$$

Note that function 1 belongs to H^∞. The Bézout equation is trivially solved as

$$G \cdot 0 + 1 \cdot 1 = 1.$$

Likewise, for $R = \mathscr{E}'(\mathbb{R})$,

$$g * 0 + \delta * \delta = \delta,$$

where g denotes the inverse Laplace transform of G. However, over $R = \mathscr{E}'(\mathbb{R}_-)$, this example exhibits a slightly different behavior. Note that the fraction (4) in itself does not give a fractional representation over $\mathscr{E}'(\mathbb{R}_-)$ since e^{-s} or its inverse Laplace transform δ_1 is not an element of $\mathscr{E}'(\mathbb{R}_-)$, i.e., the support $\operatorname{supp}\delta_1 = \{1\} \not\subset (-\infty, 0]$. We instead take an equivalent fraction

$$G(s) := \frac{e^s}{(s+1)(s(e^s-1)+e^s)}$$

or its time-domain equivalent

$$g = \delta_{-1} * (\delta' + \delta)^{-1} * [(\delta'_{-1} - \delta') + \delta_{-1}]^{-1}.$$

The pair $(\delta_{-1}, (\delta' + \delta)^{-1} * [(\delta'_{-1} - \delta') + \delta_{-1}]^{-1})$ is indeed Bézout, but it is not immediately obvious to exhibit a coprime factorization for this. We postpone its proof until Section IV.

A. Systems with a chain of poles clustering on the imaginary axis from the right

In this subsection we consider a particular neutral delay system with a single delay and a chain of poles asymptotic to the imaginary axis from the right with transfer function given by

$$G(s) = \frac{1}{(s+1)(s+1+(s+2)e^{-sT})}. \tag{5}$$

We have $(s+1)/(s+2) = \alpha - \beta/s + \gamma/s + o(s^{-2})$ with $\alpha = 1, \beta = -1, \gamma = 2$ satisfying $\gamma/\alpha - \beta^2/2 > 0$, so that from [9] we know that the chain of poles is in the right half-plane.

Proposition 3.2: Let G be the one given by (5). Then $(N(s), D(s))$ defined by

$$N(s) = \frac{1}{(-\frac{1}{3}s^2 - \frac{2}{3}s + \frac{1}{2}) + (-\frac{1}{3}s^2 - s - 1)e^{-sT}}$$

and,

$$D(s) = \frac{(s+1)(s+1+(s+2)e^{-sT})}{(-\frac{1}{3}s^2 - \frac{2}{3}s + \frac{1}{2}) + (-\frac{1}{3}s^2 - s - 1)e^{-sT}}$$

gives a Bézout factorization of G over H^∞

Proof:

Consider the quasi-polynomial $as^2 + bs + c + (ds^2 + es + f)e^{-sT}$ as a potential candidate for the denominator of $N(s)$ and $D(s)$. Write

$$N(s) = \frac{1}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}},$$

and

$$D(s) = \frac{(s+1)(s+1+(s+2)e^{-sT})}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}}.$$

We then have

$$\begin{aligned} D(s) &= \frac{(s+1)(s+1+(s+2)e^{-sT})}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}} \\ &= \frac{1}{a}\left(1 + \frac{(2a-b)s + a - c}{d(s)}\right) \\ &\quad + \frac{1}{a}\left(\frac{(a-d)s^2 + s(3a-e) + 2a - f}{d(s)}\right)e^{-sT} \end{aligned}$$

Taking $b = 2a$, $d = a$ and $e = 3a$, we see that $D(s)$ is in H^∞ provided that $as^2 + bs + c + (ds^2 + es + f)e^{-sT}$ has no zeros in the closed right-half plane.

Using the Walton-Marshall method [7] we obtain some conditions on the coefficients to ensure that $as^2 + bs + c + (ds^2 + es + f)e^{-sT}$ has no unstable poles in the closed right half-plane, for example, $a = -\frac{1}{3}, c = -\frac{1}{2}, f = -1$.

Now, it is easy to verify that (N, D) satisfies the Corona condition [3] $\inf_{\operatorname{Re} s > 0} (|N(s)| + |D(s)|) > 0$. ■

Remark 3.3: The example above is somewhat counter-intuitive. Note that the denominator of $G(s)$ has infinitely many zeros that approach the imaginary axis. Hence this denominator is very close to zero, at least very often, near the imaginary axis. If there is a coprime factorization $(\tilde{N}, \tilde{D})$, $\tilde{D}(s)$ should also vanish at these zeros. Furthermore, $G(s)$ has order s^{-2} along the imaginary axis as $j\omega$ ($s = j\omega$) goes to infinity, hence it looks like $\tilde{N}(s)$ should share this order s^{-2} along the imaginary axis. Since the zeros of $\tilde{D}(s)$ become closer to the imaginary axis, it may appear that $\tilde{N}(s)$ should also go to zero with order s^{-2} along these zeros of $\tilde{D}(s)$. Hence it may seem that $\tilde{N}$ and $\tilde{D}$ cancel at infinity, and contradict the Corona condition. From this one may conclude that there exists *no* coprime factorization for this G. However, as we have seen above, there is indeed a coprime factorization over H^∞. By plotting the values of $N(s)$ at these zeros, one sees that $N(s)$ is indeed bounded from below there. (Here N may be close to zero along the imaginary axis, but not necessarily at the zeros of D.) See Figure 1. (The zeros

of $D(s)$, which are very close to the imaginary axis, are not shown there; the figure shows that $N(s)$ assume values quite far from zero at the zeros of D.)

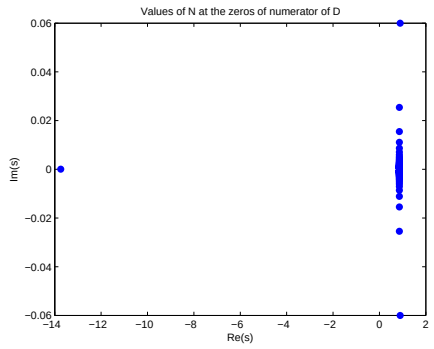


Fig. 1. Values of $N(s)$ at the zeros of $D(s)$

This example has poles clustering on the imaginary axis from the right, i.e., from the unstable side. Is there any difference if the poles approach the imaginary axis from the left? The next example considers such a case.

B. Systems with a chain of poles clustering on the imaginary axis from the left

We also give another neutral delay system with one delay and a chain of poles asymptotic to the imaginary axis from the left with transfer function given by

$$G(s) = \frac{1}{(s+1)(s-1+se^{-sT})} \tag{6}$$

We have $(s-1)/s = \alpha + \beta/s + \gamma/s + o(s^{-2})$ with $\alpha = 1, \beta = -1, \gamma = 0$ satisfying $\gamma/\alpha - \beta^2/2 > 0$, so that from [9] we see that the chain of poles is in the left half-plane. However, G may possess unstable poles of small modulus and there is also a need to determine a coprime factorization for G.

Proposition 3.4: Let G be given by (6). The following $(N(s), D(s))$

$$N(s) = \frac{1}{(s^2 - \frac{3}{2}) + (s^2 + s - \frac{3}{4})e^{-sT}}$$

and,

$$D(s) = \frac{s^2 - 1 + (s^2 + s)e^{-sT}}{(s^2 - \frac{3}{2}) + (s^2 + s - \frac{3}{4})e^{-sT}}$$

gives a coprime factorization of G over H^∞.

Proof:

Consider the quasi-polynomial $d(s) = as^2 + bs + c + (ds^2 + es + f)e^{-sT}$ as a potential candidate for the denominator of $N(s)$ and $D(s)$. Write

$$N(s) = \frac{1}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}},$$

and

$$D(s) = \frac{(s+1)(s-1+se^{-sT})}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}}.$$

We then have

$$\begin{aligned} D(s) &= \frac{(s+1)(s-1+se^{-sT})}{as^2 + bs + c + (ds^2 + es + f)e^{-sT}} \\ &= \frac{1}{a}\left(1 + \frac{-bs - c - a}{d(s)}\right) \\ &\quad + \frac{1}{a}\left(\frac{(a-d)s^2 + s(a-e) + 2a - f}{d(s)}\right)e^{-sT} \end{aligned}$$

The conditions $b = 0, d = a, e = a$ will ensure that $D(s)$ is in H^∞ if $d(s)$ has no unstable zeros.

The coefficients $a = 1, c = -\frac{3}{2}, f = -\frac{3}{4}$ ensure that $d(s)$ has no zeros in the closed right half-plane. The rest is the same as the previous subsection. ■

By plotting the values of $N(s)$ at the zeros of $D(s)$ we see again that their values are quite distant from zero. See Figure 2 below.

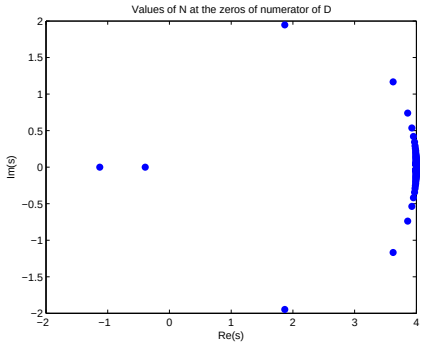


Fig. 2. Values of $N(s)$ at the zeros of $D(s)$

IV. Coprimeness over $\mathscr{E}'(\mathbb{R})$ and $\mathscr{E}'(\mathbb{R}_-)$

We first deal with $\mathscr{E}'(\mathbb{R}_-)$. The following facts summarize the results so far obtained in [17], [19] etc.

Let (p,q), $p,q \in \mathscr{E}'(\mathbb{R}_-)$ be a pseudorational pair. To state the results with some rigor, we need some preliminary definitions. Let $\Omega := \varinjlim L^2[-n,0]$ be the *inductive limit* of the spaces $\{L^2[-n,0]\}_{n>0}$. it is the space of inputs with bounded support. Dually, $\Gamma := L^2_{loc}[0,\infty)$ is the space of all *locally* Lebesgue square integrable functions with obvious family of seminorms. This is the *projective limit* of spaces $\{L^2[0,n]\}_{n>0}$. Ω is the space of past inputs, and Γ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the following natural *left shift* semigroups:

$$(\sigma_t \omega)(s) := \begin{cases} \omega(s+t), & s \le -t, \\ 0, & -t < s \le 0, \end{cases} \tag{7}$$
$$\omega \in \Omega, t \ge 0, s \le 0.$$

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \ge 0, s \ge 0. \tag{8}$$

The *input/output map* associated with (p,q) is defined as

$$f(\omega) : \Omega \to \Gamma : \omega \mapsto \pi(q^{-1} * p * \omega), \tag{9}$$

where $\pi\psi := \psi|_{[0,\infty)}$ is the truncation at 0.

Now define

$$X^q := \{x \in \Gamma | \operatorname{supp}(q * x) \subset (-\infty, 0]\}.$$

X^q is a left-shift invariant subspace of Γ, and we can take this left-shift operator as the state transition semigroup. The "A" operator is the infinitesimal generator of this semigroup, and the remaining B and C are defined suitably; see [15] for detailed definitions. We denote this system by $\Sigma^{q,p}$.

The following facts hold:

Facts 4.1: 1) Every eigenmode of $\Sigma^{q,p}$ is controllable if and only if (p,q) is spectrally coprime.

2) The pair is approximately coprime if and only if
   - (p,q) is spectrally coprime, and
   - $\max\{r(q), r(p)\} = 0$

   where $r(q)$ denotes $\sup\{t \in \operatorname{supp} q\}$.

This result immediately leads to the observation that there indeed exists a pair (p,q) that is spectrally coprime but not approximately coprime. Indeed, see the following example.

Example 4.2: Let $\hat{p} := 1$, $\hat{q} := e^s - 1$. Clearly

$$1 \cdot 1 + (e^s - 1) \cdot 0 = 1.$$

On the other hand, if we multiply e^s to both $\hat{p}$ and $\hat{q}$, they have a common factor e^s. Since e^s is not invertible in $\mathscr{E}'(\mathbb{R}_-)$, the pair $(e^s\hat{p}, e^s\hat{q})$ cannot be approximately coprime.

It is interesting to note that condition

$$r(q) = 0$$

means eigenfunction completeness of $\Sigma^{q,p}$. If this holds, spectral coprimeness obviously implies approximate controllability. In fact, the condition

$$\max\{r(q), r(p)\} = 0$$

means that system $\Sigma^{q,p}$ can be made eigenfunction complete by a unity feedback in itself [17]. Since feedback does not change controllability, this immediately implies that spectral controllability yields approximate controllability.

A. Bézout identity over $\mathscr{E}'(\mathbb{R}_-)$

The strongest notion of coprimeness is the Bézout condition. Clearly

Bézout $\Rightarrow$ approximately coprime $\Rightarrow$ spectrally coprime

and each implication is strict, i.e., the converse does not hold. As we have seen already, giving an example that approximate comprimeness does not imply a Bézout identity is not trivial. We here give some examples.

Example 4.3: Let (p,q) admit the following Bézout identity.

$$p * x + q * y = \delta.$$

This yields

$$q^{-1} = q^{-1} * p * x + y.$$

Then we have

$$q^{-1} = \pi(q^{-1} * p * x)$$

because $\pi y = 0$. In other words, with an extended input $x \in \mathscr{E}'(\mathbb{R}_-)$, q^{-1} is reachable from 0.

Now if we take p to be a C^∞ function with support in $[-T, 0]$, and let $q := \delta_{-T}$. It follows that

$$\delta_T = \pi(\delta_T * p * x),$$

but for any $x \in \mathscr{E}'(\mathbb{R}_-)$, the right-hand side is always in C^∞, hence it cannot be δ_T. Hence this pair does not admit a Bézout identity.

Example 4.4: Consider $p := e^s$, $q := e^s + 1$. The pair is Bézout. Indeed,

$$e^s \cdot (1) + (e^s - 1) \cdot (-1) = 1.$$

We also note that a "Corona-like" condition

$$\inf_{s \in \mathbb{C}} \{|e^s| + |e^s - 1|\} = 1$$

holds.

If we multiply s to e^s in the second factor, we have a more delicate example.

Example 4.5: Consider $p := e^s$, $q := se^s - 1$. Then as

$$e^s \cdot (s) + (se^s - 1) \cdot (-1) = 1 \tag{10}$$

shows, this pair is Bézout. Now note that $se^s - 1$ has infinitely many zeros that goes to infinity. Along these zeros λ_n, $n = 1, 2, \ldots$, e^{λ_n} goes to zero with order $1/\lambda_n$. Hence at these zeros λ_n,

$$\inf_{s=\lambda_n} \{|e^s| + |se^s - 1|\} = 0,$$

and a "Corona-like" condition fails.

Unlike the previous example, q has the first-order differentiation s, and this prohibits a routine application of a "Corona-like" condition. This is due to the fact that distributions are locally of finite order, and it can allow multiplication by polynomials (and other distributions of finite order). In fact,

$$|s \cdot e^s| = 1 > 0, \quad s = \lambda_n,$$

and this multiplication of s "saves" the asymptotic cancellation at infinity. The following theorem indeed assures this.

We quote the following theorem from [19]:

Theorem 4.6: Let (p,q) be pseudorational with $p, q \in \mathscr{E}'(\mathbb{R}_-)$ such that $r(q) = 0$. Suppose that the algebraic multiplicities of λ in $\Lambda := \{\lambda \in \mathbb{C} | \hat{q}(\lambda) = 0\}$, i.e., the zeros of $\hat{q}$ are globally bounded, say by m. If

$$|(1 + |\lambda|)^m \hat{p}(\lambda)| \geq c, \quad \forall \lambda \in \Lambda, \tag{11}$$

then the pair (p,q) is Bézout.

In Example 4.5, the order of q is 1, and the multiplicities of the zeros of $\hat{q}(s)$ is bounded by 1. We have just seen (11) is satisfied.

Example 4.7: Let us return to Example 3.1. Here $\hat{p}(s) = e^s$ and $\hat{q}(s) = s(e^s - 1) + e^s$. As with other examples of neutral systems, $\hat{q}(s)$ has a chain of zeros approaching the imaginary axis. Denote these zeros of $\hat{q}(s)$ by λ_n, $n = 1, 2, \ldots$, and because of this property, $\inf\{\operatorname{Re}\lambda_n\} \geq a$ for some $a \in \mathbb{R}$. On the other hand, $|e^s| \geq e^{\operatorname{Re} a} > 0$, so that by Theorem 4.6 (p,q) is a Bézout pair.

B. Coprimeness over $\mathscr{E}'(\mathbb{R})$

We now consider the case of $\mathscr{E}'(\mathbb{R})$. While $\mathscr{E}'(\mathbb{R}_-)$ is closely related to the realization $\Sigma^{q,p}$, this case $\mathscr{E}'(\mathbb{R})$ is closer to behavioral system theory. Also, while δ_a is not unimodular in $\mathscr{E}'(\mathbb{R}_-)$ unless $a=0$, it is indeed unimodular in $\mathscr{E}'(\mathbb{R})$ because its inverse is δ_{-a}. This has some important consequences for coprimeness notions. For example, spectral coprimeness is equivalent to approximate coprimeness over $\mathscr{E}'(\mathbb{R})$. Indeed, recall that the difference between the two notions is related to eigenfunction completeness, and this is further related to $r(p)$ or $r(q)$. The notion of approximate coprimeness is invariant by shifts in $\mathscr{E}'(\mathbb{R})$, one can conclude approximate coprimeness from spectral coprimeness.

We start with the following definition:

Definition 4.8: Let (p,q) be pseudorational with $p,q \in \mathscr{E}'(\mathbb{R})$. The *distributional behavior* $\mathscr{B}_{\mathscr{D}'}$ defined by (p,q) is given by

$$\mathscr{B}_{\mathscr{D}'} := \{w \in (\mathscr{D}')^2 |\ [p\ q] * w = 0\}. \tag{12}$$

For controllability, we employ the following definition:

Definition 4.9: Let $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (12). $\mathscr{B}_{\mathscr{D}'}$ is said to be *distributionally controllable* if for every pair $w_1, w_2 \in \mathscr{B}_{\mathscr{D}'}$, there exists $T \geq 0$ and $w \in \mathscr{B}_{\mathscr{D}'}$, such that $w|_{(-\infty,0)} = w_1$ on $(-\infty,0)$, and $w|_{(T,\infty)} = \sigma_{-T} w_2$ on (T,∞).

Then we have the following [21]:

Theorem 4.10: Let (p,q), $p,q \in \mathscr{E}'(\mathbb{R})$ be pseudorational, and let $\mathscr{B}_{\mathscr{D}'}$ be as defined above in (12). Then $\mathscr{B}_{\mathscr{D}'}$ is distributionally controllable if and only if (p,q) is a Bézout pair, i.e., there exist $x,y \in \mathscr{E}'(\mathbb{R})$ such that $p*x+q*y=\delta$.

This is a special case of a more general theorem given in [21]. For the proof, see [21].

Now generalizing Theorem 4.6, we have the following:

Proposition 4.11: Let p,q be as above, and suppose that

$$r(p) = a, r(q) = b. \tag{13}$$

Suppose that the multiplicities of zeros $\lambda \in \Lambda$ are globally bounded and

$$|(1+|\lambda|)^m (\delta_{-a} * p)\hat{}(\lambda)| = |(1+|\lambda|)^m e^{a\lambda} \hat{p}(\lambda)| \geq c, \quad \forall \lambda \in \Lambda \tag{14}$$

for some m and $c > 0$. Then (p,q) is a Bézout pair in $\mathscr{E}'(\mathbb{R})$.

Proof Condition (13) implies $\delta_{-a} * p, \delta_{-b} * q \in \mathscr{E}'(\mathbb{R}_-)$, and $r(\delta_{-b} * q) = 0$. Since

$$|(1+|\lambda|)^m (\delta_{-a} * p)\hat{}(\lambda)| \geq c \quad \forall \lambda \in \Lambda$$

by (14), we can apply Theorem 4.6 to $\delta_{-a} * p, \delta_{-b} * q$ to obtain

$$\delta_{-a} * p * x + \delta_{-b} * q * y = \delta$$

for some $x,y \in \mathscr{E}'(\mathbb{R}_-)$. This yields

$$p * (\delta_{-a} * x) + q * (\delta_{-b} * y) = \delta$$

as desired. □

Example 4.12: Consider the pair $p(s) = e^s$ and $q(s) := se^{2s} - e^s$. Note that there is no common zero between $p(s)$ and $q(s)$, and hence the pair is spectrally coprime. On the other hand, it is not even approximately coprime over $\mathscr{E}'(\mathbb{R}_-)$ since it has a common factor e^s, and hence not Bézout over $\mathscr{E}'(\mathbb{R}_-)$. However, it is coprime over $\mathscr{E}'(\mathbb{R})$ as

$$e^s \cdot se^{-s} + (se^{2s} - e^s) \cdot (-e^{-s}) = 1.$$

Clearly this is possible because δ_{-1} (with Laplace transform e^s) is unimodular in $\mathscr{E}'(\mathbb{R})$ (but not in $\mathscr{E}'(\mathbb{R}_-)$) with inverse δ_1.

Remark 4.13: It is worth noting that [5], [10] have proven that systems with commensurable delays are controllable if and only if $(\hat{p}(s), \hat{q}(s))$ has no cancellation in $\mathbb{C}$. In light of Proposition 4.11, this may appear strange because the proposition generally requires "no cancellation at infinity." However, the very nature of delay systems with commensurable delays makes it possible to conclude that any cancellation at infinity can be suitably "removed" by applying multiplication by a polynomial in s, which is exactly the case in Example 4.5. See [21] for a more detailed discussion.

APPENDIX

The following Paley-Wiener theorem is fundamental for the Laplace transform of elements in $\mathscr{E}'(\mathbb{R})$.

Theorem 1.1 ([11]): A complex analytic function $f(s)$ is the Laplace transform of a distribution $\phi \in \mathscr{E}'(\mathbb{R})$ if and only if $f(s)$ is an entire function that satisfies the following growth estimate for some $C > 0, a > 0$ and integer $m \geq 0$:

$$|f(s)| \leq C(1+|s|)^m e^{a|\operatorname{Re} s|}. \tag{15}$$

In particular, $f(s) = \hat{\phi}(s)$ for some $\phi \in \mathscr{E}'(\mathbb{R}_-)$ if and only if it satisfies the estimate

$$\begin{aligned} |\hat{f}(s)| &\leq C(1+|s|)^m e^{a \operatorname{Re} s}, \operatorname{Re} s \geq 0, \\ &\leq C(1+|s|)^m, \operatorname{Re} s \leq 0 \end{aligned} \tag{16}$$

for some $C > 0, a > 0$ and integer $m \geq 0$. In this case, the support of ϕ is contained in $[-a, 0]$

We will refer to (15) as the *Paley-Wiener estimate*, and the class of functions satisfying this estimate as the *Paley-Wiener class*.

The zeros of $\hat{f}(s)$ in the Paley-Wiener class are discrete, and each zero has a finite multiplicity. This in particular implies the following *Hadamard factorization* for $\hat{f}(s)$ [1] in terms of the zeros of f:

$$\hat{f}(s) = cs^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{17}$$

Since there are no finite accumulation point for $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$.

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