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Robust Output-Control for Uncertain Linear Systems: Homogeneous Differentiator-based Observer Approach*

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SUMMARY

This paper deals with the design of a robust control for linear systems with external disturbances using a homogeneous differentiator-based observer based on a Implicit Lyapunov Function approach. Sufficient conditions for stability of the closed-loop system in the presence of external disturbances are obtained and represented by Linear Matrix Inequalities. The parameter tuning for both controller and observer is formulated as a semi-definite programming problem with Linear Matrix Inequalities constraints. Simulation results illustrate the feasibility of the proposed approach and some improvements with respect to the classic linear observer approach.

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KEY WORDS: Uncertain linear systems; Closed-loop stability; Homogeneous differentiator.

1. INTRODUCTION

1.1. State of the Art and Motivation

In the last thirty years, the problems of system stability analysis and designing a controller for a given system that guarantees the closed-loop stability and, at the same time, ensures given performance requirements in presence of uncertainties, have been widely studied (see [1, 2, 3, 4]). In this sense, the *robust control* is a branch of modern control theory whose aim is to achieve robust performance and/or stability in the presence of bounded modeling errors and/or uncertainties. The classic control design, based on the frequency domain, was reasonable robust. This was the beginning of the robust control theory.

Likely the most important results of the robust control theory are H_∞ and μ analysis (see, *e.g.* the books [5, 4, 6, 7]). These methods minimize the sensitivity of a system over its frequency spectrum guaranteeing that the system will have a sufficiently small deviation from expected trajectories when disturbances affect the system. Another approach to robust controller design is the so-called *Sliding-Mode Control* technique (see, *e.g.* [8, 9, 10]). This approach has attracted the attention because it possesses important properties such as: insensitivity (more than robustness) with respect to so-called

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matched uncertainties/disturbances acting in the same channel as the control; and finite/fixed time convergence (see, *e.g.* [11, 12, 13, 14]).

Another approach is the Attractive Ellipsoid Method (AEM) [15, 16]. The AEM, originally introduced in [17] and later formalized for linear systems in [18, 19], and for nonlinear systems in [20]; is convenient for the analysis of systems with unmatched and non-vanishing perturbations (and uncertainties). An additional advantage of this method is that the design parameters can be obtained directly from a linear optimization problem subject to a set of Linear Matrix Inequalities (LMIs), which in general is desirable from the computational point of view.

Because the AEM is a Lyapunov based approach, it is natural to use the Implicit Lyapunov Function (ILF) [21] to obtain additional properties for the closed-loop system, such as finite-time convergence to the ellipsoidal set [22]. This property is crucial considering that many applications require control inputs that guarantee convergence of the closed-loop system trajectories to a set of values in a given amount of time. When dealing with the limited information problem the finite-time property is also important for the observer design [23].

It is important to mention that most of the robust output-control approaches are based on linear observers, and therefore the type of convergence is not faster than exponential. In this sense, the ILF originally presented in [24], and later revisited in [25, 21], can be used to design nonlinear observers based on a Lyapunov function defined implicitly by an algebraic equation and providing faster rate of convergence than the exponential one, *i.e.* finite-time [26]. On the other hand, the stability analysis of the error dynamics then does not require the explicit solution of this equation, instead the stability conditions can be derived directly using the implicit function theorem. Therefore, part of the motivation of this paper is founded on the necessity of regulating dynamical systems even in the presence of external disturbances, where the state estimation time is crucial and a faster convergence is required.

1.2. Main Contribution

Motivated by the features of the ILF approach and the fact that in general there does not exist a constructive way to design robust output-control based on nonlinear observers, this paper contributes to the design of a constructive robust output-control for linear systems with external disturbances that uses a homogeneous differentiator-based observer. This approach provides the following features:

1. Sufficient conditions for the stability of the closed-loop system in the presence of bounded external disturbances are given based on the ILF approach.
2. Such conditions are expressed in terms of matrix inequalities.
3. The parameter tuning for both controller and observer is formulated as the semi-definite programming problem with LMIs constraints.
4. Simulation results show some improvements with respect to the classic linear observer approach.

Structure of the Paper: The problem statement is given in Section 2 while some preliminaries are presented in Section 3. The robust control design, the ILF homogeneous differentiator observer and the conditions for the stability of the closed-loop system are described in Section 4. Some simulation results are depicted in Section 5 and concluding remarks in Section 6. Finally, the finite-time convergence proof of the ILF homogeneous differentiator, and the proof for the closed-loop stability are given in the Appendix.

2. PROBLEM STATEMENT

Consider the following class of linear dynamical systems with external disturbances, *i.e.*

$$\dot{x} = Ax + Bu + Dw(t), \tag{1}$$

$$y = Cx, \tag{2}$$

where $x \in \Re^n$, $u \in \Re^p$, $y \in \Re^m$, and $w \in \Re^q$ are the system state, control input, measurable output and the unknown external disturbance vector, respectively. The known matrices A, B, C, and D have suitable dimensions. It is assumed that all considered inputs allow the existence and extension of solutions to the whole semi-axis $t \geq 0$. The goal is to build a robust control, based on the state estimation given by a robust state observer, that can maintain closed-loop stability in spite of certain class of external disturbances.

3. PRELIMINARIES

Some preliminaries and results of *homogeneity, finite-time* stability and *Implicit Lyapunov Function* (see, *e.g.* [27, 28, 29]) are introduced in this section. Let $|q|$ denote the Euclidian norm of a vector q, and $\overline{1,r}$ a sequence of integers $1,...,r$. Let Ω be an open subset of $\Re^n$ with $0 \in \Omega$. Let us consider that for all $x_0 \in \Omega$ the solutions of a differential equation, *i.e.* $x(t, x_0)$, are defined for all $t \geq 0$.

3.1. HOMOGENEITY

In the *homogeneity* framework [27], for any $r_i > 0$, $i = \overline{1,n}$ and $\lambda > 0$, define the dilation matrix $\Lambda_r(\lambda) = \text{diag}(\lambda^{r_i})$, $i = \overline{1,n}$, and the vector of weights $r = (r_1, ..., r_n)^T$.

Definition 1. *The function $g : \Re^n \to \Re$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $x \in \Re^n$ the relation $g(\Lambda_r(\lambda)x) = \lambda^d g(x)$ holds for some $d \in \Re$ and all $\lambda > 0$. Respectively, the vector field $f : \Re^n \to \Re^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $x \in \Re^n$ the relation $f(\Lambda_r(\lambda)x) = \lambda^d \Lambda_r(\lambda) f(x)$ holds for some $d \geq -\min_{1\leq i\leq n} r_i$ and all $\lambda > 0$. In both cases, the constant d is called the degree of homogeneity.*

A dynamical system

$$\dot{x} = f(x), \; x(0) = x_0, \tag{3}$$

is called r-homogeneous of degree d if this property is satisfied for the vector field f in the sense of Definition 1.

3.2. STABILITY

Definition 2. *The origin of system (3) is said to be†: Stable if for any $\epsilon > 0$ there is $\delta(\epsilon)$ such that for any $x_0 \in \Omega$ the solutions are defined for all $t \geq 0$ and, if $|x_0| \leq \delta(\epsilon)$, then $|x(t, x_0)| \leq \epsilon$, for all $t \geq 0$; AS if it is Stable and for any $\epsilon > 0$ there exists $T(\epsilon, \kappa) \geq 0$ such that for any $x_0 \in \Omega$, if $|x_0| \leq \kappa$, then $|x(t, x_0)| \leq \epsilon$, for all $t \geq T(\epsilon, \kappa)$; FTS if it is AS and for any $x_0 \in \Omega$ there exists $0 \leq T^{x_0} < +\infty$ such that $x(t, x_0) = 0$, for all $t \geq T^{x_0}$.‡*

If $\Omega = \Re^n$, then $x = 0$ is said to be globally *Stable (GS), AS (GAS), or FTS (GFTS)*, respectively (see, for more details [30]). Now, the following result, given by [28], represents the main application of homogeneity to finite-time stability and finite-time stabilization.

Theorem 1. *Let $f : \Re^n \to \Re^n$ be a continuous r-homogeneous vector field with a negative degree. If the origin of the system (3) is locally AS then it is GFTS.*

3.3. IMPLICIT LYAPUNOV FUNCTION

The following theorems provide the background for *asymptotic* and *finite-time stability* analysis, respectively, of (3) using the *ILF Approach* [29].

†The acronyms AS and FTS correspond to Asymptotically Stable and Finite-Time Stable, respectively.
‡The function $T(x_0) = \inf\{T^{x_0} \geq 0 : x(t, x_0) = 0 \; \forall t \geq T^{x_0}\}$ is called the uniform settling time of the system (3).

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Theorem 2. *If there exists a continuous function $G : \Re_+ \times \Re^n \to \Re$, $(V, x) \mapsto G(V, x)$, satisfying the following conditions: 1) G is continuously differentiable outside the origin for all positive $V \in \Re_+$ and for all $x \in \Re^n \backslash \{0\}$; 2) for any $x \in \Re^n \backslash \{0\}$ there exists $V \in \Re_+$ such that $G(V, x) = 0$; 3) let $\Phi = \{(V, x) \in \Re_+ \times \Re^n \backslash \{0\} : G(V, x) = 0\}$, then, $\lim_{|x| \to 0} V = 0^+$, $\lim_{V \to 0} |x| = 0$, $\lim_{|x| \to \infty} V = +\infty$, for all $(V, x) \in \Phi$; 4) the inequality $\frac{\partial G(V,x)}{\partial V} < 0$ holds for all $V \in \Re_+$ and for all $x \in \Re^n \backslash \{0\}$; 5) $\frac{\partial G(V,x)}{\partial x} f(x) < 0$ holds for all $(V, x) \in \Phi$; then the origin of (3) is GAS.*

Theorem 3. *If there exists a continuous function $G : \Re_+ \times \Re^n \to \Re$ that satisfies the conditions 1-4 of Theorem 2, and there exist $c > 0$ and $0 < \mu < 1$, such that*

$$\frac{\partial G(V, x)}{\partial x} f(x) \leq c V^{1-\mu} \frac{\partial G(V, x)}{\partial V},$$

holds for all $(V, x) \in \Phi$, then the origin of (3) is GFTS and $T(x_0) \leq \frac{V_0^\mu}{c\mu}$ is the settling time function, where $G(V_0, x_0) = 0$.

4. ROBUST CONTROL DESIGN

4.1. Closed-Loop System Dynamics

For simplicity, let us start with the case $m = 1$, *i.e.* $y \in \Re$. Thus, the following assumptions are introduced.

Assumption 1. *The pair (A, C) is observable.*

Assumption 2. *The unknown external input $w(t)$ is bounded by a known positive constant w^+, i.e. $|w(t)| \leq w^+$, for all $t \geq 0$.*

Let Assumptions 1 and 2 be satisfied. Then, the observer for system (1)-(2) takes the following structure:

$$\dot{\hat{x}} = P^{-1} A_0 P \hat{x} + P^{-1} \bar{a}_1 y + Bu - P^{-1} R[y - C\hat{x}], \tag{4}$$

where $\hat{x} \in \Re^n$ is the estimation of x, the term $R : \Re \to \Re^n$ represents a nonlinear injection and it will be designed further, and the nonsingular matrix $P \in \Re^{n \times n}$ satisfies the following form[§]

$$PAP^{-1} = A_0 + \bar{a}_1 CP^{-1}, \;\; CP^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}, \tag{5}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; \bar{a}_1 = (a_1, \dots, a_n)^T,$$

and a_i, $i = \overline{1, n}$ are the coefficients of the characteristic polynomial of matrix A. Let us define the estimation error as $e := x - \hat{x}$. Hence, the estimation error dynamics is given by

$$\dot{e} = \left(A - P^{-1} A_0 P - P^{-1} \bar{a}_1 C\right) x + P^{-1} A_0 P e + Dw + P^{-1} R[Ce]. \tag{6}$$

From (5) it follows that $A = P^{-1} A_0 P + P^{-1} \bar{a}_1 C$. Thus, (6) is rewritten as follows

$$\dot{e} = P^{-1} A_0 P e + Dw + P^{-1} R[Ce]. \tag{7}$$

Now, let us introduce the following assumption that ensures the possibility for designing a state feedback control u.

[§]Note that Assumption 1 implies that the matrix P always exists.

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Assumption 3. *The pair (A, B) is controllable.*

Thus, consider that the control u is designed as a feedback control based on the state estimation, *i.e.* $u = K_u\hat{x}$, where $K_u \in \Re^{p\times n}$ is a matrix feedback gain which will be designed further. Hence, substituting $u = K_u\hat{x} = K_u(x - e)$ in (1), it follows that

$$\dot{x} = A_K x - BK_u e + Dw(t), \tag{8}$$

where $A_K = A + BK_u$. Thus, the complete closed-loop system dynamics can be written as follows

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A_K & -BK_u \\ \mathbf{0}_{n\times n} & P^{-1}A_0P \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} D \\ D \end{pmatrix} w + \begin{pmatrix} \mathbf{0}_{n\times 1} \\ P^{-1}R[Ce] \end{pmatrix}. \tag{9}$$

Note that the complete closed-loop system dynamics (9) is a cascade system since the dynamics of e is totally independent of x. Thus, its analysis can be decomposed on two separate steps:

1. Optimization of performance (ellipsoid size with respect to w) of (7) by selecting the form and shape of the nonlinear injection R.
2. Optimization of performance (ellipsoid size with respect to w) of (8) by a choice of K_u.

In the following the designs of the nonlinear injection term R and the matrix feedback gain K_u are presented, respectively.

4.2. Nonlinear Injection Design

Let us consider the estimation error dynamics (7) and the linear transformation $\varepsilon = Pe$. Thus, taking into account that Assumption 1 holds, the corresponding dynamics is given as follows

$$\dot{\varepsilon} = A_0\varepsilon + \bar{D}w + R[CP^{-1}\varepsilon], \tag{10}$$

$$e_y = CP^{-1}\varepsilon = \varepsilon_1, \tag{11}$$

where $\bar{D} := PD$. In this work the nonlinear term is designed as follows

$$R[\varepsilon_1] = \Lambda_{\tilde{r}}^{-1}(V_\varepsilon(\varepsilon_1 h))k\varepsilon_1 = \Lambda_{\tilde{r}}^{-1}(V_\varepsilon(\varepsilon_1 h))kh^T\varepsilon, \tag{12}$$

where $k := (k_1, \ldots, k_n)^T \in \Re^n$ is a vector of gains, $h := (1, 0, \ldots, 0)^T \in \Re^n$, $\Lambda_{\tilde{r}}(\cdot)$ is a dilation matrix with a vector of weights $\tilde{r} := (\mu, 2\mu, \ldots, n\mu)^T$, where $0 \leq \mu \leq 1$, and $V_\varepsilon : \Re^n \to \Re_+$ is a positive definite function implicitly defined by $G(V_\varepsilon, \varepsilon) = 0$, for any $\varepsilon \in \Re^n\backslash\{0\}$, and

$$G(V_\varepsilon, \varepsilon) := \varepsilon^T\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})\varepsilon - 1, \tag{13}$$

with a positive definite matrix $P_\varepsilon = P_\varepsilon^T \in \Re^{n\times n}$ and

$$r := (1 + (n-1)\mu, 1 + (n-2)\mu, \ldots, 1)^T \in \Re^n.$$

Note that $G(V_\varepsilon(\varepsilon_1 h), \varepsilon_1 h) = 0$ implies that $(V_\varepsilon(\varepsilon_1 h))^{2+2(n-1)\mu} = \varepsilon_1^2 h^T P_\varepsilon h$. Therefore, in an expanded way, the nonlinear injection (12) may be written as follows

$$R[\varepsilon_1] = \begin{pmatrix} k_1 p_{11}^{\frac{-\mu}{(2+2\mu(n-1))}} \lceil\varepsilon_1\rfloor^{\frac{1+\mu(n-2)}{1+\mu(n-1)}} \\ k_2 p_{11}^{\frac{-2\mu}{(2+2\mu(n-1))}} \lceil\varepsilon_1\rfloor^{\frac{1+\mu(n-3)}{1+\mu(n-1)}} \\ \vdots \\ k_n p_{11}^{\frac{-n\mu}{(2+2\mu(n-1))}} \lceil\varepsilon_1\rfloor^{\frac{1-\mu}{1+\mu(n-1)}} \end{pmatrix}, \tag{14}$$

where $p_{11} = h^T P_\varepsilon h$ and $\lceil s\rfloor^\alpha := |s|^\alpha \text{sign}(s)$, for any $s \in \Re$ and $\alpha > 0$.

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The dynamics (10) with the nonlinear injection (12) and $w = 0$ describes the error dynamics of the *homogeneous differentiator* given in [26], *i.e.*

$$\begin{aligned}\dot{\varepsilon}_1 &= \varepsilon_2 + \tilde{k}_1\lceil\varepsilon_1\rfloor^{\frac{1+\mu(n-2)}{1+\mu(n-1)}},\\ \dot{\varepsilon}_2 &= \varepsilon_3 + \tilde{k}_2\lceil\varepsilon_1\rfloor^{\frac{1+\mu(n-3)}{1+\mu(n-1)}},\\ &\vdots\\ \dot{\varepsilon}_n &= \tilde{k}_n\lceil\varepsilon_1\rfloor^{\frac{1-\mu}{1+\mu(n-1)}},\end{aligned}$$

where $\tilde{k}_i = k_i p_{11}^{\frac{-i\mu}{(2+2\mu(n-1))}}$, $i = \overline{1,n}$.

Remark 1. For the case $\mu = 1$, (12) takes the form of the HOSM differentiator *[31]. For this case, a similar observer for linear systems with external disturbances has been presented in [32], where $\hat{x}$ is a linear combination of an estimation given by a Luenberger observer and the compensation of its estimation error provided by the HOSM differentiator [31].*

Remark 2. The nonlinear injection term $R[\varepsilon_1]$ may be implemented using the implicit (12) or the explicit (14) form, in the first case V_ε can be calculated by means of the algorithm presented in [29] which is based on the bisection numerical method.

The following theorem describes the finite-time convergence of the error dynamics (10) to a certain neighborhood of the origin.

Theorem 4. *Let the nonlinear injection (12) be applied to the observer (4). Define $\varphi_0 := |\bar{D}|w^+$. If the following matrix inequalities*

$$P_\varepsilon \geq \delta \mathrm{diag}(h) P_\varepsilon \mathrm{diag}(h), \tag{15}$$

$$\begin{pmatrix} P_\varepsilon A_0 + A_0^T P_\varepsilon + P_\varepsilon k h^T + h k^T P_\varepsilon + H P_\varepsilon + P_\varepsilon H + \beta P_\varepsilon & P_\varepsilon \\ P_\varepsilon & -\gamma\varphi_0^{-1} I_n \end{pmatrix} \leq 0, \tag{16}$$

$$\frac{k^T\left(I_n - \Lambda_{\tilde{r}(\lambda)}\right)\Lambda_r(\lambda) P_\varepsilon \Lambda_r(\lambda)\left(I_n - \Lambda_{\tilde{r}(\lambda)}\right)k}{\beta^2\lambda^{2\mu} h^T P_\varepsilon h} \leq 1,\ \forall \lambda \in [0, \lambda^*],\ \lambda^* = \delta^{-1/(2+2(n-1)\mu)}, \tag{17}$$

$$P_\varepsilon H + H P_\varepsilon \geq (\gamma\varphi_0 + \beta)^{1/2} P_\varepsilon > 0,\ \ \gamma\varphi_0 + \beta < 1, \tag{18}$$

are feasible for some positive scalars β and γ, a positive definite matrix $P_\varepsilon = P_\varepsilon^T \in \Re^{n\times n}$, a matrix $H = \mathrm{diag}\{r_i\}_{i=1}^n$, and a vector $k \in \Re^n$, then the estimation error $e(t)$ converges in a finite time to a neighborhood of the origin, i.e.

$$|e(t)| \leq \frac{1}{\lambda_{\min}^{1/2}(P^T P_\varepsilon P)},\ \forall t \geq T_\varepsilon^+(\varepsilon(0)), \tag{19}$$

with the following settling time function

$$T_\varepsilon^+(\varepsilon) \leq \frac{V_\varepsilon^\mu(0) - 1}{\mu(1 - \gamma\varphi_0 - \beta)}, \tag{20}$$

where $V_\varepsilon(0) \geq 0$ such that $G(V_\varepsilon(0), 0) = 0$, and $\mu \in (0, 1)$.

Proof
See the Appendix. □

Remark 3. The main difference with respect to the HOSM observer [32] is that in this work an auxiliary Luenberger observer is not required, and moreover, the proposed observer gains are designed in a constructive way using LMIs.

Remark 4. Since most of the HOSM observers, for instance [32], do not have a Lyapunov function to prove the convergence, it is more complicated to prove the closed-loop stability when they are implemented for robust output-control design.

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4.2.1. Numerical Aspects. The set of matrix inequalities (15)-(18), presented in Theorem 1 for the *homogeneous differentiator* design, is bilinear with respect to matrices k and P_ε and nonlinear with respect to λ, γ and β.

The solution to this problem is in general complex but it is possible to apply some algorithms (see, *e.g.* [33] and [26]), to find a numerical solution for the system of nonlinear inequalities.

For some fixed $\beta \in \Re_+$, $\gamma \in \Re_+$ and $\mu, \delta \in (0,1)$, if the following system of LMIs

$$\begin{gathered}\tilde{P}_\varepsilon \geq \delta H \tilde{P}_\varepsilon H, \quad h^T \tilde{P}_\varepsilon h = 1, \\ \begin{pmatrix} \tilde{P}_\varepsilon A_0 + A_0^T \tilde{P}_\varepsilon + X h^T + h X^T + H \tilde{P}_\varepsilon + \tilde{P}_\varepsilon H + \beta \tilde{P}_\varepsilon & \tilde{P}_\varepsilon \\ \tilde{P}_\varepsilon & -\gamma \varphi_0^{-1} I_n \end{pmatrix} \leq 0, \\ \tilde{P}_\varepsilon H + H \tilde{P}_\varepsilon \geq (\gamma\varphi_0 + \beta)^{1/2} \tilde{P}_\varepsilon > 0, \ \ \gamma\varphi_0 + \beta < 1,\end{gathered} \tag{21}$$

is feasible for a positive definite matrix $\tilde{P}_\varepsilon = \tilde{P}_\varepsilon^T \in \Re^{n \times n}$ and $X \in \Re^{n \times 1}$, then the variables $P_\varepsilon = \tilde{P}_\varepsilon$ and $k = \tilde{P}_\varepsilon^{-1} X$ satisfy the set of matrix inequalities (15), (16) and (18) given in Theorem 1.

For the inequality (17) it is possible to find first a feasible solution $(\tilde{P}_\varepsilon, X)$, for the set of LMIs (21), and then evaluate numerically the inequality (17). This inequality can be easily checked on a grid with a sufficiently small step size for $\lambda = \frac{\lambda^* j}{N}$, for $j = 0, 1, 2, \ldots, N$; where N is a sufficiently large number.

4.2.2. Optimization Problem 1. Considering the implicitly defined function V_ε in (13), such that $G(V_\varepsilon, \varepsilon) = 0$, it is clear that the region to which the error ε converge is related to the size of matrix P_ε. In fact, the maximization of P_ε is equivalent to the minimization of the ellipsoidal set $\varepsilon^T \Lambda_r(V_\varepsilon^{-1}) P_\varepsilon \Lambda_r(V_\varepsilon^{-1}) \varepsilon$ for $V_\varepsilon = 1$. In order to reduce the size of this region the following optimization problem is proposed.

$$\begin{gathered} tr(P_\varepsilon) \to \max_{P_\varepsilon, X, \beta, \gamma} \\ \text{subject to (21)} \end{gathered} \tag{22}$$

If $\hat{P}_\varepsilon$ and $\hat{X}$ are the solution of the above optimization problem, then the minimizing gain is given by $\hat{k} = \hat{P}_\varepsilon^{-1} \hat{X}$.

4.3. Matrix Feedback Gain Design

Let us consider the closed-loop dynamics (8). Thus, based on [15], the following result is established.

Theorem 5. *Let the linear control $u = K_u \hat{x}$ be applied to the system (1). Let Assumption 2, 3 and the statements of the Theorem 4 hold. If the following matrix inequality*

$$\begin{pmatrix} P_x^{-1} A_K + A_K^T P_x^{-1} + 2\alpha_x P_x^{-1} & -P_x^{-1} B K_u & P_x^{-1} D \\ \star & -\alpha_x P_T & \mathbf{0}_{n \times q} \\ \star & \star & -\alpha_x Q_w \end{pmatrix} \leq 0, \tag{23}$$

is feasible for $\alpha_x \in \Re_+$, and some positive definite matrices $Q_w = Q_w^T \in \Re^{q \times q}$ and $P_x^{-1} = {P_x^{-1}}^T \in \Re^{n \times n}$, $P_T := P^T P_\varepsilon P \in \Re^{n \times n}$ with P that satisfies (5), then the ellipsoid $\mathcal{E}(P_x) := \left\{ x \in \Re^n : x^T P_x^{-1} x < 1 \right\}$ is exponentially attractive for the closed-loop system (8).

Proof
See the Appendix. □

4.3.1. Numerical Aspects. In order to apply the result given by Theorem 5 it is needed to solve the nonlinear matrix inequality (23) with respect to the variables P_x^{-1} and K_u for a positive scalar α_x. The following proposition provides one simple scheme that can be used for practical selection of the design parameters.

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Proposition 1. *Let $\alpha_x \in \Re_+$, P defined in (5). If the following LMIs*

$$\begin{pmatrix} -2P_x + R_x & I_n \\ I_n & -P_T \end{pmatrix} \leq 0, \tag{24}$$

$$\begin{pmatrix} AP_x + BY_u + (AP_x + BY_u)^T + 2\alpha_x P_x & -BY_u & D \\ \star & -\alpha_x R_x & \mathbf{0}_{n\times q} \\ \star & \star & -\alpha_x Q_w \end{pmatrix} \leq 0, \tag{25}$$

hold for some symmetric positive definite matrices $Q_w \in \Re^{q\times q}$, $P_x^{-1} \in \Re^{n\times n}$, $R_x \in \Re^{n\times n}$, $P_T \in \Re^{n\times n}$, and the matrix $Y_u \in \Re^{p\times n}$, then the matrix inequality (23) holds, and the matrix feedback gain is chosen as $K_u = Y_u P_x^{-1}$.

Proof
Post and pre-multiplying (23) by $\mathrm{diag}(P_x, P_x, I_{q\times q})$ it follows that

$$\begin{pmatrix} A_K P_x + P_x A_K^T + 2\alpha_x P_x & -BKP_x & D \\ \star & -\alpha_x P_x P_T P_x & \mathbf{0}_{n\times q} \\ \star & \star & -\alpha_x Q_w \end{pmatrix} \leq 0. \tag{26}$$

Applying the inequality $2X^T Y \leq X^T \Lambda^{-1} X + Y^T \Lambda Y$ for any positive definite matrix $\Lambda = \Lambda^T$ to the term $-P_x P_T P_x$ one obtains

$$-P_x P_T P_x \leq -2P_x + P_T^{-1}.$$

Let us assume that $-2P_x + P_T^{-1} \leq -R_x$. Then, the LMI (24) is just the last inequality written using the Schur's complement. Finally, defining $Y_u = K_u P_x$, and taking into account that $-P_x P_T P_x \leq -R_x$, the LMI (25) is provided from (26). Hence, if (24) and (25) hold, then (23) is satisfied. □

4.3.2. Optimization Problem 2. Similarly as in section 4.2.2, it is possible to minimize the ellipsoidal $\mathcal{E}(P_x)$ by minimizing the size of matrix P_x. One simple way to achive this is to minimize the trace of P_x

$$tr(P_x) \to \min_{P_x, Y_u, R_x, \alpha_x} \tag{27}$$
$$\text{subject to (24), (25)}$$

If $\hat{P}_x$ and $\hat{Y}_u$ are the solution of (27), the optimal gain can be obtained as $\hat{K}_u = \hat{Y}_u \hat{P}_x^{-1}$.

Remark 5. Both of the optimization problems presented above are bilinear with respect to the scalar variables. In order to solve them it is necessary to use a Bilinear Matrix Inequality solver, e.g. *PENBMI, or to use an iterative method ,* e.g. *Algorithm 3.1 in [15].*

4.4. CLOSED-LOOP STABILITY

Based on the previous results, the following theorem is provided for the complete closed-loop stability.

Theorem 6. *Let Assumption 1, 2 and 3 hold. Let observer (4) designed according to Theorem 4, and the linear control $u = K_u \hat{x}$ designed according to Theorem 5, be applied to the system (1). Then the trajectories of the closed-loop system satisfy*

$$|e(t)| \leq \frac{(-(1-\beta_0)\mu t + V_\varepsilon(0)^\mu)^{\frac{1+(n-1)\mu}{\mu}}}{\lambda_{\min}^{1/2}(P^T P)(\lambda_{\min}(P_\varepsilon))^{\frac{1-\mu(1+(n-1)\mu)}{2+2(n-1)\mu}}}, \ \forall\, 0 \leq t < T_\varepsilon^+(\varepsilon(0)), \tag{28}$$

$$|x(t)| \leq \epsilon_x, \ \forall t \geq 0, \tag{29}$$

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with $V_\varepsilon(0)^\mu = (\lambda_{\min}(P^TP)\lambda_{\max}(P_\varepsilon))^{\frac{\mu}{2}}|e(0)|^\mu$, $\epsilon_x > 0$ *, and*

$$|e(t)| \leq \frac{1}{\lambda_{\min}^{1/2}(P_T)}, \ \forall t \geq T_\varepsilon^+(\varepsilon(0)), \tag{30}$$

$$|x(t)| \leq \frac{1}{\lambda_{\min}^{1/2}(P_x^{-1})}, \ \text{as } t \to \infty. \tag{31}$$

Proof
Based on the statements given by Theorems 4 and 5, it is obtained that the trajectories of the closed-loop system (9) satisfy (30) and (31).

From the proof of Theorem 4, it is possible to show that

$$\dot{V}_\varepsilon \leq -(1-\beta_0){V_\varepsilon}^{1-\mu}, \ \forall V_\varepsilon > 1, \tag{32}$$

$$\lambda_{\max}^{1/(2+2\mu(n-1))}(P_\varepsilon)|\varepsilon|^{1/(1+\mu(n-1))} \leq V_\varepsilon \leq \lambda_{\max}^{1/2}(P_\varepsilon)|\varepsilon|, \tag{33}$$

where $\beta_0 = \sqrt{\gamma\varphi_0 + \beta}$ and $\mu \in (0,1)$. Therefore, ε is always bounded, and thus e is also bounded for all $t \geq 0$.

From (9), since A_K is Hurwitz, it follows that

$$\begin{aligned} x(t) &= e^{A_K t}|x(0)| + \int_0^t e^{A_K(t-\tau)}(Dw(\tau) - BK_u e(\tau))d\tau, \\ &\leq k_x e^{-\lambda_x t}|x(0)| + \frac{k_x}{\lambda_x}\left(|D|w^+ + |BK_u| \sup_{0\leq\tau\leq t}|e(\tau)|\right) \\ &\leq k_x|x(0)| + \frac{k_x}{\lambda_x}\left(|D|w^+ + |BK_u| \sup_{0\leq\tau\leq t}|e(\tau)|\right) = \epsilon_x, \end{aligned}$$

where $|e^{A_K t}| \leq k_x e^{-\lambda_x t}$, for some $k_x > 0$ and $\lambda_x > 0$. Thus, since e is bounded for all $t \geq 0$, the state x is also bounded, *i.e.* $|x(t)| \leq \epsilon_x$ for all $t \geq 0$.

Hence, the closed-loop system is forward complete [34], and thus the trajectories of the closed-loop system do not have a growth faster than exponential one.

The rest of the result is deduced from the comparison principle [30] applied to (32) and using (33) to provide the corresponding bounds for e. □

4.5. Multiple-Output Multiple-Input Case

4.5.1. Nonlinear Injection Design. The observer (4) can be applied for the case $m, p \geq 2$ under the following assumption.

Assumption 4. *The output of the system* (1)-(2) *is such that the observability index* σ_j *for each output* y_j, $j = \overline{1,m}$, *satisfies* $\sigma_1 + \ldots + \sigma_m = n$.

Then, the observer takes the following structure:

$$\dot{\hat{x}} = \mathcal{P}^{-1}\mathcal{A}_0\mathcal{P}\hat{x} + \mathcal{P}^{-1}\mathcal{A}_1 y + Bu - \mathcal{P}^{-1}\mathcal{R}[y - C\hat{x}], \tag{34}$$

where the nonsingular matrix $\mathcal{P} \in \Re^{n\times n}$ satisfies the following canonical form

$$\mathcal{P}A\mathcal{P}^{-1} = \mathcal{A}_0 + \mathcal{A}_1 C\mathcal{P}^{-1}, \tag{35}$$

where

$$\mathcal{A}_0 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\sigma_1\times\sigma_1}, \ldots, \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\sigma_m\times\sigma_m}\right) \in \Re^{n\times n},$$

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$$\mathcal{A}_1 = \mathrm{diag}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{\sigma_1} \end{bmatrix}_{\sigma_1}, \begin{bmatrix} a_{\sigma_1+1} \\ \vdots \\ a_{\sigma_1+\sigma_2} \end{bmatrix}_{\sigma_2}, \ldots, \begin{bmatrix} a_{\sigma_1+\ldots+\sigma_{m-1}} \\ \vdots \\ a_{\sigma_1+\ldots+\sigma_m} \end{bmatrix}_{\sigma_m}\right) \in \Re^{n\times m},$$

$$C\mathcal{P}^{-1} = \mathrm{diag}\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T_{\sigma_1}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T_{\sigma_2}, \ldots, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T_{\sigma_m}\right) \in \Re^{m\times n},$$

with a_i, $i = \overline{1,n}$ the coefficients of the characteristic polynomial of matrix A, and the nonlinear injection as

$$\mathcal{R}[y - C\hat{x}] = \begin{pmatrix} R_1[e_{y_1}] \\ \vdots \\ R_m[e_{y_m}] \end{pmatrix} = \begin{pmatrix} \Lambda^{-1}_{\tilde{r}_1}(V_{\bar{\varepsilon}_{\sigma_1}}(e_{y_1}h_1))k_1 e_{y_1} \\ \vdots \\ \Lambda^{-1}_{\tilde{r}_m}(V_{\bar{\varepsilon}_{\sigma_m}}(e_{y_m}h_m))k_m e_{y_m} \end{pmatrix}.$$

where $k_j := (k_{j,1}, \ldots, k_{j,\sigma_j})^T \in \Re^{\sigma_j}$ are the vectors of gains, $\Lambda_{\tilde{r}_j}(\cdot)$ are the dilation matrices with vectors of weights $\tilde{r}_j := (\mu_j, 2\mu_j, \ldots, \sigma_j\mu_j)^T$, where $0 \le \mu_j \le 1$, and $V_{\bar{\varepsilon}_{\sigma_j}} : \Re^{\sigma_j} \to \Re_+$ are positive definite functions implicitly defined by $G_j(V_{\bar{\varepsilon}_{\sigma_j}}, \bar{\varepsilon}_{\sigma_j}) = 0$, for any $\bar{\varepsilon}_{\sigma_1} := (\varepsilon_1, \ldots, \varepsilon_{\sigma_1}) \in \Re^{\sigma_1}\backslash\{0\}, \ldots, \bar{\varepsilon}_{\sigma_m} := (\varepsilon_{\sigma_1+\ldots+\sigma_{m-1}}, \ldots, \varepsilon_{\sigma_1+\ldots+\sigma_m}) \in \Re^{\sigma_m}\backslash\{0\}$, $j = \overline{1,m}$, and

$$G_j(V_{\bar{\varepsilon}_{\sigma_j}}, \bar{\varepsilon}_{\sigma_j}) := \bar{\varepsilon}^T_{\sigma_j}\Lambda_{r_j}(V^{-1}_{\bar{\varepsilon}_{\sigma_j}})P_{\bar{\varepsilon}_{\sigma_j}}\Lambda_{r_j}(V^{-1}_{\bar{\varepsilon}_{\sigma_j}})\bar{\varepsilon}_{\sigma_j} - 1,\ j = \overline{1,m},$$

with positive definite matrices $P_{\bar{\varepsilon}_{\sigma_j}} = P^T_{\bar{\varepsilon}_{\sigma_j}} \in \Re^{\sigma_j\times\sigma_j}$, and

$$r_j := (1 + (\sigma_j - 1)\mu_j, 1 + (\sigma_j - 2)\mu_j, \ldots, 1)^T \in \Re^{\sigma_j},\ j = \overline{1,m}.$$

Remark 6. Due to Assumption 4, the observer (34) takes a diagonal block form for each output error $e_{y_j} = y_j - c_j\hat{x}$, $j = \overline{1,m}$, where c_j is the $j-th$ row of the matrix C.

Then, applying the results given by Theorem 4 for each output error e_{y_j}, it is possible to form the positive definite matrix $\mathcal{P}_\varepsilon = \mathrm{diag}(P_{\bar{\varepsilon}_{\sigma_1}}, \ldots, P_{\bar{\varepsilon}_{\sigma_m}}) \in \Re^{n\times n}$, where each $P_{\bar{\varepsilon}_{\sigma_j}} = P^T_{\bar{\varepsilon}_{\sigma_j}} \in \Re^{\sigma_j\times\sigma_j}$, $j = \overline{1,m}$, is the solution of the set of LMIs (21) for each e_{y_j}, $j = \overline{1,m}$.

Based on Theorem 4, the observation results for the multiple-output case can be deduced as an extension of the results obtained for the single output case applied in a block form for every output. Therefore, the estimation error e will converge in a finite time to a neighborhood of the origin, *i.e.*

$$|e(t)| \le \frac{1}{\lambda^{1/2}_{\min}(\mathcal{P}^T\mathcal{P}_\varepsilon\mathcal{P})},\ \forall t \ge T^+_\varepsilon(\varepsilon(0)).$$

4.5.2. Matrix Feedback Gain Design. The results given by Theorem 5 can be directly applied for the case $m, p \ge 2$ defining the matrix $\mathcal{P}_T$ in (23) as $\mathcal{P}_T := \mathcal{P}^T\mathcal{P}_\varepsilon\mathcal{P} \in \Re^{n\times n}$.

Based on Theorem 5, if the matrix inequality (23) is feasible for the new $\mathcal{P}_T$, then the ellipsoid $\mathcal{E}(\mathcal{P}_x) = \left\{x \in \Re^n : x^T\mathcal{P}^{-1}_x x < 1\right\}$ will be exponentially attractive for the closed-loop system.

Thus, the matrix feedback gain is taken as $K_u = Y_u\mathcal{P}^{-1}_x$, with $\mathcal{P}^{-1}_x \in \Re^{n\times n}$ and $Y_u \in \Re^{p\times n}$ the solutions of the set of LMIs given by Proposition 1 with $\mathcal{P}_T = \mathcal{P}^T\mathcal{P}_\varepsilon\mathcal{P}$.

4.5.3. Closed-Loop Stability. Based on the previous results, the following theorem is provided for the complete closed-loop stability for the multiple-output multiple-input case.

Theorem 7. *Let Assumption 2, 3 and 4 hold. Let observer (34) designed according to Theorem 4 in a block form, and the linear control $u = K_u\hat{x}$ designed according to Theorem 5, be applied to the*

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system (1) for the case $m, p \geq 2$. Then the trajectories of the closed-loop system satisfy

$$|e(t)| \leq \frac{\left(-(1-\beta_0)\mu t + V_\varepsilon(0)^\mu\right)^{\frac{1+(n-1)\mu}{\mu}}}{\lambda_{\min}^{1/2}(\mathcal{P}^T\mathcal{P})(\lambda_{\min}(\mathcal{P}_\varepsilon))^{\frac{1-\mu(1+(n-1)\mu)}{2+2(n-1)\mu}}}, \ \forall\, 0 \leq t < T_\varepsilon^+(\varepsilon(0)), \tag{36}$$

$$|x(t)| \leq \epsilon_x, \ \forall t \geq 0, \tag{37}$$

with $V_\varepsilon(0)^\mu = (\lambda_{\min}(\mathcal{P}^T\mathcal{P})\lambda_{\max}(\mathcal{P}_\varepsilon))^{\frac{\mu}{2}}|e(0)|^\mu$, $\epsilon_x > 0$ *, and*

$$|e(t)| \leq \frac{1}{\lambda_{\min}^{1/2}(\mathcal{P}_T)}, \ \forall t \geq T_\varepsilon^+(\varepsilon(0)), \tag{38}$$

$$|x(t)| \leq \frac{1}{\lambda_{\min}^{1/2}(\mathcal{P}_x^{-1})}, \ \text{as } t \to \infty. \tag{39}$$

Proof
The proof follows the same steps as the proof of Theorem 6. □

5. SIMULATION RESULTS

5.1. Spacecraft

Consider the following linear simplified model of a spacecraft [15]

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.0028 & 0.0142 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0825 & -0.4126 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 0.0076 & 0.0037 \\ 0 & 0 \\ -0.1676 & -0.0979 \end{pmatrix} w(t),$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x,$$

where x_1 and x_2 are the rotation angle and angular velocity of the spacecraft, respectively; x_3 is the generalized coordinate and x_4 is its velocity. The initial conditions are $x_0 = (0.8, 0, 0.8, 0)^T$ and the perturbation term

$$w(t) = 0.1 \begin{pmatrix} \sin(10t) + \sin(t) + 0.5 \\ \sin(10t) + \sin(t) + 0.5 \end{pmatrix}, \ Q_w = \begin{pmatrix} 3.3667 & 0 \\ 0 & 3.3667 \end{pmatrix}.$$

It is possible to check that Assumptions 2 and 4 are satisfied, with $\varphi_0 = 0.1059$ and $\sigma_1 = \sigma_2 = 2$, respectively. The transformation matrix $\mathcal{P}$ takes the following form

$$\mathcal{P} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0.4126 & 1 \end{array}\right).$$

For this example, the two nonlinear injections R_1 and R_2 are designed in the same way. Then, selecting $\beta = 0.5$, $\gamma = 0.4250$, $\delta = 0.1$, $\mu = 0.6$, and fixing $\varphi_0 = 0.1059$, the ellipsoidal matrix and the gains, obtained from the solution of the LMI problem stated in Theorem 4 for the observer (4), are the following

$$P_{\bar{\varepsilon}_{\sigma_1}} = P_{\bar{\varepsilon}_{\sigma_2}} = \begin{pmatrix} 1.0000 & -0.6323 \\ -0.6323 & 0.4462 \end{pmatrix}, \ k_1 = k_2 = \begin{pmatrix} -10.1120 & -11.6682 \end{pmatrix}^T.$$

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For the matrix feedback gain design, let us fix $\alpha_x = 0.01$, then the solutions of the LMIs (24)-(25), given by Proposition 1, are the following

$$\mathcal{P}_x = \begin{pmatrix} 30.8996 & -5.6253 & 37.7240 & 6.1230 \\ -5.6253 & 18.1212 & -11.5245 & -6.2370 \\ 37.7240 & -11.5245 & 88.9076 & -1.9046 \\ 6.1230 & -6.2370 & -1.9046 & 15.6509 \end{pmatrix},$$

$$R_x = \begin{pmatrix} 1.3362 & 0.5635 & -0.3028 & -0.3051 \\ 0.5635 & 1.6689 & -0.3965 & -0.4024 \\ -0.3028 & -0.3965 & 1.1278 & 0.2183 \\ -0.3051 & -0.4024 & 0.2183 & 1.1299 \end{pmatrix} \times 10^3,$$

$$K_u = \begin{pmatrix} -1.5966 & -0.5595 & 0.7268 & 1.0558 \end{pmatrix}.$$

In order to have a reference for the improvements with respect to the classic linear design control schemes, the following Luenberger observer-based feedback control is implemented

$$u = K\hat{x},$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

The control gain matrix K and the observer gain matrix L are calculated by means of the AEM (see Appendix 6.1). The obtained results are the following

$$K = \begin{pmatrix} -0.8111 & -1.3041 & 0.2786 & -0.6873 \end{pmatrix}, \; L = \begin{pmatrix} 1.5322 & 0.0116 \\ 1.0192 & 0.1140 \\ 0.0115 & 1.5828 \\ 0.0644 & 2.1971 \end{pmatrix}.$$

The simulations have been done using the explicit Euler method with a sampling time equal to 0.001 seconds while the solutions for the LMIs were found using SeDuMi solver among YALMIP in MATLAB. The results are depicted in the Figures 1-4.

From Figures 1-4 one may conclude that if the control based on homogeneous differentiator is applied, then the trajectories of the system converge to a smaller region than the trajectories with the control based on AEM design. Moreover, for the control with homogeneous differentiator the trajectories of the system do not present an overshoot, and a better precision for the state estimation is achieved than with Luenberger observer.

Note that in the Figures 2 and 4 the presence of oscillations can be detected in the state and in the control signa. This effect is caused by the homogeneous differentiator structure itself and it might be seen as a trade-off with the faster convergence rate. It is noteworthy that it is possible to tune the parameters of the homogeneous differentiator, specifically μ, to reduce this effect with the cost of having a slower convergence rate.

6. CONCLUSIONS

This work deals with the design of a robust control for linear systems with external disturbances based on a *homogeneous differentiator* observer. Sufficient conditions for stability of the closed-loop system in the presence of external disturbances are obtained and represented by LMIs. The parameter tuning for both controller and observer is formulated as the semi-definite programming problem with LMIs constraints. Simulation results illustrate the feasibility of the proposed approach and some improvements with respect to the classic linear observer approach.

APPENDIX

The following auxiliary result is required.

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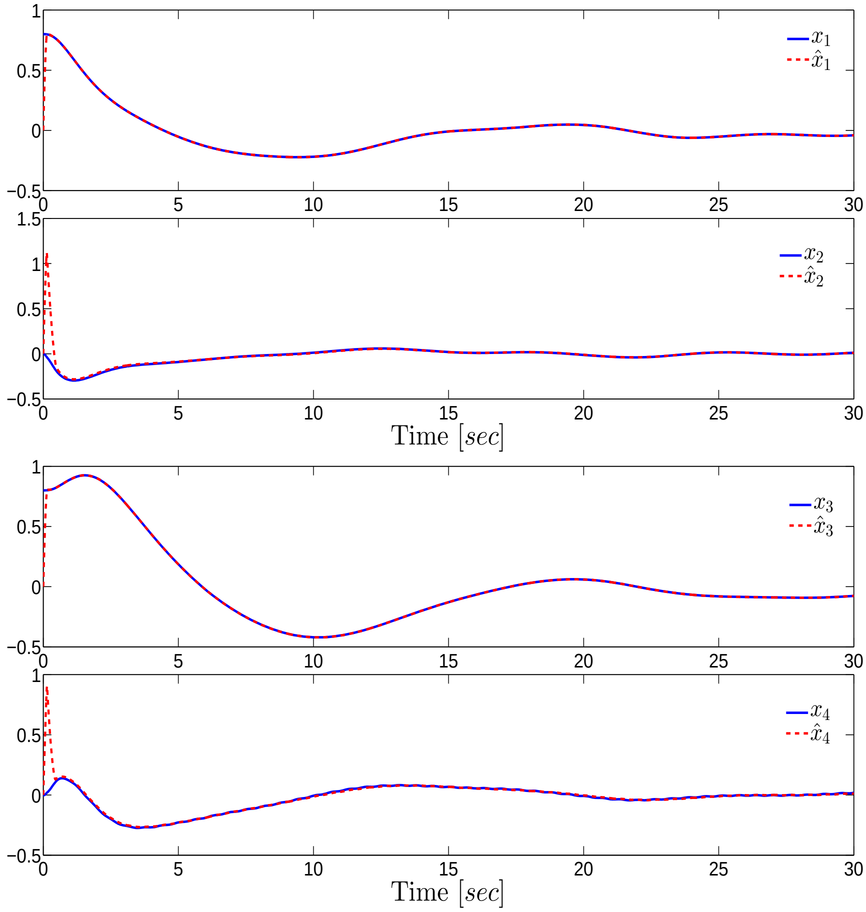

Figure 1. Real State vs Estimated State. *In this figure the state estimation given by the* homogeneous differentiator observer *is illustrated. The fast rate of converge can be seen in these graphs.*

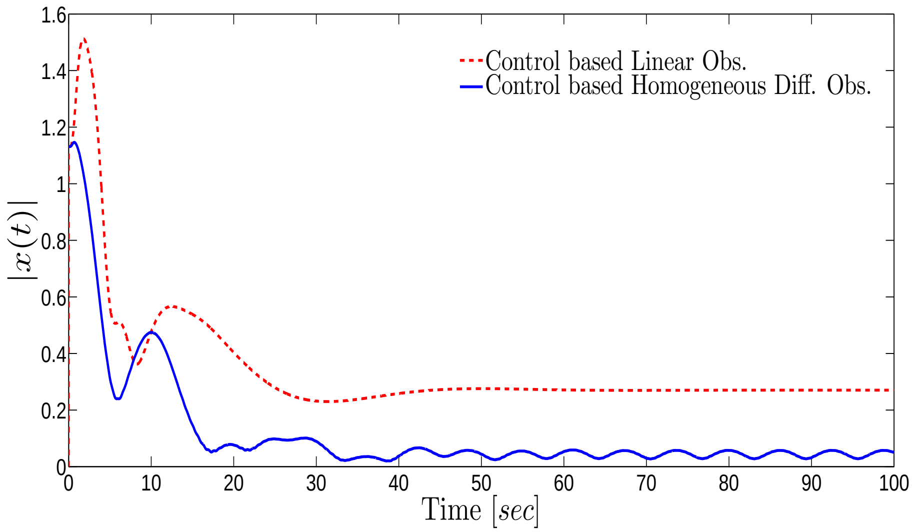

Figure 2. Trajectories of the Real System. *The trajectories of the system when the control based* homogeneous differentiator observer *is used converge to a smaller region than the trajectories for the control based Luenberger observer and there does not appear overshoot.*

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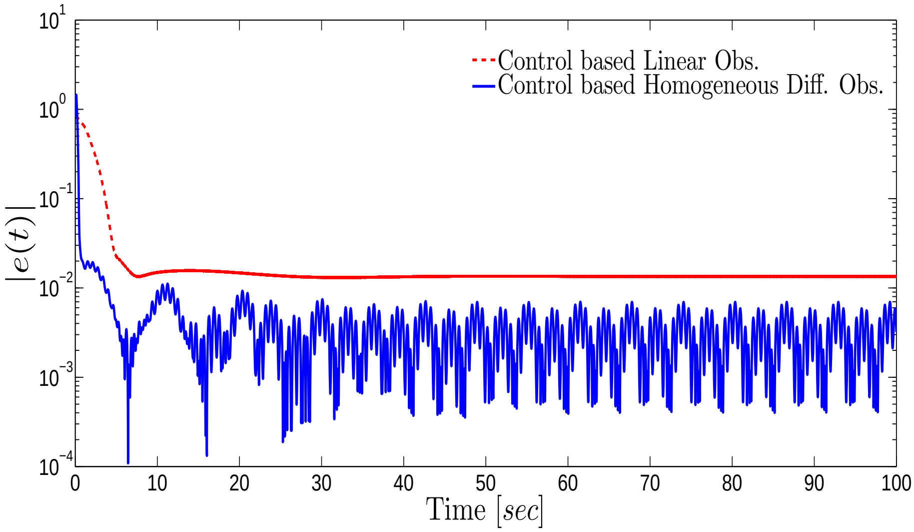

Figure 3. State Estimation Error. *It is evident that the* homogeneous differentiator observer*provides a better precision for the state estimation than the Luenberger observer based on AEM.*

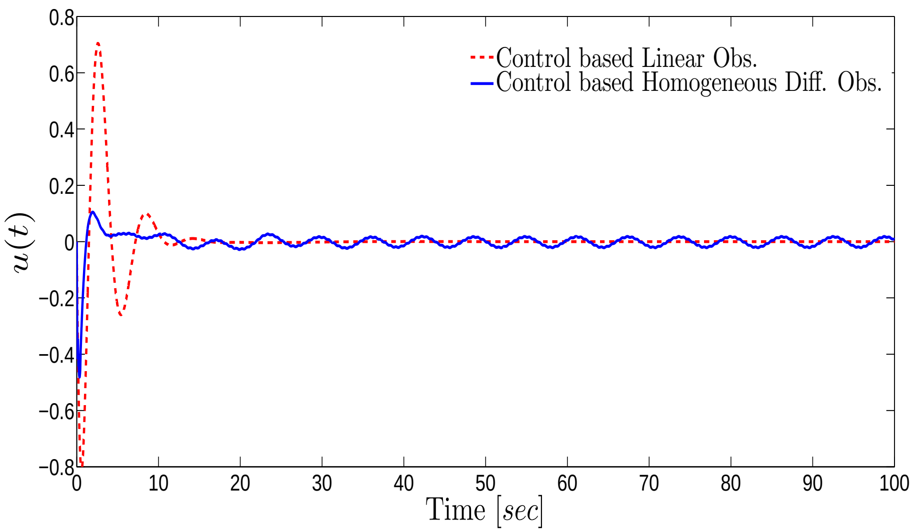

Figure 4. Control Signal. *The signal control based on the Luenberger observer spends more energy during the transient than the* homogeneous differentiator observer. *After this transient, the corresponding control signals are practically the same for both control approaches.*

Lemma 1. *Let $P_\varepsilon \in \Re^{n\times n}$, $P_\varepsilon = P_\varepsilon^T > 0$, the vectors $\varepsilon \in \Re^n$, $h = (1, 0, \ldots, 0)^T \in \Re^n$, and the scalar $\delta \in (0, 1)$. Consider the function $G(V_\varepsilon, \varepsilon)$ defined as in (13). If $G(V_\varepsilon, \varepsilon) = 0$ and the inequality (15) is fulfilled, then*

$$0 \le \frac{V_\varepsilon(\varepsilon_1 h)}{V_\varepsilon(\varepsilon)} \le \delta^{-1/(2+2(n-1)\mu)}, \tag{40}$$

with $\varepsilon_1 \in \Re$ the first component of the vector ε.

Proof
Let us introduce the notation $V_h := V_\varepsilon(\varepsilon_1 h)$ and $V_\varepsilon := V_\varepsilon(\varepsilon)$. Denote the elements of matrix P_ε as p_{ij}. It is clear that for $\varepsilon_1 = 0$ the inequality (40) holds trivially. Then, let us consider the case when $\varepsilon_1 \neq 0$. Note that $G(V_h, \varepsilon_1 h) = 0$ implies that $\varepsilon_1^2 h^T P_\varepsilon h = V_h^{2+2(n-1)\mu}$. Thus, $G(V_\varepsilon, \varepsilon) = 0$,

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for $\varepsilon_1 > 0$, can be rewritten in the following form

$$\begin{pmatrix} \frac{V_h^{1+(n-1)\mu}}{\sqrt{p_{11}}} \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}^T \Lambda_r(V_\varepsilon^{-1}) P_\varepsilon \Lambda_r(V_\varepsilon^{-1}) \begin{pmatrix} \frac{V_h^{1+(n-1)\mu}}{\sqrt{p_{11}}} \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} = 1. \tag{41}$$

The case when $\varepsilon_1 < 0$ can be analyzed analogously. Denote $p_* = \left(\frac{p_{12}}{\sqrt{p_{11}}}, \dots, \frac{p_{1n}}{\sqrt{p_{11}}} \right) \in \Re^{n-1}$ and

$$\bar{P}_\varepsilon = \begin{pmatrix} p_{22} & \cdots & p_{2n} \\ \vdots & \ddots & \vdots \\ p_{n2} & \cdots & p_{nn} \end{pmatrix} \in \Re^{(n-1)\times(n-1)}.$$

Hence, for $\varepsilon_1 > 0$, (41) can be rewritten as follows

$$\begin{pmatrix} 1 \\ \tilde{\varepsilon}_2 \\ \vdots \\ \tilde{\varepsilon}_n \end{pmatrix}^T \begin{pmatrix} 1 & p_*^T \\ p_* & \bar{P}_\varepsilon \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{\varepsilon}_2 \\ \vdots \\ \tilde{\varepsilon}_n \end{pmatrix} = \left(\frac{V_\varepsilon}{V_h} \right)^{2+2(n-1)\mu}, \tag{42}$$

where $\tilde{\varepsilon}_i = \varepsilon_i V_\varepsilon^{(1-i)\mu} / V_h^{1+(n-1)\mu}$, $i = \overline{2, n}$. Equivalently, in terms of $\tilde{\varepsilon} = (\tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_n)^T \in \Re^{n-1}$, (42) is rewritten as

$$\left(\frac{V_\varepsilon}{V_h} \right)^{2+2(n-1)\mu} = \tilde{\varepsilon}^T \bar{P}_\varepsilon \tilde{\varepsilon} + p_*^T \tilde{\varepsilon} + \tilde{\varepsilon}^T p_* + 1.$$

Since $\bar{P}_\varepsilon > 0$, the right hand side of the last equality has a minimal value. Thus, it can be shown that this value will be attained for $\tilde{\varepsilon} = -\bar{P}_\varepsilon^{-1} p_*$, *i.e.*

$$\left(\frac{V_\varepsilon}{V_h} \right)^{2+2(n-1)\mu} \geq 1 - p_*^T \bar{P}_\varepsilon^{-1} p_*.$$

Finally, the inequality (15) implies that $1 - p_*^T \bar{P}_\varepsilon^{-1} p_* \geq \delta$, with $\delta \in (0, 1)$. Therefore, (40) is obtained and the theorem is proved. □

Proof of Theorem 4: The error dynamics given in (10) is rewritten as follows *i.e.*

$$\dot{\varepsilon} = \left(A_0 + \Lambda_{\tilde{r}}(V_\varepsilon^{-1}(\varepsilon_1 h)) k h^T \right) \varepsilon + d_w + d_\varepsilon,$$

where $d_w = \bar{D} w$, and $d_\varepsilon = \left(\Lambda_{\tilde{r}}(V_h^{-1}) - \Lambda_{\tilde{r}}(V_\varepsilon^{-1}) \right) k h^T \varepsilon$. To prove the convergence of ε to zero, let us compute the time derivative of the function V_ε by means of the implicit function theorem, *i.e.*

$$\dot{V}_\varepsilon = - \left[\frac{\partial G}{\partial V_\varepsilon} \right]^{-1} \frac{\partial G}{\partial \varepsilon} \dot{\varepsilon}.$$

Then, it follows that

$$0 > \frac{\partial G}{\partial V_\varepsilon} = -V_\varepsilon^{-1} \varepsilon^T \Lambda_r(V_\varepsilon^{-1}) \left(P_\varepsilon H + H P_\varepsilon \right) \Lambda_r(V_\varepsilon^{-1}) \varepsilon,$$

and

$$\frac{\partial G}{\partial \varepsilon} \dot{\varepsilon} = 2 \varepsilon^T \Lambda_r(V_\varepsilon^{-1}) P_\varepsilon \Lambda_r(V_\varepsilon^{-1}) \left(\left(A_0 + \Lambda_{\tilde{r}}(V_\varepsilon^{-1}(\varepsilon_1 h)) k h^T \right) \varepsilon + d_w + d_\varepsilon \right).$$

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Considering the homogeneity properties of the dilation matrix, *i.e.*

$$\Lambda_r(V_\varepsilon^{-1})A_0[\Lambda_r(V_\varepsilon^{-1})]^{-1} = V_\varepsilon^{-\mu}A_0,$$
$$\Lambda_{\tilde r}(V_\varepsilon^{-1}(\varepsilon_1 h))kh^T = V_\varepsilon^{-\mu}[\Lambda_r(V_\varepsilon^{-1})]^{-1}kh^T\Lambda_r(V_\varepsilon^{-1}),$$

one obtains

$$\begin{aligned}\frac{\partial G}{\partial \varepsilon}\dot{\varepsilon} = 2V_\varepsilon^{-\mu}\Big[&\varepsilon^T\Lambda_r(V_\varepsilon^{-1})\left(P_\varepsilon A_0 + P_\varepsilon kh^T\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon \\ &+V_\varepsilon^{\mu-1}\varepsilon^T\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})d_w + V_\varepsilon^{\mu}\varepsilon\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})d_\varepsilon\Big]. \quad (43)\end{aligned}$$

Let us introduce the extended vector ξ, *i.e.*

$$\xi := \begin{pmatrix}\Lambda_r(V_\varepsilon^{-1})\varepsilon \\ \Lambda_r(V_\varepsilon^{-1})d_\varepsilon \\ \Lambda_r(V_\varepsilon^{-1})d_w\end{pmatrix}.$$

Adding and subtracting the following terms

$$V_\varepsilon^{-\mu}\varepsilon^T\Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon,$$
$$V_\varepsilon^{\mu}\beta^{-1}d_\varepsilon^T\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})d_\varepsilon,$$
$$V_\varepsilon^{\mu}\gamma\varphi_0^{-1}d_w^T\Lambda_r^2(V_\varepsilon^{-1})d_w,$$

where $\varphi := |\Lambda_r(V_\varepsilon^{-1})d_w| \leq \varphi_0 := |\bar{D}|w^+$ for $V_\varepsilon > 1$. Then, from (43), it is given that

$$\begin{aligned}\frac{\partial G}{\partial \varepsilon}\dot{\varepsilon} = V_\varepsilon^{-\mu}\big(&\xi^T\Omega\xi - \varepsilon^T\Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon \\ &+\gamma\varphi_0^{-1}\varphi^2 + \beta^{-1}V_\varepsilon^{2\mu}d_\varepsilon^T\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})d_\varepsilon\big),\end{aligned}$$

where

$$\Omega = \begin{pmatrix}P_\varepsilon A_0 + A_0^TP_\varepsilon + P_\varepsilon kh^T + hk^TP_\varepsilon + HP_\varepsilon + P_\varepsilon H & V_\varepsilon^\mu P_\varepsilon & V_\varepsilon^\mu P_\varepsilon \\ V_\varepsilon^\mu P_\varepsilon & -\beta^{-1}V_\varepsilon^{2\mu}P_\varepsilon & 0 \\ V_\varepsilon^\mu P_\varepsilon & 0 & -V_\varepsilon^{2\mu}\gamma\varphi_0^{-1}I_n\end{pmatrix}. \quad (44)$$

Applying Schur's complement to (44) and using the fact that $|\varphi| \leq \varphi_0$, one obtains LMI (16). Thus, the term $\frac{\partial G}{\partial \varepsilon}\dot{\varepsilon}$ is upper bounded for $V_\varepsilon > 1$ as follows

$$\frac{\partial G}{\partial \varepsilon}\dot{\varepsilon} \leq V_\varepsilon^{-\mu}\left[-\varepsilon^T\Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon + \gamma\varphi_0 + \beta S(\varepsilon, V_\varepsilon, V_h)\right], \quad (45)$$

with

$$S(\varepsilon, V_\varepsilon, V_h) = \beta^{-2}V_\varepsilon^{2\mu}d_\varepsilon^T\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})d_\varepsilon.$$

Taking into account that $\varepsilon_1^2h^TP_\mu h = V_\varepsilon(\varepsilon_1 h)^{2+2(n-1)\mu}$, it follows that

$$S(\varepsilon, V_\varepsilon, V_h) = \frac{k^T\left(\Lambda_{\tilde r}(V_h^{-1}) - \Lambda_{\tilde r}(V_\varepsilon^{-1})\right)\Lambda_r(V_\varepsilon^{-1})P_\varepsilon\Lambda_r(V_\varepsilon^{-1})\left(\Lambda_{\tilde r}(V_h^{-1}) - \Lambda_{\tilde r}(\varepsilon)^{-1}\right)k}{\beta^2h^TP_\varepsilon h(V_h/V_\varepsilon)^{-2-2(n-1)\mu}},$$

and since $V_h^{1+(n-1)\mu}\Lambda_{\tilde r}(V_h^{-1}) = \Lambda_r(V_h)$, denoting $\lambda = V_h/V_\varepsilon$, it is given that

$$S(\varepsilon, V_\varepsilon, V_h) = \frac{k^T\left(I_n - \Lambda_{\tilde r}(\lambda)\right)\Lambda_r(\lambda)P_\varepsilon\Lambda_r(\lambda)\left(I_n - \Lambda_{\tilde r}(\lambda)\right)k}{\lambda^{2\mu}\beta^2h^TP_\varepsilon h}.$$

Now, from Lemma 1, it follows that $\lambda \in [0, \lambda^*]$, with $\lambda^* = \delta^{-1/(2+2(n-1)\mu)}$. Then, inequality (17) implies that $S(\varepsilon, V_\varepsilon, V_h) \leq 1$, for any $\lambda \in [0, \lambda^*]$. Hence, (45) can be upper bounded for all $V_\varepsilon > 1$

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as

$$\frac{\partial G}{\partial \varepsilon}\dot{\varepsilon} \leq V_\varepsilon^{-\mu}\left(-\varepsilon^T \Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon + \gamma\varphi_0 + \beta\right). \tag{46}$$

For the time derivative of V_ε, it is given that

$$\begin{aligned}\dot{V}_\varepsilon &\leq \frac{-\varepsilon^T \Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon + \gamma\varphi_0 + \beta}{\varepsilon^T \Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon} V_\varepsilon^{1-\mu} \\ &= \left(-1 + \frac{\gamma\varphi_0 + \beta}{\varepsilon^T \Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon}\right) V_\varepsilon^{1-\mu},\ \forall V_\varepsilon > 1.\end{aligned}$$

Then, if $G(V_\varepsilon, \varepsilon) = 0$ and $\beta_0^2 := \gamma\varphi_0 + \beta$ is such that LMI (18) is satisfied, it follows that for all $V_\varepsilon > 1$

$$\dot{V}_\varepsilon \leq \left(-1 + \frac{\beta_0^2 \varepsilon^T \Lambda_r(V_\varepsilon^{-1})P_\varepsilon \Lambda_r(V_\varepsilon^{-1})\varepsilon}{\varepsilon^T \Lambda_r(V_\varepsilon^{-1})\left(HP_\varepsilon + P_\varepsilon H\right)\Lambda_r(V_\varepsilon^{-1})\varepsilon}\right) V_\varepsilon^{1-\mu} \leq -\left(1-\beta_0\right) V_\varepsilon^{1-\mu}. \tag{47}$$

Therefore, since $\varepsilon^T \Lambda_r(V_\varepsilon^{-1})P_\varepsilon \Lambda_r(V_\varepsilon^{-1})\varepsilon = 1$ it follows that $\lambda_{\min}(P_\varepsilon)\,|\varepsilon|^2 \leq \varepsilon^T P_\varepsilon \varepsilon \leq 1$. Thus, the estimation error e converges in a finite time to a neighborhood of the origin given by (19). Finally, the computation of the settling time function (20) is straightforward from (47). Hence, the theorem is proven. □

Proof of Theorem 5: Under Assumption 3 the matrix K_u always exists, then due to Assumption 2 and the statements of Theorem 4 it follows that

$$w^T Q_w w \leq 1,\ \ e^T P_T e \leq 1, \tag{48}$$

for the nonsingular matrix P defined in (5), and the positive definite matrix $P_\varepsilon = P_\varepsilon^T > 0$ defined in Theorem 4 such that $P_T = P^T P_\varepsilon P$, and $Q_w = Q_w^T > 0$ such that $w^+ = (\lambda_{\min}(Q_w))^{-1/2}$.

Let us consider the quadratic function $V_x(x) = x^T P_x^{-1} x$, with $P_x = P_x^T > 0$. Its derivative along the trajectories of the system (8) is

$$\begin{aligned}\dot{V}_x &= \begin{pmatrix} x \\ e \\ w \end{pmatrix}^T \begin{pmatrix} P_x^{-1}A_K + A_K^T P_x^{-1} + 2\alpha_x P_x^{-1} & P_x^{-1}BK_u & P_x^{-1}D \\ \star & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times q} \\ \star & \star & \mathbf{0}_{q\times q} \end{pmatrix} \begin{pmatrix} x \\ e \\ w \end{pmatrix} \\ &= \begin{pmatrix} x \\ e \\ w \end{pmatrix}^T \begin{pmatrix} P_x^{-1}A_K + A_K^T P_x^{-1} + \alpha_x P_x^{-1} & P_x^{-1}BK_u & P_x^{-1}D \\ \star & -\alpha_x P_T & \mathbf{0}_{n\times q} \\ \star & \star & -\alpha_x Q_w \end{pmatrix} \begin{pmatrix} x \\ e \\ w \end{pmatrix} \\ &\quad - 2\alpha_x x^T P_x^{-1} x + \alpha_x e^T P_T e + \alpha_x w^T Q_w w,\end{aligned}$$

for some positive scalar α_x. Taking into account the assumptions given in (48), it follows that

$$\begin{aligned}\dot{V}_x \leq &-2\alpha_x\left(x^T P_x^{-1} x - 1\right) + \\ &\begin{pmatrix} x \\ e \\ w \end{pmatrix}^T \begin{pmatrix} P_x^{-1}A_K + A_K^T P_x^{-1} + \alpha_x P_x^{-1} & P_x^{-1}BK_u & P_x^{-1}D \\ \star & -\alpha_x P_T & \mathbf{0}_{n\times q} \\ \star & \star & -\alpha_x Q_w \end{pmatrix} \begin{pmatrix} x \\ e \\ w \end{pmatrix}.\end{aligned}$$

Therefore, if the matrix inequality (23) holds, then the time derivative of V_x is negative definite outside the ellipsoid $x^T P_x^{-1} x \leq 1$, which implies by [15] that $\mathcal{E}(P_x)$ is the attractive ellipsoid of the closed-loop system (8). □

6.1. AEM Linear Case

Let us consider, based on [15], the linear version of the AEM to compare the results and the implementability of the approach presented in this paper. The linear AEM formulation consists

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in using an observer based feedback $u = K\hat{x}$, where $K \in \Re^{m\times n}$ is a constant gain and $\hat{x}$ is the state estimation whose dynamics is given by the classic Luenberger observer, *i.e.*

$$\dot{\hat{x}} = A\hat{x} + BK\hat{x} + L(y - C\hat{x}),$$

with the matrix gain $L \in \Re^{n\times m}$. Define the estimation error as $e = x - \hat{x}$. Now, let us select a quadratic storage function

$$V_c = \hat{x}^T P_c \hat{x} + e^T P_e e,$$

with $P_c = P_c^T > 0$ and $P_e = P_e^T > 0$. The time derivative of the function V_c can be written as

$$\dot{V}_c = 2\hat{x}^T P_c\left[(A+BK)\hat{x} + LCe\right] + 2e^T P_e\left[(A-LC)e + Dw\right] =$$
$$\begin{pmatrix}\hat{x}\\ e\\ \omega\end{pmatrix}^T \begin{pmatrix} P_c(A+BK)+(A+BK)^T P_c & P_cLC & 0\\ \star & P_e(A-LC)+(A-LC)^T P_e & P_eD\\ \star & \star & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ e\\ \omega\end{pmatrix}.$$

Then, adding and subtracting $\alpha_c x^T P_c x = \alpha_c(\hat{x}+e)^T P_c(\hat{x}+e)$ and $\alpha_c\omega^T Q\omega$, it is given that

$$\dot{V}_c = z_c^T W z_c - \alpha_c x^T P_c^{-1} x + \alpha_c \omega^T Q\omega,$$

where $z_c = (\hat{x}^T, e^T, w^T)^T$, and

$$W = \begin{pmatrix} P_cA_k + A_k^T P_c + \alpha_c P_c & P_cLC + \alpha_c P_c & 0\\ \star & P_eA_l + A_l^T P_e + \alpha_c P_c & P_eD\\ \star & \star & -\alpha_c Q\end{pmatrix}, \tag{49}$$

where $A_k = A + BK$ and $A_l = A - LC$. Thus, taking into account that $\omega^T Q\omega \leq 1$, if $W \leq 0$ then an upper bound for the derivative of $\dot{V}_c$ is given by

$$\dot{V}_c \leq \alpha_c(1 - x^T P_c x).$$

To facilitate the implementation it is possible to approximate the nonlinear inequality $W \leq 0$ by applying the quadratic transformation $W_1 = TWT^T$ with $T = \text{diag}(P_eP_c^{-1},\ I_n,\ I_n)$. Then, let us define $Y_1 := KP_c^{-1}$ and $Y_2 := P_eL$, and inroduce R_1 and R_2 such that

$$P_e\left(AP_c^{-1} + BY_1 + Y_1^T B^T + P_c^{-1}A^T + \alpha_c P_c^{-1}\right)P_e \leq -R_1 < 0,$$
$$0 < P_c \leq R_2.$$

Applying $\Lambda-$inequality and Schur's complement it follows that

$$\begin{pmatrix} R_1 - 2P_c^{-1} & I_n\\ I_n & AP_c^{-1} + BY_1 + (AP_c^{-1}+BY_1)^T + \alpha_c P_c^{-1}\end{pmatrix} \leq 0, \tag{50}$$

$$\begin{pmatrix} R_2 & I_n\\ I_n & P_c^{-1}\end{pmatrix} \geq 0. \tag{51}$$

Therefore, W_1 is upper estimated by W_2, *i.e.* $W_1 \leq W_2$, where

$$W_2 = \begin{pmatrix} -R_1 & Y_2C + \alpha_c Pe & 0\\ \star & P_eA - Y_2C + (P_eA - Y_2C)^T + \alpha_c R_2 & P_eD\\ \star & \star & -\alpha_c Q\end{pmatrix} \leq 0. \tag{52}$$

Hence, it is clear that if $W_2 \leq 0$, then $W \leq 0$ is true, and the ellipsoid characterized by the matrix P_c is an attractive ellipsoid for the system (1).

Now, to minimize the size of the ellipsoid, the function $tr(P_c^{-1})$ is selected as the objective, and the following optimization problem is proposed

$$tr(P_c^{-1}) \to \min_{P_c^{-1}, P_e.Y_1, Y_2, R_1, R_2}$$
$$\text{subject to (50), (51), (52)}$$

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The solution $(\hat{P}_c^{-1}, \hat{P}_e, \hat{Y}_1, \hat{Y}_2)$ is used to calculate the minimizing control gain as $\hat{K} = \hat{Y}_1 \hat{P}_c$ and the optimal observer gain as $\hat{L} = \hat{P}_e^{-1} \hat{Y}_2$.

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