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# Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

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<sup>2</sup>Université de Pau et des Pays de l'Adour

<sup>3</sup>University of the Basque Country (UPV/EHU), Leioa, Spain

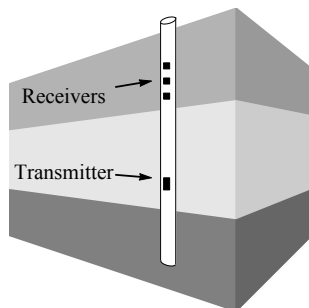
<sup>4</sup>Basque Center for Applied Mathematics (BCAM), Bilbao, Spain

<sup>5</sup>Ikerbasque (Basque Foundation for Sciences), Bilbao, Spain



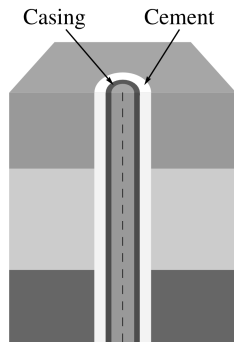
# MOTIVATION

- **Main goal:** To obtain a better characterization of the Earth's subsurface
- **How:** Recording borehole resistivity measurements
- **Procedure:**
  - Well
  - Transmitters
  - Receivers



# MOTIVATION

- **Practical Difficulties:**
  - It is not easy to drill a borehole
  - It may collapse
- **Practical Solutions:**
  - Use a metallic casing
  - Surround with a cement layer
- **Problem solved, but...**



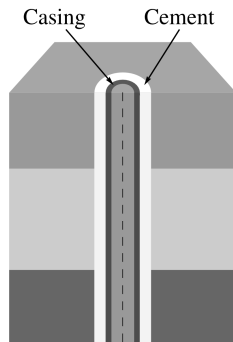
# MOTIVATION

- **Practical Difficulties:**

- It is not easy to drill a borehole
- It may collapse

- **Practical Solutions:**

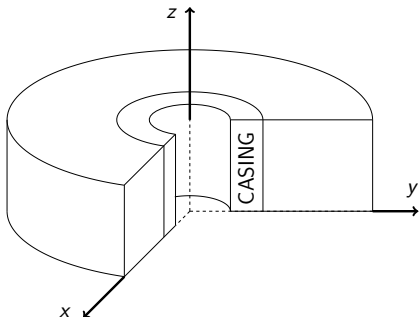
- Use a metallic casing
- Surround with a cement layer



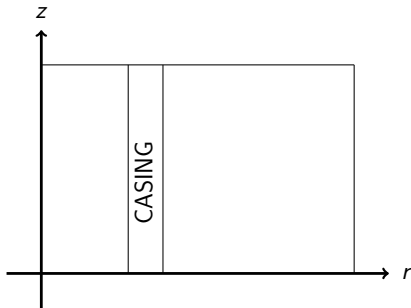
- **Problem solved, but... Numerical problems due to the high conductivity and thinness of the casing**

# CONFIGURATION OF INTEREST

## SECTIONED 3D DOMAIN



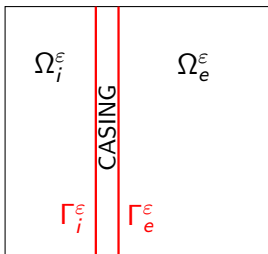
## MERIDIAN DOMAIN



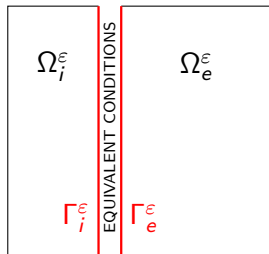
We reduce the 3D problem to a 2D problem due to the **axysimmetric** configuration.

# MAIN IDEA

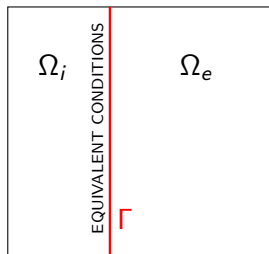
## REFERENCE MODEL



1st Approach

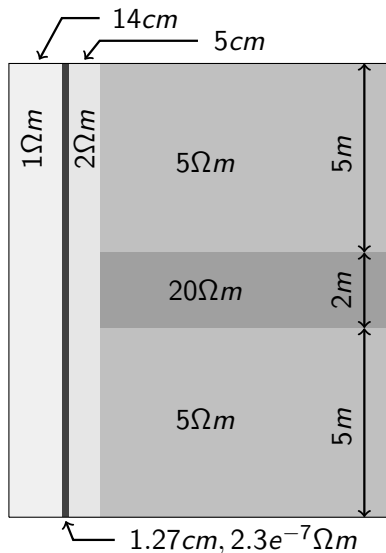


2nd Approach



**Idea:** Replace the casing by  
equivalent conditions.

# REALISTIC SCENARIO



- **Conductivity and casing width:**

$$\begin{cases} \varepsilon = 1.27e - 2m \\ \sigma_c = 4.34e6 \Omega^{-1}m^{-1} \end{cases}$$

$$\Rightarrow \sigma_c \approx \varepsilon^{-3}$$

- **Classical approach:**

$$\sigma_c = \alpha \quad \alpha \in \mathbb{R}$$

- **High conductive case:**

$$\sigma_c = \alpha \varepsilon^{-3} \quad \alpha \in \mathbb{R}$$



# REFERENCES

- [1] A.A. Kaufman. The electrical field in a borehole with a casing. *Geophysics*, Vol.55, Issue 1, pp. 29-38, 1990.
- [2] D.Pardo, C.Torres-Verdín and Z.Zhang. Sensitivity study of borehole-to-surface and crosswell electromagnetic measurements acquired with energized steel casing to water displacement in hydrocarbon-bearing layers. *Geophysics*, 73 No.6, F261-F268, 2008.
- [3] M. Duruflé, V. Péron and C. Poignard. Thin Layer Models for Electromagnetism. *Communications in Computational Physics* 16(1):213-238, 2014.
- [4] K. Schmidt, A. Chernov, Robust transmission conditions of high order for thin conducting sheets in two dimensions, *IEEE Trans. Magn.*, 50 (2014), pp. 41–44.

# OUTLINE

- 1 Equivalent Conditions
- 2 Numerical Results
- 3 Second Approach
- 4 Application
- 5 Perspectives

# OUTLINE

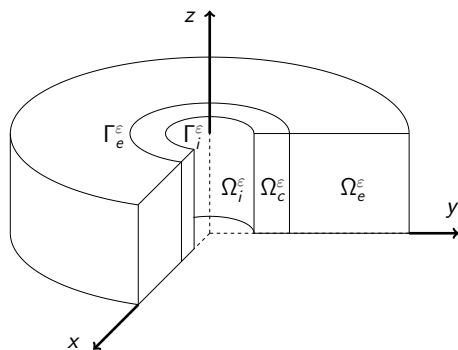
- 1 **Equivalent Conditions**
- 2 Numerical Results
- 3 Second Approach
- 4 Application
- 5 Perspectives

## MODEL PROBLEM - 3D

## STATIC ELECTRIC POTENTIAL

$$\operatorname{div}((\sigma - i\epsilon\omega)\nabla u) = f \text{ in } \Omega$$

Where  $f$  is a known data,  $\sigma$  is piecewise constant



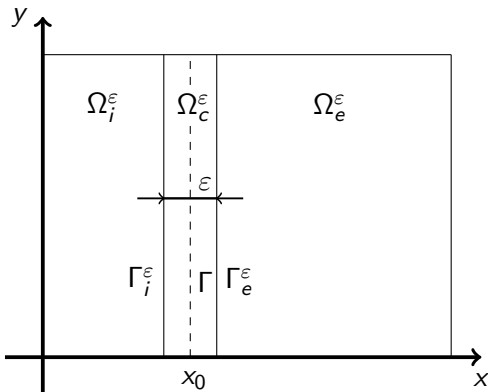
$$\sigma = \begin{cases} \sigma_i & \text{in } \Omega_i^\epsilon \\ \sigma_c = \alpha\epsilon^{-3} & \text{in } \Omega_c^\epsilon \\ \sigma_e & \text{in } \Omega_e^\epsilon \end{cases}$$

and the solution is expressed as

$$u = \begin{cases} u_i & \text{in } \Omega_i^\epsilon \\ u_c & \text{in } \Omega_c^\epsilon \\ u_e & \text{in } \Omega_e^\epsilon \end{cases}$$

# MODEL PROBLEM - 2D

Reference problem (P) set in the domain  $\Omega = \Omega_i^\varepsilon \cup \Omega_c^\varepsilon \cup \Omega_e^\varepsilon \cup \Gamma_i^\varepsilon \cup \Gamma_e^\varepsilon$ .



$$\left\{ \begin{array}{ll} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_c \Delta u_c = 0 & \text{in } \Omega_c^\varepsilon \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\ u_i = u_c & \text{on } \Gamma_i^\varepsilon \\ u_c = u_e & \text{on } \Gamma_e^\varepsilon \\ \sigma_i \partial_n u_i = \sigma_c \partial_n u_c & \text{on } \Gamma_i^\varepsilon \\ \sigma_c \partial_n u_c = \sigma_e \partial_n u_e & \text{on } \Gamma_e^\varepsilon \\ u = 0 & \text{on } \partial\Omega \end{array} \right.$$

# EXISTENCE AND UNIQUENESS

## Theorem 1

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , if  $f_i \in L^2(\Omega_i^\varepsilon)$  and  $f_e \in L^2(\Omega_e^\varepsilon)$ , then  $\exists! u \in H_0^1(\Omega)$  solution of (P) and

$$\|u\|_{1,\Omega} \leq C \left( \|f_i\|_{0,\Omega_i^\varepsilon} + \|f_e\|_{0,\Omega_e^\varepsilon} \right).$$

### Proof:

- Derive **variational formulation**, for all  $v \in H_0^1(\Omega)$ ,  $a(u, v) = b(v)$ .
- Prove that  $a$  is **coercive** in  $H_0^1(\Omega)$ .
- Apply the **Lax-Milgram** Lemma.

# DEFINITIONS

## Definition 1

We define the **jump** and **mean value** of the solution  $u$  across the casing as

$$[u] = u|_{\Gamma_e^\varepsilon} - u|_{\Gamma_i^\varepsilon}$$

$$\{u\} = \frac{1}{2} \left( u|_{\Gamma_e^\varepsilon} + u|_{\Gamma_i^\varepsilon} \right)$$

## Definition 2

Let  $u$  be the reference solution. We say an asymptotic model is of **Order  $k+1$** , if its solution  $u^{[k]}$  satisfies

$$\|u - u^{[k]}\|_{H^1} \leq C\varepsilon^{k+1} \quad \text{when } \varepsilon \rightarrow 0$$

# METHODOLOGY

- **Step1:** Derive an **Asymptotic Expansion** for  $u$  when  $\varepsilon \rightarrow 0$

- In the casing: 
$$u_c(x, y) = \sum_{n \in \mathbb{N}} \varepsilon^n U^n \left( \frac{x - x_0}{\varepsilon}, y \right)$$

- Outside the casing: 
$$u(x, y) = \sum_{n \in \mathbb{N}} \varepsilon^n u^n(x, y)$$

- **Step2:** Obtain **Equivalent Conditions** of order  $k + 1$  by identifying a simpler problem satisfied by the truncated expansion and neglecting the higher order terms in  $\varepsilon$

- $$u_{k,\varepsilon} := u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots + \varepsilon^k u^k$$

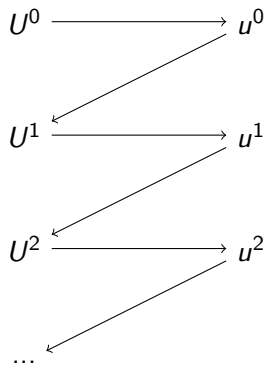
- **Step3:** **Prove Convergence** rates for asymptotic models



# MULTISCALE EXPANSION

$$\left\{ \begin{array}{ll}
 \partial_X^2 U^k + \partial_y^2 U^{k-2} = 0 & \text{in } \Omega_c^\varepsilon \\
 \sigma_i \Delta u_i^k = f_i & \text{in } \Omega_i^\varepsilon \\
 \sigma_e \Delta u_e^k = f_e & \text{in } \Omega_e^\varepsilon \\
 u_i^k = U^k & \text{on } \Gamma_i^\varepsilon \\
 U^k = u_e^k & \text{on } \Gamma_e^\varepsilon \\
 \sigma_i \partial_x u_i^{k-4} = \alpha \partial_X U^k & \text{on } \Gamma_i^\varepsilon \\
 \alpha \partial_X U^k = \sigma_e \partial_x u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\
 U^k = 0 & \text{on } \partial\Omega
 \end{array} \right.$$

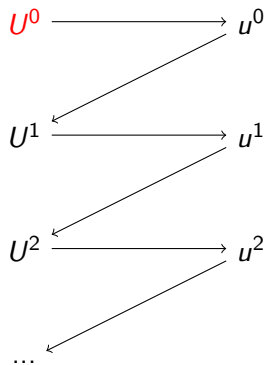
## PROCEDURE



# MULTISCALE EXPANSION

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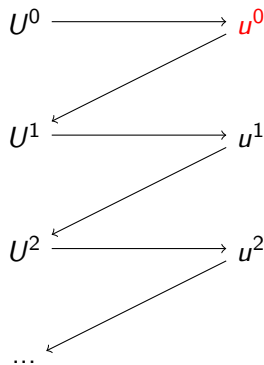
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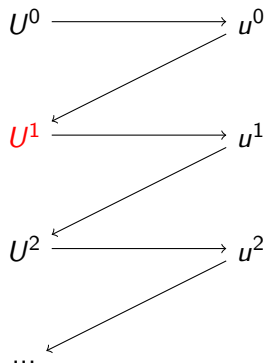
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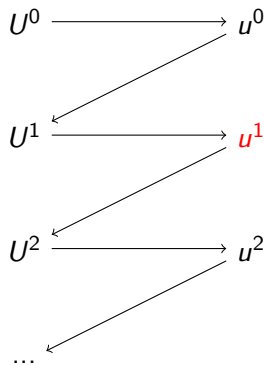
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# MULTISCALE EXPANSION

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 \alpha \partial_X U^k = \sigma_e \partial_x u_e^{k-4} & \text{on } \Gamma_e^\varepsilon \\
 U^k = 0 & \text{on } \partial\Omega
 \end{array} \right.$$

## PROCEDURE



## EQUIVALENT MODELS

## FIRST APPROACH

(Transmission conditions across the casing)

• **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\ u_i = 0 & \text{on } \partial\Omega_i^\varepsilon \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\ u_e = 0 & \text{on } \partial\Omega_e^\varepsilon \end{cases} \quad (P_2)$$

• **Order 4:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\ [u] = 0 \\ [\sigma \partial_n u] = -\frac{\alpha}{\varepsilon^2} \Delta_{\Gamma^\varepsilon} \{u\} \\ u = 0 & \text{on } \partial\Omega \cup \partial(\Omega_e^\varepsilon \cup \Omega_i^\varepsilon) \end{cases} \quad (P_4)$$

# CONVERGENCE

## Theorem 2

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , if  $f_i \in L^2(\Omega_i^\varepsilon)$  and  $f_e \in L^2(\Omega_e^\varepsilon)$ , then  $\exists! u^{[k]} \in V_{k+1}$  solution of  $(P_{k+1})$ ,  $k = 1, 3$ , and

$$\|u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left( \|f_i\|_{0, \Omega_i^\varepsilon} + \|f_e\|_{0, \Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \varepsilon^{k+1}$$

Where

$$V_2 = \{v : v_i \in H_0^1(\Omega_i^\varepsilon), v_e \in H_0^1(\Omega_e^\varepsilon)\}$$

$$V_4 = \{v : v_i \in H^1(\Omega_i^\varepsilon), v_e \in H^1(\Omega_e^\varepsilon), \nabla_{\Gamma^\varepsilon} \{v\} \in L^2(\Gamma^\varepsilon), v|_{\Gamma_i^\varepsilon} = v|_{\Gamma_e^\varepsilon}, v|_{\Gamma_0} = 0\}$$

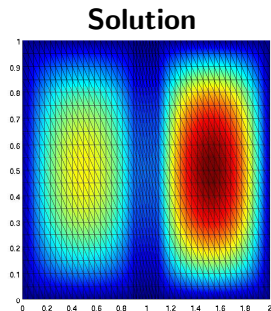
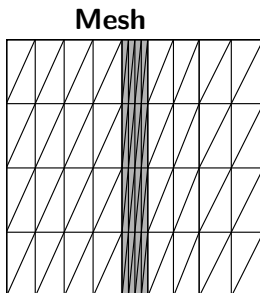
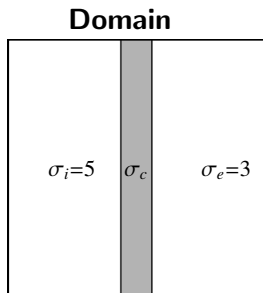
# OUTLINE

- 1 Equivalent Conditions
- 2 Numerical Results**
- 3 Second Approach
- 4 Application
- 5 Perspectives

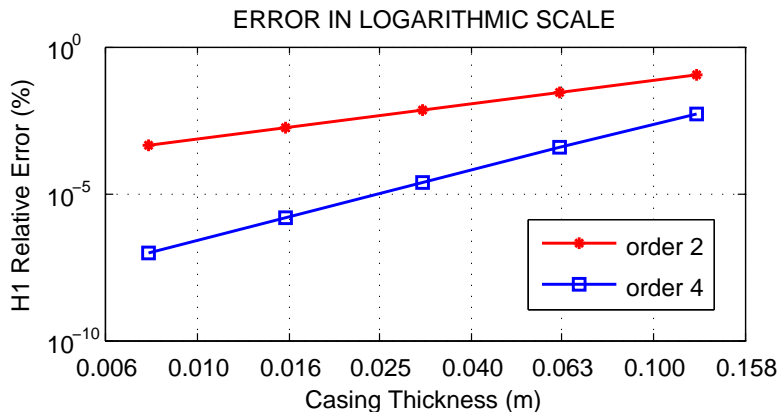


# FEM CODE

- Classic Finite Element Method code
- Straight triangular elements (h refinement)
- Lagrange shape functions of any degree (p refinement)



# CONVERGENCE RATES



Casing Thickness ( $\varepsilon$ )	0.0117	0.0234	0.0469	0.0938	Expected ( $\varepsilon \rightarrow 0$ )
Order 2 Slopes	1.9969	1.9899	1.9638	1.8624	2
Order 4 Slopes	4.0046	4.0039	3.9879	3.8992	4

# OUTLINE

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## EQUIVALENT MODELS

## SECOND APPROACH

(Transmission conditions across the interface  $\Gamma$ )

## • Order 1:

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ u_i = 0 & \text{on } \partial\Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ u_e = 0 & \text{on } \partial\Omega_e \end{cases} \quad (P_1)$$

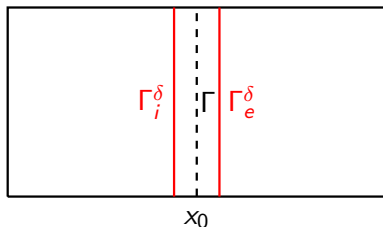
## • Order 2:

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ [u] = -\varepsilon \{\partial_n u\} & \text{on } \Gamma \\ [\partial_n u] = -\frac{4}{\varepsilon} \{u\} & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

# STABILITY

## Problem

Stability problems for the order 2 model of the second approach due to a non coercive term.



## Solution

Use of **Artificial Boundaries** to move the transmission conditions and recover stability. For  $\delta > 0.5$

$$\Gamma_i^\delta = \{(x, y) : x = x_0 - \delta\varepsilon, y \in (0, y_0)\}$$

$$\Gamma_e^\delta = \{(x, y) : x = x_0 + \delta\varepsilon, y \in (0, y_0)\}$$

## EQUIVALENT MODELS

## SECOND APPROACH

(Transmission conditions across the artificial interfaces)

$$\bullet \text{ Order 2: } \left\{ \begin{array}{ll} \sigma_i \Delta u_i = f_i & \text{in } \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in } \Omega_e \\ [u] = -\varepsilon (1 - 2\delta) \{\partial_n u\} & \\ [\partial_n u] = -\frac{4(1 - 2\delta)}{\varepsilon} \{u\} & \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \quad (P_2)$$

Stable if  $\delta > 0.5$

# CONVERGENCE

## Theorem 3

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , if  $f_i \in L^2(\Omega_i^\varepsilon)$  and  $f_e \in L^2(\Omega_e^\varepsilon)$ , then  $\exists! u^{[k]} \in V_{k+1}$  solution of  $(P_{k+1})$ ,  $k = 1, 2$ , and

$$\|u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left( \|f_i\|_{0, \Omega_i^\varepsilon} + \|f_e\|_{0, \Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1, \Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C\varepsilon^{k+1}$$

Where

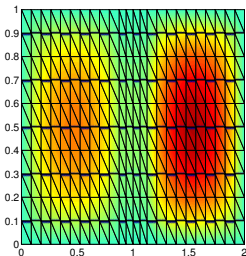
$$V_1 = \{v : v_i \in H_0^1(\Omega_i), v_e \in H_0^1(\Omega_e)\}$$

$$V_2 = \{v : v_i \in H^1(\Omega_i), v_e \in H^1(\Omega_e), v_i|_\Gamma = v_e|_\Gamma, v|_{\Gamma_0} = 0\}$$

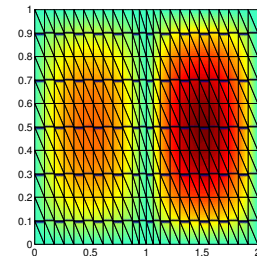
# QUALITATIVE COMPARISON

X-Y VIEW

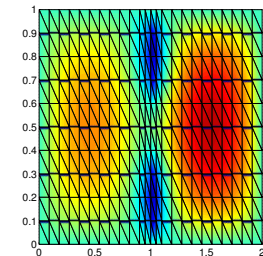
Reference Model



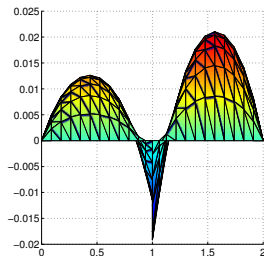
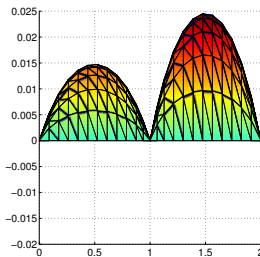
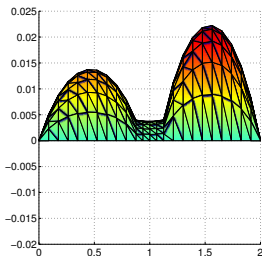
Order 1 Model



Order 2 Model

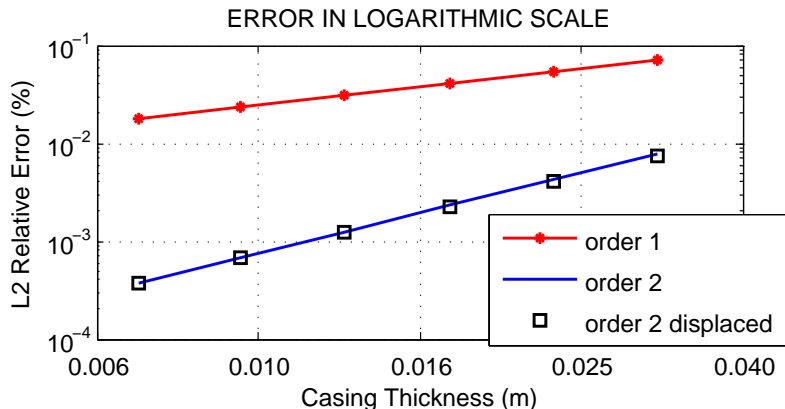


X-Z VIEW



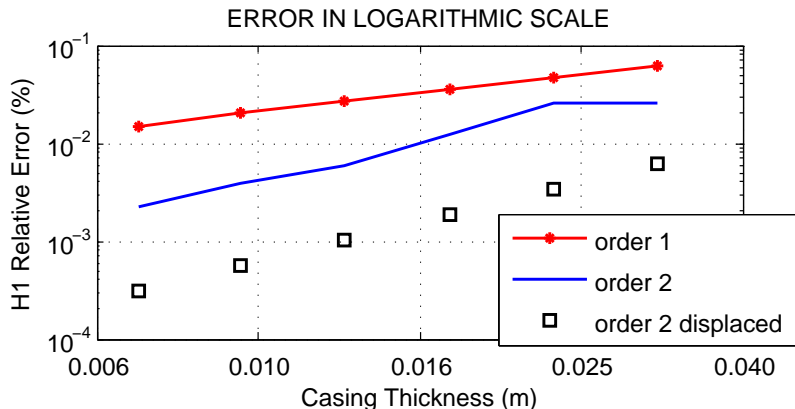


# CONVERGENCE RATES $L^2$



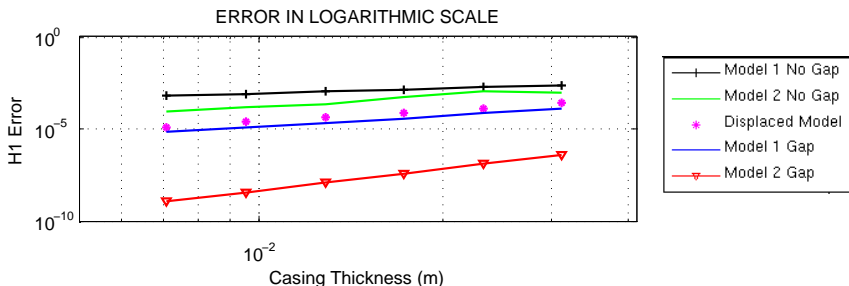
Casing Thickness ( $\varepsilon$ )	0.008	0.011	0.015	0.020	0.027	Expected ( $\varepsilon \rightarrow 0$ )
Order 1 Slopes	0.972	0.962	0.948	0.928	0.899	1
Order 2 Slopes	2.032	2.011	2.062	2.090	1.9636	2
Order 2 displaced	2.005	2.006	2.007	2.008	2.007	2

# CONVERGENCE RATES $H^1$



Casing Thickness ( $\varepsilon$ )	0.008	0.011	0.015	0.020	0.027	Expected ( $\varepsilon \rightarrow 0$ )
Order 1 Slopes	0.982	0.975	0.966	0.953	0.935	1
Order 2 Slopes	1.915	1.302	2.463	2.569	-0.125	2
Order 2 displaced	2.004	2.005	2.006	2.008	2.007	2

## COMPARISON

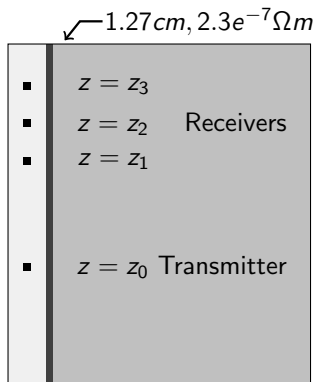


	Order	Stability	$\varepsilon$ -independent Domain
Model 1 No Gap	1	✓	✓
Model 2 No Gap	2	✗	✓
Displaced Model	2	✓	✗
Model 1 Gap	2	✓	✗
Model 2 Gap	4	✓	✗

# OUTLINE

- 1 Equivalent Conditions
- 2 Numerical Results
- 3 Second Approach
- 4 Application**
- 5 Perspectives

## APPLICATION



- **Equation:**  $\operatorname{div}(\sigma \nabla u) = f$

- **Right hand side:**

$$f = \begin{cases} 1 & \text{In the transmitter} \\ 0 & \text{Outside the transmitter} \end{cases}$$

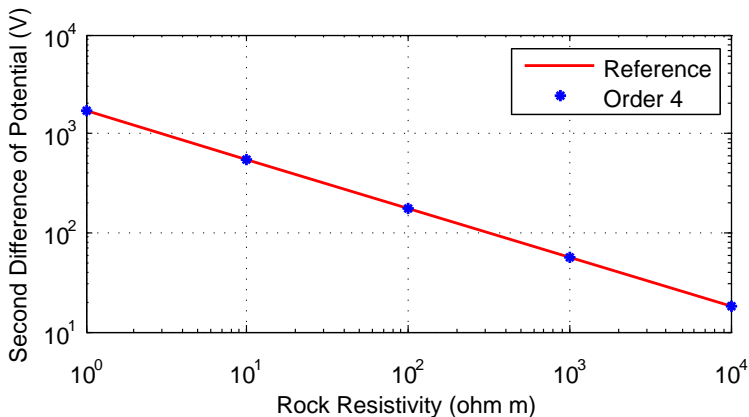
- **Objective:** Measure the second difference of potential on the Receivers

$$U_2 = u(z_1) - 2u(z_2) + u(z_3)$$

- **Expected Result:** (Model of Kaufman) Relation between second difference of potential and rock resistivity of the form

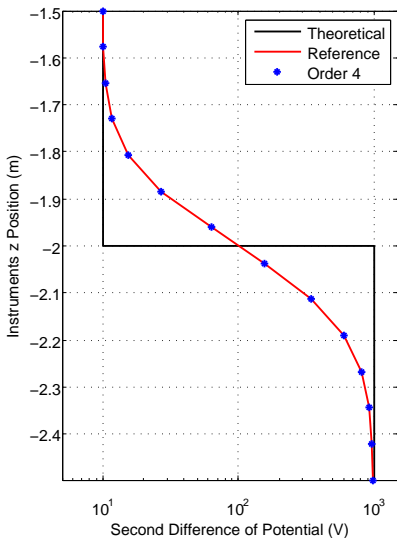
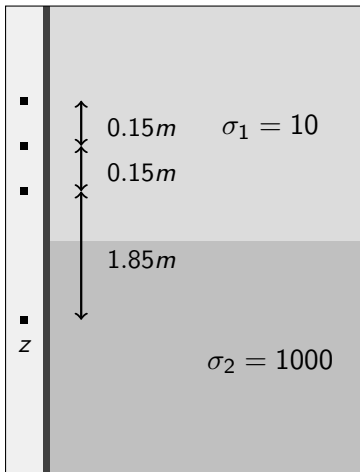
$$U_2 = k \cdot \rho_{\text{rock}}^{-\frac{1}{2}} \quad k \in \mathbb{R}$$

# VARYING ROCK CONDUCTIVITY



Resistivity	[1,10]	[10,10 <sup>2</sup> ]	[10 <sup>2</sup> ,10 <sup>3</sup> ]	[10 <sup>3</sup> ,10 <sup>4</sup> ]	Expected
Reference Model Slopes	-0.4914	-0.4924	-0.4978	-0.4991	-0.5
Order 4 Slopes	-0.4914	-0.4924	-0.4977	-0.4993	-0.5

# TWO ROCK LAYERS



# OUTLINE

- 1 Equivalent Conditions
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- 5 Perspectives**



# Perspectives

- Obtain semianalytical solutions to reduce the computational cost.
- Consider physically more realistic scenarios.
- Develop 3D electromagnetic models.
- Study highly deviated boreholes.

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**THANK YOU FOR YOUR  
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