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# Comparison of solvers performance when solving the 3D Helmholtz elastic wave equations using the Hybridizable Discontinuous Galerkin method

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<sup>1</sup> INRIA Bordeaux-Sud-Ouest, team-project Magique 3D

<sup>2</sup> INRIA Sophia-Antipolis-Méditerranée, team-project Nachos

<sup>3</sup> TOTAL Exploration-Production

# Motivations

## Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

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## Seismic imaging : time-domain or harmonic-domain ?

- ▶ **Time-domain** : **imaging condition complicated** but **quite low computational cost**
- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**

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**Memory usage**



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Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

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First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- ▶  $\mathbf{v}$  : velocity vector
- ▶  $\underline{\underline{\sigma}}$  : stress tensor
- ▶  $\underline{\underline{\varepsilon}}$  : strain tensor

# Motivations

## Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

## First order formulation of Helmholtz wave equations

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- ▶  $\rho$  : mass density
- ▶  $\underline{\underline{C}}$  : elasticity tensor
- ▶  $\mathbf{f}_s$  : source term,  $\mathbf{f}_s \in L^2(\Omega)$



# Approximation methods

## Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method

# Approximation methods

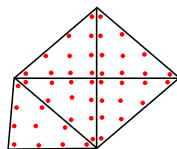
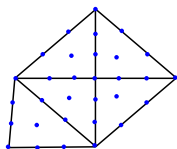
## Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
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- ✓ easily parallelizable method
- ✗ ✗ large number of DOF as compared to classical FEM

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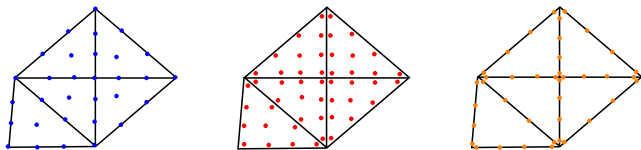
## Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods

# Approximation methods

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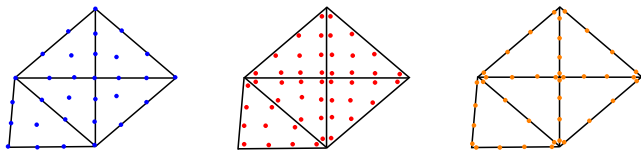
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# Approximation methods

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- ✓ same advantages as DG methods : unstructured tetrahedral meshes,  $hp$ -adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods
- ✗ time-domain increases computational costs



# Hybridizable Discontinuous Galerkin method

-  B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis*, Vol. 47 :1319-1365, 2009.
-  S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL*, 32(3)1112-1138, 2013.
-  N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics*, 230 :7151-7175, 2011
-  N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955-5988, 2012

# Contents

## Hybridizable Discontinuous Galerkin method

- Classical HDG Formulation

- Symmetric HDG formulation

- Algorithm

2D Numerical results : comparison of the two HDG formulations

3D numerical results : focus on the resolution part



# HDG formulation of the equations

## Local HDG formulation

$$\begin{cases} i\omega \rho \mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} = 0 \\ i\omega \underline{\underline{\sigma}} - \underline{\underline{C}} \varepsilon(\mathbf{v}) = 0 \end{cases}$$

# HDG formulation of the equations

## Local HDG formulation

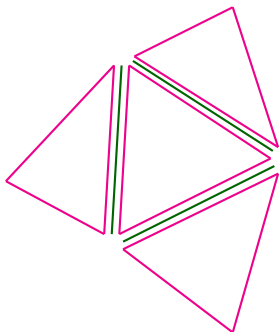
$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$  and  $\widehat{\mathbf{v}}^K$  are numerical traces of  $\underline{\underline{\sigma}}^K$  and  $\mathbf{v}^K$  respectively on  $\partial K$

# HDG formulation of the equations

We define :

$$\hat{v}^{\partial K} = \lambda^F, \quad \forall F \in \mathcal{F}_h,$$

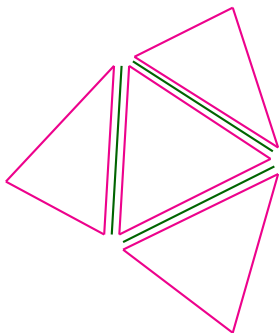


## HDG formulation of the equations

We define :

$$\begin{aligned} \hat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} &= \underline{\sigma}^K \cdot \mathbf{n} - \tau \mathbf{l} (\mathbf{v}^K - \lambda^F), & \text{on } \partial K \end{aligned}$$

where  $\tau$  is the stabilization parameter ( $\tau > 0$ )



# HDG formulation of the equations

## Local HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

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We define :

$$\underline{\underline{W}}^K = \left( \underline{V}_x^K, \underline{V}_y^K, \underline{V}_z^K, \underline{\sigma}_{xx}^K, \underline{\sigma}_{yy}^K, \underline{\sigma}_{zz}^K, \underline{\sigma}_{xy}^K, \underline{\sigma}_{xz}^K, \underline{\sigma}_{yz}^K \right)^T$$

$$\underline{\underline{\Lambda}} = \left( \underline{\Lambda}^{F_1}, \underline{\Lambda}^{F_2}, \dots, \underline{\Lambda}^{F_{n_f}} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

## Discretization of the local HDG formulation

$$\underline{\underline{A}}^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \underline{\underline{C}}^{K,F} \underline{\underline{\Lambda}} = 0$$

# HDG formulation of the equations

## Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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$$\mathbf{A}^K \underline{\underline{W}}^K + \mathbf{C}^K \underline{\underline{\Lambda}} = 0$$

# HDG formulation of the equations

## Transmission condition

In order to determine  $\lambda^F$ , the continuity of the normal component of  $\underline{\hat{\sigma}}^{\partial K}$  is weakly enforced, rendering this numerical trace conservative :

$$\int_F \llbracket \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$$



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Replacing  $(\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n})$  and summing over all faces, the transmission condition becomes :

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\hat{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

## HDG formulation of the equations

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In order to determine  $\lambda^F$ , the continuity of the normal component of  $\underline{\hat{\sigma}}^{\partial K}$  is weakly enforced, rendering this numerical trace conservative :

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### Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[ \mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

# HDG formulation of the equations

## Global HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}_{K\underline{\underline{\xi}}} \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\underline{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0 \end{array} \right.$$

# HDG formulation of the equations

## Global HDG discretization

$$\left\{ \begin{array}{l} \mathbf{A}^K \underline{W}^K + \mathbf{C}^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [\mathbf{B}^K \underline{W}^K + \mathbf{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

# HDG formulation of the equations

## Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}^K)^{-1} \underline{C}^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [\underline{B}^K \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

# HDG formulation of the equations

## Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K] \underline{\Delta} = 0$$

# Symmetric HDG formulation

## Local HDG formulation

$$\begin{cases} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} = 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\underline{\underline{\varepsilon}}(\mathbf{v}) = 0 \end{cases}$$

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## Local HDG formulation

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$\underline{\underline{C}}$  invertible and symmetric tensor, i.e for a symmetric  $\underline{\underline{\sigma}}$  :

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\underline{\underline{\varepsilon}}(\mathbf{u}) \quad \text{and} \quad \underline{\underline{\varepsilon}}(\mathbf{u}) = \underline{\underline{D}}\underline{\underline{\sigma}}$$

with  $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$  and  $\mathbf{u} = i\omega\mathbf{v}$



# Symmetric HDG formulation

## Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$  and  $\widehat{\mathbf{v}}^K$  are numerical traces of  $\underline{\underline{\sigma}}^K$  and  $\mathbf{v}^K$  respectively on  $\partial K$

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$$\underline{\underline{\xi}} = -\underline{\underline{D}}^K \underline{\underline{\xi}}'$$

# Symmetric HDG formulation

## Local HDG formulation

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## Discretization of the local HDG formulation

$$\mathbb{A}_2^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \mathbb{C}_2^{K,F} \underline{\underline{\Lambda}} = 0$$

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$\mathbb{A}_2^K$  symmetric matrix

# Symmetric HDG formulation

Transmission condition

$$\int_F \llbracket \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$$

# Symmetric HDG formulation

## Transmission condition

$$\int_F [[\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\sigma}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

# Symmetric HDG formulation

## Transmission condition

$$\int_F [[\underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\hat{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0$$

## Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0$$

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$$\sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{\underline{w}}^K + \mathbb{L}^K \underline{\underline{\Lambda}}] = 0$$

$$\mathbb{B}^K = (\mathbb{C}_2^K)^T$$

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# Symmetric HDG formulation of the equations

## Global HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ - \int_K i\omega \underline{\underline{D}}^K \underline{\underline{\sigma}}^K : \underline{\underline{\xi}}' - \int_K \mathbf{v}^K \cdot \nabla \cdot \underline{\underline{\xi}}' + \int_{\partial K} \lambda^F \cdot \underline{\underline{\xi}}' \cdot \mathbf{n} = 0 \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underline{\underline{\sigma}}^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0 \end{array} \right.$$

# Symmetric HDG formulation of the equations

## Global HDG discretization

$$\left\{ \begin{array}{l} \mathbb{A}_2^K \underline{W}^K + \mathbb{C}_2^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [(\mathbb{C}_2^K)^T \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$



# Symmetric HDG formulation of the equations

## Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}_2^K)^{-1} \underline{C}_2^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [(\underline{C}_2^K)^T \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

# Symmetric HDG formulation of the equations

## Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-(\mathbf{C}_2^K)^T (\mathbf{A}_2^K)^{-1} \mathbf{C}_2^K + \mathbf{L}^K] \underline{\Lambda} = 0$$

# Symmetric HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-(\mathbf{C}_2^K)^T (\mathbf{A}_2^K)^{-1} \mathbf{C}_2^K + \mathbf{L}^K] \underline{\underline{\Lambda}} = 0$$

$\Rightarrow$  **Symmetric linear system**

# Main steps of the HDG algorithm

---

1. Construction of the global matrix  $\mathbf{M}$

with 
$$\mathbf{M} = \sum_{K \in \mathcal{T}_h} \left[ -\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K \right]$$

**for**  $K = 1$  to  $Nb_{tri}$  **do**

    Computation of matrices  $\mathbf{B}^K$ ,  $(\mathbf{A}^K)^{-1}$ ,  $\mathbf{C}^K$  and  $\mathbf{L}^K$

    Construction of the corresponding section of  $\mathbf{M}$

**end for**

---

# Main steps of the HDG algorithm

- 
1. Construction of the global matrix  $\mathbf{M}$
  2. Construction of the right hand side  $\mathbf{S}$
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## Main steps of the HDG algorithm

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**for**  $K = 1$  to  $Nb_{tri}$  **do**  
 Compute  $\underline{\mathbf{W}}^K = -(\underline{\mathbf{A}}^K)^{-1}\underline{\mathbf{C}}^K\underline{\mathbf{\Lambda}}$   
**end for**

---



# Contents

Hybridizable Discontinuous Galerkin method

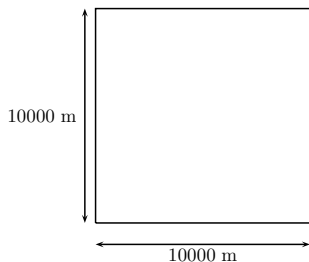
2D Numerical results : comparison of the two HDG formulations

Plane wave in an homogeneous medium

Anisotropic test case

3D numerical results : focus on the resolution part

# Plane wave



Computational domain  $\Omega$   
setting

► Physical parameters :

- $\rho = 2000 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

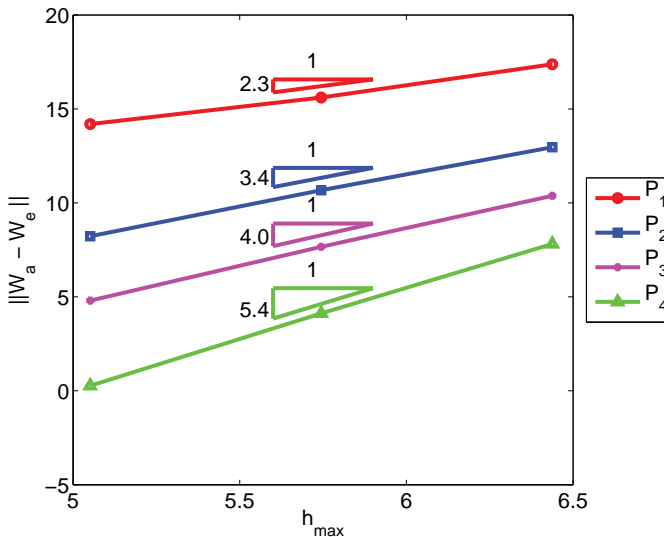
► Plane wave :

$$u = \nabla e^{i(k \cos \theta x + k \sin \theta y)}$$

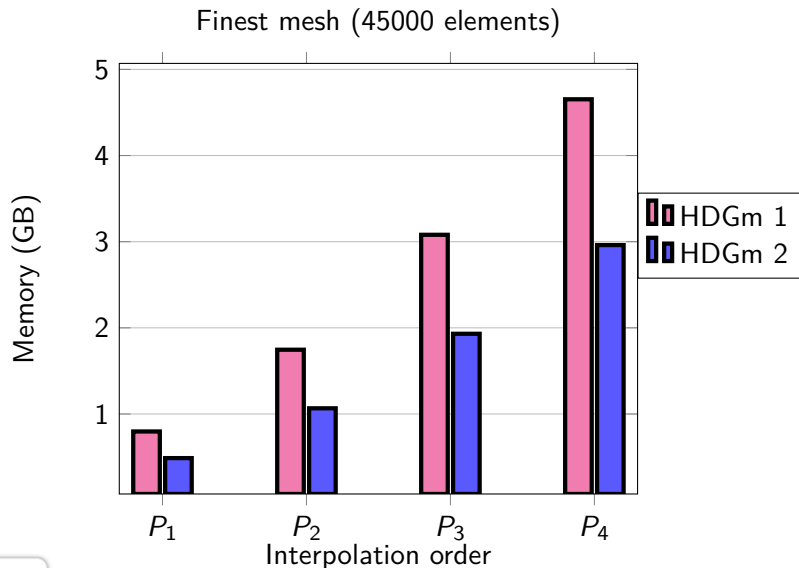
where  $k = \frac{\omega}{v_p}$

- $\theta = 0, v_p = 4000 \text{ m.s}^{-1}, \omega = 4\pi$
- Three meshes :
  - 3000 elements
  - 10000 elements
  - 45000 elements

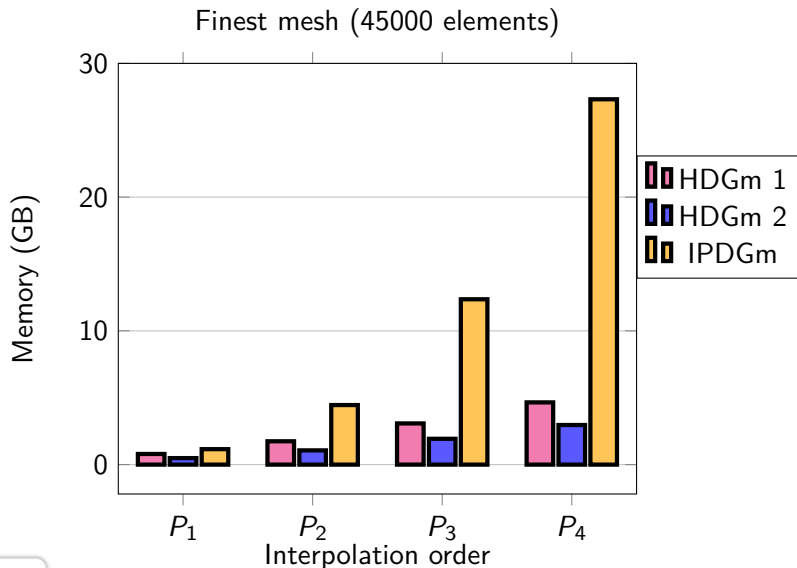
## Plane wave : Convergence order



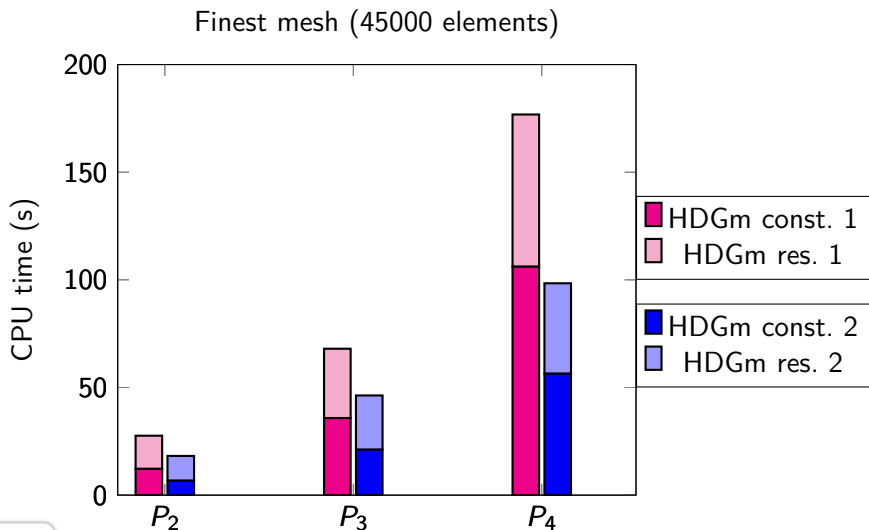
## Plane wave : Memory consumption



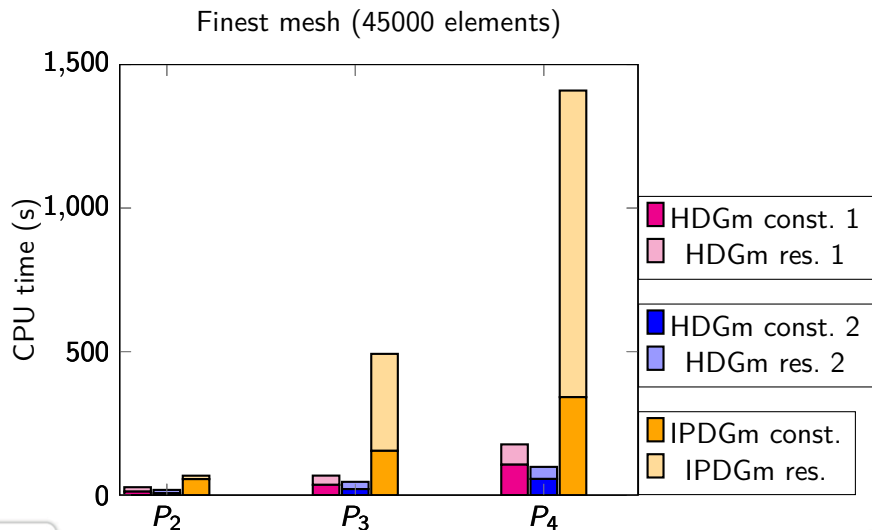
## Plane wave : Memory consumption



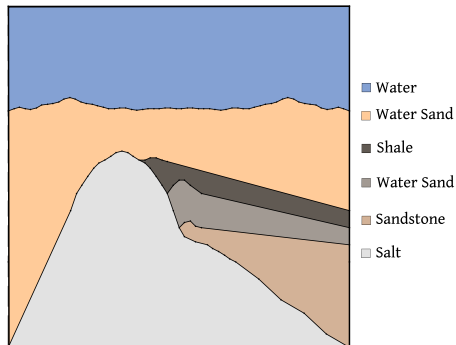
## Plane wave : CPU time



## Plane wave : CPU time



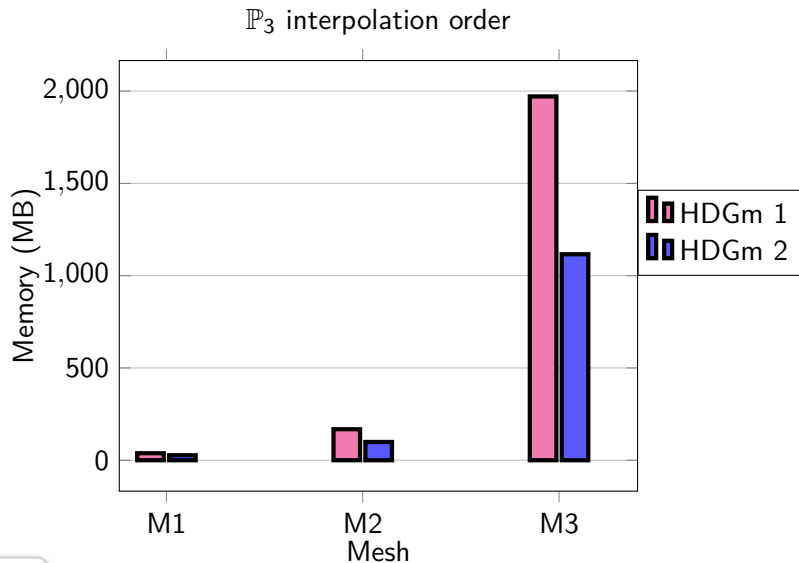
## Anisotropic test case



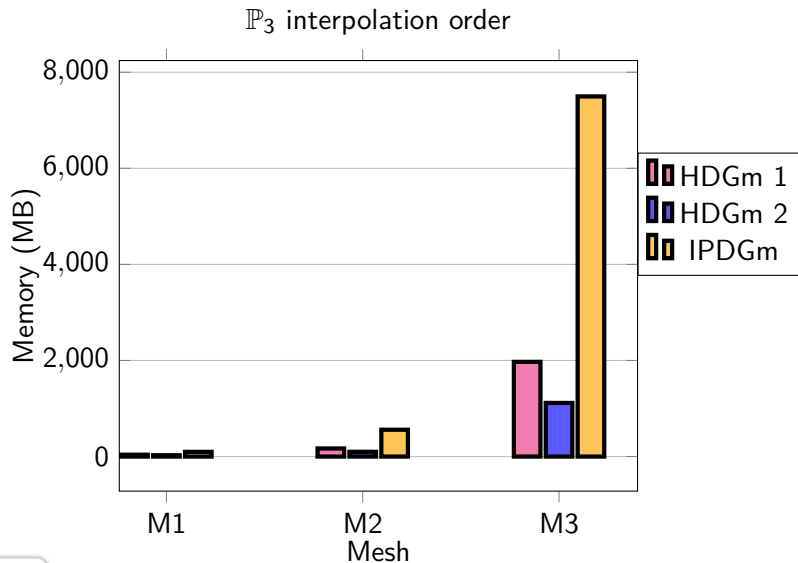
- ▶ Three meshes :
  - ▶ 600 elements
  - ▶ 3000 elements
  - ▶ 28000 elements



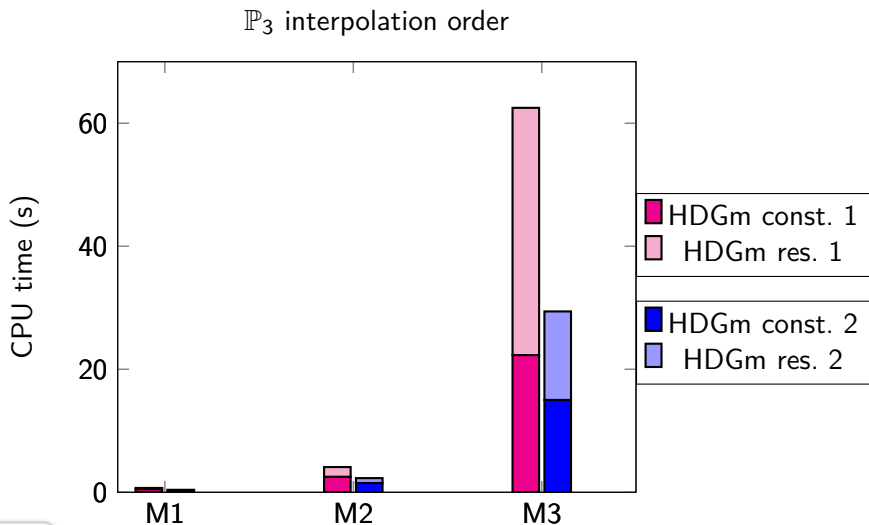
## Anisotropic case : Memory consumption



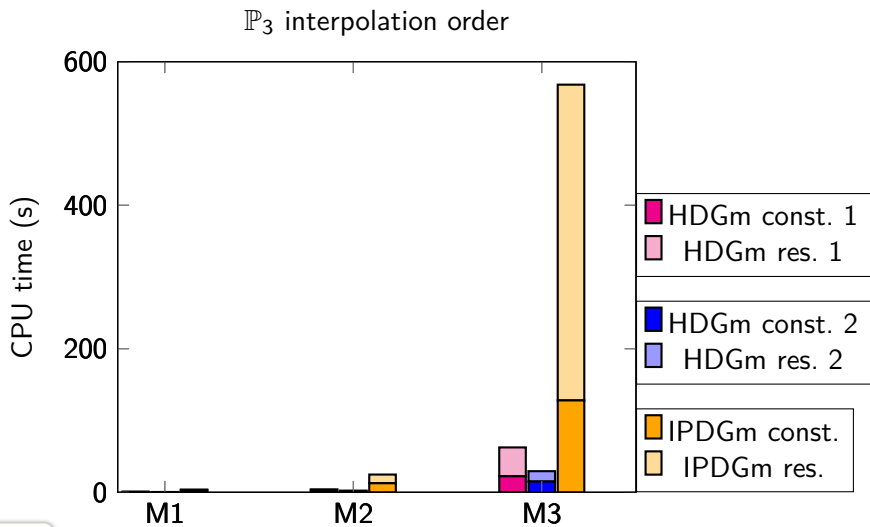
## Anisotropic case : Memory consumption



## Anisotropic case : CPU time (s)



## Anisotropic case : CPU time (s)



## Conclusion

- ▶ HDG method more efficient than classical DG methods for a same accuracy
  - ▶ Memory
  - ▶ Computational time

### 2D specific study of HDG formulation

- ▶ Anisotropic HDG algorithm without any additional computational cost
- ▶ Computational gain without loss of accuracy using  $p$ -adaptivity

# Contents

Hybridizable Discontinuous Galerkin method

2D Numerical results : comparison of the two HDG formulations

3D numerical results : focus on the resolution part

3D plane wave in an homogeneous medium

3D geophysic test-case : Epati test-case

## Main steps of the HDG algorithm

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1. Construction of the global matrix  $\mathbf{M}$
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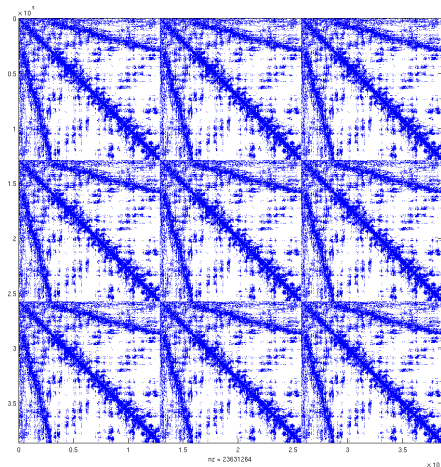
# Main steps of the HDG algorithm

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1. Construction of the global matrix  $\mathbf{M}$
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  4. Computation of the solutions of the initial problem
-



# MaPhys Vs MUMPS

Pattern of the HDG global matrix for  $\mathbb{P}_1$  interpolation and for a 3D mesh composed of 21 000 elements

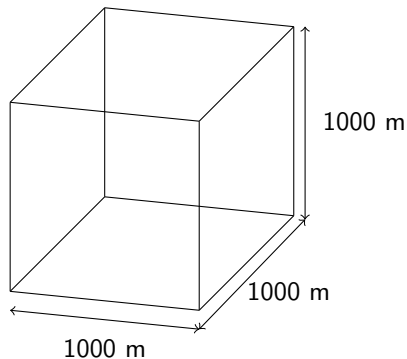


# MaPhys Vs MUMPS

Software packages for solving systems of linear equations  $Ax = b$ , where  $A$  is a sparse matrix

- ▶ MUMPS (MULTifrontal Massively Parallel sparse direct Solver) :
  - ▶ Direct factorization  $A = LU$  or  $A = LDL^T$
  - ▶ Multifrontal approach
- ▶ MaPhys (Massively Parallel Hybrid Solver) :
  - ▶ Direct and iterative methods
  - ▶ non-overlapping algebraic domain decomposition method (Schur complement method)
  - ▶ resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.

## 3D plane wave in an homogeneous medium



Configuration of the computational domain  $\Omega$ .

► Physical parameters :

- $\rho = 1 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

► Plane wave :

$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where  $k_x = \frac{\omega}{v_p} \cos \theta_0 \cos \theta_1$ ,

$k_y = \frac{\omega}{v_p} \sin \theta_0 \cos \theta_1$ , and

$k_z = \frac{\omega}{v_p} \sin \theta_1$

- $\omega = 2\pi f$ ,  $f = 8 \text{ Hz}$
- $\theta_0 = 30^\circ$ ,  $\theta_1 = 0^\circ$
- Mesh composed of 21 000 elements

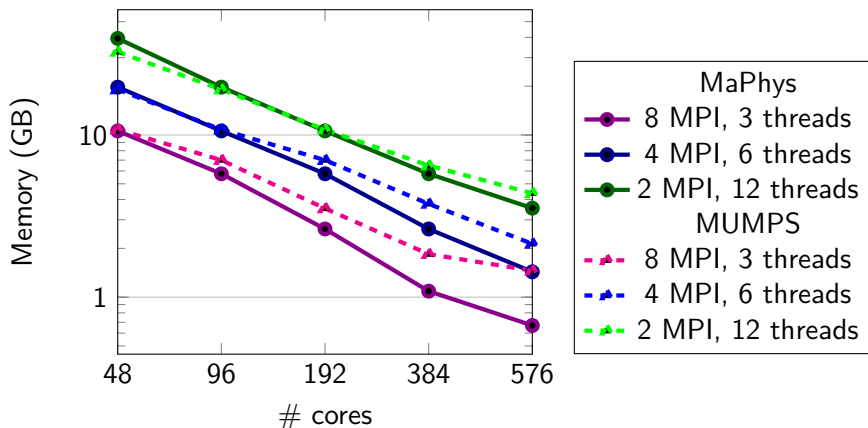
# Cluster configuration

Features of the nodes :

- ▶ 2 Dodeca-core Haswell Intel Xeon E5-2680
- ▶ Frequency : 2,5 GHz
- ▶ RAM : 128 Go
- ▶ Storage : 500 Go
- ▶ Infiniband QDR TrueScale : 40Gb/s
- ▶ Ethernet : 1Gb/s

# 3D Plane wave : Memory consumption

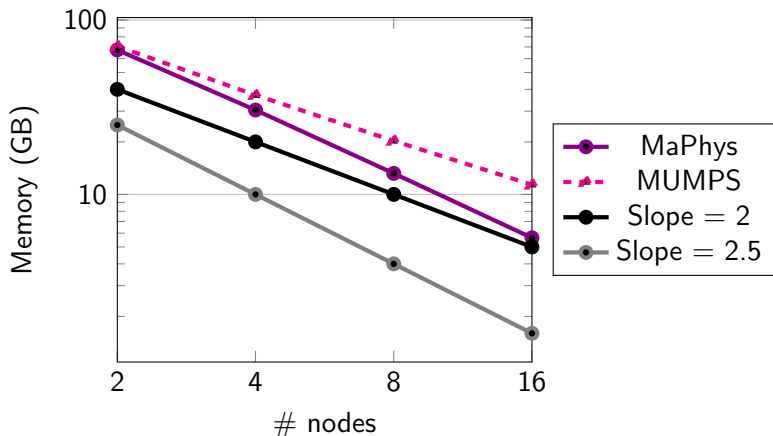
Maximum local memory for HDG- $\mathbb{P}_3$  method



(matrix order = 1 287 360, # nz=298 598 400 )

## 3D Plane wave : Memory consumption

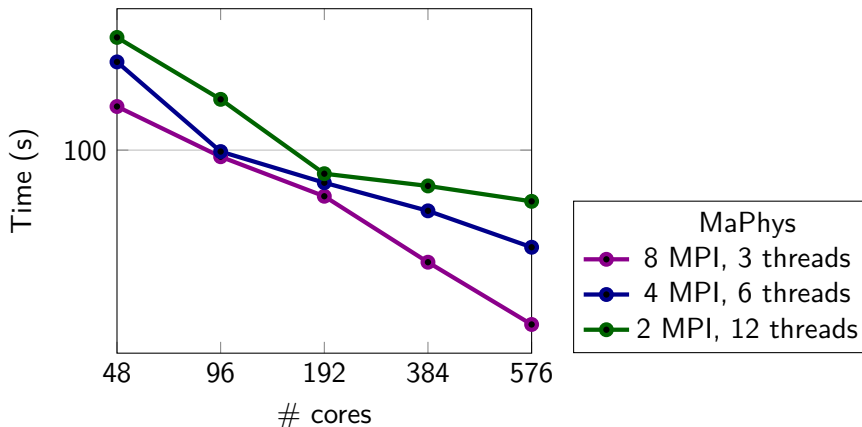
Average memory for one node (8 MPI by node and 3 threads by MPI)



(matrix order = 1 287 360, # nz=298 598 400 )

## 3D Plane wave : Execution time

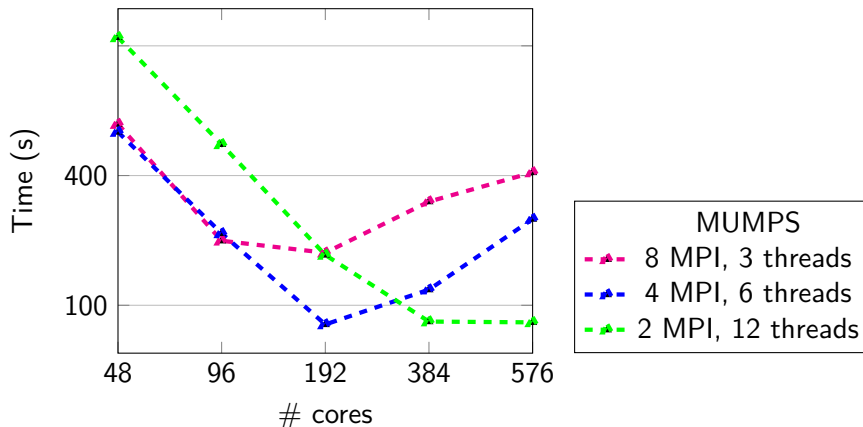
Execution time for the resolution of the HDG- $\mathbb{P}_3$  system



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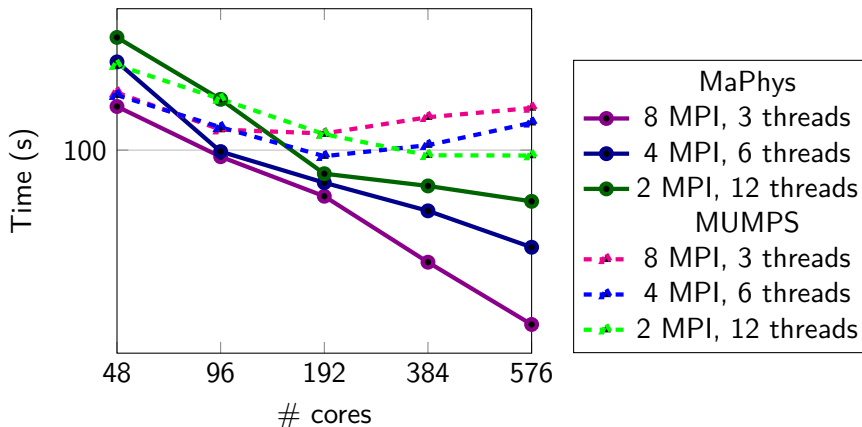


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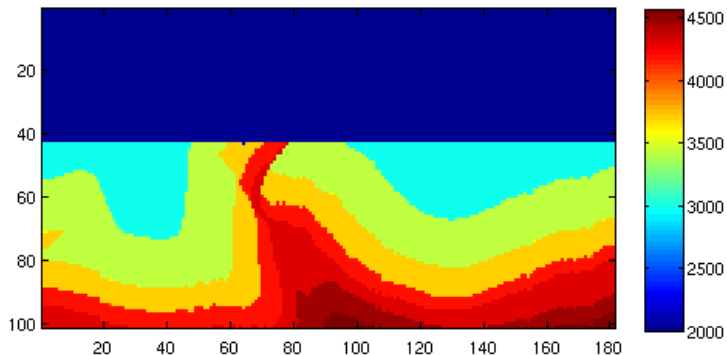
## 3D Plane wave : Execution time

Execution time for the resolution of the HDG- $\mathbb{P}_3$  system



(matrix order = 1 287 360, # nz=298 598 400 )

# Epati test-case

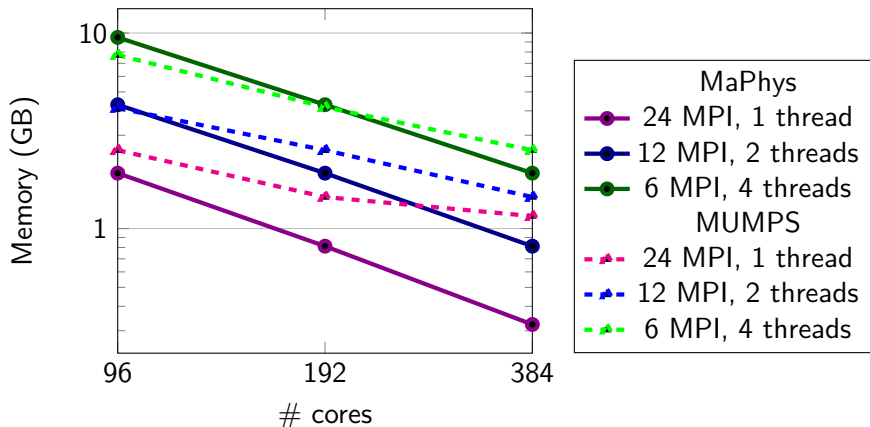


$V_p$ -velocity model ( $\text{m}\cdot\text{s}^{-1}$ ), vertical section at  $y = 700$  m

Mesh composed of 25 000 tetrahedrons

# Epati test-case : Memory consumption

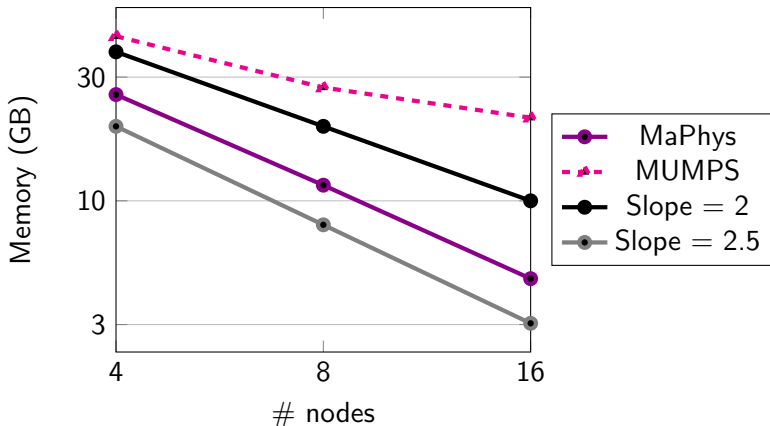
Maximum local memory for HDG- $\mathbb{P}_3$  method



(matrix order = 1 600 740, # nz=365 385 600)

## Epati test-case : Memory consumption

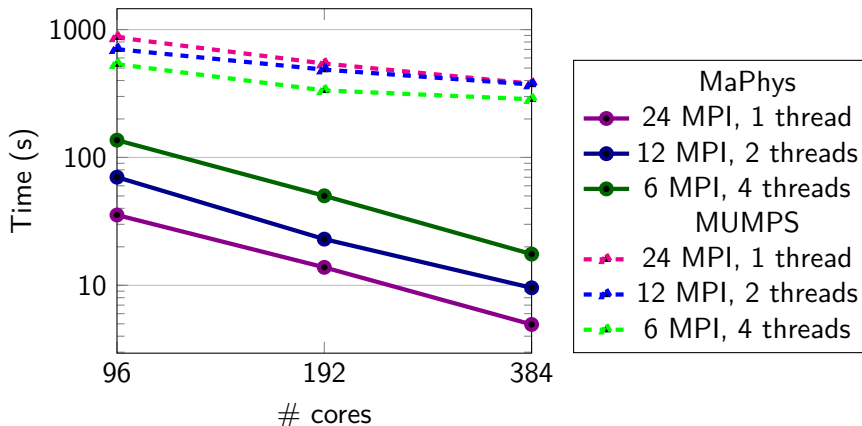
Average memory for one node (24 MPI by node and 1 thread by MPI)



(matrix order = 1 287 360, # nz=365 385 600 )

# Epati test-case : Execution time

Execution time for the resolution of the HDG- $\mathbb{P}_3$  system



(matrix order = 1 287 360, # nz=365 385 600 )

# Conclusion-Perspectives

- ▶ more detailed analysis of the comparison between MUMPS and MaPhys
  - ▶ comparison for the symmetric HDG formulation
- ▶ comparison to PaStiX solver
- ▶ extension to elasto-acoustic case
- ▶ study of the stabilization parameter  $\tau$  for the 3D case
- ▶ call for projects PRACE to test bigger test-cases

Thank you !

The logo for Inria, featuring the word "inria" in a stylized, cursive font with a color gradient from red to orange. Above the "ria" part, the words "informatiques" and "mathématiques" are written in a smaller, sans-serif font, separated by a small star-like symbol.

*inria*  
informatiques mathématiques