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# The quasiinvariants of the symmetric group

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**Abstract.** For  $m$  a non-negative integer and  $G$  a Coxeter group, we denote by  $\mathbf{QI}_m(G)$  the ring of  $m$ -quasiinvariants of  $G$ , as defined by Chalykh, Feigin, and Veselov. These form a nested series of rings, with  $\mathbf{QI}_0(G)$  the whole polynomial ring, and the limit  $\mathbf{QI}_\infty(G)$  the usual ring of invariants. Remarkably, the ring  $\mathbf{QI}_m(G)$  is freely generated over the ideal generated by the invariants of  $G$  without constant term, and the quotient is isomorphic to the left regular representation of  $G$ . However, even in the case of the symmetric group, no basis for  $\mathbf{QI}_m(G)$  is known. We provide a new description of  $\mathbf{QI}_m(S_n)$ , and use this to give a basis for the isotypic component of  $\mathbf{QI}_m(S_n)$  indexed by the shape  $[n - 1, 1]$ .

**Résumé.** Pour  $m$  un entier positif ou nul et  $G$  un groupe de Coxeter, nous notons  $QI_m(G)$  l'anneau des quasiinvariants définis par Chalykh, Feigin et Veselov. On obtient ainsi une série d'anneaux emboîtés,  $QI_0(G)$  étant l'anneau des polynômes, et la limite  $QI_\infty(G)$  l'anneau des invariants usuels. Il est remarquable que l'anneau  $QI_m(G)$  est librement généré sur l'idéal engendré par les invariants de  $G$  sans terme constant, et le quotient est isomorphe à la représentation régulière à gauche de  $G$ . Cependant, même dans le cas du groupe symétrique, aucune base de  $QI_m(G)$  n'est connue. Nous donnons une nouvelle description de  $QI_m(G)$  et l'utilisons pour obtenir une base du composant isotypique de  $QI_m(S_n)$  indexée par la partition  $(n - 1, 1)$ .

**Keywords:** symmetric group, invariants, quasiinvariants

## 1 Introduction

A permutation  $\sigma \in S_n$  acts on a polynomial in  $\mathbf{R} = \mathbb{Q}[x_1, \dots, x_n]$  by permutation of indices:

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \quad (1)$$

The  $S_n$ -invariant polynomials are known as symmetric functions, and denoted by  $\Lambda_n$ . It is well known (e.g., [Sta99]) that the elements of  $\Lambda_n$  without constant term are generated by the elementary symmetric functions  $\{e_1, \dots, e_n\}$  where

$$e_j = \sum_{i_1 < i_2 < \dots < i_j} x_{i_1} \dots x_{i_j}. \quad (2)$$

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The ring of coinvariants of  $S_n$  is the quotient

$$\mathbf{R}/\langle e_1, \dots, e_n \rangle. \quad (3)$$

As an  $S_n$ -module, the ring of coinvariants is known to be isomorphic to the left regular representation. It is also known that  $\mathbf{R}$  is freely generated over this ideal which implies that if we choose a basis of minimal degree elements  $\mathcal{B} = \{b_1, \dots, b_{n!}\}$  for the ring of coinvariants, any element  $P \in \mathbf{R}$  has a unique expansion

$$P = \sum_{i=1}^{n!} b_i g_i \quad (4)$$

with the  $g_i \in \Lambda_n$ . More information is given by the Hilbert series for the isotypic component of  $\mathbf{R}$  corresponding to  $\lambda$ , namely

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda q^{\text{charge}(T)}}{(1-q)(1-q^2) \dots (1-q^n)}. \quad (5)$$

Known bases for the ring of coinvariants with combinatorial descriptions include the Artin monomials and the Schubert polynomials.

In [CV90] and [FV02], Chalykh, Feigin and Veselov introduced a generalization of invariance known as “ $m$ -quasiinvariance”. For the symmetric group, the  $m$ -quasiinvariants are the polynomials  $P \in \mathbb{Q}[x_1, \dots, x_n]$  which have the divisibility property

$$(x_i - x_j)^{2m+1} \mid \left(1 - (i, j)\right) P \quad (6)$$

for every transposition  $(i, j)$ . We set

$$\mathbf{QI}_m = \{m\text{-quasiinvariants of } S_n\}. \quad (7)$$

A straightforward calculation (which we do not do here) shows that the  $m$ -quasiinvariants of  $S_n$  form a ring and an  $S_n$  module, and that we have the following containments:

$$\mathbf{R} = \mathbf{QI}_0 \supset \mathbf{QI}_1 \supset \dots \supset \mathbf{QI}_m \supset \dots \supset \Lambda_n. \quad (8)$$

For all  $m$ , the ring  $\mathbf{QI}_m$  was conjectured in [FV02] to be freely generated over the ideal generated by symmetric functions without constant term. The corresponding quotient  $\mathbf{QI}_m/\langle e_1, \dots, e_n \rangle$  was conjectured to be isomorphic as an  $S_n$  module to the left regular representation. These facts were proved in [EG02]. Further information was given in [FV03], where they showed the Hilbert series of the isotypic component of  $\mathbf{QI}_m$  indexed by  $\lambda$  is given by

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda q^{m \binom{n}{2} - \text{content}(\lambda(T)) + \text{charge}(T)}}{(1-q)(1-q^2) \dots (1-q^n)}. \quad (9)$$

Here  $f_\lambda$  is the number of standard tableaux of shape  $\lambda$  and *content* and *charge* are two statistics on tableaux—we will not need the precise definitions, though we note that *content* only depends on the

shape of  $T$  and thus is actually a statistic on partitions. We should note that the definitions, theorems and conjectures cited above were all phrased in terms of general Coxeter groups; it is only for simplicity that we have restricted our attention to  $S_n$ .

In light of the combinatorially interesting bases for the coinvariants in the classical (*i.e.*,  $m = 0$ ) case, the authors have looked for a basis for larger  $m$ . In [FV02], and later in [BM05], a basis was given for the case  $n = 3$ . (The work [FV02] specifically described the quasiinvariants for dihedral groups; in particular for  $D_3 \cong S_3$ .) Further, in [FV03], Felder and Veselov provide integral expressions,  $\phi^{(j)}(x)$  for  $2 \leq j \leq n$ , for the lowest degree non-symmetric  $m$ -quasiinvariants, *i.e.* those of degree  $mn + 1$ . In the present work, we give a complete basis of the isotypic component indexed by the partition  $[n - 1, 1]$ . This is accomplished by means of a new characterization of  $\mathbf{QI}_m$ :

**Theorem 1** *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} (\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R})$$

where  $ST(n)$  is the set of standard tableaux of size  $n$ ,  $\gamma_T$  is a projection operator due to Young (defined in full detail in the next section) and  $V_T$  is the polynomial given by the product over the columns of  $T$  of the associated ‘‘Vandermonde determinants’’ (this is also defined in detail below). This characterization is proved using completely elementary methods. We then give a basis for the  $[n - 1, 1]$  isotypic component, using the previous characterization to show that our basis does, in fact, consist of quasiinvariants. Precisely, for  $T$  a standard Young tableau of shape  $[n - 1, 1]$  with  $j$  the entry in the second row, we set

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt. \tag{10}$$

With this definition, we have

**Theorem 2** *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \dots, Q_T^{n-2,m}\} \tag{11}$$

*is a basis for  $\gamma_T (\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle)$ .*

The fact  $\mathbf{QI}_m$  is freely generated over  $\langle e_1, \dots, e_n \rangle$  then gives us that the set

$$\{Q_T^{k,m} e_1^{p_1} e_2^{p_2} \dots e_n^{p_n}\} \quad 0 \leq k \leq n - 2, \quad p_i \in \mathbb{N}$$

is a basis for  $\mathbf{QI}_m$ .

## 2 Acknowledgements

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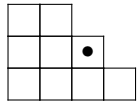
### 3 Definitions and notation

Throughout this work, we will write elements of the symmetric group  $S_n$  using cycle notation. We will perform computations in the group algebra of  $S_n$ , and as such it will be useful to have shorthand notation for many commonly occurring elements. For a given subgroup  $A$  of  $S_n$ , we set

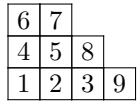
$$[A] = \sum_{\sigma \in A} \sigma \quad \text{and}$$

$$[A]' = \sum_{\sigma \in A} \text{sgn}(\sigma)\sigma.$$

The Young diagram of a partition  $\lambda$  consists of a collection of  $\lambda_i$  left-justified boxes in the  $i^{\text{th}}$  row of the positive integer lattice. These boxes are indexed by ordered pairs  $(i, j)$ , where  $i$  is the row index and  $j$  is the column index. For example, in the following Young diagram of  $[4, 3, 2]$ , the cell  $(2, 3)$  is marked:



A *tableau* of shape  $\lambda \vdash n$  is a function from the cells of the Young diagram of  $\lambda$  to the set  $\{1, \dots, n\}$ . We write the  $T(i, j)$  for the value of  $T$  at the cell  $(i, j)$ . For example, if  $T$  is the following tableau,  $T(2, 3) = 8$ :



We call a tableau *standard* if it is injective and the entries increase across the rows and up the columns. For example, the tableau above is standard. We denote the set of standard tableaux of shape  $\lambda$  by  $ST(\lambda)$  and the set of all standard tableaux with  $n$  boxes by  $ST(n)$ .

Given a tableau  $T$  we let  $C_i$  be the set of elements in the  $i^{\text{th}}$  column and we define  $R_i$  similarly for the rows. We also set

$$C(T) = \bigcup_i \{(a, b) : a, b \in C_i\} \tag{12}$$

$$N(T) = \prod_i [C_i]' \tag{13}$$

$$P(T) = \prod_i [R_i] \tag{14}$$

$$\gamma_T = \frac{f_\lambda N(T)P(T)}{n!} \tag{15}$$

$$\lambda(T) = \text{the shape of tableau } T. \tag{16}$$

Finally, we define the following useful polynomial associated with a tableau  $T$ :

$$V_T = \prod_{(i,j) \in C(T)} (x_i - x_j). \tag{17}$$

## 4 Useful Facts From Representation Theory

The following is a fundamental fact about the representation theory of the symmetric group.

**Proposition 1** For  $W$  a finite dimensional  $S_n$ -module,

$$W \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda} \quad (18)$$

where the  $V_\lambda$  are the irreducible representations of  $S_n$  and the  $m_\lambda$  are non-negative integers.

The vector space and  $S_n$ -module  $V_\lambda^{\oplus m_\lambda}$  is known as the *isotypic component* of  $V$  indexed by  $\lambda$ . Now,  $\mathbf{QI}_m$  is infinite dimensional, but it is the direct sum of homogeneous components, each of which are finite dimensional. So we have that each homogeneous component of  $\mathbf{QI}_m$  decomposes into the direct sum of irreducibles. The direct sum of all copies of  $V_\lambda$  occurring in this decomposition is still itself an  $S_n$ -module, and is still referred to as the isotypic component indexed by  $\lambda$ . However, we will find the following decomposition of  $V$  more useful.

**Proposition 2** On any  $S_n$  module  $W$ , the group algebra elements  $\{\gamma_T\}_{T \in ST(n)}$  act as projection operators. In symbols, we have the conditions

1.  $\gamma_T^2 = \gamma_T$
2.  $W = \bigoplus_{T \in ST(n)} \gamma_T W$ .

Note that in this decomposition, unlike the previous one, the direct summands are not themselves  $S_n$ -modules. We do have the following proposition, however, nicely relating the previous two.

**Proposition 3** For any  $S_n$  module  $W$ ,

$$\bigoplus_{T \in ST(\lambda)} \gamma_T W \quad (19)$$

is the isotypic component of  $W$  indexed by  $\lambda$ .

In the case of the quasiinvariants, we have the following

**Proposition 4** The  $\mathbb{Q}$ -vector space of  $m$ -quasiinvariants has the following direct sum decomposition:

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \gamma_T \mathbf{QI}_m.$$

Our goal will be to use the decomposition  $\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle = \bigoplus_T \gamma_T (\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle)$  to find a partial basis for this space.

## 5 A New Characterization of $S_n$ -Quasiinvariants

In this section we prove the following theorem:

**Theorem 1** The vector space of quasiinvariants has the following direct sum decomposition:

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} (\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}).$$

We will prove this by showing

$$\gamma_T \mathbf{QI}_m = \gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}. \quad (20)$$

Combining (20) with Proposition 4 will prove the theorem. We note first that one containment of (20) is easy:

$$\gamma_T \mathbf{QI}_m \subseteq \gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}. \quad (21)$$

By definition, any element of  $\gamma_T \mathbf{QI}_m$  is also in  $\gamma_T \mathbf{R}$ . Thus, we must only show that for  $Q \in \mathbf{QI}_m$  we have

$$\gamma_T Q \in V_T^{2m+1} \mathbf{R}. \quad (22)$$

Let  $P = \gamma_T Q = N(T)Q'$ .  $P$  must be anti-symmetric with respect to all transpositions in  $C(T)$  since it is in the image of  $N(T)$ . Thus, for any  $(a, b) \in C(T)$ ,  $\left(1 - (a, b)\right)P = 2P$ . Hence  $(x_a - x_b)^{2m+1}$  divides  $2P$  (and also  $P$ ) for all  $(a, b) \in C(T)$ . This establishes equation (21).

It remains to show that for all standard tableaux  $T$  and all  $m \geq 0$ , we have the following containment of vector spaces:

$$\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R} \subseteq \gamma_T \mathbf{QI}_m. \quad (23)$$

Since  $\gamma_T$  is an idempotent, it suffices to show that for any polynomial  $P$  in the ideal  $V_T^{2m+1} \mathbf{R}$ ,  $\gamma_T P = P$  implies that  $P$  is  $m$ -quasiinvariant. With this in mind, we let  $P$  be such that  $V_T^{2m+1} | P$  and  $\gamma_T P = P$ . We wish to show that  $\left(1 - (a, b)\right)P$  is divisible by  $(x_a - x_b)^{2m+1}$  for all transpositions  $(a, b)$ . We first consider the case where  $a$  and  $b$  are in the same column of  $T$ . In this case we have

$$(a, b)N(T) = -N(T)$$

and so

$$(a, b)P = (a, b)\gamma_T P = -\gamma_T P = -P.$$

Thus

$$\left(1 - (a, b)\right)P = 2P \in V_T^{2m+1} \mathbf{R}$$

which is divisible by the required factor of  $(x_a - x_b)^{2m+1}$ .

The remaining case, showing that  $\left(1 - (a, b)\right)P$  is divisible by  $(x_a - x_b)^{2m+1}$  when  $a$  and  $b$  are in different columns, is the only difficult part of the proof. Suppose without loss that  $a$  is in column  $i$ , and is to the left of  $b$ , which is in column  $j$ . We define  $\alpha_{i,b} \in \mathbb{Q}[S_n]$  to be the sum of all transpositions  $(c, b)$  where  $c$  is an element of column  $i$ . The key property of this element is the following:

**Lemma 1** *The operator  $\alpha_{i,b}$  preserves  $\gamma_T$ . In symbols,*

$$\alpha_{i,b}\gamma_T = \gamma_T.$$

The proof of Lemma 1, though completely elementary, is lengthy and is omitted here.

With this fact in hand, it is not difficult to complete the proof. By Lemma 1,  $P$  is preserved by  $\alpha_{i,b}$ :

$$\alpha_{i,b}P = \alpha_{i,b}\gamma_TP = \gamma_TP = P. \tag{24}$$

This fact and the definition of  $\alpha_{i,b}$  gives

$$\left(1 - (a, b)\right)P = P - (a, b)P \tag{25}$$

$$= \alpha_{i,b}P - (a, b)P \tag{26}$$

$$= \sum_{\substack{c \in C_i \\ c \neq a}} (c, b)P. \tag{27}$$

Since  $P \in V_T^{2m+1}\mathbf{R}$ , for any  $c \in C_i$  with  $c \neq a$  we can rewrite  $P$  as

$$P = (x_c - x_a)^{2m+1}(\text{other factors}). \tag{28}$$

Thus

$$(c, b)P = (x_b - x_a)^{2m+1}(\text{other factors}) \tag{29}$$

and we have

$$(x_b - x_a)^{2m+1} \text{ divides } (c, b)P \text{ for every } c \in C_i \text{ with } c \neq a. \tag{30}$$

Hence  $(x_b - x_a)^{2m+1}$  divides the right-hand side of equation 27, which completes the proof.

## 6 A Basis For The Isotypic Component $\lambda(T) = [n - 1, 1]$

In this section, we refer to the quotient  $\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle$  by the symbol  $\mathbf{QI}_m^*$ . Our object here is to describe a basis for  $\gamma_T\mathbf{QI}_m^*$  when  $T$  has a hook shape of the form  $[n - 1, 1]$ . Until otherwise specified, let  $\lambda$  be the partition  $[n - 1, 1]$  and let  $T$  be one of the  $(n - 1)$  standard tableaux of shape  $\lambda$ . In fact  $T$  is uniquely defined by the lone entry of the second row. Suppose it's  $j \in \{2, 3, \dots, n\}$ . We define

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt. \tag{31}$$

**Theorem 2** *The polynomials  $\{Q_T^{k,m}\}_{k=0}^{n-2}$  are a set of representatives for a basis of  $\gamma_T\mathbf{QI}_m^*$ .*

By equation (9) we know this consists of the right number of elements, and they are of the right degree. Thus we are faced with two tasks: showing the elements  $Q_T^{k,m}$  are, in fact,  $m$ -quasiinvariant, and showing that they are linearly independent in  $\mathbf{QI}_m^*$ . To show the  $m$ -quasiinvariance of  $Q_T^{k,m}$  we must, by Theorem 1, show these polynomials are preserved by  $\gamma_T$  and divisible by  $V_T^{2m+1}$ .



**Lemma 2** *The polynomials  $Q_T^{k,m}$  are preserved by the operator  $\gamma_T$ . That is,*

$$\gamma_T Q_T^{k,m} = Q_T^{k,m}. \quad (32)$$

**Proof:** We will prove this inductively, by showing that the  $Q_T^{k,m}$  satisfy a recursion that respects the action of  $\gamma_T$ . In what follows,  $e_i$  will denote the  $i$ th elementary symmetric function in the variables  $x_1, \dots, x_n$ , with the convention that  $e_0 = 1$ . We first state for reference a classical symmetric function identity:

$$\prod_{i=1}^n (t - x_i) = \sum_{i=0}^n (-1)^i e_i t^{n-i}. \quad (33)$$

We now show that for  $m > 1$ , we have the recurrence:

$$Q_T^{k,m} = \sum_{i=0}^n (-1)^i e_i Q_T^{n-i+k, m-1}. \quad (34)$$

This can be seen by first expanding the definition of  $Q_T^{k,m}$  to get

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left( \prod_{i=1}^n (t - x_i) \right) t^k \prod_{l=1}^n (t - x_l)^{m-1} dt. \quad (35)$$

Substituting (33) into (35) and pulling out the factors not involving  $t$  gives

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left( \sum_{i=0}^n (-1)^i e_i t^{n-i} \right) t^k \prod_{l=1}^n (t - x_l)^{m-1} dt \quad (36)$$

$$= \sum_{i=0}^n (-1)^i e_i \int_{x_1}^{x_j} t^{n-i+k} \prod_{l=1}^n (t - x_l)^{m-1} dt \quad (37)$$

$$= \sum_{i=0}^n (-1)^i e_i Q_T^{n-i+k, m-1}. \quad (38)$$

We now proceed with a proof of Lemma 2 by induction on  $m$ . From the definition of  $Q_T^{k,m}$  we have

$$Q_T^{k,0} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^0 dt \quad (39)$$

$$= \frac{1}{k+1} (x_j^{k+1} - x_1^{k+1}). \quad (40)$$

That  $x_j^{k+1} - x_1^{k+1}$  is preserved by  $\gamma_T$  is a straightforward calculation which we omit here.

We can now complete the proof by using equation (34), the inductive hypothesis, and the fact that symmetric functions commute with group algebra elements:

$$\gamma_T Q_T^{k,m} = \sum_{i=0}^n (-1)^i \gamma_T (e_i Q_T^{n-i+k,m-1}) \quad (41)$$

$$= \sum_{i=0}^n (-1)^i e_i (Q_T^{n-i+k,m-1}) \quad (42)$$

$$= Q_T^{k,m}. \quad (43)$$

□

In order to complete our task of showing that  $Q_T^{k,m} \in \mathbf{QI}_m$ , we must show that  $(x_j - x_1)^{2m+1}$  divides  $Q_T^{k,m}$ . We do so by proving the following stronger statement:

**Lemma 3** For all  $k$ ,

$$\lim_{x_1 \rightarrow x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \frac{(-1)^m m!^2}{(2m+1)!} x_j^k \prod_{\substack{i=2 \\ i \neq j}}^n (x_j - x_i)^m.$$

**Proof:** This is a straightforward calculation. Expanding  $Q_T^{k,m}$  according to the definition gives

$$\lim_{x_j \rightarrow x_1} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \lim_{x_j \rightarrow x_1} \frac{\int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt}{(x_j - x_1)^{2m+1}} \quad (44)$$

which is an indeterminate expression of the form  $\frac{0}{0}$ . Repeatedly applying L'Hopital's rule and evaluating the numerator with Leibniz's rule for differentiation under the integral gives that the expression in (44) equals

$$\lim_{x_j \rightarrow x_1} \frac{(-1)^m \cdot m! \int_{x_1}^{x_j} t^k \prod_{\substack{i=1 \\ i \neq j}}^n (t - x_i)^m dt}{(2m+1)(2m)(2m-1) \cdots (m+2)(x_j - x_1)^{m+1}}.$$

One more application of L'Hopital's rule, evaluated this time with the Fundamental Theorem of Calculus, yields

$$\lim_{x_j \rightarrow x_1} \frac{(-1)^m \cdot m! \cdot x_j^k \prod_{\substack{i=1 \\ i \neq j}}^n (x_j - x_i)^m}{(2m+1)(2m)(2m-1) \cdots (m+1)(x_j - x_1)^m}. \quad (45)$$

We now cancel the term  $(x_j - x_1)^m$  from both numerator and denominator and the Lemma is proven. □

The polynomiality of  $\lim_{x_1 \rightarrow x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}}$  immediately gives that  $(x_j - x_1)^{2m+1}$  divides  $Q_T^{k,m}$ . Thus we have established that  $Q_T^{k,m}$  is  $m$ -quasiinvariant.

We need one more lemma to establish Theorem 2.

**Lemma 4** *The set  $\{Q_T^{k,m}\}_{k=0}^{n-2}$  is linearly independent in  $\mathbf{QI}_m^*$ .*

**Proof:** As these polynomials are of different degrees, it suffices to show that they are non-zero in  $\mathbf{QI}_m^*$ . Put another way, we must show that  $Q_T^{k,m}$  is not in the ideal of  $\gamma_T \mathbf{QI}_m$  generated by  $\langle e_1, \dots, e_n \rangle$ . Equivalently, we must show that polynomials of the form

$$P_k = Q_T^{k,m} + A_1 Q_T^{k-1,m} + \dots + A_{k-1} Q_T^{1,m} + A_k Q_T^{0,m} \quad (46)$$

(where the  $A_i$  are symmetric functions of degree  $i$ ) can only equal 0 if  $k \geq n - 1$ . To accomplish this, we use the explicit formulas for  $\lim_{x_j \rightarrow x_1} Q_T^{k,m} / V_T^{2m+1}$  given by Lemma 3 to show the stronger statement

$$\lim_{x_j \rightarrow x_1} P_k / V_T^{2m+1} = 0 \implies k \geq n - 1 \quad (47)$$

regardless of the choice of the symmetric functions. Letting  $\widetilde{A}_i$  denote the limit  $x_j \rightarrow x_1$  applied to the symmetric function  $A_i$ , and assuming without loss of generality that  $j = 2$ , we have

$$\lim_{x_2 \rightarrow x_1} P_k / V_T^{2m+1} = 0 \quad (48)$$

$$\implies \left( \frac{(-1)^m m!^2}{(2m+1)!} \prod_{i=3}^n (x_1 - x_i)^m \right) \left( x_1^k + \widetilde{A}_1 x_1^{k-1} + \dots + \widetilde{A}_{k-1} x_1 + \widetilde{A}_k \right) = 0 \quad (49)$$

$$\implies x_1^k + \widetilde{A}_1 x_1^{k-1} + \dots + \widetilde{A}_{k-1} x_1 + \widetilde{A}_k = 0 \quad (50)$$

$$\implies \lim_{x_2 \rightarrow x_1} (x_1^k + A_1 x_1^{k-1} + \dots + A_k) = 0. \quad (51)$$

We set

$$Q(x_1, \dots, x_n) = x_1^k + A_1 x_1^{k-1} + \dots + A_k \quad (52)$$

and now equation (51) implies that

$$Q(x_1, \dots, x_n) = (x_2 - x_1) \cdot R(x_1, \dots, x_n). \quad (53)$$

However,  $Q$  must be symmetric with respect to all pairs of variables not involving  $x_1$ . Thus, for any  $\sigma \in S_{\{2,3,\dots,n\}}$ ,  $\sigma Q = Q$  and so

$$Q(x_1, \dots, x_n) = \sigma Q(x_1, \dots, x_n) = (x_{\sigma(2)} - x_1) \cdot \sigma R(x_1, \dots, x_n). \quad (54)$$

Hence  $\prod_{i=2}^n (x_i - x_1)$  divides  $Q(x_1, \dots, x_n)$ , and  $k$ , which is the degree of  $Q(x_1, \dots, x_n)$ , must be greater than or equal to  $n - 1$ .  $\square$

The proof of Theorem 2 now follows immediately from Lemmas 2, 3, and 4.

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