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# Enumeration of walks reaching a line

Philippe Nadeau

Laboratoire de Recherche en informatique, Bât. 490, Université Paris Sud, 91405 ORSAY Cedex, France

We enumerate walks in the plane  $\mathbb{R}^2$ , with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results ; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in  $\mathbb{R}^m$ .

**Keywords:** walk, generating function, rational language, singularity analysis

## 1 Introduction

In this work we consider primarily two classes of walks in the plane  $\mathbb{R}^2$ , noted  $\mathcal{W}_{a,\delta}^+$  and  $\mathcal{W}_{a,\delta}^-$ , defined in the following manner :

**Definition 1** Let  $a \in [0, 1[$  and  $\delta \geq 0$  be real numbers. We denote by  $\mathcal{D}_{a,\delta}$  the line of  $\mathbb{R}^2$  with slope  $-a$ , going through the point  $(\delta, 0)$ . An equation of  $\mathcal{D}_{a,\delta}$  is  $y = -a(x - \delta)$ . We denote by  $\mathcal{W}_{a,\delta}^+$  (resp.  $\mathcal{W}_{a,\delta}^-$ ) the set of walks in the plane  $\mathbb{R}^2$  starting at the origin  $O = (0, 0)$  with steps East or North, which end as soon as they reach the open (resp. closed) half plane above  $\mathcal{D}_{a,\delta}$ . The cardinalities of the sets  $\mathcal{W}_{a,\delta}^+$  and  $\mathcal{W}_{a,\delta}^-$  are denoted respectively by  $W_{a,\delta}^+$  and  $W_{a,\delta}^-$ .

These definitions are illustrated on Figure 1.

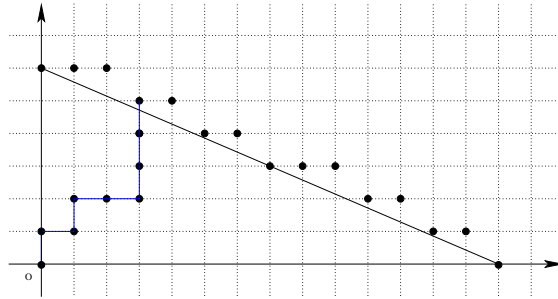
These walks stop as soon as they cross the line  $\mathcal{D}_{a,\delta}$ , those in  $\mathcal{W}_{a,\delta}^+$  having to go strictly beyond the line, whereas those in  $\mathcal{W}_{a,\delta}^-$  stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter  $\delta$ ; that is, we fix the slope  $-a$  of the line  $\mathcal{D}_{a,\delta}$ , and study the numbers  $W_{a,\delta}^+$  and  $W_{a,\delta}^-$  in function of  $\delta$ . Note that, up to a constant factor  $a$ ,  $\delta$  represents the distance of the line  $\mathcal{D}_{a,\delta}$  to the origin.

We can now state our first theorem which gives all asymptotic results for  $W_{a,\delta}^+$  and  $W_{a,\delta}^-$  when  $\delta$  goes to infinity.

**Theorem 1** Let  $a \in ]0, 1]$ , and let  $\lambda$  be the unique positive solution to the equation  $\lambda^{-1} + \lambda^{-1/a} = 1$ .

If  $a = p/q > 0$  is a fixed rational number, where  $p$  and  $q$  are relatively prime positive integers, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{p(1 - \lambda^{-1/p})} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p} \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{p(\lambda^{1/p} - 1)} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p}$$



**Fig. 1:** An example of walk in  $\mathcal{W}_{a,\delta}^+$  with  $a = \frac{3}{7}$  and  $n = 14$ .

hold when  $\delta$  goes to infinity. If  $a$  is irrational, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta$$

hold when  $\delta$  goes to infinity.

As this theorem shows, the behavior of  $W_{a,\delta}^+$  and  $W_{a,\delta}^-$  depends on the rationality of the number  $a$ ; if  $a$  is rational, then we will find the generating function of the numbers  $W_{a,n}^+$  and  $W_{a,n}^-$ . In this case, we will actually introduce another class of walks that includes  $\mathcal{W}_{a,n}^+$  and  $\mathcal{W}_{a,n}^-$  and find a bijection that sends walks to words of a rational language; various enumerative and asymptotic results derive from there. In the case of a general  $a$ , we will proceed differently, and start from an easily obtained functional equation to obtain asymptotic results. Our method is close to Erdős et al. (EHO<sup>+</sup>87), method that is also applicable to a wider class of walks defined in  $\mathbb{R}^n$ .

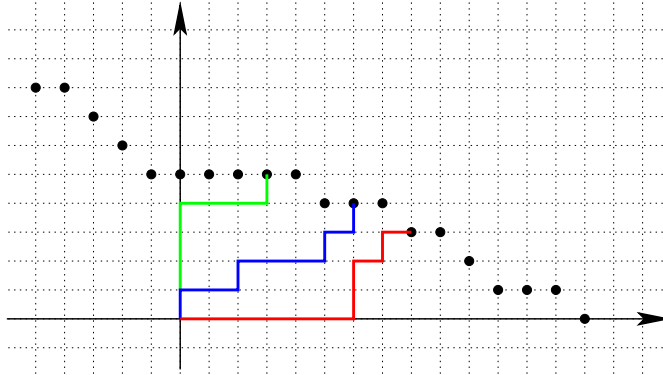
## 2 Walks reaching a set of points

As announced in the introduction, we now introduce a new class of walks that will include our original walks when the slope of  $\mathcal{D}_{a,\delta}$  is rational. The reader is advised to look at Figure 2 while reading the following definition.

**Definition 2** ( $V_{d,n}$  and  $\mathcal{W}_{d,n}$ ) Let  $d = (d_i)_{i \geq 1}$  be an infinite sequence of positive integers, and let  $e = (e_i)_{i \in \mathbb{N}}$  be the corresponding sequence of partial sums, defined by  $e_0 = 0$  and  $e_k = d_1 + d_2 + \dots + d_k$ , for  $k \geq 1$ . We associate to  $d$  a set of points  $V_d$  in the plane, with integer coordinates: the set  $V_d \subset \mathbb{Z} \times \mathbb{N}$  consists in the origin  $O$  together with, for every  $k \geq 1$ , the  $d_k$  points with  $y$ -coordinate equal to  $k$  and  $x$ -coordinate in  $\llbracket -e_k, -e_{k-1} - 1 \rrbracket$ .

For any integer  $n$ ,  $V_{d,n}$  is defined as the translated of  $V_d$  by the vector  $(n, 0)$ . That is,  $V_{d,n} = V_d + (n, 0)$ . The generalized set of walks  $\mathcal{W}_{d,n}$  consists of the walks that start at the origin  $O$ , make steps East or North, and have their last points, and no other one, in  $V_{d,n}$ .

These walks are a generalization of our walks  $\mathcal{W}_{a,n}^+$  and  $\mathcal{W}_{a,n}^-$ . Indeed, let  $d_a^+$  and  $d_a^-$  be the sequences whose  $k$ th terms are given respectively by  $\lceil \frac{k}{a} \rceil - \lceil \frac{k-1}{a} \rceil$  and  $\lfloor \frac{k}{a} \rfloor - \lfloor \frac{k-1}{a} \rfloor$ . Then we have the following proposition :



**Fig. 2:** A set  $V_{d,n}$  with examples of walks in  $W_{d,n}$ . Here  $n = 14$  and  $d = (3, 1, 2, 3, 6, 1, 1, \dots)$

**Proposition 1** For every  $n \in \mathbb{N}$  and  $a \in ]0, 1]$ , we have the equalities

$$\mathcal{W}_{a,n}^+ = \mathcal{W}_{d_a^+,n+1} \text{ and } \mathcal{W}_{a,n}^- = \mathcal{W}_{d_a^-,n}.$$

An interesting case happens when the sequence  $d$  is periodic. It is easy to see that  $d_a^+$  and  $d_a^-$  are periodic exactly when  $a$  is a rational number. If  $d = (d_1, \dots, d_p)$  is a finite sequence, we will note  $V_{d,n} = V_{\bar{d},n}$  and  $\mathcal{W}_{d,n} = \mathcal{W}_{\bar{d},n}$ , where  $\bar{d}$  is the periodic infinite sequence  $(d_1, \dots, d_p, d_1, \dots, d_p, \dots)$ .

From now on  $d$  will stand for a finite sequence  $d = (d_1, \dots, d_p)$  of positive integers. We define  $q = d_1 + \dots + d_p$ , and  $a = p/q$ . To such a sequence we attach the following language on a finite alphabet (recall that a run in a finite word is a maximal factor composed of identical letters)

**Definition 3 (Language  $\mathcal{L}_d$ )** The language  $\mathcal{L}_d$  is the set of words  $w$  on the alphabet  $\Sigma = \{a_0, a_1, \dots, a_{p-1}\}$  that satisfy the following conditions (where we set by convention  $d_0 = d_p$  and  $a_p = a_0$ ):

- C1.  $w$  is the empty word, or its initial letter belongs to  $\{a_0, a_1\}$
- C2. for all  $i$ , a run of  $a_i$  in  $w$  is terminal or is followed by a run of  $a_{i+1}$ ;
- C3. for all  $i$ , the runs of  $a_i$  in  $w$  are of length at least  $d_i$ ; this constraint does not apply to the last run, and, if  $w$  begins with  $a_0$ , it does not apply to the first run either.

We can finally state the theorem announced in the introduction:

**Theorem 2** Let  $n \geq 0$  be an integer. There exists an explicit bijection between walks in  $\mathcal{W}_{d,n}$  and words of  $\mathcal{L}_d$  of length  $n$ .

The language  $\mathcal{L}_d$  is rational, and we give an unambiguous rational expression that represents it. Then the existence of a bijection as stated in Theorem 2 allows us to explicit the generating function  $W_d(x) = \sum_{k=0}^{\infty} W_{d,k} x^k$  of the sequence  $(W_{d,n})_{n \in \mathbb{N}}$ :

**Theorem 3** The generating function  $W_d(x)$  has the following expression:

$$W_d(x) = \frac{N(x)}{(1-x)^p - x^q}, \text{ with } N(x) = (1-x)^{p-2} + \sum_{i=1}^{p-2} x^{e_i+1} (1-x)^{p-2-i} + \sum_{k=e_{p-1}+1}^{e_p-1} x^k.$$

Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers  $(W_{d,n,k})_{n,k}$  which enumerate walks in  $\mathcal{W}_{d,n}$  of length  $k$ . By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in  $\mathcal{W}_{d,n}$  is asymptotically  $C_a \cdot n$  when  $n$  goes to infinity, where  $C_a$  is positive constant depending only on  $a$ .

### 3 Asymptotic results in the general case

Let  $W_a^+$  be the function defined on  $\mathbb{R}$  by  $W_a^+(\delta) = 1$  if  $\delta < 0$ , and by  $W_a^+(\delta) = W_{a,\delta}^+$  if  $\delta \geq 0$ . Then, by decomposing walks according to their first step, one shows that  $W_a^+$  satisfies the following functional equation :

$$\forall \delta \geq 0, \quad W_a^+(\delta) = W_a^+(\delta - 1/a) + W_a^+(\delta - 1). \quad (1)$$

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO<sup>+</sup>87). This consists in interpreting Equation 1 as a “renewal equation”, so that its asymptotic behavior is given by the celebrated *Renewal Limit Theorem* (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as  $W_{a,\delta}^+$  is concerned. It is then extended to  $W_{a,\delta}^-$  by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let  $\bar{a} = (a_1, \dots, a_m)$  be a vector in  $\mathbb{R}^m$ , with all coordinates being positive, and  $\mathcal{H}_\delta$  be the hyperplane of equation  $\mathcal{H}_\delta : a_1x_1 + \dots + a_{m-1}x_{m-1} + a_mx_m - \delta = 0$ . Then define  $W_{\bar{a},\delta}^+$  (resp.  $W_{\bar{a},\delta}^-$ ) to be the numbers of walks in  $\mathbb{R}^m$  from the origin with steps in  $\{e_i\}_{1 \leq i \leq m}$  defined by the fact that their last points, and no other one, are “above  $\mathcal{H}_\delta$ ” (resp. “above or on  $\mathcal{H}_\delta$ ”).

Assume  $1 = a_m \leq a_1 \leq a_2 \leq \dots \leq a_{m-1}$ , and let  $\lambda$  designate the unique positive solution to  $\sum_{i=1}^m \lambda^{-a_i} = 1$ . If all  $a_i$  are rational numbers and we write  $a_i = p_i/q_i$  in reduced form for each  $i$ , we define  $q = \text{lcm}(q_i)$ . Then the proof of the following theorem is proved along the same lines as described above :

**Theorem 4** *Let  $\lambda$  and  $q$  be defined as above. Then we have the following asymptotics when  $\delta$  tends to  $\infty$ :*

(i) *if at least one  $a_i$  is irrational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta,$$

(ii) *and if all  $a_i$  are rational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{q(1 - \lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q} \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{q(\lambda^{1/q} - 1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q}.$$

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.

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