



HAL
open science

Enumeration of walks reaching a line

Philippe Nadeau

► **To cite this version:**

Philippe Nadeau. Enumeration of walks reaching a line. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.401-406, 10.46298/dmtcs.3449 . hal-01184438

HAL Id: hal-01184438

<https://inria.hal.science/hal-01184438v1>

Submitted on 17 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Enumeration of walks reaching a line

Philippe Nadeau

Laboratoire de Recherche en informatique, Bât. 490, Université Paris Sud, 91405 ORSAY Cedex, France

We enumerate walks in the plane \mathbb{R}^2 , with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results ; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in \mathbb{R}^m .

Keywords: walk, generating function, rational language, singularity analysis

1 Introduction

In this work we consider primarily two classes of walks in the plane \mathbb{R}^2 , noted $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$, defined in the following manner :

Definition 1 Let $a \in [0, 1[$ and $\delta \geq 0$ be real numbers. We denote by $\mathcal{D}_{a,\delta}$ the line of \mathbb{R}^2 with slope $-a$, going through the point $(\delta, 0)$. An equation of $\mathcal{D}_{a,\delta}$ is $y = -a(x - \delta)$. We denote by $\mathcal{W}_{a,\delta}^+$ (resp. $\mathcal{W}_{a,\delta}^-$) the set of walks in the plane \mathbb{R}^2 starting at the origin $O = (0, 0)$ with steps East or North, which end as soon as they reach the open (resp. closed) half plane above $\mathcal{D}_{a,\delta}$. The cardinalities of the sets $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$ are denoted respectively by $W_{a,\delta}^+$ and $W_{a,\delta}^-$.

These definitions are illustrated on Figure 1.

These walks stop as soon as they cross the line $\mathcal{D}_{a,\delta}$, those in $\mathcal{W}_{a,\delta}^+$ having to go strictly beyond the line, whereas those in $\mathcal{W}_{a,\delta}^-$ stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter δ ; that is, we fix the slope $-a$ of the line $\mathcal{D}_{a,\delta}$, and study the numbers $W_{a,\delta}^+$ and $W_{a,\delta}^-$ in function of δ . Note that, up to a constant factor a , δ represents the distance of the line $\mathcal{D}_{a,\delta}$ to the origin.

We can now state our first theorem which gives all asymptotic results for $W_{a,\delta}^+$ and $W_{a,\delta}^-$ when δ goes to infinity.

Theorem 1 Let $a \in]0, 1]$, and let λ be the unique positive solution to the equation $\lambda^{-1} + \lambda^{-1/a} = 1$.

If $a = p/q > 0$ is a fixed rational number, where p and q are relatively prime positive integers, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{p(1 - \lambda^{-1/p})} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p} \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{p(\lambda^{1/p} - 1)} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p}$$

Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers $(W_{d,n,k})_{n,k}$ which enumerate walks in $\mathcal{W}_{d,n}$ of length k . By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in $\mathcal{W}_{d,n}$ is asymptotically $C_a \cdot n$ when n goes to infinity, where C_a is positive constant depending only on a .

3 Asymptotic results in the general case

Let W_a^+ be the function defined on \mathbb{R} by $W_a^+(\delta) = 1$ if $\delta < 0$, and by $W_a^+(\delta) = W_{a,\delta}^+$ if $\delta \geq 0$. Then, by decomposing walks according to their first step, one shows that W_a^+ satisfies the following functional equation :

$$\forall \delta \geq 0, \quad W_a^+(\delta) = W_a^+(\delta - 1/a) + W_a^+(\delta - 1). \quad (1)$$

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO⁺87). This consists in interpreting Equation 1 as a “renewal equation”, so that its asymptotic behavior is given by the celebrated *Renewal Limit Theorem* (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as $W_{a,\delta}^+$ is concerned. It is then extended to $W_{\bar{a},\delta}^-$ by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let $\bar{a} = (a_1, \dots, a_m)$ be a vector in \mathbb{R}^m , with all coordinates being positive, and \mathcal{H}_δ be the hyperplane of equation $\mathcal{H}_\delta : a_1x_1 + \dots + a_{m-1}x_{m-1} + a_mx_m - \delta = 0$. Then define $W_{\bar{a},\delta}^+$ (resp. $W_{\bar{a},\delta}^-$) to be the numbers of walks in \mathbb{R}^m from the origin with steps in $\{e_i\}_{1 \leq i \leq m}$ defined by the fact that their last points, and no other one, are “above \mathcal{H}_δ ” (resp. “above or on \mathcal{H}_δ ”).

Assume $1 = a_m \leq a_1 \leq a_2 \leq \dots \leq a_{m-1}$, and let λ designate the unique positive solution to $\sum_{i=1}^m \lambda^{-a_i} = 1$. If all a_i are rational numbers and we write $a_i = p_i/q_i$ in reduced form for each i , we define $q = \text{lcm}(q_i)$. Then the proof of the following theorem is proved along the same lines as described above :

Theorem 4 *Let λ and q be defined as above. Then we have the following asymptotics when δ tends to ∞ :*

(i) *if at least one a_i is irrational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta,$$

(ii) *and if all a_i are rational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{q(1 - \lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q} \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{q(\lambda^{1/q} - 1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q}.$$

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.

Acknowledgements

I would like to thank Yves Verhoeven who is at the origin of this work and helped me on many occasions during my researches.

References

- [CG01] V. Choi and M. J. Golin. Lopsided trees. I. Analyses. *Algorithmica*, 31(3):240–290, 2001. Mathematical analysis of algorithms.
- [EHO⁺87] P. Erdős, A. Hildebrand, A. Odlyzko, P. Pudaite, and B. Reznick. The asymptotic behavior of a family of sequences. *Pacific J. Math.*, 126(2):227–241, 1987.
- [Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.
- [FK74] Michael L. Fredman and Donald E. Knuth. Recurrence relations based on minimization. *J. Math. Anal. Appl.*, 48:534–559, 1974.
- [FS] P. Flajolet and R. Sedgewick. Analytic combinatorics. Book in preparation. Preliminary version available at <http://algo.inria.fr/flajolet/Publications/books.html>.
- [Pip93] Nicholas Pippenger. An elementary approach to some analytic asymptotics. *SIAM J. Math. Anal.*, 24(5):1361–1377, 1993.

