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Reconstruction Thresholds on Regular Trees

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We consider the model of *broadcasting on a tree*, with binary state space, on the infinite rooted tree \mathbb{T}^k in which each node has k children. The root of the tree takes a random value 0 or 1, and then each node passes a value independently to each of its children according to a 2×2 transition matrix \mathbf{P} . We say that *reconstruction is possible* if the values at the d th level of the tree contain non-vanishing information about the value at the root as $d \rightarrow \infty$. Extending a method of Brightwell and Winkler, we obtain new conditions under which reconstruction is impossible, both in the general case and in the special case $p_{11} = 0$. The latter case is closely related to the *hard-core model* from statistical physics; a corollary of our results is that, for the hard-core model on the $(k + 1)$ -regular tree with activity $\lambda = 1$, the unique simple invariant Gibbs measure is extremal in the set of Gibbs measures, for any $k \geq 2$.

Keywords: broadcasting on a tree, reconstruction, hard-core model, Gibbs measure, extremality

1 Introduction

1.1 Broadcasting on a tree

We consider a model of a broadcasting on the rooted tree \mathbb{T}^k , in which every node has k children.

Let $\mathbf{P} = \{p_{ij}, i, j = 0, 1\}$ be a 2×2 stochastic matrix, which we regard as a transition matrix on the set $\{0, 1\}$. Each node $u \in \mathbb{T}^k$ will carry a value $\phi(u) \in \{0, 1\}$, generated as follows. The root takes value 0 with probability $\pi_0 = p_{10}/(p_{01} + p_{10})$ and value 1 with probability $\pi_1 = 1 - \pi_0$. Thereafter the configuration on \mathbb{T}^k is generated recursively; if a node has value $i \in \{0, 1\}$, each of its k children takes the value 0 with probability p_{i0} and the value 1 with probability p_{i1} , all choices being made independently.

We write $\phi = \{\phi(u), u \in \mathbb{T}^k\}$ for a configuration on the whole tree, and denote by μ the probability measure on $\{0, 1\}^{\mathbb{T}^k}$ resulting from this broadcasting construction.

For a node $u \in \mathbb{T}^k$, let $\mathbb{T}^k(u)$ be the subtree consisting of u and all its descendants. By the choice of π_0 , we have a translation invariance property for μ ; namely that $\mu(\phi(u) = 0) = \pi_0$ for every $u \in \mathbb{T}^k$, and so for any $u, v \in \mathbb{T}^k$, the configurations on $\mathbb{T}^k(u)$ and $\mathbb{T}^k(v)$ have the same distribution, under a natural mapping between the subtrees $\mathbb{T}^k(u)$ and $\mathbb{T}^k(v)$.

We are interested in the following question of *reconstruction*: for $d \geq 1$, how much information about the value at node u is given by the values of the d th generation of its descendants?

Questions of this sort arise in several contexts – for example genetics, communication theory and statistical physics – and have been quite widely studied in the last few years; see Mossel [Mos03] for a survey, and [EKPS00, BRZ95, Iof96, KMP01, Mos01, MP03, BW03, JM03] for a variety of approaches to this

sort of model (which can of course be considerably generalised from our particular setting of a binary state space and a regular tree).

The question above can be made precise in several (often equivalent) ways. We use the following formulation.

Let $\mathbb{W}_d(u)$ be the set of descendants of u at distance exactly d from u . For a set $S \subseteq \mathbb{T}^k$, write $\sigma(S)$ for the σ -algebra of events which depend only on the values $\{\phi(u), u \in S\}$.

Define the random variable

$$A(d, u) = \mu(\phi(u) = 0 | \sigma(\mathbb{W}_d(u))),$$

that is, the conditional probability that the value at u is 0, given only the information from the d th generation of its descendants.

From the independence structure given by the broadcasting construction, additional knowledge of any information from nodes beyond the d th generation does not change the conditional distribution of the value of u ; that is,

$$A(d, u) = \mu\left(\phi(u) = 0 \mid \sigma\left(\bigcup_{d'=d}^{\infty} \mathbb{W}_{d'}(u)\right)\right).$$

Of course, if $d_1 > d_2$, then

$$\sigma\left(\bigcup_{d'=d_1}^{\infty} \mathbb{W}_{d'}(u)\right) \subseteq \sigma\left(\bigcup_{d'=d_2}^{\infty} \mathbb{W}_{d'}(u)\right),$$

so by the backwards martingale convergence theorem (see e.g. Section 14.4 of [Wil91]), we have that $A(d, u) \rightarrow A(u)$ a.s. as $d \rightarrow \infty$, where

$$A(u) = \mu(\phi(u) = 0 | \mathcal{T}(u));$$

here $\mathcal{T}(u)$ is the *tail σ -algebra* of descendants of u , defined by

$$\mathcal{T}(u) = \bigcap_{d=1}^{\infty} \sigma\left(\bigcup_{d'=d}^{\infty} \mathbb{W}_{d'}(u)\right).$$

By the translation invariance property above, the random variable $A(u)$ has the same distribution for all $u \in \mathbb{T}^k$.

Definition: We say that *reconstruction is impossible* (for a given \mathbf{P} and k) if the random variable $A(u)$ is almost surely constant, and otherwise that *reconstruction is possible*.

A complete answer to the question of when reconstruction is possible is currently only known for the case where \mathbf{P} is symmetric. Then let $p_{00} = p_{11} = 1 - \varepsilon$; reconstruction is possible if and only if $k(1 - 2\varepsilon)^2 > 1$ (see for example [BRZ95, EKPS00, Iof96]).

In general, however, there are gaps between the best known necessary and sufficient conditions for reconstruction to be possible. In this paper we give new conditions on \mathbf{P} under which we show that reconstruction is impossible.

In Proposition 4.1 of [MP03], Mossel and Peres show that reconstruction is impossible whenever

$$\frac{(p_{00} - p_{10})^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \leq \frac{1}{k}. \quad (1)$$

We improve the bound to give the following condition:

Theorem 1. *Reconstruction is impossible whenever*

$$\left(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}}\right)^2 \leq \frac{1}{k}. \quad (2)$$

A calculation (see Section 4) shows that the LHS of (2) is always less than or equal to that of (1), with equality in the following special cases: (i) \mathbf{P} is symmetric; (ii) $p_{ij} = 0$ for some i, j ; (iii) $p_{00} = p_{10}$, $p_{01} = p_{11}$. Note that for symmetric \mathbf{P} , (2) becomes the condition that $k(1 - 2\varepsilon)^2 \leq 1$, and our proof of Theorem 1 gives another proof that reconstruction is impossible under this condition.

We then focus on the special case where $p_{11} = 0$ (of course, the case $p_{00} = 0$ is analogous). This case is closely related to the *hard-core model* from statistical physics, and has been recently studied by Brightwell and Winkler [BW03] and Rozikov and Suhov [RS03]. Certain specific properties in this case allow a more sophisticated argument which gives a much better condition than is obtained by putting $p_{11} = 0$ in Theorem 1.

1.2 Hard-core model

In this section we state our result for the case $p_{11} = 0$ and explain the correspondence with the hard-core model on a regular tree.

Following [BW03], we parametrise \mathbf{P} by the quantity $w > 0$, setting

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix}, \quad (3)$$

or equivalently by the quantity $\lambda = w(1+w)^k > 0$, whose significance we explain later; note that the correspondence between $\lambda > 0$ and $w > 0$ is one-to-one and monotonic.

Let $\lambda_c = \lambda_c(k)$ be the infimum of the set of λ such that reconstruction is possible. It follows from Proposition 12 of [Mos01] that in fact reconstruction is possible for any $\lambda > \lambda_c$ (so that λ_c is also the supremum of the set of λ such that reconstruction is impossible).

Brightwell and Winkler [BW03] show that, as $k \rightarrow \infty$,

$$\frac{1 + o(1)}{\ln k} \leq \lambda_c(k) \leq (\ln k)^2(1 + o(1)). \quad (4)$$

We improve the lower bound to give the following:

Theorem 2. $\lambda_c(k) > e - 1$ for all k .

(For the equivalent threshold value w_c with $w_c(1 - w_c)^k = \lambda_c$, one can deduce that $w_c(k) > (\ln k - \ln \ln k)/k$ for all k).

We will now describe the correspondence between the broadcasting model and the hard-core model, and explain (without proofs) the significance of Theorem 2 for the hard-core model on the $(k+1)$ -regular tree. For more details on the correspondence between the two models, see also [BW03] and its references.

We denote the $(k+1)$ -regular tree by $\tilde{\mathbb{T}}^k$. We can still regard $\tilde{\mathbb{T}}^k$ as a rooted tree, in which the root node has $k+1$ children and every other node has k children. We can then carry out the broadcasting construction on $\tilde{\mathbb{T}}^k$ in exactly the same way as we did on \mathbb{T}^k ; now the root has $k+1$ rather than k children, but the values at these $k+1$ children are chosen i.i.d. according to the value at the root and the transition matrix \mathbf{P} just as before. We will write $\tilde{\mu}$ for the probability measure on $\{0, 1\}^{\tilde{\mathbb{T}}^k}$ resulting from this construction.

The independence structure of the random walk implies that the measure $\tilde{\mu}$ is *simple*, by which we mean that, for any u , the configurations

$$\{\phi(v), v \in C_1(u)\}, \dots, \{\phi(v), v \in C_{k+1}(u)\}$$

are mutually independent given $\phi(u)$, where the C_i are the connected components of $\tilde{\mathbb{T}}^k \setminus \{u\}$. Although we have defined $\tilde{\mu}$ in an asymmetric way, it's also the case that it is *invariant*, in the sense that it is preserved by any automorphism of $\tilde{\mathbb{T}}^k$. In particular, the choice of the root is not important.

To introduce the hard-core model, we first consider the case of a finite graph with node-set S (and some neighbour relation).

We can identify a configuration $\phi \in \{0, 1\}^S$ with the subset $I_\phi := \{u \in S : \phi(u) = 1\}$ of S .

A set $I \subseteq S$ is called an *independent set* if no two neighbours in the graph are both members of I .

The *hard-core measure* on S with *activity* $\lambda > 0$ is the probability measure ν on $\{0, 1\}^S$ such that

$$\nu(I_\phi \text{ is an independent set}) = 1,$$

and such that for an independent set I_0 , $\nu(I_\phi = I_0)$ is proportional to $\lambda^{|I_0|}$. Thus in fact

$$\nu(\phi = \phi_0) = Z_\lambda^{-1} \lambda^{|I_{\phi_0}|} 1(I_{\phi_0} \text{ is an independent set}),$$

where we have the normalising factor

$$Z_\lambda = \sum_{\phi_0: I_{\phi_0} \text{ is independent}} \lambda^{|I_{\phi_0}|}.$$

When $\lambda = 1$, I_ϕ has the uniform distribution over the set of independent subsets of S .

An equivalent characterisation is that ν is the unique probability measure such that, for any $\phi_0 \in \{0, 1\}^S$ and any $u \in S$,

$$\nu(\phi(u) = 1 \mid \phi(v) = \phi_0(v) \text{ for all } v \neq u) = \frac{\lambda}{1 + \lambda} 1\{I_{\phi_0} \cup \{u\} \text{ is independent}\} \quad (5)$$

The condition (5) makes sense equally when S is infinite, except that (since conditional probabilities are only well defined up to almost sure equality) we should now only demand the condition holds for ν -almost all ϕ_0 . Putting $S = \tilde{\mathbb{T}}^k$, we say that a probability measure ν satisfying (5) (for all $u \in \tilde{\mathbb{T}}^k$ and ν -almost all ϕ_0) is a *Gibbs measure* for the hard-core model on $\tilde{\mathbb{T}}^k$ with activity λ .

It is quite straightforward to show that the measure $\tilde{\mu}$ defined above by the broadcasting construction with \mathbf{P} as in (3) is a Gibbs measure for the hard-core model with activity λ . However, now that the state space is infinite, it's no longer the case that such a measure need be unique. In fact, there is a critical point $\lambda'_c = \lambda'_c(k) = k^k / (k-1)^{(k+1)}$ (identified by Kelly [Kel85]); for $\lambda \leq \lambda'_c$, $\tilde{\mu}$ is the *only* Gibbs measure, whereas for $\lambda > \lambda'_c$, there are others. Nevertheless, for any λ , the measure $\tilde{\mu}$ is the only *simple invariant* Gibbs measure; (this can be deduced, for example, from Theorem 4.1 of [Zac83] – see also Section 5 of that paper for relevant discussion).

The set of Gibbs measures forms a simplex; that is, any mixture of Gibbs measures is also a Gibbs measure, and in particular there is a set of *extremal* Gibbs measures such that every Gibbs measure is expressible in a unique way as a mixture of extremal measures. For $\lambda > \lambda'_c$, we can therefore ask whether the measure $\tilde{\mu}$ is extremal (equivalently, not expressible as a mixture of other Gibbs measures).

It turns out that $\tilde{\mu}$ is extremal at activity λ if and only if reconstruction is impossible for the corresponding broadcasting model on \mathbb{T}^k with transition matrix \mathbf{P} . (This is a consequence of the general fact that a Gibbs measure is extremal iff it is trivial on the tail σ -algebra, and of the independence structure given by the broadcasting constructions of μ and $\tilde{\mu}$). Hence the reconstruction threshold λ_c defined after (3) is also the *extremality* threshold for $\tilde{\mu}$; Theorem 2 therefore shows that whenever $\lambda \leq e - 1$, the unique simple invariant Gibbs measure $\tilde{\mu}$ for the hard-core model with activity λ is extreme, for any k .

In particular, $\tilde{\mu}$ is extreme in the special case $\lambda = 1$ for any k .

1.3 Outline of proof

Our proof of Theorems 1 and 2 is in the same spirit as the proof by Brightwell and Winkler of the lower bound in (4) for the hard-core model [BW03].

We first develop a coupling between the distributions of the random variable $A(u)$ conditioned on two different events, with certain additional properties beyond those used in [BW03]. We then use this coupling to establish a recursion linking the distribution of $A(u)$ to those of $A(u_1), \dots, A(u_k)$, where u_1, \dots, u_k are the children of u . (Of course, we already know from the translation invariance property described in Section 1.1 that all of these distributions are the same). If the recursion relation is contractive in a suitable sense, we obtain that $A(u)$ must be a.s. constant.

In Section 2, we first prove a lemma on conditional probabilities in a more general setting. Specialising to our context, we obtain the existence of a coupling of a pair of random variables A_0, A_1 with the following properties:

- (i) The distribution of A_0 is the distribution of $A(u)$ under μ conditioned on the event $\{\phi(u) = 0\}$;
- (ii) The distribution of A_1 is the distribution of $A(u)$ under μ conditioned on the event $\{\phi(u) = 1\}$;
- (iii) With probability 1, either $A_0 = A_1$ or $A_1 \leq \pi_0 \leq A_0$;
- (iv) If $A_0 = A_1$ with probability 1, then both are equal to π_0 with probability 1, and so also $A(u) = \pi_0$ a.s. under μ .

We develop the recursion relations and complete the proofs in Section 3.

The full properties of the coupling are only needed in the hard-core case, where a particular convexity property holds for the recursion relations. The argument in the general case is not as powerful, and rather than all of property (iii) above, we use only that $A_1 \leq A_0$ with probability 1. Restricting the bound in Theorem 1 to the case $p_{11} = 0$ gives a much weaker bound than that in Theorem 2 (in fact, one obtains only the bound $\lambda_c \geq \lambda'_c$ where λ'_c is the threshold for the uniqueness of the Gibbs measure; this bound is obvious in the context of the hard-core model since if the Gibbs measure is unique it is trivially extreme).

2 Conditioned conditional probabilities

We first consider the setting of a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $B \in \mathcal{F}$ be an event with probability $\pi_0 = 1 - \pi_1$, and suppose $0 < \pi_0 < 1$. Write B^C for the complement of B . Let \mathcal{G} be a sub- σ -algebra of \mathcal{F} .

We consider the random variable $\mathbb{P}(B|\mathcal{G})$ (which is the \mathcal{G} -measurable random variable, unique up to almost sure equality, such that for all $D \in \mathcal{G}$

$$\mathbb{P}(D \cap B) = \int_D \mathbb{P}(B|\mathcal{G})(\omega) d\mathbb{P}(\omega). \tag{6}$$

See for example Chapter 9 of [Wil91] for background on conditional probabilities).

Lemma 3. *Suppose $0 \leq p_0 \leq p_1 \leq 1$, and that $D \in \mathcal{G}$ with*

$$\mathbb{P}(B|\mathcal{G})(\omega) \in [p_0, p_1] \text{ for all } \omega \in D. \quad (7)$$

Then

$$\frac{\pi_1}{\pi_0} \frac{p_0}{1-p_0} \mathbb{P}(D|B^C) \leq \mathbb{P}(D|B) \leq \frac{\pi_1}{\pi_0} \frac{p_1}{1-p_1} \mathbb{P}(D|B^C).$$

Proof. From (6) and (7) we have

$$p_0 \mathbb{P}(D) \leq \mathbb{P}(D \cap B) \leq p_1 \mathbb{P}(D)$$

and

$$(1-p_1) \mathbb{P}(D) \leq \mathbb{P}(D \cap B^C) \leq (1-p_0) \mathbb{P}(D).$$

Combining these we get

$$\frac{p_0}{1-p_0} \mathbb{P}(D \cap B^C) \leq \mathbb{P}(D \cap B) \leq \frac{p_1}{1-p_1} \mathbb{P}(D \cap B^C).$$

Since $\mathbb{P}(D|B) = \mathbb{P}(D \cap B)/\pi_0$ and $\mathbb{P}(D|B^C) = \mathbb{P}(D \cap B^C)/\pi_1$, the result follows. \square

In particular, if J is a subset of the interval $[0, \pi_0)$ then we can set $D = \{\omega : \mathbb{P}(B|\mathcal{G})(\omega) \in J\}$ to obtain

$$\mathbb{P}\left\{\mathbb{P}(B|\mathcal{G}) \in J|B\right\} \leq \mathbb{P}\left\{\mathbb{P}(B|\mathcal{G}) \in J|B^C\right\},$$

while if $J \subseteq (\pi_0, 1]$ then the inequality is reversed. In each case equality holds only if both sides are 0. Hence:

Corollary 4. *There exists a coupling of two random variables Y_0 and Y_1 , such that Y_0 has the distribution of $\mathbb{P}(B|\mathcal{G})$ conditioned on B occurring, such that Y_1 has the distribution of $\mathbb{P}(B|\mathcal{G})$ conditioned on B not occurring, and such that:*

(i) *whenever $Y_0 < \pi_0$, then $Y_1 = Y_0$, and*

(ii) *whenever $Y_1 > \pi_0$, then $Y_1 = Y_0$.*

Therefore either $Y_0 = Y_1$ or $Y_1 \leq \pi_0 \leq Y_0$.

Also the distributions of Y_0 and Y_1 are identical iff $Y_0 = Y_1 = \pi_0$ with probability 1, or equivalently iff $\mathbb{P}(B|\mathcal{G}) = \pi_0$ with probability 1.

Applying this result with $B = \{\phi(u) = 0\}$, with $\mathcal{G} = \mathcal{T}(u)$, with $\mathbb{P} = \mu$ and so with $A(u) = \mathbb{P}(B|\mathcal{G})$, we obtain the coupling of A_0, A_1 with the properties claimed in Section 1.3.

3 Recurrences for likelihood ratios

Let $u \in \mathbb{T}^k$ and let \mathbf{y} be a configuration on the set $\mathbb{W}_d(u)$ (the descendants of u at distance exactly d).

For $S \subset \mathbb{T}^k$, write $\phi \downarrow_S$ for the configuration ϕ restricted to S .

Define the “likelihood functions”

$$\begin{aligned} q^{(0)}(d, \mathbf{y}) &= \mu(\phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y} \mid \phi(u) = 0) \\ q^{(1)}(d, \mathbf{y}) &= \mu(\phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y} \mid \phi(u) = 1). \end{aligned}$$

For $i = 0, 1$, the function $q^{(i)}(d, \mathbf{y})$ gives the probability of observing the configuration \mathbf{y} on the set of descendants of u at distance d , given that the value at u itself is i . (Note that because of the translation invariance property noted in Section 1.1, the choice of u is not important).

Define also the “likelihood ratio” function

$$q(d, \mathbf{y}) = \frac{q^{(0)}(d, \mathbf{y})}{q^{(1)}(d, \mathbf{y})}.$$

Let $d \geq 2$ and let the children of u be u_1, \dots, u_k . A configuration \mathbf{y} on $\mathbb{W}_d(u)$ corresponds to a set of configurations $\mathbf{y}_1, \dots, \mathbf{y}_k$ on $\mathbb{W}_{d-1}(u_1), \dots, \mathbb{W}_{d-1}(u_k)$. We then have

$$\begin{aligned} q^{(0)}(d, \mathbf{y}) &= \prod_{j=1}^k [p_{00}q^{(0)}(d-1, \mathbf{y}_j) + p_{01}q^{(1)}(d-1, \mathbf{y}_j)] \\ q^{(1)}(d, \mathbf{y}) &= \prod_{j=1}^k [p_{10}q^{(0)}(d-1, \mathbf{y}_j) + p_{11}q^{(1)}(d-1, \mathbf{y}_j)] \end{aligned}$$

and so

$$\begin{aligned} q(d, \mathbf{y}) &= \prod_{j=1}^k \left\{ \frac{p_{00}q^{(0)}(d-1, \mathbf{y}_j) + p_{01}q^{(1)}(d-1, \mathbf{y}_j)}{p_{10}q^{(0)}(d-1, \mathbf{y}_j) + p_{11}q^{(1)}(d-1, \mathbf{y}_j)} \right\} \\ &= \left(\frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left\{ 1 + \frac{c_0 - c_1}{q(d-1, \mathbf{y}_j) + c_1} \right\}, \end{aligned} \tag{8}$$

where we define

$$c_0 = \frac{p_{01}}{p_{00}}, \quad c_1 = \frac{p_{11}}{p_{10}}.$$

Define also $a(d, \mathbf{y}) = \mu(\phi(u) = 0 \mid \phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y})$. The function a gives the conditional probability that the value at the node u is 0, given a configuration on the set of descendants of u at distance d . We have

$$\begin{aligned} a(d, \mathbf{y}) &= \frac{\pi_0 q^{(0)}(d, \mathbf{y})}{\pi_0 q^{(0)}(d, \mathbf{y}) + \pi_1 q^{(1)}(d, \mathbf{y})} \\ &= \frac{1}{1 + \frac{\pi_1}{\pi_0 q(d, \mathbf{y})}}. \end{aligned} \tag{9}$$

Returning to the random variable $A(d, u) = \mu(\phi(u) = 0 | \sigma(\mathbb{W}_d(u)))$ defined in Section 1, we have

$$A(d, u) = a(d, \phi \downarrow_{\mathbb{W}_d(u)}),$$

that is, the function $a(d, \cdot)$ applied to the actually observed values of the configuration ϕ on $\mathbb{W}_d(u)$.

Similarly define

$$Q(d, u) = q(d, \phi \downarrow_{\mathbb{W}_d(u)}).$$

From (9), we have

$$A(d, u) = \frac{1}{1 + \frac{\pi_1}{\pi_0 Q(d, u)}}, \quad Q(d, u) = \frac{\pi_1}{\pi_0} \left(\frac{1}{1 - A(d, u)} - 1 \right).$$

Recalling $A(d, u) \rightarrow A(u)$ a.s., we have $Q(d, u) \rightarrow Q(u)$ a.s., where

$$Q(u) = \frac{\pi_1}{\pi_0} \left(\frac{1}{1 - A(u)} - 1 \right).$$

From (8) we get

$$Q(d, u) = \left(\frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left(1 + \frac{c_0 - c_1}{Q(d-1, u_j) + c_1} \right),$$

and, taking $d \rightarrow \infty$,

$$Q(u) = \left(\frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left(1 + \frac{c_0 - c_1}{Q(u_j) + c_1} \right).$$

Now put

$$L(u, d) = \ln Q(u, d)$$

and

$$\begin{aligned} L(u) &= \ln Q(u) \\ &= \ln \left[\frac{\pi_1}{\pi_0} \left(\frac{1}{1 - A(u)} - 1 \right) \right]. \end{aligned}$$

We have $L(u, d) \rightarrow L(u)$ a.s. as $d \rightarrow \infty$, and

$$L(u) = k \ln \left(\frac{p_{00}}{p_{10}} \right) + \sum_{j=1}^k \ln \left(1 + \frac{c_0 - c_1}{\exp(L(u_j)) + c_1} \right). \quad (10)$$

Since $L(u)$ can be written as a strictly increasing function of $A(u)$, with $A(u) = \pi_0$ corresponding to $L(u) = 0$, we can translate the coupling of A_0, A_1 described in Section 1.3 and proved in Section 2 into a coupling of two random variables L_0, L_1 with the following properties:

- (i) The distribution of L_0 is the distribution of $L(u)$ conditioned on the event $\{\phi(u) = 0\}$;
- (ii) The distribution of L_1 is the distribution of $L(u)$ conditioned on the event $\{\phi(u) = 1\}$;
- (iii) With probability 1, either $L_0 = L_1$ or $L_1 \leq 0 \leq L_0$;
- (iv) If $L_0 = L_1$ with probability 1, then both are equal to 0 with probability 1, and then also $A(u) = \pi_0$ with probability 1.

So to conclude that $A(u)$ is a.s. constant, it's enough to show that $\mathbb{E}(L_0 - L_1) = 0$.

Returning to (10), note that $\exp(L(u_j)) \geq 0$, and so the quantity inside the second logarithm is always at least $\min(c_0/c_1, 1) > 0$. Thus the distribution of $L(u)$ has compact support; hence the same is true for L_0 and L_1 , and certainly $\mathbb{E}|L_0| < \infty$, $\mathbb{E}|L_1| < \infty$.

Again let u_1, \dots, u_k be the children of a node u . From the broadcasting construction we get the following information.

Conditional on $\phi(u) = 0$:

the $\phi(u_j)$, $j = 1, \dots, k$ are i.i.d. taking value 0 w.p. p_{00} and value 1 w.p. p_{01} . Then the $L(u_j)$ are i.i.d., and the distribution of each is a mixture of the distribution of L_0 (with weight p_{00}) and the distribution of L_1 (with weight p_{01}).

Conditional on $\phi(u) = 1$:

the $\phi(u_j)$, $j = 1, \dots, k$ are i.i.d. taking value 0 w.p. p_{10} and value 1 w.p. p_{11} . Then the $L(u_j)$ are i.i.d., and the distribution of each is a mixture of the distribution of L_0 (with weight p_{10}) and the distribution of L_1 (with weight p_{11}).

Hence from (10),

$$\mathbb{E}L_0 = -k \ln \left(\frac{p_{00}}{p_{10}} \right) + k \left[p_{00} \mathbb{E} \ln \left(1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{01} \mathbb{E} \ln \left(1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$$

and

$$\mathbb{E}L_1 = -k \ln \left(\frac{p_{00}}{p_{10}} \right) + k \left[p_{10} \mathbb{E} \ln \left(1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{11} \mathbb{E} \ln \left(1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$$

Subtracting and using the fact that $p_{00} - p_{10} = p_{11} - p_{01}$, we obtain

$$\mathbb{E}(L_0 - L_1) = k \mathbb{E} [f(L_0) - f(L_1)], \tag{11}$$

where

$$\begin{aligned} f(x) &= (p_{11} - p_{01}) \ln \left(1 + \frac{c_0 - c_1}{e^x + c_1} \right) \\ &= \frac{c_1 - c_0}{(1 + c_0)(1 + c_1)} \ln \left(1 + \frac{c_0 - c_1}{e^x + c_1} \right). \end{aligned} \tag{12}$$

3.1 General case

If $c_0 = c_1$, then the function f defined at (12) is constant. In that case, (11) shows that $\mathbb{E}(L_0 = L_1) = 0$, and reconstruction is impossible.

So assume that $c_0 \neq c_1$. Then f is strictly increasing, and one obtains that

$$f'(x) = \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{e^x}{(e^x + c_0)(e^x + c_1)}. \quad (13)$$

Putting $y = e^x$ and taking the reciprocal, one can find the value of x maximising (13) by finding the value of $y \geq 0$ minimising $(y + c_0)(y + c_1)y^{-1}$. One obtains $y = e^x = (c_0c_1)^{1/2}$, and so

$$\begin{aligned} \sup_x f'(x) &= \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{(c_0c_1)^{1/2}}{((c_0c_1)^{1/2} + c_0)((c_0c_1)^{1/2} + c_1)} \\ &= \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{1}{(\sqrt{c_1} + \sqrt{c_0})^2} \\ &= \frac{(\sqrt{c_1} - \sqrt{c_0})^2}{(1 + c_0)(1 + c_1)} \\ &= \left(\sqrt{\frac{1}{1 + c_0} \frac{c_1}{1 + c_1}} - \sqrt{\frac{c_0}{1 + c_0} \frac{1}{1 + c_1}} \right)^2 \\ &= \left(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}} \right)^2. \end{aligned} \quad (14)$$

Since we know that $L_1 \leq L_0$ with probability 1, we then have that

$$0 \leq f(L_0) - f(L_1) \leq \sup_x f'(x)(L_0 - L_1)$$

with equality on the RHS iff $L_0 = L_1$, with probability 1. Hence, from (11),

$$\mathbb{E}(L_0 - L_1) \leq k \sup_x f'(x) \mathbb{E}(L_0 - L_1),$$

with equality iff both sides are 0. So to show that $\mathbb{E}(L_0 - L_1) = 0$, and therefore that reconstruction is impossible, it's enough to show that $k \sup_x f'(x) \leq 1$. Using (14), we see that (2) indeed implies that reconstruction is impossible, and the proof of Theorem 1 is done.

3.2 Hard-core case

Recall that

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix};$$

we have also that $\pi_0 = (1 + w)/(1 + 2w)$, $\pi_1 = w/(1 + 2w)$, and $c_0 = w$, $c_1 = 0$. We have also defined $\lambda = w(1 + w)^k$. Equation (12) now becomes

$$f(x) = -\frac{w}{1 + w} \ln(1 + we^{-x}),$$

and now the function f is concave as well as strictly increasing. Hence in particular, if $x_0 < x_1$ and $y_0 < y_1$ with $x_0 \leq y_0$ and $x_1 \leq y_1$, then

$$0 \leq \frac{f(y_1) - f(y_0)}{y_1 - y_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (15)$$

Recurrence (10) now becomes

$$L(u) = -k \ln(1+w) + \sum_{j=1}^k \ln \left(1 + w e^{-L(u_j)} \right),$$

and so in particular $L(u)$ is always greater than or equal to $-k \ln(1+w)$; the same is therefore true of L_0 and L_1 also. Combining this with property (iii) of the coupling described after (10), we have that, with probability 1, either $L_0 = L_1$ or $-k \ln(1+w) \leq L_1 \leq 0 \leq L_0$. Thus, using (15), we obtain that with probability 1

$$\begin{aligned} 0 \leq f(L_0) - f(L_1) &\leq (L_0 - L_1) \frac{f(0) - f(-k \ln(1+w))}{0 - (-k \ln(1+w))} \\ &= (L_0 - L_1) \frac{w}{1+w} \frac{-\ln(1+w) + \ln(1+\lambda)}{k \ln(1+w)}, \\ &= (L_0 - L_1) \frac{w}{k(1+w)} \left(\frac{\ln(1+\lambda)}{\ln(1+w)} - 1 \right), \end{aligned} \quad (16)$$

where we have used

$$f(0) = -w \ln(1+w) / (1+w)$$

and

$$\begin{aligned} f(-k \ln(1+w)) &= -\frac{w}{1+w} \ln \left(1 + w e^{k \ln(1+w)} \right) \\ &= -\frac{w}{1+w} \ln \left(1 + w(1+w)^k \right) \\ &= -\frac{w}{1+w} \ln(1+\lambda). \end{aligned}$$

Combining (11) and (16), we get $0 \leq \mathbb{E}(L_0 - L_1) \leq \rho [\mathbb{E}(L_0 - L_1)]$, where

$$\rho = \frac{w}{1+w} \left(\frac{\ln(1+\lambda)}{\ln(1+w)} - 1 \right).$$

To obtain that $\mathbb{E}(L_0 - L_1) = 0$, and hence that reconstruction is impossible, it's enough that $\rho < 1$. But

$$\begin{aligned} \rho &< \frac{w}{1+w} \frac{\ln(1+\lambda)}{\ln(1+w)} \\ &\leq \ln(1+\lambda) \sup_{w>0} \frac{w}{(1+w) \ln(1+w)} \\ &= \ln(1+\lambda), \end{aligned}$$

(since the quantity within the sup is decreasing in w and tends to 1 as $w \downarrow 0$). So certainly if $\lambda \leq e - 1$, then $\rho < 1$ as desired. Also ρ is continuous as a function of λ (or w), so the threshold value $\lambda_2(k)$ is in fact strictly greater than $e - 1$, and the proof of Theorem 2 is done.

4 A calculation

For completeness, we here include details to show that the LHS of (2) is always less than or equal to that of (1). Define $b_0 = \min(p_{00}, p_{11}) \leq \max(p_{00}, p_{11}) = b_1$. We obtain

$$\begin{aligned} \min\{p_{00} + p_{10}, p_{01} + p_{11}\} &= \min\{b_0 + 1 - b_1, b_1 + 1 - b_0\} \\ &= b_0 + 1 - b_1 \\ &= (1 - b_0)(1 - b_1) + b_0b_1 + 2b_0(1 - b_1) \\ &\leq (1 - b_0)(1 - b_1) + b_0b_1 + 2\sqrt{b_0b_1}\sqrt{(1 - b_1)(1 - b_0)} \end{aligned} \quad (17)$$

$$= \left[\sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)} \right]^2. \quad (18)$$

(The inequality in (17) follows since $b_0 \leq b_1$ and $1 - b_1 \leq 1 - b_0$; equality holds if $b_0 = b_1$ or if one of b_0 and b_1 is 0 or 1).

Then, starting from the LHS of (2) and using (18),

$$\begin{aligned} \left(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}} \right)^2 &= \left[\sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)} \right]^2 \\ &\leq \left[\sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)} \right]^2 \frac{\left[\sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)} \right]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{[b_0b_1 - (1 - b_1)(1 - b_0)]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{(b_0 - (1 - b_1))^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{[\pm(p_{00} - p_{10})]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}}, \end{aligned}$$

which gives the LHS of (1) as required.

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