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# On the convergence of the iterates of “FISTA”.

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## Abstract

We discuss here the convergence of the iterates of the “FISTA” algorithm, which is an accelerated algorithm proposed by Beck and Teboulle for minimizing the sum  $F = f + g$  of two convex, l.s.c. and proper functions, such that one is differentiable with Lipschitz gradient, and the proximity operator of the second is easy to compute. It builds a sequence of iterates  $(x_n)_{n \in \mathbb{N}}$  for which  $F(x_n) - F(x^*) \leq O(1/n^2)$ . However the convergence of these iterates  $(x_n)_{n \in \mathbb{N}}$  is not obvious. We show here that with a small modification, we can ensure the same decay of the energy as well as the (weak) convergence of the iterates to a minimizer.

## Introduction

Let  $\mathcal{H}$  be a Hilbert space and  $f$  and  $g$  two convex, l.s.c functions from  $\mathcal{H}$  to  $\mathbb{R} \cup \{+\infty\}$  such that  $f$  is differentiable with  $L$ -Lipschitz continuous gradient, and  $g$  is “simple”, meaning that its “proximal map”

$$x \mapsto \arg \min_{y \in \mathcal{H}} g(y) + \frac{\|x - y\|^2}{2\tau}$$

can be easily computed. We consider the following minimization problem

$$\min_{x \in \mathcal{H}} F(x) := f(x) + g(x) \tag{1}$$

and we assume that this problem has at least a solution (and possibly an infinite set of solutions).

Among the many algorithms which exist to tackle such problems, the proximal splitting algorithms, which perform alternating descents in  $f$  and in  $g$ , are frequently used, because of their simplicity and relatively small per-iteration complexity. One can mention the Forward-Backward (FB) splitting, the Douglas-Rachford splitting, the ADMM (alternating direction

method of multipliers),<sup>1</sup> which all have been proved to be efficient in many imaging problem such as denoising, inpainting, deconvolution, color transfer and many others.

This work focuses on the so-called “Fast Iterative Soft Thresholding Algorithm” (FISTA) which is an accelerated variant of the Forward-Backward algorithm proposed by Beck and Teboulle [2], built upon ideas of Nesterov [12] and Güler [7].

The FB is a descent algorithm which defines a sequence  $(x_n)_{n \in \mathbb{N}}$  by performing an explicit descent in  $f$  and implicit in  $g$ . It is then shown that there exist  $C > 0$ , such that for all  $n \in \mathbb{N}$   $F(x_n) - F(x^*) \leq \frac{C}{n}$  where  $x^*$  is a minimizer of  $F$ . Moreover the sequence  $(x_n)_{n \in \mathbb{N}}$  weakly converges in  $\mathcal{H}$ . See for instance [14] or [2] for a simple derivation of this rate.

The sequence  $(x_n)_{n \in \mathbb{N}}$  defined by the accelerated variant “FISTA” [2] satisfies, on the other hand,  $F(x_n) - F(x^*) \leq \frac{C'}{n^2}$  for a suitable real number  $C'$ , however no convergence of  $(x_n)_{n \in \mathbb{N}}$  has been proved so far.

The FISTA algorithm is based on a simple over-relaxation step with varying parameter, and several choices of parameters yield roughly the same rate of convergence. This paper provides complementary results on the convergence of  $F(x_n) - F(x^*)$  for some “good” choices of these parameters, for which the weak convergence of the iterates can also be proved.

In the next section, we introduce a few definitions and our main notation. In a second part, the main result on the convergence of FISTA is recalled, and we give new results on the convergence of the values of  $F(x_n)$ , for other over-relaxation sequences. In the third part we show the convergence of the iterates. This part is strongly inspired from a recent paper of Pock and Lorenz [9], inspired by works of Alvarez and Attouch [1] and Moufadi and Oliny [10]. The last part is focused on numerical experiments.

## 1 Notation and definitions

In the following  $x^*$  denotes a solution of (1), even if this solution is not unique the value  $F(x^*)$  is uniquely defined.

A key tool of FISTA is the proximal map. To any proper, convex and l.s.c function  $h$  is associated the proximal map  $\text{Prox}_h$  which is a function from  $\mathcal{H}$  to  $\mathcal{H}$  defined by

$$\text{Prox}_h(x) = \arg \min_{y \in \mathcal{H}} h(y) + \frac{1}{2} \|x - y\|^2.$$

This function is uniquely defined and generalizes the projection on a closed convex set to convex functions.

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<sup>1</sup>See for instance [4, 8, 5, 6, 3].

In the sequel,  $\gamma$  denotes a non negative real number such that  $\gamma \leq \frac{1}{L}$  where  $L$  is the Lipschitz constant of  $\nabla f$  and  $T$  the mapping from  $\mathcal{H}$  to  $\mathcal{H}$  defined by

$$T(x) := \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)),$$

The idea of FB is to apply this mapping from any  $x_0 \in \mathcal{H}$  using Krasnosel'ski Mann iterations to get a weak convergence to a minimizer  $x^*$  of  $F$ . The idea of FISTA is to apply this mapping using a suitable extragradient rule to accelerate the convergence.

FISTA is defined by a sequence  $(t_n)_{n \in \mathbb{N}}$  of real numbers greater than 1 and a point  $x_0 \in \mathcal{H}$ . Let  $(t_n)_{n \in \mathbb{N}^*}$  be a sequence of non negative real numbers and  $x_0 \in \mathcal{H}$ , the sequences  $(x_n)_{n \in \mathbb{N}}$ ,  $(y_n)_{n \in \mathbb{N}}$  and  $(u_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  are defined by  $y_0 = u_0 = x_0$  and for all  $n \geq 1$ ,

$$x_n = T(y_{n-1}) \tag{2}$$

$$y_n = \left(1 - \frac{1}{t_{n+1}}\right) x_n + \frac{1}{t_{n+1}} u_n \tag{3}$$

$$u_n = x_{n-1} + t_n(x_n - x_{n-1}). \tag{4}$$

The point  $y_n$  may also be defined from points  $x_n$  and  $x_{n-1}$  by

$$y_n = x_n + \alpha_n(x_n - x_{n-1}) \text{ with } \alpha_n := \frac{t_n - 1}{t_{n+1}} \tag{5}$$

For suitable choices of  $(t_n)_{n \in \mathbb{N}^*}$  the sequence  $(F(x_n))_{n \in \mathbb{N}}$  converge to  $F(x^*)$ , i.e the sequence  $(w_n)_{n \in \mathbb{N}}$ , defined as follows,

$$w_n := F(x_n) - F(x^*) \tag{6}$$

tends to 0 when  $n$  goes to infinity.

Several proofs use bounds on the local variation of the sequence  $(x_n)_{n \in \mathbb{N}}$ , which we will denote  $(\delta_n)_{n \in \mathbb{N}}$ : variation :

$$\delta_n := \frac{1}{2} \|x_n - x_{n-1}\|_2^2 \tag{7}$$

The sequence  $(v_n)_{n \in \mathbb{N}}$  denoting the distance between  $u_n$  and a fixed minimizer  $x^*$  of  $F$  will also be useful:

$$v_n := \frac{1}{2} \|u_n - x^*\|_2^2. \tag{8}$$

To complete this part devoted to our notation, we define a sequence  $(\rho_n)_{n \in \mathbb{N}}$ , associated to  $(t_n)_{n \in \mathbb{N}^*}$ , whose positivity will ensure the convergence of the FISTA iterations:

$$\rho_n := t_{n-1}^2 - t_n^2 + t_n. \tag{9}$$

## 2 Some results on the FISTA method

The main result of [2] is the following Theorem :

**Theorem 1** ([2, Thm. 4.1]). *For any  $x_0 \in \mathcal{H}$ , if the sequence  $(t_n)_{n \in \mathbb{N}^*}$  satisfies*

$$\forall n \geq 2 \quad t_n^2 - t_n \leq t_{n-1}^2 \quad (10)$$

and  $t_1 = 1$ , if  $\gamma \leq \frac{1}{L}$  then the sequence  $(x_n)_{n \in \mathbb{N}}$  satisfies for all  $n \in \mathbb{N}$

$$w_n \leq \frac{1}{2\gamma t_n^2} \|x_0 - x^*\|_2^2 \quad (11)$$

for any minimizer  $x^*$  of  $F$ .

Condition (10) can also be stated using the sequence  $(\rho_n)_{n \in \mathbb{N}}$ :  $\forall n \geq 2$ ,  $\rho_n \geq 0$ .

The sequence defined by  $t_1 = 1$  and

$$\forall n \in \mathbb{N}^* \quad t_{n+1} = \sqrt{t_n^2 + \frac{1}{4}} + \frac{1}{2} \quad (12)$$

achieves the equality in (10). Also, it turns out that the sequence  $(t_n)_{n \in \mathbb{N}}$  defined by  $t_n = \frac{n+1}{2}$  satisfies condition (10). But more generally, for any  $a \geq 2$  the sequence  $(t_n)_{n \in \mathbb{N}}$  defined by  $t_n = \frac{n+a-1}{a}$  satisfies (10). Indeed,

$$\rho_n = \frac{1}{a^2} ((n+a-2)^2 - (n+a-1)^2 + a(n+a-1)) = \frac{1}{a^2} ((a-2)n + a^2 - 3a + 3) \geq 0. \quad (13)$$

An induction proves that any sequence satisfying (10) (hence an inequality in (12)) and  $t_1 = 1$  satisfies  $t_n \geq n$ . Hence for any sequence defined above, Theorem 1 ensures that

$$\forall n \in \mathbb{N} \quad w_n \leq \frac{C}{n^2} \quad (14)$$

where  $C$  depends on the exact choice of the sequence  $(t_n)_{n \in \mathbb{N}}$ .

A priori, the best constant “ $C$ ” in this bound  $\frac{C}{n^2}$  will be reached if the sequence  $(t_n)_{n \in \mathbb{N}}$  is the one achieving the equality in (10), given by (12), ensuring the highest value of  $t_n$ . This is the choice in [2], and it turns out that it is nearly optimal (since for any  $n$  there exists a problem which has lower bound of the same order, see [11, 13]). We will soon see, however, that not achieving this equality may have some advantages.

This first Theorem can easily be made more precise, as follows:

**Theorem 2.** *If the sequence  $(t_n)_{n \in \mathbb{N}}$  satisfies (10) and  $t_1 = 1$ , if  $\gamma \leq \frac{1}{L}$  then for any  $N \geq 2$ ,*

$$t_{N+1}^2 w_{N+1} + \sum_{n=1}^N \rho_{n+1} w_n \leq \frac{v_0 - v_{N+1}}{\gamma}. \quad (15)$$

Using the several choices of sequence  $(t_n)_{n \in \mathbb{N}}$  described above, Theorem 2 ensures the same decay for  $w_n$  ( $w_n \leq \frac{C}{n^2}$ ) as the previous. But using (13), we readily see that for a “good” choice of the sequence  $(t_n)_{n \in \mathbb{N}^*}$ , one obtains the following corollary:

**Corollary 1.** *Let  $a > 2$  and for  $n \geq 1$ ,  $t_n = \frac{n+a-1}{a}$ . Then the sequence  $(nw_n)_{n \in \mathbb{N}}$  belongs to  $\ell_1(\mathbb{N})$ . In particular,  $\liminf_{n \rightarrow \infty} n^2 \log nw_n = 0$ .*

The classical choice (12) for the sequence  $(t_n)_{n \in \mathbb{N}^*}$  may yield the best rate of convergence for the objective, but other sequences can give better global properties of the sequence  $(nw_n)_{n \in \mathbb{N}}$ . (also notice that the other classical choice corresponding to  $a = 2$  will not ensure this summability.) An important remark, here, is that this result is not in contradiction with the lower bounds of Nemirovski and Yudin (see [11, 13]). Indeed, they show that for any integer  $n_0$  one can build a specific problem for which one will have, after  $n_0$  iterations,  $n_0 w_{n_0} \geq C/n_0$ , however this does not mean that the sequence  $(nw_n)_n$  is not eventually summable.

**Proof of Theorem 2** The proof is similar to the one of Theorem 1 in [2], however for the ease of the reader we will sketch it here. A first (standard) technical descent Lemma is useful:

**Lemma 1.** *Let  $\gamma \in ]0, \frac{1}{L}]$ , where  $L$  is the Lipschitz constant of  $\nabla f$ ,  $\bar{x} \in \mathcal{H}$  and  $\hat{x} = T\bar{x}$ . Then*

$$\forall x \in \mathcal{H} \quad F(\hat{x}) + \frac{\|\hat{x} - x\|^2}{2\gamma} \leq F(x) + \frac{\|x - \bar{x}\|^2}{2\gamma} \quad (16)$$

*Proof.* Many proofs exist of this result, we give an elementary one which is inspired from [16]. By definition of the proximal map,  $\hat{x}$  is the minimizer of the  $\frac{1}{\gamma}$ -strongly convex function

$$z \mapsto g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{1}{2\gamma} \|z - \bar{x}\|^2$$

hence for all  $z \in \mathcal{H}$

$$\begin{aligned} g(\hat{x}) + f(\bar{x}) + \langle \nabla f(\bar{x}), \hat{x} - \bar{x} \rangle + \frac{\|\bar{x} - \hat{x}\|^2}{2\gamma} + \frac{\|z - \hat{x}\|^2}{2\gamma} \\ \leq g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{\|z - \bar{x}\|^2}{2\gamma}. \end{aligned}$$

Since  $\gamma \leq 1/L$ , it follows

$$g(\hat{x}) + f(\hat{x}) + \frac{1}{2\gamma} \|z - \hat{x}\|^2 \leq g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{1}{2\gamma} \|z - \bar{x}\|^2.$$

By convexity,  $f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle \leq f(z)$ . We deduce that (16) holds and the Lemma is proved.  $\square$

*Proof of Theorem 2.* Applying this Lemma to  $\bar{x} = y_n$ ,  $\hat{x} = x_{n+1}$  and  $x = (1 - \frac{1}{t_{n+1}})x_n + \frac{1}{t_{n+1}}x^*$ , we find

$$\begin{aligned} F(x_{n+1}) + \frac{\left\| \frac{1}{t_{n+1}}u_{n+1} - \frac{1}{t_{n+1}}x^* \right\|_2^2}{2\gamma} \\ \leq F\left( \left(1 - \frac{1}{t_{n+1}}\right)x_n + \frac{1}{t_{n+1}}x^* \right) + \frac{\left\| \frac{1}{t_{n+1}}x^* - \frac{1}{t_{n+1}}u_n \right\|_2^2}{2\gamma} \end{aligned}$$

Using the convexity of  $F$  it follows

$$\begin{aligned} F(x_{n+1}) - F(x^*) - \left(1 - \frac{1}{t_{n+1}}\right) (F(x_n) - F(x^*)) \\ \leq \frac{\|u_n - x^*\|_2^2}{2\gamma t_{n+1}^2} - \frac{\|u_{n+1} - x^*\|_2^2}{2\gamma t_{n+1}^2} \end{aligned}$$

Using definitions of  $w_n$  and  $v_n$  this inequality can be stated

$$t_{n+1}^2 w_{n+1} - (t_{n+1}^2 - t_{n+1})w_n \leq \frac{v_n - v_{n+1}}{\gamma} \quad (17)$$

Summing these inequalities from  $n = 1$  to  $n = N$  leads to

$$t_{N+1}^2 w_{N+1} + \sum_{n=1}^N \rho_{n+1} w_n \leq \frac{v_0 - v_{N+1}}{\gamma}. \quad (18)$$

which ends the proof of Theorem 2.  $\square$

We can deduce another useful corollary:

**Corollary 2.** *Let  $a > 2$  and for  $n \geq 1$ ,  $t_n = \frac{n+a-1}{a}$ . Then the sequence  $(n\delta_n)_{n \in \mathbb{N}}$  belongs to  $\ell_1(\mathbb{N})$ , in particular  $\liminf_{n \rightarrow \infty} n^2 \log n \delta_n = 0$ . In addition, there exists  $C > 0$  such that for all  $n \in \mathbb{N}^*$ ,  $\delta_n \leq \frac{C}{n^2}$ .*

This results which is a consequence of Corollary 1 is the key to prove the convergence of the sequence  $(x_n)_{n \in \mathbb{N}}$ .

*Proof.* Applying Lemma 1 to  $\bar{x} = y_n = x_n + \alpha_n(x_n - x_{n-1})$ , and  $x = x_n$  leads to

$$F(x_{n+1}) + \frac{\|x_n - x_{n+1}\|_2^2}{2\gamma} \leq F(x_n) + \frac{\alpha_n^2 \|x_n - x_{n-1}\|_2^2}{2\gamma}$$

which can be written with definitions of  $w_n$  and  $\delta_n$

$$\delta_{n+1} - \alpha_n^2 \delta_n \leq \gamma(w_n - w_{n+1})$$

If  $t_n = \frac{n+a-1}{a}$ ,  $\alpha_n = \frac{t_n-1}{t_{n+1}} = \frac{n-1}{n+a}$ .

Multiplying this inequality by  $(n+a)^2$  and summing from  $n = 1$  to  $n = N$  leads to

$$\sum_{n=1}^N (n+a)^2 (\delta_{n+1} - \alpha_n^2 \delta_n) \leq \gamma \sum_{n=1}^N (n+a)^2 (w_n - w_{n+1}),$$

which gives

$$(N+a)^2 \delta_{N+1} + \sum_{n=2}^N ((n+a-1)^2 - (n+a)^2 \alpha_n^2) \delta_n \leq \gamma \left( (a+1)^2 w_1 - (N+a)^2 w_{N+1} + \sum_{n=2}^N ((n+a)^2 - (n+a-1)^2) w_n \right)$$

that is

$$(N+a)^2 \delta_{N+1} + \sum_{n=2}^N a(2n-2+a) \delta_n \leq \gamma \left( (a+1)^2 w_1 - (N+a)^2 w_{N+1} + \sum_{n=2}^N (2n+2a-1) w_n \right)$$

By Corollary 1 and since we have assumed  $a > 2$ , the right part of the inequality is uniformly bounded independently of  $N$ , which ensures that the sequence  $(n\delta_n)_{n \in \mathbb{N}}$  belongs to  $\ell_1(\mathbb{N})$ . It also follows that  $N^2 \delta_{N+1}$  is globally bounded.  $\square$

### 3 Convergence of the iterates of FISTA

In this section, we show the following Theorem

**Theorem 3.** *Let  $a > 2$  be a positive real number, and for all  $n \in \mathbb{N}$  let  $t_n = \frac{n+a-1}{a}$ . Then the sequence  $(x_n)_{n \in \mathbb{N}}$  given by FISTA weakly converges to a minimizer of  $F$ .*

The proof of the theorem follows the ideas of Pock and Lorenz, in the proof of Theorem 1 in [9]—see also [1]. The two main differences between our setting and the setting of [9] are:

1. We do not assume the existence of  $\alpha < 1$  such that  $\forall n \geq 1, \alpha_n \leq \alpha$ ;
2. The sequence  $(\delta_n)_{n \in \mathbb{N}}$  produced by FISTA, with a good choice of the sequence  $(t_n)$ , has stronger properties than in [9].



It turns out that Corollary 2 is crucial, while classical bounds on  $\delta_n$  which only show the existence of a constant  $C > 0$  such that  $\delta_n \leq \frac{C}{n^2}$  are not sufficient.

Before giving the complete proof of this result, several remarks can be done.

1. From Corollary 2 it follows that the sequence  $(n(x_{n+1} - x_n))_{n \in \mathbb{N}}$  is bounded, moreover from (18) it follows that the sequence  $(v_n)_{n \in \mathbb{N}}$  defined in (8) is also bounded (hence  $(u_n)_{n \in \mathbb{N}}$ ). These two facts imply that the sequence  $((x_n)_{n \in \mathbb{N}})$  is bounded, hence weakly sequentially compact.
2. Assume we have a subsequence which weakly converges to a  $\tilde{x} \in \mathcal{H}$ ,  $x_\nu \rightharpoonup \tilde{x}$ : then since the sequence  $(\delta_n)_{n \in \mathbb{N}}$  tends to 0,  $y_\nu \rightharpoonup \tilde{x}$  which shows that  $\tilde{x}$  is a fixed point of the nonexpansive operator  $T$ . Hence it is a minimizer of  $F$ .

If we are able to prove that the sequence  $\|x_n - x^*\|$  has a limit for any minimizer  $x^*$  of  $F$ , Theorem 3 will follow, from points 1. and 2. above and the observation that if  $x_\nu \rightharpoonup \tilde{x}$  and  $x_{\nu'} \rightharpoonup \tilde{x}'$ , then using  $\lim_\nu \|x_\nu - \tilde{x}\|^2 = \lim_{\nu'} \|x_{\nu'} - \tilde{x}\|^2$  and the same equality with  $\tilde{x}'$ , it follows  $\|\tilde{x} - \tilde{x}'\|^2 = 0$  (this is Opial's Theorem [15]). Before proving Theorem 3, let us establish the following estimate.

**Lemma 2.** *For all  $j \geq 1$ , let us define*

$$\beta_{j,k} = \prod_{l=j}^k \alpha_l = \prod_{l=j}^k \frac{l-1}{l+a},$$

for all  $k \geq j$ , and  $\beta_{j,k} = 1$  for  $k < j$ . (Observe that since  $\alpha_1 = 0$ ,  $\forall k > 1, \beta_{1,k} = 0$ .) Then, we have for all  $j$

$$\sum_{k=j}^{+\infty} \beta_{j,k} \leq \frac{j+5}{2}. \quad (19)$$

*Proof.* Since  $a \geq 2$ ,

$$\beta_{j,k} \leq \prod_{l=j}^k \frac{l-1}{l+2}.$$

Hence, for all  $j \geq 2$  and for all  $k \geq 1$ ,  $\beta_{j,k} \leq 1$ , while if  $k - j \geq 2$ ,

$$\beta_{j,k} \leq \left( \frac{j+1}{k} \right)^3.$$

It follows that for all  $j \geq 2$ ,

$$\begin{aligned} \sum_{k=j}^{+\infty} \beta_{j,k} &\leq 2 + \sum_{k=j+2}^{+\infty} \beta_{j,k} \leq 2 + \sum_{k=j+2}^{+\infty} \left(\frac{j+1}{k}\right)^3 \leq 2 + (j+1)^3 \sum_{k=j+2}^{+\infty} \frac{1}{k^3} \\ &\leq 2 + (j+1)^3 \int_{t=j+1}^{+\infty} \frac{dt}{t^3} \leq 2 + (j+1)^3 \frac{1}{2(j+1)^2}. \end{aligned}$$

Estimate (19) follows.  $\square$

*Proof of Theorem 3.* Let us define

$$\Phi_n = \frac{1}{2} \|x_n - x^*\|_2^2 \quad \text{and} \quad \Gamma_n = \frac{1}{2} \|x_{n+1} - y_n\|_2^2$$

From the identity

$$\langle a - b, a - c \rangle = \frac{1}{2} \|a - b\|_2^2 + \frac{1}{2} \|a - c\|_2^2 - \frac{1}{2} \|b - c\|_2^2 \quad (20)$$

we have by using the definition of  $y_n$

$$\Phi_n - \Phi_{n+1} = \delta_{n+1} + \langle y_n - x_{n+1}, x_{n+1} - x^* \rangle - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x^* \rangle \quad (21)$$

Then, using the monotonicity of  $\partial g$ , we deduce that for any  $z_{n+1} \in \partial g(x_{n+1})$  and for any  $z^* \in \partial g(x^*)$

$$\langle \gamma z_{n+1} - \gamma z^*, x_{n+1} - x^* \rangle \geq 0$$

By definition of  $x^*$ ,  $-\nabla(f(x^*)) \in \partial g(x^*)$  and  $y_n - x_{n+1} - \gamma \nabla f(y_n) \in \gamma \partial g(x_{n+1})$ .

It follows

$$\begin{aligned} \langle y_n - x_{n+1} - \gamma \nabla f(y_n) + \gamma \nabla f(x^*), x_{n+1} - x^* \rangle &\geq 0 \\ \langle y_n - x_{n+1}, x_{n+1} - x^* \rangle + \gamma \langle \nabla f(x^*) - \nabla f(y_n), x_{n+1} - x^* \rangle &\geq 0 \end{aligned}$$

Combining with (21) we obtain

$$\Phi_n - \Phi_{n+1} \geq \delta_{n+1} + \gamma \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - x^* \rangle - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x^* \rangle. \quad (22)$$

From the co-coercivity of  $\nabla f$ , we have

$$\begin{aligned} \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - x^* \rangle &= \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - y_n + y_n - x^* \rangle \\ &\geq \frac{1}{L} \|\nabla f(y_n) - \nabla f(x^*)\|_2^2 + \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - y_n \rangle \\ &\geq \frac{1}{L} \|\nabla f(y_n) - \nabla f(x^*)\|_2^2 - \frac{1}{L} \|\nabla f(y_n) - \nabla f(x^*)\|_2^2 - \frac{L}{2} \Gamma_n \\ &\geq -\frac{L}{2} \Gamma_n. \end{aligned}$$

Substituting back into (22), we get

$$\Phi_n - \Phi_{n+1} \geq \delta_{n+1} - \frac{\gamma L}{2} \Gamma_n - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x^* \rangle,$$

and invoking (20) it follows that

$$\begin{aligned} \Phi_{n+1} - \Phi_n - \alpha_n(\Phi_n - \Phi_{n-1}) &\leq -\delta_{n+1} + \frac{\gamma L}{2} \Gamma_n \\ &\quad + \alpha_n(\delta_n + \langle x_n - x_{n-1}, x_{n+1} - x_n \rangle) \\ &= -\Gamma_n + \frac{\gamma L}{2} \Gamma_n + (\alpha_n + \alpha_n^2) \delta_n, \end{aligned}$$

where we have used the fact that

$$\delta_{n+1} - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x_n \rangle = \alpha_n^2 \frac{\|x_n - x_{n-1}\|^2}{2} - \frac{\|x_{n+1} - y_n\|^2}{2}.$$

Using  $\frac{\alpha_n + \alpha_n^2}{2} \leq \alpha_n$  we obtain

$$\Phi_{n+1} - \Phi_n - \alpha_n(\Phi_n - \Phi_{n-1}) \leq -\left(1 - \frac{\gamma L}{2}\right) \Gamma_n + 2\alpha_n \delta_n \quad (23)$$

with  $1 - \frac{\gamma L}{2} > 0$ .

Now defining  $\theta_n = \max(0, \Phi_n - \Phi_{n-1})$  we obtain

$$\theta_{n+1} \leq \alpha_n(\theta_n + 2\delta_n) \quad (24)$$

Applying recursively (24) it follows that for all  $n \geq 2$  ( $\alpha_1 = 0$ , and in particular  $\theta_1, \theta_2 = 0$ ).

$$\theta_{n+1} \leq 2 \sum_{j=2}^n \left( \prod_{l=j}^n \alpha_l \right) \delta_j = 2 \sum_{j=2}^n \beta_{j,n} \delta_j. \quad (25)$$

Hence (using (19)),

$$\begin{aligned} \sum_{n=2}^{+\infty} \theta_n &\leq 2 \sum_{n=1}^{+\infty} \sum_{j=2}^n \beta_{j,n} \delta_j \\ &\leq 2 \sum_{j=2}^{\infty} \delta_j \sum_{n=j}^{\infty} \beta_{j,n} \\ &\leq 2 \sum_{j=1}^{\infty} \delta_j \frac{j+5}{2}. \end{aligned}$$

From Corollary 2 the right side of the last inequality is finite if  $a > 2$ , therefore the sequence  $(\theta_n)_{n \in \mathbb{N}}$  belongs to  $\ell_1(\mathbb{N})$ .

The end of the proof follows Lorenz and Pock [9]. We set  $s_n = \Phi_n - \sum_{i=1}^n \theta_i$  and since  $\Phi_n \geq 0$  and  $\sum_{i=1}^n \theta_i$  is bounded independently of  $n$ , we see that  $s_n$  is bounded from below. On the other hand

$$s_{n+1} = \Phi_{n+1} - \theta_n - \sum_{i=1}^n \theta_i \leq \Phi_{n+1} - \Phi_{n+1} + \Phi_n - \sum_{i=1}^n \theta_i = s_n$$

and hence  $(s_n)_{n \in \mathbb{N}}$  is a non-decreasing sequence and thus is convergent. This implies that  $\Phi_n$  is convergent, which concludes the proof of Theorem 3.  $\square$

## 4 Numerical Experiments

In the previous parts, it was shown that non classical choices of the sequence  $(t_n)_{n \in \mathbb{N}}$  ensure weak convergence of iterates  $(x_n)_{n \in \mathbb{N}}$  and good properties for the sequence  $(F(x_n) - F(x^*))_{n \in \mathbb{N}}$ . On three examples, inpainting, deblurring and denoising, we compare several choices of parameters.

For each example the 4 following sequences are tested :

- $t_1 = 1$  and  $\forall n \in \mathbb{N}$ ,  $t_{n+1} = \sqrt{t_n^2 + \frac{1}{4}} + \frac{1}{2}$ ,
- $t_n = \frac{n+a+1}{a}$ ,  $\forall n \in \mathbb{N}$  with  $a = 2, 3$  and  $4$ .

For each problem, at each iteration  $n$ , the values  $\|x_n - x_{n-1}\|_2^2$  and  $F(x^n) - F(x^*)$  are computed. Since  $F(x^*)$  can not be exactly computed, the value  $F(x^*)$  is estimated by the the minimum of the values computed on 2000 iterations for the four methods. The plot of these two quantities is thus given from  $n = 1$  to  $n = 1800$ .

**Inpainting** Let us consider here a degraded image  $y^0 = Mx^0$  where  $x^0$  is an unknown source image and  $M$  a mask operator. In our example 50% of the pixels are removed. We estimate the image  $x^0$  from  $y^0$  by minimizing

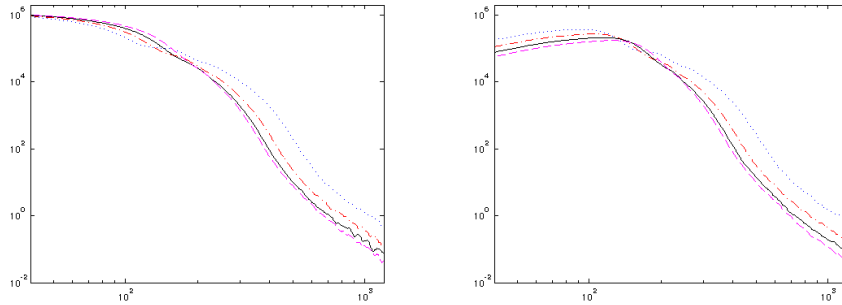
$$F(x) = \frac{1}{2} \|y^0 - Mx\|_2^2 + \lambda \|Tx\|_1 \quad (26)$$

where  $\lambda$  is a small positive parameter and  $T$  an orthogonal (Daubechies) wavelet transform.

Considering  $f(x) = \frac{1}{2} \|y^0 - Mx\|_2^2$  and  $g(x) = \lambda \|Tx\|_1$ , FISTA may be applied to minimize  $F$ .



Left: the masked image  $y^0$ . Right: an image  $\hat{x}$  estimated minimizing  $F$  with FISTA.<sup>2</sup>



Left: values of  $F(x_n) - F(x^*)$ . Right: values of  $\|x_n - x_{n-1}\|_2^2$ .  
Blue dot line classical FISTA, red dashed-dot line  $a = 2$ , black solid line  $a = 3$  and magenta dashed line  $a = 4$ .

On this example, the choices  $a = 3$  or  $a = 4$  seems better than classical choices after 100 iterations. One can notice that classical FISTA is better for a small number of iterations.

**Deblurring** In this second example  $y^0 = h \star x^0 + n$  is the noisy image of a blurred images  $x^0$ , where  $h$  is a gaussian filter and  $n$  is a random gaussian noise. The image  $x^0$  can be estimated minimizing

$$F(x) = \frac{1}{2} \|y^0 - h \star x\|_2^2 + \lambda \|Tx\|_1 \quad (27)$$

where  $\lambda$  is a small positive real number whose value depends on the noise level and  $T$  is an orthogonal (Daubechies) wavelet transform.

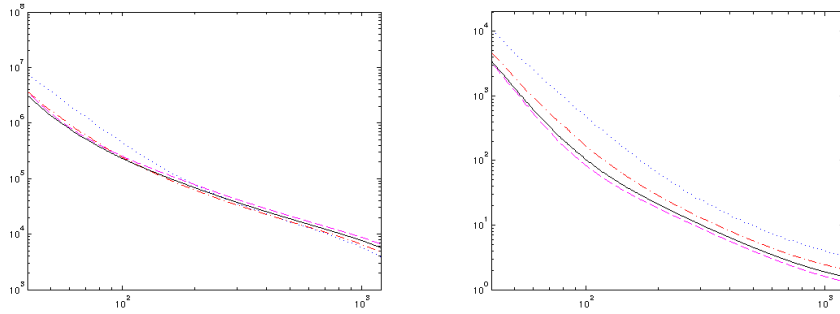
Considering  $f(x) = \frac{1}{2} \|y^0 - h \star x\|_2^2$  and  $g(x) = \lambda \|Tx\|_1$ , FISTA may be applied to minimize  $F$ .

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<sup>2</sup>The images provided by the 4 versions of FISTA look very similar, the difference appears in the values of the variation of fonctionnal  $F$  through the iterations.



Left: the blurred image  $y^0$ . Right: an image  $\hat{x}$  estimated minimizing  $F$  with FISTA.



Left: values of  $F(x_n) - F(x^*)$ . Right: values of  $\|x_n - x_{n-1}\|_2^2$ . Blue dot line classical FISTA, red dashed-dot line  $a = 2$ , black solid line  $a = 3$  and magenta dashed line  $a = 4$ .

On this example, the choices of the classical FISTA seems better after 200 iterations, but the decreasing of  $\delta_n$  seems still better for  $a = 3$  and  $a = 4$ .

**TV denoising** Let us consider now a noisy image  $y^0 = x^0 + n$ . The image  $x^0$  may be estimated from  $y^0$  minimizing

$$F(x) = \frac{1}{2} \|y^0 - x\|_2^2 + \lambda \|\nabla x\|_1 \quad (28)$$

where  $\nabla x$  is the gradient of the image  $x$  and  $\|\nabla x\|_1$  is the isotropic  $\ell_1$ -norm of the gradient. This regularization is also called Total Variation (TV) regularization. The proximal map of the function  $x \mapsto \|\nabla x\|_1$  does not have a close form and FISTA is difficult to use directly here. Nevertheless, by duality, this minimization problem is equivalent to minimize

$$G(p) = \frac{1}{2} \|y^0 + \operatorname{div} p\|_2^2 + i_{\|\cdot\|_\infty \leq \lambda}(p) \quad (29)$$

where  $i_C$  denotes the function such that  $i_C(x) = 0$  if  $x \in C$  and  $i_C(x) = +\infty$  if  $x \notin C$  and where  $x \mapsto -\operatorname{div} x$  is the divergence operator, conjugate of

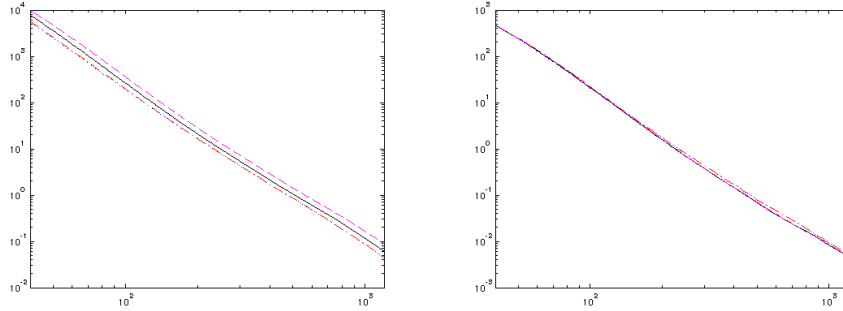
the gradient  $\nabla$ .

These two problems are equivalent and for any solution  $p^*$  of the second minimization problem,  $y^0 + \operatorname{div} p^*$  is a solution of the first minimization problem.

This second problem can be solved using FISTA since the gradient of  $p \mapsto \frac{1}{2} \|y + \operatorname{div} p\|_2^2$  is Lipschitz and the proximal map of  $p \mapsto i_{\|\cdot\|_{+\infty} \leq \lambda}(p)$  is a simple projection.



Left: noisy image  $y^0$ . Right: the image  $\hat{x}$  estimated minimizing  $F$  with FISTA.



Left: values of  $G(p_n) - G(p^*)$ . Right: values of  $\|p_n - p_{n-1}\|_2^2$ .  
Blue dot line classical FISTA, red dashed-dot line  $a = 2$ , black solid line  $a = 3$  and magenta dashed line  $a = 4$ .

On this example, the different choices of parameters seem to be equivalent.

These three examples shows that choosing a priori a sequence  $(t_n)_{n \in \mathbb{N}}$  for FISTA is difficult and that for a given problem, it would be useful to test various options. Sometimes the classical parameters proposed by Beck and Teboulle are better to get a faster minimization, sometimes the use of  $a = 3$  or  $a = 4$  is better. But on the three examples the norm of the variation  $\delta_n$  is smaller for  $a = 3$  or  $a = 4$  than  $a = 2$  or the classical FISTA, which may indicate that the convergence of the iterates is faster.

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