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Quick reachability and proper extension for problems with unbounded controls

María Soledad Aronna
IMPA, Rio de Janeiro, Brazil

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(Joint work with M. Motta & F. Rampazzo, Università di Padova, Italy)

Conference on New Trends in Optimal Control
June 2014, Tours, France

Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha}, \quad \text{on } [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \rightarrow \mathbf{R}^n$, $u : [0, T] \rightarrow U \subset \mathbf{R}^m$, $v : [0, T] \rightarrow V \subset \mathbf{R}^l$,

we rely on the notion of **LIMIT SOLUTION**,

and we investigate whether minimum problems with \mathcal{L}^1 -controls are

PROPER EXTENSIONS

of regular problems with **more regular controls** (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

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Limit solutions

Consider the Cauchy problem

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha}, \quad \text{for } t \in [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here $(\bar{x}, \bar{u}) \in \mathbf{R}^n \times U$, $u \in \mathcal{L}^1([0, T]; U)$ having $u(0) = \bar{u}$, and $v \in L^1([0, T]; V)$.

We say that $x : [0, T] \rightarrow \mathbf{R}^n$ is a **LIMIT SOLUTION** if, for every $\tau \in [0, T]$, there exists $(u_k^{\tau}) \subset AC([0, T]; U)$ such that $u_k^{\tau}(0) = \bar{u}$ and the corresponding Carathéodory solutions (x_k^{τ}) are uniformly bounded and satisfy

$$|(x_k^{\tau}, u_k^{\tau})(\tau) - (x, u)(\tau)| + \|(x_k^{\tau}, u_k^{\tau}) - (x, u)\|_1 \rightarrow 0, \quad \text{when } k \rightarrow \infty.$$

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Example: AC –reachable set $\neq \mathcal{L}^1$ –reachable set

Fix $(\bar{x}, \bar{u}) \in \mathbf{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1, x_2)(0) = (1, 1), \quad u(0) = 1, \end{cases}$$

with $U = [0, 1]$, $t \in [0, 1]$.

It is trivial to verify that if \tilde{u} is absolutely continuous then the corresponding trajectory \tilde{x} verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0, 0, 0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0, 1], \end{cases}$ we get that

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Proper extension of minimum problems

Definition

Let E be a set and let $\mathcal{F} : E \rightarrow \mathbf{R}$ be a function. A **proper extension** of a minimum problem

$$\inf_{e \in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set \hat{E} endowed with a limit notion and such that there exists an injective map $i : E \rightarrow \hat{E}$ verifying the following properties:

- (i) $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$ for all $e \in E$ and, moreover, for every $\hat{e} \in \hat{E}$ there exists a sequence (e_k) in E such that, setting $\hat{e}_k := i(e_k)$, one has

$$\lim_{k \rightarrow \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

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Proper extension with NO final constraints

For

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha}, \quad \text{on } [0, T], \quad (x, u)(0) = (\bar{x}, \bar{u}),$$

consider a cost function $\psi : \mathbf{R}^n \times U \rightarrow \mathbf{R}$.

Define the following optimal control problems depending on (\bar{x}, \bar{u}) :

(P) $\inf \psi((x, u)(T)) : (u, v) \in \mathcal{L}^1 \times L^1, u(0) = \bar{u}, x \in \Sigma[\bar{x}, u, v],$

(P_{AC}) $\inf \psi((x, u)(T)) : (u, v) \in AC \times L^1, u(0) = \bar{u}, x = x[\bar{x}, u, v],$

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u, v) , there exists a sequence $(u_k) \subset AC, u_k(0) = \bar{u}$, and $x_k := x[\bar{x}, u_k, v]$ such that $\|(x, u) - (x_k, u_k)\|_1 \rightarrow 0$, and

$$(ii) \quad \inf_{(u,v) \in \mathcal{L}^1 \times L^1} \psi((x, u)(T)) = \inf_{(u,v) \in AC \times L^1} \psi((x, u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

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BV controls - Recall: Graph completions

- For **regular** $u \in AC$ one can **reparametrize time** $t(s) = \varphi_0(s)$ with $\varphi_0 : [0, 1] \rightarrow [0, T]$, and set $\varphi(s) := u \circ \varphi_0(s)$.

The **SPACE-TIME SYSTEM**:

$$\begin{cases} y_0'(s) = \varphi_0'(s), \\ y'(s) = f(y(s), \varphi(s), \psi(s))\varphi_0'(s) + \sum_{\alpha=1}^m g_\alpha(y(s))\varphi_\alpha'(s), & s \in [0, 1], \\ (y_0, y)(0) = (0, \bar{x}), \end{cases}$$

- For **BV controls** u , let (φ_0, φ) be a **graph completion** of u :
 $(\varphi_0, \varphi) : [0, 1] \rightarrow [0, T] \times U$ Lipschitz continuous such that,
 $\forall t \in [0, T]$, there exists $s \in [0, 1]$ verifying $(t, u(t)) = (\varphi_0, \varphi)(s)$.

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Recall: Graph completion solutions

Given $(\varphi_0, \varphi) : [0, 1] \rightarrow [0, T] \times U$, let y be the solution of the SPACE-TIME system

Graph completion solution: (possibly) set-valued map $x : [0, T] \rightrightarrows \mathbb{R}^n$,

$$t \longmapsto x(t) := y \circ \varphi_0^{-1}(t).$$

Single-valued graph completion solution:

Let $\sigma : [0, T] \rightarrow [0, 1]$ be a right-inverse of φ_0 having $u(t) = \varphi \circ \sigma(t)$,

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Optimal control problems with bounded variation controls

$$(P_{gc}^K) \inf \psi((x, u)(T)) : (u, v) \in BV^K \times L^1, u(0) = \bar{u}, x \in \Sigma_{gc}^{K+T}[\bar{x}, u, v],$$

Recall: A simple limit solution $x : [0, T] \rightarrow \mathbf{R}^n$ is a **BV-SIMPLE limit solution** if (u_k^τ) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

$$(P_{BVS}^K) \inf \psi((x, u)(T)) : (u, v) \in BV^K \times L^1, u(0) = \bar{u}, x \in \Sigma_{BVS}^K[\bar{x}, u, v],$$

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma_{gc}^{K+T}[\bar{x}, u, v] = \Sigma_{BVS}^K[\bar{x}, u, v]$$

Theorem

For every initial condition $(\bar{x}, \bar{u}) \in \mathbf{R}^n \times U$, one has

$$\lim_{K \rightarrow \infty} \text{Val}(P_{gc}^K)(\bar{x}, \bar{u}) = \lim_{K \rightarrow \infty} \text{Val}(P_{BVS}^K)(\bar{x}, \bar{u}) = V(\bar{x}, \bar{u}).$$

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Optimal control problems with bounded variation controls

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The problem with final constraints

Let $\mathcal{S} \subset \mathbf{R}^n \times U$ be a closed subset. Define the problems

$$(P^c) \inf \psi((x, u)(T)) : (u, v) \in \mathcal{L}^1 \times L^1, u(0) = \bar{u}, \\ x \in \Sigma[\bar{x}, u, v], (x, u)(T) \in \mathcal{S},$$

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inf $x_2(1)$,

$$\dot{x}_1(t) = \dot{u}(t),$$

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$$(x_1, x_2)(0) = (1, 0), \quad u(0) = 1$$

$$(x_1, x_2, u)(1) \in \mathcal{S} := \left(\{(0, 0)\} \cup (\mathbf{R} \times [1, +\infty[)) \times [0, 1], \right.$$

$$\left. u(t) \in U := \left[-\frac{1}{3}, \frac{4}{3}\right] \right.$$

For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| ds > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \implies x_2(1) \geq 1, \implies \text{Val}(P_{AC}^c) \geq 1$.

On the other hand, by implementing the impulsive control

$$u(t) := \begin{cases} 1, & t = 0, \\ 0, & t \in]0, 1], \end{cases} \implies x_2(1) = 0 \implies \text{Val}(P^c) \leq 0 < 1 \leq \text{Val}(P_{AC}^c).$$

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[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that \mathcal{S} is a compact set contained in the interior of $\mathbb{R}^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S}, \rho)} \setminus \mathcal{S}$, and $\forall (p_x, p_u) \in D^*d_{\mathcal{S}}(x, u)$, we have

$$\min_{|w| \leq 1} \left\{ \left\langle p_x, \sum_{\alpha=1}^m g_\alpha(x, u) w^\alpha \right\rangle + \langle p_u, w \rangle \right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset \mathbb{R}^k$ be an open set, $F : \Omega \rightarrow \mathbb{R}$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^*F(y) := \{w \in \mathbb{R}^k : w = \lim \nabla F(y_k), y_k \in \text{DIFF} \setminus \{y\}, \lim y_k = y\}.$$

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Let us consider $(u, v) \in \mathcal{L}^1 \times L^1$ and an associated limit solution x feasible for the problem (P^c) , i.e. having

$$(x, u)(T) \in \mathcal{S}.$$

By definition of limit solution, there exists a sequence $(\hat{u}_k) \subset AC$ such that $\hat{u}_k(0) = \bar{u}$ and

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In general,

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$$\mathcal{T}_{\mathcal{S}, \eta}(y) \leq \frac{d_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]

This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \rightarrow \mathbb{R}^n \times U$, having

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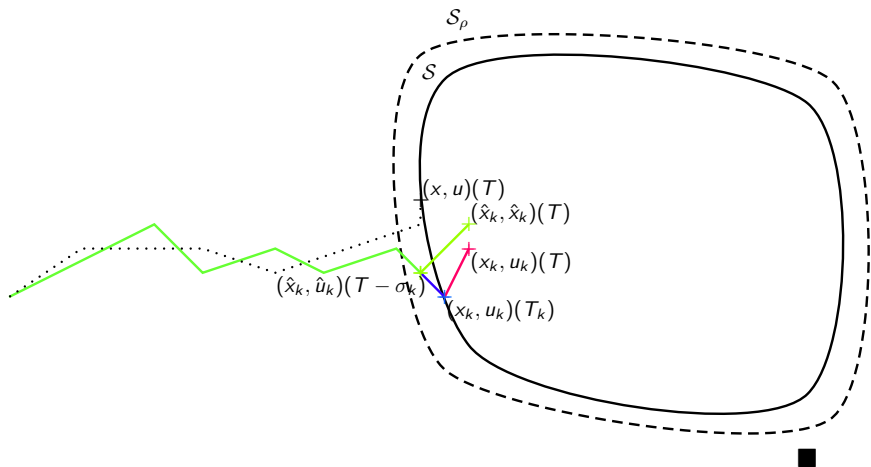
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Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with \mathcal{L}^1 controls and trajectories) with no final constraints is the limit of problems with controls of variation K with $K \rightarrow \infty$.
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THANK YOU FOR YOUR ATTENTION