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► **To cite this version:**

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan Perrussel, et al.. Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model. JSA 2013 - Journées Singulières Augmentées, Aug 2013, Rennes, France. hal-00931735

**HAL Id: hal-00931735**

**<https://inria.hal.science/hal-00931735v1>**

Submitted on 19 Mar 2021

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## Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

Monique Dauge<sup>\*</sup>   Patrick Dular<sup>♡</sup>   Laurent Krähenbühl<sup>◇</sup>  
Victor Péron<sup>♠</sup>   Ronan Perrussel<sup>†</sup>   Clair Poignard<sup>♣</sup>

<sup>\*</sup>IRMAR (Rennes), <sup>♡</sup>ACE (Liège), <sup>◇</sup>Ampère (Lyon), <sup>♠</sup>EPI Magique3D & LMAP, UPPA (Pau),  
<sup>†</sup>LAPLACE (Toulouse), <sup>♣</sup>EPI MC2 (Bordeaux)

JSA 2013,  
Conférence en l'honneur de Martin Costabel pour ses 65 ans,  
Rennes, August 26-30.

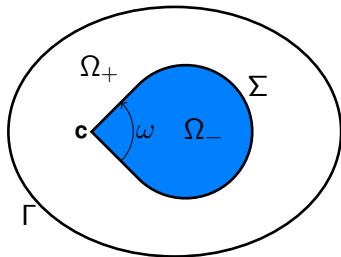
## The Configuration

$\Omega_-$  : a conducting body ( $\sigma > 0$ )

$\Omega_+$  : a dielectric medium

$\mathbf{c}$  : a corner

$\omega \in (0, 2\pi)$  : the angle of the corner



We describe the Magnetic potential in the vicinity of the corner  $\mathbf{c}$  .

# The Magnetic Potential

## The eddy-current problem

The magnetic vector potential  $\mathcal{A}$  satisfies

$$\left\{ \begin{array}{l} -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ -\Delta \mathcal{A}^+ = \mu_0 J \text{ in } \Omega_+, \\ \mathcal{A}^+ = 0 \text{ on } \Gamma, \end{array} \right. \quad \begin{array}{l} [\mathcal{A}]_\Sigma = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_\Sigma = 0, \text{ on } \Sigma. \end{array} \quad (1)$$

Here  $\zeta^2 = \kappa\mu_0\sigma/4 > 0$ ,

$J$  : a smooth data, vanishing near the corner  $\mathbf{c}$ .

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Here  $\zeta^2 = \kappa\mu_0\sigma/4 > 0$ ,

$J$  : a smooth data, vanishing near the corner  $\mathbf{c}$ .

## Proposition

*There exists a unique solution  $\mathcal{A}$  in  $H_0^1(\Omega)$  to problem (1). Moreover,  $\mathcal{A}$  belongs to  $H^{\frac{5}{2}-\varepsilon}(\Omega)$  for any  $\varepsilon > 0$ . In particular,  $\mathcal{A}$  belongs to  $\mathcal{C}^1(\overline{\Omega})$ .*

$\mathcal{A}$  possesses a **corner asymptotic** expansion near  $\mathbf{c}$ .

# Corner Asymptotics

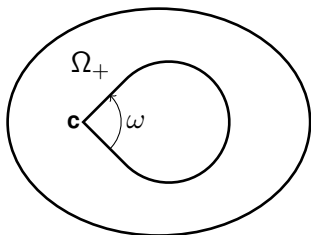
- To generalize the Taylor Expansion
- Corner Asymptotics involve
  - 1 The **singular functions** (**primal** and **dual**) :  
belong to the kernel of the considered operator in  $\mathbb{R}^2$ .
  - 2 The **singular coefficients** :  
its calculation requires the knowledge of **dual singular functions**.

**Aim** : To explicit the Corner Asymptotics of  $\mathcal{A}$  near the corner  $\mathbf{c}$ .

## Motivation : The eddy-current phenomenon

- The limit problem in large frequency/high conductivity :

$$\begin{cases} -\Delta \mathcal{A}_0^+ = \mu_0 J \text{ in } \Omega_+, \\ \mathcal{A}_0^+ = 0 \text{ on } \partial\Omega_+. \end{cases} \quad (2)$$

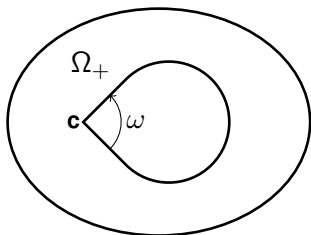




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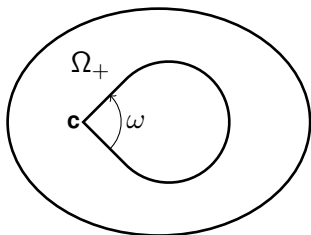


- If  $\Omega_+$  has a convex corner, i.e.  $\omega \in (0, \pi)$ , Problem (2) has **non  $\mathcal{C}^1$  singularities**

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- Problem (1) has  **$C^1$  singularities**

## Motivation : Multi-scale analysis for eddy-currents



BURET *et al* (IEEE Trans. on Mag. '12)

*Eddy currents and corner singularities*

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- When  $\delta = \sqrt{\frac{2}{\kappa\mu_0\sigma}} \ll 1$ , there holds

$$\mathcal{A}_\delta \approx \mathcal{A}_0 + a\delta^\alpha V\left(\frac{\cdot}{\delta}\right) + R_\delta.$$

Here  $V$  is a *profile* defined in  $\mathbb{R}^2$  such that

$$-\Delta V + 2iV \mathbb{1}_{\Omega_-} = 0 \quad \text{near } \mathbf{c},$$

$$\text{and } \alpha = \frac{\pi}{2\pi - \omega}.$$

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$$\text{and } \alpha = \frac{\pi}{2\pi - \omega}.$$

$\implies$  The knowledge of the *singularities* of  $-\Delta + 4i\zeta^2 \mathbb{1}_{\Omega_-}$  is "crucial".

## Main difficulty

The considered operator is

$$-\Delta + i\kappa\mu_0\sigma \mathbb{1}_{\Omega_-} = \begin{cases} -\Delta + i\kappa\mu_0\sigma & \text{in } \Omega_-, \\ -\Delta & \text{in } \Omega_+. \end{cases}$$

The *singularities* are generated by the term  $i\kappa\mu_0\sigma \mathbb{1}_{\Omega_-}$ .

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The *singularities* are generated by the term  $i\kappa\mu_0\sigma \mathbb{1}_{\Omega_-}$ .

$\implies$  the derivation of the Corner Asymptotics is not obvious.



## Our Reference



M. DAUGE *et al* (To appear in MMAS - INRIA RR 8204)

*Corner asymptotics of the magnetic potential in the eddy-current model*

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- $\mathcal{A}$  possesses a corner asymptotic expansion

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta) + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r, \theta),$$

$(r, \theta)$  : polar coordinates centered at  $\mathbf{c}$

$\mathfrak{S}^{k,p}$  : primal singular functions

$\Lambda^{k,p}$  : singular coefficients

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- We provide a constructive procedure to determine the **primal** and **dual singularities**

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$(r, \theta)$  : polar coordinates centered at  $\mathbf{c}$

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$\Lambda^{k,p}$  : singular coefficients

- We provide a constructive procedure to determine the **primal** and **dual singularities**
- We generalize the method of moments and we introduce the method of **quasi-dual functions** to determine  $\Lambda^{k,p}$

# Outline

① **The case  $\zeta = 0$  :**

We introduce the method of **moments** and the method of **dual functions**.

② **The case  $\zeta \neq 0$  :**

We provide a constructive procedure to determine the **singularities**.

We introduce the method of **quasi-dual singular functions**.

③ **Numerical simulations**

## Laplace Operator ( $\zeta = 0$ )

- We consider the solution  $\mathcal{A}$  to

$$\begin{cases} -\Delta \mathcal{A} = \mu_0 J \text{ in } \Omega \\ \mathcal{A} = 0 \text{ on } \Gamma \end{cases}$$

- $\mathcal{A}$  admits the Taylor expansion at  $\mathbf{c}$  :

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \underbrace{r^k \cos(k\theta - p\pi/2)}_{= \mathfrak{s}^{k,p}(r, \theta)}$$

$\mathfrak{s}^{k,p}$  : harmonic polynomials

## Methods to extract the coefficients $\Lambda^{k,p}$

We use dual harmonic functions :

$$\mathfrak{t}^{k,p}(r, \theta) = \begin{cases} -\frac{1}{2\pi} \log r, & \text{if } k = 0, p = 0, \\ \frac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \geq 1, p = 0, 1. \end{cases}$$

- The method of moments
- The dual function method

## The Method of Moments

For  $R > 0$ , we introduce the form  $\mathcal{M}_R$  :

$$\mathcal{M}_R(K, A) = \frac{1}{R} \int_{r=R} K A R d\theta.$$

Assume  $J$  has a support outside the ball  $\mathcal{B}(\mathbf{c}, R)$ .

### Proposition

Let  $\mathcal{A}$  be the solution to the Laplace equation. Then

$$\mathcal{M}_R(1, \mathcal{A}) = 2\pi \Lambda^{0,0} \quad \text{and} \quad \mathcal{M}_R(\mathfrak{k}^{k,p}, \mathcal{A}) = \frac{1}{2k} \Lambda^{k,p}, \quad k \geq 1, p = 0, 1.$$



# The Dual Function Method



V.G. MAZ'YA, B.A. PLAMENEVSKII (Amer. Math. Soc. Trans. (2) '84)

*On the coefficients in the asymptotic of solutions of the elliptic boundary problem in domains with conical points*

- For  $R > 0$ , let us introduce the bilinear form

$$\mathcal{J}_R(K, A) = \int_{r=R} (K \partial_r A - \partial_r K A) R d\theta.$$

## Proposition

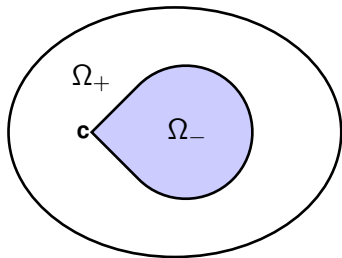
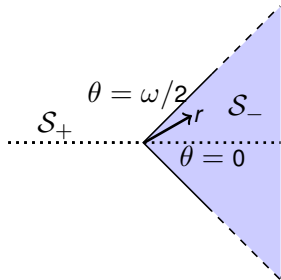
Let  $\mathcal{A}$  be the solution to the Laplace equation. Then

$$\mathcal{J}_R(\mathfrak{k}^{0,0}, \mathcal{A}) = \Lambda^{0,0} \quad \text{and} \quad \mathcal{J}_R(\mathfrak{k}^{k,p}, \mathcal{A}) = \Lambda^{k,p}, \quad k \geq 1, \quad p = 0, 1.$$

## The singularities

The singularities  $\mathfrak{U}$  of  $-\Delta + 4i\zeta^2 \mathbb{1}_{\mathcal{S}_-}$  :

$$\begin{cases} -\Delta \mathfrak{U} + 4i\zeta^2 \mathfrak{U} = 0 & \text{in } \mathcal{S}_-, \\ -\Delta \mathfrak{U} = 0 & \text{in } \mathcal{S}_+. \end{cases}$$



## Description of the *Singularities*



V.A. KONDRATEV (Trudy Moskov. Mat. Obšč. '67)

*Boundary value problems for elliptic equations in domains with conical or angular points*

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$$\mathfrak{U} = \underbrace{u_0}_{\text{leading part}} + \sum_{j \geq 1} (i\zeta^2)^j \underbrace{u_j}_{\text{shadow}},$$

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$$\mathfrak{U} = \underbrace{u_0}_{\text{leading part}} + \sum_{j \geq 1} (i\zeta^2)^j \underbrace{u_j}_{\text{shadow}},$$

- We solve

$$\Delta u_0 = 0, \quad \Delta u_1 = 4u_0 \mathbb{1}_{S_-}, \quad \dots, \quad \Delta u_j = 4u_{j-1} \mathbb{1}_{S_-},$$

$$u_j \in S^\lambda = \text{Span} \left\{ r^\lambda \log^q r \Phi(\theta), \quad q \in \mathbb{N}, \quad \Phi \in C^1(\mathbb{T}), \quad \Phi^\pm \in C^\infty(\overline{\mathbb{T}}_\pm) \right\}$$

Here  $\lambda \in \mathbb{Z}$ ,

$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ ,  $\mathbb{T}_- = (-\omega/2, \omega/2)$  and  $\mathbb{T}_+ = \mathbb{T} \setminus \overline{\mathbb{T}_-}$ .

## Existence of the shadows

### Lemma

Let  $\lambda \in \mathbb{Z}$  and  $f \in T^{\lambda-2}$ . Then, there exists  $u \in S^\lambda$  such that  $\Delta u = f$ .

Moreover

(i) If  $\lambda \in \mathbb{Z} \setminus \{0\}$ ,  $\deg u \leq \deg f + 1$ ,


(ii) If  $\lambda = 0$ ,  $\deg u \leq \deg f + 2$ .

Here,

$$T^\lambda = \text{Span} \left\{ r^\lambda \log^q r \Psi(\theta), \quad q \in \mathbb{N}, \quad \Psi \in L^2(\mathbb{T}), \quad \Psi^\pm \in C^\infty(\overline{\mathbb{T}}_\pm) \right\}.$$

The degree of  $g$  is its degree as polynomial of  $\log r$ .

## Proof of Lemma

 M. Dauge *Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Berlin 1988*

We introduce the Mellin symbol  $\mathcal{M}$  associated with  $\Delta$

$$\mathcal{M}(\mu) = (\mu^2 + \partial_\theta^2), \theta \in \mathbb{T}.$$

Setting

$$f = \sum_{q=0}^{\deg f} r^{\lambda-2} \log^q r \Psi_q \in \mathbb{T}^{\lambda-2},$$

and

$$u = \operatorname{Res}_{\mu=\lambda} \left\{ r^\mu \mathcal{M}(\mu)^{-1} \sum_{q=0}^{\deg f} \frac{q! \Psi_q}{(\mu - \lambda)^{q+1}} \right\} \in \mathbb{S}^\lambda,$$

we check that  $\Delta u = f$ .

## Primal and Dual Singularities

- *Primal singular functions*  $\mathfrak{S}$  belong to  $H^1$  in any bounded neighborhood  $\mathcal{B}$  of  $\mathbf{c}$ .
- *Dual singular functions*  $\mathfrak{R}$  do not belong to  $H^1$  in such a neighborhood  $\mathcal{B}$ .

The functions  $\mathfrak{R}$  are needed for the determination of the coefficients  $\Lambda$  involved in the asymptotics.



# 1. A basis for Primal Singularities

For  $(k, p) = (0, 0)$  and for  $(k, p) \in \mathbb{N}^* \times \{0, 1\}$ ,

$$\mathfrak{S}^{k,p} = \underbrace{\mathfrak{s}^{k,p}}_{\text{leading part}} + \sum_{j \geq 1} (i\zeta^2)^j \underbrace{\mathfrak{s}_j^{k,p}}_{\text{shadow}} .$$

## Lemma

For any  $j \geq 1$ , there exists  $\mathfrak{s}_j^{k,p} \in \mathbf{S}^{k+2j}$ , with  $\deg \mathfrak{s}_j^{k,p} \leq j$ , satisfying

$$\Delta \mathfrak{s}_j^{k,p} = 4\mathfrak{s}_{j-1}^{k,p} \mathbb{1}_{\mathcal{S}_-} .$$

Application :  $\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta) + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r, \theta)$

## 2. A basis for Dual Singularities

For  $(k, p) = (0, 0)$  and for  $(k, p) \in \mathbb{N}^* \times \{0, 1\}$ ,



$$\mathfrak{K}^{k,p} = \underbrace{\mathfrak{k}_j^{k,p}}_{\text{leading part}} + \sum_{j \geq 1} (i\zeta^2)^j \underbrace{\mathfrak{k}_j^{k,p}}_{\text{shadow}} .$$

### Lemma

For any  $j \geq 1$ , there exists  $\mathfrak{k}_j^{k,p} \in \mathfrak{S}^{-k+2j}$ , with  $\deg \mathfrak{k}_j^{k,p} \leq j + 1$ , satisfying

$$\Delta \mathfrak{k}_j^{k,p} = 4\mathfrak{k}_{j-1}^{k,p} \mathbb{1}_{\mathfrak{S}_-} .$$

### 3. The Quasi-Dual Function Method

-  M. COSTABEL *et al* (SIAM '04)  
*A quasidual function method for extracting edge stress intensity functions*
-  S. SHANNON *et al* (Preprint '12)  
*Extracting generalized edge flux intensity functions by the quasidual function method along circular 3-D edges*
- We use quasi-dual functions  $\mathcal{R}_m^{k,p}$  :

$$\mathcal{R}_m^{k,p} = \mathfrak{t}^{k,p} + \sum_{j=1}^m (i\zeta^2)^j \underbrace{\mathfrak{t}_j^{k,p}}_{\text{shadow}}$$

# The Quasi-Dual Function Method

## Extraction of coefficients

### Theorem

Let  $\mathcal{A}$  be the solution to problem (1). Let  $k \in \mathbb{N}$  and  $p \in \{0, 1\}$  ( $p = 0$  if  $k = 0$ ). Then for all  $m$  such that  $2m + 2 > k$ ,

$$\mathcal{J}_R(\mathcal{R}_m^{k,p}, \mathcal{A}) \underset{R \rightarrow 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{\lfloor k/2 \rfloor} \mathcal{J}^{k,p; k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}(R^{-k} R_0^{2m+2} \log R),$$

where  $R_0 = \zeta R (1 + \sqrt{|\log R|})$ .

# The Quasi-Dual Function Method

## Examples

- For  $k = 0$

$$\Lambda^{0,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$$

- For  $k = 1$

$$\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$$

- For  $k = 2$ , we need  $m \geq 1$

$$\Lambda^{2,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \Lambda^{0,0} + \mathcal{O}(R^{-2} R_0^4 \log R)$$

# The Quasi-Dual Function Method

Key for the Proof of Theorem

We use quasi-primal singularities  $\mathfrak{S}_m^{k,p}$  :


$$\mathfrak{S}_m^{k,p} = \mathfrak{s}^{k,p} + \sum_{j=1}^m (i\zeta^2)^j \underbrace{\mathfrak{s}_j^{k,p}}_{\text{shadow}}$$

For  $\varepsilon \in (0, R)$ , we evaluate

$$\mathcal{J}_R(\mathfrak{R}_m^k, \mathfrak{S}_{m'}^{k'}) - \mathcal{J}_\varepsilon(\mathfrak{R}_m^k, \mathfrak{S}_{m'}^{k'}) \underset{R \rightarrow 0}{=} \mathcal{O}(R^{-k+k'} R_0^{2m+2} \log R)$$

with any chosen  $k'$ , and  $m' \geq m$ .

## 4. Calculation of singularities

-  M. COSTABEL, M. DAUGE (Math. Nachr. '93)  
*Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems.*

We use appropriate complex variables  $z_{\pm}$  instead of the polar coordinates :

$$z_{-} = z \text{ (when } z \in \mathcal{S}_{-}) \text{ and } z_{+} = -z \text{ (when } z \in \mathcal{S}_{+})$$

- We use an Ansatz involving only integer powers of

$$z_{\pm}, \bar{z}_{\pm}, \log z_{\pm}, \text{ and } \log \bar{z}_{\pm}.$$

## The first shadow of primal singularities

Example : expression of  $\mathfrak{s}_1^{0,0}$

$$\mathfrak{s}_1^{0,0}(r, \theta) = \frac{\sin \omega}{\pi} r^2 \left( \log r \cos 2\theta - \theta \sin 2\theta \right) + r^2 \left( 1 - \frac{\cos \omega \cos 2\theta}{2} \right) \quad \text{in } \mathcal{S}_-$$

$$\mathfrak{s}_1^{0,0}(r, \theta) = \frac{\sin \omega}{\pi} r^2 \left( \log r \cos 2\theta - \theta_+ \sin 2\theta \right) + r^2 \frac{\cos \omega \cos 2\theta}{2} \quad \text{in } \mathcal{S}_+$$

with  $\theta_+ = \theta - \pi \operatorname{sgn} \theta$



## The first shadow of primal singularities

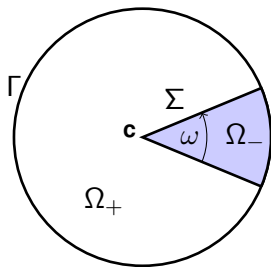
Example : Expression of  $\mathfrak{s}_1^{1,0}$

$$\begin{aligned}\mathfrak{s}_1^{1,0}(r, \theta) &= \frac{2 \sin \omega + \sin 2\omega}{6\pi} r^3 \left( \log r \cos 3\theta - \theta \sin 3\theta \right) \\ &+ r^3 \left( \frac{\cos \theta}{2} - \frac{\cos \omega \cos 3\theta}{3} \right), \quad \text{in } \mathcal{S}_-\end{aligned}$$

$$\begin{aligned}\mathfrak{s}_1^{1,0}(r, \theta) &= \frac{2 \sin \omega + \sin 2\omega}{6\pi} r^3 \left( \log r \cos 3\theta - \theta_+ \sin 3\theta \right) \\ &+ r^3 \frac{\cos 2\omega \cos 3\theta}{6}, \quad \text{in } \mathcal{S}_+\end{aligned}$$

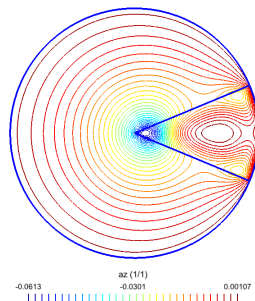
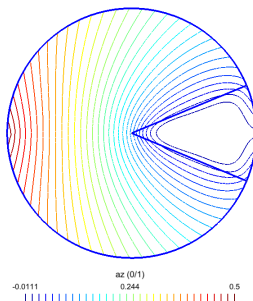
## Framework

- $\Omega$  : disk of radius 50 mm
- Conducting sector :  $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2}) \text{ mm}^{-1}$
- Source :  $\mathcal{A}^+ = \frac{|\theta|}{2\pi}$  on  $\partial\Omega$



# Finite Element Solution

We plot the real part and the imaginary part of the FE solution.

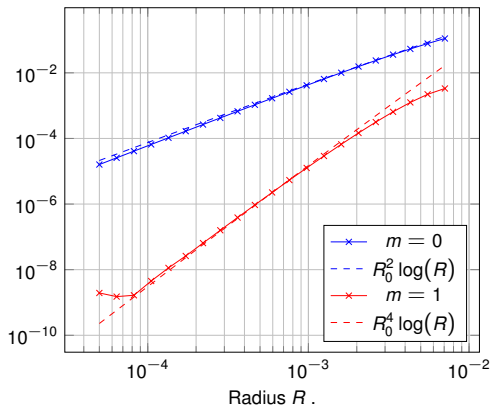


# Accuracy for the computation of $\Lambda^{0,0}$ as a function of $R$

- There holds

$$\Lambda^{0,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R).$$

- We plot  $|\mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) - \mathcal{A}|_c|/|\mathcal{A}|_c|$  as a function of  $R$  ( $m = 0, 1$ ).

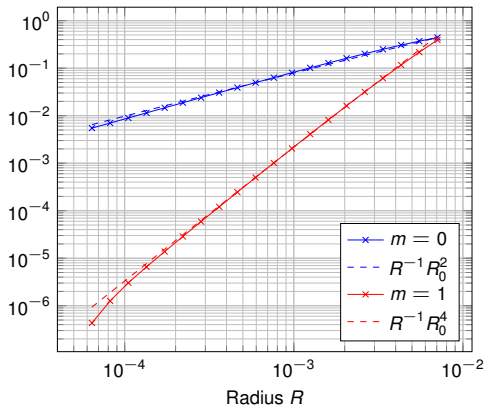


# Accuracy for the computation of $\Lambda^{1,0}$ as a function of $R$

- Recall that

$$\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2}).$$

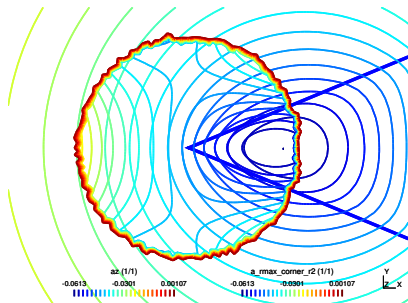
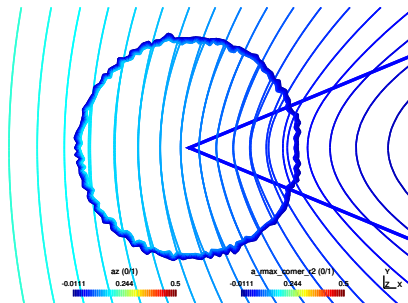
- We plot  $|\mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_S}(\mathcal{R}_1^{1,0}, \mathcal{A})| / |\mathcal{J}_{R_S}(\mathcal{R}_1^{1,0}, \mathcal{A})|$ .



# Comparison of the FE solution and of the local expansion

In the expansion, we collect the terms which behave as constant,  $r$ ,  $r^2$  and  $r^2 \log r$ .

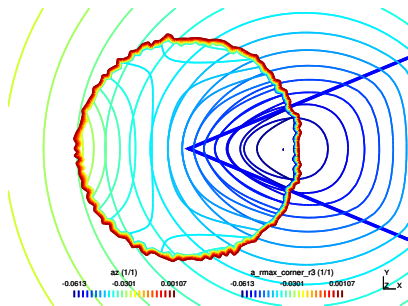
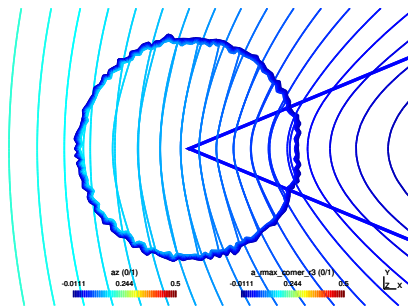
$$\begin{aligned} \mathcal{A} \underset{r \rightarrow 0}{\sim} & \mathcal{J}_{R_s}(\mathcal{K}_1^{0,0}, \mathcal{A})(1 + i\zeta^2 \mathfrak{s}_1^{0,0}) + \mathcal{J}_{R_s}(\mathcal{K}_1^{1,0}, \mathcal{A}) \mathfrak{s}_1^{1,0} \\ & + (\mathcal{J}_{R_s}(\mathcal{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \mathcal{J}_{R_s}(\mathcal{K}_1^{0,0}, \mathcal{A})) \mathfrak{s}_1^{2,0} \end{aligned}$$



# Comparison of the FE solution and of the local expansion

Then we add the terms which behave as  $r^3$  and  $r^3 \log r$ .

$$\begin{aligned} \mathcal{A} \underset{r \rightarrow 0}{\sim} & \mathcal{J}_{R_S}(\mathfrak{K}_1^{0,0}, \mathcal{A})(1 + i\zeta^2 \mathfrak{s}_1^{0,0}) + \mathcal{J}_{R_S}(\mathfrak{K}_1^{1,0}, \mathcal{A})(\mathfrak{s}_1^{1,0} + i\zeta^2 \mathfrak{s}_1^{1,0}) + \\ & (\mathcal{J}_{R_S}(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \mathcal{J}_{R_S}(\mathfrak{K}_1^{0,0}, \mathcal{A})) \mathfrak{s}_1^{2,0} + \\ & (\mathcal{J}_{R_S}(\mathfrak{K}_1^{3,0}, \mathcal{A}) - \mathcal{J}^{3,0;1,0} \mathcal{J}_{R_S}(\mathfrak{K}_1^{1,0}, \mathcal{A})) \mathfrak{s}_1^{3,0} \end{aligned}$$



Thank you for your attention