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Discontinuous Galerkin methods for solving Helmholtz isotropic wave equations for seismic applications

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September 2, 2013



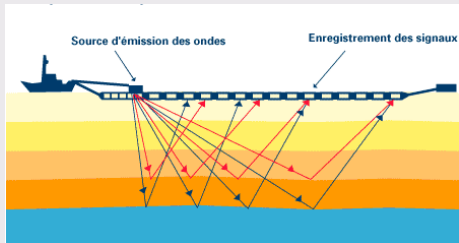
Motivation

Examples of the seismic applications



Motivation

Examples of the seismic applications



Motivation

Imaging method : the full wave inversion

- Quantitative **high resolution** images of the subsurface physical parameters

Forward problem of the inversion process

- Elastic waves propagation in harmonic domain : **Helmholtz equation**

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Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities

DG method

- Use of triangular unstructured meshes
- Flexible choice of interpolation orders (p – *adaptativity*)

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Drawback of DG method

- Important computational cost

Main objective of the thesis

- Development of an hybridizable DG (HDG) method
- Development of a reference method, a classical DG method

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- 1 2D Helmholtz isotropic elastic equations
- 2 DG formulation of the equations
 - Centered flux DG scheme
 - Upwind flux DG scheme
- 3 Numerical results
 - Plane wave in an homogeneous medium
 - Circular diffraction
 - Results for various frequencies
 - Results with the p – *adaptativity*
- 4 Conclusion-Perspectives

Contents

- 1 2D Helmholtz isotropic elastic equations
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2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2,$$

$$\begin{cases} i\omega \rho(\mathbf{x}) \mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega \underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- Free surface condition : $\underline{\underline{\sigma}} \mathbf{n} = 0$ on Γ_f
- Absorbing boundary condition : $\underline{\underline{\sigma}} \mathbf{n} = v_p(\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + v_s(\mathbf{v} \cdot \mathbf{t}) \mathbf{t}$ on Γ_a

- \mathbf{v} : velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{\mathbf{C}}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- Free surface condition : $\underline{\underline{\sigma}}\mathbf{n} = 0$ on Γ_f
- Absorbing boundary condition : $\underline{\underline{\sigma}}\mathbf{n} = v_p(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + v_s(\mathbf{v} \cdot \mathbf{t})\mathbf{t}$ on Γ_a
- ρ : mass density
- $\underline{\underline{\mathbf{C}}}$: tensor of elasticity coefficients
- \mathbf{f}_s : source term, $\mathbf{f}_s \in L^2(\Omega)$
- v_p : P-wave velocity
- v_s : S-wave velocity

2D Helmholtz isotropic elastic equations

First order formulation of Helmholtz isotropic wave equations

$$\left\{ \begin{array}{l} i\omega v_x = \frac{1}{\rho} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \right) \\ i\omega v_z = \frac{1}{\rho} \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \right) \\ i\omega \sigma_{xx} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \\ i\omega \sigma_{zz} = \lambda \frac{\partial v_x}{\partial x} + (\lambda + 2\mu) \frac{\partial v_z}{\partial z} \\ i\omega \sigma_{xz} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \end{array} \right.$$

λ and μ Lamé's constants and $v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and $v_s = \sqrt{\frac{\mu}{\rho}}$

2D Helmholtz isotropic elastic equations

Vectorial form

$$i\omega \mathbf{Q} + \mathbf{A}_x \frac{\partial \mathbf{Q}}{\partial x} + \mathbf{A}_z \frac{\partial \mathbf{Q}}{\partial z} = 0$$

where $\mathbf{Q} = (v_x, v_z, \sigma_{xx}, \sigma_{zz}, \sigma_{xz})^T$ and :

$$\mathbf{A}_x = - \begin{pmatrix} 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho} \\ \lambda + 2\mu & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_z = - \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & \lambda + 2\mu & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \end{pmatrix}$$

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DG methods in time domain for seismic applications

DG methods in time domain for seismic applications

- M. Dumbser and M. Käser, *An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes - II; The three-dimensional isotropic case*, 2006 (**upwind scheme**)
- S. Delcourte, L.Fezoui and N. Glinsky-Olivier, *A high order discontinuous Galerkin method for the seismic wave propagation*, 2009 (**centered scheme**)

Notations and definitions

Notations

- Γ_l free surface boundary
- Γ_a the absorbing boundary
- \mathcal{T}_h mesh of Ω composed of triangles K
- \mathcal{F}_h set of all faces F of \mathcal{T}_h
- \mathbf{n} the normal outward vector of an element K

Notations and definitions

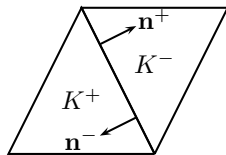
Definitions

- Jump $[[\cdot]]$ of a vector \mathbf{u} for F :

$$[[\mathbf{u}]] = \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^- = \mathbf{u}^+ \cdot \mathbf{n}^+ - \mathbf{u}^- \cdot \mathbf{n}^+$$

- Jump of a tensor $\underline{\underline{\sigma}}$ for F :

$$[[\underline{\underline{\sigma}}]] = \underline{\underline{\sigma}}^+ \mathbf{n}^+ + \underline{\underline{\sigma}}^- \mathbf{n}^- = \underline{\underline{\sigma}}^+ \mathbf{n}^+ - \underline{\underline{\sigma}}^- \mathbf{n}^+$$

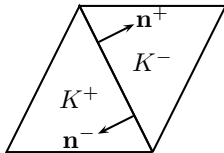


Notations and definitions

Definitions

- Average $\{\cdot\}$ of a variable u , for F :

$$\{u\} = \frac{u^+ + u^-}{2}$$



DG formulation of the original equation

Local DG formulation

$$i\omega Q + \mathbf{A}_x \frac{\partial Q}{\partial x} + \mathbf{A}_z \frac{\partial Q}{\partial z} = 0$$

DG formulation of the original equation

Local DG formulation

$$i\omega \mathbf{Q}\varphi + \mathbf{A}_x \frac{\partial \mathbf{Q}}{\partial x} \varphi + \mathbf{A}_z \frac{\partial \mathbf{Q}}{\partial z} \varphi = 0$$

DG formulation of the original equation

Local DG formulation

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x^K \frac{\partial \mathbf{Q}^K}{\partial x} \varphi - \int_K \mathbf{A}_z^K \frac{\partial \mathbf{Q}^K}{\partial z} \varphi = 0$$

DG formulation of the original equation

Local DG formulation

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A}_z^K \frac{\partial \varphi}{\partial z} + \sum_{F \in \mathcal{F}(K)} \int_F \mathbf{D}_n \mathbf{Q} \varphi = 0$$

$$\mathbf{D}_n = n_x \mathbf{A}_x + n_z \mathbf{A}_z = - \begin{pmatrix} 0 & 0 & \frac{n_x}{\rho} & 0 & \frac{n_z}{\rho} \\ 0 & 0 & 0 & \frac{n_z}{\rho} & \frac{\rho}{n_x} \\ n_x(\lambda + 2\mu) & n_z \lambda & 0 & 0 & 0 \\ n_x \lambda & n_z(\lambda + 2\mu) & 0 & 0 & 0 \\ n_z \mu & n_x \mu & 0 & 0 & 0 \end{pmatrix}$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K \in \mathcal{T}_h} \int_K i\omega \mathbf{Q} \varphi - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_x \mathbf{Q} \frac{\partial \varphi}{\partial x} - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_z \mathbf{Q} \frac{\partial \varphi}{\partial z} + \sum_{F \in \mathcal{F}_h} \int_F \llbracket \mathbf{D}_n \mathbf{Q} \varphi \rrbracket = 0$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K \in \mathcal{T}_h} \int_K i\omega \mathbf{Q} \varphi - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_x \mathbf{Q} \frac{\partial \varphi}{\partial x} - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_z \mathbf{Q} \frac{\partial \varphi}{\partial z} + \sum_{F \in \mathcal{F}_h} \int_F \llbracket \mathbf{D}_n \mathbf{Q} \varphi \rrbracket = 0$$

$$\llbracket \mathbf{D}_n \mathbf{Q} \varphi \rrbracket \simeq (\mathbf{D}_n \mathbf{Q}) \llbracket \varphi \rrbracket$$

DG formulation of the original equation

Global DG formulation

$$\sum_{K \in \mathcal{T}_h} \int_K i\omega \mathbf{Q} \varphi - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_x \mathbf{Q} \frac{\partial \varphi}{\partial x} - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{A}_z \mathbf{Q} \frac{\partial \varphi}{\partial z} + \sum_{F \in \mathcal{F}_h} \int_F (\mathbf{D}_n \mathbf{Q}) [[\varphi]] = 0$$

Centered flux DG scheme

Centered flux on a face F

$$(\mathbf{D}_n \mathbf{Q})|_F = \{\mathbf{D}_n \mathbf{Q}\} = \frac{1}{2} \left(\mathbf{D}_n^K \mathbf{Q}^K + \mathbf{D}_n^{K'} \mathbf{Q}^{K'} \right)$$

Centered flux DG scheme

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A}_z^K \mathbf{Q}^K \frac{\partial \varphi}{\partial z} + \sum_F \int_F \frac{1}{2} \left(\mathbf{D}_n^K \mathbf{Q}^K + \mathbf{D}_n^{K'} \mathbf{Q}^{K'} \right) \varphi = 0$$

Upwind flux DG scheme

Definition

$$\begin{cases} \mathbf{D}_n^+ &= \mathbf{R}_n \mathbf{\Gamma}^+ (\mathbf{R}_n)^{-1} \\ \mathbf{D}_n^- &= \mathbf{R}_n \mathbf{\Gamma}^- (\mathbf{R}_n)^{-1} \end{cases}$$

where

$$\mathbf{\Gamma}^- = - \begin{pmatrix} v_p & 0 & 0 & 0 & 0 \\ 0 & v_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{\Gamma}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_s & 0 \\ 0 & 0 & 0 & 0 & v_p \end{pmatrix}$$

Upwind flux DG scheme

Definition

$$\begin{cases} \mathbf{D}_n^+ &= \mathbf{R}_n \mathbf{\Gamma}^+ (\mathbf{R}_n)^{-1} \\ \mathbf{D}_n^- &= \mathbf{R}_n \mathbf{\Gamma}^- (\mathbf{R}_n)^{-1} \end{cases}$$

and

$$\mathbf{R}_n = \begin{pmatrix} n_x v_p & -n_z v_s & 0 & n_z v_s & -n_x v_p \\ n_z v_p & n_x v_s & 0 & -n_x v_s & -n_z v_p \\ \lambda + 2n_x^2 \mu & -2n_z n_x \mu & n_z^2 & -2n_z n_x \mu & \lambda + 2n_x^2 \mu \\ \lambda + 2n_z^2 \mu & 2n_z n_x \mu & n_x^2 & 2n_z n_x \mu & \lambda + 2n_z^2 \mu \\ 2n_z n_x \mu & \mu(n_x^2 - n_z^2) & -n_x n_z & \mu(n_x^2 - n_z^2) & 2n_z n_x \mu \end{pmatrix}$$

Upwind flux DG scheme

Upwind flux on a face F

$$(\mathbf{D}_n \mathbf{Q})|_F = (\mathbf{D}_n^K)^+ \mathbf{Q}^K + (\mathbf{D}_n^{K'})^- \mathbf{Q}^{K'}$$

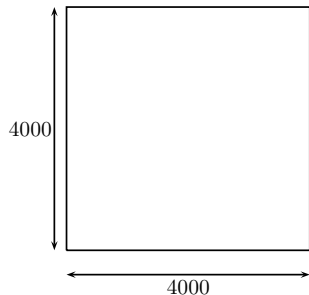
Upwind flux DG scheme

$$\int_K i\omega \mathbf{Q}^K \varphi - \int_K \mathbf{A}_x^K \mathbf{Q}^K \frac{\partial \varphi}{\partial x} - \int_K \mathbf{A}_z^K \mathbf{Q}^K \frac{\partial \varphi}{\partial z} + \sum_F \int_F \left[(\mathbf{D}_n^K)^+ \mathbf{Q}^K + (\mathbf{D}_n^{K'})^- \mathbf{Q}^{K'} \right] \varphi = 0$$

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Plane wave



Computational domain Ω
setting

- Physical parameters :
 - $\rho = 2.10^3 \text{ kg.m}^{-3}$
 - $\lambda = 1,6.10^{10} \text{ Pa}$
 - $\mu = 8.10^9 \text{ Pa}$
 - $v_p = 4.10^3 \text{ m.s}^{-1}$
 - $v_s = 2.10^3 \text{ m.s}^{-1}$
- Boundary :
 - boundary conditions on $\partial\Omega$
such as :

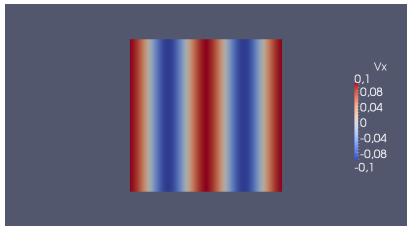
$$u = \nabla e^{i(k \cos \theta x + k \sin \theta y)}$$

$$\text{where } k = \frac{\omega}{v_p} \text{ or } k = \frac{\omega}{v_s}$$

Plane wave for a frequency $f = 2$ Hz, component V_x



Exact solution



P_2 centered flux DG formulation

Plane wave for a frequency $f = 2$ Hz, component V_x

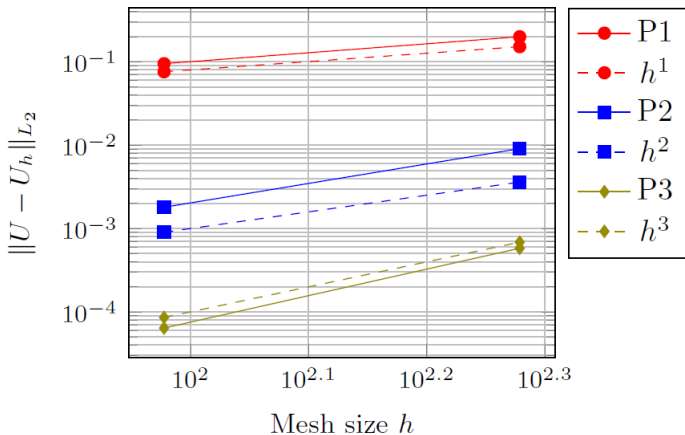


Exact solution



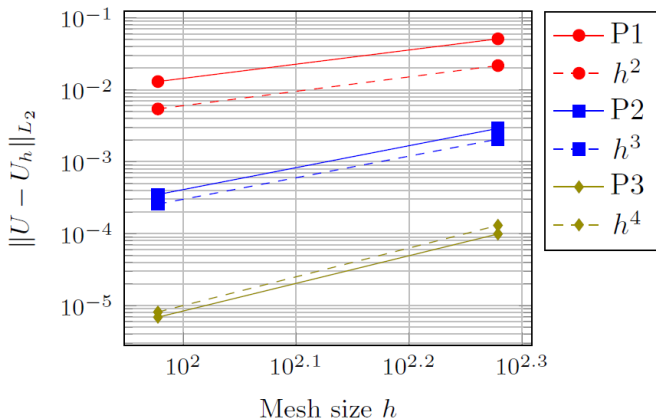
P_2 upwind flux DG formulation

Plane wave



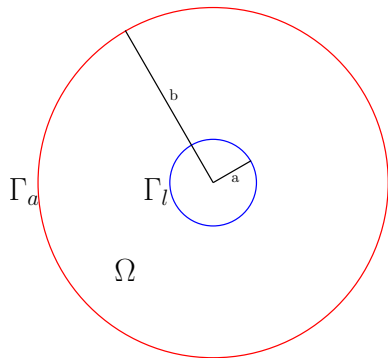
L_2 error for the centered flux

Plane wave



L_2 -error for the upwind flux

Circular diffraction

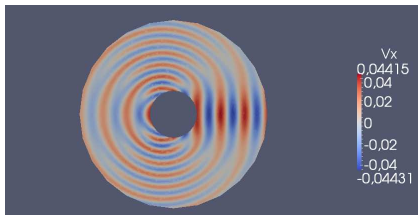


Configuration of the computational domain Ω

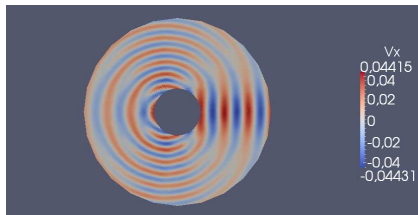
- Physical parameters :
 - $\rho = 2.10^3 \text{ kg.m}^{-3}$
 - $\lambda = 1,6.10^{10} \text{ Pa}$
 - $\mu = 8.10^9 \text{ Pa}$
 - $v_p = 4.10^3 \text{ m.s}^{-1}$
 - $v_s = 2.10^3 \text{ m.s}^{-1}$
- Boundary :
 - Γ_l is a free surface :

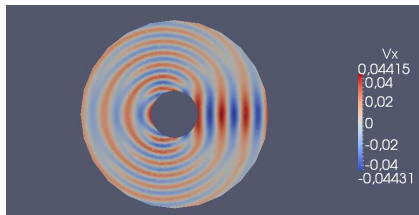
$$\underline{\underline{\sigma}} \mathbf{n} = 0$$
 - Γ_a absorbing boundary :

$$\underline{\underline{\sigma}} \mathbf{n} = v_p (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + v_s (\mathbf{v} \cdot \mathbf{t}) \mathbf{t}$$
- $a = 2000 \text{ m}$
- $b = 8000 \text{ m}$
- \mathcal{T}_h composed of 9653 elements

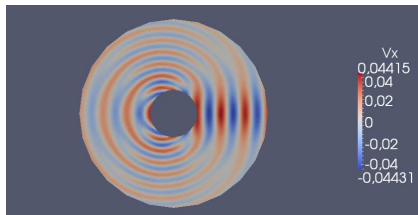
Results for various frequencies : $f = 2$ Hz

Exact solution

 P_1 centered scheme

Results for various frequencies : $f = 2$ Hz

Exact solution

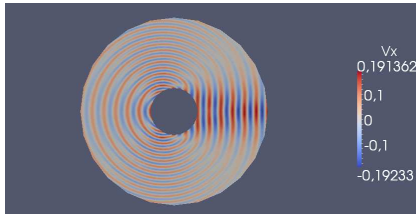
 P_1 upwind scheme

Results for various frequencies : $f = 2$ Hz

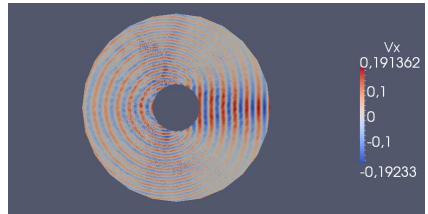
Nb dof		Centered	Upwind
144795 (P_1)	V_x L_2 -error	3.44e-01	1.49e-01
	Fact.-res. time (s)	13	16
	Memory	800	980
289590 (P_2)	V_x L_2 -error	4.41e-02	4.67e-02
	Fact.-res. time (s)	40	57
	Memory	1900	2800

Numerical statistics

Results for various frequencies : $f = 4$ Hz

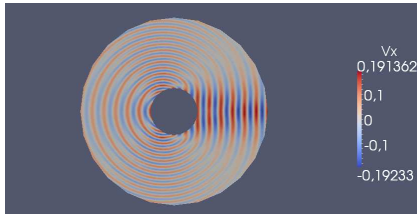


Exact solution

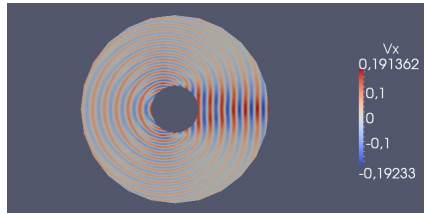


P_2 centered scheme

Results for various frequencies : $f = 4$ Hz



Exact solution



P_2 upwind scheme

Results for various frequencies : $f = 4$ Hz

Nb dof		Centered	Upwind
144795	V_x L_2 -error	1.19	0.506
	Fact.-res. time (s)	13	16
	Memory	800	970
289590	V_x L_2 -error	4.24e-01	1.66e-01
	Fact.-res. time (s)	40	57
	Memory	1900	2800

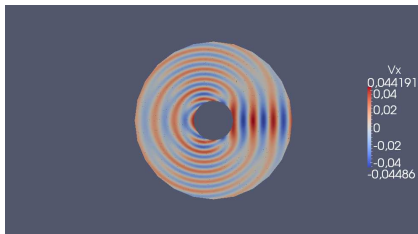
Numerical statistics

Results with p – adaptativity

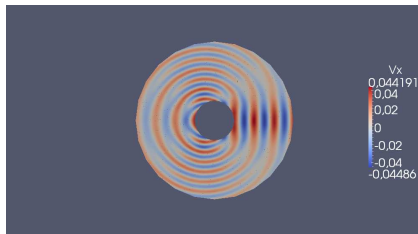
Area of the triangle	Interpolation order	Number of triangles
]0 ;10000]	0	3
]10000 ;15000]	1	1745
]15000 ;20000]	2	3999
]20000 ;25000]	3	2658
]25000 ;30000]	4	1248

Distribution of the interpolation orders

Results with p – adaptativity : $f = 2$ Hz

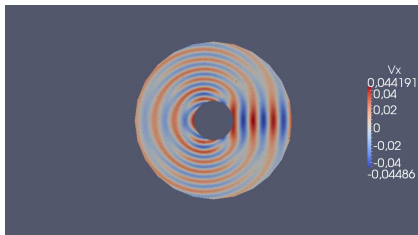


Exact solution

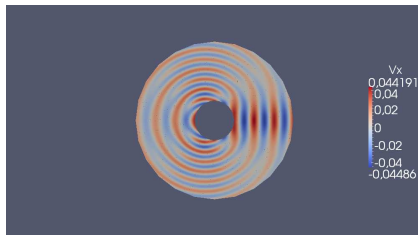


"p-local" centered scheme

Results with p – adaptativity : $f = 2$ Hz

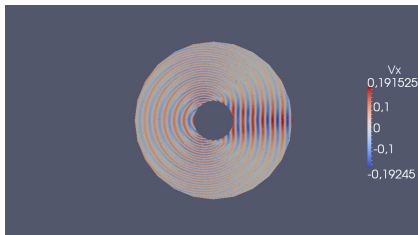


Exact solution

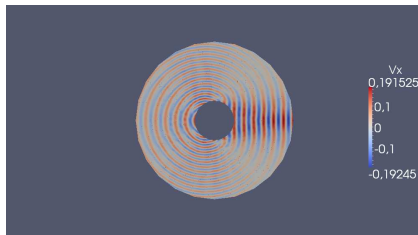


"p-local" upwind scheme

Results with p – adaptativity : $f = 4$ Hz

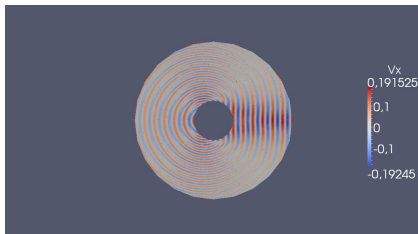


Exact solution

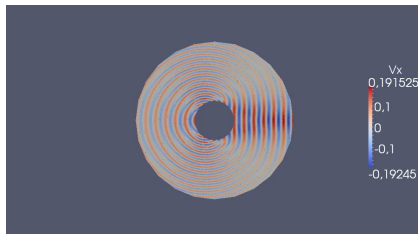


"p-local" centered scheme

Results with p – adaptativity : $f = 4$ Hz



Exact solution



“ p -local” upwind scheme

Results with p – adaptativity

	$f = 2$ Hz		$f = 4$ Hz	
	Centered	Upwind	Centered	Upwind
V_x L_2 -error	5.02e-02	6.00e-02	2.83e-01	2.76e-01
Fact.-res. time (s)	57	79	57	80
Memory	3000	3800	3000	3800

Numerical statistics for both schemes as function of the frequency

Comparison between p – adaptativity and p – global for $f = 2\text{Hz}$

	p – adaptativity		p – global	
	Centered	Upwind	Centered	Upwind
V_x L_2 -error	5.02e-02	6.00e-02	4.41e-02	4.41e-02
Fact.-res. time (s)	57	79	40	57
Memory	3000	3800	1900	2800

Comparison between p – adaptativity and p – global for $f = 4\text{Hz}$

	p – adaptativity		p – global	
	Centered	Upwind	Centered	Upwind
V_x L_2 -error	2.83e-01	2.75e-01	4.24e-01	1.66e-01
Fact.-res. time (s)	57	80	40	57
Memory	3000	3800	1900	2800

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Conclusion-Perspectives

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- With the upwind flux DG formulation we obtain one convergence order more than the centered flux DG formulation

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- Develop 3D upwind flux DG formulation for Helmholtz equations
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- Construction of a HDG formulation
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