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# Unital versions of the higher order peak algebras

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**Abstract.** We construct unital extensions of the higher order peak algebras defined by Krob and the third author in [Ann. Comb. 9 (2005), 411–430], and show that they can be obtained as homomorphic images of certain subalgebras of the Mantaci-Reutenauer algebras of type  $B$ . This generalizes a result of Bergeron, Nyman and the first author [Trans. AMS 356 (2004), 2781–2824].

**Résumé.** Nous construisons des extensions unitaires des algèbres de pics d'ordre supérieur définies par Krob et le troisième auteur dans [Ann. Comb. 9 (2005), 411–430], et nous montrons qu'elles peuvent être obtenues comme images homomorphes de certaines sous-algèbres des algèbres de Mantaci-Reutenauer de type  $B$ . Ceci généralise un résultat dû à Bergeron, Nyman et au premier auteur [Trans. AMS 356 (2004), 2781–2824].

**Keywords:** Descent algebras, Noncommutative symmetric functions, Peak algebras

## 1 Introduction

A *descent* of a permutation  $\sigma \in \mathfrak{S}_n$  is an index  $i$  such that  $\sigma(i) > \sigma(i+1)$ . A descent is a *peak* if moreover  $i > 1$  and  $\sigma(i) > \sigma(i-1)$ . The sums of permutations with a given descent set span a subalgebra of the group algebra, the *descent algebra*  $\Sigma_n$ . The *peak algebra*  $\mathring{P}_n$  of  $\mathfrak{S}_n$  is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit. Descent algebras can be defined for all finite Coxeter groups [19]. In [2], it is shown that the peak algebra of  $\mathfrak{S}_n$  can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group  $B_n$ .

The direct sum of the peak algebras turns out to be a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with **Sym**, the Hopf algebra of noncommutative symmetric functions [9]. As explained in [5], it turns out that a fair amount of results on the peak algebras can be deduced from the case  $q = -1$  of a  $q$ -identity of [11]. Specializing  $q$  to other roots of unity, Krob and the third author introduced and studied *higher order peak algebras* in [12]. Again, these are non-unital, and it is natural to ask whether the construction of [2] can be extended to this case.

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We will show that this is indeed possible. We first construct the unital versions of the higher order peak algebras by a simple manipulation of generating series. We then show that they can be obtained as homomorphic images of the *Mantaci-Reutenauer algebras* of type  $B$ . Hence no Coxeter groups other than  $B_n$  and  $\mathfrak{S}_n$  are involved in the process; in fact, the construction is related to the notion of superization, as defined in [16], rather than to root systems or wreath products.

## 2 Notations and background

### 2.1 Noncommutative symmetric functions

We will assume familiarity with the notations of [9] and with the main results of [12]. We recall a few definitions for the convenience of the reader.

The Hopf algebra of noncommutative symmetric functions is denoted by  $\mathbf{Sym}$ , or by  $\mathbf{Sym}(A)$  if we consider the realization in terms of an auxiliary alphabet  $A$ . Linear bases of  $\mathbf{Sym}_n$  are labelled by compositions  $I = (i_1, \dots, i_r)$  of  $n$  (we write  $I \models n$ ). The noncommutative complete and elementary functions are denoted by  $S_n$  and  $\Lambda_n$ , and  $S^I = S_{i_1} \cdots S_{i_r}$ . The ribbon basis is denoted by  $R_I$ . The *descent set* of  $I$  is  $\text{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}$ . The *descent composition* of a permutation  $\sigma \in \mathfrak{S}_n$  is the composition  $I = D(\sigma)$  of  $n$  whose descent set is the descent set of  $\sigma$ .

Recall from [8] that for an infinite totally ordered alphabet  $A$ ,  $\mathbf{FQSym}(A)$  is the subalgebra of  $\mathbb{C}\langle A \rangle$  spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\text{std}(w)=\sigma} w, \quad (1)$$

that is, the sum of all words in  $A^n$  whose standardization is the permutation  $\sigma \in \mathfrak{S}_n$ . The noncommutative ribbon Schur function  $R_I \in \mathbf{Sym}$  is then

$$R_I = \sum_{D(\sigma)=I} \mathbf{G}_\sigma. \quad (2)$$

This defines a Hopf embedding  $\mathbf{Sym} \rightarrow \mathbf{FQSym}$ . The Hopf algebra  $\mathbf{FQSym}$  is self-dual under the pairing  $(\mathbf{G}_\sigma, \mathbf{G}_\tau) = \delta_{\sigma, \tau^{-1}}$  (Kronecker symbol). Let  $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$ , so that  $\{\mathbf{F}_\sigma\}$  is the dual basis of  $\{\mathbf{G}_\sigma\}$ . The *internal product*  $*$  of  $\mathbf{FQSym}$  is induced by composition  $\circ$  in  $\mathfrak{S}_n$  in the basis  $\mathbf{F}$ , that is,

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma \circ \tau} \quad \text{and} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau \circ \sigma}. \quad (3)$$

Each subspace  $\mathbf{Sym}_n$  is stable under this operation, and anti-isomorphic to the descent algebra  $\Sigma_n$  of  $\mathfrak{S}_n$ . For  $f_i \in \mathbf{FQSym}$  and  $g \in \mathbf{Sym}$ , we have the splitting formula

$$(f_1 \dots f_r) * g = \mu_r \cdot (f_1 \otimes \dots \otimes f_r) *_r \Delta^r g, \quad (4)$$

where  $\mu_r$  is  $r$ -fold multiplication, and  $\Delta^r$  the iterated coproduct with values in the  $r$ -th tensor power.

### 2.2 The Mantaci-Reutenauer algebra of level 2

We denote by  $\mathbf{MR}$  the free product  $\mathbf{Sym} \star \mathbf{Sym}$  of two copies of the Hopf algebra of noncommutative symmetric functions [14]. That is,  $\mathbf{MR}$  is the free associative algebra on two sequences  $(S_n)$  and  $(S_{\bar{n}})$  ( $n \geq 1$ ). We regard the two copies of  $\mathbf{Sym}$  as noncommutative symmetric functions on two auxiliary

alphabets:  $S_n = S_n(A)$  and  $S_{\bar{n}} = S_n(\bar{A})$ . We denote by  $F \mapsto \bar{F}$  the involutive automorphism which exchanges  $S_n$  and  $S_{\bar{n}}$ . The bialgebra structure is defined by the requirement that the series

$$\sigma_1 = \sum_{n \geq 0} S_n \text{ and } \bar{\sigma}_1 = \sum_{n \geq 0} S_{\bar{n}} \quad (5)$$

are grouplike. The internal product of **MR** can be computed from the splitting formula and the conditions that  $\sigma_1$  is neutral,  $\bar{\sigma}_1$  is central, and  $\bar{\sigma}_1 * \sigma_1 = \sigma_1$ .

In [15], an embedding of **MR** in the Hopf algebra **BFQSym** of free quasi-symmetric functions of type  $B$  (spanned by colored permutations) is described. Under this embedding, left  $*$ -multiplication by  $\Lambda_n = \mathbf{G}_{n \ n-1 \dots 2, 1}$  corresponds to right multiplication by  $n \ n-1 \dots 2, 1$  in the group algebra of  $B_n$ . This implies that left  $*$ -multiplication by  $\lambda_1$  is an involutive anti-automorphism of **BFQSym**, hence of **MR**.

### 2.3 Noncommutative symmetric functions of type $B$

The hyperoctahedral analogue **BSym** of **Sym**, defined in [6], is the right **Sym**-module freely generated by another sequence  $(\tilde{S}_n)$  ( $n \geq 0$ ,  $\tilde{S}_0 = 1$ ) of homogeneous elements, with  $\tilde{\sigma}_1$  grouplike. This is a coalgebra, but not an algebra. It is endowed with an internal product, for which each homogeneous component **BSym** $_n$  is anti-isomorphic to the descent algebra of  $B_n$ .

## 3 Solomon descent algebras of type $B$

### 3.1 Descents in $B_n$

The hyperoctahedral group  $B_n$  is the group of signed permutations. A signed permutation can be denoted by  $w = (\sigma, \epsilon)$  where  $\sigma$  is an ordinary permutation and  $\epsilon \in \{\pm 1\}^n$ , such that  $w(i) = \epsilon_i \sigma(i)$ . If we set  $w(0) = 0$ , then,  $i \in [0, n-1]$  is a descent of  $w$  if  $w(i) > w(i+1)$ . Hence, the descent set of  $w$  is a subset  $D = \{i_0, i_0 + i_1, \dots, i_0 + i_1 + \dots + i_{r-1}\}$  of  $[0, n-1]$ . We then associate to  $D$  a so-called type- $B$  composition (a composition whose first part can be zero)  $(i_0 - 0, i_1, \dots, i_{r-1}, n - i_{r-1})$ . The sum of all signed permutations whose descent set is contained in  $D$  is mapped to  $\tilde{S}^I := \tilde{S}_{i_0} S^{I'}$  by Chow's anti-isomorphism [6], where  $I' = (i_1, \dots, i_r)$ .

### 3.2 Noncommutative supersymmetric functions

An embedding of **BSym** as a sub-coalgebra and sub-**Sym**-module of **MR** can be deduced from [14]. To describe it, let us define, for  $F \in \mathbf{Sym}(A)$ ,

$$F^\sharp = F(A|\bar{A}) = F(A - q\bar{A})|_{q=-1} \quad (6)$$

(the supersymmetric version of  $F$ ). The superization of  $F \in \mathbf{Sym}(A)$  can also be given by

$$F^\sharp = F * \sigma_1^\sharp. \quad (7)$$

Indeed,  $\sigma_1^\sharp$  is grouplike, and for  $F = S^I$ , the splitting formula gives

$$(S_{i_1} \cdots S_{i_r}) * \sigma_1^\sharp = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\sharp \otimes \cdots \otimes \sigma_1^\sharp)] = S^{I^\sharp}. \quad (8)$$

We have

$$\sigma_1^\sharp = \bar{\lambda}_1 \sigma_1 = \sum \Lambda_i S_j. \quad (9)$$

The element  $\bar{\sigma}_1$  is central for the internal product, and

$$\bar{\sigma}_1 * F = \bar{F} = F * \bar{\sigma}_1. \quad (10)$$

Hence,

$$\bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1 \bar{\sigma}_1 =: \sigma_1^\flat. \quad (11)$$

The basis element  $\tilde{S}^I$  of **BSym**, where  $I = (i_0, i_1, \dots, i_r)$  is a type  $B$ -composition, can be embedded as

$$\tilde{S}^I = S_{i_0}(A) S^{i_1 i_2 \dots i_r}(A | \bar{A}). \quad (12)$$

We will identify **BSym** with its image under this embedding.

### 3.3 A proof that **BSym** is $*$ -stable

We are now in a position to understand why **BSym** is a  $*$ -subalgebra of **MR**. The argument will be extended below to the case of unital peak algebras. Let  $F, G \in \mathbf{Sym}$ . We want to understand why  $\sigma_1 F^\sharp * \sigma_1 G^\sharp$  is in **BSym**. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G_{(1)}^\sharp) (F^\sharp * \sigma_1 G_{(2)}^\sharp). \quad (13)$$

We now only have to show that each term  $F^\sharp * \sigma_1 G_{(2)}^\sharp$  is in **Sym** $^\sharp$ . We may assume that  $F = S^I$ , and for any  $G \in \mathbf{Sym}$ ,

$$S^{I^\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S_{i_1}^\sharp \otimes \dots \otimes S_{i_r}^\sharp) * (\sigma_1 G_{(1)}^\sharp \otimes \dots \otimes \sigma_1 G_{(r)}^\sharp)] \quad (14)$$

so that it is sufficient to prove the property for  $F = S_n$ . Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_1 \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_1 * \sigma_1 G_{(1)}^\sharp) (\sigma_1 G_{(2)}^\sharp) \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * \sigma_1 G_{(1)}^\sharp) \cdot \sigma_1 \cdot G_{(2)}^\sharp \end{aligned} \quad (15)$$

Now,

$$\lambda_1 * \sigma_1 G_{(1)}^\sharp = (\lambda_1 * G_{(1)}^\sharp) (\lambda_1 * \sigma_1) = (\lambda_1 * G_{(1)}^\sharp) \lambda_1, \quad (16)$$

since  $\lambda_1$  is an anti-automorphism. We then get

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= \sum_{(G)} (\bar{\sigma}_1 * ((\lambda_1 * G_{(1)}^\sharp) \lambda_1) \cdot \sigma_1 \cdot G_{(2)}^\sharp) \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * G_{(1)}^\sharp) \cdot (\bar{\sigma}_1 * \lambda_1) \sigma_1 \cdot G_{(2)}^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_1 * G_{(1)}^\sharp) \cdot \sigma_1^\sharp \cdot G_{(2)}^\sharp \end{aligned} \quad (17)$$

Now, the result will follow if we can prove that  $\bar{\lambda}_1 * G^\sharp$  is in  $\mathbf{Sym}^\sharp$  for any  $G \in \mathbf{Sym}$ .

For  $G = S^I$ ,

$$\bar{\lambda}_1 * S^{I\sharp} = \lambda_1 * \bar{\sigma}_1 * S^I * \sigma_1^\sharp = \lambda_1 * S^I * \bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1 * S^I * \sigma_1^\flat. \quad (18)$$

Since left  $*$ -multiplication by  $\lambda_1$  in an anti-automorphism, we only need to prove that  $\lambda_1 * S_n^\flat$  is of the form  $G^\sharp$ . And indeed,

$$\begin{aligned} \lambda_1 * S_n^\flat &= \sum_{i+j=n} \lambda_1 * (\Lambda_i S_{\bar{j}}) \\ &= \sum_{i+j=n} (\lambda_1 * S_{\bar{j}})(\lambda_1 * \Lambda_i) \\ &= \sum_{i+j=n} \Lambda_{\bar{j}} S_i = S_n^\sharp. \end{aligned} \quad (19)$$

This concludes the proof that  $\mathbf{BSym}$  is a  $*$ -subalgebra of  $\mathbf{BFQSym}$ .

## 4 Unital versions of the higher order peak algebras

As shown in [5], much of the theory of the peak algebra can be deduced from a formula of [11] for  $R_I((1-q)A)$ , in the special case  $q = -1$ . In [12], this formula was studied in the case where  $q$  is an arbitrary root of unity, and higher order analogs of the peak algebra were obtained. In [2], it was shown that the classical peak algebra can be extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of type  $B$ . In this section, we construct unital extensions of the higher order peak algebras.

Let  $q$  be a primitive  $r$ -th root of unity. All objects introduced below will depend on  $q$  (and  $r$ ), although this dependence will not be made explicit in the notation. We denote by  $\theta_q$  the endomorphism of  $\mathbf{Sym}$  defined by

$$\tilde{f} = \theta_q(f) = f((1-q)A) = f(A) * \sigma_1((1-q)A). \quad (20)$$

We denote by  $\mathring{\mathcal{P}}$  the image of  $\theta_q$  and by  $\mathcal{P}$  the right  $\mathring{\mathcal{P}}$ -module generated by the  $S_n$  for  $n \geq 0$ . Note that  $\mathring{\mathcal{P}}$  is by definition a left  $*$ -ideal of  $\mathbf{Sym}$ .

**Theorem 4.1**  $\mathcal{P}$  is a unital  $*$ -subalgebra of  $\mathbf{Sym}$ . Its Hilbert series is

$$\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \dots - t^r}. \quad (21)$$

*Proof* – Since the internal product of homogeneous elements of different degrees is zero, it is enough to show that, for any  $f, g \in \mathbf{Sym}$ ,  $\sigma_1 \tilde{f} * \sigma_1 \tilde{g}$  is in  $\mathcal{P}$ . Thanks to the splitting formula,

$$\begin{aligned} \sigma_1 \tilde{f} * \sigma_1 \tilde{g} &= \mu[(\sigma_1 \otimes \tilde{f}) * \sum_{(g)} \sigma_1 \tilde{g}_{(1)} \otimes \sigma_1 \tilde{g}_{(2)}] \\ &= \sum_{(g)} (\sigma_1 \tilde{g}_{(1)}) (\tilde{f} * \sigma_1 \tilde{g}_{(2)}). \end{aligned} \quad (22)$$

Thus, it is enough to check that  $\tilde{f} * \sigma_1 \tilde{h}$  is in  $\mathring{\mathcal{P}}$  for any  $f, h \in \mathbf{Sym}$ . Now,

$$\tilde{f} * \sigma_1 \tilde{h} = f * \sigma_1((1 - q)A) * \sigma_1 \tilde{h}, \quad (23)$$

and since  $\mathring{\mathcal{P}}$  is a  $\mathbf{Sym}$  left  $*$ -ideal, we only have to show that  $\sigma_1((1 - q)A) * \sigma_1 \tilde{h}$  is in  $\mathring{\mathcal{P}}$ . One more splitting yields

$$\begin{aligned} \sigma_1((1 - q)A) * \sigma_1 \tilde{h} &= (\lambda_{-q} \sigma_1) * \sigma_1 \tilde{h} \\ &= \mu[(\lambda_{-q} \otimes \sigma_1) * \sum_{(h)} \sigma_1 \tilde{h}_{(1)} \otimes \sigma_1 \tilde{h}_{(2)}] \\ &= \sum_{(h)} (\lambda_{-q} * \sigma_1 \tilde{h}_{(1)}) (\sigma_1 \tilde{h}_{(2)}) \\ &= \sum_{(h)} (\lambda_{-q} * \tilde{h}_{(1)}) \lambda_{-q} \sigma_1 \tilde{h}_{(2)} \end{aligned} \quad (24)$$

(since left  $*$ -multiplication by  $\lambda_{-q}$  is an anti-automorphism, namely the composition of the antipode and  $q^{\text{degree}}$ ). The first parentheses  $(\lambda_{-q} * \tilde{h}_{(1)})$  are in  $\mathring{\mathcal{P}}$  since it is a left  $*$ -ideal. The middle term is  $\sigma_1((1 - q)A)$ , and the last one is in  $\mathring{\mathcal{P}}$  by definition.

Recall from [12, Prop. 3.5] that the Hilbert series of  $\mathring{\mathcal{P}}$  is

$$\sum_{n \geq 0} \dim \mathring{\mathcal{P}}_n t^n = \frac{1 - t^r}{1 - t - t^2 - \dots - t^r}. \quad (25)$$

From [12, Lemma 3.13 and Eq. (3.9)], it follows that  $S_n \notin \mathring{\mathcal{P}}$  if and only if  $n \equiv 0 \pmod{r}$ , so that the Hilbert series of  $\mathcal{P}$  is

$$\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \dots - t^r}. \quad (26)$$

■

## 5 Back to the Mantaci-Reutenauer algebra

The above proofs are in fact special cases of a master calculation in the Mantaci-Reutenauer algebra, which we carry out in this section.

Let  $q$  be an arbitrary complex number or an indeterminate, and define, for any  $F \in \mathbf{MR}$ ,

$$F^\sharp = F * \sigma_1(A - q\bar{A}) = F * \sigma_1^\sharp. \quad (27)$$

Since  $\sigma_1^\sharp$  is grouplike, it follows from the splitting formula that

$$F \mapsto F^\sharp \quad (28)$$

is an automorphism of  $\mathbf{MR}$  for the Hopf structure. In addition, it is clear from the definition that it is also an endomorphism of left  $*$ -modules. We refer to it as the  $\sharp$  transform.

We now define

$$\mathring{\mathcal{Q}} = \mathbf{MR}^\sharp, \quad (29)$$

the image of the  $\sharp$  transform. Since the latter is an endomorphism of Hopf algebras and of left  $*$ -modules,  $\mathring{\mathcal{Q}}$  is both a Hopf subalgebra of  $\mathbf{MR}$  and a left  $*$ -ideal. When  $q$  is a root of unity, its image under the specialization  $\bar{A} = A$  is the non-unital peak algebra  $\mathring{\mathcal{P}}$  of Section 4 (and for generic  $q$ , it is  $\mathbf{Sym}$ ).

Let  $\mathcal{Q}$  be the right  $\mathring{\mathcal{Q}}$ -module generated by the  $S_n$ , for all  $n \geq 0$ . Clearly, the identification  $\bar{A} = A$  maps  $\mathcal{Q}$  onto  $\mathcal{P}$ , the unital peak algebra of Section 4.

**Theorem 5.1**  *$\mathcal{Q}$  is a  $*$ -subalgebra of  $\mathbf{MR}$ , containing  $\mathring{\mathcal{Q}}$  as a left ideal.*

*Proof* – Let  $F, G \in \mathbf{MR}$ . As above, we want to show that  $\sigma_1 F^\sharp * \sigma_1 G^\sharp$  is in  $\mathcal{Q}$ . Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta \sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G_{(1)}^\sharp) (F^\sharp * \sigma_1 G_{(2)}^\sharp) \quad (30)$$

and we only have to show that each term  $F^\sharp * \sigma_1 G_{(2)}^\sharp$  is in  $\mathring{\mathcal{Q}}$ . We may assume that  $F = S^I$ , where  $I$  is now a bicolored composition, and for any  $G \in \mathbf{MR}$ ,

$$S^{I^\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S_{i_1}^\sharp \otimes \cdots \otimes S_{i_r}^\sharp) * (\sigma_1 G_{(1)}^\sharp \otimes \cdots \otimes \sigma_1 G_{(r)}^\sharp)] \quad (31)$$

so that it is sufficient to prove the property for  $F = S_n$  or  $S_{\bar{n}}$ . Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_{-q} \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_{-q} 1 * \sigma_1 G_{(1)}^\sharp) (\sigma_1 G_{(2)}^\sharp) \\ &= \sum_{(G)} (\bar{\lambda}_{-q} * G_{(1)}^\sharp) \cdot \sigma_1^\sharp \cdot G_{(2)}^\sharp \end{aligned} \quad (32)$$

which is in  $\mathring{\mathcal{Q}}$ , since it is a subalgebra and a left  $*$ -ideal, and similarly,

$$\begin{aligned} \bar{\sigma}_1^\sharp * \sigma_1 G^\sharp &= (\lambda_{-q} \bar{\sigma}_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\lambda_{-q} * \sigma_1 G_{(1)}^\sharp) (\bar{\sigma}_1 \bar{G}_{(2)}^\sharp) \\ &= \sum_{(G)} (\lambda_{-q} * G_{(1)}^\sharp) \cdot \bar{\sigma}_1^\sharp \cdot \bar{G}_{(2)}^\sharp \end{aligned} \quad (33)$$

is also in  $\mathring{\mathcal{Q}}$ . ■

The various algebras introduced in this paper and their interrelationships are summarized in the following diagram.

$$\begin{array}{ccccccc} \mathring{\mathcal{Q}} & \subseteq & \mathcal{Q} & \subseteq & \mathbf{MR} & \subseteq & \mathbf{BFQSym} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathring{\mathcal{P}} & \subseteq & \mathcal{P} & \subseteq & \mathbf{Sym} & \subseteq & \mathbf{FQSym} \end{array} \quad (34)$$



Note that in the special case  $q = -1$ , by the results of Section 3.3,  $\mathcal{Q}_n$  is the (Solomon) descent algebra of  $B_n$ ,  $\mathcal{Q}$  is isomorphic to **BSym**, and  $\mathcal{P}$  is the unital peak algebra of [2].

## 6 Further developments

### 6.1 Inversion of the generic $\sharp$ transform

For generic  $q$ , the endomorphism (27) of **MR** is invertible; therefore

$$\mathring{\mathcal{Q}} \sim \mathbf{MR}. \quad (35)$$

The inverse endomorphism of **MR** arises from the transformation of alphabets

$$A \mapsto (q\bar{A} + A)/(1 - q^2), \quad (36)$$

which is to be understood in the following sense:

$$\sigma_1 \left( \frac{q\bar{A} + A}{1 - q^2} \right) := \prod_{k \geq 0} \sigma_{q^{2k+1}}(\bar{A}) \sigma_{q^{2k}}(A). \quad (37)$$

Indeed,

$$\begin{aligned} \sigma_1 \left( \frac{q\bar{A} + A}{1 - q^2} \right) * \sigma_1(A - q\bar{A}) &= \prod_{k \geq 0} \sigma_{q^{2k+1}}(\bar{A} - qA) \sigma_{q^{2k}}(A - q\bar{A}) \\ &= \prod_{k \geq 0} \lambda_{-q^{2k+2}}(A) \sigma_{q^{2k+1}}(\bar{A}) \lambda_{-q^{2k+1}}(\bar{A}) \sigma_{q^{2k}}(A) \\ &= \sigma_1(A). \end{aligned} \quad (38)$$

By normalizing the term of degree  $n$  in (37), we obtain  $B_n$ -analogs of the  $q$ -Klyachko elements defined in [9]:

$$K_n(q; A, \bar{A}) := \prod_{i=1}^n (1 - q^{2i}) S_n \left( \frac{q\bar{A} + A}{1 - q^2} \right) = \sum_{I \models n} q^{2 \text{maj}(I)} R_I(q\bar{A} + A). \quad (39)$$

This expression can be completely expanded on signed ribbons. From the expression of  $R_I$  in **FQSym**, we have

$$R_I(\bar{A} + A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(\bar{A} + A) \quad (40)$$

where  $\bar{A} + A$  is the ordinal sum. If we order  $\bar{A}$  by

$$\bar{a}_1 < \bar{a}_2 < \dots < \bar{a}_k < \dots \quad (41)$$

then, arguing as in [16], we have

$$\mathbf{G}_\sigma(\bar{A} + A) = \sum_{\text{std}(\tau, \epsilon) = \sigma} \mathbf{G}_{\tau, \epsilon} \quad (42)$$

so that

$$R_I(\bar{A} + A) = \sum_{\rho(J)=I} R_J \quad (43)$$

where for a signed composition  $J = (J, \epsilon)$ , the unsigned composition  $\rho(J)$  is defined as the shape of  $\text{std}(\sigma, \epsilon)$ , where  $\sigma$  is any permutation of shape  $J$ .

Replacing  $\bar{A}$  by  $q\bar{A}$ , one obtains the expansion of the  $q$ -Klyachko elements of type  $B$ :

$$K_n(q; A, \bar{A}) = \sum_J q^{\text{bmaj}(J)} R_J \quad (44)$$

where

$$\text{bmaj}(J) = 2 \text{maj}(\rho(J)) + |\epsilon|, \quad (45)$$

where  $|\epsilon|$  is the number of minus signs in  $\epsilon$ .

For example,

$$K_2(q) = R_2 + q^2 R_{\bar{2}} + q^2 R_{11} + q^3 R_{1\bar{1}} + q R_{\bar{1}1} + q^4 R_{\bar{1}\bar{1}}. \quad (46)$$

$$\begin{aligned} K_3(q) = & R_3 + q^3 R_{\bar{3}} + q^4 R_{21} + q^5 R_{2\bar{1}} + q^2 R_{\bar{2}1} + q^7 R_{\bar{2}\bar{1}} + q^2 R_{12} + q^4 R_{1\bar{2}} \\ & + q R_{\bar{1}2} + q^5 R_{\bar{1}\bar{2}} + q^6 R_{111} + q^7 R_{11\bar{1}} + q^3 R_{1\bar{1}1} + q^8 R_{1\bar{1}\bar{1}} \\ & + q^5 R_{\bar{1}11} + q^6 R_{\bar{1}1\bar{1}} + q^4 R_{\bar{1}\bar{1}1} + q^9 R_{\bar{1}\bar{1}\bar{1}}. \end{aligned} \quad (47)$$

This major index of type  $B$  is the flag major index defined in [1].

Following [1] and considering the signed composition (where  $\epsilon$  is encoded as boolean vector for readability)

$$J = (2, 1, 1, \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) = (2113124122, 00001111110000100000) \quad (48)$$

we can take the smallest permutation of shape  $(2, 1, 1, 3, 1, 2, 4, 1, 2, 2)$ , which is

$$\alpha = 1\ 5\ 4\ 3\ 2\ 6\ 9\ 8\ 7\ 11\ 10\ 12\ 13\ 16\ 15\ 14\ 18\ 17\ 19 \quad (49)$$

sign it according to  $\epsilon$ , which yields

$$1\ 5\ 4\ 3\ \bar{2}\ \bar{6}\ \bar{9}\ \bar{8}\ \bar{7}\ \bar{1}\bar{1}\ 10\ 12\ 13\ 16\ \bar{1}\bar{5}\ 14\ 18\ 17\ 19 \quad (50)$$

whose standardized is

$$8\ 11\ 10\ 9\ 1\ 2\ 5\ 4\ 3\ 6\ 12\ 13\ 14\ 16\ 7\ 15\ 18\ 17\ 19 \quad (51)$$

and has shape  $\rho(J) = (2, 1, 1, 3, 1, 6, 3, 2)$ . The major index of  $\rho(J)$  is 55, the number of minus signs in  $\epsilon$  is 7, so  $\text{bmaj}(J) = 2 \times 55 + 7 = 117$ .

The major index of type  $B$  can be read directly on signed compositions without reference to signed permutations as follows: one can get  $\rho(J)$  by first adding the absolute values of two consecutive parts if the left one is signed and the second one is not, then remove the signs and proceed as before.

A different solution consists in reading the composition from right to left, then associate weight 0 (resp. 1) to the rightmost part if it is positive (resp. negative) and then proceed left by adding 2 to the weight if the two parts are of the same sign and 1 if not. Finally, add up the product of the absolute values of the parts with their weight.

For example, with the same  $J$  as above we have the following weights:

$$\begin{aligned} J = & (2, 1, 1, \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) \\ \text{weights} : & 14\ 12\ 10\ 9\ 7\ 5\ 4\ 3\ 2\ 0 \end{aligned} \quad (52)$$

so that we get  $2 \cdot 14 + 1 \cdot 12 + 1 \cdot 10 + 3 \cdot 9 + 1 \cdot 7 + 2 \cdot 5 + 4 \cdot 4 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 = 117$ .

This technique generalizes immediately to colored compositions with a fixed number  $c$  of colors  $0, 1, \dots, c-1$ : the weight of the rightmost cell is its color and the weight of a part is equal to the sum of the weight of the next part and the unique representative of the difference of the colors of those parts modulo  $c$  belonging to the interval  $[1, c]$ .

## 6.2 Generators and Hilbert series

For  $n \geq 0$ , let

$$S_n^\pm = S_n(A) \pm S_n(\bar{A}), \quad (53)$$

and denote by  $\mathcal{H}_n$  the subalgebra of  $\mathbf{MR}$  generated by the  $S_k^\pm$  for  $k \leq n$ . For  $n \geq 0$ , we have

$$(S_n^\pm)^\sharp \equiv (1 \mp q^n) S_n^\pm \pmod{\mathcal{H}_{n-1}}, \quad (54)$$

so that the  $(S_n^\pm)^\sharp$  such that  $1 \mp q^n \neq 0$  form a set of free generators in  $\mathbf{MR}^\sharp$ .

**Conjecture 6.1** *If  $r$  is odd, a basis of  $\mathbf{MR}^\sharp$  will be parametrized by colored compositions such that parts of color 0 are not  $\equiv 0 \pmod{r}$  and parts of color 1 are arbitrary. The Hilbert series is then*

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \dots + t^r)}. \quad (55)$$

*If  $r$  is even, there is the extra condition that parts of color 1 are not  $\equiv r/2 \pmod{r}$ . The Hilbert series is then*

$$H_r(t) = \frac{1 - t^r}{1 - 2(t + t^2 + \dots + t^r) + t^{r/2}}. \quad (56)$$

For example,

$$H_2(t) = 1 + t + 2t^2 + 4t^3 + 8t^4 + 16t^5 + 32t^6 + 64t^7 + 128t^8 + O(t^9) \quad (57)$$

$$H_3(t) = 1 + 2t + 6t^2 + 17t^3 + 50t^4 + 146t^5 + 426t^6 + 1244t^7 + 3632t^8 + O(t^9) \quad (58)$$

$$H_4(t) = 1 + 2t + 5t^2 + 14t^3 + 38t^4 + 104t^5 + 284t^6 + 776t^7 + 2120t^8 + O(t^9) \quad (59)$$

If these conjectures are correct, the Hilbert series of the right  $\mathbf{MR}^\sharp$ -modules generated by the  $S_n$  are respectively

$$\frac{1}{1 - 2(t + t^2 + \dots + t^r)}, \quad (60)$$

or

$$\frac{1}{1 - 2(t + t^2 + \dots + t^r) + t^{r/2}}. \quad (61)$$

according to whether  $r$  is odd or even.

The cases  $r = 1$  and  $r = 2$  are easily proved as follows. Assume first that  $q = 1$ . Set

$$f = 1 + (\sigma_1^+)^\sharp = (\sigma_1 + \lambda_{-1})(A - \bar{A}), \quad (62)$$

$$g = (\sigma_1^-)^\sharp - 1 = (\sigma_1 - \lambda_{-1})(A - \bar{A}). \quad (63)$$

Then,  $f^2 = g^2 + 4$ , so that

$$f = 2 \left( 1 + \frac{1}{4} g^2 \right)^{\frac{1}{2}} \quad (64)$$

which proves that the  $(S_n^+)^\sharp$  can be expressed in terms of the  $(S_m^-)^\sharp$ .

Similarly, for  $q = -1$ , one can express

$$f = \sum_{n \geq 1} (S_{2n}^+)^\sharp + \sum_{n \geq 0} (S_{2n+1}^-)^\sharp \quad (65)$$

in terms of

$$g = \sum_{n \geq 1} (S_{2n}^-)^{\sharp} + \sum_{n \geq 0} (S_{2n+1}^+)^{\sharp} \quad (66)$$

since, as is easily verified,

$$(f+2)^2 = g^2 + 4, \text{ i.e., } f = -2 + 2 \left(1 + \frac{1}{4}g^2\right)^{\frac{1}{2}}. \quad (67)$$

Apparently, this approach does not work anymore for higher roots of unity.

## 7 Appendix: monomial expansion of the $(1 - q)$ -kernel

The results of [16, 7] allow us to write down a new expansion of  $S_n((1 - q)A)$ , in terms of the monomial basis of [4]. The special case  $q = 1$  gives back a curious expression of Dynkin's idempotent, first obtained in [3].

Let  $\sigma$  be a permutation. We then define its *left-right minima* set  $\text{LR}(\sigma)$  as the values of  $\sigma$  that have no smaller value to their left. We will denote by  $\text{lr}(\sigma)$  the cardinality of  $\text{LR}(\sigma)$ . For example, with  $\sigma = 46735182$ , we have  $\text{LR}(\sigma) = \{4, 3, 1\}$ , and  $\text{lr}(\sigma) = 3$ .

Let us now decompose  $S_n((1 - q)A)$  on the monomial basis  $\mathbf{M}_\sigma$  (see [4]) of **FQSym**. Thanks to the Cauchy formula of **FQSym** [7], we have

$$S_n((1 - q)A) = \sum_{\sigma} \mathbf{S}^\sigma (1 - q) \mathbf{M}_\sigma(A), \quad (68)$$

where  $\mathbf{S}$  is the dual basis of  $\mathbf{M}$ . Given the transition matrix between  $\mathbf{M}$  and  $\mathbf{G}$ , we see that

$$\mathbf{S}^\sigma = \sum_{\tau \leq \sigma^{-1}} \mathbf{F}_\tau, \quad (69)$$

where  $\leq$  is the right weak order, e.g.,  $\mathbf{S}^{312} = \mathbf{F}_{123} + \mathbf{F}_{213} + \mathbf{F}_{231}$ . Thanks to [16], we know that  $\mathbf{F}_\sigma(1 - q)$  is either  $(-q)^k$  if  $\text{Des}(\sigma) = \{1, \dots, k\}$  or 0 otherwise. Let us define *hook permutations* of hook  $k$  the permutations  $\sigma$  such that  $\text{Des}(\sigma) = \{1, \dots, k\}$ . Now,  $\mathbf{S}^\sigma(1 - q)$  amounts to compute the list of *hook permutations* smaller than  $\sigma$ . Note that hook permutations are completely characterized by their left-right minima. Moreover, if  $\tau$  is smaller than  $\sigma$  in the right weak order, then  $\text{LR}(\tau) \subset \text{LR}(\sigma)$ .

Hence all hook permutations smaller than a given permutation  $\sigma$  belong to the set of hook permutations with left-right minima in  $\text{LR}(\sigma)$ . Since by elementary transpositions decreasing the length, one can get from  $\sigma$  to the hook permutation with the same left-right minima and then from this permutation to all the others, we have:

**Theorem 7.1** *Let  $n$  be an integer. Then*

$$S_n((1 - q)A) = \sum_{\sigma \in \mathfrak{S}_n} (1 - q)^{\text{lr}(\sigma)} \mathbf{M}_\sigma. \quad (70)$$

■

In the particular case  $q = 1$ , we recover a result of [3]:

$$\Psi_n = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=1}} \mathbf{M}_\sigma, \quad (71)$$

where  $\Psi_n$  is the noncommutative power sum associated with Dynkin's idempotent [11, Prop. 5.2].

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## References

- [1] R. ADIN and Y. ROICHMAN, *The flag major index and group actions on polynomial rings*, Europ. J. Combin. **22** (2001), 431–446.
- [2] M. AGUIAR, N. BERGERON, and K. NYMAN, *The peak algebra and the descent algebra of type B and D*, Trans. of the AMS. **356** (2004), 2781–2824.
- [3] M. AGUIAR and M. LIVERNET, *The associative operad and the weak order on the symmetric group*, J. Homotopy and Related Structures, **2** n.1 (2007), 57–84.
- [4] M. AGUIAR and F. SOTTILE, *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*, Adv. in Math. **191** (2005), 225–275.
- [5] N. BERGERON, F. HIVERT and J.-Y. THIBON, *The peak algebra and the Hecke-Clifford algebras at  $q = 0$* , J. Combinatorial Theory A **117** (2004), 1–19.
- [6] C.-O. CHOW, *Noncommutative symmetric functions of type B*, Thesis, MIT, 2001.
- [7] G. DUCHAMP, F. HIVERT, J.-C. NOVELLI, and J.-Y. THIBON, *Noncommutative Symmetric Functions VII: Free Quasi-Symmetric Functions Revisited*, preprint, math.CO/0809.4479.
- [8] G. DUCHAMP, F. HIVERT, and J.-Y. THIBON, *Noncommutative symmetric functions VI: free quasi-symmetric functions and related algebras*, Internat. J. Alg. Comput. **12** (2002), 671–717.
- [9] I.M. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V. S. RETAKH, and J.-Y. THIBON, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.
- [10] F. HIVERT and N. THIÉRY, *MuPAD-Combinat, an open-source package for research in algebraic combinatorics*, Sémin. Lothar. Combin. **51** (2004), 70p. (electronic).
- [11] D. KROB, B. LECLERC, and J.-Y. THIBON, *Noncommutative symmetric functions II: Transformations of alphabets*, Internal J. Alg. Comput. **7** (1997), 181–264.
- [12] D. KROB and J.-Y. THIBON, *Higher order peak algebras*, Ann. Combin. **9** (2005), 411–430.
- [13] I.G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
- [14] R. MANTACI and C. REUTENAUER, *A generalization of Solomon’s descent algebra for hyperoctahedral groups and wreath products*, Comm. Algebra **23** (1995), 27–56.
- [15] J.-C. NOVELLI and J.-Y. THIBON, *Free quasi-symmetric functions of arbitrary level*, preprint math.CO/0405597.
- [16] J.-C. NOVELLI and J.-Y. THIBON, *Superization and  $(q, t)$ -specialization in combinatorial Hopf algebras*, math.CO/0803.1816.
- [17] S. POIRIER, *Cycle type and descent set in wreath products*, Disc. Math., **180** (1998), 315–343.
- [18] C. REUTENAUER, *Free Lie algebras*, Oxford University Press, 1993.
- [19] L. SOLOMON, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra, **41**, (1976), 255–268.
- [20] R. P. STANLEY, *Enumerative combinatorics*, vol. 2, Cambridge University Press, 1999.