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Records of stationary processes and unimodular graphs

Bharath Roy Choudhury

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DE L'UNIVERSITÉ PSL

Préparée au Département d'Informatique de l'ENS

Records of stationary processes and unimodular graphs

Soutenue par

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Chapter 1

Introduction

Consider a navigation rule defined on a graph that maps every vertex of the graph to a vertex in such a way that the navigation rule commutes with every automorphism of the graph. It is to say that the navigation rule applied to the vertices remains the same after taking any automorphism of the graph. Such a navigation rule is said to have the covariance property. Such covariant navigation rule indexed by locally finite graphs and satisfying a measurability condition is called a vertex-shift. More generally, one can consider vertex-shifts on networks, graphs that have labels on edges and on vertices. A vertex-shift induces a dynamic on the space of locally finite rooted networks. We are interested in a navigation rule, called the record vertex-shift, defined on trajectories of the one dimensional discrete random walk that can jump only one step to the left. The main part of the thesis is focused on studying the record vertex-shift on random walks. A few results on record vertex-shift on processes with stationary increments are also derived. The thesis also contains results on more general vertex-shifts on unimodular networks (see Def. 6).

A random walk on \mathbb{Z} is called skip-free to the left (resp. right) if its i.i.d. increments take values in $\{j \in \mathbb{Z} : j \geq -1\}$ (resp. $\{j \in \mathbb{Z} : j \leq 1\}$).

Consider an i.i.d. sequence of random variables $X = (X_n)_{n \in \mathbb{Z}}$, where X_n takes values in $\{j \in \mathbb{Z} : j \geq -1\}$ and with finite mean for all $n \in \mathbb{Z}$. A two-sided skip-free random walk $(S_n)_{n \in \mathbb{Z}}$ on \mathbb{Z} with increment sequence X is given by: $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} X_k$, $\forall n \geq 0$, and $S_n = \sum_{k=n}^{-1} -X_k$, $\forall n \leq -1$. Observe that $(S_n)_{n \geq 0}$ is skip-free to the left whereas $(S_{-n})_{n \geq 0}$ is skip-free to the right. Construct the record graph of X on \mathbb{Z} by drawing a directed edge from each integer i to its record $R(i) := \min\{n > i : S_n \geq S_i\}$ if the minimum

exists, otherwise to i itself. The record graph is a random directed graph, with \mathbb{Z} as the vertex set and where the directed edges are random. Consider the connected component of 0 in the constructed record graph and root it at 0. Denote it by $[T, 0]$. This work is motivated by the following questions:

1. What is the distribution of the random rooted graph $[T, 0]$?
2. How does the distribution of $[T, 0]$ vary with the mean of the increment $\mathbb{E}[X_0]$?
3. What is the trajectory of the random walk seen from the n -th record of 0 as $n \rightarrow \infty$?
4. Is there a non-trivial measure-preserving map on the trajectory of the random walk which is compatible with the record map R (see the discussion below for the definition of ‘compatibility’)?

Concerning the second question, we show that the record graph exhibits a phase transition when the mean $\mathbb{E}[X_0]$ is varied between $-\infty$ and ∞ . There are three distinct phases, and each phase has a characteristic feature in terms of the classification of unimodular directed trees. The first phase is observed in the region $(-\infty, 0)$, in which the record graph has an infinite number of connected components with the component of the origin being a finite unimodular directed tree. The second phase occurs when $\mathbb{E}[X_0] = 0$. In this phase, the record graph is a.s. connected, and it is a one-ended unimodular directed tree. The third phase, which occurs when $\mathbb{E}[X_0] \in (0, \infty)$, has the characteristic feature that the record graph is a.s. a two-ended unimodular directed tree.

A Family Tree is a rooted directed tree whose out-degree is at most 1. By neglecting the loops, which occur only if $\mathbb{E}[X_0] < 0$, we obtain that the connected component of 0 in the record graph is a random Family Tree T rooted at 0. We address the first question by explicitly computing the distribution of the Family Tree $[T, 0]$ in the three regimes, namely when $\mathbb{E}[X_0] \in (-\infty, 0)$ (Theorem 48), $\mathbb{E}[X_0] = 0$ (Theorem 53), and $\mathbb{E}[X_0] \in (0, \infty)$ (Theorem 54, Theorem 55).

In [7], the authors introduced the notion of vertex-shifts on graphs. Informally, a vertex-shift is a collection of maps on graphs that are covariant and satisfy a measurability condition. Given a vertex-shift f , we can associate to each graph G its f -graph G^f whose vertices are the same as those

of G and with directed edges given by $\{(x, f(x)) : x \in V(G), f(x) \neq x\}$. The vertex-shift f defines an equivalence relation on $V(G)$ by saying that two vertices u and v are equivalent if there exists some positive integer n such that $f^n(u) = f^n(v)$. The equivalence class of $u \in V(G)$ is called the foil of u (denoted as $\text{foil}(u)$) and the collection of all foils of $V(G)$ is called the foliation of G . Since the connected component $G_f(u)$ of any vertex u in G^f is the set $\{v \in V(G) : f^n(v) = f^m(u) \text{ for some } n \geq 1, m \geq 1\}$, we have $\text{foil}(u) \subseteq V(G_f(u))$.

The authors showed in [7] that given a unimodular random rooted graph $[\mathbf{G}, \mathbf{o}]$, the connected component of the root in the f -graph G^f is a unimodular Family Tree, and a.s. every component of G^f belongs to one of the three classes: \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{I} , and \mathcal{I}/\mathcal{F} . The class \mathcal{F}/\mathcal{F} corresponds to the component being finite and its foils being finite. Similarly, the classes \mathcal{I}/\mathcal{I} and \mathcal{I}/\mathcal{F} correspond to the component being infinite and all foils of the component being infinite, and the component being infinite and all foils of the component being finite, respectively. It is easy to see that the record map R is a vertex-shift, which implies that the Family Tree $[T, 0]$ is unimodular, where T is the connected component of 0 in the record graph. The three phases of the record graph $[T, 0]$ for $\mathbb{E}[X_0] < 0$, $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X_0] > 0$ correspond to these three classes \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{I} and \mathcal{I}/\mathcal{F} respectively. So, the record vertex-shift is an example for which all the three classes respectively occur depending on the mean $\mathbb{E}[X_0]$.

Using the framework of [3], we view the increments of the skip-free random walk as marks associated to the edges of \mathbb{Z} , seen from the root 0. More precisely, we consider the random rooted network $[\mathbb{Z}, 0, X]$, with $X = (X_n)_{n \in \mathbb{Z}}$ being the sequence of increments and X_n being the label assigned to the edge $(n, n+1)$, for all $n \in \mathbb{Z}$. It is easy to see that $[\mathbb{Z}, 0, X]$ is a unimodular network. By the local view from the n -th record ($n \geq 1$), we mean the distribution of $[\mathbb{Z}, R^n(0), X]$. The latter is the same as $[\mathbb{Z}, 0, R^n(0)X]$, where $R^n(0)X$ is the shifted sequence $(X_{m+R^n(0)})_{m \in \mathbb{Z}}$. Using the unimodularity of $[\mathbb{Z}, 0, X]$, we show that the local view from the n -th record is absolutely continuous with respect to $[\mathbb{Z}, 0, X]$ with Radon-Nikodym derivative $d_n(0)$, which is defined as the number of integers for which 0 is the n -th record.

Further, in connection with the third question, we show that the local view seen from the n -th record as n tends to ∞ , $\lim_{n \rightarrow \infty} [\mathbb{Z}, 0, R^n(0)X]$, exists for $\mathbb{E}[X_0] < 0$, $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X_0] > 0$ respectively. Denote this limit by $[\mathbb{Z}, 0, Y]$, where $Y = (Y_n)_{n \in \mathbb{Z}}$ is the mark on the edges of \mathbb{Z} . We observe

that Y is not an i.i.d. sequence. In particular, its negative part $(Y_n)_{n<0}$ fails to be an i.i.d. sequence. However, it has a regenerative structure when $\mathbb{E}[X_0] \geq 0$.

As for the fourth question, a map R' on \mathbb{Z} is said to be foil-preserving with R if it maps every foil to itself, i.e., for every $i \in \mathbb{Z}$, $R'(i)$ belongs to the foil(i) (the foil associated to the vertex-shift R and i).

When $\mathbb{E}[X_0] \geq 0$, we define a random map R_\perp that takes i to

$$R_\perp(i) = \inf\{k > i : R^n(i) = R^n(k) \text{ for some } n \geq 1\}.$$

We show that R_\perp is foil-preserving with R and it preserves the distribution of $[\mathbb{Z}, 0, X]$, i.e., the distribution of $[\mathbb{Z}, R_\perp(0), X]$ is the same as that of $[\mathbb{Z}, 0, X]$.

The statistics of records of time series have found applications in finance, hydrology and physics. See for instance the references in the survey [13] for a non-exhaustive list of applications. These studies of records are focused on statistics such as the distribution of n -th record, the age of a record (i.e., the gap between two consecutive records), and the age of the long-lasting record ([21], [13]). When the time series under consideration is an i.i.d. sequence, it is shown in [19] that the mean of the jump between two consecutive record values conditioned on the initial record value could characterize the distribution of the i.i.d. sequence. In contrast, in the present work, the focus is on the joint local structure of the records of skip-free random walks. The structure has a natural family tree flavour with an order suggesting to term it as ‘genealogy of records’.

The relation between skip-free random walks and Galton-Watson trees was extensively discussed in, e.g., [8], [16], and [14]. In fact, the authors of [8] encode the critical Galton-Watson tree and show that the encoding gives a correspondence between records of the random walk and ancestors of the root in the Galton-Watson tree. We extend these ideas to define the backward map Φ_R that takes a record graph and maps it back to the trajectory of a random walk (see Section 5.1, Chapter 5). However, we are not restricted to finite Galton-Watson trees here. To the best of our knowledge this complete description of the record graph of skip-free random walk is new.

In Chapter 2, we give the background and the statements of the theorems that we use in this work. We introduce the canonical probability spaces in which our random objects take values and introduce the dynamics induced

by vertex-shifts. Our focus is on studying vertex-shifts on unimodular networks. We state a key theorem that we use in this regard known as Foil classification in unimodular networks Theorem. We also introduce some operations on unimodular Family Trees.

In Chapter 3, we introduce the record map with which we obtain the record vertex-shift. We study its properties on deterministic integer-valued sequences. We study the properties of the record graph associated to the sequences. Using these properties, we prove that the record graph of $[\mathbb{Z}, 0, X]$ exhibits phase transitions, where X is the i.i.d. sequence of increments of a skip-free to the left random walk. We also describe the distribution of component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ in each of these phases. We define the map R_1 and prove that it preserves the measure of $[\mathbb{Z}, 0, X]$ when $\mathbb{E}[X_0] = 0$.

In Chapter 4, we focus on general vertex-shifts on unimodular networks and study the existence and uniqueness of vertex-shift probabilities. We provide sufficient conditions under which a vertex-shift on a random network has a unique vertex-shift probability. We apply these results to compute the vertex-shift probability of unimodular Family Trees where the vertex-shift is given by the parent vertex-shift.

In Chapter 5, we apply the results of Chapter 4 to study the R -probability of $[\mathbb{Z}, 0, X]$, where R is the record vertex-shift and X is the i.i.d. sequence of increments of a ship-free to the left random walk. We also provide constructions to generate the R -probability.

In Chapter 6, we study the record graph of a stationary and ergodic sequence. We show that the component of 0 in the record graph exhibits a phase transition at 0 when the mean is varied between $-\infty$ and $+\infty$ (see Theorem 119). We provide a sufficient condition under which a unimodular ordered EFT can be represented as the component of 0 in the record graph of the network associated to a stationary sequence (Theorem 121).

Introduction in French

Considérez une règle de navigation définie sur un graphe qui associe à chaque sommet du graphe un sommet de telle sorte que la règle de navigation commute avec chaque automorphisme du graphe. Cela signifie que la règle de navigation appliquée aux sommets reste la même après avoir pris n'importe quel automorphisme du graphe. Une telle règle de navigation est

dite avoir la propriété de covariance. Une telle règle de navigation covariante indexée par des graphes localement finis et satisfaisant une condition de mesurabilité est appelée un décalage de sommet. Plus généralement, on peut considérer des décalages de sommets sur des réseaux, des graphes qui ont des étiquettes sur les arêtes et sur les sommets. Un décalage de sommet induit une dynamique sur l'espace des réseaux enracinés localement finis. Nous nous intéressons à une règle de navigation, appelée décalage de sommet des records, définie sur les trajectoires de la marche aléatoire discrète à une dimension qui ne peut sauter qu'un pas vers la gauche. La partie principale de la thèse est consacrée à l'étude du décalage de sommet des records sur ces marches aléatoires. Quelques résultats sur le décalage de sommet des records sur des processus à accroissements stationnaires sont également obtenus. La thèse contient également des résultats sur des décalages de sommets plus généraux sur des réseaux unimodulaires (voir Def. 6).

Une marche aléatoire sur \mathbb{Z} est dite sans saut vers la gauche (resp. droite) si ses accroissements i.i.d. prennent des valeurs dans $\{j \in \mathbb{Z} : j \geq -1\}$ (resp. $\{j \in \mathbb{Z} : j \leq 1\}$).

Considérons une suite i.i.d. de variables aléatoires $X = (X_n)_{n \in \mathbb{Z}}$, où X_n prend des valeurs dans $\{j \in \mathbb{Z} : j \geq -1\}$ et avec une espérance finie pour tout $n \in \mathbb{Z}$. Une marche aléatoire bilatérale sans saut vers la gauche $(S_n)_{n \in \mathbb{Z}}$ sur \mathbb{Z} associée à une suite d'accroissements X est donnée par : $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} X_k$, $\forall n \geq 0$, et $S_n = \sum_{k=n}^{-1} -X_k$, $\forall n \leq -1$. Remarquez que $(S_n)_{n \geq 0}$ est sans saut vers la gauche tandis que $(S_{-n})_{n \geq 0}$ est sans saut vers la droite. Construisez le graphe des records de X sur \mathbb{Z} en traçant une arête orientée de chaque entier i vers son record $R(i) := \min\{n > i : S_n \geq S_i\}$ si le minimum existe, sinon vers i lui-même. Le graphe des records est un graphe aléatoire orienté, avec \mathbb{Z} comme ensemble de sommets et où les arêtes orientées sont aléatoires. Considérez la composante connexe de 0 dans le graphe des records construit et enracinez-la en 0. Notez-la par $[T, 0]$. Ce travail est motivé par les questions suivantes :

1. Quelle est la distribution du graphe enraciné aléatoire $[T, 0]$?
2. Comment varie la distribution de $[T, 0]$ avec l'espérance de l'accroissement $\mathbb{E}[X_0]$?
3. Quelle est la trajectoire de la marche aléatoire vue depuis le n -ième record de 0 lorsque $n \rightarrow \infty$?

4. Existe-t-il une application non triviale qui préserve la mesure sur la trajectoire de la marche aléatoire qui est compatible avec l'application record R (voir la discussion ci-dessous pour la définition de 'compatibilité') ?

En ce qui concerne la deuxième question, nous montrons que le graphe des records présente une transition de phase lorsque la moyenne $\mathbb{E}[X_0]$ varie entre $-\infty$ et ∞ . Il y a trois phases distinctes, et chaque phase a une caractéristique propre en termes de classification des arbres dirigés unimodulaires.

La première phase est observée dans la région $(-\infty, 0)$, dans laquelle le graphe des records a un nombre infini de composantes connexes avec la composante de l'origine étant un arbre dirigé unimodulaire fini. La deuxième phase se produit lorsque $\mathbb{E}[X_0] = 0$. Dans cette phase, le graphe des records est connexe, et c'est un arbre dirigé unimodulaire à une extrémité. La troisième phase, qui se produit lorsque $\mathbb{E}[X_0] \in (0, \infty)$, a la caractéristique que le graphe des records est un arbre dirigé unimodulaire à deux extrémités.

Un arbre généalogique est un arbre dirigé enraciné dont le degré sortant est au plus 1. En négligeant les boucles, qui ne se produisent que si $\mathbb{E}[X_0] < 0$, nous obtenons que la composante connexe de 0 dans le graphe des records est un arbre généalogique aléatoire T enraciné en 0. Nous abordons la première question en calculant explicitement la distribution de l'arbre généalogique $[T, 0]$ dans les trois régimes, à savoir lorsque $\mathbb{E}[X_0] \in (-\infty, 0)$ (Théorème 48), $\mathbb{E}[X_0] = 0$ (Théorème 53), et $\mathbb{E}[X_0] \in (0, \infty)$ (Théorème 54, Théorème 55).

Dans [7], les auteurs ont introduit la notion de décalages de sommets sur les graphes. Informellement, un décalage de sommet est une collection de fonctions sur les graphes qui sont covariantes et satisfont une condition de mesurabilité. Étant donné un décalage de sommet f , nous pouvons associer à chaque graphe G son f -graphe G^f dont les sommets sont les mêmes que ceux de G et avec des arêtes dirigées données par $\{(x, f(x)) : x \in V(G)\}$. Le décalage de sommet f définit une relation d'équivalence sur $V(G)$ en disant que deux sommets u et v sont équivalents s'il existe un entier positif n tel que $f^n(u) = f^n(v)$. La classe d'équivalence de $u \in V(G)$ est appelée le feuillet de u (noté $\text{feuillet}(u)$) et l'ensemble de tous les feuillets de $V(G)$ est appelé la foliation de G . Comme la composante connexe $G_f(u)$ de tout sommet u dans G^f est l'ensemble $\{v \in V(G) : f^n(v) = f^m(u) \text{ pour certains } n \geq 1, m \geq 1\}$, nous avons $\text{feuillet}(u) \subseteq V(G_f(u))$.

Les auteurs ont montré dans [7] que, étant donné un graphe aléatoire unimodulaire enraciné $[\mathbf{G}, \mathbf{o}]$, c.-à-d. la composante connexe de la racine dans le f -graphe G^f , est un arbre généalogique unimodulaire appartenant à l'une des trois classes : \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{I} , et \mathcal{I}/\mathcal{F} . La classe \mathcal{F}/\mathcal{F} correspond au cas où la composante de \mathbf{o} est finie et ses feuillets sont finis. De même, les classes \mathcal{I}/\mathcal{I} et \mathcal{I}/\mathcal{F} correspondent au cas où la composante est infinie et tous les feuillets de la composante sont finis, respectivement. Il est facile de voir que le décalage record R est un décalage de sommet, ce qui implique que l'arbre généalogique $[T, 0]$ est unimodulaire, où T est la composante connexe de 0 dans le graphe des records. Les trois phases du graphe des records $[T, 0]$ pour $\mathbb{E}[X_0] < 0$, $\mathbb{E}[X_0] = 0$ et $\mathbb{E}[X_0] > 0$ correspondent à ces trois classes \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{I} et \mathcal{I}/\mathcal{F} respectivement. Ainsi, le décalage record est un exemple pour lequel les trois classes se produisent respectivement en fonction de la moyenne $\mathbb{E}[X_0]$.

En utilisant le cadre de [3], nous considérons les accroissements de la marche aléatoire sans saut comme des marques associées aux arêtes de \mathbb{Z} , vues depuis la racine 0 . Plus précisément, nous considérons le réseau aléatoire enraciné $[\mathbb{Z}, 0, X]$, avec $X = (X_n)_{n \in \mathbb{Z}}$ étant la suite des accroissements et X_n étant le label assigné à l'arête $(n, n+1)$, pour tout $n \in \mathbb{Z}$. Il est facile de voir que $[\mathbb{Z}, 0, X]$ est un réseau unimodulaire. Par vue locale depuis le n -ième record ($n \geq 1$), nous entendons la distribution de $[\mathbb{Z}, R^n(0), X]$. Cette dernière est la même que celle de $[\mathbb{Z}, 0, R^n(0)X]$, où $R^n(0)X$ est la suite décalée $(X_{m+R^n(0)})_{m \in \mathbb{Z}}$. En utilisant l'unimodularité de $[\mathbb{Z}, 0, X]$, nous montrons que la vue locale depuis le n -ième record est absolument continue par rapport à $[\mathbb{Z}, 0, X]$ avec une dérivée de Radon-Nikodym $d_n(0)$, qui est définie comme le nombre d'entiers pour lesquels 0 est le n -ième record.

De plus, en lien avec la troisième question, nous montrons que la vue locale vue depuis le n -ième record lorsque n tend vers ∞ , $\lim_{n \rightarrow \infty} [\mathbb{Z}, 0, R^n(0)X]$, existe pour $\mathbb{E}[X_0] < 0$, $\mathbb{E}[X_0] = 0$ et $\mathbb{E}[X_0] > 0$ respectivement. Notons cette limite par $[\mathbb{Z}, 0, Y]$, où $Y = (Y_n)_{n \in \mathbb{Z}}$ est la marque sur les arêtes de \mathbb{Z} . Nous observons que Y n'est pas une suite i.i.d. En particulier, sa partie négative $(Y_n)_{n < 0}$ n'est pas une suite i.i.d. Cependant, elle a une structure régénérative lorsque $\mathbb{E}[X_0] \geq 0$.

En ce qui concerne la quatrième question, une fonction R' sur \mathbb{Z} est dite préservant les feuillets avec R si elle envoie chaque feuillet sur lui-même, c'est-à-dire que pour tout $i \in \mathbb{Z}$, $R'(i)$ appartient au feuillet (i) (le feuillet associé au décalage de sommet R et i).

Lorsque $\mathbb{E}[X_0] \geq 0$, nous définissons une fonction aléatoire R_{\perp} qui envoie i en

$$R_{\perp}(i) = \inf\{k > i : R^n(i) = R^n(k) \text{ pour un certain } n \geq 1\}.$$

Nous montrons que R_{\perp} préserve les feuillets de R et préserve la distribution de $[\mathbb{Z}, 0, X]$, c'est-à-dire que la distribution de $[\mathbb{Z}, R_{\perp}(0), X]$ est la même que celle de $[\mathbb{Z}, 0, X]$.

Les statistiques des records des séries temporelles ont trouvé des applications en finance, hydrologie et physique. Voir par exemple les références dans l'article de synthèse [13] pour une liste non exhaustive d'applications. Ces études sur les records sont centrées sur des statistiques telles que la distribution du n -ième record, l'âge d'un record (c'est-à-dire l'écart entre deux records consécutifs), et l'âge du record le plus durable ([21], [13]). Lorsque la série temporelle considérée est une suite i.i.d., il est montré dans [19] que la moyenne du saut entre deux valeurs consécutives de record conditionnée à la valeur initiale du record peut caractériser la distribution de la suite i.i.d. En revanche, dans le présent travail, l'accent est mis sur la structure locale conjointe des records des marches aléatoires sans saut vers la gauche. La structure a une connexion naturelle avec la notion d'arbre généalogique qui justifie de parler de 'généalogie des records'.

La relation entre les marches aléatoires sans saut vers la gauche et les arbres de Galton-Watson a été largement discutée dans, par exemple, [8], [16] et [14]. En fait, les auteurs de [8] codent l'arbre de Galton-Watson critique et montrent que le codage donne une correspondance entre les records de la marche aléatoire et les ancêtres de la racine dans l'arbre de Galton-Watson. Nous étendons ces idées pour définir l'application réciproque Φ_R qui prend un graphe des records et le renvoie à la trajectoire d'une marche aléatoire (voir Section 5.1, Chapitre 5). Cependant, nous ne sommes pas limités aux arbres de Galton-Watson finis ici. À notre connaissance, cette description complète du graphe des records de la marche aléatoire sans saut est nouvelle. Nous ne sommes pas limités au cas des marches aléatoires par ailleurs.

Dans le chapitre 2, nous donnons les préliminaires et les énoncés des théorèmes que nous utilisons dans ce travail. Nous introduisons les espaces probabilisés canoniques dans lesquels nos objets aléatoires prennent leurs valeurs et introduisons la dynamique induite par les décalages de sommets. Nous nous concentrons sur l'étude des décalages de sommets sur les réseaux unimodulaires. Nous énonçons un théorème clé que nous utilisons à cet

égard, connu sous le nom de théorème de classification des feuilletés dans les réseaux unimodulaires. Nous introduisons également quelques opérations sur les arbres généalogiques unimodulaires.

Dans le chapitre 3, nous introduisons le décalage de sommet des records. Nous étudions ses propriétés sur les suites déterministes à valeurs entières. Nous étudions les propriétés du graphe des records associé aux suites. En utilisant ces propriétés, nous prouvons que le graphe des records de $[\mathbb{Z}, 0, X]$ présente des transitions de phase, où X est la suite i.i.d. des accroissements d'une marche aléatoire sans saut vers la gauche. Nous décrivons également la distribution de la composante de (0) dans le graphe des records de $[\mathbb{Z}, 0, X]$ dans chacune de ces phases. Nous définissons la fonction R_{\perp} et prouvons qu'elle préserve la loi de $[\mathbb{Z}, 0, X]$ lorsque $\mathbb{E}[X_0] = 0$.

Dans le chapitre 4, nous nous intéressons aux décalages de sommets généraux sur les réseaux unimodulaires et nous étudions l'existence et l'unicité des probabilités de décalage de sommet. Nous fournissons des conditions suffisantes sous lesquelles un décalage de sommet sur un réseau aléatoire a une probabilité de décalage de sommet unique. Nous appliquons ces résultats pour calculer la probabilité de décalage de sommet des arbres généalogiques unimodulaires où le décalage de sommet est donné par le décalage de sommet parent.

Dans le chapitre 5, nous appliquons les résultats du chapitre 4 pour étudier la R -probabilité de $[\mathbb{Z}, 0, X]$, où R est le décalage record de sommet et X est la suite i.i.d. des accroissements d'une marche aléatoire sans saut vers la gauche. Nous donnons également des constructions pour générer la R -probabilité.

Dans le chapitre 6, nous étudions le graphe des records d'une suite stationnaire et ergodique. Nous montrons que la composante de 0 dans le graphe des records présente une transition de phase en 0 lorsque la moyenne varie entre $-\infty$ et $+\infty$ (voir le théorème 119). Nous fournissons une condition suffisante sous laquelle un EFT ordonné unimodulaire peut être représenté comme la composante de 0 dans le graphe des records du réseau associé à une suite stationnaire (théorème 121).

Chapter 2

Preliminaries

In this chapter, we summarize the framework of networks given in [3] and in [7].

2.1 Canonical probability spaces for networks

A graph G with a vertex set $V = V(G)$ and an (undirected) edge set $E = E(G)$ is denoted by $G = (V, E)$. A **network** is a graph $G = (V, E)$ together with a Polish space Ξ and two maps: one from V to Ξ and the other from E to Ξ . The space Ξ is called mark space and the value assigned to a vertex (resp. an edge) is called **mark of the vertex** (resp. edge). Each edge $e = \{x, y\}$ in the network has two marks associated with its endpoints, one assigned to $\{e, x\}$ and the other assigned to $\{e, y\}$. A directed edge can be obtained from an undirected edge by assigning 0 as mark to one of its endpoint and 1 as mark to the other, with the understanding that the orientation of the directed edge is from the endpoint with mark 0 to the endpoint with mark 1. A graph can be seen as a network with a fixed mark. In this work, we focus on the networks that are connected and locally finite, i.e., degree of every vertex is finite, and whose mark space is \mathbb{Z} .

An **isomorphism** ϕ of networks G_1 and G_2 , denoted as $\phi : G_1 \rightarrow G_2$, is a bijective map $\phi_V : V(G_1) \rightarrow V(G_2)$ satisfying the three conditions: (1) for every pair of vertices $u, v \in V(G_1)$, (u, v) is an edge of G_1 if and only if $(\phi_V(u), \phi_V(v))$ is an edge of G_2 , (2) for all $u \in V(G_1)$, mark of u is the same as that of $\phi_V(u)$, (3) for all $e = (u, v) \in E(G_1)$, mark of e is same as

that of $(\phi_V(u), \phi_V(v))$. In particular, both G_1, G_2 should have the same mark space.

A **rooted network** (G, o) is a network G with a distinguished vertex $o \in V(G)$. For any $r \geq 0$, let $(G, o)_r$ be the subgraph induced by the vertices $\{u \in V(G) : d_G(u, o) \leq r\}$, where d_G denotes the graph distance in G , and for any vertex $v \in V((G, o)_r)$, let $((G, o)_r, v)$ be the rooted network obtained by rooting $(G, o)_r$ at v . The marks of $(G, o)_r$ are induced by the marks of (G, o) . A **rooted-isomorphism** ϕ of two rooted networks $(G_1, o_1), (G_2, o_2)$ is an isomorphism $\phi : G_1 \rightarrow G_2$ of networks G_1, G_2 with the additional condition that $\phi_V(o_1) = \phi_V(o_2)$. Two rooted networks (G_1, o_1) and (G_2, o_2) are said to be **equivalent**, denoted as $(G_1, o_1) \sim (G_2, o_2)$, if and only if there is a rooted-isomorphism from one to the other. An equivalent class of (G_1, o_1) is denoted by $[G_1, o_1]$. We do not distinguish between a rooted graph and its equivalent class unless explicitly stated.

A doubly rooted network (G, u, v) is a network G with two distinguished vertices $u, v \in V(G)$. For any $r \geq 0$, let $(G, u, v)_r$ be the subgraph induced by the vertices $\{w \in V(G) : d_G(w, u) \wedge d_G(w, v) \leq r\}$ with marks of edges and vertices induced from that of G , where d_G denotes the graph distance in G and \wedge denotes the minimum function. A doubly rooted-isomorphism ϕ of two doubly rooted networks (G_1, u_1, v_1) and (G_2, u_2, v_2) is an isomorphism $\phi : G_1 \rightarrow G_2$ of networks G_1, G_2 such that $\phi_V(u_1) = u_2$ and $\phi_V(v_1) = v_2$. Two doubly rooted networks (G_1, u_1, v_1) and (G_2, u_2, v_2) are said to be equivalent, denoted as $(G_1, u_1, v_1) \sim (G_2, u_2, v_2)$, if and only if there exists a doubly rooted-isomorphism between them.

We denote the **space of equivalence class of rooted locally finite and connected networks** by \mathcal{G}_* and the **space of equivalence class of doubly rooted locally finite and connected networks** by \mathcal{G}_{**} .

The space \mathcal{G}_* can be equipped with a metric in the following way: distance between two rooted networks (G_1, o_1) and (G_2, o_2) is defined as $2^{-d_{1,2}}$, where $d_{1,2}$ is the supremum of $r \geq 0$ such that there exists a network isomorphism $\phi_r : (G_1, o_1)_r \rightarrow (G_2, o_2)_r$ and that for every $u \in V(G_1), (v, w) \in E(G_1)$, the distance between marks of u and of $\phi_r(u)$ is at most $\frac{1}{r}$, and the distance between mark of (v, w) and of $(\phi_r(v), \phi_r(w))$ is at most $\frac{1}{r}$. The space \mathcal{G}_{**} can also be equipped with a metric in the similar way as in \mathcal{G}_* by defining the distance between two doubly rooted networks (G_1, u_1, v_1) and (G_2, u_2, v_2) as $2^{-d_{1,2}}$, where $d_{1,2}$ is the same as in \mathcal{G}_* but applied to $(G_1, u_1, v_1)_r, (G_2, u_2, v_2)_r$ instead of $(G_1, o_1)_r, (G_2, o_2)_r$ respectively. The topology generated on \mathcal{G}_* with the above metric is called the local topology.

Both the spaces \mathcal{G}_* and \mathcal{G}_{**} endowed with their respective metrics are Polish spaces. Let us denote the space of equivalence classes of graphs by $\hat{\mathcal{G}}_*$.

In this work, we often encounter with a special networks called Family Trees. A **Family Tree** is a locally finite directed tree that has the property that the out-degree of every vertex is at most 1. If the out-degree of every vertex is equal to 1 a.s. then it is called an **Eternal Family Tree** (EFT). A vertex u of a Family Tree is called the **parent** of a vertex v if there is a directed edge from v to u . In this case, v is called a **child** of u . An **ordered Family Tree** is a Family Tree in which children of every vertex are ordered. Although, we use similar notation for ordered and unordered, one can distinguish them depending on the context. A rooted (ordered) Family Tree is a pair (T, o) , where T is an (ordered) Family Tree and o is a distinguished vertex of T . The space of equivalence classes of rooted (ordered) Family Trees is denoted by \mathcal{T}_* , where the equivalence relation is given by rooted (ordered) isomorphisms, i.e., two rooted Family Trees are equivalent if there is a rooted isomorphism between them that preserves the order. The space \mathcal{T}_* is Polish with respect to local topology [7]. By forgetting the order of an ordered Family Tree, one obtains an unordered Family Tree. We often call a rooted Family Tree, a Family Tree.

We will also encounter marked Family Trees whose vertices have \mathbb{Z} -valued marks. Denote the space of equivalence classes of such (rooted) marked Family Trees by $\hat{\mathcal{T}}_*$.

A random (rooted) network is a measurable map from a probability space to \mathcal{G}_* . We use the bold fonts to denote a random network and normal fonts to denote a realization of a random network.

We use the following theorem to prove the convergence of random networks. For any rooted network (G, o) and $r \geq 0$, let $[G, o]_r$ denote the equivalent class $(G, o)_r$.

Theorem 1 ([17]). *A sequence of random networks $([\mathbf{G}_n, \mathbf{o}_n])_{n \geq 1}$ converges in distribution to a random network $[\mathbf{G}, \mathbf{o}]$ as $n \rightarrow \infty$ if and only if for every $r \geq 0$, the sequence of random networks $([\mathbf{G}_n, \mathbf{o}_n]_r)_{n \geq 1}$ converges in distribution as $n \rightarrow \infty$.*

Relative compactness and tightness

We recall the definitions of relative compactness and tightness. This subsection is concerned only about graphs (not networks). A subset \mathcal{A} of a topological space is said to be **relatively compact** if every sequence in \mathcal{A} has a convergent subsequence. A subset \mathcal{A} of probability measures on a measurable space is said to be **tight** if for any $\epsilon > 0$ there exists a compact set K such that $\forall \mathcal{P} \in \mathcal{A}, \mathcal{P}[K] \geq 1 - \epsilon$. In particular, if there exists an increasing sequence of compact sets $\{K_m\}_{m \in \mathbb{N}}$ such that

$$\lim_{m \rightarrow \infty} \sup_{\mathcal{P} \in \mathcal{A}} \mathcal{P}[K_m^c] = 0,$$

then \mathcal{A} is tight. Prokhorov's theorem states that if \mathcal{A} is tight then \mathcal{A} is relatively compact and if the measurable space is complete and separable, then relative compactness is equivalent to tightness, i.e., \mathcal{A} is relatively compact if and only if \mathcal{A} is tight.

For any $r \geq 0$, we define the restriction map $|_r : \mathcal{G}_* \rightarrow \mathcal{G}_*$ taking $[G, o]$ to $[G, o]_r$ by $[G, o]_r = [(G, o)|_r, o]$ for any $[G, o] \in \mathcal{G}_*$. The image of this restriction map $\mathcal{G}_*|_r$ is countable and does not have any accumulation point. So, every compact subset of $\mathcal{G}_*|_r$ is finite. Let K_m^r be those rooted graphs of $\mathcal{G}_*|_r$ that have at most m vertices, i.e., $K_m^r := \{[G, o] \in \mathcal{G}_*|_r \text{ such that } |V(G)| \leq m\}$ which is a finite set. Then, the sequence $\{K_m^r\}_{m \in \mathbb{N}}$ is an increasing sequence of compact sets.

We state a lemma that gives an equivalent condition for a sequence of random graphs to be tight in terms of their neighborhood restrictions.

Lemma 2. *A family of random graphs $\{[\mathbf{G}_i, \mathbf{o}_i], i \in I\}$ is tight if and only if for all $r \geq 1$, the family $\{[\mathbf{G}_i, \mathbf{o}_i]_r, i \in I\}$ is tight.*

Proof. Refer [17]. □

Therefore, from the above discussion and Lemma 2, we conclude that a family of random graphs $\{[\mathbf{G}_i, \mathbf{o}_i], i \in I\}$ is relatively compact if and only if for any $r \geq 1$

$$\lim_{m \rightarrow \infty} \sup_{i \in I} \mathbb{P}[[\mathbf{G}_i, \mathbf{o}_i]_r \notin K_m^r] = 0.$$

We use this conclusion to prove the relative compactness of vertex-shift-probabilities when the vertex-shift has finite orbits.

2.2 Dynamics on networks

The dynamics we look at are navigation rules known as vertex-shifts on random networks. The definitions and notations of this section are taken from [7].

Definition 3. A **vertex-shift** f is a collection of maps indexed by networks $f_G : V(G) \rightarrow V(G)$, where G is a locally finite network, satisfying the following properties:

1. *Covariance:* For any graph isomorphism $\sigma : G \rightarrow H$ of two graphs G and H , f satisfies $f_H \circ \sigma = \sigma \circ f_G$,
2. *Measurability:* The map $[G, u, v] \mapsto \mathbf{1}_{\{f_G(u)=v\}}$ from the doubly rooted space \mathcal{G}_{**} is measurable.

The first condition allows us to define a (deterministic) dynamic on the space \mathcal{G}_* in the following way: for each vertex-shift f , define a map θ_f by

$$\begin{aligned} \theta_f : \mathcal{G}_* &\longrightarrow \mathcal{G}_* \\ [G, o] &\mapsto [G, f(o)]. \end{aligned} \tag{2.2.1}$$

The map θ_f is well-defined.

The second condition allows us to define the following functions. For any non-negative measurable function $h : \mathcal{G}_{**} \rightarrow \mathbb{R}_{\geq 0}$, we define two measurable functions $h^+ : \mathcal{G}_* \rightarrow \mathbb{R}_{\geq 0}$ and $h^- : \mathcal{G}_* \rightarrow \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} h^+([G, o]) &= \sum_{u \in V(G)} h([G, o, u]) \mathbf{1}_{\{f_G(u) = o\}} \\ h^-([G, o]) &= \sum_{u \in V(G)} h([G, u, o]) \mathbf{1}_{\{f_G(o) = u\}}, \end{aligned}$$

for all $[G, o] \in \mathcal{G}_*$.

Notations

Given a vertex-shift f and a network G , the f -graph of G is the network G^f with vertex set $V(G^f) := V(G)$ and directed edges $E(G^f) = \{(u, f_G(u)) : u \in V(G) \text{ and } f_G(u) \neq u\}$. Let $G^f(u)$ denote the component of u , for any vertex $u \in V(G)$. Define the map $\Psi_f : \mathcal{G}_* \rightarrow \mathcal{T}_*$ that maps a rooted network

$[G, o]$ to $[G^f(o), o]$, where $G^f(o)$ is obtained by discarding all self-loops (if there are any) of the connected component of o in its f -graph G^f .

For every vertex-shift f on a network G , the components of the f -graph of G are Family Trees [7]. For any integer $n \geq 0$, a vertex of the form $f_G^n(u)$ is called the **ancestor of order n of u** (also called n -th ancestor of u). We denote the set of **n -th descendants** (also called descendants of order n) of vertex $u \in V(G)$ by $D_n(u) := \{v \in V(G) \setminus \{u\} : f_G^n(v) = u\}$ and its cardinality by $d_n(u) := \#D_n(u)$, for any $n \geq 1$. We follow the convention that $f^0(u) = D_0(u) = u$. Denote the set of all descendants of u by $D(u) := \{v \in V(G) \setminus \{u\} : f_G^n(v) = u \text{ for some } n > 0\} \cup \{u\}$ and its cardinality by $d(u) = \#D(u)$. For any vertex $u \in G$, its ancestor of order 1 is called the **parent of u** , its descendants of order 1 are called **children of u** , and the set $\{v \in V(G) \setminus \{u\} : f_G(v) = f_G(u)\}$ is called the **set of siblings of u** .

A vertex-shift f defines the following equivalence relation on the vertices of any network G : two vertices u and v of G are equivalent if and only if $f_G^n(u) = f_G^n(v)$ for some $n \geq 0$. The equivalence class of u is called the **f -foil of u** , and the collection of f -foils of G is called the **f -foliation of G** .

A **covariant subset** is a collection of subsets S_G indexed by network G , where $S_G \subseteq V(G)$ and the collection satisfies the following two conditions, (1) for every network isomorphism $\alpha : G \rightarrow H$, we have $\alpha(S_G) = S_H$, (2) the map $[G, o] \mapsto \mathbf{1}_{\{o \in S_G\}}$ is measurable. A **covariant partition** is a collection of partitions Π_G indexed by network G , where Π_G is a partition of $V(G)$ and the collection satisfies the following two conditions, (1) for every network isomorphism $\alpha : G \rightarrow H$, we have $\alpha \circ \Pi_G = \Pi_H$, (2) the subset $\{[G, o, u] : u \in \Pi_G(o)\}$ is a measurable subset.

Examples of covariant partitions can be constructed from any vertex-shift f . Given f , the collection of f -foils of G indexed by G is a covariant partition. Similarly, the collection of subsets $S_G = \{u \in V(G) : f_G(u) = u\}$ of $V(G)$ indexed by network G is an example of a covariant subset.

Example 4. For every Family Tree T , consider the map $F_T : V(T) \rightarrow V(T)$ that maps every vertex u to its parent if it exists, otherwise maps to itself. The vertex-shift F is called the **parent vertex-shift**.

For every vertex-shift f on G , the f -graph of G is a Family Tree [7]. Observe that the terminologies of parent and child are consistent with respect to the parent vertex-shift. The dynamic naturally induced by the parent vertex-shift is given by $\theta_F : \mathcal{T}_* \rightarrow \mathcal{T}_*$ given by $\theta_F([T, o]) = [T, F(o)]$.

Given a Family Tree T , let $l(\cdot, \cdot) = l_T(\cdot, \cdot) : V(T) \times V(T) \rightarrow \mathbb{Z}$ be the function defined by the following equations: for every $v, w \in V(T)$,

$$\begin{aligned} l(v, v) &= 0, & l(v, F(w)) &= l(v, w) - 1, \\ & & l(w, v) &= -l(v, w), \\ & & l(v, w) + l(w, z) &= l(v, z). \end{aligned}$$

For a rooted Family Tree $[T, o]$, the set $\{w \in V(T) : l(o, w) = n\}$ is the n -th generation of $[T, o]$.

Royal Line of Succession order

Let T be an ordered Family Tree, i.e., the children of every vertex are totally ordered, with an order denoted by $<$. We can extend this order on children to a total order on the vertices of T in the following way. Let u, v be two vertices of T . We say that v has precedence over u , denoted as $v > u$ (or $u < v$), if either $F^k(u) = v$ for some $k > 0$, or there exists a common ancestor $w = F^m(u) = F^n(v)$ of u and v with m, n being the smallest positive integers with this property, and $F^{m-1}(u) < F^{n-1}(v)$. In the latter case, both $F^{m-1}(u)$ and $F^{n-1}(v)$ can be compared since they are children of w .

Remark 5. For two vertices u, v with common ancestor w , distinct from u and v , $u < v$ if and only if $y_u < y_v$ for every descendant y_u of u and for every descendant y_v of v .

We call the total order $<$ the **Royal Line of Succession (RLS) order** associated with $<$. It is also known as **Depth First Search order**.

2.3 Unimodular random networks and their properties

Definition 6. A random network $[\mathbf{G}, \mathbf{o}]$ is said to be **unimodular** if it satisfies the **Mass Transport Principle**:

$$\mathbb{E} \left[\sum_{u \in V(\mathbf{G})} f([\mathbf{G}, \mathbf{o}, u]) \right] = \mathbb{E} \left[\sum_{u \in V(\mathbf{G})} f([\mathbf{G}, u, \mathbf{o}]) \right], \quad (2.3.1)$$

for every non-negative measurable function $f : \mathcal{G}_{**} \rightarrow \mathbb{R}_{\geq 0}$.

The following theorem suggests that there are a wide class of unimodular random networks. For any finite network G , we can associate a random rooted network $[G, \mathbf{o}]$ by choosing the root uniformly among the vertices of G , i.e., for any measurable set $A \subset \mathcal{G}_*$, we have

$$\mathbb{P}[[G, \mathbf{o}] \in A] = \frac{1}{\#V(G)} \sum_{u \in V(G)} \mathbf{1}_A([G, u]). \quad (2.3.2)$$

It can be easily shown that $[G, \mathbf{o}]$ is unimodular [17]. In fact every random finite network can be obtained by uniformly choosing the root [17].

Theorem 7 ([3]). *Let $[G_n, \mathbf{o}_n]$ be a sequence of unimodular networks that converge in distribution to a random network $[G, \mathbf{o}]$. Then, $[G, \mathbf{o}]$ is unimodular.*

The above theorem implies that if $(G_n)_{n \geq 1}$ is a sequence of finite deterministic networks such that the sequence of random networks $([G_n, \mathbf{o}_n])_{n \geq 1}$, obtained by applying Eq. (2.3.2), converges in distribution to a random network $[G, \mathbf{o}]$, then $[G, \mathbf{o}]$ is unimodular. In the above, we can replace deterministic finite network by random finite network. The network $[G, \mathbf{o}]$ is said to be obtained by taking **Benjamini-Schramm** limit of $(G_n)_{n \geq 1}$.

Definition 8 (Network associated to a sequence). Let $x = (x_n)_{n \in \mathbb{Z}}$ be a sequence of real numbers. The network associated to x is the network (\mathbb{Z}, x) with vertex set \mathbb{Z} , with edge set $\{(n, n+1) : n \in \mathbb{Z}\}$, and with x_n as mark of the edge $(n, n+1)$ for all $n \in \mathbb{Z}$.

See Figure 2.1 for an illustration.

Example 9 (of a unimodular network). *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence. The network $[\mathbb{Z}, 0, X]$ associated to the sequence X is unimodular.*

The network $[\mathbb{Z}, 0, X]$ in Example 9 is unimodular because the network can be shown to be the Benjamini-Schramm limit of the sequence $(G_n, X^{(n)})_{n \geq 1}$, where $(G_n, X^{(n)})$ is the network with vertex set $\{-n, -n+1, \dots, 0, 1, \dots, n\}$, edge set $\{(k, k+1) : -n \leq k \leq n-1\}$ with mark X_k assigned to the edge $(k, k+1)$ for all $-n \leq k \leq n-1$.

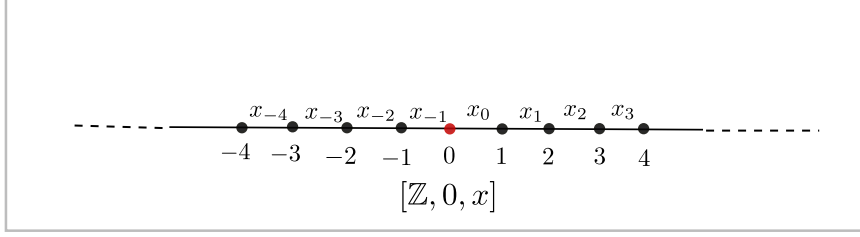


Figure 2.1: The network (\mathbb{Z}, x) associated to a sequence $x = (x_n)_{n \in \mathbb{Z}}$. The rooted network $[\mathbb{Z}, 0, x]$ denotes the equivalence class of the network (\mathbb{Z}, x) rooted at 0.

Vertex-shifts on unimodular networks

Recall the map θ_f , in Eq. (2.2.1), associated to a vertex-shift f . The following proposition will be used in the later chapters while studying measure preserving vertex-shifts.

Proposition 10 (Mecke-Thorisson Point Stationarity [7]). *Let f be a vertex-shift and $[\mathbf{G}, \mathbf{o}]$ be a unimodular measure. Then, the map θ_f preserves the distribution of $[\mathbf{G}, \mathbf{o}]$ if and only if f is a.s. bijective.*

The following two propositions will be used to show in the later chapters that a unimodular ordered Family Tree does not have a smallest nor a largest vertex.

Proposition 11 ([7]). *Let $[\mathbf{G}, \mathbf{o}]$ be unimodular and f be a vertex-shift.*

1. *If f is injective a.s., then f is bijective a.s.*
2. *If f is surjective a.s., then f is bijective a.s.*

Lemma 12 (No Infinite/Finite Inclusion [7]). *Let $[\mathbf{G}, \mathbf{o}]$ be a unimodular network, \mathfrak{S} be a covariant subset and $\mathbf{\Pi}$ be a covariant partition. Almost surely, there is no infinite element E of $\mathbf{\Pi}_{\mathbf{G}}$ such that $E \cap \mathfrak{S}_{\mathbf{G}}$ is finite and non-empty.*

The following lemma shows that probability that covariant subset is non-empty is related to the probability that the root belongs to this covariant subset.

Lemma 13 ([7]). *Let $[\mathbf{G}, \mathbf{o}]$ be a unimodular network, \mathfrak{S} be a covariant subset. Then, $\mathbb{P}[\mathfrak{S}_{\mathbf{G}} \text{ is non-empty}] > 0$ if and only if $\mathbb{P}[\mathbf{o} \in \mathfrak{S}_{\mathbf{G}}] > 0$.*

Proof. Take $f([G, u, v]) = \mathbf{1}\{v \in \mathfrak{S}_G\}$ and apply the Mass Transport Principle. \square

The following theorem gives a classification of the components of f -graph, for any vertex-shift f on a unimodular network.

Theorem 14 (Foil Classification in Unimodular Networks [7]). *Let $[\mathbf{G}, \mathbf{o}]$ be a unimodular network and f be a vertex-shift. Almost surely, every vertex has finite degree in the graph G^f . In addition, each component C of G^f has at most two ends and it belongs to one of the following three classes:*

- *Class \mathcal{F}/\mathcal{F} : C and all its foils are finite. If $n = n(C)$ is the number of foils in C , then*
 - *C has a unique f -cycle and its length is n ;*
 - *$f_G^\infty(C)$ is the set of vertices of the cycle;*
 - *Each foil of C contains exactly one vertex of the cycle.*
- *Class \mathcal{I}/\mathcal{F} : C is infinite but all its foils are finite. In this case,*
 - *The (undirected) f -graph on C is a tree;*
 - *There is a unique bi-infinite f -path in C , each foil in C contain exactly one vertex of the path, and $f_G^\infty(C)$ coincides with the set of vertices of the path;*
 - *The order of the foils of C is of type \mathbb{Z} ; that is, there is no youngest foil in C .*
- *Class \mathcal{I}/\mathcal{I} : C and all foils of C are infinite. In this case,*
 - *The (undirected) f -graph on C is a tree;*
 - *C has one end, there is no bi-infinite f -path in C , $D(v)$ is finite for every vertex $v \in C$, and $f_G^\infty(C) = \emptyset$;*
 - *The order of the foils of C is of type \mathbb{N} or \mathbb{Z} , that is, there may or may not be a youngest foil in C .*

We will see examples from each class in the next chapter. The above theorem when applied to Family Trees gives the following theorem, which will be used in the later chapters. Recall the notation that F denotes the parent vertex-shift.

Theorem 15 (Classification of unimodular Family Trees [7]). *A unimodular Family Tree almost surely belongs to one of the following three classes:*

1. *Class \mathcal{F}/\mathcal{F} : the tree is finite.*
2. *Class \mathcal{I}/\mathcal{I} : every generation is infinite, each vertex has finitely many descendants, and there is no bi-infinite path, i.e., the tree is one-ended.*
3. *Class \mathcal{I}/\mathcal{F} : every generation is finite and the set of vertices that have infinitely many descendants form a unique bi-infinite F -path, i.e., the tree has two ends.*

The following lemma gives a recipe to construct unimodular Family Trees. Recall that for a vertex-shift f and a network G , G^f denotes the f -graph of G and for any vertex $u \in V(G)$, $G^f(u)$ denotes the component of u in G^f .

Lemma 16 ([7]). *Let f be a vertex-shift and $[\mathbf{G}, \mathbf{o}]$ be a random rooted network. If $[\mathbf{G}, \mathbf{o}]$ is unimodular then the Family Tree $[\mathbf{G}^f(\mathbf{o}), \mathbf{o}]$ is unimodular.*

Unimodular ordered Eternal Galton-Watson Tree

We give an example of unimodular Family Tree of class \mathcal{I}/\mathcal{I} originally introduced in [7].

Let π be a probability distribution on $\mathbb{N} \cup \{0\}$ with mean $m(\pi) = 1$, and $\hat{\pi}$ be its size-biased distribution defined by $\hat{\pi}(k) = k\pi(k)$, $\forall k \geq 0$.

The ordered unimodular Eternal Galton-Watson Tree with offspring distribution π , $EGWT(\pi)$, is a rooted ordered Eternal Family Tree $[\mathbf{T}, \mathbf{o}]$ with the following properties. The root \mathbf{o} and all of its descendants reproduce independently with the common offspring distribution π . For all $n \geq 1$, the ancestor $F^n(\mathbf{o})$ of \mathbf{o} reproduces with distribution $\hat{\pi}$. The descendants of $F^n(\mathbf{o})$ which are not the descendants of $F^{n-1}(\mathbf{o})$ (and not $F^{n-1}(\mathbf{o})$) reproduce independently with the common offspring distribution π , with the

convention $F^0(\mathbf{o}) = \mathbf{o}$. All individuals (vertices) reproduce independently. The order among the children of every vertex is uniform.

In particular, except for the vertices $F^n(\mathbf{o})$, $n \geq 1$, which reproduce independently with distribution $\hat{\pi}$, all the remaining vertices of the tree reproduce independently with distribution π . See Figure 2.2 for an illustration. It is proved in [7] that $EGWT(\pi)$ is unimodular if and only if the mean $m(\pi)$ is 1.

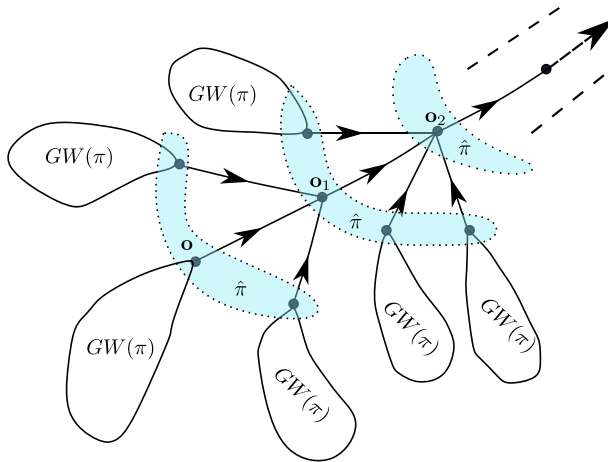


Figure 2.2: The Eternal Galton-Watson Tree with offspring distribution π . Notation: $GW(\pi)$ denotes the Galton-Watson Tree, $\hat{\pi}$ is the size-biased distribution of π . The order among the children of every vertex is from the left to the right.

Remark 17. Let $[\mathbf{T}, \mathbf{o}]$ be an $EGWT(\pi)$ with mean $m(\pi) = 1$. Then, the descendant trees of all vertices, except for the ancestors of the root \mathbf{o} , are independent critical Galton-Watson trees with offspring distribution π . Therefore, the descendant trees are finite. Hence, a realization of $EGWT(\pi)$ has the property that the descendant tree of every vertex of it is finite. So, it is of class \mathcal{I}/\mathcal{I} .

The following theorem characterizes a unimodular $EGWT$.

Theorem 18 ([7]). *A unimodular Eternal Family Tree $[\mathbf{T}, \mathbf{o}]$ is an $EGWT$ if and only if the number of children of the root $d_1(\mathbf{o})$ is independent of the non-descendant tree $D^c(\mathbf{o})$, i.e., the subtree induced by $(V(\mathbf{T}) \setminus D(\mathbf{o})) \cup \{\mathbf{o}\}$.*

Unimodular Family Trees of class \mathcal{I}/\mathcal{F}

In the following, we describe a construction given in [7] used to generate unimodular Family Trees of class \mathcal{I}/\mathcal{F} .

Let $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ be a stationary sequence of finite random rooted Family Trees. Add a directed edge $(\mathbf{o}_i, \mathbf{o}_{i+1})$ for all $i \in \mathbb{Z}$, and denote this new Eternal Family Tree (EFT) by \mathbf{T}' and let $\mathbf{o}' = \mathbf{o}_0$. The EFT $[\mathbf{T}', \mathbf{o}']$ is called the **joining** of $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$.

In addition, suppose that $\mathbb{E}[\#V(\mathbf{T}_0)] < \infty$. Choose the root uniformly in $V(\mathbf{T}_0)$ and bias the measure by the mean size of $V(\mathbf{T}_0)$. Formally, define a measure σ' by: for any measurable set A ,

$$\sigma'[A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T}_0)]} \mathbb{E} \left[\sum_{v \in V(\mathbf{T}_0)} \mathbf{1}_{[\mathbf{T}', v]}(A) \right]. \quad (2.3.3)$$

An operator similar to σ' is defined in [2]. An EFT $[\mathbf{T}, \mathbf{o}]$ whose distribution is σ' is called the **typically rooted joining** of $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$. Note that the assumption $\mathbb{E}[\#V(\mathbf{T}_0)] < \infty$ is not needed in the joining operation, however, it is essential in order to define the typically rooted joining operation. We state the following results taken from [7].

Theorem 19 ([7]). *Let $[\mathbf{T}, \mathbf{o}]$ be the typically rooted joining of a stationary sequence $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$, where $\mathbb{E}[\#V(\mathbf{T}_0)] < \infty$. Then, $[\mathbf{T}, \mathbf{o}]$ is unimodular.*

Theorem 20 ([7]). *Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular EFT of class \mathcal{I}/\mathcal{F} a.s., and $[\mathbf{T}', \mathbf{o}']$ be the Family Tree obtained by conditioning $[\mathbf{T}, \mathbf{o}]$ on the event that \mathbf{o} belongs to the bi-infinite path of \mathbf{T} . Then, $[\mathbf{T}', \mathbf{o}']$ is the joining of some stationary sequence of finite Family Trees $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ and $[\mathbf{T}, \mathbf{o}]$ is the typically rooted joining of $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$.*

Typical re-rooting operation

Definition 21. Let $[\mathbf{T}, \mathbf{o}]$ be a random rooted finite Family Tree such that $\mathbb{E}[\#V(\mathbf{T})] < \infty$. A random rooted finite Family Tree $[\mathbf{T}', \mathbf{o}']$ is said to be obtained by **re-rooting to a typical vertex of $[\mathbf{T}, \mathbf{o}]$** if

$$\mathbb{P}[[\mathbf{T}', \mathbf{o}'] \in A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T})]} \mathbb{E} \left[\sum_{u \in V(\mathbf{T})} \mathbf{1}_A([\mathbf{T}, u]) \right], \quad (2.3.4)$$

for every measurable subset A of \mathcal{T}_* .

Chapter 3

Record vertex-shift for random walk case

In this chapter, we define the record vertex-shift and study its f -graph called record graph. The record vertex-shift is defined on the networks of the form (\mathbb{Z}, x) , where $x = (x_n)_{n \in \mathbb{Z}}$ is a sequence of integers in $\{-1, 0, 1, 2, \dots\}$. The record vertex-shift is defined by the record map defined as follows: for each integer i , look at the sums $\sum_{k=i}^{n-1} x_k$, starting from i , and assign i to the smallest $n > i$ for which $\sum_{k=i}^{n-1} x_k$ is non-negative. The randomness of the network is introduced when we take i.i.d. sequence $X = (X_n)$ with a common distribution on $\{-1, 0, 1, \dots\}$ such that $0 < \mathbb{P}[X_0 = -1] < 1$ and $-\infty < \mathbb{E}[X_0] < \infty$. The f -graph of the record vertex-shift on (\mathbb{Z}, X) is called the record graph. The properties of the record map and the nature of the random walk influence the distribution of the component of 0 of the record graph. These properties are studied in Section 3.1. Using these properties, we show, in Section 3.2, that the component of 0 in the record graph exhibits phase transitions at 0 when the mean of increment $\mathbb{E}[X_0]$ is varied between $-\infty$ and ∞ . These three phases of the component of 0 corresponding to the region $(-\infty, 0)$, at 0, and the region $(0, \infty)$ are namely Typically rooted Galton-Watson Tree (*TGWT*), unimodular Eternal Galton-Watson Tree *EGWT* and the unimodularised bi-variate Eternal Kesten Tree *MEKT* respectively. The Family Tree *EGWT* was introduced in [7]. We introduce the *TGWT* and *MEKT* in Subsection 3.2. All of these Family Trees are unimodular.

(In French) Dans ce chapitre, nous définissons le décalage de sommet des records et étudions son f -graphe appelé graphe des records. Le décalage

de sommet des records est défini sur les réseaux de la forme (\mathbb{Z}, x) , où $x = (x_n)_{n \in \mathbb{Z}}$ est une suite d'entiers dans $\{-1, 0, 1, 2, \dots\}$. Le décalage de sommet des records est défini par la fonction record définie comme suit : pour chaque entier i , on regarde les sommes $\sum_{k=i}^{n-1} x_k$, en partant de i , et on assigne i au plus petit $n > i$ pour lequel $\sum_{k=i}^{n-1} x_k$ est non négatif. L'aléa du réseau est introduit lorsque nous prenons une suite i.i.d. $X = (X_n)$ avec une loi commune sur $\{-1, 0, 1, \dots\}$ telle que $0 < \mathbb{P}[X_0 = -1] < 1$ et $-\infty < \mathbb{E}[X_0] < \infty$. Le f -graphe du décalage de sommet des records sur (\mathbb{Z}, X) est appelé le graphe des records. Les propriétés de la fonction record et la nature de la marche aléatoire influencent la distribution de la composante de 0 du graphe des records. Ces propriétés sont étudiées dans la section 3.1. En utilisant ces propriétés, nous montrons, dans la section 3.2, que la composante de 0 dans le graphe des records présente des transitions de phase en 0 lorsque la moyenne des accroissements $\mathbb{E}[X_0]$ varie entre $-\infty$ et ∞ . Ces trois phases de la composante de 0 correspondant à la région $(-\infty, 0)$, en 0, et à la région $(0, \infty)$ sont respectivement l'arbre généalogique typiquement enraciné de Galton-Watson (*TGWT*), l'arbre généalogique éternel unimodulaire de Galton-Watson (*EGWT*) et l'arbre éternel bivarié de Kesten unimodularisé (*MEKT*). L'arbre généalogique *EGWT* a été introduit dans [7]. Nous introduisons le *TGWT* et le *MEKT* dans la sous-section 3.2. Tous ces arbres généalogiques sont unimodulaires.

3.1 Record map

Notations: Given an integer valued sequence $x = (x_n)_{n \in \mathbb{Z}}$ with $x_n \geq -1$ for all $n \in \mathbb{Z}$, let

$$y(n, j) := \sum_{l=n}^{j-1} x_l, \tag{3.1.1}$$

for all integers $n \leq j$, with the convention that $\sum_n^{n-1} \equiv 0$.

For any integer i , the sequence $(-y(n, i))_{n < i}$ is called the sequence of sums in the past of i seen from i , whereas, the sequence $(y(i, n))_{n \geq i}$ is called the sums in the future of i seen from i . Similarly, the sequence $(n, -y(n, i))_{n < i}$ is called the trajectory in past of i seen from i and the sequence $(n, y(i, n))_{n \geq i}$ is called the trajectory in the future of i seen from i .

Let $s^{(i)}$ denote the sequence $(s_n^{(i)})_{n \geq 0}$, where $s_n^{(i)} = y(i-n, i)$ for all $n \geq 0$. Let $s = (s_n)_{n \in \mathbb{Z}}$ be the sequence of locations associated to x , where $s_0 = 0$, $s_n = -y(n, 0)$, for all $n < 0$, and $s_n = y(0, n)$ for all $n > 0$. In particular, we have $s_n^{(0)} = -s_{-n}$ for all $n \geq 0$.

Definition 22. A sequence of i.i.d. random variables $X = (X_n)_{n \in \mathbb{Z}}$ with X_n taking values in $\mathbb{N} \cup \{-1\}$ for all $n \in \mathbb{Z}$, such that $\mathbb{P}[X_0 = -1] > 0$ and $\mathbb{E}[X_0] < \infty$, are called i.i.d. increments of a skip-free to the left random walk.

We use capital letters to denote random variables. For instance, the sums associated to $X = (X_n)_{n \in \mathbb{Z}}$ are denoted by $S = (S_n)_{n \in \mathbb{Z}}$, and $S^{(i)} = (S_n^{(i)})_{n \geq 0}$ respectively.

Remark 23. Since $x_k \geq -1$ for all $k \in \mathbb{Z}$, an obvious observation is that if $y(n, j) = m < 0$ for any two integers $n < j$, then there exist integers $n = i_{-m+1} < i_{-m} < \dots < i_2 < i_1 = j$ such that $y(i_{l+1}, i_l) = -1$ for all $1 \leq l \leq -m$.

Define the **record map** $R_x : \mathbb{Z} \rightarrow \mathbb{Z}$ to be the function that maps any $n \in \mathbb{Z}$ to

$$R_x(n) = \begin{cases} \inf\{j > n : y(n, j) \geq \max\{y(n, l) : n \leq l \leq j-1\}\} & \text{if inf exists} \\ n & \text{otherwise,} \end{cases}$$

with the convention that $y(n, n) = 0$. Equivalently, we obtain

$$R_x(n) = \begin{cases} \inf\{j > n : y(n, j) \geq 0\} & \text{if the infimum exists} \\ n & \text{otherwise.} \end{cases}$$

Denote the k -th iterate of the record map by R_x^k .

Given a real valued sequence $x = (x_n)_{n \in \mathbb{Z}}$, we consider its associated network (\mathbb{Z}, x) with label x_i on the edge $(i, i+1)$, for all $i \in \mathbb{Z}$ (see [3] for the terminology on networks). For the random variables $X = (X_n)_{n \in \mathbb{Z}}$, we consider the random rooted network $[\mathbb{Z}, 0, X]$ associated to X (with mark X and where the network (\mathbb{Z}, X) is rooted at 0). Note that $[\mathbb{Z}, 0, X]$ is unimodular since X is stationary [3].

Consider networks of the form (\mathbb{Z}, i, x) , where $i \in \mathbb{Z}$ is the root and the mark $x = (x_n)_{n \in \mathbb{Z}}$ is a real-valued sequence which assigns label x_n to edge $(n, n+1)$ for each $n \in \mathbb{Z}$. We know that any order preserving graph

isomorphism of \mathbb{Z} is a shift map T_i , for some integer i , namely $T_i(n) = n + i$, $n \in \mathbb{Z}$. The map T_i acts on a sequence $x = (x_n)_{n \in \mathbb{Z}}$ by $T_i x = (x_{n-i})_{n \in \mathbb{Z}}$. So, any network (\mathbb{Z}, i, x) is root-isomorphic to $(\mathbb{Z}, 0, T_{-i}x)$.

Consider the collection of networks $\{(\mathbb{Z}, x) : x = (x_n)_{n \in \mathbb{Z}}, x_n \in \mathbb{Z}_{\geq -1}\}$. Then, by taking the record map R on this collection and the identity map on the rest of the networks, we obtain a vertex-shift. Indeed, R satisfies the first condition, i.e., R is covariant for all i because

$$\begin{aligned} R_{T_i x}(k+i) &= \begin{cases} \inf\{j > k+i : \sum_{l=k+i}^{j-1} x_{l-i} \geq 0\} & \text{if infimum is defined,} \\ k+i & \text{otherwise} \end{cases} \\ &= \begin{cases} i + \inf\{j > k : \sum_{l=k}^{j-1} x_l\} & \text{if infimum is defined} \\ i+k & \text{otherwise} \end{cases} \\ &= i + R_x(k). \end{aligned}$$

The record map R also satisfies the measurability condition since the map $[\mathbb{Z}, i, j, x] \mapsto 1_{R_x(i)=j}$ is a function of $(x_i, x_{i+1}, \dots, x_{j-1})$.

Properties of the record map

Let $x := (x_n)_{n \in \mathbb{Z}}$ be an arbitrary integer-valued sequence with $x_n \geq -1$, $\forall n \in \mathbb{Z}$, and let (\mathbb{Z}, x) be its associated network with x_n as the mark on the edge $(n, n+1)$ for $n \in \mathbb{Z}$. Define the function $L : \mathbb{R}^{\mathbb{Z}} \times \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$ in the following way:

$$L_x(i) = \begin{cases} \inf\{j < i : y(k, i) \geq 0, \forall j \leq k < i\}, & \text{if inf exists;} \\ i & \text{otherwise.} \end{cases} \quad (3.1.2)$$

The integer $L_x(i)$ is the largest integer (if it exists) up to which the sums in the past of i and seen from i are non-positive, equivalently, $L_x(i)$ is the largest integer up to which the trajectory in the past of i seen from i stays below level 1. Note that $-\infty \leq L_x(i) < i$ if and only if the sum $-y(j, i) \leq 0$ for all integers j such that $L_x(i) \leq j < i$. It could happen that $L_x(i) = i$. This is possible only if $x_{i-1} = -1$.

For the sequence x and any integer $i \in \mathbb{Z}$, the set $D_1(i) := \{j < i : R_x(j) = i\}$ is called the set of children of i and $D(i) := \{j < i : R_x^n(j) = i \text{ for some } n > 0\}$ is called the set of descendants of i . The following lemma shows that the set of descendants of any integer i is the set of integers that lie between $L_x(i)$ and i (including $L_x(i)$ and excluding i).

Lemma 24. *Let $x = (x_n)_{n \in \mathbb{Z}}$ be an integer-valued sequence with $x_n \geq -1$, $\forall n \in \mathbb{Z}$, and R be the record vertex-shift on the network (\mathbb{Z}, x) . The set of descendants of any integer i is given by*

$$D(i) = \{j \in \mathbb{Z} : L_x(i) \leq j < i\}.$$

Proof. We first prove that $\{j \in \mathbb{Z} : L_x(i) \leq j < i\} \subset D(i)$. If $L_x(i) = i$, then there is nothing to prove. So, assume that $L_x(i) < i$ (note that $L_x(i)$ can be $-\infty$). Consider any integer j such that $L_x(i) \leq j < i$. Since $y(j, i) \geq 0$, we have,

$$j < R_x(j) := \inf\{k > j : y(j, k) \geq 0\} \leq i.$$

By iteratively applying the same argument to $R^l(j)$ for each $l > 0$, we find the smallest non-negative integer m such that $R_x^m(j) = i$. Such an m exists as there are only finitely many integers between j and i . This implies that j is a descendant of i . Therefore, $\{j : L_x(i) \leq j < i\} \subseteq D(i)$.

We now prove that $D(i) = \{j \in \mathbb{Z} : L_x(i) \leq j < i\}$, i.e., if $-\infty < j < L_x(i)$, then j is not a descendant of i . For any $-\infty < j < L_x(i)$, we claim that, either $R_x(j) < L_x(i)$ or $R_x(j) > i$. Assume that the claim is true. If $R_x(j) < L_x(i)$, then by applying the claim again to $R_x(j)$, we obtain that, either $R_x^2(j) < L_x(i)$ or $R_x^2(j) > i$. Iterate this process several times until we find the largest non-negative integer n such that $R_x^n(j) < L_x(i)$. Such an n exists as there are only finitely many integers between j and $L_x(i)$. As n is the largest integer satisfying this condition, by applying the claim to $R_x^n(j)$, we obtain that $R_x^{n+1}(j) > i$. This implies that none of the descendants of $R_x^n(j)$ (including j) is a descendant of i , which completes the proof.

We now prove the last claim. Let $k := L_x(i) - 1$ and let j be as in the claim. Assume that $i \geq R_x(j) \geq L_x(i)$, i.e., $i \geq R_x(j) > k$. Then, $y(k, i) < 0$, by the definition of $L_x(i)$. Since $R_x(j) > k$, we have $y(j, k) \leq 0$ (equality holds only if $j = k$). So, $y(j, i) = y(j, k) + y(k, i) < 0$. Therefore, for any $L_x(i) \leq m \leq i$, $y(j, m) = y(j, i) - y(m, i) < 0$, since $y(m, i) \geq 0$. So, $R_x(j) > i$ and the claim is proved. \square

Lemma 25 (Interval property). *Let $x = (x_n)_{n \in \mathbb{Z}}$ be an integer-valued sequence with $x_n \geq -1$, $\forall n \in \mathbb{Z}$, and R be the record vertex-shift on the network (\mathbb{Z}, x) . If $i \in \mathbb{Z}$ satisfies $R_x(i) > i$, then the set of descendants of $R_x(i)$ contains $\{j : i \leq j < R_x(i)\}$. There could be more descendants of $R_x(i)$ that are less than i .*

Proof. By Lemma 24, it is enough to show that $L_x(R_x(i)) \leq i$. For any integer m such that $i < m < R_x(i)$, since $y(i, m) < 0$ and $y(i, R_x(i)) \geq 0$, we have

$$y(m, R_x(i)) = y(i, R_x(i)) - y(i, m) \geq 0.$$

Thus, $L_x(R_x(i)) \leq i$. □

Remark 26. Although, Lemma 24 and Lemma 25 are proved under the assumption that $x = (x_n)_{n \in \mathbb{Z}}$ is an integer-valued sequence satisfying the condition $x_n \geq -1 \forall n \in \mathbb{Z}$, the condition does not play any role in the proofs of these two lemmas. So the statements are indeed valid for any integer-valued sequence.

Properties of the record map on skip-free to the left random walk

The following two lemmas focus on computing the probability of certain events of skip-free random walks. They are used in the later sections for describing distribution of the record graph associated to skip-free random walks. Although, these computations can be found in [18], we prove them here for the sake of completeness and notational consistency.

We use the following notation in this and the remaining sections thereafter. Let $X = (X_n)_{n \in \mathbb{Z}}$ be i.i.d. increments of a skip-free to the left random walk, and $[\mathbb{Z}, 0, X]$ be its associated network. Let $(S_n)_{n \in \mathbb{Z}}$ be the random walk associated to the increment sequence X , i.e., $S_0 = 0$, $S_n = S_0 + \sum_{i=0}^{n-1} X_i$ for $n \geq 1$ and $S_n = S_0 + \sum_{i=n}^{-1} -X_i$ for $n \leq -1$. Clearly, $(S_n)_{n \geq 0}$ is a skip-free to the left random walk, and $(S_{-n})_{n \geq 0}$ is a skip-free to the right random walk. For any $j \in \mathbb{Z}$, let

$$\eta_j := \begin{cases} \inf\{n \geq 0 : S_n = j\} & \text{if } S_n = j \text{ for some } n \geq 0 \\ \infty & \text{otherwise.} \end{cases} \quad (3.1.3)$$

For an event A , let $\mathbb{P}_k[A]$ denote the probability of the event A when the random walk starts at $k \in \mathbb{Z}$. Let $\mathbb{P} := \mathbb{P}_0$, and $c := \mathbb{P}[\eta_{-1} < \infty]$.

Lemma 27. For $0 \leq j \leq k$, $\mathbb{P}[\eta_{j-k} < \infty] = \mathbb{P}_k[\eta_j < \infty] = c^{k-j}$.

Proof. The first equality (in the statement) is obvious since the probability is invariant to shift of the starting point of the random walk. The second

equality follows because of the skip free property which prohibits the random walk from taking jumps smaller than -1 . We prove it by induction on $k-j$. If $j = k$, then $\mathbb{P}_j[\eta_j < \infty] = 1$. So, assume that $0 \leq j < k$. By induction, we have $\mathbb{P}_{k'}[\eta_{j'} < \infty] = c^{k'-j'}$ for all $0 \leq j' \leq k'$ such that $k' - j' < k - j$. Then,

$$\mathbb{P}_k[\eta_j < \infty] = \mathbb{P}_k[\eta_{k-1} < \infty] \mathbb{P}_{k-1}[\eta_j < \infty] = \mathbb{P}[\eta_{-1} < \infty] c^{k-1-j} = c^{k-j}.$$

The second equality follows by applying the inductive statement to $k' = k-1$, and $j' = j$. \square

Let $\tau := \inf\{n > 0 : S_n \geq 0\}$, with $\tau = \infty$ if $S_n < 0 \quad \forall n > 0$, be the weak upper record (epoch), S_τ be the weak upper record height, and $X_{\tau-1}$ be its last increment (both are defined to be arbitrary when $\tau = \infty$).

Lemma 28 ([8]). *For all integers j, k such that $0 \leq j \leq k$,*

$$\mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] = \mathbb{P}[X_0 = k] c^{k-j}.$$

Proof. Note the equality of the following events:

$$\{S_\tau = X_{\tau-1}, \tau < \infty\} = \{\tau = 1\} = \{S_1 \geq 0\}.$$

Therefore, for $0 \leq j = k$, we have $\mathbb{P}[S_\tau = j, X_{\tau-1} = j, \tau < \infty] = \mathbb{P}[S_1 = j]$.

So, for $0 \leq j < k$,

$$\begin{aligned} \mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] &= \sum_{n=2}^{\infty} \mathbb{P}[\tau = n, S_n = j, X_{n-1} = k] \\ &= \sum_{n=2}^{\infty} \mathbb{P}[S_1 < 0, \dots, S_{n-1} < 0, S_n = j, S_n - S_{n-1} = k]. \end{aligned}$$

The sum in the first equation starts from $n = 2$ since we assumed that $j < k$ ($\{\tau = 1\}$ occurs if and only $j = k$ which is already covered). By the duality principle (also known as reflection principle), the last expression is equal to

$$\sum_{n=2}^{\infty} \mathbb{P}[S_n - S_{n-1} < 0, S_n - S_{n-2} < 0, \dots, S_n - S_1 < 0, S_n = j, S_n - (S_n - S_1) = k].$$

So, we obtain:

$$\begin{aligned}
 \mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] &= \sum_{n=2}^{\infty} \mathbb{P}[S_n = j, S_{n-1} > j, S_{n-2} > j, \dots, S_1 > j, S_1 = k] \\
 &= \sum_{n=2}^{\infty} \mathbb{P}[S_1 = k] \mathbb{P}_k[S_1 > j, \dots, S_{n-2} > j, S_{n-1} = j] \\
 &= \mathbb{P}[S_1 = k] \sum_{n=2}^{\infty} \mathbb{P}_k[\eta_j = n - 1] \\
 &= \mathbb{P}[S_1 = k] \mathbb{P}_k[0 < \eta_j < \infty] \\
 &= \mathbb{P}[S_1 = k] c^{k-j} = \mathbb{P}[X_0 = k] c^{k-j}.
 \end{aligned}$$

The second equation follows by the Markov property. The second to last equation follows from Lemma 27 and the fact that $\mathbb{P}[\eta_{j-k} = 0] = \delta_j(k)$, where δ is the Dirac function. \square

Remark 29. If the random walk has positive drift, i.e., $0 < \mathbb{E}X_0 < \infty$, then $S_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. So, $\tau < \infty$ a.s.. Therefore, $\mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] = \mathbb{P}[S_\tau = j, X_{\tau-1} = k]$ for $0 \leq j \leq k$.

If the random walk has negative drift, i.e., $\mathbb{E}[X_0] < 0$, then $S_n \rightarrow -\infty$ a.s. as $n \rightarrow \infty$. Therefore, $c = 1$, which implies that $\mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] = \mathbb{P}[X_0 = k]$, and

$$\mathbb{P}[\tau < \infty, X_{\tau-1} = k] = \sum_{j=0}^k \mathbb{P}[\tau < \infty, S_\tau = j, X_{\tau-1} = k] = (k+1)\mathbb{P}[X_0 = k].$$

Let us assume that $\mathbb{E}[X_0] > 0$. The following lemma shows that the random walk conditioned to hit -1 is also a random walk with a different increment distribution. Recall that η_{-1} is the first time the random walk $(S_n)_{n \geq 0}$ hits -1 (see Eq. (3.1.3)). Let \bar{S} has the conditional distribution of $(S_{n \wedge \eta_{-1}})_{n \geq 0}$ given that $\eta_{-1} < \infty$. Let $p_k := \mathbb{P}[X_0 = k]$, $\forall k \geq -1$. Observe that $\sum_{k=-1}^{\infty} p_k c^{k+1} = c$, since the sum on the left can be written as

$$\begin{aligned}
 \sum_{k=-1}^{\infty} p_k c^{k+1} &= \sum_{k=-1}^{\infty} \mathbb{P}[X_0 = k] c^{k+1} \\
 &= \sum_{k=-1}^{\infty} \mathbb{P}[X_0 = k] \mathbb{P}_k[\eta_{-1} < \infty] \\
 &= \mathbb{P}_0[\eta_{-1} < \infty] = c.
 \end{aligned}$$

Therefore, we have

$$\sum_{k=-1}^{\infty} p_k c^k = 1. \quad (3.1.4)$$

Let $\hat{X} = (\hat{X}_n)_{n \geq 0}$ be i.i.d. sequence of a skip-free to the left random walk whose common distribution is given by:

$$\mathbb{P}[\hat{X}_0 = k] = \mathbb{P}[X_0 = k]c^k = p_k c^k, \quad \forall k \geq -1, \quad (3.1.5)$$

which is a valid distribution by Eq. (3.1.4). Let $(\hat{S}_n)_{n \geq 0}$ be the random walk whose increment sequence is \hat{X} , i.e., $\hat{S}_n = \sum_{k=0}^{n-1} \hat{X}_k$ for all $k \geq 0$, and $\hat{S} = (\hat{S}_{n \wedge \hat{\eta}_{-1}})_{n \geq 0}$ be the stopped random walk.

Remark 30. In fact, the process $(\hat{S}_n)_{n \geq 0}$ is the Doob h -transform of p , where p is the distribution of X_0 and h is the harmonic function defined on $\mathbb{Z}_{\geq 0}$ by $h(i) = \mathbb{P}_i[\eta_{-1} < \infty]$ for all $i \geq 0$.

Lemma 31. *Let $\mathbb{E}[X_0] > 0$. Then the process \bar{S} has the same distribution as the stopped skip-free to the left random walk \hat{S} .*

Proof. Let (x_0, x_1, \dots, x_n) be the first $n + 1$ increments of a sample of the process with $x_k \geq -1, \forall k \leq n, x_1 + \dots + x_m \geq 0, \forall m < n$, and $x_1 + \dots + x_n \geq -1$. Then, for all such n and x ,

$$\begin{aligned} & \mathbb{P}[S_0 = 0, S_1 = x_1, S_2 - S_1 = x_2, \dots, S_{(n \wedge \eta_{-1})} - S_{(n \wedge \eta_{-1}-1)} = x_n, \eta_{-1} \geq n | \eta_{-1} < \infty] \\ &= \frac{\mathbb{P}[S_1 = x_1] \mathbb{P}_{x_1}[S_1 = x_2, S_2 - S_1 = x_3, \dots, S_{n-1} - S_{n-2} = x_n, \eta_{-1} < \infty]}{\mathbb{P}[\eta_{-1} < \infty]} \\ &= \frac{(\prod_{i=1}^n \mathbb{P}[S_1 = x_i]) \mathbb{P}_{x_1 + \dots + x_n}[\eta_{-1} < \infty]}{\mathbb{P}[\eta_{-1} < \infty]} = \frac{(\prod_{i=1}^n \mathbb{P}[S_1 = x_i]) c^{x_1 + x_2 + \dots + x_{n+1}}}{c} \\ &= \prod_{i=1}^n (\mathbb{P}[X_0 = x_i] c^{x_i}) \\ &= \mathbb{P}[\hat{S}_1 = x_1, \hat{S}_2 - \hat{S}_1 = x_2, \dots, \hat{S}_{(n \wedge \hat{\eta}_{-1})} - \hat{S}_{(n \wedge \hat{\eta}_{-1}-1)} = x_n, \hat{\eta}_{-1} \geq n]. \end{aligned}$$

Thus, the conditioned random walk is a stopped random walk, stopped at the stopping time $\hat{\eta}_{-1}$, whose increments are distributed as $\mathbb{P}[S_1 = j]c^j, \forall j \geq -1$. \square

We now show that the random walk $(\hat{S}_n)_{n \geq 0}$ has negative drift.

Lemma 32. *Assume $\mathbb{E}[X_0] > 0$. Then the mean of \hat{X}_0 is negative.*

Proof. Consider the function $\phi(x) = \sum_{k=0}^{\infty} (k+1)x^k p_k - 1$ on the interval $[0, 1]$. We have

$$\begin{aligned} \phi(c) &= \sum_{k=0}^{\infty} (k+1)c^k p_k - 1 = \sum_{k=0}^{\infty} k c^k p_k + \left(\sum_{k=0}^{\infty} c^k p_k \right) - 1 \\ &= \sum_{k=0}^{\infty} k c^k p_k + 1 - \frac{p-1}{c} - 1 = \sum_{k=-1}^{\infty} k c^k p_k = \mathbb{E}[\hat{X}]. \end{aligned}$$

The function ϕ has the following properties on $[0, 1]$:

- $\phi(0) < 0$: Since $\phi(0) = p_0 - 1 < 0$.
- $\phi(1) > 0$: Since $\phi(1) = \sum_{k=0}^{\infty} (k+1)p_k - 1 = \sum_{k=0}^{\infty} k p_k + 1 - p_{-1} - 1 = \mathbb{E}[X] > 0$.
- ϕ is strictly increasing since $\phi'(x) > 0, \forall x \in (0, 1)$.

Therefore, there exists a unique $c' \in (0, 1)$ such that $\phi(c') = 0$. The proof is complete if we show that $c < c'$, as this implies that $\phi(c) < \phi(c') = 0$ by monotonicity of ϕ , and the fact that $\phi(c) = \mathbb{E}[\hat{X}]$.

We now show that $c < c'$. Consider the function $\psi(x) = \sum_{k=0}^{\infty} x^{k+1} p_k + p_{-1} - x$ on the interval $[0, 1]$. Observe that $\psi'(x) = \sum_{k=0}^{\infty} (k+1)x^k p_k - 1 = \phi(x)$,

$$\psi(c) = \sum_{k=0}^{\infty} c^{k+1} p_k + p_{-1} - c = c \left(\sum_{k=0}^{\infty} c^k p_k + \frac{p-1}{c} - 1 \right) = 0,$$

and $\psi(1) = \sum_{k=0}^{\infty} p_k + p_{-1} - 1 = 0$. As $\psi(c) = \psi(1) = 0$, there exists a $d \in (c, 1)$ such that $\psi'(d) = 0$. Since $\phi(d) = \psi'(d) = 0$, we have $d = c'$. Thus, $c < c'$. \square

Relation between RLS order and the order on integers

Let $X = (X_n)_{n \in \mathbb{Z}}$ be the i.i.d. increments of a skip-free to the left random walk (see Def. 22) with $\mathbb{E}[X_0] = 0$ and let $[\mathbb{Z}, 0, X]$ be its associated network. Let $[\mathbf{T}, 0]$ denote the connected component of 0 (rooted at 0) in the R -graph (see notations of Chapter 2) of the network $[\mathbb{Z}, 0, X]$. Since the vertices of \mathbf{T} are integers, we can order the children of every vertex using the total order $<$ of \mathbb{Z} . Then, the RLS order $<$ gives a total order on the vertices of \mathbf{T} . Thus, we have two total orders $<$ and $<$ on the vertices of \mathbf{T} . The former

is deterministic whereas the latter is random (it depends on the realization of X). Using the interval property (Lemma 25) of the record vertex-shift, we show in the following lemma that these two orders are in fact equivalent in the following sense:

Lemma 33. *Let $x = (x_n)_{n \in \mathbb{Z}}$ be a real-valued sequence, $[\mathbb{Z}, 0, x]$ be its associated network. Let i, j be two integers having a smallest common ancestor w , with $w = R^m(i) = R^k(j)$, where m and k are the smallest non-negative integers satisfying this property. Then, $i < j$ (with respect to the order on \mathbb{Z}) if and only if $i < j$.*

Proof. ($i < j \implies i < j$): Since $R^n(i)$ is an increasing sequence for $0 < n < m$, there exists a largest $n_0 \leq m$ such that $R^{n_0}(i) \leq j$. If $R^{n_0}(i) = j$, then $i < j$, which is the desired relation. So, let us assume that $R^{n_0}(i) < j$ and $R^{n_0+1}(i) > j$. By Lemma 25, both $R^{n_0}(i)$ and j are descendants of $R^{n_0+1}(i)$, which implies that $R^{n_0+1}(i) = w$. Hence, $R^{m_0}(j) = R^{n_0+1}(i) = w$ for some $m_0 > 0$. Since $R^{n_0}(i) < j \leq R^{m_0-1}(j)$, we have $R^{n_0}(i) < R^{m_0-1}(j)$, and by Remark 5, we have $i < j$.

($i < j \implies i < j$): The case where j is an ancestor of i is trivial. So, let us assume that j is not an ancestor of i . Then we have $k > 0$, and $R^{m-1}(i) < R^{k-1}(j)$. Since $R^{m-1}(i)$ is not a descendant of $R^{k-1}(j)$, but j is a descendant of $R^{k-1}(j)$, by Lemma 24, we have $i < R^{m-1}(i) < L_x(R^{k-1}(j)) \leq j$. \square

Relation between offspring count and increment in the record graph

Let T be an ordered Family Tree, $o \in V(T)$, and $u_n < u_{n-1} < \dots < u_1$ be the children of o (totally ordered by $<$). Define the order of any child u_j among its siblings as j . That is, u_j is the j -th child of o . Recall the notation $y(j, k) := \sum_{l=j}^{k-1} x_l$ for integers $j < k$ and for a given sequence $x = (x_n)_{n \in \mathbb{Z}}$.

The skip-free property of the increments of the random walk gives us the following lemma.

Lemma 34. *Let $x = (x_k)_{k \in \mathbb{Z}}$ be a sequence where $x_k \in \mathbb{N} \cup \{0, -1\}$. Let $[\mathbb{Z}, 0, x]$ be the associated network of x , and T be the R -graph of $[\mathbb{Z}, 0, x]$. Let i be an integer such that $L_x(i) > -\infty$. Then,*

1. *The number of children $d_1(i)$ of i in T is given by $d_1(i) = x_{i-1} + 1$.*

2. Let $d_1(i) = n > 0$ and $i_n < i_{n-1} < \dots < i_1$ be the positions on \mathbb{Z} of the n children of i . Then, the position i_m of the m -th child satisfies the relation $m = x_{i-1} + 1 - y(i_m, i)$.
3. Let $d_1(i) = n > 0$, and $m \in \{1, 2, \dots, n\}$. Then, an integer $i' < i$ is the m -th child of i if and only if i' is the largest integer among $\{L_x(i), \dots, i-1\}$ that satisfies $y(i', i) = x_{i-1} + 1 - m$.

Proof. If $x_{i-1} = -1$, then $L_x(i) = i$. So, by Lemma 24, i has no descendant. Then, (1.) holds as $d_1(i) = x_{i-1} + 1 = 0$, whereas the other statements are empty.

So, let us assume that $x_{i-1} \geq 0$. Clearly $i-1$ is then a child of i as i is the record of $i-1$. Therefore, $d_1(i) \geq 1$. We show below that the sum of increments between any two successive children is -1 .

Let $i_n < i_{n-1} < \dots < i_1$ be the positions on \mathbb{Z} of the children of i , with $i_1 = i-1$. Consider a child i_m with $m < n$. If $i_{m+1} = i_m - 1$, then $x_{i_{m+1}} = -1$. Therefore, $y(i_{m+1}, i_m) = -1$. Consider now the case where $i_{m+1} < i_m - 1$, and let i' be an integer with $i_{m+1} < i' < i_m$. As i' is not a child of i , and i_m is a child of i , by Lemma 25, we have $R_x(i') \leq i_m$, and there exists a smallest $k > 0$ such that $R_x^k(i') = i_m$. Therefore, $y(i', i_m) = \sum_{l=0}^{k-1} y(R_x^l(i'), R_x^{l+1}(i')) \geq 0$.

In particular, $y(i_{m+1} + 1, i_m) \geq 0$. But, $y(i_{m+1}, i_m) < 0$, because the record of i_{m+1} is i (not i_m). Therefore, the only possibility is that $x_{i_{m+1}} = -1$. This also implies that $y(i_{m+1} + 1, i_m) = 0$. Indeed, if $y(i_{m+1} + 1, i_m) \geq 1$, then $y(i_{m+1}, i_m) = x_{i_{m+1}} + y(i_{m+1} + 1, i_m) = -1 + y(i_{m+1} + 1, i_m) \geq 0$, which gives a contradiction that i_m is the record of i_{m+1} . Hence, $y(i_{m+1}, i_m) = x_{i_{m+1}} + y(i_{m+1} + 1, i_m) = -1$. Thus, we have shown that the sum of increments between two successive children is -1 . We now use this to prove the three statements of the lemma.

The proof of the second statement follows because, for any child i_m of i , we have

$$y(i_m, i) = y(i_m, i_1) + x_{i-1} = y(i_m, i_{m-1}) + \dots + y(i_2, i_1) + x_{i-1} = -(m-1) + x_{i-1}. \quad (3.1.6)$$

We now prove the third statement. Since i_m is a child of i , $y(i_m, i) \geq 0$, $\forall 1 \leq m \leq n$. Further, for any $i_m < i' \leq i-1$, we have $y(i', i) = y(i', i_j) + y(i_j, i)$, with $i_j \geq i'$ being the unique smallest integer which is a child of i . Such an i_j always exists since $i-1$ is a child of i . Moreover, we have $j < m$. Since

$y(i', i_j) \geq 0$, we have

$$y(i', i) = y(i', i_j) + x_{i-1} + 1 - j \geq x_{i-1} + 1 - j > x_{i-1} + 1 - m,$$

where we used Eq. (3.1.6) to get the first equality. This proves the forward implication of the third statement: if h is the m -th child of i then $h < i$ is the largest integer that satisfies $y(h, i) = x_{i-1} + 1 - m$.

As for the backward implication of the third statement, if $i' < i$ is the largest integer satisfying $y(i', i) = x_{i-1} + 1 - m \geq 0$ for some $m \in \{1, 2, \dots, n\}$, then $y(i', j) = y(i', i) - y(j, i) < 0$, for all $i' < j \leq i - 1$. So, i' is a child of i . But, by Eq. (3.1.6), there is only one child that satisfies this condition, namely, the m -th child of i . This proves the third statement.

We now show that $y(L_x(i), i) = 0$. Indeed, from the definition of $L_x(i)$, it follows that $y(L_x(i) - 1, i) = x_{L_x(i)-1} + y(L_x(i), i) < 0$. But $y(L_x(i), i) \geq 0$, implying that $x_{L_x(i)-1} = -1$ (the only possibility for $x_{L_x(i)-1}$). Hence, $y(L_x(i), i) = 0$. This implies that $i_n = L_x(i)$ since no integer smaller than $L_x(i)$ can be a descendant of i by Lemma 24. To prove the first statement, observe that

$$y(i_n, i) = x_{i-1} + \sum_{k=1}^{n-1} y(i_k, i_{k+1}) = x_{i-1} - (n - 1).$$

Therefore, $y(i_n, i) = 0 = x_{i-1} + 1 - n$ which gives $d_1(i) = n = x_{i-1} + 1$, proving the first statement. \square

Remark 35. From the last part of the proof of Lemma 34, it follows that, if $L_x(i) > -\infty$, then $L_x(i)$ is the smallest child of i .

The type function and its properties

Let $x = (x_n)_{n \in \mathbb{Z}}$ be an integer-valued sequence with $x_n \geq -1$ for all $n \in \mathbb{Z}$, $[\mathbb{Z}, 0, x]$ be the rooted network associated to x , and T be the record graph of the network (\mathbb{Z}, x) . For any integer i , denote the number of children of i in T by $d_1(i, T)$.

The **type function** associated to x is the map $t : \mathbb{Z} \rightarrow \mathbb{Z}$ that assigns to each integer $i \in \mathbb{Z}$ the integer $t(i) \in \{-1, 0, 1, \dots\}$ defined as

$$t(i) := t_x(i) = \inf\{n \geq -1 : y(m, i) = n \text{ for some } m < i\}. \quad (3.1.7)$$

The integer $t(i)$ is called the **type** of i . Note that the type function is a deterministic function of x . The type $t_x(i)$ of an integer i can also be written

$$t_x(i) = \inf\{y(m, i) \vee -1 : m < i\}, \quad (3.1.8)$$

where \vee is the max function.

The type function is analogous to the Loynes construction [5].

We show, in the later sections, that when $\mathbb{E}[X_0] > 0$ the record graph together with the type function encodes the trajectory of the random walk. This allows one to move back and forth between a trajectory of the random walk and its record graph together with the type function.

Remark 36. Let X be the i.i.d. increments of a skip-free to the left random walk. Then, for any integer $i \in \mathbb{Z}$, the sequence $S^{(i)} = (S_m^{(i)})_{m \geq 0} := (\sum_{k=i-m}^{i-1} X_k)_{m \geq 0}$ is a skip-free to the left random walk. So, it can only take negative steps of size 1 at a time (i.e., $S_m^{(i)} < 0$ for some $m \geq 1$ if and only if $S_j^{(i)} = -1$ for some $j \leq m$). The type of vertex i is -1 if and only if $S_m^{(i)} = -1$ for some $m \geq 1$, and it is equal to $n \geq 0$ if and only if $\min_{m \geq 1} \{S_m^{(i)}\} = n$, i.e., the random walk $(S_m^{(i)})_{m \geq 1}$ reaches n but stays above $n - 1$. In particular, $t(i)$ is well-defined for all $i \in \mathbb{Z}$.

Define the function $l_x : \mathbb{Z} \rightarrow \mathbb{Z}$ that maps an integer i to

$$l(i) := l_x(i) = \sup\{m < i : y(m, i) = t_x(i)\}. \quad (3.1.9)$$

Recall the function L defined in Eq. (3.1.2). This function is related to the function l in the following way. First note that for every $i \in \mathbb{Z}$, we have $-\infty < l(i) < i$ since $t_x(i) = \min\{y(m, i) \vee -1 : m < i\}$ and $l(i)$ is the largest m for which $y(m, i)$ attains $t_x(i)$, when m is varied in $\mathbb{Z}_{<i}$. If $t_x(i) = -1$, then $l(i) = L(i) - 1$. For any integer i , $t_x(i) \geq 0$ if and only if $L(i) = -\infty$.

Lemma 37. *For any integer i , we have the following dichotomy:*

- if $t_x(i) = -1$, then $L(i)$ is the smallest child of i in T ,
- if $t_x(i) \geq 0$, then $l(i)$ is the smallest child of i in T .

Proof. If $t_x(i) = -1$, then $L(i) > -\infty$. So, by Remark 35, $L(i)$ is the smallest child of i in T .

So, let us assume that $t_x(i) \geq 0$. In this case, i is the record of $l(i)$, i.e., $i = R(l(i))$. This follows because, for any $l(i) < m < i$, we have $y(m, i) >$

$t_x(i)$ by the definition of $t_x(i)$ and $l(i)$. Since $y(l(i), i) = t_x(i)$, we have $y(l(i), m) = y(l(i), i) - y(m, i) < 0$. Thus, $R(l(i)) = i$. Further, if $t_x(i) \geq 0$, then $y(m, l(i)) \geq 0$ for all $m < l(i)$. Indeed, for $m < l(i)$, since $y(m, i) \geq t_x(i)$, we have $y(m, l(i)) = y(m, i) - y(l(i), i) \geq t_x(i) - t_x(i) = 0$. So, $R(m) \leq l(i)$ for all $m < l(i)$. This implies that $l(i)$ is the smallest among the children of i as none of the integers smaller than $l(i)$ are children of i . \square

The following lemma describes the relation between the number of children of any integer i , its associated increment x_i and its type $t_x(i)$ when $t_x(i) \geq 0$. When $t_x(i) = -1$, the type does not play any role in determining this relation.

Lemma 38. *Let $i \in \mathbb{Z}$ be an integer. We have the following dichotomy:*

- If $t_x(i) = -1$, then the number of children of i in T is given by $d_1(i, T) = x_{i-1} + 1$.
- If $t_x(i) \geq 0$, then the number of children of i in T is given by $d_1(i, T) = x_{i-1} + 1 - t_x(i)$.

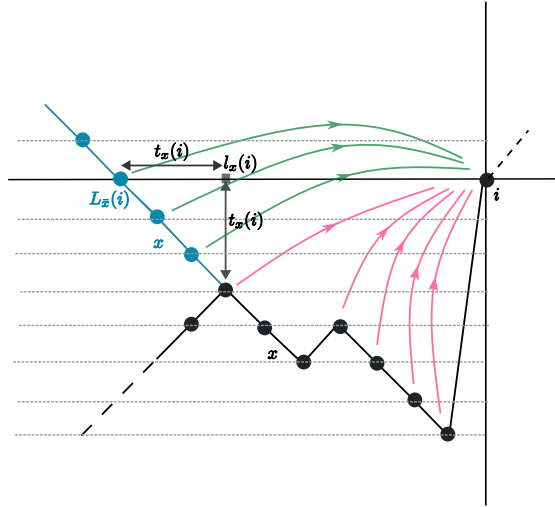


Figure 3.1: The trajectories of $x = (x_n)_{n \in \mathbb{Z}}$ (in black) and $\bar{x} = (\bar{x}_n)_{n \in \mathbb{Z}}$ (in blue). The children of i in T are drawn in red, and the additional children of i in \bar{T} are drawn in green. See Lemma 38.

Proof. If $t_x(i) = -1$, then $L_x(i) > -\infty$ (see the paragraph above Lemma 37), which is the condition needed to apply Lemma 34. The first part of the statement follows from part 1 of Lemma 34.

We now prove the second part of the statement. To follow the proof, see Fig. 3.1. Define a new sequence $\bar{x} = (\bar{x}_n)_{n \in \mathbb{Z}}$ by

$$\bar{x}_n = \begin{cases} -1, & \forall n < l_x(i) \\ x_n, & \forall n \geq l_x(i). \end{cases}$$

Let $\bar{y}(j, k) = \sum_{l=j}^{k-1} \bar{x}_l$ for all integers $j < k$. Then, $y(j, k) = \bar{y}(j, k)$ for all $l_x(i) \leq j < k$. Thus, we have

$$\bar{y}(k, i) = y(k, i) \geq t_x(i) > 0 \quad \forall l_x(i) \leq k < i, \quad (3.1.10)$$

and

$$\bar{y}(k, i) = k - l_x(i) + t_x(i) \quad \forall k < l_x(i).$$

Therefore, (see Eq.(3.1.2) for the definition of L)

$$L_{\bar{x}}(i) = l_x(i) - t_x(i) > -\infty. \quad (3.1.11)$$

Let \bar{T} denote the record graph of the network (\mathbb{Z}, \bar{x}) , $D_1(i, \bar{T})$ denote the set of children of i in \bar{T} and let $d_1(i, \bar{T}) = \#D_1(i, \bar{T})$. Then, by the first part of Lemma 34, we obtain that $d_1(i, \bar{T}) = \bar{x}_{i-1} + 1$. Since $l_x(i) < i$, we have $x_{i-1} = \bar{x}_{i-1}$ which implies that $d_1(i, \bar{T}) = x_{i-1} + 1$.

Note that $\bar{y}(l_x(i), i) = t_x(i)$ and $\bar{y}(j, i) > t_x(i)$ for all $l_x(i) < j < i$. Therefore, by the third part of Lemma 34, $l_x(i)$ is the $x_{i-1} + 1 - t_x(i)$ -th child of i in \bar{T} . Since $\bar{x}_j = -1$ for all $j < l_x(i)$, by (3) of Lemma 34, we obtain that

$$\{j \in \mathbb{Z} : L_{\bar{x}}(i) \leq j < l_x(i)\} \subseteq D_1(i, \bar{T}).$$

By Eq. (3.1.11), $\#\{j \in \mathbb{Z} : L_{\bar{x}}(i) \leq j < l_x(i)\} = l_x(i) - L_{\bar{x}}(i) = t_x(i)$. Observe that $D_1(i, \bar{T}) \cap \{j \in \mathbb{Z} : l_x(i) \leq j < i\} = D_1(i, T)$ by Eq. (3.1.10). Finally, Lemma 37 implies that $l_x(i)$ is the smallest child of i in T . Thus, we have

$$D_1(i, T) = D_1(i, \bar{T}) \setminus \{j \in \mathbb{Z} : L_{\bar{x}} \leq j < l_x(i)\},$$

which gives

$$d_1(i, T) = d_1(i, \bar{T}) - t_x(i) = x_{i-1} + 1 - t_x(i).$$

□

3.2 Classification theorem and phase transitions

Recall the foil classification theorem 14. It states that a.s. any unimodular Family Tree belongs to either class \mathcal{F}/\mathcal{F} or class \mathcal{I}/\mathcal{F} or class \mathcal{I}/\mathcal{I} . These three classes are characterized by the following distinctive properties. Every Family Tree of class \mathcal{F}/\mathcal{F} has finitely many vertices. Every Family Tree of class \mathcal{I}/\mathcal{I} is eternal, one-ended and all of its vertices have finitely many descendants. In contrast, every Family Tree of class \mathcal{I}/\mathcal{F} is eternal, two ended, and all the vertices that lie on its unique bi-infinite path have infinitely many descendants and all the remaining vertices have finitely many descendants. In addition, if a unimodular Family Tree is of class \mathcal{I}/\mathcal{F} a.s., then it is obtained by the operation called typically rooted joining of a stationary sequence of finite Family Trees.

In Subsection 3.2, we will use the typically rooted joining construction to define a unimodular Family Tree of class \mathcal{I}/\mathcal{F} .

We first describe the class (according to the classification theorem) of the connected component of 0 in the R -graph of $[\mathbb{Z}, 0, X]$, where $X = (X_n)_{n \in \mathbb{Z}}$ are the i.i.d. increments of a skip-free to the left random walk (see Def. 22). Let $(S_n)_{n \in \mathbb{Z}}$ be the two-sided random walk associated to X , where $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} X_k$ for $n > 0$, and $S_n = \sum_{k=n}^{-1} -X_k$ for $n < 0$. Let \mathbb{Z}^R be the record graph of (\mathbb{Z}, X) and $\mathbf{T} = \mathbb{Z}^R(0)$ be the connected component of 0 in \mathbb{Z}^R . Since $[\mathbb{Z}, 0, X]$ is unimodular, $[\mathbf{T}, 0]$ is a unimodular Family Tree (see Lemma 16 taken from [7]).

The phase transitions occur not only for the i.i.d. sequence of random variables that have the support in $\mathbb{Z}_{\geq -1}$, but also for more general stationary and ergodic integer-valued sequence of random variables (See Theorem 119). In the i.i.d. case, the component of 0 in the record graph exhibits three phases (\mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{I} and \mathcal{I}/\mathcal{F} resp.) corresponding to the conditions $\mathbb{E}[X_0] < 0$, $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X_0] > 0$ respectively. But for more general stationary and ergodic case, one can construct examples for which there are only two phases: \mathcal{F}/\mathcal{F} when $\mathbb{E}[X_0] < 0$ and \mathcal{I}/\mathcal{F} when $\mathbb{E}[X_0] \geq 0$ (see Example 120 and Theorem 119).

Proposition 39. *If $\mathbb{E}[X_0] < 0$, then the unimodular Family Tree $[\mathbf{T}, 0]$ is of class \mathcal{F}/\mathcal{F} .*

Proof. Since $\mathbb{E}[X_0] < 0$, the random walk $(S_n)_{n \geq 0}$ drifts to $-\infty$, i.e., $S_n \rightarrow$

$-\infty$ a.s.. So, a.s., 0 has finitely many ancestors in \mathbf{T} . However, the random walk $(S_{-n})_{n \geq 0}$ drifts to ∞ . So, $L_X(0) > -\infty$. Therefore, by Lemma 24, 0 has finitely many descendants. Thus, $[\mathbf{T}, 0]$ is of class \mathcal{F}/\mathcal{F} . \square

Proposition 40. *If $\mathbb{E}[X_0] = 0$, then the record graph \mathbb{Z}^R is connected and the unimodular Family Tree $[\mathbf{T}, 0]$ is of class \mathcal{I}/\mathcal{I} .*

Proof. Since $\mathbb{E}[|X_n|] < \infty$, by the Chung-Fuchs theorem [15], the random walk starting at 0 with increments $(X_n)_{n \geq i}$ is recurrent for any $i \in \mathbb{Z}$. This implies that for any $i \in \mathbb{Z}$, almost surely, the sequence $(R_X^n(i))_{n \in \mathbb{N}}$ (of higher order record epochs of i) is strictly increasing, i.e., $R_X^n(i) \neq R_X^{n+1}(i), \forall n \geq 1$.

Let A_i be the event that the integer i is in the connected component of 0 in the R -graph. We claim that the event $A = \bigcap_{i \in \mathbb{Z}} A_i$ occurs with probability 1.

Let $i < 0$ (the proof for $i > 0$ is obtained by interchanging i and 0). Since the sequence $(R_X^n(i))_{n \in \mathbb{Z}}$ is strictly increasing a.s., there exists a smallest (random) integer $k > 0$ such that $R_X^{k-1}(i) < 0 \leq R_X^k(i)$ a.s.. By Lemma 25 (applied to $R_X^{k-1}(i)$), 0 is a descendant of $R_X^k(i)$ a.s.. Thus, a.s., 0 and i are in the same connected component. The claim follows since i is arbitrary.

The above argument also implies that, a.s., $[\mathbf{T}, 0]$ is an EFT, and Lemma 16 implies that it is unimodular. We prove that $[\mathbf{T}, 0]$ is of class \mathcal{I}/\mathcal{I} by showing that the descendant tree of every vertex of \mathbf{T} is finite. For any integer i , the random walk starting at 0 with increments $(-X_{(i-1-n)})_{n \geq 0}$ is recurrent by the Chung-Fuchs theorem. Therefore, $L_X(i) > -\infty$ a.s., implying that the set of descendants of i is a.s. finite by Part 1 of Lemma 34. This completes the proof. \square

Proposition 41. *If $0 < \mathbb{E}[X_0] < \infty$, then the record graph \mathbb{Z}^R is connected and the unimodular Family Tree $[\mathbf{T}, 0]$ is of class \mathcal{I}/\mathcal{F} .*

Proof. We show that there exists a vertex of \mathbf{T} that has infinitely many descendants, which implies that $[\mathbf{T}, 0]$ is of class \mathcal{I}/\mathcal{F} . Since the random walk has positive drift ($0 < \mathbb{E}[X_0] < \infty$), the positively indexed part of the random walk $(S_n)_{n \geq 0}$ drifts to $+\infty$ a.s.. So, a.s., the n -th record epoch of 0 is always larger than the $(n-1)$ -th record epoch of 0 for all $n \geq 1$, i.e., $R^n(0) > R^{n-1}(0), \forall n \geq 1$. This implies that 0 has an infinite number of ancestors. The negatively indexed part $(S_{-n})_{n \geq 0}$ drifts to $-\infty$ a.s.. In particular, a.s., the random variable $m_{-1} := \sup_{n \geq 1} \{S_{-n}\}$ is finite and the random non-positive integer $u^* := -\min\{n \geq 1 : S_{-n} = m_{-1}\}$ exists. Since

$R(u^*) \geq 0$, there exists the smallest ancestor $u \in \mathbb{Z}$ of 0 such that $R(u^*) = u$. Since $S_n \leq m_{-1}$ for all $n \leq u^*$, u^* has infinitely many descendants by Lemma 24. Therefore, by the interval property (Lemma 25), it follows that the record graph \mathbb{Z}^R is connected. Further, by the classification theorem of unimodular Family Trees [7], $[\mathbf{T}, 0]$ is of class \mathcal{I}/\mathcal{F} . \square

Instances of unimodular Family Trees

In this subsection, we describe three instances of unimodular Family Trees, one from each of the classes \mathcal{F}/\mathcal{F} , \mathcal{I}/\mathcal{F} and \mathcal{I}/\mathcal{I} , which will be used in the next sections. The first instance is the Eternal Galton-Watson tree with offspring distribution π . It was studied in [7], in which it is shown that the tree is unimodular if and only if the mean of π is 1. It is of class \mathcal{I}/\mathcal{I} when π is non degenerate. We introduce two new unimodular Family Trees, namely, the typically rooted Galton-Watson tree; which is of class \mathcal{F}/\mathcal{F} ; and the unimodularised marked ECS ordered bi-variate Eternal Kesten tree, which is of class \mathcal{I}/\mathcal{F} . The former is shown to be obtained from Galton-Watson Tree by re-rooting the root to a typically chosen vertex of the Galton-Watson Tree.

Typically rooted Galton-Watson Tree (TGWT)

Let π be a probability distribution on $\{0, 1, 2, 3, \dots\}$ such that its mean $0 < m(\pi) < 1$ and $\hat{\pi}$ be the size-biased distribution of π , given by $\hat{\pi}(k) = \frac{k\pi(k)}{m(\pi)}$ for all $k \in \{0, 1, 2, \dots\}$. The **Typically rooted Galton-Watson Tree** ($TGWT(\pi)$) with offspring distribution π is a Family Tree $[\mathbf{T}, \mathbf{o}]$ defined in the following way.

- Add a parent to the root \mathbf{o} with probability $m(\pi)$. Independently iterate the same to the parent of \mathbf{o} and to all of its ancestors. So, the number of ancestors of the root \mathbf{o} has a geometric distribution with the success probability $1 - m(\pi)$.
- Let Z be a random variable that has distribution $\hat{\pi}$ (the size-biasing of π). To each of the ancestors of \mathbf{o} attach independently children with distribution the same as that of $Z - 1$.
- To each of these new children and to the root \mathbf{o} , attach independently ordered Galton-Watson trees with offspring distribution π .

See Figure 3.2 for an illustration.

Remark 42. The $TGWT(\pi)$ is a finite unimodular ordered Family Tree (see Proposition 52).

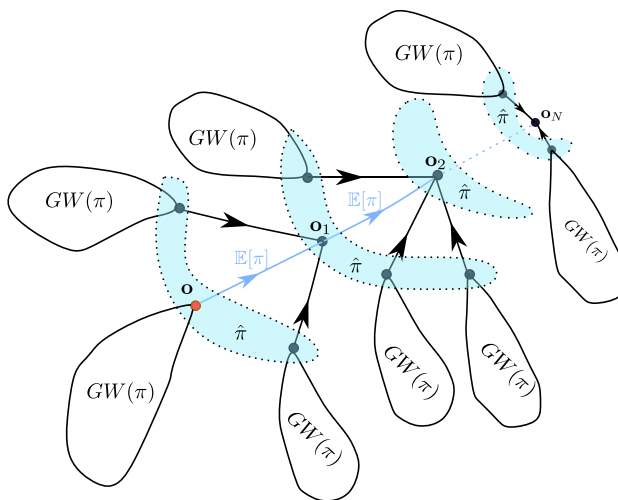


Figure 3.2: Typically rooted Galton-Watson Tree with offspring distribution π (\mathbf{o} is the root), $\mathbb{E}[\pi]$ is the probability that the root \mathbf{o} has a parent, $\hat{\pi}$ is the offspring distribution of the parent of \mathbf{o} , N is a geometric random variable with the success probability $1 - \mathbb{E}[\pi]$.

The nomenclature “typically rooted” in the typically rooted Galton-Watson Tree suggests that it is obtained by re-rooting to a typical vertex of a Galton-Watson tree. It is indeed true, we prove this fact in Proposition 50. We also give a characterizing condition for $TGWT$ in Proposition 52.

Bi-variate Eternal Kesten Tree

In the following, we give an instance of a unimodular tree of class \mathcal{I}/\mathcal{F} using the typically rooted joining operation. We first introduce a Family Tree which we call it the bi-variate Eternal Kesten Tree $EKT(\alpha, \beta)$ with offspring distributions α and β . It is unordered, unlabelled and parametrized by α and β . Using this Family Tree, we introduce a class of marked and ordered Family Trees namely marked ECS ordered bi-variate Eternal Kesten

Trees. None of these trees are unimodular, but we can obtain a unimodularised version from them using an operation called typically rooted joining. In order to perform this operation, it is essential that the mean of offspring distribution β satisfies $m(\beta) < 1$. But, to define $EKT(\alpha, \beta)$, it is sufficient that the mean of the offspring distributions satisfy $m(\alpha) < \infty$ and $m(\beta) \leq 1$.

Let α, β be two probability distributions on $\{0, 1, 2, \dots\}$ such that their means satisfy $m(\alpha) < \infty$ and $m(\beta) \leq 1$. A **bi-variate Eternal Kesten Tree** with offspring distributions α, β , denoted as $EKT(\alpha, \beta)$, is a random Family Tree $[\mathbf{T}', \mathbf{o}']$ consisting of a unique bi-infinite F -path $(\mathbf{o}_n)_{n \in \mathbb{Z}}$, where $\mathbf{o}_0 = \mathbf{o}'$, with the following property: the sequence of Family Trees $([D(\mathbf{o}_n) \setminus D(\mathbf{o}_{n-1}), \mathbf{o}_n])_{n \in \mathbb{Z}}$ is i.i.d. with the following common distribution: the offspring distribution of the root is α and the descendant trees of the children of the root are i.i.d. Galton-Watson trees with offspring distribution β (denoted as $GW(\beta)$). The Family Trees $\{[D(\mathbf{o}_n) \setminus D(\mathbf{o}_{n-1}), \mathbf{o}_n] : n \in \mathbb{Z}\}$ are called the bushes of $EKT(\alpha, \beta)$ and the Family Tree $[D(\mathbf{o}_0) \setminus D(\mathbf{o}_{-1}), \mathbf{o}_0]$ is called the **bush of the root**.

Note that $EKT(\alpha, \beta)$ is the joining of the i.i.d. sequence $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$, where $T_i = D(\mathbf{o}_i) \setminus D(\mathbf{o}_{i-1})$. Keep in mind that $d_1(\mathbf{o}_i, \mathbf{T}_i) = d_1(\mathbf{o}_i, \mathbf{T}') - 1$, where $d_1(\mathbf{o}_i, \mathbf{T}_i)$ denotes the offspring count of \mathbf{o}_i in \mathbf{T}_i .

In addition, suppose that $m(\beta) < 1$. Then, $GW(\beta)$ is subcritical, which implies that $\mathbb{E}[\#V(\mathbf{T}_0)] < \infty$. So, in this case, we can take the typically rooted joining of the i.i.d. Family Trees $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$. Let $[\mathbf{T}, \mathbf{o}]$ be the typically rooted joining of $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ (as in Eq. (2.3.4)). Then, by Theorem 19, the EFT $[\mathbf{T}, \mathbf{o}]$ is unimodular. We call the distribution of $[\mathbf{T}, \mathbf{o}]$ the unimodularised bi-variate $EKT(\alpha, \beta)$. The unimodularised $EKT(\alpha, \beta)$ is of class \mathcal{I}/\mathcal{F} .

An **Every Child Succeeding (ECS) order** on $[\mathbf{T}', \mathbf{o}']$ is obtained by declaring that \mathbf{o}_n is the smallest among $D_1(\mathbf{o}_{n+1})$ and using the uniform order on the remaining children for all $n \in \mathbb{Z}$. The name ‘‘ECS’’ order comes from the fact that the succession line (see Def. 87) starting from the root reaches all the vertices on the bi-infinite path (i.e., the succession line is an order preserving bijection from \mathbb{Z} to the vertices of the tree). The unimodularised ECS ordered $EKT(\alpha, \beta)$ is an example of a unimodular ordered EFT of class \mathcal{I}/\mathcal{F} . Later, we show that the unimodularised ECS ordered $EKT(\alpha, \beta)$ is the record graph of the network $[\mathbb{Z}, 0, X]$, where X is as in Def. 22 and satisfies $\mathbb{E}[X_0] > 0$.

We focus on the ordered Family Trees whose vertices have marks that take values in \mathbb{Z} , as will be shown in the later sections that they encode

information about the trajectory of a skip-free to the left random walk when the mean of the random walk is positive. Let $\hat{\mathcal{T}}_*$ denote the space of equivalence classes of rooted ordered Family Trees whose mark function assigns \mathbb{Z} -valued labels to its vertices. The space $\hat{\mathcal{T}}_*$ is Polish [3]. We give an instance of a unimodular marked ordered Family Tree of class \mathcal{I}/\mathcal{F} . It is important to note that the distribution of the marks need not be independent of the tree structure. For instance, marks can be a deterministic function of the structure of a tree.

Let α, β be probability distributions on $\{0, 1, 2, \dots\}$ such that their means satisfy $m(\alpha) < \infty$ and $m(\beta) < 1$. A **bi-variate ECS ordered marked Eternal Kesten Tree** $MEKT(\alpha, \beta)$ with parameters α, β is a random ordered marked rooted Family Tree $[\mathbf{T}', \mathbf{o}', Z']$ that has the following properties:

1. The Family Tree $[\mathbf{T}', \mathbf{o}']$ obtained by forgetting the marks is the ECS ordered $EKT(\alpha, \beta)$. Let $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ be the i.i.d. sequence of Family Trees whose joining is $[\mathbf{T}', \mathbf{o}']$ (with $\mathbf{o}' = \mathbf{o}_0$).
2. The sequence of marked Family Trees $([\mathbf{T}_i, \mathbf{o}_i, Z_i])_{i \in \mathbb{Z}}$ is i.i.d., where Z_i is the mark function on \mathbf{T}_i obtained by restricting Z' to \mathbf{T}_i .

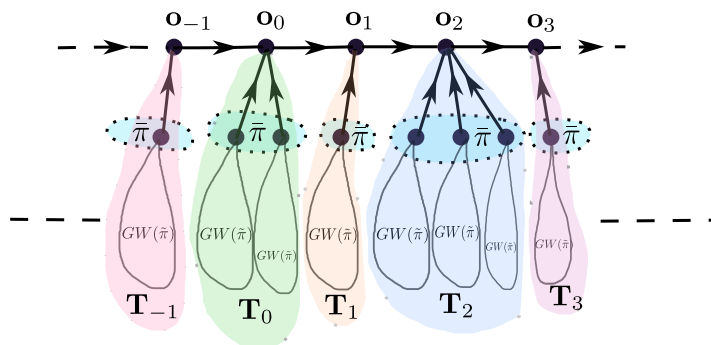


Figure 3.3: Bi-variate ECS ordered EKT with distributions $\bar{\pi}, \tilde{\pi}$ (\mathbf{o}_0 is the root), $\bar{\pi}$ is the offspring distribution of \mathbf{o}_0 in \mathbf{T}_0 , $\tilde{\pi}$ is the offspring distribution of every child of \mathbf{o}_0 in \mathbf{T}_0 , see Eq. (3.2.8) for the expressions of $\bar{\pi}, \tilde{\pi}$.

Remark 43. Because of the independence condition involved in the second part of the construction of $MEKT(\alpha, \beta)$, it follows that $EKT(\alpha, \beta)$ is the ECS ordered joining of an i.i.d. sequence of finite marked Family Trees $([\mathbf{T}_i, \mathbf{o}_i, Z_i])_{i \in \mathbb{Z}}$, where the distribution of $[\mathbf{T}_0, \mathbf{o}_0]$ is as follows: the root \mathbf{o}_0 does not have parent, the offspring distribution of \mathbf{o}_0 is α , the descendant trees of the children of \mathbf{o}_0 are independent subcritical ordered Galton-Watson trees with offspring distribution β .

It follows, from the second part of the construction of $MEKT(\alpha, \beta)$, that the sequence of random variables $(Z_i(\mathbf{o}_i))_{i \in \mathbb{Z}}$ is i.i.d. (see the Figure 3.3). Note that Z_i need not be independent of $[\mathbf{T}_i, \mathbf{o}_i]$ for any $i \in \mathbb{Z}$.

Remark 44. It is also clear from the construction of $MEKT(\alpha, \beta)$ that the distribution of $[\mathbf{T}_0, \mathbf{o}_0, Z_0]$ characterizes $MEKT(\alpha, \beta)$.

Remark 45. Let $[\mathbf{T}', \mathbf{o}', Z']$ be an $MEKT(\alpha, \beta)$ with α, β same as the above, and let $[\mathbf{T}, \mathbf{o}, Z]$ be the random ordered marked Family Tree obtained by taking σ' on $[\mathbf{T}', \mathbf{o}', Z']$, i.e., for any measurable subset A of $\hat{\mathcal{T}}_*$,

$$\sigma'[A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T}_0)]} \mathbb{E} \left[\sum_{v \in V(\mathbf{T}_0)} \mathbf{1}_{[\mathbf{T}', v, Z']}(A) \right]. \quad (3.2.1)$$

Then, by Theorem 19, the marked Family Tree $[\mathbf{T}, \mathbf{o}, Z]$ is unimodular.

The distribution of the marked Family Tree $[\mathbf{T}, \mathbf{o}, Z]$ is called the **unimodularised** $MEKT(\alpha, \beta)$.

Distribution of the record graph in the three phases

An $n + 1$ -tuple of integers of the form $(0, s_1, \dots, s_{n-1}, 1)$ with $s_j \leq 0$ for all $1 \leq j \leq n - 1$ is called an **excursion set**. Consider the skip-free to the right random walk $(S_{-n})_{n \geq 0}$ associated to the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$. The following proposition describes the relation between an excursion set of this random walk and the descendant tree of 0. Let $\delta = \max\{n < 0 : S_n \geq 1\} = \max\{n < 0 : S_n = 1\}$ (by the skip-free property).

Proposition 46. *Assume $\mathbb{E}[X_0] \leq 0$. Consider the rooted network $[\mathbb{Z}, 0, X]$. Then, the descendant tree of the root 0 in the connected component of 0 of the R-graph of $[\mathbb{Z}, 0, X]$ is a sample of the ordered Galton-Watson tree with offspring distribution $\pi \stackrel{\mathcal{D}}{=} X_0 + 1$.*

Proof. Since the random walk $(S_{-n})_{n \geq 0}$ drifts to ∞ , we have $\delta > -\infty$. It follows from the definition of $L_X(0)$ (see Eq. (3.1.2)) that $L_X(0) = \delta - 1$ since $y(\delta, 0) = -S_\delta = -1$ and for any $\delta - 1 \leq j < 0$, $y(j, 0) = -S_j > -1$. Thus, $L_X(0) > -\infty$ a.s., so that the condition required to apply Lemma 34 is satisfied. The first part of Lemma 34 implies that $d_1(0) = X_{-1} + 1 \stackrel{\mathcal{D}}{=} \pi$.

We now show that, conditioned on the number of children of 0, the subtrees of these children are jointly independent Galton-Watson trees with offspring distribution π .

For each $m \geq 1$, let $i_m = \max\{n < 0 : y(i_m, 0) = X_{-1} + 1 - m\}$ and let $i_0 = 0$. By the third part of Lemma 34, $i_1 > i_2 > \dots > i_{X_{-1}+1}$ are the children of 0, and $i_{X_{-1}+1} = L_X(0)$. The fact that, for each $m \geq 1$, i_m is a stopping time for the random walk $(S_{-n})_{n \geq 0}$, and the third part of Lemma 34, together imply the following: the excursions $((S_{i_k} - S_{i_k}, S_{i_{k-1}} - S_{i_k}, \dots, S_{i_{k+1}} - S_{i_k}))_{k=1}^{X_{-1}}$, with the notation $S_{i_0} = 0$, are independent and identically distributed as $(S_0, S_{-1}, \dots, S_\delta)$.

By Lemma 24, for any $1 \leq k \leq X_{-1} + 1$, the descendant tree of i_k is completely determined by

$$(S_{i_k} - S_{i_{k-1}}, S_{i_{k-1}} - S_{i_{k-1}}, \dots, S_{i_{k+1}} - S_{i_{k-1}}) \stackrel{\mathcal{D}}{=} (S_0, S_{-1}, \dots, S_\delta).$$

This completes the proof. □

Remark 47. Consider the i.i.d. sequence $\hat{X} = (\hat{X}_n)_{n \in \mathbb{Z}}$ with the distribution of \hat{X}_0 given by Eq. (3.1.5). Since $\mathbb{E}[\hat{X}_0] < 0$ by Lemma 32, the assumption of Proposition 46 is satisfied. This implies that the descendant tree of 0 in the record graph of $[\mathbb{Z}, 0, \hat{X}]$ is a Galton-Watson tree with offspring distribution the same as that of $\hat{X}_0 + 1$. This will be used in the last part of Step 2 of Theorem 54.

Negative mean

Theorem 48. *Let $[T, o]$ be the connected component of 0 in the R-graph of $[\mathbb{Z}, 0, X]$ and rooted at $o = 0$. If $\mathbb{E}[X_0] < 0$, then $[T, o]$ is the Typically rooted Galton-Watson Tree with offspring distribution π , where π is the distribution of $X_0 + 1$.*

Proof. We prove the theorem in several steps.

In Step 1, we show that the descendant tree of the root o is a Galton-Watson Tree with offspring distribution π . In Step 2, we compute the

probability that o has a parent. One way to compute it is by using Kemperman's formula (see [9, Proposition 3.7]). Alternatively, one could use the unimodularity of $[T, o]$. We use the latter in this step. Step 3 involves computing the offspring distribution of $F(o)$ conditioned that o has a parent. In Step 4, we give the distribution of the order of o among its siblings conditioned on the fact that o has a parent. In Step 5, we show that the descendant trees of the siblings of o conditioned on the fact that o has a parent are independent Galton-Watson trees with offspring distribution π .

The final step (Step 6) involves showing that the offspring distribution of $F^2(o)$ conditioned on the fact that o has an ancestor of order 2 is the same as the offspring distribution of $F(o)$ conditioned on the fact that o has a parent. Similarly, we show that the descendant trees of siblings of $F(o)$ and its order among its siblings conditioned on the fact that o has an ancestor of order 2 have the same distribution as that of o conditioned on the fact that o has a parent.

Step 1: Since the random walk $(S_{-n})_{n \geq 0}$ has positive drift, $-\infty < L(0)$. By Proposition 46, the descendant tree of o is $\text{GW}(\pi)$.

Step 2: Since $[T, o]$ is the connected component of 0 in the R -graph of a unimodular network $[\mathbb{Z}, 0, X]$, it is also unimodular by Lemma 16. Therefore,

$$\begin{aligned} \mathbb{P}[o \text{ has a parent}] &= \mathbb{E} \left[\sum_{u \in V(T)} \mathbf{1}\{F(o) = u\} \right] = \mathbb{E} \left[\sum_{u \in V(T)} \mathbf{1}\{F(u) = o\} \right] \\ &= \mathbb{E}[d_1(o)] = m(\pi) = \mathbb{E}[X_0] + 1, \end{aligned}$$

where $m(\pi)$ is the mean of π . In particular,

$$\mathbb{P}[S_n < 0 \quad \forall n > 0] = \mathbb{P}[o \text{ does not have parent}] = -\mathbb{E}[X_0].$$

Step 3: Note that o has a parent if and only if $\tau < \infty$ (see the paragraph above Lemma 28 for the definition of τ). So, $\mathbb{P}[\tau < \infty] = m(\pi)$. Since $(S_{-n})_{n \geq 0}$ drifts to $+\infty$, part (1) of Lemma 34 implies the equality of the two events $\{d_1(F(o)) = n, \tau < \infty\} = \{X_{\tau-1} = n-1, \tau < \infty\}$, for any $n > 0$. Therefore, for any $n > 0$,

$$\begin{aligned} \mathbb{P}[d_1(F(o)) = n, \tau < \infty] &= \sum_{j=0}^{n-1} \mathbb{P}[\tau < \infty, X_{\tau-1} = n-1, S_\tau = j] \\ &= \sum_{j=0}^{n-1} \mathbb{P}[S_1 = n-1] c^{j-n+1} = n\pi(n). \end{aligned} \quad (3.2.2)$$

Eq. (3.2.2) follows from the previous equation by Lemma 28. The last equation follows since $c = \mathbb{P}[S_n = -1 \text{ for some } n > 0] = 1$ as the random walk $(S_n)_{n \geq 0}$ drifts to $-\infty$. Therefore,

$$\mathbb{P}[d_1(F(o)) = n | \tau < \infty] = \frac{n\pi(n)}{m(\pi)} = \hat{\pi}(n),$$

for all $n > 0$.

Step 4: Let $c_k(u)$ denote k -th child of u for any vertex u and positive integer k . Note that the equality of the two events $\{\tau < \infty, c_k(F(o)) = o, d_1(o) = n\} = \{\tau < \infty, X_{\tau-1} = n-1, S_\tau = n-k\}$ for any $0 < k \leq n$ follows from part (3) of Lemma 34 (with i_m as 0 and i as $F(0)$). Therefore, using Lemma 28 and $c = 1$, we obtain for any $0 < k \leq n$,

$$\mathbb{P}[\tau < \infty, c_k(F(o)) = o, d_1(o) = n] = \mathbb{P}[X_0 = n-1].$$

Thus, for any $0 < k \leq n$

$$\mathbb{P}[c_k(F(o)) = o | d_1(o) = n, \tau < \infty] = \frac{\mathbb{P}[X_0 = n-1]}{n\mathbb{P}[X_0 = n-1]} = \frac{1}{n},$$

which is the uniform order among the siblings of o .

Step 5: On the event $\{\tau < \infty, d_1(F(o)) = n, c_k(F(o)) = o\}$, let $i_n < i_{n-1} < \dots < i_1$ be the positions of the children of $F(o)$. Then, conditioned on $\{\tau < \infty, d_1(F(o)) = n, c_k(F(o)) = o\}$, for each $1 \leq j \leq n-1$, the part of the random walk $(0, S_{i_{j-1}} - S_{i_j}, S_{i_{j-2}} - S_{i_j}, \dots, S_{i_{j+1}} - S_{i_j})$ is an excursion set by Lemma 34 (part 3). Further, these excursion sets are independent of one another because the times i_j (for $1 \leq j \leq n$) are stopping times. Indeed, for all $1 \leq j \leq n-1$, $S_{i_{j+1}} - S_{i_j} = 1$ and $S_l < S_{i_{j+1}}$ for all $i_{j+1} < l \leq i_j$. Therefore, for each $1 \leq j \leq n-1$, $(0, S_{i_{j-1}} - S_{i_j}, S_{i_{j-2}} - S_{i_j}, \dots, S_{i_{j+1}} - S_{i_j})$ is the skip-free to the right random walk $(S_{-n})_{n \geq 0}$ conditioned on $\eta_1 < \infty$, and stopped at η_1 , where $\eta_1 = \min\{n > 0 : S_{-n} = 1\}$. But $(S_{-n})_{n \geq 0}$ drifts to $+\infty$. So, $\eta_1 < \infty$ a.s.. Hence, $(0, S_{i_{j-1}} - S_{i_j}, S_{i_{j-2}} - S_{i_j}, \dots, S_{i_{j+1}} - S_{i_j})$ has the same distribution as $(0, S_{-1}, \dots, S_{\eta_1})$. Therefore, by Proposition 46, for each $1 \leq j \leq n$, the descendant trees are independent $GW(\pi)$.

Step 6: Observe that the distribution of $(S_n - S_\tau)_{n \geq \tau}$ conditioned on $\tau < \infty$ is the same as that of $(S_n)_{n \geq 0}$ by the strong Markov property. Therefore,

$$\mathbb{P}[F(o) \text{ has a parent} | \{o \text{ has a parent}\}] = \mathbb{P}[o \text{ has a parent}] = m(\pi).$$

Since $(S_{-n})_{n \geq 0}$ drifts to $+\infty$ a.s., it reaches $S_{R^2(0)+1}$ a.s.. Therefore, by part (1) of Lemma 34, the distribution of $d_1(F^2(o))$ conditioned on the event $\{o \text{ has a grandparent}\}$ is equal to the distribution of $X_{R^2(0)-1} + 1$ conditioned on $\{R^2(0) > R(0)\}$ (i.e., o has a grandparent) which has the same distribution as that of $X_{R(0)-1} + 1$ conditioned on $R(0) > 0$ (i.e., $X_{\tau-1}$ conditioned on $\tau < \infty$). In particular, the distribution of $[D(F^2(o)) \setminus D(F(o)), o]$ conditioned on $\{o \text{ has a grandparent}\}$ is the same as that of $[D(F(o)) \setminus D(o), o]$ conditioned on $\{o \text{ has a parent}\}$. By induction, it follows that $[D(F^n(o)) \setminus D(F^{n-1}(o)), o]$ conditioned on $\{o \text{ has } n\text{-th ancestor}\}$ has the same distribution as that of $[D(F(o)) \setminus D(o), o]$ conditioned on $\{o \text{ has a parent}\}$. \square

We now show that the Typically rooted Galton-Watson Tree is obtained by re-rooting to a typical vertex of a Galton-Watson Tree. Let us continue to use the notations in Theorem 48 and let $\{F(\mathbf{o}) = \mathbf{o}\}$ denote the event that \mathbf{o} does not have a parent. Let $[\mathbf{T}', \mathbf{o}']$ be the Family Tree obtained from $[\mathbf{T}, \mathbf{o}]$ by conditioning on the event that \mathbf{o} does not have a parent. We first show that the distribution of $[\mathbf{T}', \mathbf{o}']$ is $GW(\pi)$. Since $m(\pi) < 1$, then $GW(\pi)$ has finite mean size. So, we could apply Eq. 2.3.4 to $GW(\pi)$. We then prove that $[\mathbf{T}, \mathbf{o}]$ is obtained from $[\mathbf{T}', \mathbf{o}']$ by re-rooting to a typical vertex of \mathbf{T}' .

Lemma 49. *With the notations as in the above paragraph, the distribution of $[\mathbf{T}', \mathbf{o}']$ is $GW(\pi)$.*

Proof. Note that the events $\{F(\mathbf{o}) = \mathbf{o}\}$ and $\{S_n < 0 \forall n > 0\}$ are one and the same because the latter event is the same as $\{R(0) = 0\}$. The random variables $\{S_n : n \leq 0\}$ are independent of the event $\{S_n < 0 \forall n > 0\}$, and the descendant tree of 0 is a measurable function of $\{S_n : n \leq 0\}$. This implies that the distribution of $[D(\mathbf{o}), \mathbf{o}]$ conditioned on the event $\{F(\mathbf{o}) = \mathbf{o}\}$ is the same as $[[D(\mathbf{o}), \mathbf{o}]]$ (the unconditioned Family Tree). Therefore,

$$[\mathbf{T}', \mathbf{o}'] = [D(\mathbf{o}), \mathbf{o}] | \{F(\mathbf{o}) = \mathbf{o}\} \stackrel{\mathcal{D}}{=} [D(\mathbf{o}), \mathbf{o}].$$

By Proposition 46, the distribution of $[D(\mathbf{o}), \mathbf{o}]$ is $GW(\pi)$. \square

Proposition 50. *Let α be a probability measure on $\{0, 1, 2, \dots\}$ such that its mean $m(\alpha) < 1$. Then, the TGWT(α) is obtained by re-rooting to a typical vertex of $GW(\alpha)$.*

Proof. Take an i.i.d. sequence of random variables $X' = (X'_n)_{n \in \mathbb{Z}}$ such that their common distribution is given by $\mathbb{P}[X'_0 = j] = \alpha(j+1)$ for all $j \in \{-1, 0, 1, 2, \dots\}$. Let $[\mathbf{T}, \mathbf{o}]$ be the component of 0 in the record graph of $[\mathbb{Z}, 0, X']$ and let $[\mathbf{T}', \mathbf{o}']$ be the Family Tree obtained from $[\mathbf{T}, \mathbf{o}]$ by conditioning on the event that \mathbf{o} does not have a parent. By Theorem 48, the distribution of $[\mathbf{T}, \mathbf{o}]$ is $TGWT(\alpha)$, which is unimodular; and by Lemma 49, the distribution of $[\mathbf{T}', \mathbf{o}']$ is $GW(\alpha)$, which has finite mean size. We show that $[\mathbf{T}, \mathbf{o}]$ is obtained from $[\mathbf{T}', \mathbf{o}']$ by re-rooting to a typical vertex of \mathbf{T}' .

For any measurable subset A of \mathcal{T}_* , consider the function $g_A : \mathcal{T}_{**} \rightarrow \mathbb{R}_{\geq 0}$ given by $g_A([T, u, v]) = \mathbf{1}_A([T, u])\mathbf{1}\{F(v) = v\}$. Then, for any measurable set A , we have,

$$\mathbb{E} \left[\sum_{u \in V(\mathbf{T})} g_A([\mathbf{T}, \mathbf{o}, u]) \right] = \mathbb{E}[\mathbf{1}_A([\mathbf{T}, \mathbf{o})] = \mathbb{P}[[\mathbf{T}, \mathbf{o}] \in A],$$

since there is only one vertex that has no parent. Similarly,

$$\begin{aligned} \mathbb{E} \left[\sum_{u \in V(\mathbf{T})} g_A([\mathbf{T}, u, \mathbf{o})] \right] &= \mathbb{E} \left[\mathbf{1}\{F(\mathbf{o}) = \mathbf{o}\} \sum_{u \in V(\mathbf{T})} \mathbf{1}_A([\mathbf{T}, u]) \right] \\ &= \mathbb{E} \left[\sum_{u \in V(\mathbf{T}')} \mathbf{1}_A([\mathbf{T}', u]) \right] \mathbb{P}[F(\mathbf{o}) = \mathbf{o}]. \end{aligned}$$

The last step follows since $[\mathbf{T}', \mathbf{o}']$ is the Family Tree obtained from $[\mathbf{T}, \mathbf{o}]$ by conditioning on the event $\{F(\mathbf{o}) = \mathbf{o}\}$.

By the unimodularity of $[\mathbf{T}, \mathbf{o}]$, we have

$$\mathbb{P}[[\mathbf{T}, \mathbf{o}] \in A] = \mathbb{E} \left[\sum_{u \in V(\mathbf{T}')} \mathbf{1}_A([\mathbf{T}', u]) \right] \mathbb{P}[F(\mathbf{o}) = \mathbf{o}].$$

By taking $A = \mathcal{T}_*$, we obtain $\mathbb{P}[F(\mathbf{o}) = \mathbf{o}]\mathbb{E}[\#V(\mathbf{T})] = 1$. Thus, for any measurable set A , we have

$$\mathbb{P}[[\mathbf{T}, \mathbf{o}] \in A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T})]} \mathbb{E} \left[\sum_{u \in V(\mathbf{T}')} \mathbf{1}_A([\mathbf{T}', u]) \right].$$

□

We now give a characterizing condition for *TGWT*. The proof of this proposition depends on the following lemma which says that in order to completely describe a unimodular Family Tree, it is sufficient to describe the distribution of the descendant tree of the root.

Lemma 51. *Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular Family Tree. Then, $[\mathbf{T}, \mathbf{o}]$ is completely characterized by $[D(\mathbf{o}), \mathbf{o}]$.*

Proof. Observe that $[D(F^n(\mathbf{o})), \mathbf{o}]$ converges weakly to $[\mathbf{T}, \mathbf{o}]$ as $n \rightarrow \infty$. Indeed, $[D(F^n(\mathbf{o})), \mathbf{o}]_r \stackrel{D}{=} [\mathbf{T}, \mathbf{o}]_r$ for all $n > r$.

We have, for any measurable set A ,

$$\mathbb{P}[[D(F^n(\mathbf{o})), \mathbf{o}] \in A] = \mathbb{E} \left[\sum_{u \in V(\mathbf{T})} \mathbf{1}_A[D(u), \mathbf{o}] \mathbf{1}\{u = F^n(\mathbf{o})\} \right].$$

By unimodularity, the equation on the right-hand side is equal to

$$\mathbb{E} \left[\sum_{u \in V(\mathbf{T})} \mathbf{1}_A[D(\mathbf{o}), u] \mathbf{1}\{\mathbf{o} = F^n(u)\} \right].$$

Therefore, we have

$$\mathbb{P}[[D(F^n(\mathbf{o})), \mathbf{o}] \in A] = \mathbb{E} \left[\sum_{u \in D_n(\mathbf{o})} \mathbf{1}_A[D(\mathbf{o}), u] \right].$$

□

Lemma 51 is closely related to [2, Proposition 10]. In the language of this work, the latter proposition proves that if the descendant tree $[D(\mathbf{o}), \mathbf{o}]$ of a rooted Family Tree $[\mathbf{T}, \mathbf{o}]$ is a fringe distribution, then $[\mathbf{T}, \mathbf{o}]$ is completely described by $[D(\mathbf{o}), \mathbf{o}]$. Its connection to Lemma 51 follows from the observation: for any unimodular tree $[\mathbf{T}, \mathbf{o}]$ of class \mathcal{I}/\mathcal{I} , its descendant tree $[D(\mathbf{o}), \mathbf{o}]$ is a fringe distribution. However, Lemma 51 is also applicable to the unimodular Family trees of class \mathcal{I}/\mathcal{F} . For more details on this connection, see [7, Bibliographical Comments (Section 6.3)].

The following proposition and its proof are analogous to the characterization of Eternal Galton-Watson Tree of [7]. However, the following proposition differs from the former as its statement is about finite Family Trees, whereas the former is about EFTs.

In the following proposition, only the random objects are denoted in bold letters. We use the notations of Section 2.1, Chapter 2. For a (deterministic) Family Tree $[T, o]$, the non-descendant tree of the root o is the Family Tree $[D^c(o), o]$, where $D^c(o)$ is the subtree induced by $(V(T) \setminus D(o)) \cup \{o\}$. For any vertex u of T , let $c_j(u)$ denote the j -th child of u .

Proposition 52 (Characterization of *TGWT*). *A random finite Family Tree $[\mathbf{T}, \mathbf{o}]$ is a Typically rooted Galton-Watson Tree (*TGWT*) if and only if*

1. *it is unimodular, and*
2. *the number of children of the root $d_1(\mathbf{o})$ is independent of the non-descendant tree of the root \mathbf{o} .*

Proof. Let $[\mathbf{T}, \mathbf{o}]$ be a random Family Tree whose distribution is $TGWT(\pi)$, where π is a probability measure on $\{0, 1, 2, 3, \dots\}$ with $m(\pi) < 1$. Consider an i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$ of random variables whose common distribution is given by $\mathbb{P}[X_0 = n] = \pi(n + 1)$ for all $n \in \{-1, 0, 1, 2, \dots\}$, and let $[\mathbb{Z}, 0, X]$ be its network. Since $\mathbb{E}[X_0] < 0$, by Theorem 48, the connected component of 0 in the R -graph of $[\mathbb{Z}, 0, X]$ is $TGWT(\pi)$. Therefore, $[\mathbf{T}, \mathbf{o}]$ is unimodular by Lemma 16. The second condition is satisfied by $TGWT$, which follows from its construction.

We now show that if a random finite Family Tree $[\mathbf{T}, \mathbf{o}]$ satisfies the above conditions 1 and 2, then it is a *TGWT*.

Let E denote the event that o has a parent. For a positive integer k , let $A = (A'; A_1, A_2, \dots, A_k)$ be an event of the form

$$d_1(o) = k, D^c(o) \in A', D(c_1(o)) \in A_1, \dots, D(c_k(o)) \in A_k. \quad (3.2.3)$$

By Lemma 51, any unimodular Family Tree is characterized by the descendant tree of its root. So, it is sufficient to prove that

$$\mathbb{P}[A] = \mathbb{P}[d_1(\mathbf{o}) = k] \mathbb{P}[D^c(\mathbf{o}) \in A'] \left(\prod_{i=1}^k \mathbb{P}[D(\mathbf{o}) \in A_i] \right), \quad (3.2.4)$$

for any such event A of the form given by Eq (3.2.3). Assume further that the events A_1, A_2, \dots, A_k depend only up to n -th generation of o , which is the union of $D_1(o), D_2(o), \dots, D_n(o)$. It suffices to show Eq. (3.2.4) for such events because of the local topology.

We prove the result by induction on n . For $n = 0$, we have $\mathbb{P}[A] = \mathbb{P}[d_1(\mathbf{o}) = k] \mathbb{P}[D^c(\mathbf{o}) \in A']$ by condition 2 of the hypothesis. So, assume that the result is true for $n-1$. Let $n \geq 1$ and $1 \leq j \leq k$. For any Family Tree $[T, o]$, let $h_j([T, o]) := \mathbf{1}_A[T, F(o)] \mathbf{1}\{o = c_j(F(o))\}$. Then, for any $1 \leq j \leq k$, we have

$$\mathbf{1}_A[T, o] = \sum_{u \in D_1(o)} \mathbf{1}_A[T, F(u)] \mathbf{1}\{u = c_j(o)\} = \sum_{u \in D_1(o)} h_j([T, u]). \quad (3.2.5)$$

Note that, since $k \geq 1$, the event $\{[T, o] \in A\}$ is a subset of the event $\{D_1(o) \neq \emptyset\}$. Applying Eq. (3.2.5) for $j = 1$, we obtain

$$\begin{aligned} \mathbb{P}[A] &= \mathbb{E} \left[\sum_{u \in V(\mathbf{T})} \mathbf{1}_A[\mathbf{T}, F(u)] \mathbf{1}\{u = c_1(\mathbf{o})\} \mathbf{1}\{u \in D_1(\mathbf{o})\} \right] \\ &= \mathbb{E} \left[\sum_{u \in V(\mathbf{T})} \mathbf{1}_A[\mathbf{T}, F(\mathbf{o})] \mathbf{1}\{\mathbf{o} = c_1(u)\} \mathbf{1}\{\mathbf{o} \in D_1(u)\} \right] \end{aligned} \quad (3.2.6)$$

$$\begin{aligned} &= \mathbb{E} [\mathbf{1}_A[\mathbf{T}, F(\mathbf{o})] \mathbf{1}\{\mathbf{o} \in E\} \mathbf{1}\{\mathbf{o} = c_1(F(\mathbf{o}))\}] \\ &= \mathbb{P}[\mathbf{o} \in E, \mathbf{o} = c_1(F(\mathbf{o})), [\mathbf{T}, F(\mathbf{o})] \in A] \\ &= \mathbb{P}[D(\mathbf{o}) \in A_1, \mathbf{o} \in E, \mathbf{o} = c_1(F(\mathbf{o})), [\mathbf{T}, F(\mathbf{o})] \in A(A'; \mathcal{T}_*, A_2, \dots, A_k)]. \end{aligned} \quad (3.2.7)$$

In the above, Eq. (3.2.6) follows from unimodularity. Note that $D(o) \in A_1$ depends on one generation less than that of $[T, F(o)] \in A$. Therefore, by induction, we obtain,

$$\begin{aligned} \mathbb{P}[A] &= \mathbb{P}[D(\mathbf{o}) \in A_1] \mathbb{P}[\mathbf{o} \in E, \mathbf{o} = c_1(F(\mathbf{o})), [\mathbf{T}, F(\mathbf{o})] \in A(A'; \mathcal{T}_*, A_2, \dots, A_k)] \\ &= \mathbb{P}[D(\mathbf{o}) \in A_1] \mathbb{P}[A(A'; \mathcal{T}_*, A_2, \dots, A_k)]. \end{aligned}$$

The last equation is obtained by Eq. (3.2.7).

Applying Eq.(3.2.5) with $j = 2$ to $A(A'; \mathcal{T}_*, A_2, \dots, A_k)$, we obtain

$$\begin{aligned} \mathbb{P}[A(A'; \mathcal{T}_*, A_2, \dots, A_k)] &= \mathbb{P}[D(\mathbf{o}) \in A_2] \mathbb{P}[\mathbf{o} \in E, \mathbf{o} = c_1(F(\mathbf{o})), [\mathbf{T}, F(\mathbf{o})] \in A(A'; \mathcal{T}_*, \mathcal{T}_*, \dots, A_k)] \\ &= \mathbb{P}[D(\mathbf{o}) \in A_2] \mathbb{P}[A(A'; \mathcal{T}_*, \mathcal{T}_*, \dots, A_k)]. \end{aligned}$$

By iterating the same procedure for $j = 3, 4, \dots$, we obtain,

$$\begin{aligned} \mathbb{P}[A] &= \left(\prod_{i=1}^k \mathbb{P}[D(\mathbf{o}) \in A_i] \right) \mathbb{P}[d_1(\mathbf{o}) = k, D^c(\mathbf{o}) \in A'] \\ &= \left(\prod_{i=1}^k \mathbb{P}[D(\mathbf{o}) \in A_i] \right) \mathbb{P}[d_1(\mathbf{o}) = k] \mathbb{P}[D^c(\mathbf{o}) \in A']. \end{aligned}$$

Thus, we showed Eq. (3.2.4), which completes the proof. \square

Zero mean

Theorem 53. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a sequence of random variables as in Def. 22 with $\mathbb{E}[X_0] = 0$, $[\mathbb{Z}, 0, X]$ be its associated network, and $[\mathbf{T}, 0]$ be the connected component of 0 in the R-graph of $[\mathbb{Z}, 0, X]$. Then, $[\mathbf{T}, 0]$ is the ordered Eternal Galton-Watson Tree EGWT(π) with offspring distribution $\pi \stackrel{D}{=} X_0 + 1$.*

Proof. We apply Theorem 18. We show that $[\mathbf{T}, 0]$ satisfies the second condition of Theorem 18. Observe that $L_X(0)$ is a stopping time for the random walk with increments $(X_{-n})_{n \geq 1}$ and $-\infty < L_X(0)$ a.s.. By the strong Markov property, the random variables $Y = (X_{-n})_{n > -L_X(0)}$ are independent of $Z = (X_{-n})_{n=1}^{n=L_X(0)}$. Let $(\mathbf{T}', 0)$ denote the non-descendant tree of o , i.e., the tree generated by $(\mathbf{T} \setminus D(0)) \cup \{0\}$. By Lemma 24, the subtree $(D(0), 0)$ is a function of Z and the subtree $(\mathbf{T}', 0)$ is a function of $Y \cup (X_n)_{n \geq 0}$. So, the unordered tree $(D(0), 0)$ is independent of the unordered tree $(\mathbf{T}', 0)$. The second part of Lemma 34 implies that, conditionally on the event that 0 has children, the order of a child is independent of $Y \cup (X_n)_{n \geq 0}$, whereas the order of any vertex in $(\mathbf{T}', 0)$ is a function of $Y \cup (X_n)_{n \geq 0}$. Thus, the ordered subtree $(D(0), 0)$ is independent of the ordered subtree $(\mathbf{T}', 0)$. So, the sufficient condition of Theorem 18 is satisfied by $[\mathbf{T}, 0]$.

The first statement of Lemma 34 implies that $d_1(0) = X_{-1} + 1$, and thus completes the proof. \square

Positive mean

Let $X = (X_n)_{n \in \mathbb{Z}}$ be the i.i.d. increments of a skip-free random walk with mean $0 < \mathbb{E}[X_0] < \infty$, and $[\mathbb{Z}, 0, X]$ be its associated network.

Let $\eta_{-1} = \inf\{n > 0 : S_n = -1\}$ if the infimum exists and ∞ if $S_n > -1 \forall n \geq 1$, and let $c = \mathbb{P}[\eta_{-1} < \infty]$. Let $[\mathbf{T}, \mathbf{o}]$ be the connected component of 0 in the R -graph of $[\mathbb{Z}, 0, X]$, with root $0 = \mathbf{o}$.

Let $\tilde{\pi}, \bar{\pi}$ be the probability distributions on $\{0, 1, 2, \dots\}$ given by:

$$\begin{aligned}\tilde{\pi}(k) &= c^{k-1} \mathbb{P}[X_0 = k - 1], \\ \bar{\pi}(k) &= \mathbb{P}[X_0 \geq k] c^k.\end{aligned}\tag{3.2.8}$$

Let $[\mathbf{T}_0, \mathbf{o}_0]$ be a Family Tree whose distribution is given by the following: the offspring distribution of \mathbf{o}_0 is $\bar{\pi}$. The descendant trees of the children of \mathbf{o}_0 are independent Galton-Watson trees with offspring distribution $\tilde{\pi}$, and they are independent of the offspring distribution of \mathbf{o}_0 . Let $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ be an i.i.d. sequence of Family Trees. Recall that the distribution of the Family Tree obtained from the typically rooted joining of $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$ is the unimodularised $EKT(\bar{\pi}, \tilde{\pi})$.

Theorem 54. *If $0 < \mathbb{E}[X_0] < \infty$, then the distribution of $[\mathbf{T}, \mathbf{o}]$ is the unimodularised $EKT(\bar{\pi}, \tilde{\pi})$.*

Proof. Since the random walk has a finite positive drift, the unimodular EFT $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{F} , see Theorem 41. Let $\mathbf{o} \in \mathcal{Z}$ denote the event that \mathbf{o} belongs to the bi-infinite path of \mathbf{T} . Let $[\mathbf{T}', \mathbf{o}']$ denote the Family Tree obtained from $[\mathbf{T}, \mathbf{o}]$ by conditioning on the event $\mathbf{o} \in \mathcal{Z}$, denoted as $[\mathbf{T}, \mathbf{o}]|_{\mathbf{o} \in \mathcal{Z}}$. By Theorem 20, $[\mathbf{T}', \mathbf{o}']$ is the joining of some stationary sequence of Family Trees $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \in \mathbb{Z}}$. The proof is complete once we prove that this sequence is i.i.d., and $[\mathbf{T}'_1, \mathbf{o}'_1] \stackrel{\mathcal{D}}{=} [\mathbf{T}_0, \mathbf{o}_0]$.

Step 1: $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \in \mathbb{Z}}$ is an i.i.d. sequence: Since the increments have positive mean, $S_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $S_n \rightarrow -\infty$ as $n \rightarrow -\infty$. By Lemma 24, $\mathbf{o} \in \mathcal{Z}$ if and only if $L_X(0) = \infty$, and the latter condition holds if and only if $S_n \leq 0, \forall n \leq -1$.

For any $u \in \mathbb{Z}$, let $\mathfrak{S}(u)$ denote the (random) subtree of the descendants of u (including u) in \mathbf{T} . The distribution of $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \geq 1}$ is the same as that of

$$([\mathfrak{S}(F^n(\mathbf{o})) \setminus \mathfrak{S}(F^{n-1}(\mathbf{o})), F^n(\mathbf{o}))]_{n \geq 1} | \mathbf{o} \in \mathcal{Z}$$

(distribution obtained by conditioning on $\mathbf{o} \in \mathcal{Z}$), where F is the parent vertex-shift on \mathbf{T} with the convention that $F^0(\mathbf{o}) = \mathbf{o}$ (and $F^n(\mathbf{o})$ is the parent of order n of \mathbf{o}). On the event $\mathbf{o} \in \mathcal{Z}$, each subtree $\mathfrak{S}(F^n(\mathbf{o})) \setminus \mathfrak{S}(F^{n-1}(\mathbf{o}))$ is completely determined by the part of the random walk

$$(S_{R^{n-1}(0)+1} - S_{R^{n-1}(0)}, \dots, S_{R^n(0)} - S_{R^{n-1}(0)}),$$

for each $n \geq 1$ (see Figure 3.4 for an illustration). The distribution of the sequence

$$((S_{R^{n-1}(0)+1} - S_{R^{n-1}(0)}, \dots, S_{R^n(0)} - S_{R^{n-1}(0)})_{n \geq 1}$$

conditioned on $\{S_n \leq 0 : n \leq -1\}$ is the same as that of the unconditioned sequence (the same sequence without the condition) because the sequence is independent of the event on which it is conditioned. Moreover, it is an i.i.d. sequence by the strong Markov property of the random walk. This implies that $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \geq 1}$ is an i.i.d. sequence.

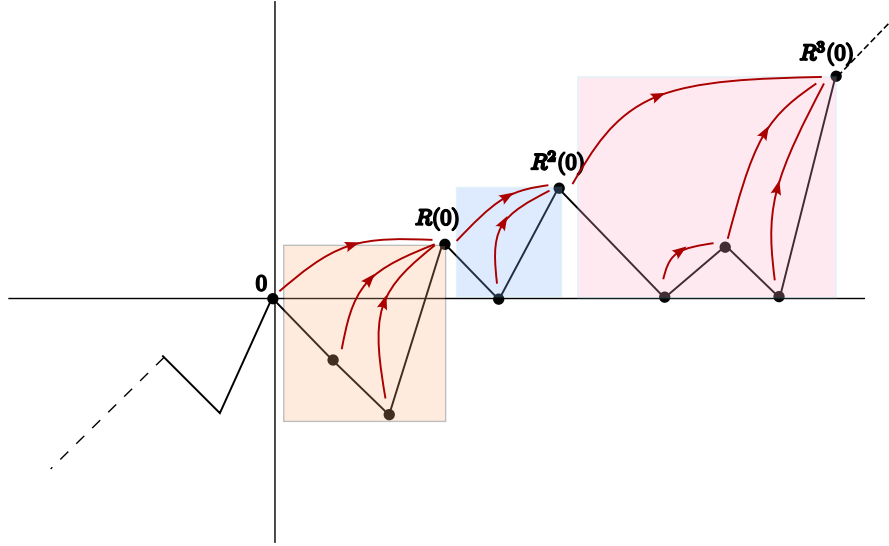


Figure 3.4: Illustration of the subtrees of ancestors of 0 on a trajectory conditioned on the event that $S_n \leq 0$ for all $n \leq 0$ when $\mathbb{E}[X_0] > 0$. Each box represents the subtrees $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \dots$ (see Step 1 of Theorem 54).

Further, the stationarity of $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \in \mathbb{Z}}$ implies that $([\mathbf{T}'_i, \mathbf{o}'_i])_{i \in \mathbb{Z}}$ is an i.i.d. sequence (the index set here is \mathbb{Z} , whereas in the previous sentence it is $i \geq 1$).

Step 2: $[\mathbf{T}'_1, \mathbf{o}'_1] \stackrel{\mathcal{D}}{=} [\mathbf{T}_0, \mathbf{o}_0]$: We first show that the offspring distribution of \mathbf{o}'_1 is $\bar{\pi}$. Let $\tau = \inf\{n > 0 : S_n \geq 0\}$ and ∞ if $S_n < 0, \forall n > 0$, and $X_{\tau-1}, S_\tau$ be the increment and location of the random walk at this time. Since the increments have positive mean, $\tau < \infty$ a.s., and therefore $R(0) = \tau$ a.s. (if $\tau = \infty$ then $R(0)$ is defined to be 0).

We also have the equality of the following events,

$$\{d_1(F(\mathbf{o})) = 1\} = \{\tau = 1\} = \{X_{\tau-1} = S_\tau\} = \{S_1 \geq 0\}.$$

Therefore,

$$\mathbb{P}[d_1(F(\mathbf{o})) = 1, \mathbf{o} \in \mathcal{Z}'] = \sum_{j=0}^{\infty} (\mathbb{P}[X_0 = j] \mathbb{P}[\mathbf{o} \in \mathcal{Z}']) = \mathbb{P}[X_0 \geq 0] \mathbb{P}[\mathbf{o} \in \mathcal{Z}'].$$

We now study the event $d_1(F(\mathbf{o})) = k$ with $k > 1$ (since 0 is the child of τ , the event $\mathbb{P}[d_1(\tau) = 0] = 0$). Observe that, on the event $\mathbf{o} \in \mathcal{Z}'$, no negative integer is a child of $F(\mathbf{o})$. To see this, it is sufficient to show that $L_X(0) = -\infty$. Because, the latter condition implies that all the non-negative integers are the descendants of 0 by Lemma 24. So, none of them are the children of τ (see Figure 3.4). The latter condition is equivalent to $S_n \leq 0, \forall n < 0$, which is further equivalent to $\mathbf{o} \in \mathcal{Z}'$. Thus, the observation follows. This observation implies that 0 is the last child of τ . So, $l_X(\tau) = 0$ and $t_X(\tau) = S_\tau \geq 0$. Using Lemma 38, we obtain the relation $d_1(\tau) = X_{\tau-1} + 1 - S_\tau$. Use this relation to get

$$\mathbb{P}[d_1(F(\mathbf{o})) = k, \mathbf{o} \in \mathcal{Z}'] = \sum_{j=k-1}^{\infty} \mathbb{P}[X_{\tau-1} = j, S_\tau = j - (k-1)] \mathbb{P}[\mathbf{o} \in \mathcal{Z}'].$$

We apply Lemma 28 to the first term inside the sum to get

$$\mathbb{P}[X_{\tau-1} = j, S_\tau = j - (k-1)] = \mathbb{P}[X_0 = j] c^{k-1}, 0 < k-1 \leq j.$$

Using this, we obtain

$$\begin{aligned} \mathbb{P}[d_1(F(\mathbf{o})) = k, \mathbf{o} \in \mathcal{Z}'] &= \sum_{j=k-1}^{\infty} \mathbb{P}[X_0 = j] c^{k-1} \mathbb{P}[\mathbf{o} \in \mathcal{Z}'] \\ &= \mathbb{P}[X_0 \geq k-1] c^{k-1} \mathbb{P}[\mathbf{o} \in \mathcal{Z}']. \end{aligned}$$

Thus, the offspring distribution of \mathbf{o}'_1 in the tree T'_1 is given by: for any $k \geq 1$,

$$\begin{aligned} \mathbb{P}[d_1(\mathbf{o}'_1, T'_1) = k-1] &= \mathbb{P}[d_1(\mathbf{o}'_1, T') = k] = \mathbf{P}[d_1(F(\mathbf{o})) = k | \mathbf{o} \in \mathcal{Z}'] \\ &= \mathbb{P}[X_0 \geq k-1] c^{k-1}, \end{aligned}$$

which proves the first result.

We now show that conditioned on $\{d_1(\mathbf{o}'_1, T') = k\}$, the descendant subtrees of the children of \mathbf{o}'_1 are independent Galton-Watson trees with offspring distribution $\tilde{\pi}$.

On $\{d_1(F(\mathbf{o})) = k\}$, let $0 = i_k < i_{k-1} < \dots < i_1$ be the children of $F(\mathbf{o})$. Observe that, on the event $\{d_1(F(\mathbf{o})) = k\}$, for each $1 \leq j \leq k$, the part of the random walk $(0, S_{i_{j-1}} - S_{i_j}, S_{i_{j-2}} - S_{i_j}, \dots, S_{i_{(j+1)}} - S_{i_j})$ is an excursion set because, by the third part of Lemma 34, $S_{i_{(j+1)}} - S_{i_j} = 1$ and $S_l \leq S_{i_{(j+1)}}$ for all $i_{(j+1)} < l \leq i_j$. Further, these excursion sets are mutually independent because the times i_j , for $1 \leq j \leq k$, are stopping times. Therefore, for each j chosen above, $(0, S_{i_{j-1}} - S_{i_j}, S_{i_{j-2}} - S_{i_j}, \dots, S_{i_{(j+1)}} - S_{i_j})$ obtained by conditioning on $\{d_1(F(\mathbf{o})) = k\}$ is the conditioned skip-free to the right random walk of $(S_{-n})_{n \geq 0}$ conditioned on $\eta_1 < \infty$ and stopped at η_1 , where $\eta_1 = \min\{n > 0 : S_{-n} = 1\}$ and $-\infty$ if $S_{-n} < 1$ for all $n > 0$. By Lemma 31 (and applying the same to $(S_n)_{n \geq 0}$ and η_{-1}), the conditioned random walk is an unconditioned random walk $(\hat{S}_n)_{n \geq 0}$ whose increments $(\hat{X}_n)_{n \geq 1}$ have distribution $\mathbb{P}[\hat{X}_1 = k] = \mathbb{P}[X_1 = k]c^k$, for all $k \geq -1$. Moreover, by Lemma 32, the random walk $(\hat{S}_n)_{n \geq 0}$ has negative drift. Therefore, by Proposition 46, the descendant subtree of the children of \mathbf{o}'_1 is a Galton-Watson tree with offspring distribution $\tilde{\pi}$. In particular, the mean $m(\tilde{\pi}) < 1$ by Lemma 32. So, $GW(\tilde{\pi})$ is a.s. finite. □

We now focus on the distribution of the type function associated to a trajectory of the random walk and on the order of the record graph when $\mathbb{E}[X_0] > 0$. We enrich the record graph by assigning the type function as the mark function to the record graph. It will be shown later that the knowledge of the type function and of the record graph of a trajectory is sufficient to construct the negative part of the trajectory. The following theorem shows that the distribution of the record graph with type function as its mark function is a bi-variate ECS ordered marked Eternal Kesten Tree. The latter is characterized by the joint distribution of the mark function restricted to the bush of the root and the distribution of the (unlabelled) bush of the root by Remark 44 and the discussion above it.

Theorem 55. *Let $0 < \mathbb{E}[X_0] < \infty$. Let $[\mathbf{T}, \mathbf{o}, t]$ be the component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ with t , the type function associated to X , as the mark function. Let $[\mathbf{T}', \mathbf{o}', t']$ be the marked Family Tree obtained from $[\mathbf{T}, \mathbf{o}, t]$ by conditioning on $\mathbf{o} \in \mathcal{Z}^r$. Then, the distribution of $[\mathbf{T}', \mathbf{o}', t']$*

is $MEKT(\bar{\pi}, \tilde{\pi})$ where the marked bush $[\hat{\mathbf{T}}_0, \hat{\mathbf{o}}_0, m_0]$ of the root has the following distribution: the offspring distribution of $\hat{\mathbf{o}}_0$ is $\bar{\pi}$; the descendant trees of the children of $\hat{\mathbf{o}}_0$ are i.i.d. $GW(\tilde{\pi})$ and they are independent of $d_1(\hat{\mathbf{o}}_0)$ (see Eq. (3.2.8) for the definitions of $\bar{\pi}$ and $\tilde{\pi}$); the mark of every vertex other than $\hat{\mathbf{o}}_0$ is -1 ; the mark of $\hat{\mathbf{o}}_0$ is independent of the descendant subtrees of children of $\hat{\mathbf{o}}_0$, i.e.,

$$\mathbb{P}[m_0(\hat{\mathbf{o}}_0)|[\hat{\mathbf{T}}_0, \hat{\mathbf{o}}_0]] = \mathbf{P}[m_0(\hat{\mathbf{o}}_0)|d_1(\hat{\mathbf{o}}_0)];$$

and the joint distribution of the mark of $\hat{\mathbf{o}}_0$ and $d_1(\hat{\mathbf{o}}_0)$ is given by

$$\mathbb{P}[m_0(\hat{\mathbf{o}}_0) = n, d_1(\hat{\mathbf{o}}_0) = k] = \mathbb{P}[X_0 = n + k]c^k, \forall n \geq 0, k \geq 0. \quad (3.2.9)$$

Proof. Forgetting the marks and the order of vertices of \mathbf{T}' , it follows from Theorem 54 that $[\mathbf{T}', \mathbf{o}']$ is the joining of the i.i.d. sequence $([\mathbf{T}_i, \mathbf{o}_i])_{i \in \mathbb{Z}}$, where $[\mathbf{T}_0, \mathbf{o}_0]$ has the same distribution as that of $[\hat{\mathbf{T}}_0, \hat{\mathbf{o}}_0]$, the random Family Tree obtained after forgetting the marks of $V(\hat{\mathbf{T}}_0)$. Let t_i be the restriction of the type function t' to $V(\mathbf{T}_i)$ for all $i \in \mathbb{Z}$.

Observe that an integer j belongs to the bi-infinite path of \mathbf{T} if and only if $t(j) \geq 0$. This follows because, for any $j \in \mathbb{Z}$, its type $t(j)$ is non-negative if and only if $y(n, j) \geq 0$ for all $n < j$. The latter condition is satisfied if and only if $L(j) = -\infty$. By Lemma 24, $L(j) = -\infty$ if and only if j has infinitely many descendants. In view of this, it is sufficient to focus on the type of vertices that lie on the bi-infinite path since the remaining vertices have fixed type -1 .

By the above argument, if the sequence of random variables $(t(\mathbf{o}_i), d_1(\mathbf{o}_i))_{i \in \mathbb{Z}}$ is i.i.d., then the sequence $([\mathbf{T}_i, \mathbf{o}_i, t_i])_{i \in \mathbb{Z}}$ is i.i.d.. So, the statement of the theorem is proved once we show that

- (Step 1:) the order of \mathbf{o}_i is the smallest among the children of \mathbf{o}_{i+1} , and
- (Step 2:) the sequence $((t(\mathbf{o}_i), d_1(\mathbf{o}_i))_{i \in \mathbb{Z}})$ is i.i.d. with the common distribution given by Eq. (3.2.9).

Let $\mathbf{o} \in \mathcal{P}$ denote the event that the root \mathbf{o} belongs to the bi-infinite path of \mathbf{T} .

Step 1: For any $i \in \mathbb{Z}$ with $t(i) \geq 0$, let $l(i) := l_X(i) = \max\{m < i : y(m, i) = t(i)\}$ and $l^n(i) := l^{n-1}(l(i))$ for all $n > 1$. By Lemma 37, it follows that $l(i)$ is the smallest among the children of i .

On the event $\{S_n \leq 0 \quad \forall n < 0\}$ (which is equivalent to $\mathbf{o} \in \mathcal{Z}$), we have $t(0) \geq 0$. Since $S_m \leq S_{l^n(0)}$ for all $m \leq l^n(0), n \geq 1$, we have $t(l^n(0)) \geq 0$ for all $n \geq 1$. So, by the above discussion, on the event $\mathbf{o} \in \mathcal{Z}$, we get that $R(l^n(0)) = l^{n-1}(0)$ for all $n \geq 1$ (with the notation $l^0(0) = 0$). Moreover, on the same event, the vertices of the bi-infinite F -path of \mathbf{T} are $\{l^n(0) : n \geq 1\} \cup \{0\} \cup \{R^m(0) : m \geq 1\}$. Since, on the event $\mathbf{o} \in \mathcal{Z}$, for any $m \geq 1$, $S_{R^m(0)} - S_n > S_{R^m(0)} - S_{R^{m-1}(0)}$ for all $R^{m-1}(0) < n < R^m(0)$, we have $l(R^m(0)) = R^{m-1}(0)$ (with the notation that $R^0(0) = 0$). Thus, \mathbf{o}_i has the smallest order among the children of \mathbf{o}_{i+1} for all $i \in \mathbb{Z}$. Since the order of $(\mathbf{o}_i)_{i \in \mathbb{Z}}$ characterizes the order on $[\mathbf{T}', \mathbf{o}']$, this completes Step 1.

Step 2: Firstly, we compute the joint distribution of $(t(\mathbf{o}_0), d_1(\mathbf{o}_0))$. Note that $d_1(\mathbf{o}_0) = d_1(\mathbf{o}') - 1 = (d_1(\mathbf{o}) - 1) | \mathbf{o} \in \mathcal{Z}$. By Lemma 38, we have, for $n \geq 0, k \geq 0$,

$$\mathbb{P}[\mathbf{o} \in \mathcal{Z}, t(\mathbf{o}) = n, d_1(\mathbf{o}) = k + 1] = \mathbb{P}[\mathbf{o} \in \mathcal{Z}, X_{-1} = k + n, t(\mathbf{o}) = n].$$

The right-hand term in the above is equal to

$$\mathbb{P}[X_{-1} = k + n] \mathbb{P}_{-(k+n)}[S_m = -n \text{ for some } m \leq 0] \mathbb{P}_{-n}[S_m \leq -n \quad \forall m \leq 0],$$

where the second term in the above product is the probability that the skip-free to the right random walk starting from $-(n+k)$ hits $-n$, whereas the third term is the probability that the same random walk starting from $-n$ never hits $-n+1$. Lemma 27 implies that the second term is equal to c^k and the third term is equal to $(1-c)$. Therefore, for $n \geq 0$,

$$\mathbb{P}[\mathbf{o} \in \mathcal{Z}, t(\mathbf{o}) = n, d_1(\mathbf{o}) = k + 1] = \mathbb{P}[X_{-1} = n + k] c^k (1 - c).$$

Since $\mathbb{P}[\mathbf{o} \in \mathcal{Z}] = 1 - c$, we obtain, for $n \geq 0$,

$$\begin{aligned} \mathbb{P}[t(\mathbf{o}_0) = n, d_1(\mathbf{o}_0) = k] &= \mathbb{P}[t(\mathbf{o}') = n, d_1(\mathbf{o}') = k + 1] \\ &= \mathbb{P}[t(\mathbf{o}) = n, d_1(\mathbf{o}) = k + 1 | \mathbf{o} \in \mathcal{Z}] \\ &= \mathbb{P}[X_{-1} = n + k] c^k. \end{aligned}$$

Secondly, observe that the distribution of $(S_{-m+\mathbf{o}_{-1}} - S_{\mathbf{o}_{-1}})_{m \geq 0}$ conditioned on $\{\mathbf{o} \in \mathcal{Z}, t(\mathbf{o}) = n\}$ is the same as that of $(S_{-m})_{m \geq 0}$ conditioned on $\mathbf{o} \in \mathcal{Z}$. This follows because

$$(S_{-m+\mathbf{o}_{-1}} - S_{\mathbf{o}_{-1}})_{m \geq 0} \cap \{\mathbf{o} \in \mathcal{Z}, t(\mathbf{o}) = n\}$$

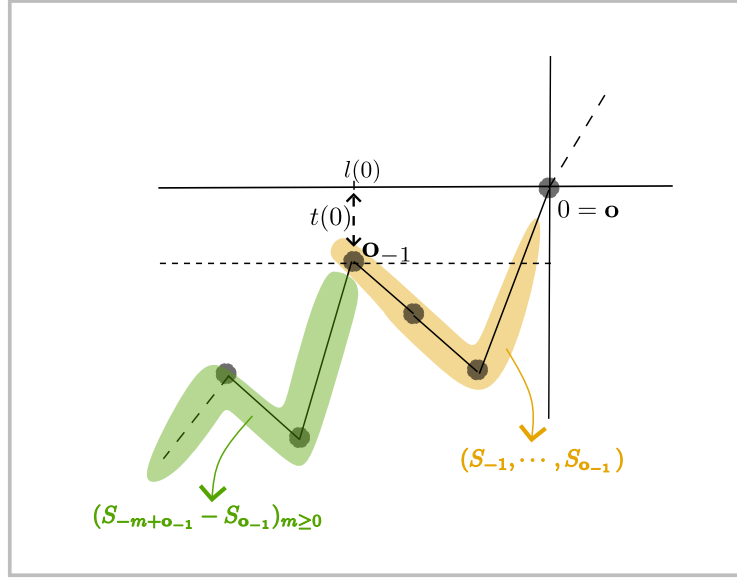


Figure 3.5: The conditioned part of the trajectory. An illustration showing that the distribution of $(S_{-m+o_{-1}} - S_{o_{-1}})_{m \geq 0} | \{\mathbf{o} \in \mathcal{I}, t(\mathbf{o}) = n\}$ is the same as that of $(S_{-m})_{m \geq 0} | \{\mathbf{o} \in \mathcal{I}\}$ and independent of $(S_{-1}, \dots, S_{o_{-1}}) | \{\mathbf{o} \in \mathcal{I}\}$.

is a conditioned skip-free to the right random walk starting at 0 conditioned on the event that the random walk stays below 1 (see Figure 3.5). Further, this conditioned random walk is independent of $(S_0, S_{-1}, \dots, S_{o_{-1}}) | \mathbf{o} \in \mathcal{I}$ (the process conditioned on the event $\mathbf{o} \in \mathcal{I}$). The joint-distribution of $(t(\mathbf{o}_0), d_1(\mathbf{o}_0))$, however, is a function of $(S_0, S_{-1}, \dots, S_{o_{-1}})$. Thus, by iteratively applying the above argument to $(S_{-m+o_{-1}} - S_{o_{-1}})_{m \geq 0}$ conditioned on $\{\mathbf{o} \in \mathcal{I}, t(\mathbf{o}) = n\}$, we obtain that the sequence of random variables $((t(\mathbf{o}_k), d_1(\mathbf{o}_k)))_{k \leq 0}$ is i.i.d.

Thirdly, the distribution of the sequence of random variables $((t(\mathbf{o}_n), d_1(\mathbf{o}_n)))_{n \geq 1}$ (conditioned on $\mathbf{o} \in \mathcal{I}$) is same as that of

$$((S_{R^n(0)} - S_{R^{n-1}(0)}, X_{(R^n(0)-1)} - (S_{R^n(0)} - S_{R^{n-1}(0)})))_{n \geq 1} | \{\mathbf{o} \in \mathcal{I}\}$$

(conditioned on the event $\mathbf{o} \in \mathcal{I}$), with the notation that $R^0(0) = 0$. Note that the latter process has the same distribution as that of unconditioned process $((S_{R^n(0)} - S_{R^{n-1}(0)}, X_{(R^n(0)-1)} - (S_{R^n(0)} - S_{R^{n-1}(0)})))_{n \geq 1}$ as it is independent of the event $\mathbf{o} \in \mathcal{I}$. But $((S_{R^n(0)} - S_{R^{n-1}(0)}, X_{(R^n(0)-1)} - (S_{R^n(0)} - S_{R^{n-1}(0)})))_{n \geq 1}$

$S_{R^{n-1}(0)})_{n \geq 1}$ is i.i.d. by the strong Markov property (since $R(0)$ is a stopping time). Thus, $((t(\mathbf{o}_n), d_1(\mathbf{o}_n)))_{n \geq 1}$ is an i.i.d. sequence.

The proof is complete once we show that

$$\mathbb{P}[t(\mathbf{o}_1) = n, d_1(\mathbf{o}_1) = k] = \mathbb{P}[t(\mathbf{o}_0) = n, d_1(\mathbf{o}_0) = k],$$

for all $n \geq 0, k \geq 0$. For $n \geq 0, k \geq 0$,

$$\begin{aligned} \mathbb{P}[t(\mathbf{o}_1) = n, d_1(\mathbf{o}_1) = k] &= \mathbb{P}[S_{R(0)} = n, X_{(R(0)-1)} - S_{R(0)} = k | \mathbf{o} \in \mathcal{Z}'] \\ &= \mathbb{P}[S_{R(0)} = n, X_{(R(0)-1)} - S_{R(0)} = k] \text{ (by independence)} \\ &= \mathbb{P}[S_{R(0)} = n, X_{R(0)-1} = n + k] \\ &= \mathbb{P}[X_0 = n + k] c^k = \mathbb{P}[t(\mathbf{o}_0) = n, d_1(\mathbf{o}_0) = k]. \end{aligned}$$

We used Lemma 28 in the last equation. This concludes the proof. \square

3.3 Record foil structure and a measure preserving map

In this section, we define the vertex-shift R_\perp on $[\mathbb{Z}, 0, X]$, where X is the i.i.d. sequence, such that its induced map θ_{R_\perp} preserves the distribution of $[\mathbb{Z}, 0, X]$. This question is of great interest in the field of stochastic geometry, see [20] for more details.

R-orthogonal map (R_\perp): Let $[\mathbb{Z}, 0, X]$ be a network, where $X = (X_n)_{n \in \mathbb{Z}}$ as in Def. 22 and satisfies $\mathbb{E}[X_0] = 0$. Consider the R -foliation of (\mathbb{Z}, X) (see Figure 3.6 for an illustration). Observe that the R -foliation is the same as the F -foliation of $\Psi_R([\mathbb{Z}, 0, X])$, where F is the Parent vertex-shift and $\Psi_R([\mathbb{Z}, 0, X])$ is the connected component of 0 in the R -graph of $[\mathbb{Z}, 0, X]$. The order of \mathbb{Z} naturally induces an order on an R -foil as the latter is a subset of the former. This order is the same as the RLS order by Lemma 33. This motivates us to define a new map R_\perp on the vertices of \mathbb{Z} in the following way: for an integer u , let $R_\perp(u)$ be the smallest integer of the R -foil of u which is larger than u (if no such integer exists then define $R_\perp(u) = u$). Since R is a vertex-shift, R_\perp is a vertex-shift.

On the random network (\mathbb{Z}, X) , the map R_\perp is the same as the following map,

$$i \mapsto \begin{cases} \inf\{k > i : R_X^n(i) = R_X^n(k) \text{ for some } n \geq 1\} & \text{if infimum exists} \\ i & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{Z}$. In Proposition 56, we show that R_\perp is measure preserving.

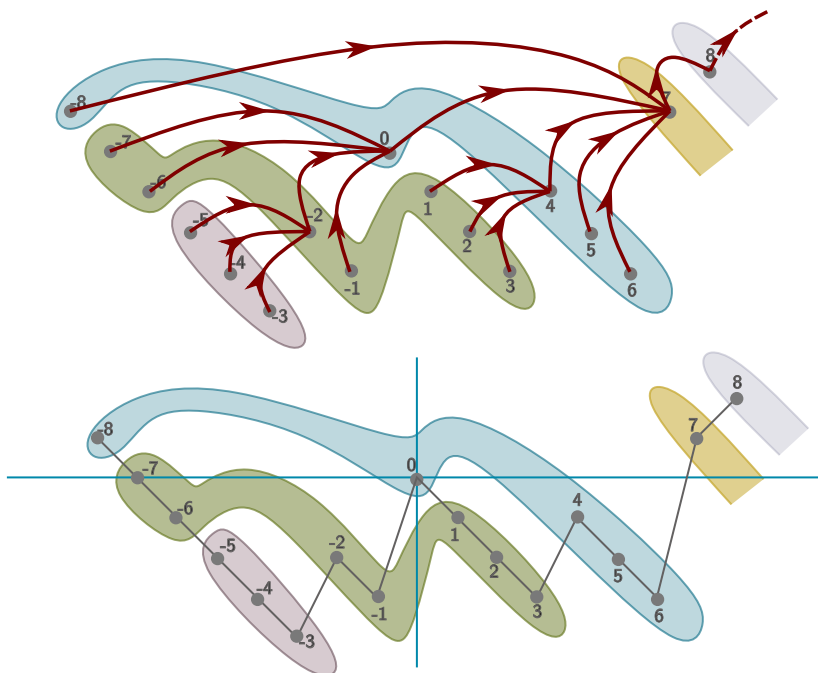


Figure 3.6: Illustration of R -foils in R -graph (above) and in trajectory (below) when the mean $\mathbb{E}[X_0] = 0$. Vertices that belong to the same R -foil are grouped together.

Proposition 56. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be the sequence of random variables as in Def. 22 and satisfies $\mathbb{E}[X_0] = 0$. The map R_\perp preserves the distribution of X , i.e., $[\mathbb{Z}, R_\perp(0), X] \stackrel{\mathcal{D}}{=} [\mathbb{Z}, 0, X]$.*

Proof. We show that R_\perp is bijective and use the fact that bijective maps preserve every unimodular measure (in our case, it is the distribution of $[\mathbb{Z}, 0, X]$). Firstly, we show that a.s. $R_\perp(i) \neq i$ for all $i \in \mathbb{Z}$. By Proposition

40, the component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ is of class \mathcal{I}/\mathcal{I} . Therefore, all R -foils are infinite. In order to show that a.s., $R_{\perp}(i) > i$ for all $i \in \mathbb{Z}$, it is enough to show that a.s., there does not exist an R -foil with a largest element (i.e., for each integer i , there exists an integer j such that $j > i$ and j belongs to the R -foil of i). There can be at most one largest element in an R -foil as the foil is totally ordered. Let S_T be the set of integers that are the largest in their R -foils. Then, S is a covariant subset. The R -foliation Π_X of $[\mathbb{Z}, 0, X]$ is a covariant partition. Since every foil $E \in \Pi$ is infinite, and $E \cap S$ has at most one element, by No Infinite/Finite inclusion Lemma 12, $E \cap S$ is empty. In particular, R -foils do not have any largest element a.s..

Secondly, since each R -foil is totally ordered as it is a subset of \mathbb{Z} , the map R_{\perp} is injective. Then, it is bijective almost surely by Proposition 11. Therefore, by Mecke-Thorisson Point Stationarity (Prop. 10) R_{\perp} preserves the distribution of $[\mathbb{Z}, 0, X]$. \square

Remark 57. In the proof of Proposition 56, we showed that, almost surely, R -foils of $[\mathbb{Z}, 0, X]$ do not have the largest vertex (according to RLS order). Similarly, by taking S_T to be set of vertices that are smallest in their R -foils, it can be shown that almost surely, none of the R -foils has the smallest vertex.

Note that it is not obvious to see that R_{\perp} is a measure preserving map without using the framework of R -foil structure and Mecke-Thorisson point Stationarity.

Chapter 4

Vertex-shift probabilities

In this chapter, we consider vertex-shifts on random rooted networks and study the existence and uniqueness of certain limits induced by them on the random networks. These limits are called vertex-shift probabilities. In Section 4.1, we introduce them and study their properties. Results analogous to those in Section 4.2-4.4 for point-shifts can be found in [6]. Given a random network and a vertex-shift f , one obtains the f -graph of the random network. The component of the root in the f -graph is a random rooted Family Tree. One may study the existence and uniqueness of vertex-shift probabilities of this Family Tree under parent vertex-shift and may wonder if this study helps to understand the vertex-shift probabilities of the random network. This question is addressed by Lemma 61, which plays a role in the next chapter. Under different circumstances, discussed in the later sections, existence of vertex-shift probabilities can be shown. The last section focuses on computing the F -probabilities of examples discussed in the previous chapter. Some of these examples will be used in the next chapter to show the existence of a unique vertex-shift probability induced by the record vertex-shift on the network $[\mathbb{Z}, 0, X]$ when X is an i.i.d. sequence as in Def. 22 and satisfy $\mathbb{E}[X_0] \geq 0$.

(In French) Dans ce chapitre, nous considérons les décalages de sommets sur les réseaux aléatoires enracinés et nous étudions l'existence et l'unicité de certaines limites induites par eux sur les réseaux aléatoires. Ces limites sont appelées probabilités de décalage de sommet. Dans la section 4.1, nous les introduisons et étudions leurs propriétés. Des résultats analogues à ceux de la section 4.2-4.4 pour les décalages de points peuvent être trouvés dans [6]. Étant donné un réseau aléatoire et un décalage de

sommet f , on obtient le f -graphe du réseau aléatoire. La composante de la racine dans le f -graphe est un arbre généalogique aléatoire enraciné. On peut étudier l'existence et l'unicité des probabilités de décalage de sommet de cet arbre généalogique associée au le décalage de sommet des parents et se demander si cette étude aide à comprendre les probabilités de décalage de sommet du réseau aléatoire. Cette question est abordée par le lemme 61, qui joue un rôle dans le chapitre suivant. Dans différentes circonstances, discutées dans les sections suivantes, l'existence des probabilités de décalage de sommet peut être montrée. La dernière section se concentre sur le calcul des F -probabilités des exemples discutés dans le chapitre précédent. Certains de ces exemples seront utilisés dans le chapitre suivant pour montrer l'existence d'une probabilité unique de décalage de sommet induite par le décalage de sommet des records sur le réseau $[\mathbb{Z}, 0, X]$ lorsque X est une suite i.i.d. comme dans la définition 22 et qui satisfait la condition $\mathbb{E}[X_0] \geq 0$.

4.1 Vertex-shift probabilities

Let $[\mathbf{G}, \mathbf{o}]$ be a random network, i.e., $[\mathbf{G}, \mathbf{o}]$ is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the space of connected and locally finite rooted networks \mathcal{G}_* . Let \mathcal{P}_0 be the probability measure on \mathcal{G}_* induced by the random network $[\mathbf{G}, \mathbf{o}]$. Let f be a vertex-shift. The map f induces a map $\theta_f : \mathcal{G}_* \rightarrow \mathcal{G}_*$, defined by $\theta_f([G, o]) = [G, f(o)]$. This determines an action of \mathbb{N} on \mathcal{G}_* defined by the iterates of θ_f , i.e., for each $n \geq 1$, we have the map $\theta_f^n : \mathcal{G}_* \rightarrow \mathcal{G}_*$ defined by $\theta_f^n([G, o]) = [G, f^n(o)]$, for all $[G, o] \in \mathcal{G}_*$. The push-forward measures of \mathcal{P}_0 under the iterates of θ_f are called the **vertex-shift probabilities** of $[\mathbf{G}, \mathbf{o}]$ under f .

Let \mathcal{P}_n be the push-forward measure of \mathcal{P}_0 under the n -th iterate θ_f^n of θ_f , i.e., $\mathcal{P}_n = (\theta_f^n)_* \mathcal{P}_0$. Therefore, for any measurable subset $A \subset \mathcal{G}_*$, we have

$$\mathcal{P}_n[A] = (\theta_f^n)_* \mathcal{P}_0[A] = \mathcal{P}_0[\{[G, o] : [G, f^n(o)] \in A\}] = \mathbb{P}[[\mathbf{G}, f^n(\mathbf{o})] \in A].$$

Given a random network \mathcal{P}_0 , we are interested to study the conditions under which the weak limit points of the sequence $\mathcal{A}_{\mathcal{P}_0} = \{\mathcal{P}_0, \mathcal{P}_0^{f,1}, \mathcal{P}_0^{f,2}, \dots\}$ exist. Any probability measure which is a weak limit of $\mathcal{A}_{\mathcal{P}_0}$ is called an **f -probability** of the random network $[\mathbf{G}, \mathbf{o}]$. If the sequence $(\mathcal{P}_0^{f,n})_{n \geq 0}$ (with the understanding that $\mathcal{P}_0 = \mathcal{P}_0^{f,0}$) converges weakly to a probability measure

\mathcal{P}_0^f then, we call \mathcal{P}_0^f **the f -probability** of \mathcal{P}_0 . Note that a probability measure \mathcal{P} on \mathcal{G}_* is a weak limit point of $\mathcal{A}_{\mathcal{P}_0}$ if and only if there exists a subsequence $\{\mathcal{P}_0^{f, n_k}\}$ such that

$$\int_{[G, u] \in \mathcal{G}_*} h([G, u]) \mathcal{P}_0^{f, n_k}(d[G, u]) \longrightarrow \int_{[G, u] \in \mathcal{G}_*} h([G, u]) \mathcal{P}(d[G, u])$$

for all continuous and bounded functions h on \mathcal{G}_* .

In ergodic theory, one studies measure preserving transformations. Seen from this perspective, Mecke's Point Stationarity ([7]) tells that a vertex-shift is unimodular measure preserving if and only if the vertex-shift is bijective almost surely.

Lemma 58. *Let $[\mathbf{G}, \mathbf{o}]$ be a unimodular network with distribution \mathcal{P} and f be a vertex-shift. Then, the distribution $\mathcal{P}_0^{f, n}$ of $[\mathbf{G}, f^n(\mathbf{o})]$ is absolutely continuous with respect to the initial distribution \mathcal{P}_0 with the Radon-Nikodym derivative*

$$\frac{d\mathcal{P}_0^{f, n}}{d\mathcal{P}}[G, o] = d_n(o).$$

Proof. Let h be a non-negative function on \mathcal{G}_* . Then,

$$\begin{aligned} \mathbb{E}[h([\mathbf{G}, f^n(\mathbf{o})])] &= \mathbb{E} \left[\sum_{u \in V(\mathbf{G})} h([\mathbf{G}, u]) \mathbf{1}_{u=f^n(\mathbf{o})} \right] \\ &= \mathbb{E} \left[\sum_{u \in V(\mathbf{G})} h([\mathbf{G}, \mathbf{o}]) \mathbf{1}_{\mathbf{o}=f^n(u)} \right] \quad (\text{by unimodularity}) \\ &= \mathbb{E}[h([\mathbf{G}, \mathbf{o}])d_n(\mathbf{o})]. \end{aligned}$$

□

Observe that, $d_1(\mathbf{o}) = 1$ a.s. if and only if f is bijective a.s.. Combining this observation with the above lemma, we obtain that a vertex-shift f on a unimodular network $[\mathbf{G}, \mathbf{o}]$ is bijective a.s. if and only if it is measure preserving (i.e., $[\mathbf{G}, f(\mathbf{o})] \stackrel{\mathcal{D}}{=} [\mathbf{G}, \mathbf{o}]$). This is Mecke-Thorisson point stationarity (see Prop. 10).

Lemma 58 implies that the probability measure $\mathcal{P}_0^{f, n}$ is supported on the networks that have n -th descendants. The following proposition compares this measure with the measure obtained from \mathcal{P}_0 by conditioning on the event that the root has n -th descendants.

Proposition 59. *Let $[\mathbf{G}, \mathbf{o}]$ be a unimodular network and \mathcal{P}_0 be the probability measure induced by $[\mathbf{G}, \mathbf{o}]$ on \mathcal{G}_* . Let f be a vertex-shift and $\{\mathcal{P}_0^{f,n}\}_{n \in \mathbb{N}}$ be the vertex-shift probabilities of \mathcal{P}_0 under f . Then, for any measurable subset A of \mathcal{G}_* ,*

$$\mathcal{P}_0^{f,n}[A] = \mathcal{E}_0 \left[\frac{d_n(o)}{\mathcal{E}_0[d_n(o) | d_n(o) > 0]} \mathbf{1}_A \mid d_n(o) > 0 \right],$$

where \mathcal{E}_0 is the expectation with respect to \mathcal{P}_0 .

Proof. By Lemma 58,

$$\mathcal{P}_0^{f,n}[A] = \mathcal{E}_0[d_n(o) \mathbf{1}_A] = \mathcal{E}_0[d_n(o) \mathbf{1}_A \mathbf{1}_{d_n(o) > 0}].$$

In particular, for $A = \mathcal{G}_*$, we obtain

$$\mathcal{P}_0^{f,n}[\mathcal{G}_*] = \mathcal{E}_0[d_n(o) \mathbf{1}_{d_n(o) > 0}].$$

Since $\mathcal{P}_0^{f,n}[\mathcal{G}_*] = 1$, we obtain,

$$1 = \mathcal{E}_0[d_n(o) \mathbf{1}_{d_n(o) > 0}] = \mathcal{E}_0[d_n(o) | d_n(o) > 0] \mathcal{P}_0[d_n(o) > 0].$$

From this, we obtain,

$$\mathcal{P}_0[d_n(o) > 0] = \frac{1}{\mathcal{E}_0[d_n(o) | d_n(o) > 0]}.$$

Therefore,

$$\begin{aligned} \mathcal{P}_0^{f,n}[A] &= \mathcal{E}_0[d_n(o) \mathbf{1}_A \mathbf{1}_{d_n(o) > 0}] = \mathcal{E}_0[d_n(o) \mathbf{1}_A | d_n(o) > 0] \mathcal{P}_0[d_n(o) > 0] \\ &= \mathcal{E}_0 \left[\frac{d_n(o)}{\mathcal{E}_0[d_n(o) | d_n(o) > 0]} \mathbf{1}_A \mid d_n(o) > 0 \right]. \end{aligned}$$

□

From the above proposition, $\mathcal{P}_0^{f,n}$ can be seen to be absolutely continuous with respect to the conditional probability $\mathcal{P}_0[\cdot | d_n(o) > 0]$.

Corollary 60. *Consider the conditions mentioned in Proposition 59. If the f -probability \mathcal{P}_0^f of \mathcal{P}_0 exists, then the following hold:*

1. For any bounded and continuous function $h : \mathcal{G}_* \rightarrow \mathbb{R}$,

$$\int_{\mathcal{G}_*} h([G, o]) d\mathcal{P}_0^f = \lim_{n \rightarrow \infty} \int_{\mathcal{G}_*} h([G, o]) d_n(o) d\mathcal{P}_0.$$

2. For any measurable subset A of \mathcal{G}_* for which $\mathcal{P}_0^{f,n}[A]$ tends to $\mathcal{P}_0^f[A]$ as n tends to infinity,

$$\mathcal{P}_0^f[A] = \lim_{n \rightarrow \infty} \mathcal{E}_0 \left[\frac{d_n(o)}{\mathcal{E}_0[d_n(o) | d_n(o) > 0]} \mathbf{1}_A \mid d_n(o) > 0 \right].$$

Proof. Follows from the definition of the f -probability of \mathcal{P}_0 . \square

Forward Backward Lemma

Consider a vertex-shift f and a unimodular random network $[\mathbf{G}, \mathbf{o}]$. Recall that the map $\theta_f : \mathcal{G}_* \rightarrow \mathcal{G}_*$ takes $[G, o]$ to $[G, f(o)]$, and the forward map $\Psi_f : \mathcal{G}_* \rightarrow \mathcal{T}_*$ takes $[G, o]$ to $[G^f(o), o]$, where $G^f(o)$ is the connected component of o (without any loops) in the f -graph G^f .

We know that $\Psi_f([\mathbf{G}, \mathbf{o}])$ is a unimodular Family Tree (Lemma 16). Consider the parent vertex-shift F on this Family Tree. If $\Psi_f([\mathbf{G}, \mathbf{o}])$ admits an F -probability, does it imply the existence and uniqueness of the f -probability? The following lemma gives sufficient conditions, under which we obtain a unique f -probability from the F -probability.

Lemma 61 (Forward Backward (FB)). *Let \mathcal{P} be a unimodular measure on \mathcal{G}_* , and f be a vertex-shift. Let $\mathcal{P}_n := \mathcal{P} \circ \theta_f^{-n}$, $n \geq 1$, be the push-forward measure of \mathcal{P} under the n -th iterate of the vertex-shift map θ_f . Let $\mathcal{Q} := \mathcal{P} \circ \Psi_f^{-1}$ and $\forall n \geq 1$, let $\mathcal{Q}_n := \mathcal{Q} \circ \theta_F^{-n}$ be the push-forward measure of \mathcal{Q} under the n -th iterate of the Parent vertex-shift map θ_F .*

Suppose there exists a map $\Phi_f : \mathcal{T}_ \rightarrow \mathcal{G}_*$ and a probability measure \mathcal{Q}_F on \mathcal{G}'_* such that*

1. $\Phi_f \circ \Psi_f = I$, \mathcal{P} -a.s., where I is the identity map.
2. \mathcal{Q}_F is the F -probability of \mathcal{Q} , i.e., $\mathcal{Q}_n \rightarrow \mathcal{Q}_F$ weakly as $n \rightarrow \infty$.
3. Φ_f is continuous \mathcal{Q}_F -a.s..

Then, $\mathcal{Q}_F \circ \Phi_f^{-1}$ is the f -probability of \mathcal{P} .

Proof. Observe that $\Psi_f \circ \theta_f = \theta_F \circ \Psi_f$. Indeed, for any $[G, o] \in \mathcal{G}_*$,

$$\begin{aligned} \Psi_f \circ \theta_f([G, o]) &= \Psi_f([G, f(o)]) = [G^f(f(o)), f(o)] \\ &= [G^f(o), f(o)] = [G^f(o), F(o)] = \theta_F \circ \Psi_f([G, o]). \end{aligned}$$

The third equality in the above equation follows from the fact that the connected component of o in G^f is the same as that of $f(o)$. Using this inductively we obtain, for any $n \geq 1$,

$$\mathcal{Q}_n = \mathcal{Q} \circ \theta_F^{-n} = \mathcal{P} \circ \Psi_f^{-1} \circ \theta_F^{-n} = \mathcal{P} \circ \theta_f^{-n} \circ \Psi_f^{-1} = \mathcal{P}_n \circ \Psi_f^{-1}.$$

We know that \mathcal{Q} is unimodular because \mathcal{P} is unimodular by Lemma 16. By Lemma 58, \mathcal{Q}_n (resp. \mathcal{P}_n) is absolutely continuous with respect to \mathcal{Q} (resp. \mathcal{P}) for all $n \geq 1$. So, it follows from the first condition that $\Phi_f \circ \Psi_f = I$ \mathcal{P}_n -a.s.. Using this, we obtain

$$\mathcal{P}_n = \mathcal{P}_n \circ (\Phi_f \circ \Psi_f)^{-1} = \mathcal{P}_n \circ \Psi_f^{-1} \circ \Phi_f^{-1} = \mathcal{Q}_n \circ \Phi_f^{-1}.$$

Since \mathcal{Q}_n weakly converges to \mathcal{Q}_F and Φ_f is continuous \mathcal{Q}_F -a.s., by the Mann-Wald theorem [9], $\mathcal{Q}_n \circ \Phi_f^{-1}$ converges weakly to $\mathcal{Q}_F \circ \Phi_f^{-1}$ as $n \rightarrow \infty$. Thus, $\lim_{n \rightarrow \infty} \mathcal{P}_n = \mathcal{Q}_F \circ \Phi_f^{-1}$. \square

In general, the existence of the map Φ_f is not guaranteed since the f -graph may not contain the needed information about the structure of the network. For instance, consider the identity vertex-shift. In this case, the connected components of its f -graph are just singletons. So, no information can be obtained from the f -graph. We may however attach more information of a network to its f -graph as marks. In the case of the record vertex-shift R on the random network $[\mathbb{Z}, 0, X]$ (where X is the sequence of i.i.d. increments of a skip-free to the left random walk), it suffices to consider the order of \mathbb{Z} when $\mathbb{E}[X_0] = 0$ and both the order and the type function t when $\mathbb{E}[X_0] > 0$ as we shall see in the next chapter. Note that, in FB Lemma 61, the backward map Φ_f may depend on the initial distribution \mathcal{P} .

4.2 Finite orbits and relative compactness

We use the notations of Subsection 2.1. In this section, we are only focused on unlabelled graphs. Recall that $\hat{\mathcal{G}}_*$ denotes the space of rooted graphs.

Definition 62. A vertex-shift f is said to have **finite orbits** on a random graph $[\mathbf{G}, \mathbf{o}]$ if the set $\{[G, f^n(o)], n \in \mathbb{N}\}$ is finite for $-\mathcal{P}_0$ -a.s. $[G, o] \in \hat{\mathcal{G}}_*$, where \mathcal{P}_0 is the probability measure induced by $[\mathbf{G}, \mathbf{o}]$.

Proposition 63. Let $[\mathbf{G}, \mathbf{o}]$ be a random graph which induces the probability measure \mathcal{P}_0 on $\hat{\mathcal{G}}_*$, f be a vertex-shift and $\mathcal{A}_{\mathcal{P}_0} = \{\mathcal{P}_0^{f,n}, n \in \mathbb{N}\}$ be the vertex-shift probabilities induced by f . If f has finite orbits on $[\mathbf{G}, \mathbf{o}]$ then $\mathcal{A}_{\mathcal{P}_0}$ is relatively compact.

Proof. Let $r \geq 1$ be fixed. We show that,

$$\limsup_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P}[[\mathbf{G}, f^n(\mathbf{o})]_r \notin K_m^r] = 0. \quad (4.2.1)$$

For any fixed $m \geq 1$ and for all $n \geq 1$, the event $\{[\mathbf{G}, f^n(\mathbf{o})]_r \in \hat{\mathcal{G}}_* \setminus K_m^r\}$ is a subset of $\{orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset\}$, where $orb([\mathbf{G}, \mathbf{o}])_r = \{[\mathbf{G}, f^n(\mathbf{o})]_r, n \in \mathbb{N}\}$. This implies that

$$\sup_{n \in \mathbb{N}} \mathbb{P}[[\mathbf{G}, f^n(\mathbf{o})]_r \in \hat{\mathcal{G}}_* \setminus K_m^r] \leq \mathbb{P}[orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset].$$

Since K_m^r is an increasing sequence in m , $\hat{\mathcal{G}}_* \setminus K_m^r$ is a decreasing sequence of sets, which implies that the events $\{orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset\}$ are decreasing in m . Therefore,

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P}[[\mathbf{G}, f^n(\mathbf{o})]_r \in \hat{\mathcal{G}}_* \setminus K_m^r] \\ & \leq \lim_{m \rightarrow \infty} \mathbb{P}[orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset] \\ & = \mathbb{P} \left[\bigcap_{m \in \mathbb{N}} \{orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset\} \right]. \end{aligned}$$

But $orb([\mathbf{G}, \mathbf{o}])_r$ is a.s. finite. So, it can have non-empty intersection with $\hat{\mathcal{G}}_* \setminus K_m^r$ only for finitely many m . Therefore,

$$\mathbb{P} \left[\bigcap_{m \in \mathbb{N}} \{orb([\mathbf{G}, \mathbf{o}])_r \cap (\hat{\mathcal{G}}_* \setminus K_m^r) \neq \emptyset\} \right] = 0,$$

which gives the desired equality in Equation (4.2.1). \square

4.3 Evaporation

The n -th vertex-shift-probability measure $\mathcal{P}_0^{f,n}$ of a probability measure \mathcal{P}_0 is supported on the rooted network $[G, o]$ whose root o has n -th descendants, i.e., $d_n(o) > 0$. This leads us to ask, “are all f -probabilities, assuming they exist, supported only on the rooted networks whose root has infinitely many descendants, i.e., $d_n(o) > 0, \forall n \in \mathbb{N}$?”. The answer is affirmative under the sufficient conditions mentioned in Lemma 65 or in Lemma 66.

Let us denote $I := \{[G, o] \in \mathcal{G}_* \text{ such that } d_n(o) > 0 \forall n \geq 1\}$, the set of rooted networks whose root has infinitely many descendants. Let $A_m = \{[G, o] \in \mathcal{G}_* \text{ such that } d_m(o) = 0\}$ and $J = \cup_{m \geq 1} A_m$. Then, J is the set of rooted networks whose root has finitely many descendants, and we have

$$J^c = \cap_{m \geq 1} A_m^c = \cap_{m \geq 1} \{[G, o] \in \mathcal{G}_* : d_m(o) > 0\} = I.$$

Definition 64. We say that a random network $[\mathbf{G}, \mathbf{o}]$ **evaporates** on a vertex-shift f if $\mathbb{P}[[\mathbf{G}, \mathbf{o}] \in J] = 1$.

Lemma 65. *Let f be a vertex-shift and $[\mathbf{G}, \mathbf{o}]$ be a unimodular random network. If Q is an f -probability of $[\mathbf{G}, \mathbf{o}]$, then $Q[I] = 1$.*

Proof. For any $n \in \mathbb{N}$, let us denote $B_n = \{[G, o] \in \mathcal{G}_* \text{ such that } d_n(o) > 0\}$. Then B_n is a non-increasing sequence of subsets whose intersection is I , i.e., $I = \cap_n B_n$ and $B_n \searrow I$. Therefore,

$$Q[I] = \lim_{n \rightarrow \infty} Q[B_n].$$

Since Q is a f -probability, there exists a subsequence $\{\mathcal{P}_0^{f, n_k}\}$ of the vertex-shift probabilities $\{\mathcal{P}_0^{f, n}\}$ that converges weakly to Q . Equivalently, for any continuity set B , $\mathcal{P}_0^{f, n_k}[B] \rightarrow Q[B]$ as k tends to infinity.

For all $n \geq 1$, the map $[G, o] \mapsto \mathbf{1}_{d_n(o) > 0}$ is continuous on \mathcal{G}_* since for any sequence $([G_m, o_m])_{m \geq 1}$ that converges to a network $[G', o']$ as $m \rightarrow \infty$, the sequence $(d_n(o_m))_{m \geq 1}$ is eventually a constant sequence, i.e., there exists $k \geq 1$ such that $d_n(o_m) = d_n(o')$ for all $m \geq k$. Therefore, B_n is a Q -continuity set, i.e., $\mathcal{P}_0^{f, n_k}[B_n] \rightarrow Q[B_n]$ as k tends to infinity. But, notice that $\mathcal{P}_0^{f, n_k}[B_{n_k}] = 1$, as it is only supported on the roots that have n_k -th descendants, and $B_{n_k} \subseteq B_n$ for all $n_k \geq n$. So, $\mathcal{P}_0^{f, n_k}[B_n] = 1$ for all $n_k \geq n$ which implies that $Q[B_n] = 1$. This gives the desired result that $Q[I] = 1$. \square

Lemma 66. *Let f be a vertex-shift and \mathcal{Q} be a probability measure on \mathcal{G}_* which satisfies Mecke's invariant measure equation, i.e., $(\theta_f)^*\mathcal{Q} = \mathcal{Q}$. Then $\mathcal{Q}[I] = 1$.*

Proof. We show that $\mathcal{Q}[J] = 0$. For any $m \in \mathbb{N}$, let us denote $A_m = \{[G, o] \in \mathcal{G}_* : d_m(o) = 0\}$. Since $(\theta_f)_*\mathcal{Q} = \mathcal{Q}$, we have $(\theta_f^m)_*\mathcal{Q} = \mathcal{Q}$ for any $m \in \mathbb{N}$. Therefore,

$$\begin{aligned} \mathcal{Q}[A_m] &= (\theta_f^m)_*\mathcal{Q}[A_m] = \mathcal{Q}[(\theta_f^m)^{-1}(A_m)] = \mathcal{Q}[[G, o] \in \mathcal{G}_* : [G, f^m(o)] \in A_m] \\ &= 0 \text{ (since no element of } A_m \text{ has } m\text{-th descendant)}. \end{aligned}$$

Since $\mathcal{Q}[A_m] = 0$ for all $m \in \mathbb{N}$, we have

$$\mathcal{Q}[J] = \mathcal{Q}\left[\bigcup_{m \in \mathbb{N}} A_m\right] = 0.$$

Therefore, $\mathcal{Q}[I] = 1$. □

Corollary 67. *Let $[\mathbf{G}, \mathbf{o}]$ be a random network which evaporates under the action of a vertex-shift f . Let \mathcal{P}_0^f be a f -probability of \mathcal{P}_0 (the distribution of $[\mathbf{G}, \mathbf{o}]$) which satisfies Mecke's invariant measure equation, i.e., $(\theta_f)^*\mathcal{P}_0^f = \mathcal{P}_0^f$. Then, \mathcal{P}_0^f is singular with respect to the distribution \mathcal{P}_0 of $[\mathbf{G}, \mathbf{o}]$.*

Proof. Since $[\mathbf{G}, \mathbf{o}]$ evaporates under f , $\mathcal{P}_0[J] = 1$. By Lemma 66, $\mathcal{P}_0^f[J] = 0$. Therefore, \mathcal{P}_0^f is singular with respect to \mathcal{P}_0 . □

Corollary 68. *Let $[\mathbf{G}, \mathbf{o}]$ be a random network which evaporates under the action of a vertex-shift f . Assume that the vertex-shift probabilities converge to a unique f -probability \mathcal{P}_0^f and θ_f is continuous at \mathcal{P}_0^f . Then, \mathcal{P}_0^f is singular with respect to the distribution \mathcal{P}_0 of $[\mathbf{G}, \mathbf{o}]$.*

Proof. Let $\{\mathcal{P}_0^{f,n}\}$ be the vertex-shift-probabilities of \mathcal{P}_0 . Since, for all $n \geq 1$, $(\theta_f)_*\mathcal{P}_0^{f,n} = \mathcal{P}_0^{f,n+1}$, the sequence $((\theta_f)_*\mathcal{P}_0^{f,n})_n$ converges to \mathcal{P}_0^f . Since θ_f is continuous at \mathcal{P}_0^f , we have $(\theta_f)_*\mathcal{P}_0^f = \mathcal{P}_0^f$. The statement follows from Corollary 67. □

4.4 Periodicity

In the previous Section, we saw conditions under which an f -probability is singular with respect to the initial distribution. In this section, we show that

if the vertex-shift is periodic then f -probabilities are absolutely continuous with respect to the initial distribution (see Theorem 72, Remark 74).

Definition 69. A vertex-shift f is said to be **periodic** if for all G and $o \in V(G)$, there exists positive integers $k = k(o)$ and $p = p(o)$ such that

$$\forall n \geq k, f_G^{n+p}(o) = f^n(o).$$

Definition 70. A vertex-shift f is said to be **p -periodic** if p does not depend on $[G, o]$.

A vertex $o \in V(G)$ is called a **trap** of G with respect to a vertex-shift f if $f_G(o) = o$. For the identity vertex-shift, every vertex of every network is a trap. Note that, for any vertex-shift f , the collection of set of traps of networks $S_G := \{u \in V(G) : f_G(u) = u\}$ is a covariant subset.

Lemma 71. *If a vertex-shift f is 1-periodic a.s. on a random network $[\mathbf{G}, \mathbf{o}]$ then the limit $\lim_{n \rightarrow \infty} [\mathbf{G}, f^n(\mathbf{o})]$ exists and is supported on the traps, i.e., $\mathbb{P}[\lim_{n \rightarrow \infty} [\mathbf{G}, f^n(\mathbf{o})] \in B] = 1$, where $B := \{[G, o] \in \mathcal{G}_* : o \text{ is a trap}\}$.*

Proof. Since f is 1-periodic a.s., for \mathcal{P}_0 -a.s. $[G, o] \in \mathcal{G}_*$ there exists $k = k(o)$ such that

$$\forall n \geq k, f_G^{n+1}(o) = f_G^n(o) = f_G^k(o).$$

Therefore, $f_G^k(o)$ is a trap. Hence, $\mathbb{P}[\lim_{n \rightarrow \infty} [\mathbf{G}, f^n(\mathbf{o})] \in B] = 1$. \square

In view of the above lemma, for a random network $[\mathbf{G}, \mathbf{o}]$ and a 1-periodic vertex-shift f , let $[\bar{\mathbf{G}}, \bar{\mathbf{o}}] = \lim_{n \rightarrow \infty} [\mathbf{G}, f^n(\mathbf{o})]$ and \mathcal{P}_0^∞ be the distribution of $[\bar{\mathbf{G}}, \bar{\mathbf{o}}]$.

Theorem 72. *Let $[\mathbf{G}, \mathbf{o}]$ be a random network, \mathcal{P}_0 be its distribution and f be an a.s. 1-periodic vertex-shift on $[\mathbf{G}, \mathbf{o}]$. Then the f -probability \mathcal{P}_0^f of \mathcal{P}_0 exists and \mathcal{P}_0^f is the distribution of $[\bar{\mathbf{G}}, \bar{\mathbf{o}}]$.*

In addition, if $[\mathbf{G}, \mathbf{o}]$ is unimodular then the f -probability \mathcal{P}_0^f is absolutely continuous with respect to the initial distribution \mathcal{P}_0 . The limit function $d_\infty(o) = \lim_{n \rightarrow \infty} d_n(o)$ exists and it is finite for \mathcal{P}_0^f -a.s. $[G, o]$. The Radon-Nikodym derivative of \mathcal{P}_0^f with respect to \mathcal{P}_0 is given by

$$\frac{d\mathcal{P}_0^f}{d\mathcal{P}_0}[G, o] = d_\infty(o) \mathbf{1}\{o \text{ is a trap}\}.$$

Proof. We prove that \mathcal{P}_0^∞ (the distribution of $[\bar{\mathbf{G}}, \bar{\mathbf{o}}]$) is the f -probability of \mathcal{P}_0 . For any bounded and continuous function $h : \mathcal{G}_* \rightarrow \mathbb{R}$,

$$\begin{aligned} \int_{\mathcal{G}_*} h[G, o] d\mathcal{P}_0^\infty &= \mathbb{E}[h[\bar{\mathbf{G}}, \bar{\mathbf{o}}]] = \mathbb{E}[h(\lim_{n \rightarrow \infty} [\mathbf{G}, f^n(\mathbf{o})])] \\ &= \mathbb{E}[\lim_{n \rightarrow \infty} h[\mathbf{G}, f^n(\mathbf{o})]] \text{ (by continuity of } h\text{)} \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[h([\mathbf{G}, f^n(\mathbf{o})])]. \end{aligned}$$

In the last step of the above equation, we used the boundedness of h and the dominated convergence theorem. Thus, \mathcal{P}_0^∞ is the f -probability of \mathcal{P}_0 .

We now assume that $[\mathbf{G}, \mathbf{o}]$ is unimodular.

The limit function d_∞ :

The sequence of functions $(d_n(o))_{n \in \mathbb{N}}$ is non-decreasing on the set of traps due to the following reason. Let $o \in V(G)$ be a trap, and $u \in D_n(o)$ for some $n \in \mathbb{N}$. Then, since $f_G^{n+1}(u) = f_G^n(u) = o$, we have $D_n(o) \subseteq D_{n+1}(o)$ and $d_n(o) \leq d_{n+1}(o)$.

Since the random network $[\bar{\mathbf{G}}, \bar{\mathbf{o}}]$ is supported on the traps, the random variable

$$d_\infty([\bar{\mathbf{G}}, \bar{\mathbf{o}}]) = \lim_{n \rightarrow \infty} d_n([\bar{\mathbf{G}}, \bar{\mathbf{o}}])$$

exists a.s.

As for the finiteness of the random variable $d_\infty([\bar{\mathbf{G}}, \bar{\mathbf{o}}])$, let

$$\mathcal{A} := \{[G, o] \in \mathcal{G}_* : f_G^{n+1}(o) = f_G^n(o) \text{ for some } n = n(o) \in \mathbb{N}\}.$$

Then, for any $[G, o] \in \mathcal{A}$, the root o has finitely many ancestors in the f -graph. Therefore, by the unimodular classification theorem ([7]), the connected component $C(o)$ of o is of type \mathcal{F}/\mathcal{F} as it is the only case in which vertices have finitely many ancestors. This implies that $C(o)$ is finite. But the connected component of $f^{n(o)}(o)$ in the f -graph is also $C(o)$ and $f^{n(o)}(o)$ is a trap. Thus, $d_\infty(\lim_{m \rightarrow \infty} [G, f^m(o)]) = \#D(f^{n(o)}(o)) = \#C(o)$ is finite.

Radon-Nikodym derivation of \mathcal{P}_0^f with respect to \mathcal{P}_0 :

Let $h : \mathcal{G}_* \rightarrow \mathbb{R}_{\geq 0}$ be any measurable function. Then,

$$\begin{aligned}
\int_{\mathcal{G}_*} h([G, o]) d\mathcal{P}_0^f &= \mathbb{E}[h([\bar{\mathbf{G}}, \bar{\mathbf{o}}])] = \mathbb{E}\left[\sum_{u \in V(\bar{\mathbf{G}})} h([\bar{\mathbf{G}}, u]) \mathbf{1}\{\bar{\mathbf{o}} = u\}\right] \\
&= \mathbb{E}\left[\sum_{u \in V(\mathbf{G})} h([\mathbf{G}, u]) \mathbf{1}\{f^{n(\mathbf{o})}(\mathbf{o}) = u\} \mathbf{1}\{f^{n(\mathbf{o})}(\mathbf{o}) \text{ is a trap}\}\right] \\
&= \mathbb{E}\left[\sum_{u \in V(\mathbf{G})} h([\mathbf{G}, \mathbf{o}]) \mathbf{1}\{f^{n(u)}(u) = \mathbf{o}\} \mathbf{1}\{f^{n(u)}(u) \text{ is a trap}\}\right] \quad (4.4.1) \\
&= \mathbb{E}[h([\mathbf{G}, \mathbf{o}]) \#D(\mathbf{o}) \mathbf{1}\{\mathbf{o} \text{ is a trap}\}] \\
&= \mathbb{E}\left[h([\mathbf{G}, \mathbf{o}]) \lim_{m \rightarrow \infty} d_m(\mathbf{o}) \mathbf{1}\{\mathbf{o} \text{ is a trap}\}\right] \\
&= \mathbb{E}[h([\mathbf{G}, \mathbf{o}]) d_\infty(\mathbf{o}) \mathbf{1}\{\mathbf{o} \text{ is a trap}\}].
\end{aligned}$$

We used unimodularity to get Eq. (4.4.1). Thus, we obtain the Radon-Nikodym derivative of \mathcal{P}_0^f with respect to \mathcal{P}_0 ,

$$\frac{d\mathcal{P}_0^f}{d\mathcal{P}_0}[G, o] = d_\infty(o) \mathbf{1}\{o \text{ is a trap}\}.$$

□

Remark 73. From the proof of Theorem 72, it follows that $d_\infty(\mathbf{o}) = \#D(\mathbf{o})$.

Remark 74. Let p be a positive integer. If a vertex-shift f is a.s. p -periodic on a random network $[\mathbf{G}, \mathbf{o}]$ then for \mathcal{P}_0 -a.s. $[G, o] \in \mathcal{G}_*$, there exists a $k = k(o) \in \mathbb{N}$ such that $f_G^{n+p}(o) = f_G^n(o)$ for all $n \geq k$. In particular, $f_G^{np+i+p}(o) = f_G^{np+i}(o)$ for all $n \geq k$ and for any $i = 0, 1, \dots, p-1$. So, the vertex-shift f^p is a.s. 1-periodic on the random networks $\{[\mathbf{G}, \mathbf{o}], [\mathbf{G}, f(\mathbf{o})], \dots, [\mathbf{G}, f^{p-1}(\mathbf{o})]\}$. By Theorem 72, for each $i = 0, 1, \dots, p-1$, $[\mathbf{G}, f^i(\mathbf{o})]$ has unique f^p -probability. So, $[\mathbf{G}, \mathbf{o}]$ can have at most p f -probabilities. In addition, if $[\mathbf{G}, \mathbf{o}]$ is unimodular, then all of its f -probabilities are absolutely continuous with respect to \mathcal{P}_0 by Lemma 58 and by Theorem 72.

4.5 Examples and their F-probabilities

In this section, we compute the F -probabilities of some Family Trees. A few remarks are given in the respective subsections about the application

of results of the previous section to the examples studied here. Among the examples, the Canopy tree and the unimodular Eternal Galton-Watson Tree are taken from [7]. We have already introduced two new unimodular Family Trees namely, the typically rooted Galton-Watson Tree and the unimodularised bi-variate Eternal Kesten Tree. The F -probability of the latter is shown to be the bi-variate Eternal Kesten Tree in Proposition 82. We introduce Eternal Kesten Tree, a special instance of bi-variate Eternal Kesten Tree, and prove that it is the F -probability of the Eternal Galton-Watson Tree (in Proposition 80). We also introduce the size-biased Galton-Watson Tree, and show that it is the F -probability of the typically rooted Galton-Watson Tree.

The results from this section together with FB Lemma 61 will be used in the next chapter to compute the R -probability.

Canopy tree

The Canopy tree with the number of children d , C_d is an Eternal Family Tree with vertices partitioned into infinitely many levels L_0, L_1, L_2, \dots such that, for any $i > 0$, every vertex v in L_i has d children in L_{i-1} , and a unique parent in L_{i+1} ; the vertices of L_0 do not have any children but each vertex of L_0 has a unique parent in L_1 . The tree is deterministic, but the root is chosen randomly to obtain a random rooted Family Tree $[C_d, \mathbf{o}]$.

Since, for any two vertices u, v of the same level L_i (for any $i \geq 0$), the equivalent class of (C_d, u) and (C_d, v) are one and the same, the set of probabilities $\{p_i = \mathbb{P}[\mathbf{o} \in L_i] : i \geq 1\}$ fully characterizes the distribution of $[C_d, \mathbf{o}]$. Thus, in order to describe the distribution of the random root \mathbf{o} of C_d , it is enough to define $\mathbb{P}[\mathbf{o} \in L_i]$. Let

$$p_i = \mathbb{P}[\mathbf{o} \in L_i], \quad i \geq 0.$$

Let T_{d+1} be the directed tree whose vertices have one out-degree and d in-degree, i.e., T_{d+1} is the $d+1$ regular tree with one selected end. Consider the parent vertex-shift on C_d . In Proposition 76, we show that the randomly rooted Canopy tree $[C_d, \mathbf{o}]$ has a unique F -probability T_{d+1} .

Recall the notations of Section 2.1 of Chapter 2.

Lemma 75. *Let u be a vertex of the canopy tree C_d that belongs to level L_i . Then*

$$((C_d, u)_r, u) \sim ((T_{d+1}, v)_r, v) \text{ for all } 0 \leq r \leq i.$$

Proof. We prove the lemma by induction on r . For $r = 0$, $((C_d, u)_r, u) \sim ((T_{d+1}, v)_r, v)$. Let us assume that the statement is true for $r - 1 \geq 0$ and let $\rho : (C_d, u)_{r-1} \rightarrow (T_{d+1}, v)_{r-1}$ be a network isomorphism such that $\rho(u) = v$. To show $((C_d, u)_r, u) \sim ((T_{d+1}, v)_r, v)$, it is enough to extend the isomorphism ρ to every vertex in $(C_d, u)_r \setminus (C_d, u)_{r-1}$.

If $w \in (C_d, u)_r \setminus (C_d, u)_{r-1}$ then w can only be one of the three cases: an ancestor of u , or a descendant of u , or a descendant of an ancestor of u .

- If w is an ancestor of u then $w = F^r(u)$ and $F^{r-1}(u) \in (C_d, u)_{r-1}$. We extend ρ to w by assigning $\rho(w)$ to the parent of $\rho(F^{r-1}(u))$ in T_{d+1} .
- If w is a descendant either of u or, of an ancestor of u then $F(w) \in (C_d, u)_{r-1}$. As $i \geq r$, the vertex $F(w) \notin L_0$. Therefore, $F(w)$ has d -children and we can obtain a bijection between the children of $F(w)$ and the children of $\rho(F(w))$. We extend ρ to the children of $F(w)$ by using this bijection.

□

Let $[C_d, u_i]$ be the equivalence class of any u_i vertex in L_i . By the above lemma, the distance between $[C_d, u_i]$ and $[T_{d+1}, v]$ in \mathcal{T}_* is bounded above by $\frac{1}{2^i}$. Thus, $[C_d, u_i] \rightarrow [T_{d+1}, v]$ in \mathcal{G}_* as i tends to infinity. We now show that the F -probability of a randomly rooted Canopy tree $[C_d, \mathbf{o}]$ is $[T_{d+1}, v]$, for any vertex v of T_{d+1} .

Proposition 76. *Let $[C_d, \mathbf{o}]$ be the canopy tree with \mathbf{o} the root \mathbf{o} chosen randomly and $[C_d, \mathbf{o}_i] \in \mathcal{T}_*$ be the canopy tree with random root \mathbf{o}_i , where $\mathbf{o}_i = F^i(\mathbf{o})$, $i \geq 0$. Then $[C_d, \mathbf{o}_i]$ converges in distribution to $[T_{d+1}, v]$ as $i \rightarrow \infty$, for any vertex v of T_{d+1} .*

Proof. Let g be a continuous bounded function on \mathcal{G}_* , $u_k \in L_k$ be a vertex of C_d , $g_k := g([C_d, u_k])$, and $a := g([T_{d+1}, v])$. As $[C_d, u_k] \rightarrow [T_d, v]$ in \mathcal{G}_* when k tends to infinity, and by the continuity of the function g , g_k converges to a .

The probability, $\mathbb{P}[\mathbf{o}_i \in L_k] = \mathbb{P}[F^i(\mathbf{o}) \in L_k] = \mathbb{P}[\mathbf{o} \in L_{k-i}]$ is given by

$$\mathbb{P}[\mathbf{o}_i \in L_k] = \begin{cases} 0 & k - i < 0 \\ p_{k-i} & k - i \geq 0. \end{cases}$$

and we have

$$\mathbb{E}[g([C_d, \mathbf{o}_i])] = \sum_{k=0}^{\infty} g([C_d, u_k]) \mathbb{P}[\mathbf{o}_i \in L_k] = \sum_{k=i}^{\infty} g_k p_{k-i}.$$

Since g_i converges to a as i tends to infinity, given $\epsilon > 0$ there exists N such that for all $i \geq N$, $|g_i - a| < \epsilon$. Therefore, for such i ,

$$\begin{aligned} |\mathbb{E}[g([C_d, \mathbf{o}_i])] - g([T_{d+1}, v])| &= \left| \sum_{k=i}^{\infty} g_k p_{k-i} - a \right| = |p_0(g_i - a) + p_1(g_{i+1} - a) + \dots| \\ &\leq p_0|g_i - a| + p_1|g_{i+1} - a| + \dots + p_k|g_{i+k} - a| + \dots \\ &< \epsilon \left(\sum_{k=0}^{\infty} p_k \right) < \epsilon. \end{aligned}$$

□

Remark 77. Observe that the map θ_F induced by the parent vertex-shift F is continuous on \mathcal{T}_* . It is also clear that the randomly rooted Canopy tree $[C_d, \mathbf{o}]$ evaporates under F as the root \mathbf{o} has a.s. finitely many descendants. Therefore, by Corollary 68, the F -probability of $[C_d, \mathbf{o}]$ is singular with respect to the distribution of $[C_d, \mathbf{o}]$. This is indeed true since the directed regular tree $[T_{d+1}, u]$ has infinitely many descendants.

Typically rooted Galton-Watson Tree (TGWT)

Let π be a probability distribution on $\{0, 1, 2, \dots\}$ with mean $m(\pi) < 1$ and $[\mathbf{T}', \mathbf{o}']$ be a random Family Tree distributed as $GW(\pi)$. The **Size-Biased Galton-Watson Tree** with offspring distribution π , denoted as $SBGW(\pi)$ is a random Family Tree $[\bar{\mathbf{T}}, \bar{\mathbf{o}}]$ whose distribution is given by:

$$\mathbb{P}[[\bar{\mathbf{T}}, \bar{\mathbf{o}}] \in A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T}')] } \mathbb{E}[\#V(\mathbf{T}') \mathbf{1}_A([\mathbf{T}', \mathbf{o}'])], \quad (4.5.1)$$

for every measurable set A . Note that $\mathbb{E}[\#V(\mathbf{T})]$ is finite since $m(\pi) < 1$.

We show that the F -probability of $TGWT(\pi)$ is $SBGW(\pi)$ using Theorem 72.

Proposition 78. *Let $m(\pi) < 1$. Then, the F -probability of $TGWT(\pi)$ is $SBGW(\pi)$.*

Proof. Let $[\mathbf{T}, \mathbf{o}]$ be a random Family Tree distributed as $TGWT(\pi)$ and let $[\mathbf{T}', \mathbf{o}']$ be a random Family Tree distributed as $GW(\pi)$. Since, the root \mathbf{o} has a.s. finitely many ancestors, $[\mathbf{T}, \mathbf{o}]$ is a.s. 1-periodic with respect to the parent vertex-shift F . By Theorem 72, the F -probability of $[\mathbf{T}, \mathbf{o}]$ exists. Let $[\bar{\mathbf{T}}, \bar{\mathbf{o}}]$ be the F -probability of $[\mathbf{T}, \mathbf{o}]$. Since $TGWT(\pi)$ is unimodular (see Proposition 52), by Theorem 72, we have

$$\mathbb{P}[[\bar{\mathbf{T}}, \bar{\mathbf{o}}] \in A] = \mathbb{E}[\#V(\mathbf{T})\mathbf{1}\{\mathbf{o} \text{ is a trap}\}\mathbf{1}_A([\mathbf{T}, \mathbf{o}])],$$

for every measurable set A . By Proposition 50, we have

$$\mathbb{P}[[\mathbf{T}, \mathbf{o}] \in A] = \frac{1}{\mathbb{E}[\#V(\mathbf{T}')]}\mathbb{E}\left[\sum_{u \in V(\mathbf{T}')} \mathbf{1}_A([\mathbf{T}', u])\right],$$

for every measurable set A . Combining the above two equations, we obtain

$$\begin{aligned} \mathbb{P}[[\bar{\mathbf{T}}, \bar{\mathbf{o}}] \in A] &= \frac{1}{\mathbb{E}[\#V(\mathbf{T}')]}\mathbb{E}\left[\#V(\mathbf{T}') \sum_{u \in V(\mathbf{T}')} \mathbf{1}\{\mathbf{u} \text{ is a trap}\}\mathbf{1}_A([\mathbf{T}', u])\right] \\ &= \frac{1}{\mathbb{E}[\#V(\mathbf{T}')]}\mathbb{E}[\#V(\mathbf{T}')\mathbf{1}_A([\mathbf{T}', \mathbf{o}'])], \end{aligned}$$

for every measurable set A . The last step follows since the root is the only vertex that does not have a parent and thus the statement is proved. \square

Unimodular Eternal Galton-Watson Tree

In the following, we introduce a new EFT called the Eternal Kesten Tree (EKT) and show that it is the F -probability of the unimodular Eternal Galton-Watson Tree. The EKT is an eternal version of Kesten's tree which was defined in [1].

Let $[\mathbf{T}, \mathbf{o}]$ be a rooted Family Tree. A vertex of \mathbf{T} is called **common**(π) if its offspring distribution is π and all of its descendants reproduce independently with offspring distribution π . A vertex of \mathbf{T} is **special**(π) if it satisfies the following three properties: (1) its offspring distribution is $\hat{\pi}$ (the size-biased distribution of π), (2) all of its descendants reproduce independently, (3) for all $n \geq 1$, exactly one of its n -th descendants reproduces with distribution $\hat{\pi}$ and the rest (if any) are **common**(π).

The ordered Eternal Kesten Tree: Let π be a probability distribution on $\{0, 1, 2, \dots\}$ with mean $m(\pi) = 1$. The ordered Eternal Kesten Tree with

offspring distribution π , denoted as $EKT(\pi)$, is defined as follows: it is a rooted Eternal Family Tree $[\mathbf{S}, \mathbf{u}]$ containing two types of individuals (vertices) namely $\text{common}(\pi)$ and $\text{special}(\pi)$. It has a unique bi-infinite F -path $(\mathbf{o}_n)_{n \in \mathbb{Z}}$ of $\text{special}(\pi)$ individuals with $\mathbf{o}_0 = \mathbf{u}$, i.e., $\forall n \in \mathbb{Z}$, \mathbf{o}_n is a parent of \mathbf{o}_{n-1} and \mathbf{o}_{n-1} is the $\text{special}(\pi)$ child of \mathbf{o}_n . All the remaining vertices are $\text{common}(\pi)$. The order among the children of every vertex is uniform. All individuals reproduce independently.

The relation between the Eternal Kesten Tree and the bi-variate Eternal Kesten Tree (defined in Subsection 3.2) is as follows: Let π be a probability distribution on $\{0, 1, 2, \dots\}$ with mean $m(\pi) = 1$, $\hat{\pi}$ be the size-biased distribution of π , and α be the distribution given by $\alpha(k) = \hat{\pi}(k + 1)$ for all $k \geq 0$. Then, the unordered $EKT(\pi)$ is the same as the unordered $EKT(\alpha, \pi)$, and the ordered $EKT(\pi)$ is obtained by taking the uniform order on $EKT(\alpha, \pi)$.

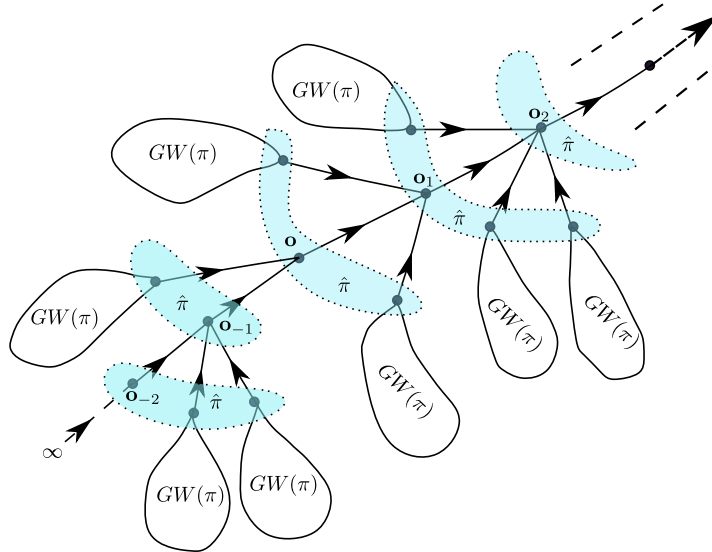


Figure 4.1: The Eternal Kesten tree with distribution π . Notation: $GW(\pi)$ denotes Galton-Watson tree; $\hat{\pi}$ is the size-biased distribution of π .

Remark 79. For all $n \in \mathbb{Z}$, the descendant trees of siblings of \mathbf{o}_n are independent critical Galton-Watson trees with offspring distribution π , and thus they are all finite trees. Therefore, every realization of $EKT(\pi)$ has

a unique bi-infinite F -path passing through the root. See Figure 4.1 for an illustration.

We follow the notations of Section 2.1, Chapter 2.

Proposition 80. *The F -probability of the unimodular (ordered) Eternal Galton-Watson Tree $[\mathbf{T}, \mathbf{o}]$ with offspring distribution π is the (ordered) Eternal Kesten Tree $[\mathbf{S}, \mathbf{u}]$ with distribution $EKT(\pi)$.*

Proof. Let $r > 0$, and $\mathbf{u}_{-r}, \mathbf{u}_{-r+1}, \dots, \mathbf{u}_{-1}$ be the special descendants of $\mathbf{u}_0 = \mathbf{u}$ in \mathbf{S} . Note that $F^n(\mathbf{o})$ has at least one k -th descendant, for all $1 \leq k \leq n$. For all $n > r$, the distributions of $[\mathbf{T}, F^n(\mathbf{o})]_r$ and $[\mathbf{S}, \mathbf{u}]_r$ are the same. Indeed, the offspring distribution of $F^n(\mathbf{o})$ is the same as that of \mathbf{u} which is $\hat{\pi}$. The special child of \mathbf{u} has the same offspring distribution $\hat{\pi}$ as that of $F^{n-1}(\mathbf{o})$ and the remaining other children of u follow the offspring distribution π . By induction, $F^{n-r+1}(\mathbf{o})$ has the same distribution as that of the special $n-r+1$ -th descendant of \mathbf{u} . The same is true for the remaining vertices of $[\mathbf{T}, F^n(\mathbf{o})]_r$ and $[\mathbf{S}, \mathbf{u}]_r$. Therefore,

$$[\mathbf{S}, \mathbf{u}] \stackrel{D}{=} \lim_{n \rightarrow \infty} [\mathbf{T}, F^n(\mathbf{o})].$$

In addition, if $[\mathbf{T}, \mathbf{o}]$ is an ordered $EGWT(\pi)$ and $[\mathbf{S}, \mathbf{u}]$ is an ordered $EKT(\pi)$, then the order of $F^i(\mathbf{o})$ is uniform among the children of $F^{i+1}(\mathbf{o})$ for $n-r+1 \leq i \leq n-1$, which has the same distribution as the order of u_{i-n} among the children of u_{i-n+1} . The same is true for the order of the remaining vertices of $[\mathbf{T}, F^n(\mathbf{o})]_r$ and $[\mathbf{S}, \mathbf{u}]_r$. Thus, the F -probability of the unimodular ordered $EGWT(\pi)$ is the ordered $EKT(\pi)$. \square

Remark 81. Consider a realization of $EGWT(\pi)$, where π has mean $m(\pi) = 1$ and satisfies $\pi(1) < 1$. Then, its root has finitely many descendants, i.e., $EGWT(\pi)$ evaporates under the parent vertex-shift. In contrast, the root of a realization of $EKT(\pi)$ has infinitely many descendants. Therefore, $EKT(\pi)$ is singular with respect to $EGWT(\pi)$. This is another instance where the F -probability is singular with respect to the initial distribution. This fact can also be inferred from Lemma 66 and Corollary 68, since the parent vertex-shift is continuous on \mathcal{T}_* .

Unimodularised bi-variate marked Eternal Kesten Tree

Let α, β be probability distributions on $\{0, 1, 2, \dots\}$ such that their means satisfy $m(\alpha) < \infty$ and $m(\beta) < 1$. Let $[\mathbf{T}', \mathbf{o}', Z']$ be a random marked Family Tree whose distribution is $MEKT(\alpha, \beta)$, where Z' is a mark. Let $[\mathbf{T}', \mathbf{o}', Z']$ be the joining of $([\mathbf{T}_i, \mathbf{o}_i, Z_i])_{i \in \mathbb{Z}}$, the i.i.d. sequence of random marked Family Trees defined in Remark 43.

Let $[\mathbf{T}, \mathbf{o}, Z]$ denote the size-biased joining of $([\mathbf{T}_i, \mathbf{o}_i, Z_i])_{i \in \mathbb{Z}}$. Recall that the distribution of $[\mathbf{T}, \mathbf{o}, Z]$ is given by:

$$\mathbb{P}[[\mathbf{T}, \mathbf{o}, Z] \in A] = \frac{\mathbb{E}[\sum_{u \in V(\mathbf{T}_0)} \mathbf{1}_A([\mathbf{T}', u, Z'])]}{\mathbb{E}[\#V(\mathbf{T}_0)]}, \quad (4.5.2)$$

for any measurable set A . Observe that $\mathbb{E}[\#V(\mathbf{T}_0)] < \infty$ because $\beta < 1$, and the descendant trees of the children of \mathbf{o}_0 (in \mathbf{T}_0) are subcritical Galton-Watson trees.

Recall that $\hat{\mathcal{T}}_*$ denotes the space of rooted marked Family Trees. For any $[T, u, z] \in \hat{\mathcal{T}}_*$ that has a unique bi-infinite F -path $\varkappa(T)$, for any $w \in V(T)$, let

$$M(w) := \inf\{n \geq 0 : F^n(w) \in \varkappa(T)\},$$

with the convention that $F^0(w) = w$ and let $w^* = F^{M(w)}(w)$ be the smallest ancestor of w that lies on the unique bi-infinite F -path of T . Let $[T, u, z]_r$ denote the subtree induced by the vertices of T that are at a graph distance of at most r from u and with the mark function restricted to these vertices.

Proposition 82. *The F -probability of the unimodularized $MEKT(\alpha, \beta)$ is $MEKT(\alpha, \beta)$.*

Proof. Fix $r > 0$. Let A be a measurable subset of $\hat{\mathcal{T}}_*$, and let $n > r$. Then,

$$\begin{aligned} \mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A] &= \mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A, M(\mathbf{o}) < n - r] \\ &\quad + \mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A, M(\mathbf{o}) \geq n - r]. \end{aligned}$$

The second sum is bounded by $\mathbb{P}[M(\mathbf{o}) \geq n - r]$. As $n \rightarrow \infty$, $\mathbb{P}[M(\mathbf{o}) \geq n - r] \rightarrow 0$ since $M(\mathbf{o})$ is a.s. finite, which follows from the finiteness of the tree \mathbf{T}_0 . Therefore, the second sum tends to 0 as $n \rightarrow \infty$.

Consider the first sum. We have,

$$\begin{aligned} \mathbb{P}[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A, M(\mathbf{o}) < n - r] \\ &= \sum_{k=0}^{n-r-1} \mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A, M(\mathbf{o}) = k] \\ &= \sum_{k=0}^{n-r-1} \mathbb{P}[[\mathbf{T}, F^{n-k}(\mathbf{o}^*), Z]_r \in A, M(\mathbf{o}) = k]. \end{aligned}$$

The last equation follows by the definition of $\mathbf{o}^* := F^{M(\mathbf{o})}(\mathbf{o})$. By the definition of $[\mathbf{T}, \mathbf{o}, Z]$ (see Eq. (4.5.2)), the last equation is equal to

$$\sum_{k=0}^{n-r-1} \frac{\mathbb{E} \left[\sum_{u \in V(\mathbf{T}_0)} \mathbf{1}_A([\mathbf{T}', F^{n-k}(u^*), Z']_r) \mathbf{1}\{M(u) = k\} \right]}{\mathbb{E}[\#V(\mathbf{T}_0)]}. \quad (4.5.3)$$

On the tree \mathbf{T}' , we have $F^{n-k}(u^*) = \mathbf{o}_{n-k}$, for all $u \in V(\mathbf{T}_0)$ and for all $0 \leq k \leq n - r - 1$. Since $([\mathbf{T}_i, \mathbf{o}_i, Z_i])_{i \in \mathbb{Z}}$ is an i.i.d. sequence, the distribution of $[\mathbf{T}', \mathbf{o}_{n-k}, Z']_r$ is independent of $[\mathbf{T}_0, \mathbf{o}_0, Z_0]$ for all $0 \leq k \leq n - r - 1$. By applying this fact to Eq. (4.5.3), it is equal to

$$\begin{aligned} &\sum_{k=0}^{n-r-1} \frac{\mathbb{E} \left[\sum_{u \in V(\mathbf{T}_0)} \mathbf{1}_A([\mathbf{T}', \mathbf{o}_{n-k}, Z']_r) \mathbf{1}\{M(u) = k\} \right]}{\mathbb{E}[\#V(\mathbf{T}_0)]} \\ &= \sum_{k=0}^{n-r-1} \frac{\mathbb{P}[[\mathbf{T}', \mathbf{o}_{n-k}, Z']_r \in A] \mathbb{E} \left[\sum_{u \in V(\mathbf{T}_0)} \mathbf{1}\{M(u) = k\} \right]}{\mathbb{E}[\#V(\mathbf{T}_0)]}. \end{aligned}$$

The distribution of $[\mathbf{T}', \mathbf{o}_{n-k}, Z']_r$ is identical to that of $[\mathbf{T}', \mathbf{o}', Z']_r$ for all $n > 2r + k$, by the construction of $MEKT(\alpha, \beta)$. Using this and applying Eq. (4.5.2) to the last equation, we get

$$\begin{aligned} \mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A, M(\mathbf{o}) < n - r] &= \mathbb{P}[[\mathbf{T}', \mathbf{o}', Z']_r \in A] \sum_{k=0}^{n-r-1} \mathbb{P}[M(\mathbf{o}) = k] \\ &= \mathbb{P}[[\mathbf{T}', \mathbf{o}', Z']_r \in A] \mathbb{P}[M(\mathbf{o}) < n - r]. \end{aligned}$$

As $n \rightarrow \infty$, $\mathbb{P}[M(\mathbf{o}) < n - r] \rightarrow 1$. Therefore,

$$\mathbb{P}[[\mathbf{T}, F^n(\mathbf{o}), Z]_r \in A] \xrightarrow{n \rightarrow \infty} \mathbb{P}[[\mathbf{T}', \mathbf{o}', Z']_r \in A].$$

Since this is true for any $r \geq 1$, we have

$$[\mathbf{T}, F^n(\mathbf{o}), Z] \xrightarrow{n \rightarrow \infty} [\mathbf{T}', \mathbf{o}', Z'].$$

□

Remark 83. From the construction of $MEKT(\alpha, \beta)$, it follows that both $[\mathbf{T}', F(\mathbf{o}'), Z']$ and $[\mathbf{T}', \mathbf{o}', Z']$ have the same distribution. So, the marked EFT $[\mathbf{T}', \mathbf{o}', Z']$ satisfies the Mecke's invariant measure equation. Therefore, by Lemma 66, \mathbf{o}' has a.s. infinitely many descendants, which can also be seen directly from the construction of $MEKT(\alpha, \beta)$.

Chapter 5

Record vertex-shift probability for the random walk case

In this chapter, we focus again on the record vertex-shift on the network $[\mathbb{Z}, 0, X]$, where X is an i.i.d. sequence as in Def. 22 and satisfies $\mathbb{E}[X_0] \geq 0$. We show the existence of a unique R -probability (i.e., a vertex-shift probability under record vertex-shift) when $\mathbb{E}[X_0] \geq 0$. Although one could show the same when $\mathbb{E}[X_0] < 0$, we did not consider it since the component of 0 can only give partial information of the trajectory of random walk associated to X .

We use the FB Lemma 61 to prove the existence and uniqueness of the R -probability. We define two maps namely the backward map Φ_R and the extended backward map $\hat{\Phi}_R$ that satisfy the hypotheses of the FB Lemma when $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X_0] > 0$ respectively. This is done in several sections. We also give constructions for the R -probability in these two cases and prove that the constructions indeed generate the R -probability.

(In French) Dans ce chapitre, nous nous intéressons à nouveau au décalage de sommet des records sur le réseau $[\mathbb{Z}, 0, X]$, où X est une suite i.i.d. comme dans la définition 22 et qui satisfait $\mathbb{E}[X_0] \geq 0$. Nous montrons l'existence d'une R -probabilité unique (c'est-à-dire une probabilité de décalage de sommet associée au décalage de sommet des records) lorsque $\mathbb{E}[X_0] \geq 0$. Bien que l'on puisse montrer la même chose lorsque $\mathbb{E}[X_0] < 0$, nous ne l'avons pas considéré car la composante de 0 ne peut donner qu'une information partielle sur la trajectoire de la marche aléatoire associée à X .

Nous utilisons le lemme FB 61 pour prouver l'existence et l'unicité de la R -probabilité. Nous définissons deux fonctions nommées la fonction

réciroque Φ_R et la fonction réciroque étendue $\hat{\Phi}_R$ qui satisfont les hypothèses du lemme FB lorsque $\mathbb{E}[X_0] = 0$ et $\mathbb{E}[X_0] > 0$ respectivement. Ceci est fait dans plusieurs sections. Nous donnons également des constructions pour la R -probabilité dans ces deux cas et prouvons que les constructions génèrent bien la R -probabilité.

5.1 The backward map of the record vertex-shift

In this section, we define a backward map $\Phi_R : \mathcal{T}'_* \rightarrow \mathcal{G}_*$, where \mathcal{T}'_* is the subset of \mathcal{T}_* consisting of all ordered Family Trees that have at most one bi-infinite F -path. Recall that the forward map $\Psi_R : \mathcal{G}_* \rightarrow \mathcal{T}'_*$ maps a network of the form $[\mathbb{Z}, 0, x]$ to $[\mathbb{Z}^R(0), 0]$, where x is a sample of the i.i.d. sequence X with $\mathbb{E}[X_0] = 0$, and $\mathbb{Z}^R(0)$ is the connected component of 0 of the record graph \mathbb{Z}^R of $[\mathbb{Z}, 0, x]$. For any ordered Family Tree that has at most one bi-infinite path, using the depth-first search order and the tree structure, we can define a sequence in such a way that the record graph associated to the sequence is the Family Tree. This map essentially depends on a sequence of vertices of the Family Tree called succession line, which can be seen as a factor map from \mathbb{Z} to the vertices of a unimodular Family Tree ([11]).

This map extends the usual depth-first search encoding of finite ordered trees, defined in [14, Chapter 5], [16], to infinite ordered Family Trees that have at most one bi-infinite path.

In the next sections, we show that the map Φ_R satisfies the three conditions of the FB Lemma 61 when $\mathbb{E}[X_0] = 0$. However, this map fails to be bijective when $\mathbb{E}[X_0] > 0$. So, in Section 5.3, we extend this map to the space of marked Family Trees in order to obtain a bijective map.

In Subsection 3.2 of Chapter 3, we show that the R -graph of $[\mathbb{Z}, 0, X]$ when $\mathbb{E}[X_0] = 0$ is a unimodular ordered Eternal Galton-Watson Tree $EGWT(\pi)$ with offspring distribution π , where π is the distribution of $X_0 + 1$. The parent vertex-shift probability of $EGWT(\pi)$ is shown to be the Eternal Kesten Tree $EKT(\pi)$ (see Subsection 4.5 for the definition of $EKT(\pi)$). In contrast to $EGWT(\pi)$, when $\pi(1) < 1$, a realization of $EKT(\pi)$ has a unique bi-infinite path. So, the map Φ_R should be defined such that it is continuous on the support of $EKT(\pi)$ in order to satisfy the hypotheses of FB Lemma 61. Therefore, our first goal is to define Φ_R on

ordered Family Trees that have at most one bi-infinite path. We achieve this goal using the properties of the RLS order. Then, we show that the defined map Φ_R is continuous $EKT(\pi)$ -a.s..

Let T be an ordered Family Tree. Consider the RLS order $<$ on the vertices of T and recall that for any two vertices u and v , we say u succeeds v if $u < v$ (equivalently v precedes u).

Definition 84. For any vertex $u \in T$, let $B(u) := \{v \in T : v < u\}$ and define $b(u) = \max\{v \in B(u)\}$ if it exists, where \max is taken with respect to $<$. Similarly, let $A(u) := \{v \in T : v > u\}$ and $a(u) = \min\{u \in A(u)\}$ if it exists, where \min is taken with respect to $<$.

According to the RLS order, $b(u)$ has the highest precedence over all those vertices that succeed u , and $a(u)$ has the smallest precedence among those vertices that precede u .

Lemma 85. *Let T be an ordered Family Tree and $u \in T$. If u is not the smallest vertex, then $b(u)$ exists.*

Proof. By hypothesis, $B(u)$ is not empty. So, either of the following two cases occur:

- If u has children, then since it could only have finitely many children, $b(u)$ is the largest(oldest) child of u .
- If u does not have any child, let $w = F^m(u)$ (for some $m \geq 1$) be the smallest ancestor of u that has a child smaller than $F^{m-1}(u)$, with the notation $F^0(u) = u$. Such an m exists because $B(u)$ is non-empty. Then, $b(u)$ is the largest among those siblings of $F^{m-1}(u)$ that are smaller than $F^{m-1}(u)$.

□

Lemma 86. *Let T be an ordered Family Tree, u be a non-maximal vertex of T . If every sibling of u that precedes u has finitely many descendants, then $a(u)$ exists.*

Proof. If u does not have a sibling that precedes u , then $a(u) = F(u)$ (by hypothesis $F(u) \neq u$ in this case). Otherwise, let w be the smallest sibling of u larger than u . It has finitely many descendants by hypothesis. Then $a(u)$ is the minimal vertex of $D(w)$ (recall that $D(w) = \{v \in V(T) \setminus \{w\} : F^m(v) = w \text{ for some } m > 0\} \cup \{w\}$). □

Definition 87. Consider an ordered rooted Eternal Family Tree (T, o) that has at most one bi-infinite F -path passing through o . With the RLS order, we call **succession line** of (T, o) the sequence $U((T, o)) = (u_n)_{n \in \mathbb{Z}}$ iteratively given by $u_0 = o$ and,

$$u_n = \begin{cases} b(u_{n+1}) & \text{if it exists} \\ u_{n+1} & \text{otherwise} \end{cases} \quad \text{for } n < 0, \quad (5.1.1)$$

$$u_n = \begin{cases} a(u_{n-1}) & \text{if it exists} \\ u_{n-1} & \text{otherwise} \end{cases} \quad \text{for } n > 0.$$

Observe that this is the unique sequence that satisfies $u_0 = o$, and u_{n-1} is the immediate successor of u_n , when they are distinct, for all $n \in \mathbb{Z}$. With the above preparation, we are ready to define the backward map which plays a crucial role in the rest of the chapter.

Definition 88. Define the **backward map of the record vertex-shift** $\Phi_R : \mathcal{T}'_* \rightarrow \mathcal{G}_*$ as the function that maps $[T, o] \in \mathcal{T}'_*$ to

$$\Phi_R([T, o]) = [\mathbb{Z}, 0, x((T, o))], \quad (5.1.2)$$

where $x((T, o)) = (x_n)_{n \in \mathbb{Z}}$ is given by $x_n = d_1(u_{n+1}) - 1$ and $(u_n)_{n \in \mathbb{Z}} = U((T, o))$.

Note that Φ_R is well-defined: if $(T, o) \sim (T', o')$ be two isomorphic ordered Family Trees and $(u_n)_{n \in \mathbb{Z}}$ and $(u'_n)_{n \in \mathbb{Z}}$ be their respective succession lines passing through o, o' respectively, then $d_1(u_n) - 1 = d_1(u'_n) - 1$ for all $n \in \mathbb{Z}$, which implies that $x((T, o)) = x((T', o'))$. In general the succession line of a rooted Family Tree $[T, o]$ passing through o may not contain all the vertices of T . But, if $[T, o]$ is a sample of component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ with $\mathbb{E}[X_0] \geq 0$, then we show that the succession line of $[T, o]$ contains every vertex of T (see Lemma 89 and Remark 90).

Lemma 89. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be the i.i.d. increments of a skip-free to the left random walk with $\mathbb{E}[X_0] \geq 0$ and $[\mathbb{Z}, 0, X]$ be the network associated to X . Let $[\mathbf{T}, 0]$ be the component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ rooted at 0. Then, a.s., the succession line $U((\mathbf{T}, \mathbf{o}))$ consists of distinct entries.*

Proof. Observe that the set of minimal vertices of \mathbf{T} is a covariant subset. It can only be either empty or a singleton, because \mathbf{T} is totally ordered by the

RLS order. But $[\mathbf{T}, \mathbf{o}]$ is unimodular (Lemma 16), and $\#V(\mathbf{T})$ is infinite since $\mathbb{E}[X_0] \geq 0$, in fact $V(\mathbf{T}) = \mathbb{Z}$ since the record graph is connected when $\mathbb{E}[X_0] \geq 0$ (see Proposition 40 and Proposition 41). Therefore, by No Infinite/Finite inclusion Lemma 12, a.s., the minimal vertex does not exist. Therefore, by Lemma 85, almost surely, the successor of u , $b(u)$ exists for all $u \in V(\mathbf{T})$. Since $a(b(u)) = u$ for all $u \in V(\mathbf{T})$, $a(v)$ exists for all v in the range of the map b . But b is injective a.s. and hence surjective a.s. (by the unimodularity of $[\mathbf{T}, \mathbf{o}]$). Therefore, the range of b is $V(\mathbf{T})$ a.s. and hence, a.s., $a(u)$ exists for all $u \in V(\mathbf{T})$. Thus, the map $n \mapsto u_n$ from \mathbb{Z} to $V(\mathbf{T})$ is injective, which implies that all the elements of the sequence $U((\mathbf{T}, \mathbf{o}))$ are distinct. \square

Remark 90. Observe that $U(\Psi_R([\mathbb{Z}, 0, X])) = (u_n)_{n \in \mathbb{Z}} = (n)_{n \in \mathbb{Z}}$, when the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$ satisfies $\mathbb{E}[X_0] \geq 0$. This follows by combining Lemma 33 with the facts that the R -graph \mathbb{Z}^R of $[\mathbb{Z}, 0, X]$ is a.s. connected, and the map $n \mapsto u_n$ from \mathbb{Z} to $V(\mathbb{Z}^R) = \mathbb{Z}$ is injective. Indeed, by definition, $B(0) = \{i \in \mathbb{Z} : i < 0\}$. But, by Lemma 33, $i \in B(0) \iff i < 0$. Therefore, $B(0) = \{i \in \mathbb{Z} : i < 0\}$. Hence, $b(0) = -1$ and $u_{-1} = -1$. By induction, assume $u_{-n} = -n$. Consider $B(u_{-n}) = \{i \in \mathbb{Z} : i < -n\} = \{i \in \mathbb{Z} : i < -n\}$. Therefore, $u_{-(n+1)} = -(n+1)$. Similarly, we could show that $a(u_n) = n, \forall n \geq 0$.

Lemma 91. *Let $[\mathbf{T}', \mathbf{o}']$ be a random ordered Family Tree whose distribution is $EKT(\pi)$, where π is a probability distribution on $\{0, 1, 2, \dots\}$ with mean $m(\pi) = 1$. Then, the succession line $U((\mathbf{T}', \mathbf{o}'))$ consists of distinct entries.*

Proof. Let $[T', o']$ be a realization of $EKT(\pi)$. Since T' has a unique bi-infinite F -path passing through the root o , by Lemma 85, $b(u)$ exists for all $u \in V(T')$. If u satisfies the property that every sibling of u that precedes u has finitely many descendants, then, by Lemma 86, $a(u)$ exists. In this case, $a(u)$ also satisfies the same property. Since the root o satisfies this property, the map $n \mapsto u_n$ from \mathbb{Z} to $V(T')$ is injective. Thus, all the elements of $U(\mathbf{T}', \mathbf{o}')$ are distinct. \square

The following remark is used in the later sections to describe the R -probability from EKT .

Remark 92. Let $[T', o']$ be a realization of $EKT(\pi)$ and $\varkappa([T', o'])$ denote its bi-infinite F -path. Since $o' \in \varkappa([T', o'])$, we could define the following. The **positive part** of $[T', o']$ is the rooted Family Tree $[T'^+, o']$, where T'^+

is the subtree of T' induced by the vertices $\{u \in V(T') : u > v \text{ for some } v \in \mathcal{L}([T', o'])\}$. Observe that the set of elements of $U((T', o'))$ is equal to $V(T')$.

5.2 Bijectivity and continuity of the backward map

We first show that $\Psi_R \circ \Phi_R([T, o]) = [T, o]$ a.s. for every realization $[T, o]$ of the ordered $EGWT(\pi)$, where the mean $m(\pi) = 1$, using the following lemma.

Lemma 93. *Let T be a finite ordered tree and $v_1 < v_2 < \dots < v_n$ be the vertices of T ordered according to the RLS order. Then,*

1. $\sum_{i=1}^n (d_1(v_i) - 1) = -1$.
2. $\sum_{i=1}^k (d_1(v_i) - 1) < 0$, for all $1 < k \leq n$.

Proof. The first equation follows from the fact that for a finite tree T ,

$$\sum_{u \in V(T)} d_1(u) = \#V(T) - 1.$$

We prove the second equation by induction on k . If $k = 1$ then v_1 is a leaf and hence $d_1(v_1) - 1 = -1$. Now, assume that the statement is true for $1 \leq m \leq k$, where $k > 1$. Observe that by the RLS order, either v_{k+1} is a leaf or v_{k+1} is the parent of v_k . In the latter case, there exists $1 \leq j \leq k$ such that $v_j < v_{j+1} < \dots < v_k$ (written according to the RLS order) are all the descendants of v_{k+1} . If v_{k+1} is a leaf then, by induction,

$$\sum_{i=1}^{k+1} (d_1(v_i) - 1) = \sum_{i=1}^k (d_1(v_i) - 1) + d_1(v_{k+1}) - 1 < -1.$$

The only other possibility is that v_{k+1} is the parent of v_k . Then,

$$\sum_{i=1}^{k+1} (d_1(v_i) - 1) = \sum_{i=1}^{j-1} (d_1(v_i) - 1) + \sum_{i=j}^{k+1} (d_1(v_i) - 1) \leq 0 + (-1).$$

The last step follows from the fact that the first sum on the right-hand side is at most 0 (by induction), while the second sum is on the descendant tree of v_{k+1} which is equal to -1 by the first result of the lemma. \square

5.2. BIJECTIVITY AND CONTINUITY OF THE BACKWARD MAP

Remark 94. Let $v_1 < v_2 < \dots < v_n$ be the vertices of a finite ordered tree, ordered according to RLS order. Then using Lemma 93 we obtain, for any $1 < k < n$,

$$\sum_{i=k+1}^n (d_1(v_i) - 1) = \sum_{i=1}^n (d_1(v_i) - 1) - \sum_{i=1}^k (d_1(v_i) - 1) \geq -1 - (-1) \geq 0.$$

Proposition 95. *Almost surely, every realisation $[T, o]$ of the ordered EGWT(π), where the mean $m(\pi) = 1$, satisfies $\Psi_R \circ \Phi_R([T, o]) = [T, o]$.*

Proof. Let $(u_n)_{n \in \mathbb{Z}} = U((T, o))$ be the succession line of (T, o) and $[\mathbb{Z}, 0, x] = \Phi_R([T, o])$, where $x = (x_n)_{n \in \mathbb{Z}}$ is such that $x_n = d_1(u_n) - 1$ is the mark of the edge $(n, n + 1)$, for all $n \in \mathbb{Z}$.

First, observe that the map $\alpha : \mathbb{Z} \rightarrow V(T)$ defined by $\alpha(n) = u_n$, for all $n \in \mathbb{Z}$ is bijective. This follows because, every vertex v of T has finitely many descendants. So, there are only finitely many vertices between o and v according to the RLS order, and hence $u_n = v$ for some $n \in \mathbb{Z}$. Therefore, α is surjective. The injectivity of the map α follows from Lemma 89 because $[T, o]$ is a realisation of the component of 0 in the record graph of $[\mathbb{Z}, 0, X]$ when $\mathbb{E}[X_0] = 0$. We now prove that the bijective map $\alpha : \mathbb{Z} \rightarrow V(T)$ defined by $\alpha(n) = u_n$, for all $n \in \mathbb{Z}$, induces a rooted network isomorphism α from $\Psi_R([\mathbb{Z}, 0, x])$ to (T, o) . Note that, if u_j is a parent of u_i in T for some integers i and j then $j > i$, which follows because $(u_n)_{n \in \mathbb{Z}}$ is the succession line of (T, o) . In view of this, to prove that α induces a rooted network isomorphism, it is enough to show that for all $i \in \mathbb{Z}$, if u_j is the parent of u_i in T for some $j > i$, then $j = R_x(i)$ (since every vertex has at most one parent).

Let $i \in \mathbb{Z}$ and u_j be the parent of u_i in T for some integer $j > i$. If $j = i + 1$, then $x_i = d_1(u_{i+1}) - 1 \geq 0$, because u_i is a child of u_{i+1} . Hence, $R_x(i) = i + 1$.

So, let us assume that $j > i + 1$. This implies that either u_{i+1} is a leaf and a descendant of a sibling of u_i , or u_{i+1} is a leaf and a sibling of u_i . The descendant tree of u_j is a finite ordered tree, as $[T, o]$ is a realization of EGWT(π). Let $u_{i_1} < u_{i_2} < \dots < u_{i_n}$ be all the elder siblings of u_i , and T_1, T_2, \dots, T_n be their descendant trees respectively. For any $i < k < j$, let $l(k)$ be the smallest element of the set $\{1, 2, \dots, n\}$ such that $\{u_{i+1}, u_{i+2}, \dots, u_k\} \subset V(T_1) \cup V(T_2) \cup \dots \cup V(T_{l(k)})$, and u_{k_1} be the smallest vertex of $T_{l(k)}$.

Then, the sum $y(i, k) = \sum_{m=i}^{k-1} x_m = \sum_{m=i+1}^k (d_1(u_m) - 1)$ can be written as

$$\begin{aligned} y(i, k) &= \left(\sum_{u \in V(T_1)} d_1(u) - 1 \right) + \cdots + \left(\sum_{u \in V(T_{i(k)-1})} d_1(u) - 1 \right) + \sum_{m=k_1}^k (d_1(u_m) - 1) \\ &< (-1) + \cdots + (-1) + \sum_{m=k_1}^k (d_1(u_m) - 1) < 0, \end{aligned}$$

with the notation that $V(T_0)$ is just an empty set. The last steps follow from the first and the second statements of Lemma 93.

On the other hand, the sum $y(i, j) = \sum_{m=i+1}^j (d_1(u_m) - 1)$ can be written as

$$\begin{aligned} y(i, j) &= \left(\sum_{u \in V(T_1)} d_1(u) - 1 \right) + \cdots + \left(\sum_{u \in V(T_n)} d_1(u) - 1 \right) + d_1(u_j) - 1 \\ &= -n + d_1(u_j) - 1 \geq 0. \end{aligned}$$

The above steps follow from the first part of Lemma 93, and by the assumption that u_j has at least $n + 1$ children, namely $u_i, u_{i_1}, \dots, u_{i_n}$. Thus, $R_x(i) = j$, completing the proof. \square

The following proposition shows that the backward map Φ_R satisfies the first condition of the FB lemma (Lemma 61).

Proposition 96. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be as in Def. 22 and satisfies $\mathbb{E}[X_0] = 0$. Let $x = (x_n)_{n \in \mathbb{Z}}$ be a realization of X , and $[\mathbb{Z}, 0, x]$ be its associated network. Then, $\Phi_R \circ \Psi_R([\mathbb{Z}, 0, x]) = [\mathbb{Z}, 0, x]$.*

Proof. Let $(u_n)_{n \in \mathbb{Z}} = U(\Psi_R([\mathbb{Z}, 0, x]))$ be the succession line of $\Psi_R([\mathbb{Z}, 0, x])$. By Remark 90, $u_n = n$ for all $n \in \mathbb{Z}$. Let $[\mathbb{Z}, 0, y] := \Phi_R \circ \Psi_R([\mathbb{Z}, 0, x])$, where $y = (y_n)_{n \in \mathbb{Z}}$ with $y_n := d_1(u_{n+1}) - 1 = d_1(n+1) - 1$ for all $n \in \mathbb{Z}$. From the first statement of Lemma 34, $x_n = d_1(n+1) - 1$, for all $n \in \mathbb{Z}$. Therefore, $x_n = y_n$ for all $n \in \mathbb{Z}$. \square

We now show that the map Φ_R is continuous on the set of ordered family trees that have at most one bi-infinite F -path.

Lemma 97. *Let $[T, o]$ be an ordered EFT that has at most one bi-infinite F -path, and such that the bi-infinite path passes through o . If $([T_n, o_n])_{n \geq 1}$ is any sequence of rooted Family Trees such that $[T_n, o_n] \rightarrow [T, o]$ as $n \rightarrow \infty$, then $\Phi_R([T_n, o_n]) \rightarrow \Phi_R([T, o])$ as $n \rightarrow \infty$ (both convergences are in local sense).*

Proof. For $r > 0$, denote $x_r((T, o)) = (x_{-r}, x_{-r+1}, \dots, x_0, \dots, x_{r-1}, x_r)$. We need to show that given $r > 0$, there exists N_0 such that for all $n > N_0$, $x_r((T_n, o_n)) = x_r((T, o))$. Since $(T_n, o_n) \rightarrow (T, o)$, it is enough to show that, given $r > 0$, there exists $R > 0$ such that $x_r((T, o)_R) = x_r(T, o)$, where $(T, o)_R$ is the restriction of T to the graph ball of radius R centered at o .

Let $(o_{-r-1}, o_{-r}, \dots, o, o_1, \dots, o_{r+1})$ be the $r+1$ restriction of the F -bi-infinite path of (T, o) passing through o . If such a path does not exist (take for instance, a sample of the non-trivial unimodular EGWT), then we can find the largest $k < 0$ such that $o_l = o_k$ for all $-r-1 \leq l \leq k$.

Let h_i be the largest graph distance from o_i to the descendants of o_i which are not the descendants of o_{i-1} for $-r-1 \leq i \leq r+1$. Then, h_i is finite for all i since the descendants of o_i that are not the descendants of o_{i-1} are finite for all i (otherwise this contradicts the fact that T has at most one bi-infinite F -path). Take $R = \max\{h_i : -r-1 \leq i \leq r+1\}$. Then, we have $U((T, o)_R) = U((T, o))$. Thus, $x_r((T, o)_R) = x_r((T, o))$. \square

5.3 The extended backward map of record vertex-shift

In this section, we define a map $\hat{\Phi}_R$ on the space of marked Family Trees and show that it satisfies all the conditions of FB Lemma 61. The map $\hat{\Phi}_R$ will be used later to show that the R -probability of $[\mathbb{Z}, 0, X]$ exists. We call the map $\hat{\Phi}_R$ **the extended backward map**.

Let \mathcal{T}'_* (resp. $\hat{\mathcal{T}}'_*$) be the space of rooted ordered Family Trees (resp. space of rooted marked ordered Family Trees whose marks take values in \mathbb{Z}) that have at most one bi-infinite F -path.

We have defined earlier in Eq. (5.1.1) the succession line of a rooted ordered Family Tree $[T, o]$ that has at most one bi-infinite F -path, and using this we defined in Eq. (5.1.2) the backward map $\Phi_R : \mathcal{T}'_* \rightarrow \mathcal{G}_*$.

In order to use the FB lemma (Lemma 61), we need this backward map to satisfy

$$\Phi_R \circ \Psi_R = I \text{ } \mathcal{P}\text{-a.s.}, \tag{5.3.1}$$

where \mathcal{P} is the distribution of $[\mathbb{Z}, 0, X]$, where $X = (X_n)_{n \in \mathbb{Z}}$ satisfies $\mathbb{E}[X_0] \geq 0$.

When $\mathbb{E}[X_0] > 0$, the random walk $(S_{-n})_{n \geq 0}$ starting at 0 drifts to $-\infty$ (where $S_{-n} = \sum_{k=-n}^{-1} -X_k$ for all $n > 0$ and $S_0 = 0$). In this case, Φ_R does not satisfy Eq. (5.3.1) since Ψ_R fails to be injective. For instance, consider the set $A = \{[\mathbb{Z}, 0, x = (x_n)_{n \in \mathbb{Z}}] : x_{-1} = 3 \text{ and } s_{-n} \rightarrow -\infty \text{ as } n \rightarrow \infty\}$. The set A has positive measure since we assumed that $\mathbb{E}[X_0] > 0$. For each $[\mathbb{Z}, 0, x] \in A$, define a new element $[\mathbb{Z}, 0, \tilde{x}]$, where the sequence $\tilde{x} = (\tilde{x}_n)_{n \in \mathbb{Z}}$ is given by the following: $\tilde{x}_n = x_n$ for all $n \neq -1$ and $\tilde{x}_{-1} = 4$. In comparison to the sums $(s_n)_{n \in \mathbb{Z}}$ associated to x , the sums $(\tilde{s}_n)_{n \in \mathbb{Z}}$ associated to the sequence \tilde{x} are shifted by -1 for all $n \leq -1$. Both the networks $[\mathbb{Z}, 0, x]$ and $[\mathbb{Z}, 0, \tilde{x}]$ have the same record graphs. So, Ψ_R is not injective with positive probability, see Figure 5.1.

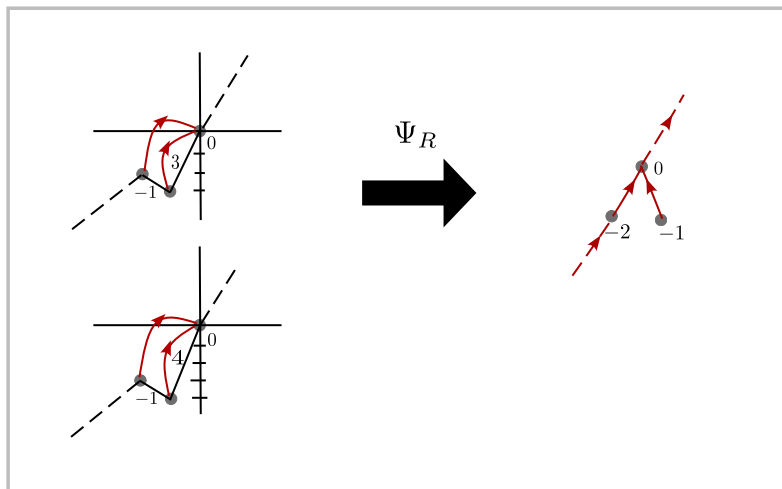


Figure 5.1: Example showing that when $\mathbb{E}[X_0] > 0$, two samples of $[\mathbb{Z}, 0, X]$ can have the same record graph.

To obtain the injectivity, we enrich the record graph $[\mathbb{Z}, 0, X]$ with the type function of X as a mark function on the vertices of the record graph. This gives the forward map $\hat{\Psi}_R : \mathcal{G}_* \rightarrow \hat{\mathcal{T}}'_*$ given by $\hat{\Psi}_R([\mathbb{Z}, 0, x]) = [T, 0, t]$, where $[T, 0] = \Psi_R([\mathbb{Z}, 0, x])$ is the component of 0 in the record graph of $[\mathbb{Z}, 0, x]$, t is the type function associated to x , and $x = (x_n)_{n \in \mathbb{Z}}$ is a sample of the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$ when $0 < \mathbb{E}[X_0] < \infty$. We define a backward map $\hat{\Phi}_R$ (associated to X) on $\hat{\mathcal{T}}'_*$ such that it satisfies Eq. (5.3.1) with $\hat{\Psi}_R$ as the forward map.

Definition 98. Let $[T, o, M]$ be a rooted marked Family Tree that has at most one bi-infinite F -path and $M : V(T) \rightarrow \mathbb{Z}$ be a marked function such that $M(u) \geq -1$ for all $u \in V(T)$. Let $u((T, o, M)) = (u_n)_{n \in \mathbb{Z}}$ be its succession line with $u_0 = o$. Define

$$\hat{\Phi}_R([T, o, M]) = [\mathbb{Z}, 0, x], \quad (5.3.2)$$

where $x = (x_n)_{n \in \mathbb{Z}}$ is given by $x_n = (d_1(u_{n+1}) - 1)\mathbf{1}\{u_{n+1} \notin \mathcal{L}\} + (d_1(u_{n+1}) + M(u_{n+1}) - 1)\mathbf{1}\{u_{n+1} \in \mathcal{L}\}$ for all $n \in \mathbb{Z}$.

The map $\hat{\Phi}_R$ is defined keeping in mind the relation between increment, offspring count and type as given in Lemma 38.

5.4 Bijectivity and continuity of the extended backward map

Let $\hat{\Psi}_R : \mathcal{G}_* \rightarrow \hat{\mathcal{T}}'_*$ be the forward map given by $\hat{\Psi}_R([\mathbb{Z}, 0, x]) = [T, 0, t]$, where $[T, 0] = \Psi_R([\mathbb{Z}, 0, x])$ is the component of 0 in the record graph of $[\mathbb{Z}, 0, x]$, t is the type function associated to x (see subsection 3.1 for the definition of the type function), and $x = (x_n)_{n \in \mathbb{Z}}$ is a sample of the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$ when $0 < \mathbb{E}[X_0] < \infty$.

Proposition 99. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence as in Def. 22 with $0 < \mathbb{E}[X_0] < \infty$, and \mathcal{P} be the distribution of $[\mathbb{Z}, 0, X]$ on \mathcal{G}_* . Then, $\hat{\Phi}_R \circ \hat{\Psi}_R = I \mathcal{P}$ - a.s.*

Proof. Let $x = (x_n)_{n \in \mathbb{Z}}$ be a sample of X , $[\mathbb{Z}, 0, x]$ be its network, and let $[T, 0, t] = \hat{\Psi}_R([\mathbb{Z}, 0, x])$ be the marked random rooted tree, where the mark function is give by the type function t associated to x . We know that $[T, 0, t]$ has at most one bi-infinite path and T is eternal. So, we obtain its succession $u([T, 0, t]) = (u_n)_{n \in \mathbb{Z}}$. Note that $u_n = n$ for all $n \in \mathbb{Z}$ by Remark 90. Let $[\mathbb{Z}, 0, y] = \hat{\Phi}_R([T, 0, t])$, where $y = (y_n)_{n \in \mathbb{Z}}$ is the sequence given by $y_n = (d_1(u_{n+1}) - 1)\mathbf{1}\{u_{n+1} \notin \mathcal{L}\} + (d_1(u_{n+1}) + t_x(u_{n+1}) - 1)\mathbf{1}\{u_{n+1} \in \mathcal{L}\}$ for all $n \in \mathbb{Z}$. By Lemma 38 and the fact that for any $n \in \mathbb{Z}$, $u_n \in \mathcal{L}$ if and only if $t(u_n) > -1$, it follows that $x_n = (d_1(n+1) - 1)\mathbf{1}\{t_x(n+1) = -1\} + (d_1(n+1) + t(n+1) - 1)\mathbf{1}\{t(n+1) \geq 0\} = y_n$ for all $n \in \mathbb{Z}$. Thus, $[\mathbb{Z}, 0, x] = [\mathbb{Z}, 0, y]$. \square

Lemma 100. *Let $[T, o, m] \in \hat{\mathcal{T}}'_*$. If a sequence $([T_n, o_n, m_n])_{n \geq 1}$ in $\hat{\mathcal{T}}'_*$ converges to $[T, o, m]$ as $n \rightarrow \infty$, then $\hat{\Phi}_R([T_n, o_n, m_n])$ converges to $\hat{\Phi}_R([T, o, t])$ as $n \rightarrow \infty$.*

Proof. Since the marks take values in \mathbb{Z} , which is a discrete set, the proof of the statement is the same as in Lemma 97, but the underlined space is $\hat{\mathcal{T}}'_*$ instead of \mathcal{T}'_* . \square

5.5 Existence of R -probability for non-negative mean

Let $X = (X_n)_{n \in \mathbb{Z}}$ be the i.i.d. sequence of increments of a skip-free to the left random walk and let $[\mathbb{Z}, 0, X]$ be its associated network. Let π be the distribution of $X_0 + 1$.

Theorem 101. *If $\mathbb{E}[X_0] = 0$, then the R -probability of $[\mathbb{Z}, 0, X]$ exists. Its distribution is equal to the push-forward of $EKT(\pi)$ under the map Φ_R , where π is the probability distribution of $X_0 + 1$.*

Proof. Let \mathcal{P} be the distribution of $[\mathbb{Z}, 0, X]$, and $\mathcal{Q} := \mathcal{P} \circ \Psi_R^{-1}$. We show that \mathcal{P} and Φ_R satisfy all the hypotheses of FB Lemma 61. Theorem 53 shows that $\mathcal{Q} = EGWT(\pi)$, where the offspring distribution $\pi \stackrel{\mathcal{D}}{=} X_0 + 1$. Proposition 96 implies the first condition of Lemma 61. In Lemma 80, we showed that the F -probability \mathcal{Q}_F of $EGWT(\pi)$ exists and it is equal to $EKT(\pi)$, satisfying the second condition of Lemma 61. The third condition of Lemma 61 is shown to be satisfied by Φ_R in Lemma 97. Thus, the R -probability of $[\mathbb{Z}, 0, X]$ exists, and its distribution is $\mathcal{Q}_F \circ \Phi_R^{-1}$. \square

Assume $0 < \mathbb{E}[X_0] < \infty$. Let $\bar{\pi}$ and $\tilde{\pi}$ be the probability distributions defined in Eq. (3.2.8). Consider the bi-variate ECS ordered marked Eternal Kesten Tree $MEKT(\bar{\pi}, \tilde{\pi})$ whose marked bush has the distribution given in Theorem 55.

Theorem 102. *If $0 < \mathbb{E}[X_0] < \infty$, then the R -probability of $[\mathbb{Z}, 0, X]$ exists, and its distribution is $MEKT(\bar{\pi}, \tilde{\pi}) \circ \hat{\Phi}_R^{-1}$, the push-forward measure of $MEKT(\bar{\pi}, \tilde{\pi})$ under the extended backward map $\hat{\Phi}_R$.*

Proof. Let $[\mathbf{T}, 0, t] = \hat{\Psi}_R([\mathbb{Z}, 0, X])$, where t is the type function associated to X , and \mathcal{P} be the distribution of $[\mathbb{Z}, 0, X]$. We apply FB Lemma 61. By Proposition 99, the extended map $\hat{\Phi}_R \circ \hat{\Psi}_R = Id$ \mathcal{P} - *a.s.*. By Theorem 55, $[\mathbf{T}, 0, t]$ is the unimodularised $MEKT(\bar{\pi}, \tilde{\pi})$ (in the sense of Eq. (2.3.3)). By Proposition 82, $MEKT(\bar{\pi}, \tilde{\pi})$ is the F -probability of the unimodularised $MEKT(\bar{\pi}, \tilde{\pi})$. By Lemma 100, the extended backward map $\hat{\Phi}_R$ is continuous.

So, all the three conditions of the FB lemma (Lemma 61) are satisfied by the extended backward map $\hat{\Phi}_R$ and the probability measure \mathcal{P} . Thus, R -probability $[\mathbb{Z}, 0, X]$ exists, and its distribution is equal to $MEKT(\bar{\pi}, \tilde{\pi}) \circ \hat{\Phi}_R^{-1}$. \square

Remark 103. When $0 < \mathbb{E}[X_0] < \infty$, $MEKT(\bar{\pi}, \tilde{\pi})$ is the distribution of the record graph of the conditioned network $[\mathbb{Z}, 0, X]$ conditioned on $\{S_n \leq 0, \forall n \leq 0\}$ which is equivalent to $\{0 \in \mathcal{Z}'\}$ (i.e., 0 belongs to the bi-infinite path of the record graph) by Theorem 55. This implies that the R -probability of $[\mathbb{Z}, 0, X]$ is the conditioned network $[\mathbb{Z}, 0, X]$ conditioned on $\{S_n \leq 0, \forall n \leq 0\}$.

5.6 Construction of the R-probability for zero mean

We have seen in the previous sections that the R -probability of $[\mathbb{Z}, 0, X]$ exists when $\mathbb{E}[X_0] = 0$. We now provide two constructions for the R -probability and prove in Proposition 104 that the constructions indeed generate the R -probability when $\mathbb{E}[X_0] = 0$. The second construction is useful to prove several properties of the R -probability, see Section 5.7.

Construction 1: We now construct a sequence of random variables $Y = (Y_n)_{n \in \mathbb{Z}}$ and show, in Proposition 104, that its associated network $[\mathbb{Z}, 0, Y]$ is the R -probability of $[\mathbb{Z}, 0, X]$ when $\mathbb{E}[X_0] = 0$. In particular, we show that $[\mathbb{Z}, 0, Y] = \Phi_R([\mathbf{T}, \mathbf{o}])$, where $[\mathbf{T}, \mathbf{o}]$ is distributed as $EKT(\pi)$ with π being the distribution of $X_0 + 1$.

Define the non-negatively indexed subsequence $(Y_n)_{n \geq 0}$ to be i.i.d. random variables with $Y_n \stackrel{\mathcal{D}}{=} X_n, n \geq 0$. The construction for the negatively indexed subsequence $(Y_n)_{n < 0}$ is given in the following steps. Recall that the size-biased distribution $\hat{\pi}$ of π is given by $\hat{\pi}(k) = k\pi(k), k \in \{0, 1, 2, \dots\}$. In particular, $\hat{\pi}(\{1, 2, 3, \dots\}) = 1$.

Let λ be the distribution of X_0 .

1. Let $l = -1$.
2. Generate a random variable Z with distribution $\hat{\pi}$ and assign $Y_l := (Z - 1)$.
3. Let K be an integer chosen uniformly among $\{0, 1, \dots, Y_l\}$.
4. Generate an i.i.d. sequence of random variables $(Z_n)_{n < 0}$ with distribution λ .
5. Let $\tau = \sup\{n < 0 : -Y_l - \sum_{m=n}^{-1} Z_m = -K\}$. Assign $Y_{n+l} := Z_n$ for $\tau \leq n < 0$.
6. Assign $l := l + \tau - 1$.
7. Repeat the procedure from step 2.

Let τ_n, l_n denote τ and l of the n -th ($n \geq 1$) iteration in the above construction, with the notation that $\tau_0 = 0, l_0 = 0$. Then, for all $n \geq 2$, the following equality of joint distribution holds because of the iterative nature of the construction:

$$(Y_{\tau_n+l_n}, Y_{\tau_n+l_n+1}, \dots, Y_{\tau_{n-1}+l_{n-1}-1}) \stackrel{\mathcal{D}}{=} (Y_{\tau_{n-1}+l_{n-1}}, Y_{\tau_{n-1}+l_{n-1}+1}, \dots, Y_{\tau_{n-2}+l_{n-2}-1}). \quad (5.6.1)$$

Proposition 104. *Let $[\mathbb{Z}, 0, Y]$ be the network associated to the sequence of random variables $Y = (Y_n)_{n \in \mathbb{Z}}$ obtained from Construction 1. Then $[\mathbb{Z}, 0, Y]$ is the R -probability of $[\mathbb{Z}, 0, X]$, where $X = (X_n)_{n \in \mathbb{Z}}$ is as in Def. 22 with $\mathbb{E}[X_0] = 0$.*

Proof. Let π be the distribution of $X_0 + 1$, $[\mathbf{T}, \mathbf{o}]$ and $[\mathbf{T}', \mathbf{o}']$ be random rooted trees with distributions $EKT(\pi)$ (see Fig. 5.2) and $EGWT(\pi)$ (see Fig. 5.3) respectively. We show that $[\mathbb{Z}, 0, Y] = \Phi_R([\mathbf{T}, \mathbf{o}])$. Let $(u_n)_{n \in \mathbb{Z}}$ and $(w_n)_{n \in \mathbb{Z}}$ be the succession lines of $[\mathbf{T}, \mathbf{o}]$ and $[\mathbf{T}', \mathbf{o}']$ respectively, with $u_0 = \mathbf{o}$ and $w_0 = \mathbf{o}'$. It follows from the constructions of $EGWT(\pi)$ and $EKT(\pi)$ that the distribution of the sequence of offspring counts of $(u_n)_{n \geq 1}$ is the same as that of $(w_n)_{n \geq 1}$. Since the map Φ_R is bijective $EGWT(\pi)$ -a.s., by Proposition 96 we have $(d_1(w_n) - 1)_{n \geq 1} \stackrel{\mathcal{D}}{=} (X_{n-1})_{n \geq 1}$. Therefore, $(d_1(u_n) - 1)_{n \geq 1} \stackrel{\mathcal{D}}{=} (d_1(w_n) - 1)_{n \geq 1} \stackrel{\mathcal{D}}{=} (X_{n-1})_{n \geq 1} \stackrel{\mathcal{D}}{=} (Y_{n-1})_{n \geq 1}$.

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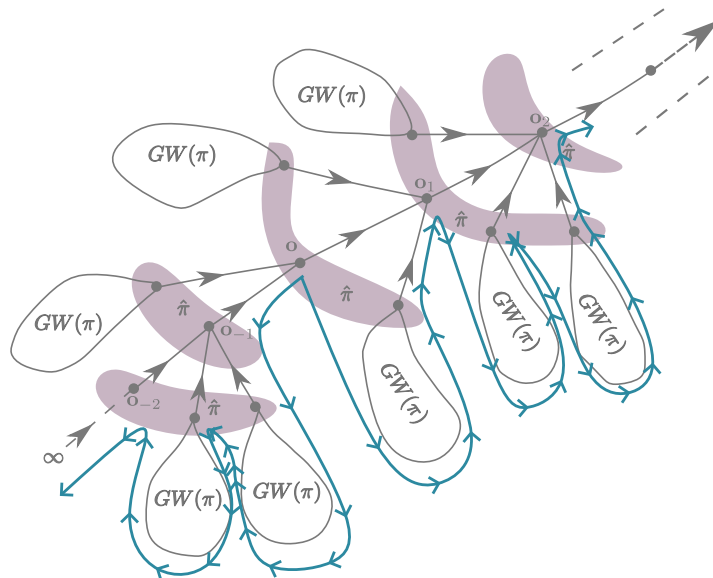


Figure 5.2: Distribution of $EKT(\pi)$ and its succession line starting from the root \circ (drawn in blue).

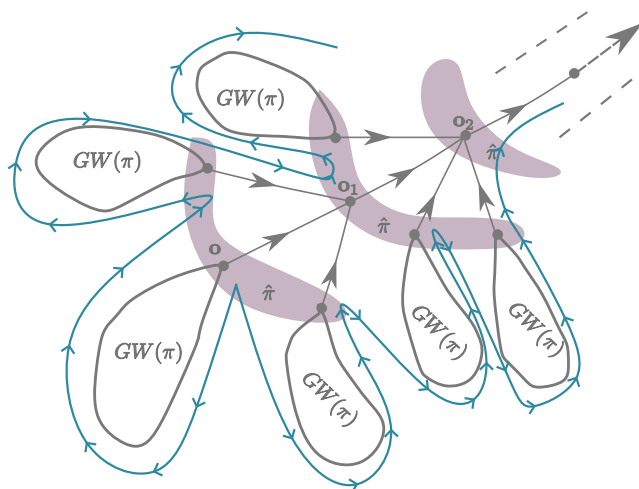


Figure 5.3: Distribution of $EGWT(\pi)$ and its succession line starting from the root \circ (drawn in blue).

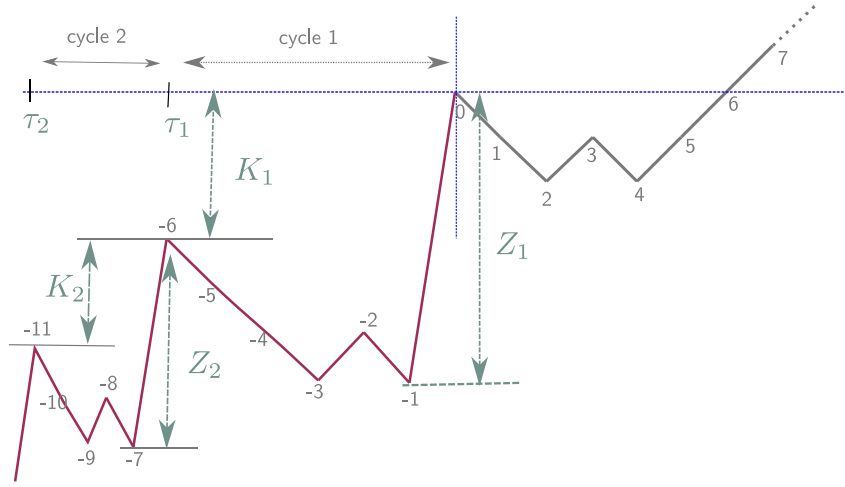


Figure 5.4: Sample trajectory of the sums of the R -probability sequence when $\mathbb{E}[X_0] = 0$ (the R -probability sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ is the sequence of increments of this trajectory).

We now show that $(Y_{-n})_{n \geq 1} \stackrel{\mathcal{D}}{=} (d_1(u_{-n+1}) - 1)_{n \geq 1}$. Consider the random variable τ defined in the first iteration of Construction 1. Let N be a positive random variable such that u_{-N} is the unique child of \mathbf{o} that belongs to the bi-infinite path of \mathbf{T} , i.e., $u_{-N} = \mathbf{o}_{-1}$. By the construction of $EKT(\pi)$, $(d_1(u_{-N-n}))_{n \geq 0} \stackrel{\mathcal{D}}{=} (d_1(u_{-n}))_{n \geq 0}$. Therefore, by Eq. (5.6.1), it is enough to show that

$$\tau \stackrel{\mathcal{D}}{=} -N + 1 \tag{5.6.2}$$

and the equality of the following conditioned distributions

$$(Y_{-n}, Y_{-n+1}, \dots, Y_{-1}) |_{\tau = -n+1} \stackrel{\mathcal{D}}{=} (d_1(u_{-n+1}) - 1, d_1(u_{-n+2}) - 1, \dots, d_1(u_0) - 1) |_{N=n}. \tag{5.6.3}$$

By the construction of $EKT(\pi)$ and Construction 1, Eq. (5.6.3) follows. It remains to show Eq. (5.6.2). We show it using the following inequality. For a realization $[T, o]$ of $EKT(\pi)$, and any M such that $0 < M \leq N - 1$ (N as above), let $u' := u'(M)$ denote the youngest child of u_0 such that $u > u'$ implies that $u \in \{u_0, u_{-1}, \dots, u_{-M}\}$. Then, we have

$$\sum_{i=0}^M (d_1(u_{-i}) - 1) \geq \#\{u \in D_1(u_0) : u < u_{-N}\}, \tag{5.6.4}$$

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with equality if and only if $M = N - 1$. Indeed,

$$\begin{aligned} \sum_{i=0}^M (d_1(u_{-i}) - 1) &= d_1(u_0) - 1 \\ &+ \sum_{\{u \in D_1(u_0) : u > u'\}} \sum_{v \in T_u} (d_1(v) - 1) + \sum_{\{u \in T_{u'} : u \geq u_{-M}\}} (d_1(u) - 1), \end{aligned}$$

where T_u in the above equation denotes the descendant tree of u (with root as u). The above equation is obtained by rearranging the sum on the left-hand side into sums on the descendant trees of the children of u_0 whose descendants are contained in $\{u_0, u_{-1}, \dots, u_{-M}\}$ and the remaining part. By applying part 1 of Lemma 93 to the second sum and Remark 94 to the third sum of the above equation respectively, the right-hand side of the above equation is

$$\geq d_1(u_0) - 1 + \sum_{\{u \in D_1(u_0) : u > u'\}} (-1) + 0.$$

Since $u' \geq u_{-N}$, the sum in the above equation is lower bounded by $\#\{u \in D_1(u_0) : u > u_{-N}\}$, so we obtain,

$$\begin{aligned} \sum_{i=0}^M (d_1(u_{-i}) - 1) &\geq d_1(u_0) - 1 - \#\{u \in D_1(u_0) : u > u_{-N}\} \\ &= \#\{u \in D_1(u_0) : u < u_{-N}\}. \end{aligned}$$

Equality in the above occurs if and only if $u' = u_{-N}$, i.e., $M = N - 1$.

Since the order of \mathbf{o}_{-1} ($= u_{-N}$) is uniform among the children of \mathbf{o} in $EKT(\pi)$, and $d_1(u_0) > 0$ (because $d_1(u_0) \stackrel{\mathcal{D}}{=} \hat{\pi}$ and $\hat{\pi}(\mathbb{N}) = 1$), it follows that the distribution of $\#\{u \in D_1(u_0) : u < u_{-N}\}$ conditioned on $\{d_1(u_0) = \ell\}$ is the uniform distribution on $\{0, 1, 2, \dots, \ell - 1\}$. Therefore, $\#\{u \in D_1(u_0) : u < u_{-N}\} \stackrel{\mathcal{D}}{=} K$, where K is the random variable in the first iteration of Construction 1.

By the argument following Eq. (5.6.4), it follows that $M = N - 1$ is the smallest non-negative integer for which equality is attained in Eq. (5.6.4). Therefore, $\tau \stackrel{\mathcal{D}}{=} -(N - 1)$. Thus,

$$\begin{aligned} (Y_{-N}, Y_{-N+1}, \dots, Y_{-1}) &\stackrel{\mathcal{D}}{=} (Z_\tau, Z_{\tau+1}, \dots, Z_{-1}, Y_{-1}) \\ &\stackrel{\mathcal{D}}{=} (d_1(u_{-N+1}) - 1, d_1(u_{-N+2}) - 1, \dots, d_1(u_0) - 1). \end{aligned}$$

□

Construction 2: In the following steps, we provide an alternative construction of the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ (also illustrated in Figure 5.4) from the sequence $X = (X_n)_{n \in \mathbb{Z}}$, where X is as in Def. 22 with $\mathbb{E}[X_0] = 0$. This construction is used in Section 5.9 to compare the foil structures of $[\mathbb{Z}, 0, X]$ and $[\mathbb{Z}, 0, Y]$ under the record vertex-shift. Let π be the distribution of $X_0 + 1$, and $\hat{\pi}$ be the size-biased distribution of π .

1. Take $(Y_n)_{n \geq 0} \stackrel{\mathcal{D}}{=} (X_n)_{n \geq 0}$.
2. Let $(Z_n)_{n \geq 1}$ be an i.i.d. sequence of random variables, independent of $(Y_n)_{n \geq 0}$, that take values in $\mathbb{Z}_{\geq -1}$ and whose common distribution is given by $\mathbb{P}[Z_1 = b] = \hat{\pi}(b + 1)$ for all $b \geq -1$. Note that $\mathbb{P}[Z_1 = -1] = 0$.
3. Let $(K_n)_{n \geq 1}$ be an i.i.d. sequence of random variables such that K_n is independent of Z_m whenever $m \neq n$, and $\mathbb{P}[K_n = a | Z_n = b] = \frac{1}{b+1} \mathbf{1}_{0 \leq a \leq b}$, for all $(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and for all $n \geq 1$.
4. Consider an array $(X_m^{(n)})_{m, n \in \mathbb{N}}$ of i.i.d. random variables, independent of all the above random variables, such that $X_1^{(1)} \stackrel{\mathcal{D}}{=} X_1$.
5. Define a sequence of random times $(\tau_n)_{n \geq 1}$ by $\tau_n := \inf\{m \geq 1 : -Z_n - \sum_{l=1}^m X_m^{(n)} \geq -K_n\}$ for all $n \geq 1$. Note that $\tau_n < \infty$ almost surely for all $n \geq 1$ by Chung-Fuchs theorem. Since $X_1^{(1)}$ cannot be smaller than -1 , we have $-Z_n - \sum_{l=1}^{\tau_n} X_m^{(n)} = -K_n$ for all $n \geq 1$.

Finally, consider the following independent sequence of random-sized tuples

$$\begin{aligned} (Y_{-1}, Y_{-2}, \dots, Y_{-(\tau_1+1)}) &\stackrel{\mathcal{D}}{=} (Z_1, X_1^{(1)}, X_2^{(1)}, \dots, X_{\tau_1}^{(1)}), \\ (Y_{-(\tau_1+2)}, \dots, Y_{-(\tau_1+\tau_2+2)}) &\stackrel{\mathcal{D}}{=} (Z_2, X_1^{(2)}, \dots, X_{\tau_2}^{(2)}), \dots, \\ (Y_{-(\tau_1+\dots+\tau_{n-1}+n)} + \dots + Y_{-(\tau_1+\dots+\tau_{n-1}+\tau_n+n)}) &\stackrel{\mathcal{D}}{=} (Z_n, X_1^n, \dots, X_{\tau_n}^n), \dots \end{aligned}$$

We call the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ the **R -probability sequence** associated to $X = (X_n)_{n \in \mathbb{Z}}$.

Remark 105. Note that K_1 is a proper random variable since $\mathbb{P}[K_1 = a] = \sum_{l=a+1}^{\infty} \pi(l)$ for all non-negative integers $a \geq 0$ and $\mathbb{P}[K_n < \infty] = \sum_{a=0}^{\infty} \pi((a, \infty)) = m(\pi) = 1$. Observe that $\mathbb{P}[K_1 > 0] > 0$ since $\pi(0) < 1$.

From the construction, it is clear that $(K_n, \tau_n) \stackrel{\mathcal{D}}{=} (K, \tau)$, where K, τ are the random variables defined in Construction 1.

5.7 Properties of the R-probability sequence for zero mean

In this section, we discuss the properties of the R-probability sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of $X = (X_n)_{n \in \mathbb{Z}}$, defined in the last section, when $\mathbb{E}[X_0] = 0$. Recall that $(Y_n)_{n \geq 0}$ is an i.i.d. sequence whose common distribution is given by $Y_0 \stackrel{\mathcal{D}}{=} X_0$. So, we focus on the properties of $(Y_n)_{n < 0}$.

For two integers $m < n$, let $w(m, n) := \sum_{k=m}^{n-1} Y_k$ and $W_n = -w(-n, 0) = -\sum_{k=-n}^{-1} Y_k$, for $n \geq 1$. We continue to use the notations of the last section in the following lemma and proposition. From the definitions of τ_n and K_n , it follows that $W_{\tau_1 + \tau_2 + \dots + \tau_n + n} = -(K_1 + K_2 + \dots + K_n)$ for all $n \geq 1$.

Lemma 106. *Almost surely $W_m < -(K_1 + \dots + K_n + K_{n+1})$ for all positive integers m, n such that $n \geq 1$ and $\tau_1 + \tau_2 + \dots + \tau_n + n < m < \tau_1 + \tau_2 + \dots + \tau_{n+1} + n + 1$, where $(\tau_i)_{i \geq 1}$ are as in Construction 2.*

Proof. The proof is by induction on n . For $n = 1$ and $\tau_1 + 1 < m < \tau_1 + \tau_2 + 2$, we have

$$W_m = W_{\tau_1 + 1} - w(-m, -(\tau_1 + 1)) = -K_1 - w(-m, -(\tau_1 + 1)).$$

But, by the definition of τ_2 ,

$$-w(-m, -(\tau_1 + 1)) = -Y_{-(\tau_1 + 2)} - w(-m, -(\tau_1 + 2)) \stackrel{\mathcal{D}}{=} -Z_2 - \sum_{l=1}^{m - (\tau_1 + 2)} X_l^{(2)} < -K_2.$$

Therefore, $W_m < -K_1 - K_2$ for $n = 1$.

Assume that the statement is true for $n - 1$. For $\tau_1 + \dots + \tau_n + n < m < \tau_1 + \tau_2 + \dots + \tau_{n+1} + n + 1$, we have

$$\begin{aligned} W_m &= W_{\tau_1 + \dots + \tau_n + n} - w(-m, -(\tau_1 + \dots + \tau_n + n)) \\ &\stackrel{\mathcal{D}}{=} W_{\tau_1 + \dots + \tau_n + n} - Z_{n+1} - \sum_{l=1}^{m - (\tau_1 + \dots + \tau_n + n + 1)} X_l^{(n+1)} \\ &< -(K_1 + \dots + K_n + K_{n+1}). \end{aligned}$$

Last step follows by induction, and by the definition of τ_{n+1} . \square

Proposition 107. *Almost surely $W_n \rightarrow -\infty$ as $n \rightarrow \infty$.*

Proof. From Lemma 106, and the fact that $K_n \geq 0, \forall n \geq 1$, it follows that $W_m < -(K_1 + K_2 + \dots + K_n)$, for all $n \geq 1$ and $m > \tau_1 + \dots + \tau_n + n$. Therefore, the event $\{W_m \xrightarrow{m \rightarrow \infty} -\infty\}$ contains the event $\{-(K_1 + \dots + K_n) \xrightarrow{n \rightarrow \infty} -\infty\}$. So, it is enough to show that $\mathbb{P}[-(K_1 + K_2 + \dots + K_n) \xrightarrow{n \rightarrow \infty} -\infty] = 1$. Since K_n is non-negative for all $n \geq 1$, it is enough to show that $\mathbb{P}[K_n > 0 \text{ infinitely often}] = 1$. The events $\{K_n > 0\}$ satisfy $\sum_{n=1}^{\infty} \mathbb{P}[K_n > 0] = \infty$ since, $\mathbb{P}[K_1 > 0] > 0$ and $(K_n)_{n \geq 1}$ are i.i.d. random variables. Therefore, $\mathbb{P}[K_n > 0 \text{ i.o.}] = 1$ by the Borel-Cantelli Lemma. \square

The following corollary is used to describe the record foils of the R -probability (see Proposition 112).

Corollary 108. *For any integer j , almost surely*

$$-w(j - n, j) = - \sum_{k=j-n}^{j-1} Y_k \xrightarrow{n \rightarrow \infty} -\infty.$$

Proof. For $n > \max\{0, j\}$,

$$-w(j - n, j) = \begin{cases} W_n - w(0, j) & \text{if } j > 0 \\ W_n + w(j, 0) & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 107, a.s. $-w(j - n, j) \rightarrow -\infty$ as $n \rightarrow \infty$. \square

5.8 Construction of R -probability for positive mean

In the following steps, we construct a sequence $\tilde{Y} = (\tilde{Y}_n)_{n \in \mathbb{Z}}$ and later show that $[\mathbb{Z}, 0, \tilde{Y}]$ is the R -probability of $[\mathbb{Z}, 0, X]$ when $0 < \mathbb{E}[X_0] < \infty$. Recall the definitions of $\tilde{\pi}, \bar{\pi}$ and c from Eq. (3.2.8) in Section 3.2.

1. The random variables $(\tilde{Y}_n)_{n \geq 0}$ are i.i.d. with the common distribution the same as that of X_0 . That is, $\tilde{Y}_n \stackrel{D}{=} X_0$ for all $n \geq 0$.
2. Let $((K_n, Z_n))_{n \geq 1}$ be an i.i.d. sequence of random variables taking values in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ whose common joint-distribution is given by $\mathbb{P}[K_1 = m, Z_1 = l] = \mathbb{P}[X_0 = m + l]c^l$ for all $m \geq 0, l \geq 0$.

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3. Define an array $(X_m^{(n)})_{m,n \in \mathbb{N}}$ of i.i.d. sequence of random variables whose common distribution is given by $\mathbb{P}[X_1^{(1)} = k] = \tilde{\pi}(k+1)$ for all $k \geq 0$.
4. Define the sequence of random times $(\tau_n)_{n \geq 1}$ by $\tau_n = \inf\{m \geq 1 : -Z_n - K_n - \sum_{l=1}^m X_m^{(n)} = -K_n\} = \inf\{m \geq 1 : -Z_n - \sum_{l=1}^m X_m^{(n)} = 0\}$ if the infimum is attained and $\tau_n = \infty$ otherwise, for all $n \geq 1$. Note that $\tau_1 < \infty$ a.s. since the random walk $(\tilde{S}_m)_{m \geq 0}$ starting at 0 drifts to ∞ by Lemma 32, where $\tilde{S}_m = -\sum_{l=1}^m X_m^{(1)}$ for all $m \geq 1$.
5. Define $(\tilde{Y}_{-n})_{n \geq 0}$ by

$$\begin{aligned} (\tilde{Y}_{-1}, \tilde{Y}_{-2}, \dots, \tilde{Y}_{-(\tau_1+1)}) &= (Z_1 + K_1, X_1^{(1)}, X_2^{(1)}, \dots, X_{\tau_1}^{(1)}), \\ (\tilde{Y}_{-(\tau_1+2)}, \tilde{Y}_{-(\tau_1+3)}, \dots, \tilde{Y}_{-(\tau_1+\tau_2+2)}) &= (Z_2 + K_2, X_1^{(2)}, X_2^{(2)}, \dots, X_{\tau_2}^{(2)}), \dots, \\ \dots, (\tilde{Y}_{-(\tau_1+\dots+\tau_{n-1}+n)}, \dots, \tilde{Y}_{-(\tau_1+\dots+\tau_n+n)}) &= (Z_n + K_n, X_1^{(n)}, \dots, X_{\tau_n}^{(n)}), \dots \end{aligned}$$

Note that the R -probability of $[\mathbb{Z}, 0, X]$ exists by Theorem 102. We now prove that the sequence constructed above is the R -probability of $[\mathbb{Z}, 0, X]$ when $0 < \mathbb{E}[X_0] < \infty$. The proof of this statement together with Remark 103 implies that $[\mathbb{Z}, 0, \tilde{Y}]$ has the same distribution as that of the conditioned network $[\mathbb{Z}, 0, X]$ conditioned on the event that $\{S_n \leq 0 \ \forall n \leq 0\}$, where $(S_n)_{n \in \mathbb{Z}}$ denotes the skip-free random walk associated to the increment sequence $X = (X_n)_{n \in \mathbb{Z}}$.

For any Family Tree T that has a unique bi-infinite path, denote its bi-infinite path by $\not\prec(T)$.

Proposition 109. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be the i.i.d. sequence of increments of a skip-free to the left random walk that satisfies $0 < \mathbb{E}[X_0] < \infty$ and $[\mathbb{Z}, 0, X]$ be its associated network. Let $[\mathbb{Z}, 0, Y]$ be the R -probability of $[\mathbb{Z}, 0, X]$, where $Y = (Y_n)_{n \in \mathbb{Z}}$. Then, $(Y_n)_{n \geq 0}$ is an i.i.d. sequence of random variables independent of $(Y_n)_{n < 0}$ and their common distribution is the same as that of X_0 .*

Proof. Let $[\mathbf{T}, 0, t]$ be the record graph of $[\mathbb{Z}, 0, X]$ with the type function t as the mark function and $[\mathbf{T}', \mathbf{o}', t']$ be the random Family Tree obtained from $[\mathbf{T}, 0, t]$ by conditioning on the event $0 \in \not\prec$ (denoted as $[\mathbf{T}, 0, t]|0 \in \not\prec$). Note that $[\mathbf{T}', \mathbf{o}', t']$ is the F -probability of $[\mathbf{T}, 0, t]$ (see Proposition 82 and Theorem 55). By Theorem 54 and Theorem 55, the distribution of $[\mathbf{T}', \mathbf{o}', t']$

is the $MEKT(\bar{\pi}, \tilde{\pi})$ where the marked bush of the root is distributed as in Eq. (3.2.9) and $\bar{\pi}, \tilde{\pi}$ are as in Eq. (3.2.8).

Let $U((\mathbf{T}', \mathbf{o}')) = (u_n)_{n \in \mathbb{Z}}$ be the succession line of $(\mathbf{T}', \mathbf{o}')$ with $u_0 = \mathbf{o}'$, and $U((\mathbf{T}, 0)) = (w_n)_{n \in \mathbb{Z}}$ be the succession line of $(\mathbf{T}, 0)$ with $w_0 = 0$.

Since $[\mathbf{T}', \mathbf{o}', t'] \stackrel{\mathcal{D}}{=} [\mathbf{T}, 0, t] | 0 \in \mathcal{L}'$, we have the equality in distribution of the following two sequence of random variables:

$$\begin{aligned} & ((d_1(u_n) - 1)\mathbf{1}\{u_n \notin \mathcal{L}'\} + (d_1(u_n) + t(u_n) - 1)\mathbf{1}\{u_n \in \mathcal{L}'\})_{n \geq 1} \quad (5.8.1) \\ & \stackrel{\mathcal{D}}{=} \\ & ((d_1(w_n) - 1)\mathbf{1}\{w_n \notin \mathcal{L}'\} + (d_1(w_n) + t(w_n) - 1)\mathbf{1}\{w_n \in \mathcal{L}'\})_{n \geq 1} | 0 \in \mathcal{L}' \end{aligned}$$

Since $[\mathbb{Z}, 0, Y]$ is the push-forward of the $MEKT(\bar{\pi}, \tilde{\pi})$ (with t as the mark function) under the extended backward map $\hat{\Phi}_R$, we have $(d_1(u_n) - 1)\mathbf{1}\{u_n \notin \mathcal{L}'\} + (d_1(u_n) + t(u_n) - 1)\mathbf{1}\{u_n \in \mathcal{L}'\} = Y_{n-1}, \forall n \in \mathbb{Z}$. From the construction of the $MEKT(\bar{\pi}, \tilde{\pi})$ with t as the mark function, the subtrees generated by $\{u_n : n \geq 1\}$ and $\{u_n : n \leq 0\}$ are independent of each other. This implies that the sequence $(Y_n)_{n \geq 0}$ and $(Y_n)_{n < 0}$ are independent of each other.

Recall that the extended backward map $\hat{\Phi}_R$ is bijective, and we have $(d_1(w_n) - 1)\mathbf{1}\{w_n \notin \mathcal{L}'\} + (d_1(w_n) + t(w_n) - 1)\mathbf{1}\{w_n \in \mathcal{L}'\} = X_{n-1}$ for all $n \geq 1$. Observe that the event $\{0 \in \mathcal{L}'\}$ is equal to $\{S_n \leq 0 \forall n < 0\}$.

Since $(X_n)_{n \geq 0}$ is independent of the event $\{S_n \leq 0 \forall n < 0\}$, the conditioned process $(X_n)_{n \geq 0}$ conditioned on this event has the same distribution as that of the unconditioned process $(X_n)_{n \geq 0}$. Therefore, by Eq. (5.8.1), we have

$$(Y_n)_{n \geq 0} \stackrel{\mathcal{D}}{=} (X_n)_{n \geq 0} | 0 \in \mathcal{L}' = (X_n)_{n \geq 0} | \{S_n \leq 0, \forall n < 0\} \stackrel{\mathcal{D}}{=} (X_n)_{n \geq 0}.$$

Thus,

$$(Y_n)_{n \geq 0} \stackrel{\mathcal{D}}{=} (X_n)_{n \geq 0}.$$

□

Consider the sequence $\tilde{Y} = (\tilde{Y}_n)_{n \in \mathbb{Z}}$ generated by the above construction of this subsection. Proposition 109 proves that the non-negatively indexed part $(\tilde{Y}_n)_{n \geq 0}$ has the same distribution as that of the non-negatively indexed part $(Y)_{n \geq 0}$, where $Y = (Y_n)_{n \in \mathbb{Z}}$ is the R -probability sequence associated to

5.8. CONSTRUCTION OF R -PROBABILITY FOR POSITIVE MEANS

X . In the following proposition, we show that the negatively indexed part $(\tilde{Y}_n)_{n<0}$ has the same distribution of the negatively indexed part $(Y_n)_{n<0}$. These two propositions together show that the above construction generates the R -probability sequence of X .

Proposition 110. *Let $[\mathbb{Z}, 0, Y]$ be the R -probability of $[\mathbb{Z}, 0, X]$, where $Y = (Y_n)_{n \in \mathbb{Z}}$ where X is as in Proposition 109. Then, $(\tilde{Y}_n)_{n<0} \stackrel{\mathcal{D}}{=} (Y_n)_{n<0}$.*

Proof. Let $[\mathbf{T}, 0, t]$ be the record graph of $[\mathbb{Z}, 0, X]$ with the type function t associated to X as the mark function, and $[\mathbf{T}', \mathbf{o}', t']$ be the F -probability of $[\mathbf{T}, 0, t]$. Note that the distribution of $[\mathbf{T}', \mathbf{o}', t']$ is $MEKT(\bar{\pi}, \tilde{\pi})$ whose bush of the root is distributed as in Eq. (3.2.9) (see Proposition 82 and Theorem 55). Let $U((\mathbf{T}', \mathbf{o}', t')) = (u_n)_{n \in \mathbb{Z}}$ be the succession line of $(\mathbf{T}', \mathbf{o}', t')$ with $u_0 = \mathbf{o}'$.

Let $N > 0$ be a positive integer such that u_{-N} is the smallest child of u_0 . By the construction of $[\mathbf{T}', \mathbf{o}', t']$, u_{-N} lies on the bi-infinite F -path of \mathbf{T}' . We know that $[\mathbf{T}', \mathbf{o}', t']$ is obtained as the joining of i.i.d. Family Trees $[\mathbf{T}_i, \mathbf{o}_i, t_i]_{i \in \mathbb{Z}}$, where t_i is the restriction of t' to $V(\mathbf{T}_i)$ (see Subsection 3.2). So, we have $\mathbf{o}_{-1} = u_{-N}$, $\mathbf{o}_0 = u_0$. Therefore, u_0, u_{-N} lie on the bi-infinite path and none of $\{u_{-1}, \dots, u_{-N+1}\}$ lie on the bi-infinite path.

Since $[\mathbb{Z}, 0, Y] = \hat{\Phi}_R([\mathbf{T}', \mathbf{o}', t'])$ (see Theorem 102), we have

$$(Y_n)_{n<0} = ((d_1(u_{n+1})-1)\mathbf{1}\{u_{n+1} \notin \mathcal{A}\} + (d_1(u_{n+1})+t(u_{n+1})-1)\mathbf{1}\{u_{n+1} \in \mathcal{A}\})_{n<0}.$$

In particular, we have $(Y_{-1}, Y_{-2}, \dots, Y_{-N}) \stackrel{\mathcal{D}}{=} (d_1(u_0) + t(u_0) - 1, d_1(u_{-1}) - 1, \dots, d_1(u_{-N+1}) - 1)$.

Consider τ_1 of the construction of this subsection. It is defined as $\tau_1 = \inf\{m \geq 1 : -Z_1 - \sum_{l=1}^m X_m^{(1)} = 0\}$, where $Z_1, X_m^{(1)}$ are as defined in the construction. In the construction, we generate the sequence $(\tilde{Y}_n)_{n<0}$ iteratively. The first iteration gives the sequence $(\tilde{Y}_{-1}, \tilde{Y}_{-2}, \tilde{Y}_{-3}, \dots, \tilde{Y}_{-(\tau_1+1)}) = (Z_1 + K_1, X_1^{(1)}, X_2^{(1)}, \dots, X_{\tau_1}^{(1)})$.

Because of the i.i.d. joining involved in the construction of $[\mathbf{T}', \mathbf{o}', t']$ and the iterative nature of the algorithm, it is sufficient to show that the joint distribution of the random variables associated to the subtree $[\mathbf{T}_0, \mathbf{o}_0, t_0]$ (i.e., $(\tilde{Y}_{-1}, \tilde{Y}_{-2}, \tilde{Y}_{-3}, \dots, \tilde{Y}_{-(\tau_1+1)})$) is the same as that of those that are generated in the first iteration of the construction. That is, it is sufficient to show that

$$N \stackrel{\mathcal{D}}{=} \tau_1 + 1 \tag{5.8.2}$$

and

$$(Z_1 + K_1, X_1^{(1)}, X_2^{(1)}, \dots, X_{n-1}^{(1)})|_{\tau_1=n-1} \stackrel{\mathcal{D}}{=} (d_1(u_0) + t(u_0) - 1, d_1(u_{-1}) - 1, \dots, d_1(u_{-n+1}) - 1)|_{N=n}. \quad (5.8.3)$$

The joint-distribution of (K_1, Z_1) is chosen such that $\mathbb{P}[K_1 = m, Z_1 = l] = \mathbb{P}[X_0 = m + l]c^l = \mathbb{P}[t_0(\mathbf{o}_0) = m, d_1(\mathbf{o}_0) = l]$ (see Eq. (3.2.9)). Since $d_1(u_0) = d_1(\mathbf{o}_0) - 1$, we have $Z_1 + K_1 \stackrel{\mathcal{D}}{=} d_1(u_0) - 1 + t(u_0)$. On the event that $N = n$, u_{-k} does not belong to the bi-infinite path of \mathbf{T}' and they have the same offspring distribution $\tilde{\pi}$, for all $1 \leq k \leq n - 1$. Since, $(X_i^{(1)} + 1)_{i \geq 1}$ are i.i.d. random variables with the common distribution $\tilde{\pi}$, Eq. (5.8.3) follows.

We now show Eq. (5.8.2). Consider the finite ordered Family Tree $[\mathbf{T}_0, \mathbf{o}_0]$. The vertices of \mathbf{T}_0 are $u_{-N+1} < \dots < u_{-1} < u_0$ listed according to the RLS order. For any $0 \leq k \leq N - 1$, by Remark 94 and Lemma 93, we have

$$\sum_{i=-k}^0 (d_1(u_i) - 1) \geq 0$$

and

$$\sum_{i=-N+1}^0 (d_1(u_i) - 1) = -1.$$

Rewriting the above two equations, we obtain

$$\sum_{i=-k}^{-1} (d_1(u_i) - 1) + d_1(u_0) \geq 1$$

and

$$\sum_{i=-N+1}^{-1} (d_1(u_i) - 1) + d_1(u_0) = 0.$$

Note that $Z_1 \stackrel{\mathcal{D}}{=} d_1(u_0)$ (in \mathbf{T}_0) and $d_1(u_i) - 1 \stackrel{\mathcal{D}}{=} X_{-i}^{(1)}$ for all $-N+1 \leq i \leq -1$. Thus, we obtain

$$\sum_{i=-k}^{-1} X_{-i}^{(1)} + Z_1 \geq 1$$

and

$$\sum_{i=-N+1}^{-1} X_{-i}^{(1)} + Z_1 = 0.$$

In particular, $\tau_1 \stackrel{\mathcal{D}}{=} N - 1$. This completes the proof. □

5.9 R-foils of R-probability for zero mean

Let $Y = (Y_n)_{n \in \mathbb{Z}}$ be the R -probability sequence of $X = (X_n)_{n \in \mathbb{Z}}$, where X is as in Def. 22 with $\mathbb{E}[X_0] = 0$, and $[\mathbb{Z}, 0, Y], [\mathbb{Z}, 0, X]$ be their associated networks respectively. We study the record vertex-shift on $[\mathbb{Z}, 0, Y]$ and compare it with that of $[\mathbb{Z}, 0, X]$.

Remark 111. Observe that the record vertex-shift R is continuous $[\mathbb{Z}, 0, Y]$ -a.s.. This implies that the map θ_R preserves the distribution of $[\mathbb{Z}, 0, Y]$ (see the proof of Corollary 68).

In contrast to Remark 57, we prove in this section that, almost surely, every R -foil of $[\mathbb{Z}, 0, Y]$ has a minimal element. This fact can be guessed by comparing the R -graph of $[\mathbb{Z}, 0, X]$ with the R -graph of $[\mathbb{Z}, 0, Y]$. Let us call the rooted tree $[\mathbf{T}', \mathbf{o}]$ obtained by trimming off the branches that lie above the bi-infinite F -path shown in Figure 5.2 as the **positive half of the Eternal Kesten Tree** (with the root same as before). As shown in Figure 5.2, it is the subtree induced on the vertices that belong to the succession line of the Eternal Kesten Tree passing through the root. Observe that the R -graph of $[\mathbb{Z}, 0, Y]$ is the positive half of the Eternal Kesten Tree. From the construction of Y , it is clear that a.s. $R_Y^n(0) > R_Y^{(n-1)}(0)$ for all $n \geq 1$. This and the fact that $-w(n, 0) \rightarrow -\infty$ (see Corollary 108) as $n \rightarrow \infty$ imply that the R -graph of $[\mathbb{Z}, 0, Y]$ is a.s. connected. Every generation (i.e. F -foils) of \mathbf{T}' has a unique vertex that belongs to the bi-infinite path. These unique vertices are the smallest elements among their respective foils.

Proposition 112. *Consider the R -foliation of $[\mathbb{Z}, 0, Y]$. For any integer $j \in \mathbb{Z}$, the R -foil of j has a minimal element.*

Proof. By Corollary 108, a.s. the partial sums $-w(j-n, j) \rightarrow -\infty$ as $n \rightarrow \infty$. So, there exists a (random) unique integer $k \leq j$ such that $-w(n, j) < 0$ for all $n < k$ and $w(k, j) \leq 0$. Indeed, $k = \sup\{n \leq j : -w(l, j) < 0 \forall l < n\}$. This implies that $-w(n, k) = -w(n, j) + w(k, j) < 0$ for all $n < k$. Therefore, by Lemma 24, every integer $n < k$ is a descendant of k . Since the R -graph of $[\mathbb{Z}, 0, Y]$ is connected a.s., there exist smallest integers $a, b \geq 0$ such that $R^a(k) = R^b(j)$.

If $b = 0$, then every integer smaller than j is a descendant of j by Lemma 24 and from the discussion in the above paragraph. Hence, j is the smallest element of its R -foil.

So, let us assume that $b \geq 1$. In this case, we show that only finitely many descendants of k belong to the R -foil of j . This implies that the R -foil of j has a smallest element because all the integers smaller than k are descendants of k and there are only finitely many integers between k and j . If no descendant of k belongs to the R -foil of j , then nothing to prove. Suppose that there exists an integer $l < k$ that belongs to the R -foil of j . Since l is a descendant of k , we have $R^N(l) = R^N(j)$, for some smallest $N > a$. Further, $N = b$, since $b > 0$ is the smallest for which $R^b(j) = R^a(k)$. Therefore, l is a $(b - a)$ -th descendant of k . Thus, every descendant of k that belongs to the R -foil of j is a $(b - a)$ -th descendant of k . But, k has finitely many $(b - a)$ -th descendants. Hence, finitely many descendants of k belong to the R -foil of j , and this completes the proof. \square

Chapter 6

Vertex-shifts on processes with stationary increments

In this chapter, we focus on record vertex-shift on stationary sequences and representation of unimodular EFTs as record graphs of a stationary sequence. We use the following stationary framework taken from [5, 10]. The main results of this chapter are Theorem 119 and Theorem 121. Theorem 119 shows that the component of 0 in the record graph of the network associated to a stationary sequence exhibits a phase transition at mean 0 and Theorem 121 provides sufficient conditions under which a unimodular ordered EFT can be represented as the component of 0 in the record graph of the network associated to a stationary sequence.

(In French) Dans ce chapitre, nous nous intéressons au décalage de sommet des records sur les suites stationnaires et à la représentation des EFT unimodulaires comme des graphes des records de suites stationnaires. Nous utilisons le cadre stationnaire suivant, tiré de [5, 10]. Les principaux résultats de ce chapitre sont le théorème 119 et le théorème 121. Le théorème 119 montre que la composante de 0 dans le graphe des records du réseau associé à une suite stationnaire présente une transition de phase à la moyenne 0 et le théorème 121 fournit des conditions suffisantes sous lesquelles un EFT ordonné unimodulaire peut être représenté comme la composante de 0 dans le graphe des records du réseau associé à une suite stationnaire.

6.1 Record vertex-shift on processes with stationary and ergodic increments

In this section, we first show that if $X = (X_n)_{n \in \mathbb{Z}}$ is a stationary and ergodic sequence, then the event $\{\mathbb{Z}_X^R(0) \text{ is of class } \star\}$ is trivial (i.e., it occurs with probability either 0 or 1), where $\star \in \{\mathcal{I}/\mathcal{I}, \mathcal{I}/\mathcal{F}, \mathcal{F}/\mathcal{F}\}$ and $\mathbb{Z}_X^R(0)$ is the component of 0 in the record graph of $[\mathbb{Z}, 0, X]$. This is shown in Proposition 114 and in Corollary 115. In Theorem 119, we show that $\mathbb{Z}_X^R(0)$ exhibit a phase transition at 0 when the mean of X_0 is varied between $-\infty$ and $+\infty$.

Definition 113. Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence, where $X_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $\mathbb{E}[|X_0|] < \infty$.

Define the sequence of sums $(S_n)_{n \in \mathbb{Z}}$ by $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} X_k$, for all $n > 0$, and $S_n = -\sum_{k=n-1}^{-1} X_k$, for all $n < 0$. For any sequence $x = (x_n)_{n \in \mathbb{Z}}$ of integers, let \mathbb{Z}_x^R denotes the record graph of the network (\mathbb{Z}, x) , and for any $i \in \mathbb{Z}$, let $\mathbb{Z}_x^R(i)$ denotes the component of i in \mathbb{Z}_x^R . Let T be the shift map that maps every sequence $x = (x_n)_{n \in \mathbb{Z}}$ to $Tx = (x_{n+1})_{n \in \mathbb{Z}}$. The shift map naturally maps any network of the form $[\mathbb{Z}, 0, x]$ to $T([\mathbb{Z}, 0, x]) = [\mathbb{Z}, 0, Tx]$. For any integer $i \in \mathbb{Z}$, consider the following events:

$$\begin{aligned} E_i &:= \{[\mathbb{Z}, 0, x] : \mathbb{Z}_x^R(i) \text{ is of class } \mathcal{I}/\mathcal{I}\} \\ F_i &:= \{[\mathbb{Z}, 0, x] : \mathbb{Z}_x^R(i) \text{ is of class } \mathcal{I}/\mathcal{F}\}, \end{aligned}$$

and let $T^{-1}E_i := \{[\mathbb{Z}, 0, x] : [\mathbb{Z}, 0, Tx] \in E_i\}$, $T^{-1}F_i := \{[\mathbb{Z}, 0, x] : [\mathbb{Z}, 0, Tx] \in F_i\}$. Then, for any $i \in \mathbb{Z}$, we have

$$\begin{aligned} T^{-1}E_i &= \{[\mathbb{Z}, 0, x] : \mathbb{Z}_{Tx}^R(i) \text{ is of class } \mathcal{I}/\mathcal{I}\} \\ &= \{[\mathbb{Z}, 0, x] : \mathbb{Z}_x^R(i+1) \text{ is of class } \mathcal{I}/\mathcal{I}\} \\ &= E_{i+1}, \end{aligned} \tag{6.1.1}$$

where in Eq.(6.1.1), we used the fact that $[\mathbb{Z}, i, Tx] = [\mathbb{Z}, i+1, x]$. Similarly, we can show that $T^{-1}F_i = F_{i+1}$.

Observe that if $[\mathbb{Z}, 0, x] \in E_i$ (similarly F_i), then, by the interval property (Lemma 25 and Remark 26), $i+1$ and i are in the same connected component, i.e., $\mathbb{Z}_x^R(i) = \mathbb{Z}_x^R(i+1)$, which implies that $[\mathbb{Z}, 0, x] \in E_{i+1}$ (similarly F_{i+1}). Thus, we have $E_i \subseteq T^{-1}(E_i)$. Similarly, we have $F_i \subseteq T^{-1}(F_i)$. Using the fact that, if (\mathbb{P}, T) is ergodic and A is a measurable subset such

that $A \subseteq T^{-1}A$, then $\mathbb{P}[A] \in \{0, 1\}$ (see [10, Theorem 16.1.9]), we obtain the following proposition.

Proposition 114. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence as in Def. 113 and ergodic. Then,*

$$\begin{aligned} \mathbb{P}[\mathbb{Z}_X^R(0) \text{ is of class } \mathcal{I}/\mathcal{I}] &\in \{0, 1\}, \\ \mathbb{P}[\mathbb{Z}_X^R(0) \text{ is of class } \mathcal{I}/\mathcal{F}] &\in \{0, 1\}. \end{aligned}$$

Corollary 115. *Under the assumptions of Proposition 114, the event $\{\mathbb{Z}_X^R(0) \text{ is of class } \mathcal{F}/\mathcal{F}\}$ has trivial probability measure.*

Proof. By the Foil classification Theorem 14, $\mathbb{Z}_X^R(0)$ is either of class \mathcal{I}/\mathcal{I} or \mathcal{I}/\mathcal{F} or \mathcal{F}/\mathcal{F} . By Proposition 114, the first two events have trivial probability measure which implies that the third event has trivial probability measure. \square

The proof of the following lemma is similar to that of Proposition 40. The proof essentially relies on the interval property of record vertex-shift.

Lemma 116. *Let X be a stationary sequence as in Def. 113, $[\mathbb{Z}, 0, X]$ be its associated network, and \mathbb{Z}_X^R be the record graph of the network (\mathbb{Z}, X) . If 0 has a.s. infinitely many ancestors in \mathbb{Z}_X^R , then the record graph \mathbb{Z}_X^R is a.s. connected.*

Proof. For any $i \in \mathbb{Z}$, let A_i be the event that i has infinitely many ancestors in \mathbb{Z}_X^R . By the stationarity of X , every event A_i occurs with probability 1 for each $i \in \mathbb{Z}$ since A_0 occurs with probability 1. Therefore, the event $A = \cap_{i \in \mathbb{Z}} A_i$ occurs with probability 1.

We now show that every integer $i < 0$ belongs to the component of 0 in \mathbb{Z}^R . Similarly, by interchanging the role of i and 0, we could show that every integer $i > 0$ belongs to the component of 0.

Let $i < 0$. Since A occurs with probability 1, the sequence $(R_X^n(i))_{n \in \mathbb{Z}}$ is strictly increasing a.s. So, there exists a smallest (random) integer $k > 0$ such that $R_X^{k-1}(i) < 0 \leq R_X^k(i)$ a.s.. By Lemma 25 (applied to $R_X^{k-1}(i)$) and by Remark 26, 0 is a descendant of $R_X^k(i)$ a.s.. Thus, a.s., 0 and i are in the same connected component. \square

Lemma 117. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence of random variables as in Def. 113, $[\mathbb{Z}, 0, X]$ be its associated network and for all $n \in \mathbb{Z}$, $\mathbb{Z}_X^R(n)$ be the component of n in the record graph of (\mathbb{Z}, X) . Then, we have*

$$\mathbb{P}[\mathbb{Z}_X^R(0) \text{ is finite}] = \mathbb{P}[\mathbb{Z}_X^R(n) \text{ is finite}, \forall n \geq 1] = \mathbb{P}[\mathbb{Z}_X^R(n) \text{ is finite}, \forall n \leq -1]. \quad (6.1.2)$$

Proof. Let A be the event given by $A = \{\mathbb{Z}_X^R(0) \text{ is finite}\}$. By Poincaré's recurrence lemma, $\mathbb{P}[A] = \mathbb{P}[\{\mathbb{Z}_{T^n X}^R(0) \text{ is finite for infinitely many } n \geq 1\}]$. We show that for any $\omega \in A$, if $\mathbb{Z}_{T^n X(\omega)}^R(0)$ is finite for infinitely many $n \geq 1$, then $\mathbb{Z}_{X(\omega)}^R(n)$ is finite for all $n \geq 1$. This implies the first equality in Eq. (6.1.2). The second equality of Eq. (6.1.2) follows by taking the shift $\tilde{T} = T^{-1}$ instead of T , and by using the equality of the events $\{\mathbb{Z}_{\tilde{T}^n X}^R(0) \text{ is finite}\} = \{\mathbb{Z}_X^R(-n) \text{ is finite}\}$ for all $n \geq 1$.

Since X is stationary, the network $[\mathbb{Z}, 0, X]$ is unimodular. So, we apply the classification theorem (Theorem 14) to the record graph \mathbb{Z}_X^R to obtain the following result: for any integer n , its component $\mathbb{Z}_X^R(n)$ is finite if and only if there exists an integer $i \in V(\mathbb{Z}_X^R(n))$ that has only finitely many ancestors.

We claim that if $\mathbb{Z}_{X(\omega)}^R(n_1)$ is finite for any positive integer n_1 , then $\mathbb{Z}_{X(\omega)}^R(k)$ is finite for all $k < n_1$. This follows because $\mathbb{Z}_{X(\omega)}^R(n_1)$ is finite if and only if the sequence $(S_{n+n_1})_{n \geq 0}$ attains a maximum value a_{n_1} finitely many times, i.e., $S_{n'+n_1} = a_{n_1}$ for some $n' \geq 0$ and $S_{n+n_1} < a_{n_1}$ for all $n > n'$ (equivalently, from the discussion in the above paragraph, n_1 has finitely many ancestors). So, if $k \leq n' + n_1$, then the sequence $(S_{n+k})_{n \geq 0}$ attains the maximum value a_{n_1} at $N = n' + n_1 - k$ and never attains it for all $n > N$. Thus, $\mathbb{Z}_{X(\omega)}^R(k)$ is finite and the claim is proved.

The above claim implies that if there exists a subsequence $(n_k)_{k \geq 1}$ of non-negative integers such that $n_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\mathbb{Z}_{X(\omega)}^R(n_k)$ is finite for all $k \geq 1$, then $\mathbb{Z}_{X(\omega)}^R(n)$ is finite for all $n \in \mathbb{Z}$. Using this and the fact that $\mathbb{Z}_{T^n X}^R(0) = \mathbb{Z}_X^R(n)$ for all $n \geq 0$, we obtain that the events $\{\mathbb{Z}_{T^n X}^R(0) \text{ is finite for infinitely many } n \geq 1\}$ and $\{\mathbb{Z}_X^R(n) \text{ is finite for all } n \geq 1\}$ are one and the same. \square

Remark 118. By the above lemma, it follows that if $\mathbb{Z}_X^R(0)$ is finite a.s., then a.s., $\mathbb{Z}_X^R(n)$ is finite for all $n \in \mathbb{Z}$.

Theorem 119 (Phase transition). *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence as in Def. 113 and ergodic. Let $[\mathbb{Z}_X^R(0), 0]$ be the component of 0 in the record graph \mathbb{Z}_X^R of the network (\mathbb{Z}, X) .*

1. *If $\mathbb{E}[X_0] < 0$, then a.s. every component of \mathbb{Z}_X^R is of class \mathcal{F}/\mathcal{F} .*
2. *If $\mathbb{E}[X_0] > 0$, then \mathbb{Z}_X^R is connected, and it is of class \mathcal{I}/\mathcal{F} a.s.*
3. *If $\mathbb{E}[X_0] = 0$, then a.s., \mathbb{Z}_X^R is connected, and it is either of class \mathcal{I}/\mathcal{F} or of class \mathcal{I}/\mathcal{I} .*

Proof. (Proof of 1.) Let $\mathbb{E}[X_0] < 0$. Since, by ergodicity, $\frac{S_n}{n} \rightarrow \mathbb{E}[X_0]$ a.s., we have $S_n \rightarrow -\infty$ a.s. Therefore, the sequence $(S_n)_{n \geq 0}$ attains a maximum value a_0 at some (random) $n' \geq 0$ and $S_n < a_0$ for all $n > n'$. This implies that $[\mathbb{Z}_X^R(0), 0]$ is finite a.s. and hence it is of class \mathcal{F}/\mathcal{F} . By Remark 118, every component of \mathbb{Z}_X^R is finite and hence it is of class \mathcal{F}/\mathcal{F} .

(Proof of 2.) Let $\mathbb{E}[X_0] > 0$. Since, by ergodicity, $\frac{S_n}{n} \rightarrow \mathbb{E}[X_0]$ a.s., we have $S_n \rightarrow \infty$ a.s. and $S_{-n} \rightarrow -\infty$ a.s. Therefore, 0 has infinitely many ancestors and by the interval property of record vertex-shift, some ancestor of 0 has infinitely many descendants. Hence, \mathbb{Z}_X^R has a single component (by Lemma 116) and it is of class \mathcal{I}/\mathcal{F} .

(Proof of 3.) Let $\mathbb{E}[X_0] = 0$ and A be the event

$$A := \{[\mathbb{Z}_X^R(0), 0] \text{ is of class } \mathcal{F}/\mathcal{F}\}.$$

We show that $\mathbb{P}[A] = 0$. Suppose that $\mathbb{P}[A] > 0$. Then, by Corollary 115, $\mathbb{P}[A] = 1$. Lemma 117 implies that $\mathbb{P}[\mathbb{Z}_X^R(n) \text{ is finite } \forall n \in \mathbb{Z}] = 1$. Note that $\mathbb{Z}_{X(\omega)}^R(n)$ is finite for all $n \in \mathbb{Z}$ if and only if there exists a (random) subsequence $(n'_k)_{k \in \mathbb{Z}}$ of \mathbb{Z} such that $n'_k \rightarrow +\infty$, $n'_{-k} \rightarrow -\infty$ as $k \rightarrow \infty$ and for every $k \in \mathbb{Z}$, $S_n(\omega) < S_{n'_k}(\omega)$ for all $n > n'_k$. In particular, a.s., $S_n \rightarrow -\infty$ and $S_{-n} \rightarrow \infty$ as $n \rightarrow \infty$. Consider the covariant subset Q consisting of peak points

$$Q := \left\{ k \in \mathbb{Z} : \sum_{i=k}^{k+n} X_i \leq -1 \forall n \geq 0 \right\}.$$

The covariant set Q is a subset of integers such that the sums starting from these integers are always negative. Let us enumerate the integers of the set Q as $Q = (n_i)_{i \in \mathbb{Z}}$, where $n_i < n_{i+1}$ for all $i \in \mathbb{Z}$ and $n_0 > 0$ is the smallest positive integer that belongs to the set Q . Such an enumeration is possible

because $S_n \rightarrow -\infty$ and thus, $n_i \rightarrow \infty$ as $i \rightarrow \infty$. Note that the intermediate sums between two consecutive integers of Q satisfy

$$\sum_{k=n_i}^{n_{i+1}-1} X_k \leq -1, \forall i \in \mathbb{Z}. \quad (6.1.3)$$

Since $\mathbb{P}[Q \text{ is non-empty}] = 1$, the intensity $\mathbb{P}[0 \in Q]$ of the set Q is positive (by Lemma 13). By Birkhoff's pointwise ergodic theorem, we have a.s., $\frac{S_n}{n} \rightarrow \mathbb{E}[X_0]$ as $n \rightarrow \infty$. Since $n_i \rightarrow \infty$ as $i \rightarrow \infty$, we have a.s.,

$$\lim_{i \rightarrow \infty} \frac{S_{n_i}}{n_i} = \lim_{n \rightarrow \infty} \frac{S_n}{n} = \mathbb{E}[X_0].$$

For all $i \geq 1$, we have

$$\begin{aligned} \frac{S_{n_i}}{n_i} &= \frac{X_0 + \cdots + X_{n_0-1} + \sum_{k=0}^{i-1} (X_{n_k} + \cdots + X_{n_{k+1}-1})}{n_i} \\ &\leq \frac{X_0 + \cdots + X_{n_0-1} + \sum_{k=0}^{i-1} (-1)}{n_i}, \end{aligned} \quad (6.1.4)$$

where Eq. (6.1.4) is obtained using Eq. (6.1.3). Taking the limit $i \rightarrow \infty$ on both sides of Eq. (6.1.4), we obtain a.s.,

$$\mathbb{E}[X_0] \leq -\mathbb{P}[0 \in Q],$$

since $\frac{-i}{n_i} = -\left(\frac{(\sum_{k=1}^{n_i} \mathbf{1}_{\{k \in Q\}}) - 1}{n_i}\right)$ and the latter converges a.s. to $-\mathbb{P}[0 \in Q]$ as $i \rightarrow \infty$ by the cross-ergodic theorem for point processes (see [5, Section 1.6.4]). Since, the intensity $\mathbb{P}[0 \in Q] > 0$ and $\mathbb{E}[X_0] = 0$, we obtain a contradiction. Hence, $\mathbb{P}[[\mathbb{Z}^R(0), 0]$ is of class \mathcal{I}/\mathcal{F} = 0. \square

We have already seen an instance when the component is of class \mathcal{I}/\mathcal{I} . When $X = (X_n)_{n \in \mathbb{Z}}$ is i.i.d. with mean 0, we showed that the record graph is of class \mathcal{I}/\mathcal{I} (see Proposition 40). We now give an example of class \mathcal{I}/\mathcal{F} when mean is 0.

Example 120 (stationary, ergodic, mean 0 but the record graph is of class \mathcal{I}/\mathcal{F}). See Figure 6.1 for an illustration of this example. Consider an $M/M/1/\infty$ queue with arrival rate λ and service rate μ satisfying $\lambda < \mu$. Let $(N_n)_{n \geq 1}$ be the number of customers in the queue observed at all changes of

state. The Markov chain $(N_n)_{n \geq 1}$ has a unique stationary distribution η (see [4]). The Markov chain $(N_n)_{n \geq 1}$ starting with the stationary distribution η is ergodic and $N_n = 0$ for infinitely many $n \geq 1$. Note that $N_n \geq 0$ and $N_{n+1} - N_n \in \{-1, +1\}$, for all $n \geq 1$. Consider the stationary version $(-N_n)_{n \in \mathbb{Z}}$ of $(-N_n)_{n \geq 1}$ (which can be done by extending the probability space, see [10, Theorem 5.1.14]). Let $Z_n = -N_n$ and $X_n = Z_{n+1} - Z_n$ for all $n \in \mathbb{Z}$. The sequence $X = (X_n)_{n \in \mathbb{Z}}$ is a stationary sequence of random variables taking values in $\{-1, +1\}$ and with mean $\mathbb{E}[X_0] = \mathbb{E}[Z_1 - Z_0] = 0$ (since the sequence $(Z_n)_{n \in \mathbb{Z}}$ is stationary). Consider the record vertex-shift on the network $[\mathbb{Z}, 0, X]$. Let $(S_n)_{n \in \mathbb{Z}}$ be the sums of X given by $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} X_k$ for all $n > 0$, and $S_n = \sum_{k=n}^{-1} -X_k$ for all $n < 0$. The relation between S_n and N_n is given by $S_n = -N_n + N_0$, for all $n \in \mathbb{Z}$. Since $N_n \geq 0$ for all $n \in \mathbb{Z}$ and $N_n = 0$ a.s. for infinitely many $n \geq 1$, we have $S_n \leq N_0$ for all $n \in \mathbb{Z}$ and $S_n = N_0$ for infinitely many $n \geq 1$. Therefore, 0 has infinitely many ancestors (since some k -th record epoch of 0 satisfies $S_{R^k(0)} = N_0$ and $S_{R^{(k+i)}(0)} = N_0$, for all $i \geq 1$). Similarly, since $S_n \leq N_0$ for all $n < 0$, some ancestor of 0 has infinitely many descendants (by the interval property of the record vertex-shift). Thus, the component $[\mathbb{Z}^R(0), 0]$ of the record graph of $[\mathbb{Z}, 0, X]$ is of class \mathcal{I}/\mathcal{F} .

6.2 Record representable Eternal Family Trees

In this section, we focus on the following question. Given a unimodular ordered EFT $[\mathbf{T}, \mathbf{o}]$, does there exist a stationary sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of random variables taking values in $\{-1, 0, 1, 2, \dots\}$ such that $[\mathbf{T}, \mathbf{o}]$ is the component of 0 in the record graph of the network (\mathbb{Z}, X) ? If there exists such a sequence, then we call $[\mathbf{T}, \mathbf{o}]$ a **record representable EFT**. In Theorem 121, we provide sufficient conditions for $[\mathbf{T}, \mathbf{o}]$ to be record representable. In particular, we show that every unimodular ordered EFT of class \mathcal{I}/\mathcal{I} is record representable. As for the unimodular ordered EFTs of class \mathcal{I}/\mathcal{F} , we show that those EFTs of this class that have a unique succession line are record representable. We prove Theorem 121 in the following way: we observe that on any unimodular ordered EFT, the functions a and b are well-defined. The function a maps a vertex to its immediate predecessor, whereas the function b maps it to its immediate successor in an ordered

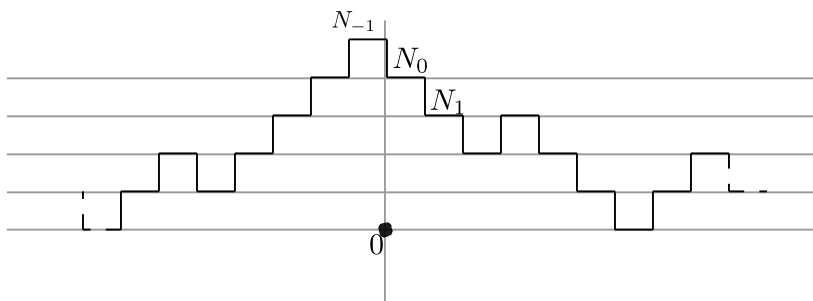


Figure A

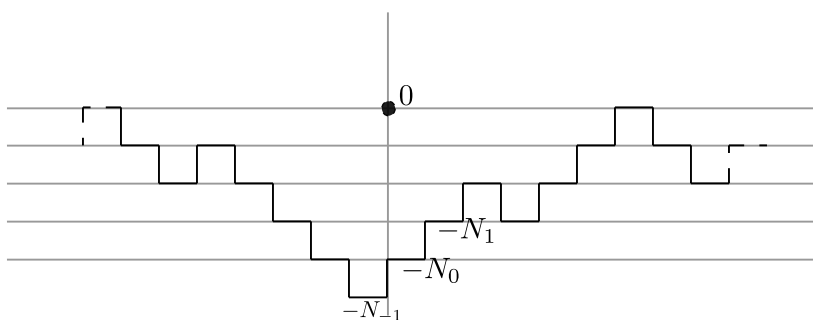


Figure B

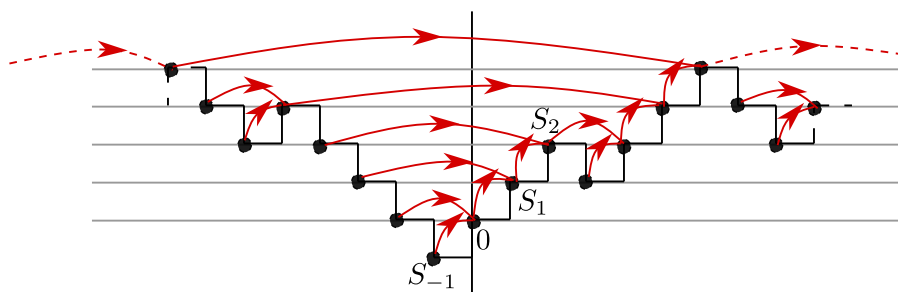


Figure C

Figure 6.1: An Illustration of Example 120, consisting in the construction of a stationary sequence $X = (X_n)_{n \in \mathbb{Z}}$ with $\mathbb{E}[X_0] = 0$ such that the component of 0 of the record graph of the network $[\mathbb{Z}, 0, X]$ is of class \mathcal{I}/\mathcal{F} . Figure A depicts the stationary sequence $(N_n)_{n \in \mathbb{Z}}$, Figure B gives the stationary sequence $(-N_n)_{n \in \mathbb{Z}}$, and Figure C is obtained from Figure B by shifting the x-axis to $-N_0$. In Figure C, the increment sequence of $(S_n)_{n \in \mathbb{Z}}$ is X , the record graph is drawn in red arrows and the sequence of top most arrows is the bi-infinite path.

EFT. We then proceed to show that every unimodular ordered EFT has at most two succession lines. The backward map Φ_R is bijective on the set of ordered EFTs that have a unique succession line. The stationary sequence Y is the push-forward of $[\mathbf{T}, \mathbf{o}]$ under the backward map Φ_R . The stationarity of Y follows by Mecke-Thorisson point stationarity 10.

An immediate corollary of Theorem 121 is that if $X = (X_n)_{n \in \mathbb{Z}}$ is a stationary sequence of random variables taking values in \mathbb{Z} such that $\mathbb{E}[|X_0|] < \infty$, $[\mathbb{Z}, 0, X]$ be its associated network, f is a vertex-shift on $[\mathbb{Z}, 0, X]$ and $[\mathbb{Z}_X^f(0), 0]$ be the component of 0 in the f -graph of (\mathbb{Z}, X) such that $[\mathbb{Z}_X^f(0), 0]$ has a unique succession line, then there exist a stationary sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of random variables taking values in $\{-1, 0, 1, 2, \dots\}$ such that the component $[\mathbb{Z}_Y^R(0), 0]$ of 0 in the record graph of the network (\mathbb{Z}, Y) has the same distribution as that of $[\mathbb{Z}_X^f(0), 0]$. In particular, if $[\mathbb{Z}_X^f(0), 0]$ is of class \mathcal{I}/\mathcal{I} , then such a stationary sequence Y exists. The vertex-shift f on $[\mathbb{Z}, 0, X]$ is said to be **record representable** if $[\mathbb{Z}_X^f(0), 0]$ is record representable.

Theorem 121. *Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular ordered EFT such that \mathbf{T} has a unique succession line. Then, there exists a stationary sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of random variables taking values in $\{-1, 0, 1, 2, \dots\}$ such that $[\mathbf{T}, \mathbf{o}] \stackrel{\mathcal{D}}{=} [\mathbb{Z}_Y^R(0), 0]$, where R is the record vertex-shift and $\mathbb{Z}_Y^R(0)$ is the component of 0 in the record graph \mathbb{Z}_Y^R of the network (\mathbb{Z}, Y) . (In particular, if $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{I} , then such a stationary sequence Y exists.)*

Corollary 122. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary sequence that satisfies $\mathbb{E}[|X_0|] < \infty$ and $[\mathbb{Z}, 0, X]$ be the network associated to X . Let f be a vertex-shift on $[\mathbb{Z}, 0, X]$ such that 0 has a.s. infinitely many ancestors in the f -graph \mathbb{Z}_X^f and the component $\mathbb{Z}_X^f(0)$ of 0 in \mathbb{Z}_X^f has a unique succession line. Then, there exists a stationary sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of random variables taking values in $\{-1, 0, 1, 2, \dots\}$ such that $[\mathbb{Z}_X^f(0), 0] \stackrel{\mathcal{D}}{=} [\mathbb{Z}_Y^R(0), 0]$, where R is the record vertex-shift and $\mathbb{Z}_Y^R(0)$ is the component of 0 in the record graph \mathbb{Z}_Y^R of the network (\mathbb{Z}, Y) .*

Proof. Since X is stationary, the network $[\mathbb{Z}_X^f(0), 0]$ is unimodular. Apply Theorem 121 to get the result. \square

We first recall the notation of Section 5.1 and prove a few lemmas before proving Theorem 121.

Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular ordered EFT. Denote the RLS (total) order obtained on the vertices of \mathbf{T} by $<$ (see Section 2.2). The set of minimal vertices of \mathbf{T} is a covariant subset. It can be either empty or a singleton. Since \mathbf{T} has infinitely many vertices, by No Infinite/ Finite Inclusion (Lemma 12), a.s. \mathbf{T} does not have any minimal vertices. Therefore, the map $b : V(\mathbf{T}) \rightarrow V(\mathbf{T})$ given by $b(u) = \max\{v : v < u\}$ (if max exists) is a.s. well-defined. Since b is injective on $V(\mathbf{T})$ and $[\mathbf{T}, \mathbf{o}]$ is unimodular, b is bijective a.s. by Proposition 11. But $a(b(u)) = u$, for all $u \in V(\mathbf{T})$, where $a(v) = \min\{w : w > v\}$. Since the range of b is $V(\mathbf{T})$, the map a is well-defined on $V(\mathbf{T})$. Thus, we have the following lemma.

Lemma 123. *Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular ordered EFT. Then, the maps $a : V(\mathbf{T}) \rightarrow V(\mathbf{T})$, $b : V(\mathbf{T}) \rightarrow V(\mathbf{T})$ are a.s. well-defined.*

Let T be an ordered EFT. For any vertex $u \in V(T)$, let $U((T, u)) = (u_n)_{n \in \mathbb{Z}}$ be the succession line passing through u (see Def. 87). The set $\{u_n : n \in \mathbb{Z}\}$ is called the succession line of u . It is easy to check that if two succession lines intersect then they are one and the same.

Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular ordered EFT. Lemma 123 implies that $U((\mathbf{T}, u))$ has distinct entries for all $u \in V(\mathbf{T})$. In Lemma 124, we show that a unimodular ordered EFT $[\mathbf{T}, \mathbf{o}]$ has at most two succession lines. Note that by the classification theorem, $[\mathbf{T}, \mathbf{o}]$ is either of class \mathcal{I}/\mathcal{I} or of class \mathcal{I}/\mathcal{F} .

Lemma 124. *Let $[\mathbf{T}, \mathbf{o}]$ be a unimodular ordered EFT.*

- *If $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{I} , then \mathbf{T} has a unique succession line and it contains all the vertices of \mathbf{T} .*
- *If $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{F} , then \mathbf{T} has at most two succession lines; together they contain all the vertices of \mathbf{T} .*

Proof. If $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{I} , then every vertex of \mathbf{T} has finitely many descendants. Let $v \in V(\mathbf{T})$. Consider the smallest common ancestor w of v and \mathbf{o} . The descendant tree of w is finite and it contains both v and \mathbf{o} . Thus, the succession lines of v and \mathbf{o} intersect, which implies that they are one and the same.

Let us assume that $[\mathbf{T}, \mathbf{o}]$ is of class \mathcal{I}/\mathcal{F} . Let \nearrow denote the unique bi-infinite path of \mathbf{T} . Consider the set

$$W = \{u \in V(\mathbf{T}) : u < v \ \forall v \in \nearrow\}. \quad (6.2.1)$$

Denote its complement $V(\mathbf{T}) \setminus W$ by W' . We first show that W' has a unique succession line and all the vertices of W' belong to this succession line. See Figure 6.2 for an illustration.

If $u \in W'$, then there exists a vertex $v \in \not\prec$ such that $u > v$ (we may safely assume it because if $u \in \not\prec$ then u has a child $v' \in \not\prec$ and $u > v'$). The vertex $a(u) = \min\{w : w > u\}$ also satisfies $a(u) > v$ since $a(u) > u$, implying that $a(u) \in W'$. The vertex $b(u) = \max\{w : w < u\}$ satisfies $b(u) \geq v$ since $v \in \{w : w < u\}$, implying that $b(u) \in W'$. Therefore, for every vertex $u \in W'$, the succession line of u is contained in W' . For any two vertices u, v in W' , there are finitely many in between them, i.e., the set $\{w : u < w < v\}$ is finite (assuming that $u < v$). Therefore, the succession line of u and v are one and the same. Thus, W' has a unique succession line and this succession line contains all the vertices of W' .

We now show that if W is non-empty, then it has a unique succession line and all the vertices of W belong to this succession line. Let $u \in W$. The vertex $a(u) = \min\{w : w > u\}$ also belongs to W because $\not\prec \subset \{w : w > u\}$ and $a(u) \notin \not\prec$. The vertex $b(u) = \max\{w : w < u\}$ also belongs to W since $b(u) < u$ and $u \in W$. Therefore, the succession line of u is contained in W . For any two vertices u, v in W , there are finitely many vertices between them. Thus, W has a unique succession line (if W is non-empty). This completes the proof. \square

Remark 125. If $[\mathbf{T}, \mathbf{o}]$ is unimodular ordered EFT of class \mathcal{I}/\mathcal{F} that has a unique succession line, then it has the ECS (Every Child Succeeding) order (see Subsection 3.2 for the definition).

Consider the following subset

$$\tilde{\mathcal{T}}_* := \{[T, o] : T \text{ is an ordered EFT with a unique succession line}\} \quad (6.2.2)$$

of EFTs. It follows from Lemma 123 and Lemma 124 that for all $[T, o] \in \tilde{\mathcal{T}}_*$, the succession line $U((T, o)) = (u_n)_{n \in \mathbb{Z}}$ passing through o with $u_0 = o$ is a bijection from \mathbb{Z} to $V(T)$ that maps n to u_n , for all $n \in \mathbb{Z}$. Recall the backward map Φ_R defined on the set \mathcal{T}'_* of Family Trees that have at most one bi-infinite F -path. For all $[T, o] \in \mathcal{T}'_*$, we have $\Phi_R([T, o]) = [\mathbb{Z}, 0, y]$, where $y = (y_n)_{n \in \mathbb{Z}}$ is the sequence given by $y_n = d_1(u_{n+1}) - 1$, for all $n \in \mathbb{Z}$ and $(u_n)_{n \in \mathbb{Z}} = U((T, o))$ is the succession line passing through o . The sequence $y = (y_n)_{n \in \mathbb{Z}}$ takes values in $\{-1, 0, 1, 2, \dots\}^{\mathbb{Z}}$. We show that $\Psi_R \circ \Phi_R = I$ on $\tilde{\mathcal{T}}_*$, where $\Psi_R([\mathbb{Z}, 0, y]) = [\mathbb{Z}_y^R(0), 0]$ is the component of 0 in the record

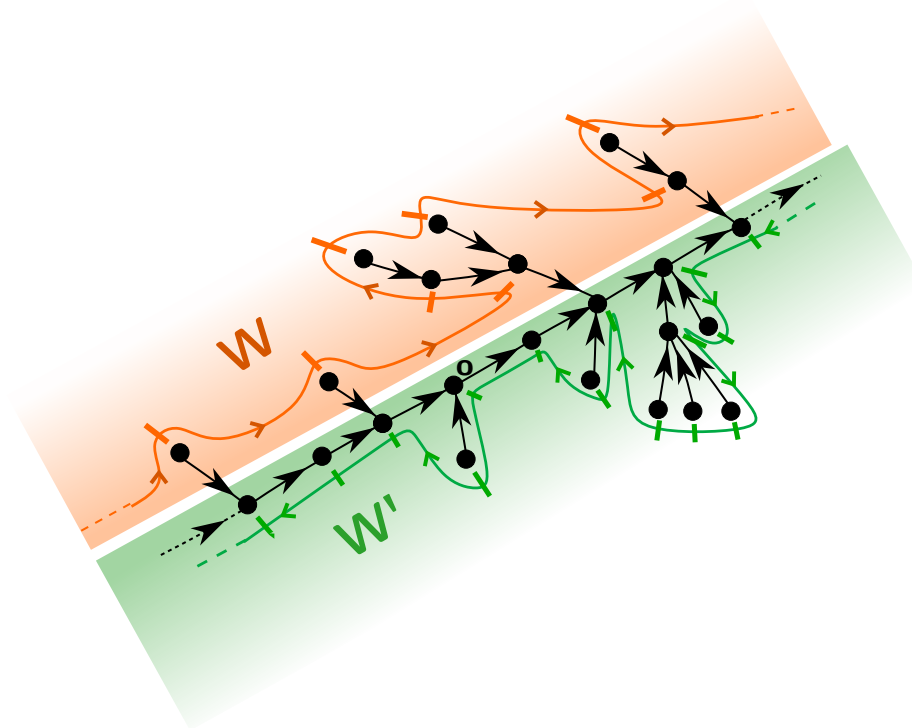


Figure 6.2: An Illustration of the \mathcal{I}/\mathcal{F} case in the proof of Lemma 124. The set W and its complement W' are highlighted. Each one of these set of vertices contains a unique succession line passing through all of its vertices. The arrow direction of the succession line indicates the images under the map b , i.e., the immediate successor of the previous vertex. In this figure, the root \mathfrak{o} belongs to the succession line containing the bi-infinite path.

graph of the network (\mathbb{Z}, y) . The proof of the following lemma is the same as that of Proposition 95 except for a few modifications. The proof relies on the property that every elder sibling of any vertex (if they exist) of a Family Tree with a unique succession line has finitely many descendants.

Lemma 126. For all $[T, \mathfrak{o}] \in \tilde{\mathcal{T}}_*$, $\Psi_R \circ \Phi_R([T, \mathfrak{o}]) = [T, \mathfrak{o}]$.

Proof. Let $(u_n)_{n \in \mathbb{Z}} = U((T, \mathfrak{o}))$ be the succession line of (T, \mathfrak{o}) and $[\mathbb{Z}, 0, y] = \Phi_R([T, \mathfrak{o}])$, where $y = (y_n)_{n \in \mathbb{Z}}$.

We prove that the bijective map $\alpha : \mathbb{Z} \rightarrow V(T)$ defined by $\alpha(n) = u_n$, for all $n \in \mathbb{Z}$, induces a rooted network isomorphism α from $[\mathbb{Z}_y^R(0), 0] =$

$\Psi_R([\mathbb{Z}, 0, y])$ to (T, o) . Note that, if u_j is a parent of u_i in T for some integers i and j then $j > i$, which follows because $(u_n)_{n \in \mathbb{Z}}$ is the succession line of (T, o) . In view of this, to prove that α induces a rooted network isomorphism, it is enough to show that for all $i \in \mathbb{Z}$, if u_j is the parent of u_i in T for some $j > i$, then $j = R_y(i)$. It is enough to show the latter because (assuming the latter implication) if $j = R_y(i)$ and u_k is a parent of u_i for some $k > i$, then $R_y(i) = k$, which implies that $j = k$, and $u_k = u_j$.

Let $i \in \mathbb{Z}$ and u_j be the parent of u_i in T for some integer $j > i$. If $j = i + 1$, then $x_i = d_1(u_{i+1}) - 1 \geq 0$, because u_i is a child of u_{i+1} . Hence, $R_y(i) = i + 1$.

So, let us assume that $j > i + 1$. This implies that either u_{i+1} is a leaf and a descendant of a sibling of u_i or u_{i+1} is a sibling of u_i . Every sibling v of u_i such that $v > u_i$ has finitely many descendants. This follows because if there exists a sibling $v > u_i$ of u_i that has infinitely many descendants, then $u_i \in W$ (see Eq. 6.2.1). But W is empty since T has a unique succession line.

Let $u_{i_1} < u_{i_2} < \dots < u_{i_n}$ be the elder siblings of u_i (i.e., $u_i < u_{i_1}$), and T_1, T_2, \dots, T_n be their descendant trees respectively. For $i < k < j$, let $l(k)$ be the smallest element of the set $\{1, 2, \dots, n\}$ such that $\{u_{i+1}, u_{i+2}, \dots, u_k\} \subset V(T_1) \cup V(T_2) \cup \dots \cup V(T_{l(k)})$, and u_{k_1} be the smallest vertex of $T_{l(k)}$.

Then, the sum $z(i, k) = \sum_{m=i}^{k-1} y_m = \sum_{m=i+1}^k (d_1(u_m) - 1)$ can be written as

$$\begin{aligned} z(i, k) &= \left(\sum_{u \in V(T_1)} d_1(u) - 1 \right) + \dots + \left(\sum_{u \in V(T_{l(k)-1})} d_1(u) - 1 \right) + \sum_{m=k_1}^k (d_1(u_m) - 1) \\ &< (-1) + \dots + (-1) + \sum_{m=k_1}^k (d_1(u_m) - 1) < 0, \end{aligned}$$

with the notation that $V(T_0)$ is an empty set. The last steps follow from the first and the second statements of Lemma 93.

On the other hand, the sum $z(i, j) = \sum_{m=i+1}^j (d_1(u_m) - 1)$ can be written as

$$\begin{aligned} z(i, j) &= \left(\sum_{u \in V(T_1)} d_1(u) - 1 \right) + \dots + \left(\sum_{u \in V(T_n)} d_1(u) - 1 \right) + d_1(u_j) - 1 \\ &= -n + d_1(u_j) - 1 \geq 0. \end{aligned}$$

The above steps follow from the first part of Lemma 93, and by the assumption that u_j has at least $n + 1$ children, namely $u_i, u_{i_1}, \dots, u_{i_n}$. Thus, $R_x(i) = j$, completing the proof. \square

Proof of Theorem 121. Let $[\mathbb{Z}, 0, Y] = \Phi_R([\mathbf{T}, \mathbf{o}])$. Since \mathbf{T} has a unique succession line, we have a.s., $\Psi_R \circ \Phi_R([\mathbf{T}, \mathbf{o}]) = [\mathbf{T}, \mathbf{o}]$ by Lemma 126. Therefore, a.s., $[\mathbb{Z}_Y^R(0), 0] = [\mathbf{T}, \mathbf{o}]$.

We now show that the sequence Y is stationary. Let $(u_n)_{n \in \mathbb{Z}} = U([\mathbf{T}, \mathbf{o}])$ be the succession line with $u_0 = \mathbf{o}$. For any $i \in \mathbb{Z}$, the map $u_n \mapsto u_{n+i}, \forall n \in \mathbb{Z}$ is bijective. Therefore, the map $[\mathbf{T}, \mathbf{o}] \mapsto [\mathbf{T}, u_i]$ is measure-preserving, for any $i \in \mathbb{Z}$, by Mecke-Thorisson point stationarity (Proposition 10). Since, $[\mathbb{Z}, 0, T_i Y] = \Phi_R([\mathbf{T}, u_i])$, the map $(Y_n)_{n \in \mathbb{Z}} \mapsto (Y_{n+i})_{n \in \mathbb{Z}}$, for any $i \in \mathbb{Z}$, is measure-preserving. Thus, Y is stationary. \square

Examples of record representable vertex-shifts

Example 127 (Strict record vertex-shift on an i.i.d. sequence (class \mathcal{I}/\mathcal{I})). An illustration of this example can be seen in Figure 6.3. The strict record vertex-shift SR is defined on the set of networks of the form $[\mathbb{Z}, 0, x]$, where $x = (x_n)_{n \in \mathbb{Z}}$ is a sequence of integers. It is defined by

$$SR(i) = \begin{cases} \inf\{n > i : \sum_{k=i}^{n-1} x_k > 0\} & \text{if the infimum exists,} \\ i & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{Z}$. Let $X = (X_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence of random variables taking values in \mathbb{Z} . One can show that the component of 0 in the SR -graph is of class \mathcal{I}/\mathcal{I} if $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X_0^2] > 0$, because $\limsup_{n \rightarrow \infty} S_{-n} = \infty$, which implies that the descendant tree of 0 is a.s. finite. By Theorem 121, there exists a stationary sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ of random variables taking values in $\{-1, 0, 1, 2, \dots\}$ such that $[\mathbb{Z}_X^{SR}(0), 0] \stackrel{\mathcal{D}}{=} [\mathbb{Z}_Y^R(0), 0]$.

Example 128 (Climbing point vertex-shift that depends on the future (class \mathcal{I}/\mathcal{F})). An illustration of this example is shown in Figure 6.4. We define a vertex-shift C (C for climbing point) based on an example taken from [12, Example 1(a)]. Let $X = (X_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence of random variables taking values in $\{-1, 0, 1\}$ with probabilities given by: $\mathbb{P}[X_0 = 1] = p, \mathbb{P}[X_0 = -1] = q$ and $\mathbb{P}[X_0 = 0] = 1 - p - q$, where $0 < q < p$ and $p + q < 1$. Define the vertex-shift on the networks of the form $[\mathbb{Z}, 0, x]$,

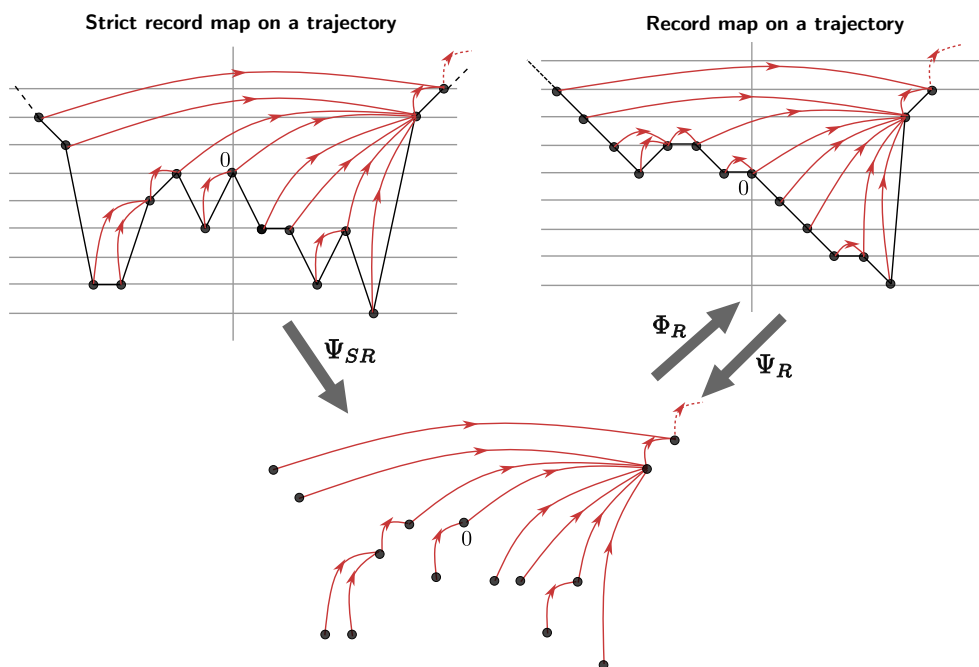


Figure 6.3: An Illustration of Example 127. The top left figure shows the trajectory of a realisation of the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$, where $X_n \in \mathbb{Z}, \forall n$. The SR -graph (strict record graph) is drawn in red on the trajectory of the top left figure. The component of 0 in the SR -graph is drawn in the bottom figure. The top right figure shows the trajectory of the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ which is obtained by applying the backward map Φ_R to the rooted tree of the bottom figure. The component of 0 in the record graph of the trajectory in the top right figure is the same as the component of 0 in the SR -graph.

where $x = (x_n)_{n \in \mathbb{Z}}$ is a sequence with $x_n \in \{-1, 0, 1\}, \forall n$ by

$$C(i) = \begin{cases} \inf\{n > i : \sum_{k=n}^m x_k \geq 0, \forall m > n\} & \text{if the infimum exists,} \\ i & \text{otherwise,} \end{cases}$$

for all $i \in \mathbb{Z}$. Note that $C(i) = k$ for some $k > i$ if and only if k is the smallest integer larger than i such that $S_k = \min\{S_n : n > i\}$. Since $\mathbb{P}[X_0] = p - q > 0$, the random walk $(S_n)_{n \geq 0}$ drifts to $+\infty$, where $S_0 = 0, S_n = \sum_{k=0}^{n-1} X_k$. Therefore, the event $C(i) > i$ for all $i \in \mathbb{Z}$ occurs with probability 1. Since, $S_{-n} = \sum_{k=-n}^{-1} -X_k$ converges to $-\infty$ as $n \rightarrow \infty$, the random variable $\max\{S_{-n} : n \geq 0\}$ is finite a.s. Thus, some ancestor of 0 in the C -graph has infinitely many descendants. So, the component $[\mathbb{Z}_X^C(0), 0]$ of 0 in the C -graph is of class \mathcal{I}/\mathcal{F} . One can show that the smallest child of every vertex which lies on the bi-infinite path of $\mathbb{Z}_X^C(0)$ belongs to the bi-infinite path. Therefore, $[\mathbb{Z}_X^C(0), 0]$ is of class \mathcal{I}/\mathcal{F} and has unique succession line (as it has ECS order).

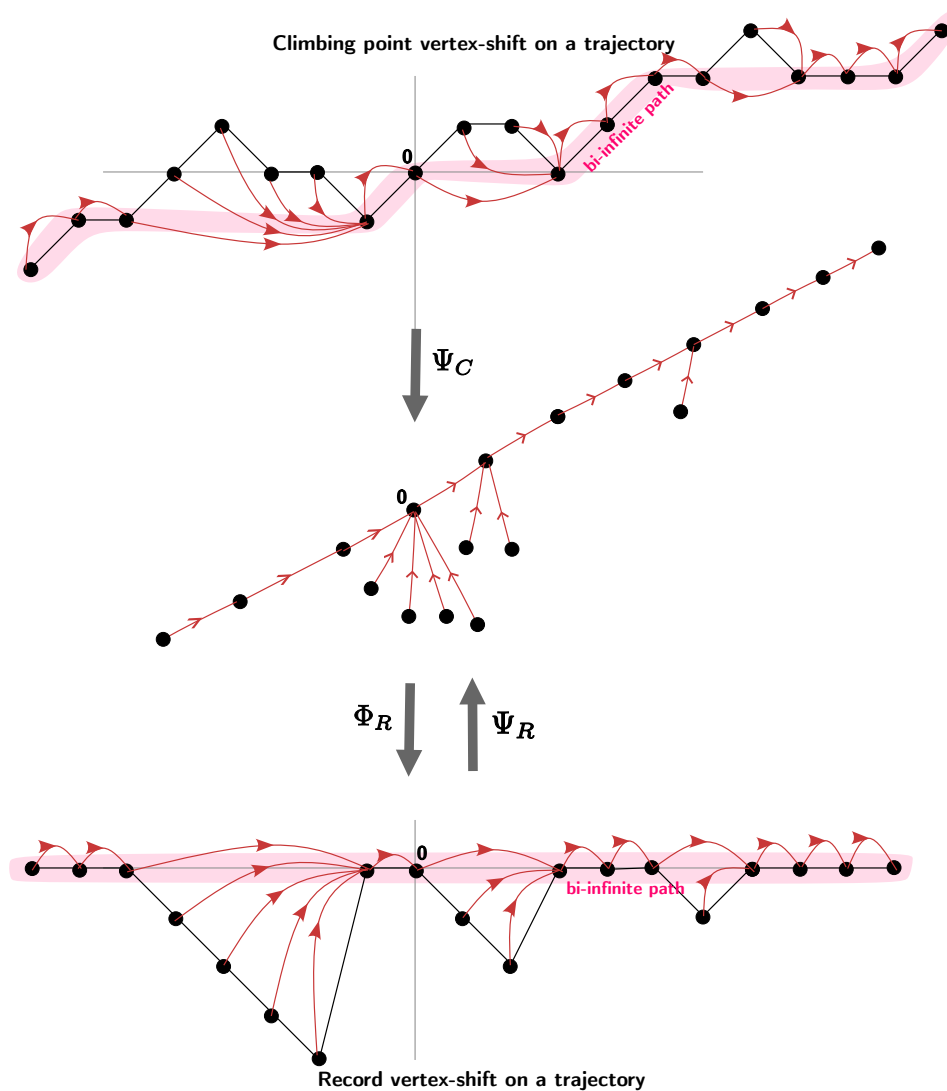


Figure 6.4: An Illustration of Example 128. The top figure shows the trajectory of a realisation of the i.i.d. sequence $X = (X_n)_{n \in \mathbb{Z}}$, where $X_n \in \mathbb{Z}, \forall n$. The C -graph (C for climbing point vertex-shift) is drawn in red on the trajectory of the top figure. The vertices on the bi-infinite path are highlighted. The component of 0 in the C -graph is drawn in the middle figure. The bottom figure shows the trajectory of the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ which is obtained by applying the backward map Φ_R to the rooted tree shown in the middle figure. The component of 0 in the record graph of the trajectory in the bottom figure is the same as the component of 0 in the C -graph.

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RÉSUMÉ

Considérons une règle de navigation définie sur un graphe qui mappe chaque sommet du graphe à un sommet de telle manière que la règle de navigation commute avec chaque automorphisme du graphe. C'est à dire que la règle de navigation appliquée aux sommets reste la même après prise d'un éventuel automorphisme du graphe. Une telle règle de navigation est dite avoir la propriété de covariance. Cette étude se penche sur un ensemble de telles règles de navigation covariantes, indexées par des graphes localement finis et soumises à une condition de mesurabilité. Cet ensemble de règles est appelé déplacement de sommet. Plus généralement, on peut considérer les déplacements de sommets sur les réseaux, des graphes qui ont des étiquettes sur les arêtes et sur les sommets. Un déplacement de sommet induit une dynamique sur l'espace des réseaux enracinés localement finis. L'objectif central de ce travail réside dans l'étude de la dynamique associée à une règle de navigation spécifique appelée record vertex-shift. Il est défini sur les trajectoires de toute marche aléatoire discrète unidimensionnelle dont les incréments ont une moyenne finie et la marche aléatoire ne peut sauter que d'un pas vers la gauche. De plus, les travaux incluent plusieurs résultats notables concernant les déplacements de sommets d'enregistrement appliqués aux processus à incréments stationnaires. La thèse contient également des résultats sur les déplacements de sommets plus généraux sur les réseaux unimodulaires.

MOTS CLÉS

unimodularité, graphe des records, limites locales, réseaux aléatoires

ABSTRACT

Consider a navigation rule defined on a graph that maps every vertex of the graph to a vertex in such a way that the navigation rule commutes with every automorphism of the graph. It is to say that the navigation rule applied to the vertices remains the same after taking any automorphism of the graph. Such a navigation rule is said to have the covariance property. This study delves into a collection of such covariant navigation rules, indexed by locally finite graphs, and subject to a measurability condition. This ensemble of rules is termed a vertex-shift. More generally, one can consider vertex-shifts on networks, graphs that have labels on edges and on vertices. A vertex-shift induces a dynamic on the space of locally finite rooted networks. The central focus of this work lies in investigating the dynamic associated to a specific navigation rule called the record vertex-shift. It is defined on the trajectories of any one dimensional discrete random walk whose increments have finite mean and the random walk can jump only one step to the left. Additionally, the work includes several notable results concerning record vertex-shifts applied to processes with stationary increments. The thesis also contains results on more general vertex-shifts on unimodular networks.

KEYWORDS

unimodularity, record graph, local limits, random networks