



HAL
open science

Linear Geometric Homogeneity for Control and Estimation in Finite Time

Andrey Polyakov

► **To cite this version:**

Andrey Polyakov. Linear Geometric Homogeneity for Control and Estimation in Finite Time. Automatic Control Engineering. Université de Lille, 2018. tel-04426543

HAL Id: tel-04426543

<https://inria.hal.science/tel-04426543>

Submitted on 30 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License



Mémoire
présenté pour obtenir le diplôme
d'Habilitation à diriger des recherches

Spécialité: Génie informatique, automatique et traitement du signal

Date de soutenance: le 29/11/2018

**Homogénéité géométrique linéaire pour le contrôle et l'estimation
en temps fini**

Andrey POLYAKOV
Non-A POST, Inria Lille

Jury

Reporters:

Yacine CHITOUR, Professor, Université Paris-Sud
Christophe PRIEUR, Director of Research at CNRS, GIPSA-Lab, Grenoble
Isabelle QUEINNEC, Director of Research at CNRS, LAAS, Toulouse

Jury Members:

Jean-Michel CORON, Professor, Université Pierre et Marie Curie
Francoise LAMNABHI-LAGARRIGUE, Director of Research at CNRS, L2S
Jaime Alberto MORENO PÉREZ, Professor, UNAM, Mexico
Wilfrid PERRUQUETTI, Professor, Centrale Lille
Jean-Pierre RICHARD, Professor, Centrale Lille

Summary

The thesis is presented in order to obtain the "Habilitation à diriger des recherches" degree in automation and industrial informatics.

The author received the Ph.D. degree in System Analysis, Control and Information Processing from Voronezh State University, Russia in 2005. From 2004 to 2007 and from 2008 to 2010 he was with the same university as lecturer and associate professor, respectively. From 2007-2008 he was research assistant (post-doc) with CINVESTAV-IPN, Mexico. From 2010 to 2012 he has served as a leader researcher at the Institute of Control Sciences of Russian Academy of Sciences in Moscow. In 2012 he joined NON-A team of Inria Lille, France. His main research interests include sliding mode control, time delay system, set-theoretic and homogeneity-based methods in control. During the scientific carrier he produced more than 150 publication.

The thesis deals with one of the main research direction of the applicant in the last 5 years: design of finite/fixed-time controllers and observers by means of a canonical homogeneous norm induced by linear dilations. Such a norm simplifies the analysis and design of homogeneous control systems, and allows a linear control and state estimation algorithms to be extended easily to some non-linear cases. Canonical homogeneous norms appear in the control and systems theory in the form of controllability functions and implicit Lyapunov functions.

The first chapter of the thesis presents the scientific background and the experience of the author. In the second chapter, the concept of homogeneity (symmetry) with respect to linear dilations (i.e. linear geometric homogeneity) is introduced and some related notions such as the canonical homogeneous norm are studied. Solutions to several control and estimation problems for linear and non-linear systems in a Banach space \mathbb{B} and in the Euclidean space \mathbb{R}^n are presented. In the third chapter, the future directions of the research of the author are discussed. Finally, the list of publications of the author and the texts of four selected papers, which are closely related with the topic of the thesis but not surveyed in the chapter 2, are presented in Appendix.

Acknowledgments

I am grateful to *my family* for their patience and the personal sacrifices which they have given to my work. I also acknowledge *my colleagues* and *friends* from France, Russia, Mexico, Israel, UK, Ireland and other countries for their strong support, for fruitful ideas, suggestions, criticism and always interesting discussions.

Contents

Notation	5
1 Scientific Background and Experience	7
1.1 Diplomas and Grades	7
1.2 Positions	7
1.3 Research and Publication Activities	8
1.4 Research projects	10
1.5 Supervision	12
1.6 Teaching	13
1.7 Prizes and Distinctions	13
2 Linear Geometric Homogeneity in Systems and Control	15
2.1 Introduction	15
2.2 Homogeneous Systems	23
2.2.1 Linear Dilations	23
2.2.2 Homogeneous Mappings and Equations	34
2.3 Control Design and State Estimation Using Linear Dilations	40
2.3.1 Homogeneous Stabilization in a Hilbert Space	40
2.3.2 Quadratic Stabilizability of Homogeneous Systems in \mathbb{R}^n	52
2.3.3 Homogeneous Observers in \mathbb{R}^n	61
2.4 Conclusion remarks	65
3 Future Directions of Research	69
3.1 Fundamental research	69
3.1.1 Short term perspective	69
3.1.2 Long term perspective	70
3.2 Applications	71
3.3 ControlHub Platform	71
Bibliography	72
Appendix	79

Notation

- \mathbb{N} is the set of natural numbers; \mathbb{Z} is the set of integers; \mathbb{R} is the set of reals; $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and $\mathbb{R}_+ = [0, +\infty)$; \mathbb{C} is the set of complex numbers.
- \mathcal{I} denotes one of the following intervals: $[a, b]$, (a, b) , $[a, b)$ or $(a, b]$, where $a < b$, $a, b \in \mathbb{R}$ or $a = -\infty$, and/or $b \in +\infty$.
- The Cartesian product of sets is denoted by \times .
- \mathbb{B} is a real Banach space with a norm $\|\cdot\|$ and \mathbb{H} is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$.
- $S = \{x \in \mathbb{B} : \|x\| = 1\}$ is the unit sphere in \mathbb{B} .
- $\mathcal{L}(X, Y)$ is the space of linear bounded operators $X \rightarrow Y$, where X and Y are Banach spaces, and

$$\|A\|_{\mathcal{L}(X, Y)} = \sup_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X} \quad \text{and} \quad [A]_{\mathcal{L}(X, Y)} = \inf_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}, \quad A \in \mathcal{L}(X, Y).$$

If $X = Y$ we use notations $\|A\|$ and $[A]$ for shortness.

- If $P = P^\top \in \mathbb{R}^{n \times n}$ then $P \succ 0$ means that the matrix P is positive definite.
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denotes minimum and maximum eigenvalues of symmetric matrix $P = P^\top \in \mathbb{R}^n$.
- \overline{X} denotes the closure of the set X of a metric space.
- $\partial\Omega$ is the boundary set of a $\Omega \subset \mathbb{R}^n$.
- $C_0^\infty(\Omega, \mathbb{R}^m)$ is the set of infinitely differentiable (smooth) functions having compact support in Ω , where $\Omega \subset \mathbb{R}^m$ is an open set.
- $H^n(\Omega, \mathbb{R}^m) = W^{n,2}(\Omega, \mathbb{R}^m)$ is the Sobolev space of functions $\Omega \subset \mathbb{R}^k \rightarrow \mathbb{R}^m$.
- $H_0^n(\Omega, \mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega, \mathbb{R}^m)$ with respect to the norm in H^n .
- $L^2(\Omega, \mathbb{R}^m) = H^0(\Omega, \mathbb{R}^m)$ is the Lebesgue space of quadratically integrable functions $\Omega \rightarrow \mathbb{R}^m$.
- The set consisting of elements x_1, x_2, \dots, x_n is denoted by $\{x_1, x_2, \dots, x_n\}$.

- The sign function is defined by

$$\text{sign}_\sigma(\rho) := \begin{cases} 1 & \text{if } \rho > 0, \\ -1 & \text{if } \rho < 0, \\ \sigma & \text{if } \rho = 0, \end{cases} \quad (1)$$

where $\sigma \in \mathbb{R} : -1 \leq \sigma \leq 1$. If a concrete value of σ is not important for considerations, we use the notation $\text{sign}(\rho)$.

- If $V(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R})$ then $\nabla V(x) := \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right)^T$. If $s \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $s(\cdot) = (s_1(\cdot), \dots, s_m(\cdot))^T$ then $\nabla s(x)$ is the matrix $\mathbb{R}^{n \times m}$ of the partial derivatives $\frac{\partial s_j}{\partial x_i}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.
- Let $\mathbb{F}_d(\mathbb{R}^n)$ (resp. $\mathbb{H}_d(\mathbb{R}^n)$) be the set of continuous vector fields $\mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^n$ (resp. functions $\mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$) homogeneous with respect to the dilation \mathbf{d} .
- Let $\text{deg}_{\mathbb{F}_d}(f)$ (resp. $\text{deg}_{\mathbb{H}_d}(h)$) denote the homogeneity degree of $f \in \mathbb{F}_d(\mathbb{R}^n)$ (resp. $h \in \mathbb{H}_d(\mathbb{R}^n)$).

Chapter 1

Scientific Background and Experience

In this chapter, my scientific background, the main achievements and scientific carrier are expressed shortly. Next, in the second chapter, the main results obtained in the field of homogeneity-based control and estimation are presented with more details. Some planned directions of research are briefly described in the third chapter.

1.1 Diplomas and Grades

Both my degrees I have obtained from Voronezh State University, Russia:

- December 2005: PhD in Physics and Mathematics (specialization in *System Analysis, Control and Information Processing*)
Title: Control of Neutral and Unstable Plants Using Relay Delayed Feedback
Supervisor: Prof. Vadim Strygin
Reporters: Prof. A. Kobrin and Prof. G. Lozgachev
- July 2003: Masters Degree in Applied Mathematic and Informatics
Supervisor: Prof. V. Strygin

1.2 Positions

I was working in 4 scientific centers (3 different countries) during my career:

- 2012-present: France, Inria Lille - Nord Europe, Non-A team project, presently Non-A Post. Position: Researcher (CR) - permanent (since 2013)

- 2010-2012: Russia, Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia; Lab.7 "Robust and Adaptive Control"; Position: Leader Researcher - permanent
- 2008-2010 (and 2004-2007): Russia, Voronezh State University, Faculty of Applied Mathematics and Mechanics; Position: Associate Professor (Lecturer in 2004-2007) - permanent
- 2007 -2008: Mexico, CINVESTAV-IPN, Department of Automatic Control. Position: Postdoc - temporary.

1.3 Research and Publication Activities

Research topics

My research activities in the field of control theory can be split in four topics

- *Homogeneity in Systems and Control* (since 2012).

Homogeneity is a sort of a symmetry allowing a lot of useful properties to be discovered in the context of systems theory and control design. For example, asymptotic stability of homogeneous system ensures its robustness (in the sense of input to state stability) with respect to a rather large class of perturbations and it implies finite-time stability if the homogeneity degree is negative. Last years, I have dealt with control and observer design based on homogeneity technique as well as with both quantitative and qualitative analysis of homogeneous systems. In particular, the methods of implicit Lyapunov functions, linear matrices inequalities and consistent discretizations have been elaborated for homogeneous systems. Some homogeneity based methods have been successfully extended to infinite dimensional systems. A few recent results are reported in the next chapter.

- *Set-Theoretic Methods in Control and Estimation* (since 2007).

In practice, asymptotic (or finite-time) stability and as well as exact estimation are hard to be realized due to technical nonidealities of actuators and sensors, uncertainties in system models, etc. In this case set-theoretic methods (e.g. stability of sets) become proper tools for quantitative control system analysis and design. Attractive ellipsoids method is one of the most efficient tools in this context. It allows the linear control/ observer design approaches to be applied for a class of quasi-Lipschitz nonlinear models. This methodology is summarized in the book (co-authored by me) "Attractive Ellipsoids in Robust Control", Birkhauser, 2014. In the last years, I have also dealt with minimax and interval observers techniques as well as observer-based control design for both finite and infinite dimensional systems.

- *Sliding Mode Control* (since 2000).

Being introduced in the early 60s, sliding mode is one of the first methods of robust control design. However, a mathematically rigorous analysis of the sliding mode control system is always complicated due to the discontinuous nature of the control laws. In this context I have proposed a unified approach to Lyapunov function-based stability analysis of the so-called second order sliding mode systems. I have also dealt with extension of attractive ellipsoids and homogeneity methods to sliding mode systems and with application of sliding mode control design methodology in flow control systems.

- *Time Delay Systems* (since 2000).

Delay effects accompany any real control systems. Analysis and design of control systems in the case of input and state delays is important for control practice. In this context, I was involved in development of relay and sliding mode control design for systems with delays. Analysis of robustness of homogeneous systems with respect to delays. Control design for bi-linear time switched systems with delays. Interval state estimation and interval control design for uncertain delay systems.

Structure of publications

The complete list of publications is given in Appendix A. This section gives just general overview of my publication activity. I have more than 150 scientific publications:

- *International Journal Papers*: more than **50**
(including 20 journal papers in the top-ranked control journals: Automatica and IEEE Transactions on Automatic Control)
- *International Conference Papers*: more than **80**
(including about 35 papers in top-ranked control conferences: IEEE Conference on Decision and Control and IFAC World Congress)
- *Books*: **1**
(A. Poznyak, A. Polyakov, V. Azhmyakov “*Attractive Ellipsoids in Robust Control*”, Birkhauser, 2014)
- *Book chapters*: **4**
- *National Journals and Conferences* (in Russian): more than **10**

My publication activity in the research topics is summarized in the next table.

Research Topic	International Journals	International Conferences	Books and Chapters	National Journals and Conferences	Total
Homogeneity	19	31	1	3	54
Set-Theoretic	11	19	2	0	32
Sliding Mode	16	17	1	1	35
Time Delays	13	15	1	7	36

Patent

- M. Feingesicht, A.Polyakov F. Kerherve, J.P. Richard, 'Dispositif de controle actif du recollement d'un ecoulement sur un profil', Patent (brevet) INPI deposit date June 15th, 2017, Nb 1755440, Patent (brevet)

1.4 Research projects

The research projects I have been involved in the last 5 years (in France):

- ANR PRC DIGITSLID (2018-2021)
Subject: "Digital set-valued and homogeneous sliding mode control and differentiators: the implicit approach"
Coordinator: Bernard Brogliato
Partners: Inria Grenoble, Inria Lille, CNRS LS2N (Nantes)

The project is devoted to the development of consistent discretization schemes for sliding mode observers and controllers. The methodology of the implicit discretization for some classes of sliding mode systems are planned to be developed. I am responsible for research activities of NON-A Post(future Valse) team of Inria in the context of this project. We will develop an innovative schemes for consistent discretization of homogeneous control and estimation algorithms.

- ANR PRC Finite4SoS (2015-2020)
Subject: "Finite-time Control and Estimation for Systems of Systems"
Coordinator: Wilfrid Perruquetti
Partners: Inria Lille, Université Pierre et Marie Curie, MINES ParisTech

The project is aimed at a creation of a unified framework for analysis and design of controlled system of systems based on the finite-time stability concept. The key feature of the project is heterogeneity of the inter-connected subsystems modeled by differential law of different types (ODEs, PDEs, time-delay systems, differential inclusions). In this project, I have developed a homogeneity concept for operators in Banach spaces and a universal controller for finite-time stabilization of an evolution equations. Homogeneous observers for ODEs have been developed by PhD Student Francisco Lopez under my supervision.

- ANR JCJC ROCC-SYS (2014-2018)
Subject: "Robust control of cyber-physical systems"
Coordinator: Laurentiu Hetel
People involved: Andrey Polyakov, Christophe Fiter, Alexandre Kruszewski

The project deals with control problems for plants governed by digital controllers. The sampled, switched and networked nature of control devices

arise some specific issues studied in this project. In particular, I have developed algorithms of control design for relay systems. The methods of convex embedding, implicit Lyapunov function and attractive ellipsoids have been utilized for successful solving of the robust stabilization problem for such control systems.

- ANR PRC Chaslim (2011-2015)
Subject: "Chattering-free sliding modes"
Coordinator: Bernard Brogliato
Partners: Inria Grenoble, Inria Lille, CNRS IRCCyN

In this project some methods of chattering reduction of sliding mode control were developed. They include novel discretization schemes for the first order and the second order systems as well as homogeneity-based design of high order sliding mode algorithms allowing chattering reduction. In this project, I have developed the implicit Lyapunov function method for high order sliding mode control design and proposed an efficient scheme of a digital control implementation allowing completely remove the chattering.

- CPER DATA, project "ControlHub" (2016-Now)
Coordinator: Andrey Polyakov
Participants: Dagher Roudy, Dherbomez Gerald, Jean-Pierre Richard, Denis Efimov

ControlHub is the platform for a rapid prototyping of control algorithms for cyber-physical systems. It allows the End-User to have a remote access to real experimental setups in order to validate mathematical control algorithms. ControlHub is under construction by Non-A Post(future Valse) team of Inria Lille Nord Europe.

- CPER ELSAT2020, project "Contratech" (2015-Now)
Coordinator: Jean-Marc Foucaut
Partners: Univ. Valenciennes, Centrale Lille, LMFL, CNRS LAMIH, ONERA, Inria Lille, CNRS CRISAL

The Contratech platform joins research groups of different institutions around the important and challenging problem of active flow control.

- "RECoT" North-European Inria Associate team with IBM Research in Dublin, Ireland
Subject: "Robust Estimation and Control with Time Constraints"
Coordinator: Andrey Polyakov (Inria)

The project bridges the research team of NON-A Post (future Valse), Inria Lille, France and IBM Research Lab in Dublin. It is aimed at development of minimax control and estimation algorithms based on homogeneity technique.

- "HoTSMoCE" Inria Associate team with UNAM, Mexico
Subject: "Homogeneity Tools for Sliding Mode Control and Estimation"
Coordinator: Denis Efimov (Inria)

The project "HoTSMoCE" bridges the research team of NON-A Post (future Valse), Inria Lille, France and Sliding Mode Control Lab of National Autonomous University of Mexico (UNAM).

1.5 Supervision

I was also involved (or I am still involved) in supervision of the following Postdocs and PhD students in France:

Postdocs

- Nicolas Espitia (10.2017-now; co-supervisors: Prof. Wilfrid Perruquetti and Dr. Denis Efimov)
- Tonametl Sanchez (10.2017-now; co-supervisor: Prof. Jean-Pierre Richard)
- Manuel Mera (15.07.2014-15.07.2015)

PhD Students

- Maxime Feingesicht (01.01.2015-11.12.2017; Co-supervisors: Jean-Pierre Richard, Andrey Polyakov and Franck Kerherve)
- Francisco Lopez Ramirez (10.2015-19.11.2018; Co-supervisors: Wilfrid Perruquetti, Andrey Polyakov, Denis Efimov,)
- Youness Braidiz (since 10.2017; Co-supervisors: Wilfrid Perruquetti, Denis Efimov, Andrey Polyakov)
- Siyuan Wang (since 10.2017; Co-supervisors: Gang Zheng, Andrey Polyakov)
- Alex Dos Reis de Souza (since 10.2018; Co-supervisors: Denis Efimov, Jean-Luc Gouze, Andrey Polyakov)

I was also involved (or I am still involved) in supervision of **PhD students** in Mexico and Russia, e.g. *Nadhynne Martinez Fonesca* (since 12.2015; UPIBI-IPN, Mexico; Co-supervisor: Dr. Isaac Chairez); *Konstantin Zimenko* (10.2015-22.11.2018, ITMO University, Russia; Supervisor: Artem Kremlev) *Tatyana Kharkovskaya* (since 10.2016, ITMO University and Centrale Lille in co-tutelle; Co-supervisors: Jean-Pierre Richard, Artem Kremlev).

Masters Students

- Erwin Jezequel (Internship, 2018, Centrale Lille)
- Marc Sanmillan (Internship, 2018, Centrale Lille)
- Dmitrii Kondratovich (2012; Institute of Control Sciences of Russian Academy of Sciences)

- Elena Maslennikova (2009, Voronezh State University)
- Aleksej Baushev (2009, Voronezh State University)
- Andrey Solov'ev (2009, Voronezh State University)
- Artem Batuev (2009, Voronezh State University)
- Evgenii Orazaev (2009, Voronezh State University)
- Sergej Kosinov (2007, Voronezh State University)
- Leonid Kukushkin (2007, Voronezh State University)
- Konstantin Chernyshev (2007, Voronezh State University)

1.6 Teaching

The courses were given in Voronezh State University (2004-2007 and 2008-2010; more than 3000 hours):

- Algebra & Analytical Geometry (for bachelors, 1st year);
- Programming Languages and Algorithms (for bachelors, 1st and 2nd years);
- Algorithms for Sparse Matrices (for bachelors, 3rd year);
- Mathematical Software (for bachelors, 3th year);
- Algorithms of Computer Graphics (for bachelors, 4th year);
- Relational Database Management Systems (for bachelors, 4th year);
- Computational Mathematics (for bachelors, 3rd and 4th years);
- Internet Programming (for masters, 1st year);
- Linear Matrix Inequalities (for masters, 2nd year);

1.7 Prizes and Distinctions

Prizes

- Letov Prize 2011 (Institute of Control Sciences, Russian Academy of Sciences)

Membership of Editorial Boards

- International Journal of Robust and Nonlinear Control (2014-2018)
- Journal of The Franklin Institute (2014-2016)
- Journal of Optimization Theory and Applications (2014-2018)
- Automation and Remote Control (2015-now)

Invited Talks

Conferences:

- "Active Flow Control: Sliding Mode Control Approach", Workshop on Active Drag Reduction Aachen, Germany, 15-16 March 2018
- "Sliding mode control of flow separation", 2nd Workshop on Machine Learning Control, Valenciennes, France, July 5th-6th , 2017
- "Robust Stabilization with Time Constraints: Implicit Lyapunov Function Approach", International Conference "Optimization And Applications in Control and Data Science" on the occasion of Boris Polyak's 80th birthday, Moscow, Russia, 13-15 May 2015

Tutorial courses:

- Tutorial course for 6th IEEE International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE-2009), November 10-13, 2009, Toluca, Mexico. (6 hours, title: "Lyapunov Function Method for Nonlinear and Discontinuous Systems")
- Tutorial course for Traditional Whole-Russian School "Control, Information and Optimization" in 2011 (2 hours; title: "Lyapunov Function Design for High Order Sliding Modes") and in 2015 (4 hours; "Stability Notions and Lyapunov Functions for Sliding Mode Control Systems")
- Tutorial course "Stability Notions and Lyapunov Functions for Sliding Mode Control Systems" (Seminar of NON-A/SyNeR teams, Inria, 20 and 21 October 2013, 2 hours)

I gave talks at *seminars* of several research institutions: CNRS L2S (2017), Inria (2014), Institute of Control Sciences of Russian Academy of Sciences (2013, 2009), CINVESTAV-IPN, Mexico (2012, 2010, 2008), UNAM, Mexico(2012), Institute for Problems in Mechanics of Russian Academy of Sciences (2010, 2005).

Reviewing activity

- Outstanding reviewer of Automatica (2016).
- I also serve as a reviewer in high ranked journals/conferences : IEEE Transactions of Automatic Control, Systems & Control Letters, International Journal of Robust and Nonlinear, Conference on Decision and Control etc.

Chapter 2

Linear Geometric Homogeneity in Systems and Control

2.1 Introduction

Dilation Symmetry

Symmetry is a certain invariance when some characteristics of an object do not change under a set of transformations. Symmetry occurs in many branches of mathematics. The simplest examples of the symmetry can be found in the geometry, for example, as an invariance of a figure with respect to such transformations as rotation, translation, reflection and dilation. It is well-known that the rotation remains invariant both size and shape of the figure, while the dilation changes its size leaving invariant only the shape.

The symmetry with respect to the dilation can also be discovered for the level set of the so-called homogeneous function. In mathematical analysis, symmetry of a function f with respect to the *uniform dilation* of its argument

$$x \rightarrow \lambda x,$$

where $\lambda > 0$ is the scaling factor, is known as *homogeneity*. In the simplest case, it is defined as follows

$$f(\lambda x) = \lambda f(x), \quad \lambda > 0.$$

In other words, *homogeneity is a dilation symmetry*. The latter always needed for linearity, which additionally asks for the central symmetry $f(-x) = -f(x)$ and the additivity $f(x + y) = f(x) + f(y)$. Thus all linear functions are homogeneous with respect to the uniform dilation. The class of *homogeneous mappings* is placed somewhere in-between essentially non-linear and linear ones.

In 18th century the *homogeneity* with respect to the uniform dilation (standard homogeneity) has been studied by Leonhard Euler. His notion of homogeneity

is also well known in the context of polynomial functions.

Definition. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be homogeneous if there exists a number $\nu \in \mathbb{R}$ such that $f(\lambda x) = \lambda^\nu f(x), \forall \lambda > 0$. The number ν is called homogeneity degree of the function f .

According to this definition, any linear function has homogeneity degree 1, but, the quadratic one

$$x = (x_1, x_2)^\top \xrightarrow{f} x_1^2 + x_1 x_2 + x_2^2$$

is homogeneous of degree 2. Its level sets are ellipsoids centered at the origin. Obviously, they are symmetric with respect to the uniform dilation.

Euler's *Homogeneous Function Theorem* is one of the famous results underlying the modern homogeneity theory:

Theorem. A continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k if and only if $\frac{\partial f}{\partial x} x = k f(x), x \in \mathbb{R}^n$.

From this theorem, in particular, we conclude that homogeneity of a function is inherited by its derivatives. In fact, the dilation symmetry can be inherited by other mathematical objects induced by homogeneous functions. In particular, this chapter deals with a symmetry of non-linear generalized homogeneous differential equations and some related properties important for control systems design.

Homogeneous Differential Equations

Let us consider the simplest scalar ordinary differential equation (ODE) with the homogeneous right-hand side

$$\dot{x} = -x^\nu, \quad x \in \mathbb{R},$$

where $\nu = p/q$, p is an odd integer and q is an even natural number. Its solution with the initial condition $x(0) = x_0$ is given by

$$x(t, x_0) = \frac{x_0}{(1 + (\nu - 1)x_0^{\nu-1}t)^{1/(\nu-1)}}.$$

Hence, we easily derive a symmetry of solutions with respect to simultaneous dilation of the initial condition and the time variable t :

$$x(\lambda^{1-\nu}t, \lambda x_0) = \lambda x(t, x_0), \quad \lambda > 0.$$

The time scaling factor depends on homogeneity degree ν . In fact, such a symmetry of solutions holds for any homogeneous differential equation [78].

Theorem. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous homogeneous vector field of degree $\nu \in \mathbb{R}$ such that the Cauchy problem

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

admits a solution $x(t, x_0)$ defined for all $t > 0$. Then

$$x(\lambda^{1-\nu}t, \lambda x_0) = \lambda x(t, x_0), \quad \lambda > 0,$$

where $x(\cdot, \lambda x_0)$ is a solution to the same problem with the scaled initial condition $x(0) = \lambda x_0$.

Homogeneity simplifies qualitative analysis and design of control systems, since homogeneous dynamical systems behave similarly (in a certain sense) to linear ones. For example, $x = 0$ is always an equilibrium of continuous homogeneous system. If there exists another equilibrium $x^* \neq 0$ then the set $M = \{x \in \mathbb{R}^n : x = \lambda x^*, \lambda > 0\}$ is invariant and $f(x) = 0$ for all $x \in M$. From the latter theorem we immediately conclude that local property (e.g. stability of the origin) always implies the global one. Robustness of homogeneous control systems can be granted by its stability in the disturbance-free case [69], [4], [8]. *In other words, homogeneous systems have similar properties to linear ODE models (see Table 2.1 and, for example, [79], [67], [34], [69], [9], [46], [55] for proofs of the mentioned properties of homogeneous systems).*

Homogeneity and Convergence Rate

Homogeneity degree allows an additional qualitative analysis to be provided easily. For example, finite-time or *fixed-time stability* (see e.g. [68], [10], [48], [4], [54], [63] for more details) can be derived from homogeneity degree. Indeed, if $\nu = 1/3$ then the solution of the scalar homogeneous system $\dot{x} = -x^\nu$ is given by

$$x(t, x_0) = \begin{cases} \left(|x_0|^{\frac{2}{3}} - \frac{2}{3}t\right)^{\frac{3}{2}} & \text{if } t \in \left[0, \frac{3}{2}|x_0|^{\frac{2}{3}}\right), \\ 0 & \text{if } t \geq \frac{3}{2}|x_0|^{\frac{2}{3}}. \end{cases}$$

Obviously, it converges to zero in a finite-time. However, if $\nu > 1$ then any trajectory converges to a unit ball in a fixed time independently of the initial condition. Indeed, the formula for its solution (see the beginning of this section) implies that $|x(t, x_0)| \leq 1$ for $t > \frac{1}{\nu-1}$ independently of $x_0 \in \mathbb{R}$. In fact, these properties can be established in a more general case. Therefore, homogeneity degree specifies the convergence rate of solutions to the origin. Namely, if the origin of homogeneous ODE is asymptotically stable then each trajectory vanishes in a finite-time provided that $\nu < 1$ but if $\nu > 1$ then all trajectories convergence in a fixed-time to a neighborhood of the origin [46].

	Linear System $\dot{x} = Ax$ $A \in \mathbb{R}^{n \times n}$	Homogeneous System $\dot{x} = f(x)$ $f(\lambda x) = \lambda^\nu f(x)$
Trajectory Scaling	$x(t, \lambda x_0) = \lambda x(t, x_0)$	$x(t, \lambda x_0) = \lambda x(\lambda^{\nu-1}t, x_0)$
Local \Leftrightarrow Global (existence of solutions, stability, etc)	✓	✓
Invariance \Leftrightarrow Stability	✓	✓
Stability \Rightarrow Robustness (Input-to-State Stability)	$\dot{x} = Ax + Dw$ $w \in L^\infty$	$\dot{x} = f(x, w)$ $w \in L^\infty$
Convergence Rate	Exponential	Finite ($\nu < 1$) Exponential ($\nu = 1$) Fixed-time ($\nu > 1$)
Lyapunov Function	Quadratic $V = x^\top P x,$ $P > 0$	Homogeneous ("Quadratic-like") $V = x^\top \Xi^\top(x) P \Xi(x) x,$ $P > 0, \Xi(\lambda x) = \Xi(x) \in \mathbb{R}^{n \times n},$ $\frac{\partial \Xi}{\partial x_i} x = 0, \det(\Xi(x)) \neq 0$

Table 2.1: Some properties of linear and homogeneous systems

Locally Homogeneous Systems

An approximation by means of homogeneous functions (*homogeneous approximation* [29], [4]) can be utilized for local analysis of dynamical systems if, for example, linearization is non-informative (for stability analysis) or simply impossible. For example, the system $\dot{x} = -x(\sin^2(x) + x^2)$ is globally uniformly asymptotically stable. The corresponding homogeneous approximation at the origin $\dot{x} = -2x^3$ is also globally asymptotically stable, while its linearization $\dot{x} = 0$ is not.

Let us consider mechanical system consisting of a rigid body moving laterally on a contact surface and in a viscous environment (fluid). A simplest real-life example of such mechanical system is a car moving on a flat road. Let z be the position of the center of mass of the body in an inertial frame. The equation describing motion of this system has the form

$$\dot{z}(t) = v(t), \quad m \dot{v}(t) = F(t), \quad t > 0, \quad z(t) \in \mathbb{R},$$

where $v(t)$ is the velocity, m is the mass of the body, but F is the sum of external forces.

We study only the deceleration phase for this mechanical system assuming that at the initial instant of time it has some non-zero velocity $\dot{z}(0) = v(0) \neq 0$. Dissipation of the energy may be caused by several external forces. To discover some fast (non-asymptotic) transition it is sufficient to consider only two of them:

- *drag force* (fluid resistance) is proportional to the square of the velocity [22]

$$F_{drag}(t) = -k_{drag} v^2(t) \text{ sign}(v(t)),$$

where $k_{drag} > 0$ is the coefficient of fluid (air) resistance and the sign function is given by

$$\text{sign}(\rho) = \begin{cases} 1 & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0, \\ -1 & \text{if } \rho < 0; \end{cases}$$

- *dry friction force* is almost independent of a velocity magnitude and given by the model [5]

$$F_{dry}(t) = -k_{dry} \text{sign}(v(t)),$$

where $k_{dry} > 0$ is the coefficient of dry friction.

The friction models also may contain some linear terms (proportional to velocity). We skip them for simplicity of analysis, since they will not effect the final conclusions about local homogeneity degree and convergence rate.

The sum of external forces $F(t)$ can be represented as follows

$$F(t) = F_{drag}(t) + F_{dry}(t) = -(k_{dry} + k_{drag} v^2(t)) \text{sign}(v(t)).$$

and the differential equation describing evolution of the velocity of the body has the form:

$$m\dot{v}(t) = -(k_{dry} + k_{drag} v^2(t)) \text{sign}(v(t)).$$

It is easy to see that $v = 0$ is the equilibrium of the last equation, which is globally stable, $v(t) \rightarrow 0$ as $t \rightarrow +\infty$. The solution can be found explicitly :

$$v(t) = \tan \left(\arctan(|v(0)|) - \frac{\sqrt{k_{dry} k_{drag}}}{m} t \right) \text{sign}(v(0)).$$

This immediately implies $v(t) = 0$ for $t \geq \frac{m \arctan(|v(0)|)}{\sqrt{k_{dry} k_{drag}}}$. Since arctan is bounded, we conclude that *independently of initial velocity the motion terminates no later than the instant of time*

$$T_{\max} = \frac{m\pi}{2\sqrt{k_{dry} k_{drag}}}.$$

The model of friction force F is not homogeneous function of the velocity v , but locally (close to the origin or close to infinity) it is. Namely,

$$F \approx -k_{dry} \text{sign}(v) \text{ as } v \rightarrow 0 \quad \text{and} \quad F \approx -k_{drag} v^2 \text{sign}(v) \text{ as } v \rightarrow \infty.$$

Therefore, the approximation at zero is a homogeneous function with degree $\nu = 0$, but the approximation at infinity is a homogeneous function with degree $\nu = 2$. The first one allows us to conclude finite-time convergence of the velocity to zero, while the second one guarantees fixed-time convergence to the neighborhood of zero. Such a combination implies fixed-time deceleration of the mechanical system independently of the initial velocity. This property is known as *fixed-time stability* (or *fixed-time convergence*) [54], [63].

Weighted Dilation

Standard (uniform) homogeneity considered above has been introduced by means of uniform dilation $x \rightarrow \lambda x$, $\lambda > 0$. It is clear that changing of this dilation to some non-uniform one defines another type of homogeneity. The *weighted dilation* (introduced by V. Zubov in 1958, [79]) of the vector $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$ is the simplest case of the so-called generalized (non-uniform) dilation:

$$(x_1, x_2, \dots, x_n) \rightarrow (\lambda^{r_1} x_1, \lambda^{r_2} x_2, \dots, \lambda^{r_n} x_n),$$

where $\lambda > 0$ as before is the scaling factor, but the positive numbers r_1, r_2, \dots, r_n are the weights specifying the dilation rates along different coordinates. If $r_1 = r_2 = \dots = r_n$ then the weighted dilation becomes uniform. The introduced transformation of coordinates is a linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by the dilation matrix:

$$x \rightarrow \Lambda x, \quad \text{where} \quad \Lambda = \begin{pmatrix} \lambda^{r_1} & 0 & \dots & 0 \\ 0 & \lambda^{r_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda^{r_n} \end{pmatrix}.$$

The symmetry (homogeneity) of a function with respect to the weighted dilation can be identified analogously to the uniform case.

Definition. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous with respect to the weighted dilation Λ if

$$f(\Lambda x) = \lambda^\nu f(x), \quad x \in \mathbb{R}^n, \quad \lambda > 0,$$

where $\nu \in \mathbb{R}$ is the homogeneity degree.

The weighted dilation extends the class of homogeneous functions under consideration. For example, the polynomial function

$$x = (x_1, x_2)^\top \xrightarrow{f} x_1^2 + x_1 x_2^2 + x_2^4$$

is homogeneous with respect to the weighted dilation $(x_1, x_2) \rightarrow (\lambda^2 x_1, \lambda x_2)$, but it is not homogeneous with respect to the uniform one $(x_1, x_2) \rightarrow (\lambda x_1, \lambda x_2)$.

Weighted homogeneous functions demonstrate similar properties to the standard homogeneous case. In particular, symmetry of solutions of homogeneous differential equations can be established easily (see, e.g. [79], [29]). Weighted homogeneous models frequently appear in control theory and applications. For example, the most of high order sliding mode control algorithms are weighted homogeneous [41], [48]. The classical minimum time control problem

$$T \rightarrow \min_u$$

subject to

$$\begin{cases} \dot{x}_1 = x_2, & u \in L^\infty((0, T), \mathbb{R}) \text{ such that } |u(t)| \leq 1 \\ \dot{x}_2 = u, & x_1(T) = x_2(T) = 0, \end{cases}$$

has solution (see, e.g. [13]) in the form of the weighted homogeneous feedback

$$u = -\text{sign}(|x_2|x_2 + 2x_1).$$

Indeed, $u(\lambda^2 x_1, \lambda x_2) = u(x_1, x_2)$, i.e. u is the weighted homogeneous function with the zero degree.

Below, for simplicity, the scaling factor $\lambda > 0$ is denoted by e^s with $s \in \mathbb{R}$, where e is the Euler number.

Linear Dilation

From the previous section we conclude that once a homogeneity of a function with respect to a group of transformations (dilations) is established, many properties useful for analysis and design of dynamical control system modeled by the corresponding ODE can be discovered. In fact, the same conclusion can be done for infinite-dimensional dynamical models [60].

The different types of dilations have been studied in systems and control theory. The brief overview of different homogeneity concepts is given below

- *Uniform dilation* - L. Euler 17th century

$$x \rightarrow e^s x, \quad s \in \mathbb{R};$$

standard homogeneous ODEs (V. Zubov 1957 [78], W. Hahn [26]);

- *Weighted dilation* - V. Zubov 1958, [79]

$$(x_1, x_2, \dots, x_n) \rightarrow (e^{r_1 s} x_1, e^{r_2 s} x_2, \dots, e^{r_n s} x_n), \quad s \in \mathbb{R};$$

weighted homogeneous ODEs (V. Zubov 1958, [79]); weighted homogeneous PDEs (G. Folland 1975, [23]);

- *Geometric dilation* induced by a flow of a vector field - V. Khomenuk 1961, [36]; M. Kawski 1991 [33]; L. Rosier 1993, [67];

In this chapter we deal with the group of **linear dilations** in a Banach space \mathbb{B} :

$$x \rightarrow \mathbf{d}(s)x, \quad x \in \mathbb{B}, \quad s \in \mathbb{R},$$

where \mathbf{d} is a strongly continuous group (i.e. $\mathbf{d}(\cdot)x : \mathbb{R} \rightarrow \mathbb{B}$ is continuous for any $x \in \mathbb{B}$) of **linear bounded operators** $\mathbb{B} \rightarrow \mathbb{B}$ and $s \in \mathbb{R}$ is the group parameter. For

$\mathbb{B} = \mathbb{R}^n$, the linear dilation includes uniform and weighted dilations as particular cases, but it is less general than the geometric dilation, which allows the group \mathbf{d} to be nonlinear. In other words, the geometric homogeneity (geometric dilation) can be split in two sub-classes: linear and non-linear. In this chapter we restrict ourself to **linear geometric homogeneity** only. The detailed study of linear dilations is presented in the next section.

Obviously that all linear operators are homogeneous with respect to the uniform dilation $\mathbf{d}(s) = e^s I$, where I is an identity operator in $\mathcal{L}(\mathbb{B}, \mathbb{B})$. Moreover, the most of nonlinear models of mathematical physics are generalized homogeneous under a proper selection of the dilation group $\mathbf{d}(s)$, e.g. Navier stocks, KdV equation, Fast diffusion equation, Burgers equation, Saint-Venant equation are homogeneous in the generalized sense [60].

The homogeneity allows many important results to be obtained, for example, in the context of controllability, stabilizability and observability of nonlinear systems [32], [34], [70], [71], [25], controllers and observers design [3], [17], [52], [4], [61], [44], [55], [58]. It simplifies robustness analysis for both delay-free [69], [4], [8] and time-delay systems [2], [76]. The most of high order sliding mode algorithms are homogeneous [41], [48], [7]. Below we introduce the linear geometric homogeneity of vector fields in \mathbb{R}^n and (unbounded) operators in the Banach space \mathbb{B} ; discovers some important properties of homogeneous systems and study some control/estimation problems in order to demonstrate how homogeneity can be utilized for analysis and design of both finite and infinite dimensional control systems.

2.2 Homogeneous Systems

2.2.1 Linear Dilations

Let \mathcal{L} be a space of linear bounded operators $\mathbb{B} \rightarrow \mathbb{B}$, where \mathbb{B} is a real Banach space with a norm $\|\cdot\|$.

Definition 2.1. A map $\mathbf{d} : \mathbb{R} \rightarrow \mathcal{L}$ is said to be *dilation* in \mathbb{B} if

- *(Group property)*

$$\mathbf{d}(0) = I \in \mathcal{L} \text{ and } \mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t) \text{ for } t, s \in \mathbb{R};$$

- *(Limit property)*

$$\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)z\| = 0 \text{ and } \lim_{s \rightarrow +\infty} \|\mathbf{d}(s)z\| = \infty \text{ uniformly on } z \in S,$$

where $S = \{z \in \mathbb{B} : \|z\| = 1\}$ is the unit sphere in \mathbb{B} .

Obviously, the dilation \mathbf{d} is a group of linear bounded and invertible operators

$$\mathbf{d}(-s) = (\mathbf{d}(s))^{-1}.$$

The limit property given above is just a version of the so-called Teresaka's condition (see, for example, [33] for more details) that is required for a group \mathbf{d} to be a dilation in an abstract space.

Definition 2.2. A dilation \mathbf{d} is

- *strongly continuous* if $\mathbf{d}(\cdot)z : \mathbb{R} \rightarrow \mathbb{B}$ is continuous for any $z \in \mathbb{B}$;

- *uniformly continuous* if $\mathbf{d}(\cdot) : \mathbb{R} \rightarrow \mathcal{L}$ is continuous in the operator norm

$$\|\mathbf{d}(\cdot)\| := \sup_{z \neq 0} \frac{\|\mathbf{d}(\cdot)z\|}{\|z\|}.$$

The strong continuity plays an important role for dilations in infinite dimensional spaces, where the uniform continuity is sometimes a too restrictive property. In a finite dimensional space (e.g. $\mathbb{B} = \mathbb{R}^n$) any strongly continuous dilation is always uniformly continuous.

Example 2.1.

- Let \mathbb{B} be the space of bounded uniformly continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ with the supremum norm. The dilation \mathbf{d} in \mathbb{B} can be defined as

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(x + \beta s),$$

where $s \in \mathbb{R}$ is the group parameter, $z \in \mathbb{B}$, $x \in \mathbb{R}$ and $\alpha > 0$ and $\beta \in \mathbb{R}$ are constant parameters.

- Let $\mathbb{B} = L^2(\mathbb{R}, \mathbb{R})$ and $\mathbf{d} : \mathbb{R} \rightarrow \mathcal{L}$ be defined as

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x),$$

where $s \in \mathbb{R}$ is the group parameter, $z \in L^2(\mathbb{R}, \mathbb{R})$, $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ are constant parameters. If

$$\beta < 2\alpha$$

then \mathbf{d} is a strongly continuous dilation in $L^2(\mathbb{R}, \mathbb{R})$.

- Let $\mathbb{B} = \mathbb{R}^n$ and Q is a Hurwitz matrix then $\mathbf{d}(s) = e^{-sQ}$ is a dilation in \mathbb{R}^n .

Let us recall the following notations

$$\|\mathbf{d}(s)\| = \sup_{u \in S} \|\mathbf{d}(s)u\| = \sup_{u \neq 0} \frac{\|\mathbf{d}(s)u\|}{\|u\|} \quad \text{and} \quad \lfloor \mathbf{d}(s) \rfloor = \inf_{u \in S} \|\mathbf{d}(s)u\| = \inf_{u \neq 0} \frac{\|\mathbf{d}(s)u\|}{\|u\|}.$$

These quantities characterize a rate of dilation in \mathbb{B} .

Proposition 2.1. [60] *If \mathbf{d} is a strongly continuous dilation in \mathbb{B} then*

- 1) $0 < \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| < +\infty, \quad s \in \mathbb{R};$
- 2) $\|\mathbf{d}(s)\| \rightarrow 0$ as $s \rightarrow -\infty$ and $\lfloor \mathbf{d}(s) \rfloor \rightarrow +\infty$ as $s \rightarrow +\infty;$
- 3) $\lfloor \mathbf{d}(s) \rfloor < 1$ for $s < 0$ and $\|\mathbf{d}(s)\| > 1$ for $s > 0;$
- 4) $\mathbf{d}(s) \neq I \in \mathcal{L}$ for $s \neq 0.$

If \mathbf{d} is uniformly continuous then $\|\mathbf{d}(\cdot)\| : \mathbb{R} \rightarrow \mathbb{R}$ and $\lfloor \mathbf{d}(\cdot) \rfloor : \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

It is well known that any nontrivial element from \mathbb{B} can be projected to the unit sphere using the uniform dilation. The corresponding projector $P : \mathbb{B} \setminus \{\mathbf{0}\} \rightarrow S$ is defined as follows

$$P(u) = \frac{u}{\|u\|} = \mathbf{d}(-\ln \|u\|)u,$$

where $\mathbf{d}(s) = e^s I$ is the uniform dilation, I is the identity operator on \mathbb{B} . In fact, the similar projection can be constructed using the linear dilation \mathbf{d} given above.

Proposition 2.2. [60] *If \mathbf{d} is a strongly continuous dilation in \mathbb{B} then for any $z \in \mathbb{B} \setminus \{\mathbf{0}\}$ there exist $s_0 \in \mathbb{R}$ and $z_0 \in S$ such that*

$$z_0 = \mathbf{d}(s_0)z.$$

The point z_0 defined in the latter proposition is called *homogeneous projection* of z to the unit sphere S . The homogeneous projection may be non-unique even in the Euclidean space \mathbb{R}^2 .

Example 2.2. *Let us consider the weighted dilation*

$$\mathbf{d}(s) = \begin{pmatrix} e^{\alpha s} & 0 \\ 0 & e^s \end{pmatrix}, \quad s \in \mathbb{R}, \quad \alpha > 1 + \frac{2}{\sqrt{5}-1},$$

in the Euclidean space $\mathbb{B} = \mathbb{R}^2$ with the norm

$$\|z\| = \sqrt{z^\top P z}, \quad z \in \mathbb{R}^2, \quad P = P^\top \in \mathbb{R}^{n \times n}, \quad P \succ 0,$$

where

$$P = P_0 + \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \succ 0, \quad P_0 = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & -1 \\ -1 & \frac{2}{3-\sqrt{5}} \end{pmatrix}, \quad \varepsilon > 0.$$

Let us show that for sufficiently small $\varepsilon > 0$, the function $\xi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\xi(s) = \left\| \mathbf{d}(s) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|^2 - 1, \quad s \in \mathbb{R}$$

has at least two zeros: $\xi(s_1) = \xi(s_2) = 0$ and $s_1 \neq s_2$. Let us denote

$$\xi_0(s) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \mathbf{d}^\top(s) P_0 \mathbf{d}(s) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1.$$

Notice that

$$\xi_0(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top P_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 = \frac{3-\sqrt{5}}{2} + \frac{2}{3-\sqrt{5}} - 3 = 0.$$

The function $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}$ is continuous differentiable on \mathbb{R} and decays in the neighborhood of the point $s = 0$, since

$$\xi_0'(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^\top P_0 + P_0 \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} \alpha(3-\sqrt{5}) & -1-\alpha \\ -1-\alpha & \frac{4}{3-\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} < 0.$$

This means that $\xi_0(s)$ is negative for small $s > 0$ and there exists $\delta > 0$ such that

$$\xi_0(s) < 0 \quad \text{for all } 0 < s < \delta.$$

Since $\xi_0(s) \rightarrow +\infty$ as $s \rightarrow +\infty$ and $\xi_0(0) = 0$ then for sufficiently small $\varepsilon > 0$ the function ξ given by

$$\xi(s) = \xi_0(s) + \varepsilon e^{2s}$$

has at least two zeros. Therefore, the point $z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ admits at least two different homogeneous projections $\mathbf{d}(s_1)z$ and $\mathbf{d}(s_2)z$, $s_1 \neq s_2$ to the unit sphere S .

Uniqueness of homogeneous projection may depend on the unit sphere S , i.e. on the norm in \mathbb{B} . Notice that in the case of uniform dilation $\mathbf{d}(s) = e^s I$, the homogeneous projection is always unique due to homogeneity of the norm: $\|\gamma u\| = |\gamma| \|u\|$, $\gamma \in \mathbb{R}$, $u \in \mathbb{B}$. The problem of uniqueness of homogeneous projection in a more general case is studied below, where the so-called monotone dilation is introduced.

Geometric Structures Induced By Dilation

Linear dilation allows a sort of norm-topology to be introduced in \mathbb{B} by means of the so-called canonical homogeneous norm studied in the next section. Here we just consider some related geometric structures (balls, spheres and cones).

Definition 2.3. *The set*

$$S_d(r, z_0) = \{z \in \mathbb{B} : \|\mathbf{d}(-\ln(r))(z - z_0)\| = 1\},$$

is called \mathbf{d} -homogeneous sphere of the radius $r > 0$ with the center at $z_0 \in \mathbb{B}$ and the set

$$B_d(r, z_0) = \{z \in \mathbb{B} : \|\mathbf{d}(-\ln(r))(z - z_0)\| < 1\},$$

is called open \mathbf{d} -homogeneous ball of the radius $r > 0$ with the center at $z_0 \in \mathbb{B}$. If $z_0 = \mathbf{0}$ we use the notations $S_d(r)$ and $B_d(r)$.

Obviously, the unit homogeneous sphere $S_d(1)$ (resp. ball $B_d(1)$) coincides with the unit sphere S (resp. ball B) in \mathbb{B} , so

$$S_d(r) = \mathbf{d}(\ln r)S \quad \text{and} \quad B_d(r) = \mathbf{d}(\ln r)B.$$

Note also that $\mathbf{0} \in B_d(r)$ for any $r > 0$.

Corollary 2.1. *The following set-theoretic relations hold*

$$1) B_d(r) \setminus \{\mathbf{0}\} \subseteq \bigcup_{0 < \delta < r} S_d(\delta); \quad 2) \mathbb{B} \setminus \{\mathbf{0}\} = \bigcup_{r > 0} S_d(r).$$

Recall that the homogeneous projection is not unique in the general case. This means that the homogeneous sphere $S_d(r)$ may have intersections with the unit sphere for $r \neq 1$.

Definition 2.4. *A nonempty set $\mathcal{D} \subseteq \mathbb{B}$ is said to be \mathbf{d} -homogeneous cone if \mathcal{D} is invariant with respect to dilation $\mathbf{d} : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{B}, \mathbb{B})$, i.e.*

$$\mathbf{d}(s)z \in \mathcal{D} \text{ for } z \in \mathcal{D} \text{ and } s \in \mathbb{R}.$$

Any point $z \in \mathcal{D}$ belongs to the homogeneous cone \mathcal{D} together with the

continuous curve (homogeneous curve)

$$\Gamma_{\mathbf{d}}(z) = \{y \in \mathbb{B} : y = \mathbf{d}(s)z, s \in \mathbb{R}\}.$$

In particular, if \mathbf{d} is the uniform dilation (i.e. $\mathbf{d}(s) = e^s I, s \in \mathbb{R}$), then \mathcal{D} becomes the conventional positive cone in \mathbb{B} .

Example 2.3. Let $\mathbb{B} = C([0, p], \mathbb{R})$ and the dilation in \mathbb{B} be defined as $(\mathbf{d}(s)z)(x) = e^{s-0.5sx/p}z(x)$, where $s \in \mathbb{R}, x \in [0, p]$. The set

$$\mathcal{D} = \{z \in C([0, p], \mathbb{R}) : z(0) = z^2(p)\}$$

is \mathbf{d} -homogeneous cone in $C([0, p], \mathbb{R})$. Indeed, if $z \in \mathcal{D}$ then $z(0) = z^2(p)$ and

$$(\mathbf{d}(s)z)(0) = e^s z(0) = e^s z^2(p) = (e^{0.5s} z(p))^2 = ((\mathbf{d}(s)z)(p))^2.$$

Generators of Dilations

It is well known (see e.g. [50]) that any strongly continuous group of linear bounded operators on \mathbb{B} can be identified by means of an infinitesimal generator.

Definition 2.5. An operator $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{B} \rightarrow \mathbb{B}$ defined as

$$G_{\mathbf{d}}z = \lim_{s \rightarrow 0^+} \frac{\mathbf{d}(s)z - z}{s}$$

is called an infinitesimal generator of a strongly continuous dilation \mathbf{d} . The domain of the generator $G_{\mathbf{d}}$ is given by

$$\mathcal{D}(G_{\mathbf{d}}) = \left\{ z \in \mathbb{B} : \lim_{s \rightarrow 0^+} \frac{\mathbf{d}(s)z - z}{s} \text{ exists} \right\}.$$

The generator is a linear possibly unbounded operator [50]. The next result follows, basically, from the fact that $\mathbf{d}(s)$ is a strongly continuous group of linear bounded operators.

Proposition 2.3. Let \mathbf{d} be a dilation in \mathbb{B} then

1) there exists $M \geq 1, \omega^+ \in \mathbb{R}$ and $\omega^- \in \mathbb{R}$ such that

$$\|\mathbf{d}(s)\|_{\mathcal{L}} \leq M \begin{cases} e^{\omega^+ s} & \text{if } s \geq 0, \\ e^{\omega^- s} & \text{if } s \leq 0; \end{cases} \quad (2.1)$$

2) if the operator G_d^- is defined as follows

$$G_d^- u = \lim_{s \rightarrow 0^-} \frac{\mathbf{d}(s)u - u}{s}, \quad u \in \mathbb{B}$$

with the domain

$$\mathcal{D}(G_d^-) = \left\{ u \in \mathbb{B} : \lim_{s \rightarrow 0^-} \frac{\mathbf{d}(s)u - u}{s} \text{ exists} \right\}$$

then $G_d^- = G_d$, i.e. $G_d^- z = G_d z$ for any $z \in \mathcal{D}(G_d^-) = \mathcal{D}(G_d)$;

3) $\mathcal{D}(G_d)$ is a \mathbf{d} -homogeneous cone and it is a linear subspace dense in \mathbb{B} ;

4) if $z \in B$ then

$$\lim_{h \rightarrow 0} \int_t^{t+h} \mathbf{d}(s)z \, ds = \mathbf{d}(t)z, \quad \int_0^t \mathbf{d}(s)z \, ds \in \mathcal{D}(G_d)$$

and

$$G_d \left(\int_0^t \mathbf{d}(s)z \, ds \right) = \mathbf{d}(t)z - z;$$

5) if $z \in \mathcal{D}(G_d)$ then

$$\frac{d}{ds} \mathbf{d}(s)z = G_d \mathbf{d}(s)z = \mathbf{d}(s)G_d z, \quad s \in \mathbb{R};$$

6) if the operator G_d is bounded then the dilation \mathbf{d} is uniformly continuous and

$$\frac{d}{ds} \mathbf{d}(s) = G_d \mathbf{d}(s) = \mathbf{d}(s)G_d, \quad s \in \mathbb{R};$$

$$\mathbf{d}(s) = e^{sG_d} := \sum_{n=0}^{+\infty} \frac{(sG_d)^n}{n!}, \quad s \in \mathbb{R};$$

$$\|\mathbf{d}(s)\|_{\mathcal{L}} \leq e^{s\|G_d\|_{\mathcal{L}}}, \quad s \in \mathbb{R}.$$

In the above proposition, all integrals are understood in the sense of Brochner (see e.g. [20, page 187]). Most of the proofs of these properties can be found in [50, Theorem 2.4, page 4]. The others are straightforward consequences of the introduced definitions.

Example 2.4.

- Let \mathbb{B} be the space of bounded uniformly continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ with the supremum norm. In Example 2.1 the dilation in \mathbb{B} is defined as $(\mathbf{d}(s)z)(x) = e^{\alpha s}z(x + \beta s)$, $s \in \mathbb{R}$, $z \in \mathbb{B}$, $x \in \mathbb{R}$ and $\alpha > 0$, $\beta \in \mathbb{R}$ are constant parameters. The generator of this dilation is defined as follows

$$(G_{\mathbf{d}}z)(x) = \alpha z(x) + \beta z'(x), \quad x \in \mathbb{R}$$

with the domain

$$\mathcal{D}(G_{\mathbf{d}}) = \{z \in \mathbb{B} : z' \in \mathbb{B}\}.$$

- In Example 2.1 the dilation in $\mathbb{B} = L^2(\mathbb{R}, \mathbb{R})$ is defined as $(\mathbf{d}(s)z)(x) = e^{\alpha s}z(e^{\beta s}x)$, $s \in \mathbb{R}$, $z \in \mathbb{B}$, $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, $\beta < 2\alpha$ are some constant parameters. The generator of this dilation is defined as follows

$$(G_{\mathbf{d}}z)(x) = \alpha z(x) + \beta xz'(x), \quad x \in \mathbb{R}$$

with the domain

$$\mathcal{D}(G_{\mathbf{d}}) = \{z \in L^2(\mathbb{R}, \mathbb{R}) : xz' \in L^2(\mathbb{R}, \mathbb{R})\}.$$

where z' denotes the weak derivative of $z \in L^2(\mathbb{R}, \mathbb{R})$ (see e.g. [20, page 623]).

- If $\mathbb{B} = \mathbb{R}^n$ then any continuous dilation in \mathbb{R}^n is defined as follows

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}}, \quad G_{\mathbf{d}} \in \mathbb{R}^{n \times n}.$$

Moreover, the limit property of the dilation implies that all eigenvalues of the matrix $G_{\mathbf{d}}$ are placed in the right complex half-plane. Indeed, otherwise there exists an eigenvalue λ_* : $\Re(\lambda_*) \leq 0$ and there exists a vector x_{λ_*} belonging to an eigensubspace of λ_* such that $\|e^{sG_{\mathbf{d}}}x_{\lambda_*}\|$ is bounded for any $s \geq 0$.

Monotonicity of Dilations

The notion of monotonicity of dilation introduced below is very important for analysis and design of homogeneous control systems. In particular, it allows uniqueness of homogeneous projection and existence of the so-called canonical homogeneous norm (see below).

Definition 2.6. A dilation \mathbf{d} is **monotone** in \mathbb{B} if it is a strong contraction for any $s < 0$, i.e.

$$\|\mathbf{d}(s)\| < 1, \quad \forall s < 0. \quad (2.2)$$

Note that if $\mathbf{d}(s)$ is a strong contraction, then due to group property we conclude that $\mathbf{d}(-s) = \mathbf{d}(s)^{-1}$ is a strong expansion, i.e.

$$[\mathbf{d}(s)] > 1 \quad \text{for any } s > 0. \quad (2.3)$$

Indeed, if $z \in S$ then $1 = \|z\| = \|\mathbf{d}(-s)\mathbf{d}(s)z\| \leq \|\mathbf{d}(-s)\| \cdot \|\mathbf{d}(s)z\|$ for any $s > 0$. Hence, $\|\mathbf{d}(s)z\| \geq \frac{1}{\|\mathbf{d}(-s)\|}$ for any fixed $s > 0$ and arbitrary $z \in S$. Due to strong contraction condition we derive

$$1 < \frac{1}{\|\mathbf{d}(-s)\|} \leq \inf_{z \in S} \|\mathbf{d}(s)z\| = \lfloor \mathbf{d}(s) \rfloor$$

for any $s > 0$.

Proposition 2.4. [56] *If \mathbf{d} is a strongly continuous monotone dilation then*

- 1) *homogeneous projection is unique, i.e. for any $z \in \mathbb{B}$ there exists a **unique** pair $(s_0, z_0) \in \mathbb{R} \times S$ such that $z = \mathbf{d}(s_0)z_0$;*
- 2) *the function $\|\mathbf{d}(\cdot)z\| : \mathbb{R} \rightarrow \mathbb{R}_+$ (with $z \in \mathbb{B} \setminus \{\mathbf{0}\}$) is continuous and strictly increasing on \mathbb{R} ;*
- 3) *the function $\|\mathbf{d}(\cdot)\| : \mathbb{R} \rightarrow \mathbb{R}_+$ (resp. $\lfloor \mathbf{d}(\cdot) \rfloor : \mathbb{R} \rightarrow \mathbb{R}_+$) is lower(upper) semicontinuous and strictly increasing, moreover,*

$$\lfloor \mathbf{d}(0^-) \rfloor \cdot \|\mathbf{d}(0^+)\| = \|\mathbf{d}(0^-)\| = \lfloor \mathbf{d}(0^+) \rfloor = 1.$$

According the latter proposition, the monotonicity of the dilation implies the uniqueness of the homogeneous projection. This property is important for the later constructions, in particular, for the proof of an existence/uniqueness of solutions of non-linear homogeneous evolution systems, stability analysis and homogeneous control design.

To specify the rate of monotone dilation the next notion is utilized.

Definition 2.7. *The dilation \mathbf{d} is **strictly monotone** in \mathbb{B} if $\exists \beta > 0$ such that*

$$\|\mathbf{d}(s)\| \leq e^{\beta s}, \quad \forall s \leq 0. \quad (2.4)$$

Let us note that if the set $\mathcal{D} \subset \mathbb{B}$ is dense in \mathbb{B} and

$$\sup_{z \in \mathcal{D} \cap S} \|\mathbf{d}(s)z\| < 1 \quad (\text{resp. } \sup_{z \in \mathcal{D} \cap S} \|\mathbf{d}(s)z\| \leq e^{\beta s} \text{ with } \beta > 0)$$

for $s < 0$ then \mathbf{d} is monotone (resp. strictly monotone). Moreover, repeating the considerations given right after Definition 2.6 we derive

$$\lfloor \mathbf{d}(s) \rfloor \geq e^{\beta s}, \quad \forall s \geq 0.$$

Proposition 2.5. *A strongly continuous dilation \mathbf{d} in a real Hilbert space \mathbb{H} is strictly monotone if and only if there exists $\beta > 0$ such that*

$$\langle G_{\mathbf{d}}z, z \rangle \geq \beta \quad \text{for any } z \in S \cap \mathcal{D}, \quad (2.5)$$

where $G_{\mathbf{d}}$ is the generator of \mathbf{d} and $\mathcal{D} \subset \mathcal{D}(G_{\mathbf{d}})$ is an arbitrary \mathbf{d} -homogeneous cone dense in $\mathcal{D}(G_{\mathbf{d}})$ and S is the unit sphere in \mathbb{B} .

Proof. The sufficiency has been proven in [58].

Necessity. Let \mathbf{d} be strictly monotone and $z \in S \cap \mathcal{D}(G_{\mathbf{d}})$. Using Cauchy-Swartz inequality for $s > 0$ we derive

$$1 = \langle z, z \rangle = \langle \mathbf{d}(-s)\mathbf{d}(s)z, z \rangle \leq \|\mathbf{d}(-s)\| \cdot |\langle \mathbf{d}(s)z, z \rangle| \leq e^{-\beta s} \|\mathbf{d}(s)z\|.$$

Hence we have

$$e^{2\beta s} - 1 \leq \langle \mathbf{d}(s)z, \mathbf{d}(s)z \rangle - \langle z, z \rangle = \langle \mathbf{d}(s)z - z, \mathbf{d}(s)z \rangle + \langle z, \mathbf{d}(s)z - z \rangle.$$

Dividing both sides of the last inequality by s taking the limit $s \rightarrow 0^+$ we obtain

$$2\beta \leq \langle G_{\mathbf{d}}z, z \rangle + \langle z, G_{\mathbf{d}}z \rangle = 2\langle G_{\mathbf{d}}z, z \rangle.$$

■

Example 2.5. Let $\mathbb{B} = L^2(\mathbb{R}, \mathbb{R})$ and $\mathbf{d} : \mathbb{R} \rightarrow \mathcal{L}$ be defined as

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x),$$

where $s \in \mathbb{R}$ is the group parameter, $z \in L^2(\mathbb{R}, \mathbb{R})$, $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ are constant parameters. If

$$\beta < 2\alpha$$

then this dilation is strongly continuous and strictly monotone since

$$\|\mathbf{d}(s)z\| = e^{(\alpha - 0.5\beta)s} \|z\|.$$

It is worth stressing that monotonicity of dilation depends on the norm $\|\cdot\|$. The next corollary is the straightforward consequence of Proposition 2.5 in the case $\mathbb{B} = \mathbb{R}^n$.

Corollary 2.2. [55] *Let \mathbf{d} be a dilation in the Euclidean space $\mathbb{B} = \mathbb{R}^n$ with the scalar product*

$$\langle u, v \rangle = u^\top P v, \quad u, v \in \mathbb{R}^n,$$

where $0 \prec P = P^\top \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix. The dilation \mathbf{d} is strictly monotone in \mathbb{R}^n equipped with the norm $\|z\| = \sqrt{\langle z, z \rangle}$ if and only if the following linear matrix inequality holds

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad (2.6)$$

where $G_{\mathbf{d}} \in \mathbb{R}^n$ is the generator of the dilation \mathbf{d} . Moreover, one has

$$\begin{aligned} e^{\alpha s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\beta s} \quad \text{if } s \leq 0, \\ e^{\beta s} \leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\alpha s} \quad \text{if } s \geq 0, \end{aligned} \quad (2.7)$$

where

$$\alpha = \frac{1}{2} \lambda_{\max} \left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}} \right) > 0$$

and

$$\beta = \frac{1}{2} \lambda_{\min} \left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}} \right) > 0.$$

Therefore, any continuous dilation $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ in the Euclidean space $\mathbb{B} = \mathbb{R}^n$ is strictly monotone provided that the inner product in \mathbb{R}^n and the weighted Euclidean norm, respectively, are selected according to the inequality (2.6).

Notice also that the parameter α given in (2.7) may not exist for infinite dimensional case, since the operator $G_{\mathbf{d}}$ may be unbounded.

Canonical Homogeneous Norm

The next definition introduces the so-called homogeneous norm in \mathbb{B} .

Definition 2.8. A continuous functional $p : \mathbb{B} \rightarrow \mathbb{R}_+$ is said to be \mathbf{d} -homogeneous norm in \mathbb{B} if

- $p(u) \rightarrow 0$ as $u \rightarrow \mathbf{0}$ and
- $p(\mathbf{d}(s)u) = e^s p(u) > 0$ for $u \in \mathbb{B} \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$.

The functional p may not satisfy the triangle inequality $p(u+v) \leq p(u) + p(v)$, so it is not even a semi-norm. However, many authors (see e.g. [32], [4], [21], [6]) call some functionals satisfying the above definition by the "homogeneous norm". We follow this tradition and consider the **canonical homogeneous norm** introduced by means of the following formula

$$\|u\|_{\mathbf{d}} = e^{s_u}, \text{ where } s_u \in \mathbb{R} \text{ is such that } \|\mathbf{d}(-s_u)u\| = 1. \quad (2.8)$$

The next lemma proves that the canonical homogeneous norm is well defined for monotone dilations.

Lemma 2.1. [58]

- If the dilation \mathbf{d} is monotone on \mathbb{B} then the functional $\|\cdot\|_{\mathbf{d}}: \mathbb{B} \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}_+$ defined by (2.8) is single-valued and positive. It can be prolonged to zero by continuity since $\|u\|_{\mathbf{d}} \rightarrow 0$ as $\|u\| \rightarrow 0$.
- If \mathbf{d} is strictly monotone on \mathbb{B} then $\|\cdot\|_{\mathbf{d}}$ is **locally Lipschitz continuous** on $\mathbb{B} \setminus \{\mathbf{0}\}$.

In [56] such a homogeneous norm was called canonical since it is induced by the canonical norm $\|\cdot\|$ in \mathbb{B} . The canonical homogeneous norm has several obvious but rather important properties:

- $\|u\|_{\mathbf{d}} = 1 \Leftrightarrow \|u\| = 1$;
- $\|\mathbf{d}(s)u\|_{\mathbf{d}} = e^s \|u\|_{\mathbf{d}}$;
- $\|u\|_{\mathbf{d}} = r \Leftrightarrow u \in S_{\mathbf{d}}(r)$, where $r \geq 0$
- for any $u \in \mathbb{B}$ one has

$$\lfloor \mathbf{d}(\ln(\|u\|_{\mathbf{d}})) \rfloor \leq \|u\| \leq \lceil \mathbf{d}(\ln(\|u\|_{\mathbf{d}})) \rceil \quad (2.9)$$

The canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is not a norm in the usual sense, but it defines a sort of norm topology in \mathbb{B} in terms of homogeneous spheres and homogeneous balls defined above.

Lemma 2.2. [58] Let \mathbf{d} be a strictly monotone dilation in a Hilbert space \mathbb{H} then the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is Fréchet differentiable on $\mathcal{D}(G_{\mathbf{d}}) \setminus \{\mathbf{0}\}$ and the **Fréchet derivative** of $\|\cdot\|_{\mathbf{d}}$ at $u \in \mathcal{D}(G_{\mathbf{d}}) \setminus \{\mathbf{0}\}$ is given by

$$(D\|u\|_{\mathbf{d}})(\cdot) = \frac{\langle \mathbf{d}(-\ln \|u\|_{\mathbf{d}}) \cdot, \mathbf{d}(-\ln \|u\|_{\mathbf{d}})u \rangle}{\langle G_{\mathbf{d}} \mathbf{d}(-\ln \|u\|_{\mathbf{d}})u, \mathbf{d}(-\ln \|u\|_{\mathbf{d}})u \rangle} \|u\|_{\mathbf{d}}. \quad (2.10)$$

To compute a homogeneous norm in practice an appropriate numerical procedure is required [61], [55]. In some particular cases it can be calculated explicitly.

Example 2.6. Sometimes, for $\mathbb{B} = \mathbb{R}^2$ the canonical homogeneous norm can be calculated explicitly. Indeed, if

$$\mathbf{d}(s) = e^{sG_d} \quad \text{with} \quad G_d = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and the norm of $u = (u_1, u_2)^\top \in \mathbb{R}^2$ is defined as

$$\|u\| = \sqrt{u^\top P u}, \quad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2} : P \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} P \succ 0$$

then the equation (2.8) becomes

$$u^\top \begin{pmatrix} e^{-2su} & 0 \\ 0 & e^{-su} \end{pmatrix} P \begin{pmatrix} e^{-2su} & 0 \\ 0 & e^{-su} \end{pmatrix} u = 1,$$

or, equivalently,

$$V^4 = aV^2 + bV + c,$$

where $V = e^{su} = \|u\|_d$ and

$$a = p_{22}u_2^2, \quad b = 2p_{12}u_1u_2, \quad c = p_{11}u_1^2.$$

Hence, using the Ferrari formulas we can solve the quartic equation to derive an explicit representation of the canonical homogeneous norm.

2.2.2 Homogeneous Mappings and Equations

Homogeneous operators

Operators to be studied in this chapter are introduced by the next definition, which extends the homogeneity concept proposed in [23].

Definition 2.9 (Homogeneous operator). An operator $f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}$ (a functional $g : \mathcal{D}(g) \subset \mathbb{B} \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree ν if $\mathcal{D}(f)$ (resp. $\mathcal{D}(g)$) is a \mathbf{d} -homogeneous cone and

$$f(\mathbf{d}(s)u) = e^{\nu s} \mathbf{d}(s)f(u), \quad s \in \mathbb{R}, \quad u \in \mathcal{D}(f), \quad (2.11)$$

$$\text{(resp. } g(\mathbf{d}(s)u) = e^{\nu s} g(u), \quad s \in \mathbb{R}, \quad u \in \mathcal{D}(g)) \quad (2.12)$$

where \mathbf{d} is a dilation in \mathbb{B} and $\nu \in \mathbb{R}$ is a homogeneity degree.

Any linear operator is always \mathbf{d} -homogeneous of zero degree if $\mathbf{d}(s) = e^s I \in \mathcal{L}(\mathbb{B})$. However, in some cases the same linear operator may also have negative or positive homogeneity degree dependently of the dilation group as shown in the following example.

Example 2.7. [60] The operator $A = \frac{\partial^2}{\partial x^2} : H^2(\mathbb{R}, \mathbb{R}) \subset H^0(\mathbb{R}, \mathbb{R}) \rightarrow H^0(\mathbb{R}, \mathbb{R})$ is \mathbf{d} -homogeneous of degree $\nu \in \mathbb{R}$ provided that the dilation \mathbf{d} is defined as follows

$$(\mathbf{d}(s)u)(x) = e^{\alpha s}u(e^{0.5\nu s}x), \quad s \in \mathbb{R}, \quad u \in H^0(\mathbb{R}, \mathbb{R}), \quad x \in \mathbb{R}$$

with $\nu < 4\alpha$. Indeed,

$$(A\mathbf{d}(s)u)(x) = \frac{\partial^2}{\partial x^2}(\mathbf{d}(s)u)(x) = \frac{\partial^2}{\partial x^2}e^{\alpha s}u(e^{0.5\nu s}x) = e^{\nu s}(A\mathbf{d}(s)u)(x).$$

Example 2.8. [55] Let $\mathbb{B} = \mathbb{R}^n$ and $f(x) = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Then the vector field f is \mathbf{d} -homogeneous of the degree $\nu \in \mathbb{R}$ if and only if

$$AG_{\mathbf{d}} - G_{\mathbf{d}}A = \nu A,$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is the generator of the dilation. Indeed, to fulfill the homogeneity condition $A\mathbf{d}(s) = e^{\nu s}\mathbf{d}(s)A$ with $s \in \mathbb{R}$, $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$, the mentioned identity is, obviously, necessary and sufficient.

Example 2.9. [60] The **Saint-Venant equation** is an example of a system of conservation laws studied in [14]. In the field of hydraulics, it represents the flow in open-channels by the following model

$$\begin{aligned} \frac{\partial H}{\partial t} &= -\frac{\partial}{\partial x}(HV), \\ \frac{\partial V}{\partial t} &= -\frac{\partial}{\partial x}\left(\frac{1}{2}V^2 + gH\right), \end{aligned} \tag{2.13}$$

where H and V are scalar functions of time and space variables. The quantity $H(t, x)$ is the water level at the instant of time $t \in \mathbb{R}_+$ in the point $x \in \mathbb{R}$, and $V(t, x)$ is the water velocity in the same position. The parameter g denotes the gravitation constant. Let us consider the case when the water channel is supported by two overflow spillways, which adjust an input and output flows in a pool (between spillways). The space argument is restricted on the segment $[0, 1]$, where $x=0$ and $x=1$ are positions of spillways, and the equation (2.13) is supported with the boundary conditions [14]:

$$\begin{aligned} H(t, 0)V(t, 0) - (Z_0 - L_0)^{3/2} &= 0, \\ H(t, 1)V(t, 1) - (H(t, 1) - L_1)^{3/2} &= 0, \end{aligned}$$

where Z_0 is the water level above the pool and L_0, L_1 are spillways heights.

Let us show that for $L_0 = Z_0$ and $L_1 = 0$ the corresponding evolution equation is homogeneous. Let us consider the operator $f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B} := C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$ defined on the domain

$$\mathcal{D}(f) = \left\{ (u_1, u_2) \in C^1([0, 1], \mathbb{R}_+) \times C^1([0, 1], \mathbb{R}) : \begin{array}{l} u_1(0)u_2(0) = 0; \\ u_1(1)u_2(1) = u_1^{3/2}(1) \end{array} \right\}$$

as follows

$$f(u) = \begin{pmatrix} -\frac{\partial}{\partial x}(u_1 u_2) \\ -\frac{\partial}{\partial x}(g u_1 + \frac{1}{2} u_2^2) \end{pmatrix},$$

where $u = (u_1, u_2) \in \mathcal{D}(f)$. The operator f is \mathbf{d} -homogeneous of degree $\nu = 1$ with respect to the weighted dilation $\mathbf{d}(s)u = (e^{2s}u_1, e^s u_2)$, where $u = (u_1, u_2) \in C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$ and $s \in \mathbb{R}$. Indeed,

$$\begin{aligned} f(\mathbf{d}(s)u) &= \begin{pmatrix} -\frac{\partial}{\partial x}(e^{2s}u_1 e^s u_2) \\ -\frac{\partial}{\partial x}(g e^{2s}u_1 + \frac{1}{2}[e^s u_2]^2) \end{pmatrix} = \\ &= \begin{pmatrix} -e^{3s} \frac{\partial}{\partial x}(u_1 u_2) \\ -e^{2s} \frac{\partial}{\partial x}(g u_1 + \frac{1}{2} u_2^2) \end{pmatrix} = e^s \mathbf{d}(s) f(u). \end{aligned}$$

Finally, the equality $(\mathbf{d}(s)u)(x) = (e^{2s}u_1(x), e^s u_2(x))$, $x \in \mathbb{R}$ implies that for any $u \in \mathcal{D}(f)$ one has $\mathbf{d}(s)u \in \mathcal{D}(f)$, i.e., $\mathcal{D}(f)$ is a \mathbf{d} -homogeneous cone.

Remark 2.1 (On a set-valued homogeneous operator). Definition 2.9 is applicable to a set-valued operator $F: \mathcal{D}(F) \subset \mathbb{B} \rightrightarrows \mathbb{B}$ provided that the identity (2.11) is understood in the set-theoretic sense (as equality of sets). It remains meaningful even for unbounded sets (values of F) since $\mathbf{d}(s)$ is a linear bounded invertible operator on \mathbb{B} . In [58] it was introduced the relaxed set-theoretic condition of homogeneity of the operator F given by the inclusion

$$e^{\nu s} \mathbf{d}(s)F(u) \subset F(\mathbf{d}(s)u), \quad s \in \mathbb{R}, \quad u \in \mathcal{D}(F). \quad (2.14)$$

Evolution equations with \mathbf{d} -homogeneous operators were studied in [60] and [56]. Below we basically use the next property: *if a uniformly asymptotically stable evolution system is \mathbf{d} -homogeneous with negative degree then it is finite-time stable.*

Homogeneous Extension

Homogeneous extension (as well as homogeneous approximation) allows some homogeneity-based methods to be applied for non-homogeneous systems.

Definition 2.10. (Homogeneous extension) A set-valued \mathbf{d} -homogeneous operator $F: \mathcal{D}(F) \subset \mathbb{B} \rightrightarrows \mathbb{B}$ is said to be \mathbf{d} -homogeneous extension of an operator $f: \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}$ if $f(u) \in F(u)$ for any $u \in \mathcal{D}(f) \subset \mathcal{D}(F)$.

Set-valued homogeneous extensions appear, for example, as a result of Filippov regularization procedure applied to discontinuous vector field [48], [41]. It also allows us to simplify robust stability analysis of uncertain non-linear affine system

$$s^{(n)} = a + b\xi,$$

$s \in \mathbb{R}$, $|a| < C$, $0 < b_{\min} < b < b_{\max}$ provided that the extended differential inclusion

$$s^{(n)} \in [-C, C] + [b_{\min}, b_{\max}]\xi$$

is homogeneous, where $\xi = \xi(s, \dot{s}, \dots, s^{(n-1)})$ is a homogeneous feedback law [48], [41].

Below we deal with the following homogeneous extension

$$F(u) = \bigcup_{s \in \mathbb{R}: \mathbf{d}(s)u \in \mathcal{D}(f)} \{e^{-\nu s} \mathbf{d}(-s) f(\mathbf{d}(s)u)\}, \quad (2.15)$$

having the domain $\mathcal{D}(F) = \bigcup_{s \in \mathbb{R}} \mathbf{d}(s) \mathcal{D}(f)$.

By construction the set-valued operator F is \mathbf{d} -homogeneous of degree ν . It satisfies (2.11) in the set-theoretic sense (see, Remark 2.1).

Formally, such a set-valued homogeneous extension is defined for any fixed $\nu \in \mathbb{R}$. For example, if $f(u) = -(1 + 0.5 \sin(u))u$ with $u \in \mathbb{R}$ then $F(u) = -u \cdot (0, +\infty)$ for $\nu \neq 0$ and $F(u) = -u \cdot (\frac{1}{2}, \frac{3}{2})$ for $\nu = 0$. The issue of an appropriate selection of the parameter ν can be solved for a concrete f .

It is worth stressing that a properly defined homogeneous extension remains informative and useful even when a homogeneous approximation [79], [4] simply does not exist.

Example 2.10. Let $\mathbb{B} = \mathbb{R}$ and $f(x) = (\cos(1/x) - 2)x^{1/3}$, $x \in \mathbb{R}$, $\mathbf{d}(s) = e^s$. Obviously, a homogeneous approximation of the function f at zero (see, [4]) does not exist, but the formula (2.15) gives

$$F(x) = [-3, -1]x^{1/3}.$$

and the stability of the original system can be easily analysed using the homogeneous extension.

Remark 2.2. Below we utilize the homogeneous extension for Lyapunov analysis of a well-posed non-homogeneous evolution system

$$\dot{u}(t) = f(u(t))$$

having classical (or strong) solutions $u(t) \in D(f) \subset \mathbb{H}$. Notice that in the latter case we do not need to prove existence of solution of the extended evolution inclusion

$$\dot{u}(t) \in F(u(t))$$

but we simply use the fact $f(u(t)) \in F(u(t))$ in order to estimate the time derivative of the Lyapunov-like functional $V : \mathcal{D}(f) \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}_+$ defined as

$$V(u) = \|u\|_{\mathbf{d}}.$$

Indeed, if $\mathcal{D}(f) \subset \mathcal{D}(G_{\mathbf{d}})$ and \mathbf{d} is a strictly monotone dilation then the formula (2.10) gives

$$\begin{aligned} \dot{V}(u) &= (D\|u\|_{\mathbf{d}})(f(u)) = \frac{\langle \mathbf{d}(-\ln\|u\|_{\mathbf{d}})f(u), \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u, \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle} \|u\|_{\mathbf{d}} \\ &\leq \sup_{y \in F(u)} \frac{\langle \mathbf{d}(-\ln\|u\|_{\mathbf{d}})y, \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u, \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle} \|u\|_{\mathbf{d}}. \end{aligned}$$

for $u \in \mathcal{D}(f) \setminus \{0\}$. Since F is \mathbf{d} -homogeneous operator then $e^{\nu s} \mathbf{d}(s)F(u) \subset F(\mathbf{d}(s)u)$ for any $s \in \mathbb{R}$ and

$$\dot{V}(u) \leq \|u\|_{\mathbf{d}}^{1+\nu} \sup_{v \in F(\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u)} \frac{\langle v, \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u, \mathbf{d}(-\ln\|u\|_{\mathbf{d}})u \rangle}.$$

Using $\|\mathbf{d}(-\ln\|u\|_{\mathbf{d}})u\| = 1$ we conclude that the inequality

$$\alpha \langle G_{\mathbf{d}}z, z \rangle + \sup_{v \in F(z)} \langle v, z \rangle \leq 0, \quad \forall z \in \mathcal{D}(f) : \|z\| = 1 \quad (2.16)$$

implies

$$\dot{V}(u) \leq -\alpha V^{1+\nu}(u), \quad \alpha > 0, \quad u \in \mathcal{D}(f) \setminus \{0\}. \quad (2.17)$$

In the latter case the functional V decays along any classical solution $u(t) \in \mathcal{D}(f)$ and vanishes in a finite time provided that $\nu < 0$.

Example 2.11. The above considerations may remain consistent for the case of unbounded sets $F(u), u \in \mathcal{D}(f)$. For example, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$f(u) = -(1 + |u|)\sqrt{|u|} \operatorname{sign}(|u|), \quad u \in \mathbb{R}$$

and $\mathbf{d}(s) = e^s, \nu = -1/2$, then the formula (2.15) gives the set-valued \mathbf{d} -homogeneous extension with

$$F(u) = -\sqrt{|u|} \operatorname{sign}(u) \cdot (1, +\infty).$$

The differential equation $\dot{x}(t) = f(x(t))$, obviously, has continuous right-hand side so its solutions are well-defined (at least locally). For the uniform dilation $\mathbf{d}(s) = e^s$ one has $G_{\mathbf{d}} = 1$ and $V(u) = \|u\|_{\mathbf{d}} = |u|$. Hence, from (2.16) and (2.17) we derive $\dot{V} \leq -V^{1/2}$ since $z^2 + \sup_{y \in F(z)} yz \leq 0$ for all $z \in \mathbb{R} : |z| = 1$. We refer the reader to [61], [62] for more details about application of these ideas in the finite-dimensional case.

Homogeneous Evolution Equations

Let us consider the following nonlinear evolution equation

$$\dot{x} = f(x) := Ax + g(x), \quad (2.18)$$

with the initial condition

$$x(0) = x_0 \in \mathbb{B},$$

where $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ is a linear densely defined closed operator, which generates a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} ; $g : \mathcal{D}(g) \subset \mathbb{B} \rightarrow \mathbb{B}$ is a nonlinear mapping, such that $\mathcal{D}(g) \subset \mathcal{D}(A)$. We refer the reader, for example, to [50] for more details about existence of solutions for the evolution equations like (2.18). Notice that for $\mathbb{B} = \mathbb{R}^n$ and $A = \mathbf{0}$ the latter equation defines a non-linear ODE.

In this chapter we deal with the *classical solutions*¹ to (2.18) and the so-called *mild solutions* satisfying

$$x_{x_0}(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)g(x_{x_0}(s)) ds, \quad (2.19)$$

where the integral is understood in the sense of Brochner (see e.g. [20, page 187]). The next results can be proven for both classical and mild solutions.

Theorem 2.1. [60] *Let the nonlinear operator $f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}$ be \mathbf{d} -homogeneous of degree $\nu \in \mathbb{R}$ and let $x_{x_0} : [0, T) \rightarrow \mathbb{B}$ be a solution to (2.18) with the initial condition $x(0) = \mathbf{d}(s)x_0$ then*

$$x_{\mathbf{d}(s)x_0}(t) = \mathbf{d}(s)x_{x_0}(e^{\nu s}t), \quad s \in \mathbb{R}, \quad t \in [0, e^{-\nu s}T),$$

is a solution to (2.18) with the initial condition $x(0) = \mathbf{d}(s)x_0$.

This simple result has a lot of corollaries useful for analysis and design of homogeneous control systems [78], [79], [66], [9]. Below we use the next one.

Corollary 2.3. [60] *Let the origin of the system (2.18) be globally uniformly asymptotically stable and the operator $f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}$ be \mathbf{d} -homogeneous of degree $\nu < 0$, then the origin is **globally uniformly finite-time stable**, i.e. there exists $T(x_0) : \mathbb{B} \rightarrow \mathbb{R}_+$ such that any solution x_{x_0} of (2.18) with $x_{x_0}(0) = x_0$ satisfies $x_{x_0}(t) = 0$ for all $t \geq T(x_0)$.*

This corollary introduces a general way for finite-time control and observation, namely, to design a stable homogeneous system with negative degree.

¹A function $x_{x_0} : [0, T) \rightarrow \mathbb{B}$ is a classical solution to (2.18) if $x_{x_0}(0) \in \mathcal{D}(A)$, $x_{x_0}(t) \in \mathcal{D}(A)$ and $\dot{x}_{x_0}(t) = g(x_{x_0}(t))$ for all $t \in (0, T)$.

2.3 Control Design and State Estimation Using Linear Dilations

2.3.1 Homogeneous Stabilization in a Hilbert Space

Let us consider the following control system

$$\dot{u}(t) = Au(t) + B\xi(u(t)), \quad t > 0 \quad (2.20)$$

$$u(0) = u_0 \in \mathcal{D}(A), \quad (2.21)$$

where $A : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is a (possibly unbounded) closed linear operator with the domain $\mathcal{D}(A)$ dense in \mathbb{H} , $B : \mathbf{X} \rightarrow \mathbb{H}$ is a linear bounded operator, \mathbf{X} is a real Banach space, \mathbb{H} is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, $u(t)$ is the system state, $\xi : \mathbb{H} \rightarrow \mathbf{X}$ is a (*locally or globally*) bounded feedback control.

It is well known that algorithms for a linear feedback design $\xi = Ku$ with $K : \mathbb{H} \rightarrow \mathbf{X}$ for the systems (2.20) are developed for both finite-time dimensional (see e.g. [75]) and infinite dimensional (see e.g. [73], [12]) cases. One of the most popular technique is based on linear matrix inequalities [11]. The closed-loop system is exponentially stable in the linear case.

Our aim is to design a control which steers any solution of the closed-loop system to zero in a finite time. The control design procedure is assumed to be based on the homogeneity with negative degree.

The problem of finite-time stabilization is studied since 1960 [24], where some finite-time control algorithms have been derived as solutions to certain optimization problems. In particular, the well-known time-optimal feedback control is a finite-time controller (see, e.g. [13]). Lyapunov function method for finite-time stability analysis is introduced in [68]. Finite-time controllers have been also studied in [27], [74], [64], [9], [41], [48].

The finite-time control of PDEs is also the topic of intensive research [38], [15, 16, 30, 51]. It is related with controllability analysis of evolution systems [73] as well as with sliding mode control method [47], [53].

In [61], [62] finite-time control is designed for linear finite-dimensional plants using implicit Lyapunov function method and homogeneity. First, a linear stabilizing feedback and a corresponding quadratic Lyapunov function are designed. Next, homogeneous dilation is applied in order to derive both a finite-time stabilizing feedback and a corresponding implicit Lyapunov function. Below we follow the same idea and use the canonical norm $\|\cdot\|_{\mathbf{a}}$ defined by (2.8) as a Lyapunov function. Alternative approaches of finite-time stabilization of linear systems in \mathbb{R}^n can be found, for example, in [64], [37], [9].

Theorem 2.2. [58] If $A: \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is a generator of a strongly continuous semigroup on \mathbb{H} , $B: \mathbf{X} \rightarrow \mathbb{H}$ is a linear bounded operator and

- A) \mathbf{d}_1 is a uniformly continuous group of linear bounded operators in \mathbb{H} such that $\mathbf{d}_1(s)\mathcal{D}(A) \subset \mathcal{D}(A)$, $s \in \mathbb{R}$ and $\gamma\|z\|^2 \leq \langle G_{\mathbf{d}_1}z, z \rangle$, $\forall z \in \mathbb{H}$ with $\gamma \geq 0$,
- B) \mathbf{d} be a dilation on \mathbb{H} such that $\beta\|z\|^2 \leq \langle G_{\mathbf{d}}z, z \rangle$ for $z \in \mathcal{D}(G_{\mathbf{d}})$ and $\beta > \|G_{\mathbf{d}_1}\|$,
- C) $\exists K \in \mathcal{L}(\mathbb{H}, \mathbf{X})$ such that $BK(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and there exists a set $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(G_{\mathbf{d}})$ dense in \mathbb{H} such that

$$\langle (A + BK + \alpha G_{\mathbf{d}})z, z \rangle \leq 0, \quad \forall z \in \mathcal{D}$$

where $\alpha > 0$ is a positive number,

- D) the operator A be \mathbf{d} -homogeneous of degree $\mu < 0$ and

$$\mathbf{d}(s)BK\mathbf{d}_1(s) = BK\mathbf{d}(s), \quad s \in \mathbb{R},$$

then the feedback control $\xi: \mathbb{H} \rightarrow \mathbb{H}$ defined by

$$\xi(u) = \begin{cases} \|u\|_{\mathbf{d}}^{\mu} K \mathbf{d}_1(-\ln(\|u\|_{\mathbf{d}}))u & \text{if } u \neq 0, \\ 0 & \text{if } u = 0 \end{cases} \quad (2.22)$$

is locally Lipschitz continuous on $\mathbb{H} \setminus \{\mathbf{0}\}$, Fréchet differentiable on $\mathcal{D}(G_{\mathbf{d}}) \setminus \{\mathbf{0}\}$ and $B\xi(\mathcal{D}(A)) \subset \mathcal{D}(A)$.

Moreover, if $\exists M \geq 1$ such that $\|\mathbf{d}(s)\| \leq Me^{(\mu+\gamma)s}$ for $s > 0$ then $\|\xi(\cdot)\| \leq M\|K\|_{\mathcal{L}}$.

For any $u_0 \in \mathbb{H} \setminus \{\mathbf{0}\}$ the closed-loop system (2.20), (2.21), (2.22) has a mild solution $u: [0, T(u_0)) \rightarrow \mathbb{H}$ such that $\|u(t)\| \rightarrow 0$ as $t \rightarrow T(u_0)$ and

$$\frac{d}{dt}\|u(t)\|_{\mathbf{d}} \stackrel{a.e.}{\leq} -\alpha\|u(t)\|_{\mathbf{d}}^{1-\mu}, \quad t \in [0, T(u_0)), \quad (2.23)$$

and the settling time admits the estimate $T(u_0) \leq \frac{\|u_0\|_{\mathbf{d}}^{-\mu}}{-\alpha\mu}$.

Sketch of the proof. 1) The local Lipschitz continuity/differentiability of the feedback (2.22) follows from the local Lipschitz continuity/differentiability of $\|\cdot\|_{\mathbf{d}}$. The inclusion $B\xi(\mathcal{D}(A)) \subset \mathcal{D}(A)$ follows from Conditions C) and D). Indeed, $\mathbf{d}(s)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $BK\mathcal{D}(A) \subset \mathcal{D}(A)$ imply $\mathbf{d}(-s)BK\mathbf{d}(s)\mathcal{D}(A) \subset \mathcal{D}(A)$. According to Condition D) we have $BK\mathbf{d}_1(s) = \mathbf{d}(-s)BK\mathbf{d}(s)$.

2) Since ξ is locally Lipschitz continuous then the closed-loop system has a mild solution any $x_0 \in \mathbb{H} \setminus \{\mathbf{0}\}$ [50, Theorem 5.3.1, page 135]. The classical solutions can be obtained using [50, Theorem 5.4.6, page 146].

The finite-time extinction of classical solutions obviously follows from the inequality (2.23), which can be derived using conditions of Theorem 2.2. Indeed, Condition D) asks for \mathbf{d} -homogeneity of the operator

$$f := A + B\xi$$

with negative degree $\mu < 0$, so one has

$$e^{\mu s} \mathbf{d}(s) f(u) = f(\mathbf{d}(s)u) = (A + BK) \mathbf{d}(s)u$$

for $u \in \mathcal{D}(A)$. Hence, for any classical solution, Condition C) and the formula (2.10) yield (2.23):

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{\mathbf{d}} &= (D_u \|u(t)\|_{\mathbf{d}})(\dot{u}(t)) = \frac{\|u(t)\|_{\mathbf{d}} \langle \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) \dot{u}(t), \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t) \rangle}{\langle G_{\mathbf{d}} \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t), \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t) \rangle} \\ &= \frac{\|u(t)\|_{\mathbf{d}}^{1+\mu} \langle (A + BK) \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t), \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t) \rangle}{\langle G_{\mathbf{d}} \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t), \mathbf{d}(-\ln \|u(t)\|_{\mathbf{d}}) u(t) \rangle} \leq -\alpha \|u(t)\|_{\mathbf{d}}^{1+\mu}. \end{aligned}$$

One can be shown that the latter inequality holds for the mild solutions almost everywhere. The latter inequality immediately implies that $\|u(t)\|_{\mathbf{d}}$ vanishes to zero in a finite time $T(u_0) \leq \frac{\|u_0\|_{\mathbf{d}}^{-\mu}}{-\alpha\mu}$. ■

Example 2.12 (Distributed Finite-time Control for a Wave Equation on \mathbb{R}). *Let*

$$\mathbb{H} = H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$$

with an inner product to be defined below.

Let us consider a wave equation represented in the form (2.20) using the notations:

$$A = \begin{pmatrix} O & I \\ \frac{\partial^2}{\partial z^2} & O \end{pmatrix} : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H},$$

where

$$\mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R}) \times H^1(\mathbb{R}, \mathbb{R}),$$

$$O \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}), L^2(\mathbb{R}, \mathbb{R})) \quad \text{and} \quad I \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}), L^2(\mathbb{R}, \mathbb{R}))$$

are zero and identity operators, respectively, and

$$B = \begin{pmatrix} O \\ I \end{pmatrix} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{H}.$$

Let the inner product in \mathbb{H} be defined as follows

$$\langle x, \tilde{x} \rangle = \int_{\mathbb{R}} x^\top(z) P \tilde{x}(z) + p_{22} \frac{\partial x_1}{\partial z} \frac{\partial \tilde{x}_1}{\partial z} dz,$$

where

$$P := \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = Q^{-1}$$

with

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

being a positive definite solution of the system of linear matrix inequalities (LMIs) and equations:

$$\begin{pmatrix} \nu-1.5\mu & 1 \\ 0 & \nu-0.5\mu \end{pmatrix} Q + Q \begin{pmatrix} \nu-1.5\mu & 0 \\ 1 & \nu-0.5\mu \end{pmatrix} + y^\top b^\top + by = 0, \quad (2.24)$$

$$\begin{pmatrix} \nu-1.5\mu & 0 \\ 0 & \nu-0.5\mu \end{pmatrix} Q + Q \begin{pmatrix} \nu-1.5\mu & 0 \\ 0 & \nu-0.5\mu \end{pmatrix} \succ 0, \quad Q \succ 0, \quad (2.25)$$

where $\nu > \max\{0, 1.5\mu\}$, $b = (0 \ 1)^\top \in \mathbb{R}^2$, $y = (y_1 \ y_2) \in \mathbb{R}^{1 \times 2}$.

The operator A is a generator of a strongly continuous semigroup of linear bounded operators on \mathbb{H} (see e.g. [50, Section 7.4] for more details).

This system of LMIs is always feasible with respect to Q and y . Since from (2.24) we conclude

$$(\nu - 1.5\mu)q_{11} + q_{12} = 0$$

then taking into account $P = Q^{-1} \succ 0$ it is easy to see that

$$0 < (\nu - 1.5\mu)p_{22} = p_{12}.$$

Let the operator $K : \mathbb{H} \rightarrow L^1(\mathbb{R}, \mathbb{R})$ be defined as

$$Kx := yQ^{-1}x, \quad x \in \mathbb{H},$$

where the pair (Q, y) is a solution of the LMIs (2.24), (2.25).

A) Let the uniformly continuous semigroup \mathbf{d}_1 of linear bounded operators on \mathbb{H} be defined as follows

$$\mathbf{d}_1(s)x = \begin{pmatrix} e^{-\mu s} & 0 \\ 0 & 1 \end{pmatrix} x.$$

Its generator

$$G_{\mathbf{d}_1}x = \begin{pmatrix} -\mu & 0 \\ 0 & 0 \end{pmatrix} x,$$

obviously, satisfies the condition A) of Theorem 2.2 with $\gamma_{\min} = \min\{0, -\mu\}$ and $\gamma_{\max} = \max\{0, -\mu\} \geq 0$.

B) Let us introduce the dilation \mathbf{d} on \mathbb{H} as follows

$$(\mathbf{d}(s)x)(z) = \begin{pmatrix} e^{s(\nu-\mu)} & 0 \\ 0 & e^{s\nu} \end{pmatrix} x(e^{\mu s}z), \quad x \in \mathbb{H}, \quad z \in \mathbb{R}.$$

The dilation \mathbf{d} has the generator $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{H} \rightarrow \mathbb{H}$ defined as

$$G_{\mathbf{d}}x = \begin{pmatrix} \nu-\mu & 0 \\ 0 & \nu \end{pmatrix} x + \mu z \frac{\partial x}{\partial z},$$

where $\mathcal{D}(G_{\mathbf{d}}) = \{x \in \mathbb{H} : z \frac{\partial x}{\partial z} \in \mathbb{H}\}$.

According to Proposition 2.5 the dilation \mathbf{d} is strictly monotone on \mathbb{H} if there exists $\beta > 0$ such that $\langle G_{\mathbf{d}}x, x \rangle \geq \beta \|x\|^2$ for any $x \in C_c^\infty(\mathbb{R}, \mathbb{R}) \times C_c^\infty(\mathbb{R}, \mathbb{R})$, where $\|x\|^2 = \sqrt{\langle x, x \rangle}$. Using integration by parts we derive

$$\begin{aligned} \langle G_{\mathbf{d}}x, x \rangle &= \int_{\mathbb{R}} x^\top(z) P \left(\begin{pmatrix} \nu - \mu & 0 \\ 0 & \nu \end{pmatrix} x(z) + \mu z \frac{\partial x(s)}{\partial z} \right) dz + \\ & p_{22} \int_{\mathbb{R}} (\nu - \mu) \left(\frac{\partial x_1(z)}{\partial z} \right)^2 + \mu \frac{\partial x_1(z)}{\partial z} \frac{\partial}{\partial z} \left(z \frac{\partial x_1(z)}{\partial z} \right) dz = \\ & \int_{\mathbb{R}} x^\top(z) P \begin{pmatrix} \nu - \mu & 0 \\ 0 & \nu \end{pmatrix} x(z) dz + \frac{\mu}{2} \int_{\mathbb{R}} z \frac{\partial}{\partial z} \left(x^\top(z) P x(z) \right) dz + \\ & p_{22} \nu \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z} \right)^2 dz + \frac{\mu}{2} \int_{\mathbb{R}} z \frac{\partial}{\partial z} \left(\frac{\partial x_1(z)}{\partial z} \right)^2 dz = \\ & \int_{\mathbb{R}} x^\top(z) P \begin{pmatrix} \nu - 1.5\mu & 0 \\ 0 & \nu - 0.5\mu \end{pmatrix} x(z) + p_{22}(\nu - 0.5\mu) \left(\frac{\partial x_1(z)}{\partial z} \right)^2 dz \geq \beta \|x\|^2, \end{aligned}$$

where

$$\beta \geq \min \left\{ \lambda_{\min} \left(P^{\frac{1}{2}} \begin{pmatrix} \nu - \frac{3\mu}{2} & 0 \\ 0 & \nu - \frac{\mu}{2} \end{pmatrix} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} \begin{pmatrix} \nu - \frac{3\mu}{2} & 0 \\ 0 & \nu - \frac{\mu}{2} \end{pmatrix} P^{\frac{1}{2}} \right), \nu - \frac{\mu}{2} \right\}$$

The LMI (2.25) implies $\beta > 0$.

C) Given $x = (x_1, x_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}) \times C_c^\infty(\mathbb{R}, \mathbb{R})$ and $A_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ we have

$$\langle (A + BK + G_{\mathbf{d}})x, x \rangle = \langle (A_0 + BK + G_{\mathbf{d}})x, x \rangle + \left\langle \begin{pmatrix} 0 \\ \frac{\partial^2 x_1}{\partial z^2} \end{pmatrix}, x \right\rangle.$$

Using (2.24) we derive

$$\langle x, (A_0 + BK + G_{\mathbf{d}})x \rangle = p_{22} \int_{\mathbb{R}} \frac{\partial x_1(z)}{\partial z} \frac{\partial x_2(z)}{\partial z} dz + p_{22}(\nu - 0.5\mu) \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z} \right)^2 dz.$$

Using integration by parts we derive

$$\left\langle x, \begin{pmatrix} 0 \\ \frac{\partial^2 x_1}{\partial z^2} \end{pmatrix} \right\rangle = \int_{\mathbb{R}} -p_{12} \left(\frac{\partial x_1(z)}{\partial z} \right)^2 - p_{22} \frac{\partial x_1(z)}{\partial z} \frac{\partial x_2(z)}{\partial z} dz,$$

Therefore, taking into account the identity $0 < (\nu - 1.5\mu)p_{22} = p_{12}$ we conclude

$$\langle x, (A + BK + G_{\mathbf{d}})x \rangle = \mu p_{22} \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z} \right) dz.$$

For $\mu \leq 0$ we, obviously, have $\langle x, (A + BK + G_{\mathbf{d}})x \rangle \leq 0$. On the other hand, if $\nu \geq 2.5\mu > 0$ then using the representation for $\langle x, G_{\mathbf{d}}x \rangle$ we derive

$$\langle x, (A + BK + 0.5G_{\mathbf{d}})x \rangle \leq -\frac{p_{22}(\nu - 2.5\mu)}{2} \int_{\mathbb{R}} \left(\frac{\partial x_1(z)}{\partial z} \right) dz \leq 0.$$

Therefore, the condition C) of Theorem 2.2 holds for any $\mu \in \mathbb{R}$ and any $\nu \geq \max\{0, 2.5\mu\}$.

D) Finally, $\mathbf{d}(s)BK\mathbf{d}_1(s) = BK\mathbf{d}(s)$ for any $s \in \mathbb{R}$ and the operator A is \mathbf{d} -homogeneous of the degree μ . Indeed,

$$(\mathbf{A}\mathbf{d}(s)x)(z) = A \begin{pmatrix} e^{(\nu-\mu)s}x_1(e^{\mu s}z) \\ e^{\nu s}x_2(e^{\mu s}z) \end{pmatrix} = \begin{pmatrix} e^{\nu s}x_2(e^{\mu s}z) \\ e^{(\nu+\mu)s} \frac{\partial^2 x_1(q)}{\partial q^2} \Big|_{q=e^{\mu s}z} \end{pmatrix} = e^{\mu s}(\mathbf{d}(s)Ax)(z)$$

and the condition D) of Theorem 2.2 is fulfilled.

Therefore, the homogeneous feedback control of the form (2.22) steers all trajectories of the wave system to the origin in a finite time if $\mu < 0$.

In the general case, the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is defined implicitly, see (2.8). For $\mu = -1$, $\nu = \frac{1}{2}$ the canonical homogeneous norm can be found as a unique positive solution of the quartic equation

$$V^4 = aV^2 + bV + c$$

with $V = e^{sx} = \|x\|_{\mathbf{d}}$,

$$a = p_{22} \int_{\mathbb{R}} x_2^2(z) + (x_1'(z))^2 dz, \quad b = 2p_{12} \int_{\mathbb{R}} x_1(z)x_2(z) dz, \quad c = p_{11} \int_{\mathbb{R}} x_1^2(z) dz.$$

In this case $\|x\|_{\mathbf{d}}$ can be computed using Ferrari formulas, i.e., the homogeneous finite-time stabilizing feedback (2.22) for the wave equation admits an explicit representation.

Non-homogeneous operators allowing homogeneous extensions

The above examples introduce dilations for systems on unbounded domains. If the domain is bounded then the mentioned dilations are not applicable and the previous kind of operators on bounded domains may become non-homogeneous. If the condition D) in Theorem 2.2 does not hold (*non-homogeneous case*) then the finite-time control can be still be designed for an operator A allowing a homogeneous extension.

Corollary 2.4. [58] *Theorem 2.2 remains true if D) is replaced with D*) there exists a set-valued operator $F : \mathcal{D}(F) \subset \mathbb{H} \rightrightarrows \mathbb{H}$ such that*

$$Au + B\xi(u) \in F(u) \text{ for } u \in \mathcal{D}(A) \subset \mathcal{D}(F),$$

F is \mathbf{d} -homogeneous of degree $-\mu < 0$ and there exists $D \subset \mathcal{D}(F) \cap \mathcal{D}(G_{\mathbf{d}})$ dense in $\mathcal{D}(A)$,

$$\alpha \langle G_{\mathbf{d}}z, z \rangle + \sup_{y \in F(z)} \langle y, z \rangle \leq 0, \quad z \in \mathcal{D} : \|z\| = 1$$

for some $\alpha > 0$.

The operator F is the *homogeneous extension* of the operator $A + B\xi$ (see Definition 2.10). Recall that we do not need to study an evolution system $\dot{u} \in F(u)$. The existence of classical solutions of the closed-loop system (2.20) comes from Conditions A)-C) of Theorem 2.2. The inclusion $Au + B\xi(u) \in F(u)$ is utilized only for analysis of finite-time extinction of these solutions (see Remark 2.2 for more details).

This corollary allows any mild solution of the closed-loop system to be prolonged to zero by continuity $u(t) = \mathbf{0}$ for $t \geq T(u_0)$ since the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the closed-loop system implying uniqueness and finite-time stability of the zero solution.

Finite-time stabilization by means of switched linear feedback

Realization of the control law (2.22) needs calculation of the homogeneous norm $\|u\|_{\mathbf{d}}$, which is defined implicitly by the functional equation (2.8). Sometimes the homogeneous norm can be calculated explicitly (see, [61], [56] and the above example). The control law (2.22) can also be applied in a sampled way by means of replacing $\|u(t)\|_{\mathbf{d}}$ in (2.22) with $\|u(t_j)\|_{\mathbf{d}}$ for $t \in [t_j, t_{j+1})$. The value $\|u(t_j)\|_{\mathbf{d}}$ can be found by means of an appropriate numerical procedure (see, e.g. [61]). The control (2.31) becomes linear on each interval of time $[t_j, t_{j+1})$.

Let us consider the following switched linear feedback

$$\xi_{\sigma(t)}(u(t)) = r_{\sigma(t)}^{-\mu} K \mathbf{d}_1(-\ln(r_{\sigma(t)}))u(t), \quad (2.26)$$

with state dependent switching function σ such that $\sigma(0) \in \mathbb{Z}$,

$$\begin{aligned} \sigma(t) &= S(\sigma(t^-), u(t^-)), \quad S : \mathbb{Z} \times \mathbb{H} \rightarrow \mathbb{Z}, \\ S(\sigma, u) &:= \begin{cases} i-1 & \text{if } \sigma=i \text{ and } \|u\|_{\mathbf{d}} \geq r_{i-1}, \\ i & \text{if } \sigma=i \text{ and } r_{i+1} \leq \|u\|_{\mathbf{d}} \leq r_{i-1}, \\ i+1 & \text{if } \sigma=i \text{ and } r_{i+1} \leq \|u\|_{\mathbf{d}}, \end{cases} \end{aligned} \quad (2.27)$$

where $r_i \in \mathbb{R}_+ : r_{i+1} < r_i, i \in \mathbb{Z}$ and $\Omega = \bigcup_{i \in \mathbb{Z}} \{r_i\}$. The switching variable σ has its own evolution equation given by (2.27) so the pair $(u(t), \sigma(t))$ can be treated as an augmented state of the closed-loop switched system having a strong solution². We refer the reader to [43] for more details about switched systems.

Corollary 2.5 (On switched linear finite-time feedback law). [58] *If all conditions of Theorem 2.2 hold and the control is given by (2.26), (2.27) with*

$$r_0 = 1, \quad r_{i+1} = e^{-q_i} r_i, \quad i \in \mathbb{Z}$$

²A continuous function $u : [0, T) \rightarrow \mathbb{H}$ which is differentiable almost everywhere on $[0, T)$ such that $\dot{u} \in L^1((0, T), \mathbb{H})$ is called strong solution to the initial value problem (2.20), (2.21), (2.26), (2.27) if $u(t) \in \mathcal{D}(A)$ for $t \in (0, T)$, $u(0) = u_0$ and there exists a (locally) piecewise constant function $\sigma : [0, T) \rightarrow \mathbb{Z}$ such that σ satisfies (2.27) on $(0, T)$ and $\dot{u}(t) = Au(t) + B\xi_{\sigma(t)}(u(t))$ almost everywhere on $(0, T)$.

and $q_i > 0$ such that

$$\sum_{i=0}^{+\infty} q_i = +\infty \quad \text{and} \quad \lim_{i \rightarrow +\infty} q_{i+1} e^{-\mu q_i} / q_i < 1,$$

then

- for $u_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ the closed-loop system (2.20), (2.21), (2.26), (2.27) has a unique strong solution and

$$\|u(t)\| \rightarrow 0 \text{ as } t \rightarrow T(u_0),$$

where the settling time functional $T : \mathbb{H} \rightarrow \mathbb{R}$ admits the estimate

$$T(u_0) \leq \alpha^{-1} \sum_{j=i_0}^{\infty} r_j^\mu q_j < +\infty$$

provided that

$$\|u_0\|_a \in (r_{i_0+1}, r_{i_0});$$

- the time instants $t_j : \|u(t_j)\|_a = r_j$ are isolated and the set $\{t_j\}_{j=i_0+1}^{+\infty}$ has the unique accumulation point $T(u_0)$;
- the control is a linear continuous feedback on each (t_j, t_{j+1}) and $\|\xi_{\sigma(t)}(t)\| \leq M \|K\|_{\mathcal{L}}$ if $\|\mathbf{d}(s)\|_{\mathbb{H}}^{\mathbb{H}} \leq M e^{(\mu+\gamma)s}$ for $s > 0$.

Existence, uniqueness and finite-time extinction of mild solutions to the closed-loop system (2.20), (2.21), (2.26), (2.27) with $u_0 \in \mathbb{H} \setminus \{\mathbf{0}\}$ can be shown repeating the proof of Corollary ??.

Corollary 2.5 proposes a scheme for homogeneity-based finite-time control design in the form of switched linear feedback. Its applicability to boundary control case looks like a promising research direction, especially, in the context of the papers [16], [57], where a finite-time linear switched boundary control has been designed for heat system using the back-stepping approach [39].

Example 2.13 (Distributed Finite-time Control for Heat System on a Segment). *To demonstrate a possible way of application of Corollary 2.4, we design a distributed finite-time control for the following heat system on $[0, 1]$:*

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \phi(z)u(t, z), \quad x(t, 0) = x(t, 1) = 0, \quad x(0, z) = x_0(z),$$

where x is the system state, u is the distributed control, $\phi : [0, 1] \rightarrow \mathbb{R}_+$ is a continuous function such that

$$cz^2 \leq \phi(z) \quad \text{for } z \in [0, 1], \quad (2.28)$$

for some $c > 0$. The function ϕ represents possibly non-uniform feedback gain on $[0, 1]$. The design procedure is realized in several steps

- First, we extend the considered system virtually to \mathbb{R}_+ (with the zero operator on $(1, +\infty)$);
- Next, we design a feedback control using conditions A)-C) of Theorem 2.2) and Condition D^*) of Corollary 2.4.
- Finally, we reduce the extended closed-loop system to the original one assuming that the initial condition of the extended system has a support only in $(0, 1)$.

Homogeneous finite-time control design on \mathbb{R}_+ . Let $\mathbb{H} = H^0(\mathbb{R}_+, \mathbb{R}) = L^2(\mathbb{R}_+, \mathbb{R})$ and $x_{(a,b)}$ denotes a restriction of a function $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ to the interval (a, b) . For any $x \in \mathbb{H}$ we have $x_{(0,1)} \in H^0((0, 1), \mathbb{R})$ and $x_{(1,+\infty)} \in H^0((1, +\infty), \mathbb{R})$.

Let the operator $A : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ be defined as the connection of the Laplace operator with Dirichlet boundary conditions on $[0, 1]$ and the zero operator on $[1, +\infty)$, i.e. $(Ax)(z) = \frac{\partial^2 x(z)}{\partial z^2}$ if $z \in (0, 1)$ and $(Ax)(z) = 0$ if $z \geq 1$ for

$$x \in \mathcal{D}(A) := \{x \in \mathbb{H} : x_{(0,1)} \in H_0^1((0, 1), \mathbb{R}) \cap H^2((0, 1), \mathbb{R})\}.$$

The operator A is closed and densely-defined. It generates the strongly continuous semigroup Φ of linear bounded operators on the Hilbert space \mathbb{H} :

$$(\Phi(t)u)(x) = \begin{cases} 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) u(y) dy & \text{if } x \in (0, 1), \\ u(x) & \text{if } x \in [1, +\infty). \end{cases}$$

Let us consider the equation (2.20) with A given above and $B : \mathbb{H} \rightarrow \mathbb{H}$ defined as follows $(Bx)(z) = \phi(z)x(z)$ if $z \in (0, 1)$ and $(Bx)(z) = \phi(1)x(z)$ if $z \geq 1$ for $x \in \mathbb{H}$.

Let us show that conditions A)-C) of Theorem 2.2) and the condition D^*) of Corollary 2.4) hold for $K = -\tilde{k}I \in \mathcal{L}(\mathbb{H})$ with a sufficiently large scalar $\tilde{k} > 0$.

A) Let a uniformly continuous group \mathbf{d}_1 be defined as follows

$$(\mathbf{d}_1(s)x)(z) = e^{-\mu s} x(z) \quad \text{if } z \in (0, 1) \quad \text{and} \quad (\mathbf{d}_1(s)x)(z) = x(z) \quad \text{if } z \geq 1,$$

where $\mu < 0$. Obviously, $\gamma_{\min} = 0$ and $\gamma_{\max} = -\mu$, where γ_{\min} and γ_{\max} are defined in Theorem 2.2).

B) Let the dilation $\mathbf{d}(s)$ in \mathbb{H} be defined as

$$(\mathbf{d}(s)x)(z) = e^s x(e^{0.5\mu s} z).$$

It has the generator $G_{\mathbf{d}}x = x + 0.5\mu z \frac{\partial x}{\partial z}$ with the domain $\mathcal{D}(G_{\mathbf{d}}) = \{x \in \mathbb{H} : x + 0.5\mu z \frac{\partial x}{\partial z} \in \mathbb{H}\}$. Obviously,

$$\|\mathbf{d}(s)x\| = \sqrt{\int_{\mathbb{R}_+} e^{2s} x^2(e^{0.5\mu s} z) dz} = e^{(1-0.25\mu)s} \|x\|$$

and $(1 - 0.25\mu)\|x\|^2 = \langle G_d x, x \rangle$ if $x \in \mathcal{D}(G_d)$, where, as before, $\|\cdot\|$ denotes the norm in \mathbb{H} . Hence, we derive

$$\|x\|_d = \|x\|^{\frac{1}{1-0.25\mu}}$$

and the formula (2.22) gives

$$u(x) = -\tilde{k}\|x\|_d^\mu \mathbf{d}_1(-\ln\|x\|_d)x = -\tilde{k}\|x\|^{\frac{2\mu}{1-0.25\mu}}x.$$

Theorem 2.2, guarantees that this feedback control is locally Lipschitz continuous on $\mathbb{H} \setminus \{\mathbf{0}\}$. Moreover, it is continuous at $\mathbf{0}$ if $-4/7 < \mu < 0$ and globally bounded if $\mu = -4/7$.

C)-D*) Let us consider the homogeneous extension (2.15):

$$F(x) = \bigcup_{s \in \mathbb{R}: \mathbf{d}(s)x \in \mathcal{D}(A)} \{A_s x + g_s(x)\},$$

where $A_s = e^{-\mu s} \mathbf{d}(-s) A \mathbf{d}(s)$ and $g_s(x) = e^{-\mu s} \mathbf{d}(-s) B u(\mathbf{d}(s)x)$. It is easy to see that $(A_s x)(z) = \frac{\partial^2 x(z)}{\partial z^2}$ if $z \in (0, e^{0.5\mu s})$ and $(A_s x)(z) = 0$ if $z \geq e^{0.5\mu s}$,

$$(g_s(x))(z) = -\tilde{k} \begin{cases} \phi(e^{-0.5\mu s} z) e^{\mu s} \frac{x(z)}{\|x\|_d^{2\mu}}, & z \in (0, e^{0.5\mu s}), \\ \phi(1) \frac{x(z)}{\|x\|_d^\mu}, & z \geq e^{0.5\mu s}. \end{cases}$$

The set-valued operator $F : \mathcal{D}(F) \subset \mathbb{H} \rightrightarrows \mathbb{H}$ is \mathbf{d} -homogeneous of degree $\mu < 0$ and $Ax + Bu(x) \in F(x)$ for any $x \in \mathcal{D}(A)$, where $\mathcal{D}(F) = \bigcup_{s \in \mathbb{R}} \mathbf{d}(s) \mathcal{D}(A)$.

Let us denote $q_s = e^{0.5\mu s}$ and consider the set

$$\tilde{\mathcal{D}}_s = \left\{ x \in \mathbb{H} : \begin{array}{l} x_{(0, q_s)} \in C_c^\infty((0, q_s), \mathbb{R}), \\ x_{(q_s, +\infty)} \in C_c^\infty((q_s, +\infty), \mathbb{R}). \end{array} \right\}$$

Obviously, $\tilde{\mathcal{D}}_0 \subset \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$ and $\mathbf{d}(s)x \in \tilde{\mathcal{D}}_0 \Rightarrow x \in \tilde{\mathcal{D}}_s$.

Let us denote $\langle x, y \rangle_{(a,b)} = \int_a^b x(z)y(z)dz$. Using integration by parts for any $s \in \mathbb{R}$ and for any $x \in \tilde{\mathcal{D}}_s$ we derive

$$\left\langle z \frac{\partial x}{\partial z}, x \right\rangle_{(0, q_s)} = \frac{-\langle x, x \rangle_{(0, q_s)}}{2}$$

and

$$\left\langle z \frac{\partial x}{\partial z}, x \right\rangle_{(q_s, +\infty)} = \frac{-\langle x, x \rangle_{(q_s, +\infty)}}{2}.$$

Due to (2.28) one has

$$\phi(e^{-0.5\mu s} z) e^{\mu s} \geq cz^2 \quad \text{if } z \in [0, q_s]$$

and

$$\langle \phi(e^{-0.5\mu s} z) e^{\mu s} x, x \rangle_{(0, e^{0.5\mu s})} \geq c \langle zx, zx \rangle_{(0, q_s)}.$$

Therefore, for any $x \in \tilde{\mathcal{D}}_s, \|x\| = 1$ we obtain

$$\begin{aligned} & \langle (A_s x + g_s(x) + \rho G_d)x, x \rangle \leq \\ & -\langle \frac{\partial x}{\partial z}, \frac{\partial x}{\partial z} \rangle_{(0, q_s)} - 2\rho(1 - 0.25\mu) \langle z \frac{\partial x}{\partial z}, x \rangle_{(0, q_s)} - \tilde{k}c \langle zx, zx \rangle_{(0, q_s)} \\ & - (\tilde{k}\phi(1) - \rho(1 - 0.25\mu)) \langle x, x \rangle_{(q_s, +\infty)} \leq 0 \end{aligned}$$

provided that $\tilde{k} \geq \max \left\{ \frac{\rho(1-0.25\mu)}{\phi(1)}, \frac{\rho^2(1-0.25\mu)^2}{c} \right\}$.

The condition D^* of Corollary 2.4 holds for

$$\mathcal{D} = \bigcup_{s \in \mathbb{R}} \tilde{\mathcal{D}}_s$$

since

- \mathcal{D} is a \mathbf{d} -homogeneous cone by construction;
- \mathcal{D} contains the set $\tilde{\mathcal{D}}_0$ dense in $\mathcal{D}(A)$;
- $x \in \mathcal{D} : \mathbf{d}(s)x \in \mathcal{D}(A) \Rightarrow x \in \tilde{\mathcal{D}}_s$.

Therefore, the feedback law (2.22) steers any solution of the extended evolution system (2.20) to zero in a finite time $T \leq (-\rho\mu)^{-1} \|x_0\|_{\mathbf{d}}^{-\mu}$, provided that $\mu \in [-4/7, 0)$.

Restriction of the finite-time control to the segment $[0, 1]$. If $\text{supp}(x) \subset [0, 1]$ and $x \in \mathbb{H} \setminus \{0\}$ then $\text{supp}(\Phi(t)x) \subset [0, 1]$, $\text{supp}(u(x)) \subset [0, 1]$ and $\|x\| = \sqrt{\int_0^1 x^2(z) dz}$, where $\text{supp}(x)$ denotes the support of a function $x \in \mathcal{D}(A)$. Therefore, any solution of the closed-loop system (2.20), (2.22) with initial condition

$$x_0 \in \{x \in \mathcal{D}(A) : \text{supp}(x) \subset [0, 1]\}$$

coincides on $[0, 1]$ with the solution to the equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \phi(z) \frac{-\tilde{k}}{\|x\|^{\frac{-2\mu}{1-0.25\mu}}} x, \quad \begin{array}{l} x(t, 0) = x(t, 1) = 0, \\ x(0, z) = u_0(x), \quad z \in [0, 1]. \end{array}$$

Consequently, any solution of the latter heat system vanishes in a finite time

$$T(x_0) \leq (-\rho\mu)^{-1} \left(\int_0^1 x_0^2(z) dz \right)^{-0.5\mu/(1-0.25\mu)}.$$

Piecewise linear finite-time control on $[0, 1]$. Since $\|x\|_{\mathbf{d}} = \|x\|^{\frac{1}{1-0.25\mu}}$ then the parameters of Lemma 2.1 are $M = 1, \alpha = \beta = 1 - 0.25\mu$. Applying Corollary 2.5 for $q_i = \ln 2$ we derive $u(t) = -\lambda(t)x(t)$, where the switched gain $\lambda(t) =$

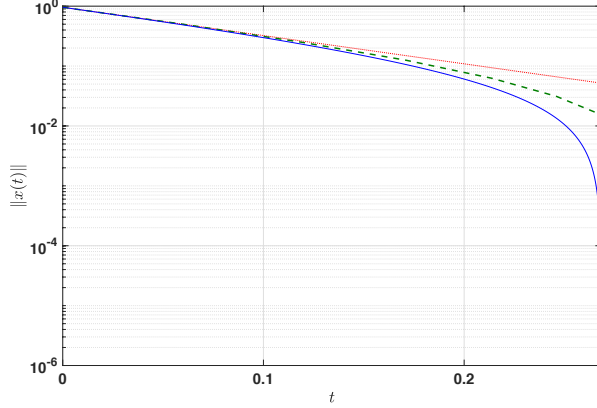


Figure 2.1: Evolution of L^2 -norm of the heat equation with a linear (red dotted line), switched linear (green dashed line) and a non-linear (blue solid line) homogeneous control (plotted in the logarithmic scale)

$\tilde{k}/r_{\sigma(t)}^{-2\mu/1-0.25\mu}$ with $r_i = 2^{-i}$ and $\sigma(t)$ defined by (2.27). Therefore, for any $x_0 \in H_0^1((0, 1), \mathbb{R}) \cap H^2((0, 1), \mathbb{R})$ the equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} - \lambda(t)\phi(z)x, \quad \begin{array}{l} x(t,0)=x(t,1)=0, \\ x(0,z)=u_0(z), \quad z \in [0,1]. \end{array} \quad (2.29)$$

has a unique strong solution which vanishes in a finite-time

$$T(x_0) \leq \frac{\ln 2}{\rho} \sum_{j=i_0}^{+\infty} 2^{\mu j}$$

for $\|x_0\|_d \in (r_{i_0+1}, r_{i_0}]$. Notice that in contrast to the linear switched feedback presented in [16], the settling time of our algorithm depends on the initial condition.

Numerical simulations. The simulation results of the system (2.29) (for $\phi(z) = z^2$) with homogeneous and switched linear finite-time controls ($\mu = -0.5, \tilde{k} = 1$) as well as with the static linear feedback ($\lambda = \tilde{k}$) are depicted at Fig. 2.1. The initial condition is given by $x_0(z) = 5.25z(1 - z)$. For numerical simulations, the system has been discretized by means of divided differences on a uniform grid with the step $h = 0.005$ for the space variable. The discretization with respect to time has been done using implicit scheme with the step size $\tau = h^2$. The norm $\|x\|$ is calculated by Simpson rule. Numerical simulations show that the heat system with homogeneous control has a fast decay (typical for finite-time stable systems) with the settling time $T(x_0) \leq 0.3$.

Example 2.14. Let us consider the linear control system in \mathbb{R}^n

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Let $x \rightarrow Ax$ be \mathbf{d} -homogeneous of degree $-\mu < 0$, i.e.

$$AG_{\mathbf{d}} = (G_{\mathbf{d}} - \mu I_n)A,$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is a generator of the dilation $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$. Let the Euclidean space \mathbb{R}^n be equipped with the inner product

$$\langle x, y \rangle = x^\top Py, \quad 0 \prec P \in \mathbb{R}^{n \times n}, \quad x, y \in \mathbb{R}^n.$$

and the dilation \mathbf{d} be monotone with respect to the norm $\|x\| = \sqrt{x^\top Px}$, i.e.

$$G_{\mathbf{d}}^\top P + PG_{\mathbf{d}} \succ 0.$$

If $G_{\mathbf{d}}B = \mu B$ and there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$(A + BK)^\top P + P(A + BK) + \alpha(G_{\mathbf{d}}^\top P + PG_{\mathbf{d}}) \preceq 0, \quad \alpha > 0,$$

then all conditions of Theorem 2.2 hold for $G_{\mathbf{d}_1} = G_{\mathbf{d}} - \mu I_n$, $\mathbf{d}_1(s) = e^{sG_{\mathbf{d}_1}}$ and the homogeneous globally bounded feedback

$$u(t) = K\mathbf{d}(-\ln \|x(t)\|_{\mathbf{d}})x(t),$$

stabilizes the origin of the closed-loop system in a finite time. If the pair $\{A, B\}$ is controllable then the obtained system of matrix inequalities is always feasible and the homogeneous feedback can always be designed (see, [62] for more details).

Other examples of an application of Theorem 2.2 for $\mathbb{H} = \mathbb{R}^n$ can be found in [61], [62] and in the next section.

2.3.2 Quadratic Stabilizability of Homogeneous Systems in \mathbb{R}^n

Problem Statement

Let $\mathbb{B} = \mathbb{R}^n$ and the model of a control system is given by

$$\dot{\sigma}(t) = g(\sigma(t), u(t)), \tag{2.30}$$

where $t \in \mathbb{R}_+$ is time variable, $\sigma(t) \in \mathbb{R}^n$ is the vector of state variables at the time instant t , $u(t) \in \mathbb{R}^m$ is the control input, $g \in C(\mathbb{R}^{n+m}, \mathbb{R}^n)$, $g(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We assume that the whole state vector σ can be measured on-line and utilized for control purposes. In this section we deal (mainly) with the problem of a dynamical feedback control design

$$\dot{u}(t) = k(\sigma(t), u(t)), \quad k \in C(\mathbb{R}^{n+m} \setminus \{\mathbf{0}\}, \mathbb{R}^m). \tag{2.31}$$

and we restrict ourselves with the class of the *homogeneous nonlinear systems*. The main goal is to derive a **necessary and sufficient** condition of (finite-time or fixed-time) stabilizability of the system (2.30) by means of a homogeneous dynamical feedback (2.31). In particular, we are interested in a (more or less) *constructive* stabilizability criteria allowing a unified scheme for control design. Obviously, under assumption

$$f = \begin{pmatrix} g \\ k \end{pmatrix} \in C(\mathbb{R}^{n+m} \setminus \{\mathbf{0}\}, \mathbb{R}^{n+m}), \quad (2.32)$$

the closed-loop system (2.30), (2.31) may have discontinuity only at the origin similarly to the unit control [74] or "quasi-continuous" control [41]. It is also worth stressing that sliding mode control method frequently uses dynamical feedbacks as a part of the control law (see, e.g. super-twisting algorithm [40]).

Let $\mathbb{F}_d(\mathbb{R}^n)$ (resp. $\mathbb{H}_d(\mathbb{R}^n)$) be the set of vector fields $\mathbb{R}^n \rightarrow \mathbb{R}^n$ (resp. functions $\mathbb{R}^n \rightarrow \mathbb{R}$) satisfying the identity (2.11) (resp. (2.12)), which are continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Let $\deg_{\mathbb{F}_d}(f)$ (resp. $\deg_{\mathbb{H}_d}(h)$) denote the homogeneity degree of $f \in \mathbb{F}_d(\mathbb{R}^n)$ (resp. $h \in \mathbb{H}_d(\mathbb{R}^n)$).

Quadratic Stability

The further constructions are essentially based on the theorem given below for the system

$$\dot{\xi} = f(\xi), \quad f \in \mathbb{F}_d(\mathbb{R}^n). \quad (2.33)$$

Theorem 2.3. [55] *The next five claims are equivalent*

- 1) *The origin of the system (2.33) is asymptotically stable.*
- 2) *There exists a Lyapunov function $V \in \mathbb{H}_d(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$;*
- 3) *The origin of the system*

$$\dot{z} = \|z\|^{1+\deg_{\mathbb{F}_d}(f)} \left(\frac{(I_n - G_d)z^\top z P}{z^\top P G_d z} + I_n \right) f \left(\frac{z}{\|z\|} \right) \quad (2.34)$$

is asymptotically stable, where $\|z\| = \sqrt{z^\top P z}$ with P satisfying (2.6).

- 4) *For any matrix $P \in \mathbb{R}^{n \times n}$ satisfying (2.6) there exists $\Psi \in \mathbb{F}_d(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$, $\deg_{\mathbb{F}_d}(\Psi) = 0$ such that Ψ is diffeomorphism on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, homeomorphism on \mathbb{R}^n , $\Psi(\mathbf{0}) = \mathbf{0}$ and*

$$\frac{\partial(\Psi^\top(\xi)P\Psi(\xi))}{\partial \xi} f(\xi) < 0 \quad \text{if} \quad \Psi^\top(\xi)P\Psi(\xi) = 1. \quad (2.35)$$

Moreover, $\|\Psi\|_{\mathbf{d}} \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$ is Lyapunov function to the system (2.33), where $\|\cdot\|_{\mathbf{d}}$ is the canonical homogeneous norm induced by $\|\xi\| = \sqrt{\xi^\top P \xi}$.

5) For any matrix $P \in \mathbb{R}^{n \times n}$ satisfying (2.6) there exists a map

$$\Xi \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^{n \times n})$$

such that

$$\det(\Xi(z)) \neq 0, \quad \frac{\partial \Xi(z)}{\partial z_i} z = 0, \quad \Xi(e^s z) = \Xi(z)$$

for $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$, $i = 1, \dots, n$ and

$$z^\top \Xi^\top(z) P \Xi(z) \left(\frac{(I_n - G_d) z z^\top P}{z^\top P G_d z} + I_n \right) f \left(\frac{z}{\sqrt{z^\top P z}} \right) < 0. \quad (2.36)$$

This theorem proves three important facts:

- Any homogeneous (with respect to linear dilations) system is homeomorphic on \mathbb{R}^n and diffeomorphic on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to a standard homogeneous one.
- Any asymptotically stable homogeneous system is homeomorphic on \mathbb{R}^n and diffeomorphic on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to a **quadratically stable** one. Indeed, making the change of variables $z = \Psi(\xi)$ we derive

$$\dot{z} = \tilde{f}(z),$$

where

$$\tilde{f}(z) = \left. \frac{\partial \Psi(\xi)}{\partial \xi} f(\xi) \right|_{\xi = \Psi^{-1}(z)},$$

but the criterion (2.35) implies that

$$z^\top P \dot{z} < 0 \quad \text{if } z^\top P z = 1,$$

so the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the latter system. Finally, the change of variable

$$x = \|z\|_{\mathbf{d}} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z$$

gives $\|z\|_{\mathbf{d}} = \|x\|$, so the transformed system

$$\dot{x} = \hat{f}(x)$$

is quadratically stable. The fact that the Lyapunov function for the homogeneous system has a level set diffeomorphic to the unit sphere, has been

noticed in [33]. The canonical homogeneous norm has allowed us to construct the required transformation in the simple explicit form

$$\Psi(\xi) = \mathbf{d} \left(\ln \frac{V^{\frac{1}{\deg_{\mathbb{H}_d}(V)}}(\xi)}{\|\xi\|_{\mathbf{d}}} \right) \xi.$$

- The formula (2.36) gives a more or less constructive stability criterion. *Since $\Xi(z) = \Xi(e^s z)$ for $z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$ then the map Ξ is constant along any straight line $\{e^s z : s \in \mathbb{R}\}$ if $z \neq \mathbf{0}$. In addition, the property $\frac{\partial \Xi(z)}{\partial z_i} z = 0$ motivates the conjecture that a quadratic Lyapunov function ($\Xi \equiv \text{const}$) always exists for asymptotically stable standard homogeneous system. Such assumption was also studied in [3]. However, in the view of [6, Proposition 5.2] this conjecture seems to be wrong. Therefore, the condition*

$$\begin{aligned} z^\top Q_0 \left(\frac{(I_n - G_d) z z^\top P}{z^\top P G_d z} + I_n \right) f \left(\frac{z}{\sqrt{z^\top P z}} \right) < 0 \text{ if } z \neq \mathbf{0}, \\ P G_d + G_d^\top P > 0, \quad P > 0, \quad Q_0 > 0 \end{aligned} \quad (2.37)$$

derived from (2.36) is just sufficient stability condition. For $Q_0 = P$ (or, equivalently, $\Xi = I_n$) we derive the more simplified stability condition

$$z^\top P f \left(\frac{z}{\sqrt{z^\top P z}} \right) < 0.$$

Finite-time Stabilization of a Linear System in \mathbb{R}^n

As an example to both Theorem 2.2 and Theorem 2.3 let us consider the system

$$\dot{\sigma} = A\sigma + Bu(\sigma), \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}, \quad (2.38)$$

$$u(\sigma) = \sum_{i=1}^n \frac{k_i \sigma_i}{N(\sigma)^{n-i+1}}, \quad \sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)^\top \in \mathbb{R}^n, \quad (2.39)$$

and the weighted dilation

$$\mathbf{d}(s) = e^{G_d s} \quad \text{with} \quad G_d = \begin{pmatrix} n & 0 & \dots & 0 & 0 \\ 0 & n-1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where $N : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous positive definite function that is \mathbf{d} -homogeneous of degree 1, i.e. $N(\sigma) > 0$ for $\sigma \neq 0$ and

$$N(\mathbf{d}(s)\sigma) = e^s N(\sigma), \quad \sigma \in \mathbb{R}^n, \quad s \in \mathbb{R}.$$

In the view of Definition 2.8 the function N is simply an arbitrary homogeneous norm and the control law can be rewritten in the form

$$u(\sigma) = K\mathbf{d}(-\ln N(\sigma))\sigma,$$

where

$$K = (k_1 \ k_2 \ \dots \ k_{n-1} \ k_n)^\top \in \mathbb{R}^n$$

is a vector of control parameters. Obviously, if N is a canonical homogeneous norm then the considered control law becomes the form (2.31).

The control $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is \mathbf{d} -homogeneous of degree zero:

$$\begin{aligned} u(\mathbf{d}(s)\sigma) &= K\mathbf{d}(-\ln N(\mathbf{d}(s)\sigma))\mathbf{d}(s)\sigma = K\mathbf{d}(-\ln(e^s N(\sigma)))\mathbf{d}(s)\sigma \\ &= K\mathbf{d}(-s - \ln N(\sigma))\mathbf{d}(s)\sigma = K\mathbf{d}(-\ln N(\sigma))\mathbf{d}(-s)\mathbf{d}(s)\sigma = u(\sigma), \end{aligned}$$

$\sigma \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$. Therefore, u is a version of the so-called **quasi-continuous homogeneous sliding mode control** [42]. Indeed, it is homogeneous of degree 0, continuous outside the origin and globally bounded:

$$\begin{aligned} \|Bu(\sigma)\| &\leq \|BK\|_{\mathcal{L}} \|\mathbf{d}(-\ln N(\sigma))\sigma\| \leq \|BK\|_{\mathcal{L}} \left\| \mathbf{d}\left(-\ln \frac{N(\sigma)}{\|\sigma\|_{\mathbf{d}}}\right) \right\|_{\mathcal{L}} \|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma\| \\ &= \|BK\|_{\mathcal{L}} \left\| \mathbf{d}\left(-\ln \frac{N(\sigma)}{\|\sigma\|_{\mathbf{d}}}\right) \right\|_{\mathcal{L}} \leq C < +\infty, \quad \sigma \in \mathbb{R}^n, \end{aligned}$$

since the function $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$, $\chi(\sigma) = \frac{N(\sigma)}{\|\sigma\|_{\mathbf{d}}}$ is homogeneous of degree 0, and, consequently, globally bounded. Similarly, for $f(\sigma) = A\sigma + Bu(\sigma)$ we derive

$$\begin{aligned} f(\mathbf{d}(s)\sigma) &= A\mathbf{d}(s)\sigma + Bu(\mathbf{d}(s)\sigma) = e^{-s}\mathbf{d}(s)A\sigma + Bu(\sigma) = e^{-s}\mathbf{d}(s)(A\sigma + Bu(\sigma)) \\ &= e^{-s}\mathbf{d}(s)f(\sigma), \quad \sigma \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \quad s \in \mathbb{R}, \end{aligned}$$

i.e. the closed loop system has negative homogeneity degree: $\deg_{\mathbb{F}_d}(f) = -1$.

Let us find some sufficient conditions (to the vector K and to the function N) allowing the finite-time stability of the closed-loop system. Selecting $Q_0 = P$ in (2.37) we derive the following sufficient stability condition

$$z^\top P \tilde{f}(z) < 0, \quad \tilde{f}(z) = \frac{1}{\|z\|} \left(Az + BK\mathbf{d}(-\ln \tilde{N}(z))z \right), \quad z \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \quad (2.40)$$

where $\tilde{N}(z) = N\left(\frac{z}{\|z\|}\right)$, $\|z\| = \sqrt{z^\top P z}$ with P satisfying (2.6). Since N is a continuous and positive definite function then

$$0 < \min_{\xi \in \mathbb{R}^n : \xi^\top P \xi = 1} N(\xi) \leq \tilde{N}(z) \leq \max_{\xi \in \mathbb{R}^n : \xi^\top P \xi = 1} N(\xi) < +\infty$$

and there exist numbers δ_{\min} and δ_{\max} such that

$$0 < \delta_{\min} \leq \tilde{N}(z) \leq \delta_{\max}. \quad (2.41)$$

Obviously, if $N(\sigma) = \|\sigma\|_{\mathbf{d}}$ is the canonical homogeneous norm then $\tilde{N}(z) = 1, z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and the stability condition becomes

$$z^T P(A + BK)z < 0, \quad z \in \mathbb{R}^n \setminus \{\mathbf{0}\},$$

i.e., K can be selected as a gain matrix of a linear stabilizing feedback. Since the pair $\{A, B\}$ is controllable then appropriate K and $P \succ 0$ can always be found (see, e.g. [11]) to guarantee the required inequality, while the parameter γ can be selected sufficiently large to fulfill (2.6). Moreover, Theorem 2.3 implies that $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the closed-loop system.

Therefore, the usage of the canonical homogeneous norm allows the very simple rule to be applied for selection of control parameters. However, the canonical homogeneous $\|\cdot\|_{\mathbf{d}}$ is defined implicitly, and for simplicity of practical realization it is worth to know if $\|\cdot\|_{\mathbf{d}}$ can be replaced with some explicitly defined homogeneous norm N (see also [61], [77]).

Proposition 2.6. [55] *Let for some $0 < \delta_{\min} \leq \delta_{\max}$ the system of LMIs*

$$\begin{pmatrix} AX + XA^T + BY + Y^T B^T - X & \sqrt{\ln\left(\frac{\delta_{\max}}{\delta_{\min}}\right)}BY & X\mathbf{d}\left(\ln\frac{\delta_{\max}}{\delta_{\min}}\right) \\ \sqrt{\ln\left(\frac{\delta_{\max}}{\delta_{\min}}\right)}Y^T B^T & -(G_{\mathbf{d}}^{-1}X + XG_{\mathbf{d}}^{-1}) & \mathbf{0} \\ \mathbf{d}\left(\ln\frac{\delta_{\max}}{\delta_{\min}}\right)X & \mathbf{0} & -X \end{pmatrix} \prec 0, \quad X \succ 0, \quad (2.42)$$

be feasible with respect to $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{1 \times n}$.

If a function $N : \mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$ is continuous, positive definite, $N \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$, $\deg_{H_{\mathbf{d}}}(N) = 1$ and it satisfies the inequalities (2.41) with $P = X^{-1}$ and $0 < \delta_{\min} \leq \delta_{\max}$ then the closed-loop system (2.38), (2.39) with

$$K = YX^{-1}\mathbf{d}(\ln\delta_{\max})$$

is globally uniformly finite-time stable and the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is its Lyapunov function.

The system of LMIs (2.42) is always feasible provided that δ_{\min} and δ_{\max} are sufficiently close to each other. Indeed,

$$\ln(\delta_{\max}/\delta_{\min}) \rightarrow 0$$

and

$$\mathbf{d}(\ln(\delta_{\max}/\delta_{\min})) \rightarrow I_n$$

as $\delta_{\max}/\delta_{\min} \rightarrow 1$, so the LMI (2.42) becomes feasible.

An explicitly-defined homogeneous norm N can be selected according to [25], [4] or [77]. Also it can be defined as follows

$$N(\sigma) = (\phi^\top(\sigma)Q\phi(\sigma))^{\frac{1}{2n}}, \quad (2.43)$$

where $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix and

$$\phi(\sigma) = \begin{pmatrix} |\sigma_1|^{r_1} \text{sign}(\sigma_1) \\ \dots \\ |\sigma_n|^{r_n} \text{sign}(\sigma_n) \end{pmatrix}, \quad r_i = \frac{n}{n-i+1}, \quad i = 1, 2, \dots, n.$$

The only problem is to select N such that (2.41) is fulfilled.

To illustrate the proposition 2.6 we consider the system (2.38), (2.39), (2.43) for $n = 2$. Selecting $\delta_{\min} = 1, \delta_{\max} = 1.4$ we solve the LMI (2.42):

$$P = \begin{pmatrix} 1.1214 & 0.4247 \\ 0.4247 & 0.2327 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -10.8695 & -6.2376 \end{pmatrix}.$$

One can be checked numerically (e.g. using a sufficiently dense grid on the unit sphere $\{z \in \mathbb{R}^n : z^\top Pz = 1\}$) that the homogeneous norm (2.43) with

$$Q = \begin{pmatrix} 1.9403 & 0.4410 \\ 0.4410 & 0.1330 \end{pmatrix}$$

satisfies $1 \leq N(z) \leq 1.4$ for $z^\top Pz = 1$, i.e. the condition (2.41) holds.

The similar control design scheme can be developed for an arbitrary matrix A (see, e.g. [62]) provided that the pair $\{A, B\}$ is controllable.

Dynamic Feedback for Non-Linear Plants

Let us consider now the problem of homogeneous stabilization of the (2.30) in the disturbance free case ($\omega = 0$), i.e.

$$\dot{\sigma}(t) = g(\sigma(t), u(t)),$$

by means of the dynamical feedback (2.31):

$$\dot{u}(t) = k(\sigma(t), u(t)).$$

Based on the scheme of the universal stabilizing control design given in [72] we derive the following corollary of Theorem 2.3.

Theorem 2.4 (On Homogeneous Dynamical Feedback). [55] *Let d_x and d_u be dilations in \mathbb{R}^n and \mathbb{R}^m , respectively,*

$$d := \begin{pmatrix} d_x(s) & 0 \\ 0 & d_u(s) \end{pmatrix} \quad \text{and} \quad \tilde{f} = \begin{pmatrix} g \\ 0 \end{pmatrix} \in \mathbb{F}_d(\mathbb{R}^{n+m}).$$

The origin of the system (2.30) is globally asymptotically stabilizable by means of the homogeneous dynamical feedback (2.31) with

$$\begin{pmatrix} g \\ k \end{pmatrix} \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^{n+m})$$

if and only if there exist a number $\gamma \geq 0$, a symmetric matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$ satisfying (2.6) and a map $\Xi \in C^\infty(\mathbb{R}^{n+m} \setminus \{\mathbf{0}\}, \mathbb{R}^{(n+m) \times (n+m)})$:

$$\det(\Xi(z)) \neq 0, \quad \frac{\partial \Xi(z)}{\partial z_i} z = 0, \quad \Xi(e^s z) = \Xi(z)$$

for $z = (z_1, \dots, z_{n+m}) \in \mathbb{R}^{n+m} \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$, $i = 1, \dots, n+m$

such that

$$a(z) < \gamma \sqrt{b^\top(z)b(z)} \quad \text{for } z \in S, \quad (2.44)$$

where

$$\begin{aligned} a(z) &= z^\top W(z) \tilde{f}(z), & b^\top(z) &= z^\top W(z) \begin{pmatrix} \sigma \\ I_m \end{pmatrix}, \\ W(z) &= \Xi^\top(z) P \Xi(z) \begin{pmatrix} (I_n - G_d) z^\top z P \\ z^\top P G_d z \end{pmatrix} + I_n \end{aligned}$$

and S is the unit sphere in \mathbb{R}^{n+m} with $\|z\| = \sqrt{z^\top P z}$. Moreover, the corresponding stabilizing homogeneous feedback law can be designed as follows

$$k(\xi) = \|\xi\|_{\mathbf{d}}^{\deg_d(\tilde{f})} \mathbf{d}_u(\ln \|\xi\|_{\mathbf{d}}) k_0(\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi) \quad \text{with } \xi = \begin{pmatrix} \sigma \\ u \end{pmatrix}, \quad (2.45)$$

where

$$k_0(\cdot) = \begin{cases} -\frac{a(\cdot) + \sqrt{a^2(\cdot) + (b^\top(\cdot)b(\cdot))^2}}{b^\top(\cdot)b(\cdot)} b(\cdot) & \text{if } b(\cdot) \neq \mathbf{0} \\ 0 & \text{if } b(\cdot) = \mathbf{0} \end{cases} \quad (2.46)$$

and $\|\cdot\|_{\mathbf{d}}$ is the canonical homogeneous norm induced by $\|\xi\| = \sqrt{\xi^\top P \xi}$.

The control design scheme consists of two steps: first, we design a stabilizing control using Sontag's formula (2.46), next, we apply the homogeneous dilation in order to derive the homogeneous stabilizing control (2.45) allowing the closed-loop system to be homogeneous with a required degree.

For $\Xi(z) = I_n$ the inequality (2.44) gives the following sufficient condition of quadratic stabilizability:

$$\begin{aligned} z^\top P \tilde{f}\left(\frac{z}{\sqrt{z^\top P z}}\right) &\leq \gamma \sqrt{z^\top P \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times m} \\ \mathbf{0}_{m \times n} & I_m \end{pmatrix} P z}, \quad z \neq \mathbf{0} \\ PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P &\succ 0, \quad P \succ 0. \end{aligned} \quad (2.47)$$

Taking into account Corollary 2.3 we derive criterion of finite-time stabilizability of the system (2.30) by means of the feedback (2.31).

Example 2.15 (Quadratic Dynamical Stabilization in a Finite Time). *Let us consider the stabilization problem for the system*

$$\dot{\sigma}_1 = |\sigma_1|^{\frac{1}{3}}u + \sigma_2, \quad \dot{\sigma}_2 = u, \quad (2.48)$$

where $\sigma_1, \sigma_2, u \in \mathbb{R}$. The vector field $\tilde{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\tilde{f}(\xi) = \begin{pmatrix} |\xi_1|^{\frac{1}{3}}\xi_3 + \xi_2 \\ \xi_3 \\ 0 \end{pmatrix},$$

$\xi = (\xi_1, \xi_2, \xi_3)^\top = (\sigma_1, \sigma_2, u)^\top$ is \mathbf{d} -homogeneous of degree -1 with respect to the dilation \mathbf{d} in \mathbb{R}^3 generated by $G_{\mathbf{d}} = \text{diag}\{3, 2, 1\}$. Hence, the sufficient stabilizability condition (2.47) becomes

$$z^\top PA(z)z < \gamma \sqrt{z^\top Pz} \sqrt{z^\top Pe_3e_3^\top Pz},$$

$P > 0$, $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P > 0$, where $z = (z_1, z_2, z_3)^\top \in \mathbb{R}^3$ and

$$A(z) = \begin{pmatrix} 0 & 1 & \left(\frac{|z_1|}{\sqrt{z^\top Pz}}\right)^{\frac{1}{3}} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $X \in \mathbb{R}^{3 \times 3}$ and $\gamma_0 > 0$ satisfy the linear matrix inequalities

$$XA_i + A_i^\top X - 2\gamma_0 e_3 e_3^\top < 0, \quad XG_{\mathbf{d}} + G_{\mathbf{d}}X > 0, \quad X > 0,$$

$A_i = \begin{pmatrix} 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$, $i = 0, 1$, $e_3 = (0, 0, 1)^\top \in \mathbb{R}^3$. Since $A(z) = \alpha(z)A_1 + (1 - \alpha(z))A_0$,

where $\alpha(z) = \left(\frac{|z_1|}{\sqrt{z^\top Pz}}\right)^{\frac{1}{3}}$ and $\alpha(z) \in [0, 1]$ provided that $p_{11} = 1$, then for $P = \tilde{P}/(\tilde{p}_{11})$, $\tilde{P} = X^{-1}$ the stabilizability condition (2.47) holds. So, the considered system can be stabilized in a finite time (due to $\deg_{\mathbf{d}}(f) = -1$) to zero by means of the \mathbf{d} -homogeneous dynamical feedback

$$\dot{u} = k_0 (\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\xi), \quad (2.49)$$

where k_0 is given by (2.46) with $a(z) = z^\top P\tilde{f}(z)$ and $b(z) = p_{13}z_1 + p_{23}z_2 + p_{33}z_3$. For example, an appropriate matrix P obtained using the above LMIs is

$$P = \begin{pmatrix} 1.0000 & 0.8930 & 0.8846 \\ 0.8930 & 1.8679 & 1.3659 \\ 0.8846 & 1.3659 & 1.7523 \end{pmatrix}.$$

Remark 2.3. The canonical homogeneous norm is defined implicitly by the formula (2.8). Therefore, to implement the designed control law either the canonical homogeneous norm must be calculated numerically on-line (like in [61], [62]) or it must

be approximated with some explicitly defined homogeneous norm N (for example, using the formula (2.43) or the results of [77]). The construction of an appropriate approximation N of the canonical homogeneous norm $\|\cdot\|$, in the general case, is a non-trivial theoretical problem needed to be studied in future.

The problems of robustness analysis and sliding mode control design using the presented methodology are studied in [55].

2.3.3 Homogeneous Observers in \mathbb{R}^n

Let us consider the perturbed linear control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \quad (2.50)$$

where $x \in \mathbb{R}^n$ is the state variable, $y \in \mathbb{R}^k$ is the measured output, $u : \mathbb{R} \rightarrow \mathbb{R}^s$ is the control input, $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B \in \mathbb{R}^{n \times s}$ is the matrix of input gains and the matrix $C \in \mathbb{R}^{k \times n}$ is the output matrix which links the measured outputs to the state variables. The pair (A, C) is assumed to be observable and $\text{rank}(C) = k$.

The goal is to design two dynamic observers that estimate the state of the non-perturbed system (2.50) in a finite time (or in a fixed, defined *a priori*, time under the assumption that the domain of initial conditions is unknown).

The observers' design starts with a decomposition of the considered system into an appropriate block canonical form.

Lemma 2.3. *Consider the system (2.30) with the pair (A, C) being observable and $\text{rank}(C) = k$. Then there exists a nonsingular transformation Φ such that*

$$\Phi A \Phi^{-1} = F \tilde{C} + \tilde{A}, \quad C \Phi^{-1} = (c_0 \ 0 \ \dots \ 0),$$

$$\tilde{C} = [I_k \ 0] \in \mathbb{R}^{k \times n}, \quad \tilde{A} = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{m-1 m} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad C_0 \in \mathbb{R}^{k \times k}, \quad F \in \mathbb{R}^{n \times k},$$

where m is an integer, $A_{j-1 j} \in \mathbb{R}^{n_{j-1} \times n_j}$, $n_j = \text{rank}(A_{j-1 j})$, $j = 2, \dots, m$, so that $n_1 = \text{rank}(C) = k$ and $\sum_{i=1}^m n_i = n$.

This lemma is a trivial consequence of well known results on block observability and controllability forms (see [75], [18], [19], [45]).

If $k = 1$ then $m = n$, $n_i = 1$ and Φ transforms the matrix A into the canonical Brunovsky form. It is also worth stressing that canonical forms and related transformations also exist for nonlinear systems (see for instance [31], [35]). Therefore, the observer design algorithms given below can be adapted to the nonlinear case.

Let us consider the following nonlinear observer

$$\frac{d}{dt} \hat{x}(t) = A \hat{x}(t) + Bu(t) - g_{FT}(y(t) - C \hat{x}(t)), \quad (2.51)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector and the function $g_{FT} : \mathbb{R}^k \rightarrow \mathbb{R}^n$ is defined as

$$g_{FT}(\sigma) = \Phi^{-1} \left[\mathbf{d}_{\tilde{r}} \left(\ln \|\tilde{P}C_0^{-1}\sigma\|^{-1} \right) L_{FT} - F \right] C_0^{-1} \sigma, \quad (2.52)$$

where $\sigma \in \mathbb{R}^k$, the matrices $\Phi \in \mathbb{R}^{n \times n}$, $C_0 \in \mathbb{R}^{k \times k}$ and $F \in \mathbb{R}^{n \times k}$ are defined in Lemma 2.3, $\mathbf{d}_{\tilde{r}}(\cdot)$ is the weighted dilation

$$\mathbf{d}(s) = e^{\text{diag}(r)s} \in \mathbb{R}^{n \times n}$$

with

$$\tilde{r} = \left(\frac{\mu}{1+(m-1)\mu}, \frac{2\mu}{1+(m-1)\mu}, \dots, \frac{m\mu}{1+(m-1)\mu} \right)^T, \quad \mu \in (0, 1], \quad (2.53)$$

and $L_{FT} \in \mathbb{R}^{n \times k}$ and $\tilde{P} \in \mathbb{R}^{k \times k}$ are matrices of observer gains, to be determined. The error equation has the form

$$\dot{e} = \left(\tilde{A} + \mathbf{d}_{\tilde{r}} \left(\ln \|\tilde{P}\tilde{C}e\|^{-1} \right) L_{FT}\tilde{C} \right) e, \quad (2.54)$$

where $e = \Phi(x - \hat{x})$, $\tilde{A} \in \mathbb{R}^{n \times n}$ and $\tilde{C} \in \mathbb{R}^{k \times n}$ are defined in Lemma 2.3. Obviously, if $\mu = 0$ then $\mathbf{d}_{\tilde{r}}(s) = I_n$ for all $s \in \mathbb{R}$ and the presented observer becomes the classical Luenberger one.

Remark 2.4. *If the term $\|\tilde{P}C_0^{-1}\sigma\|$ in (2.52) is replaced with $\epsilon^{(1+(m-1)\mu)/\mu}$ where $\epsilon > 0$ is a small constant, then the system (2.51)-(2.52) becomes a high-gain observer [65], [35] with the error dynamics given by*

$$\dot{e} = \left(\tilde{A} + \text{diag}(\epsilon^{-1}I_{n_1}, \epsilon^{-2}I_{n_2}, \dots, \epsilon^{-m}I_{n_m}) L_{FT}\tilde{C} \right) e.$$

In our algorithms the gain factor ϵ depends on the available part of the observation error, namely, on $\sigma = y - C\hat{x} = Ce$. This allows the finite-time and fixed-time observers to be less sensitive with respect to noises.

Let us denote

$$H_r = \text{diag}(r_1 I_{n_1}, r_2 I_{n_2}, \dots, r_m I_{n_m}) \in \mathbb{R}^{n \times n},$$

$$r = \left[1 + \frac{\mu}{1 + (m-1)\mu} \right] (1, 1, \dots, 1)^T - \tilde{r}$$

and

$$\Xi(\lambda) = \lambda(D_{\tilde{r}}(\lambda^{-1}) - I_n).$$

Theorem 2.5. [44] *Let for some $\mu \in (0, 1]$, $\alpha > 0$, $\xi > 0$ and $\tau \geq 1$ the system of matrix inequalities*

$$\left(\begin{array}{c} P\tilde{A} + \tilde{A}^T P + \tilde{C}^T Y^T + Y\tilde{C} + \xi P + \alpha(PH_r + H_r P) \\ P \end{array} \quad \begin{array}{c} P \\ -\xi Z \end{array} \right) \preceq 0, \quad (2.55a)$$

$$P \succ 0, \quad Z \succ 0, \quad X \succ 0, \quad (2.55b)$$

$$\begin{pmatrix} \tau X & Y^T \\ Y & P \end{pmatrix} \succeq 0, \quad (2.55c)$$

$$PH_r + H_r P \succ 0, \quad (2.55e)$$

$$P \succeq \tilde{C}^T X \tilde{C}, \quad (2.55d)$$

$$\Xi(\lambda)Z\Xi(\lambda) \preceq \frac{1}{\tau}P, \quad \forall \lambda \in [0, 1], \quad (2.55f)$$

be feasible for some $P, Z \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{k \times n}$ and $X \in \mathbb{R}^{k \times k}$. Then the error equation (2.54) with $L_{F_T} = P^{-1}Y$ and $\tilde{P} = X^{1/2}$ is globally finite-time stable with settling time

$$T \leq \frac{\|e(0)\|_d^\rho}{\alpha\rho}, \quad \rho = \frac{\mu}{1+(m-1)\mu}.$$

In other words, this theorem claims that any solution of the observer system (2.51) converges to a solution of the real system (2.30) in a finite time T , which is dependent on the initial estimation error $e(0) \in \mathbb{R}^n$. As before the main idea of the proof is to show that the canonical homogeneous norm $\|\cdot\|_d$ is a Lyapunov function of the error equation.

Corollary 2.6. *The system of matrix inequalities (2.55) is feasible for sufficiently small $\mu > 0$.*

Indeed, observability of the pair (A, C) implies that the pair $(\tilde{A} + 0.5(\alpha H_r + \xi I_n), \tilde{C})$ is also observable. Hence, it can be easily shown that the inequality (2.55a) is feasible with some positive definite matrices $P, Z \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{k \times n}$. The matrix inequalities (2.55d) and (2.55e) are also feasible for some $X \in \mathbb{R}^{k \times k}$ and sufficiently large $\tau \geq 1$. Since $\|\Xi(\lambda)\| \rightarrow 0$ uniformly on $\lambda \in [0, 1]$ as $\mu \rightarrow 0$, then the inequalities (2.55e), (2.55f) will hold for sufficiently small $\mu > 0$.

Proposition 2.7. *The parametric inequality (2.55f) holds if*

$$\Xi(q_i)Z\Xi(q_i) + \frac{1}{2}(q_i^2 - q_{i-1}^2)M \prec \frac{1}{\tau}P, \quad i = 1, \dots, N, \quad (2.56a)$$

$$P \succ 0, \quad Z \succ 0, \quad M \succ 0 \quad (2.56b)$$

$$\begin{pmatrix} (I_n - H_{\tilde{r}})Z + Z(I_n - H_{\tilde{r}}) & -ZH_{\tilde{r}} \\ -H_{\tilde{r}}Z & M \end{pmatrix} > 0, \quad (2.56c)$$

where $0 = q_0 < q_1 < \dots < q_N = 1$, $H_{\tilde{r}} = \text{diag}(\tilde{r}_1 I_{n_1}, \tilde{r}_2 I_{n_2}, \dots, \tilde{r}_m I_{n_m})$, $P, M, Z \in \mathbb{R}^{n \times n}$.

The provided result allows the implementation of a simple algorithm to solve the parametrized system of matrix inequalities (2.55) with fixed α, ξ, τ, μ .

Algorithm 2.1. [44]

Initialization: $N = 1, q_0 = 0, q_N = 1, \Sigma = \{q_0, q_N\}$.

Loop: While the system of LMI (2.55a)-(2.55d), (2.56a)-(2.56c) is not feasible, do

$\Sigma \leftarrow \Sigma \cup \left\{ \frac{q_{i-1} + q_i}{2} \right\}_{i=1}^N$ and $N \leftarrow 2N$.

Since the matrix inequality $(I_n - H_{\bar{r}})Z + Z(I_n - H_{\bar{r}}) > 0$ is obviously feasible for sufficiently small $\mu > 0$, then, in the view of Corollary 2.6, the presented algorithm always finds the required solution if μ is sufficiently small.

The fixed-time observer for linear plants is designed similarly in [44]. It is based on combination of systems with positive and negative degrees.

Example 2.16. *The numerical simulations aim to show the main properties of the finite-time observers. In the example, an inverted-cart pendulum model will be used. The robustness of the finite-time observer will be studied by applying it to the nonlinear plant and a comparison with a high-gain observer will be included.*

The state vector is given by $\mathbf{x} = [x, \dot{x}, \theta, \dot{\theta}]^T$, where as usual (x, \dot{x}) represents the position and the velocity of the cart and $(\theta, \dot{\theta})$ the angle (from the vertical down position) and the angular velocity of the pendulum. The model parameters are $M = 0.5$ Kg - mass of the cart, $m_c = 0.2$ Kg - mass of the pendulum, $b = 0.1$ N/m/s - cart friction coefficient, $l = 0.3$ m - length to pendulum center of mass, $I = 0.006$ Kg - moment of inertia of the pendulum.

It is assumed that only position and angle can be measured directly. The nonlinear equations describing the system motion are given by

$$\begin{aligned} (M + m_c)\ddot{x} + b\dot{x} + m_c - l\ddot{\theta}\cos\theta - m_cl\dot{\theta}^2\sin\theta &= F_{in} \\ (I + m_cl^2)\ddot{\theta} + m_cgl\sin\theta + m_cl\ddot{x}\cos\theta &= 0, \end{aligned}$$

where F_{in} represents the input force. A simple proportional control law $F_{in} = K_p \frac{d}{dt}\hat{\theta}(t)$ is used to stabilize the pendulum around the downward position, $\frac{d}{dt}\hat{\theta}(t)$ is an estimate of the angular velocity to be obtained using the observers. The linearization of the model around the downward equilibrium point $\mathbf{x}_0 = 0 \in \mathbb{R}^4$ gives the following parameters for (2.30):

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.672 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4545 & -31.181 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1.818 \\ 0 \\ -4.545 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The first step of the observer design is to transform the linearized model into the observable canonical form, given in Lemma 2.3, by obtaining the matrix

$$\Phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}, \text{ where } a = -0.1818 \text{ and } b = -0.4545.$$

Comparison between finite-time and high-gain observers. *In order to make a fair comparison we adjusted the parameters of both observers to have similar time response for the initial condition $x_0 = (0, -2, \pi/4, 1)$; that is to say, the norm of the estimation errors is admitted to be less than 0.15 for $t \geq 0.5$ s. Using Algorithm 2.1 with $(\mu, \alpha, \tau, \xi) = (0.25, 2.5, 100, 4)$ we design the finite-time observer (2.51), (2.52) with*

$$L_{FT} = \begin{pmatrix} -9.9080 & 0 & 0 & -46.8452 \\ 0 & -9.9080 & -46.8452 & 0 \end{pmatrix}^\top, \tilde{P} = 0.0969 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and we compare it with the high-gain observer, designed according to Remark 2.4. Namely, the term $\|\tilde{P}C_0^{-1}\sigma\|$ in (2.52) is replaced with $\epsilon^{(1+(m-1)\mu)/\mu}$.

Note that, the gain factor $\epsilon = 0.3$ (and correspondingly the admissible estimation error $\|e(0.5)\| \leq 0.15$) is selected sufficiently large since the high-gain observer becomes more sensitive with respect to measurement noises as ϵ decreases. The estimation error of the finite-time observer turned out to be 10 times less³ ($\|e(t)\| \leq 0.01$) for $t \geq 0.5$ s.

Figure 2.2 depicts the evolution of the observation errors of finite-time and high-gain observers for the noise-free case. Although the observers were designed using a linearized model, the system remains stable for the complete nonlinear model. It is also worth noting that the finite-time observer demonstrates a smaller peaking during transients.

To compare the observers for the case of noisy measurements, a band-limited white noise of power 10^{-5} has been added to the output signal during the simulation. The corresponding results are presented at Figure 2.3. They show almost twice better precision (in both $L^2_{(0.5,1.5)}$ and $L^\infty_{(0.5,1.5)}$ norms of the error) of the finite-time observer with respect to the high-gain one. This fact has a simple explanation in the context of high-gain observer theory: since the gain factor ϵ of the finite-time observer depends on the available part of the observation error (see Remark 2.4), namely,

$$\epsilon = \epsilon(\sigma) = \|\tilde{P}C_0^{-1}\sigma\|^{\mu/(1+(m-1)\mu)}, \quad \sigma = y - C\hat{x} = Ce,$$

then its value is automatically adapted to the noises of different magnitude (the larger the noise magnitude, the smaller the gain). In the noisy case the convergence time of the finite-time observer slightly increases, allowing a better estimation precision.

2.4 Conclusions

In this chapter it is shown that the method of linear dilation and the canonical homogeneous norm simplify the analysis and design of homogeneous control systems. In particular, they allow simple extensions of linear control and state estimation algorithms to the non-linear homogeneous cases. Notice that the canonical homogeneous norm implicitly appears in the theory of controllability functions introduced in [37]. It also corresponds the radius vector of "homogeneous" spheric coordinates studied in [64]. The canonical homogeneous norm also defines the implicit Lyapunov functions considered in [1], [61], [62]. It allows to unify the finite-time control design methodology for linear models in both finite and infinite dimensional spaces. A further elaboration of this method looks promising for other control problems (like observer design and robustness analysis for evolution equations in Hilbert spaces, consistent discretization of homogeneous models, etc).

³The numerical simulations have been done using the explicit Euler method with sampling period 10^{-4} .

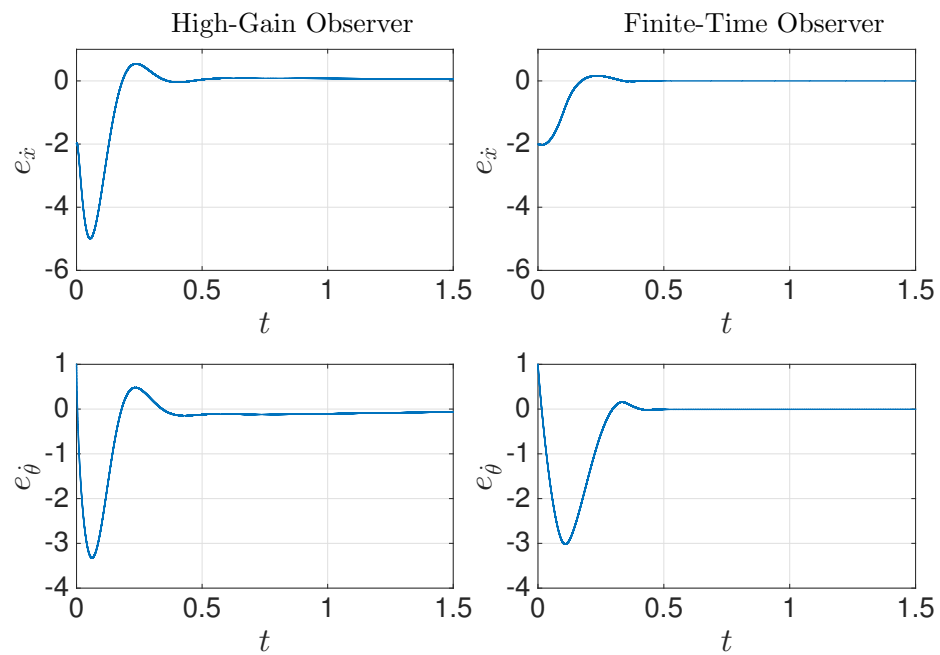


Figure 2.2: Simulation plots of $e_{\dot{x}}$ and $e_{\dot{\theta}}$ for the HG and finite-time observers applied to the nonlinear plant with IC $x(0) = (0, -2, \frac{\pi}{4}, 1)$ and $K_p = 10$.

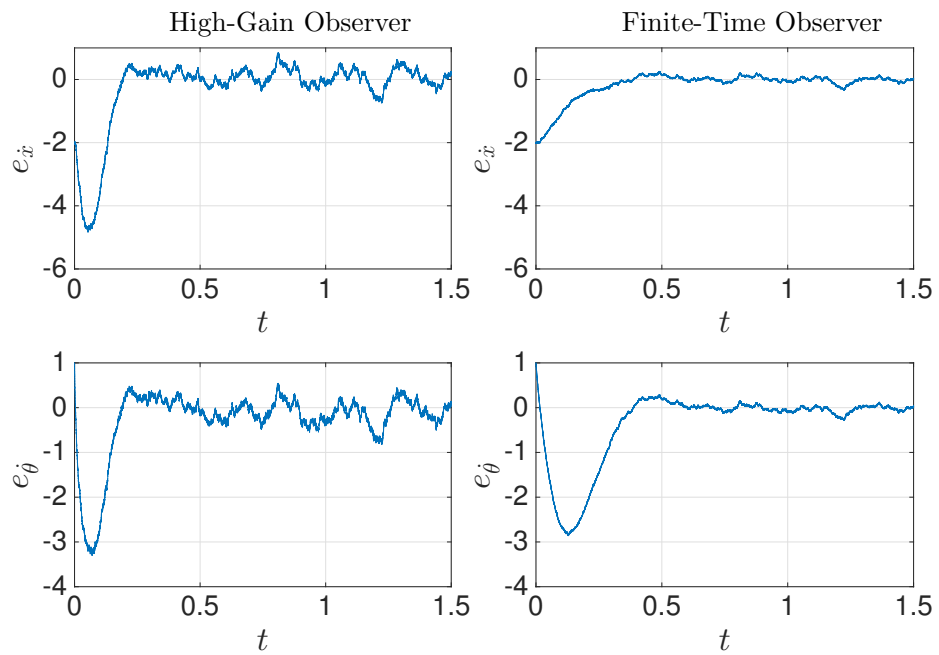


Figure 2.3: Simulation plots of $e_{\dot{x}}$ and $e_{\dot{\theta}}$ for the high-gain and finite-time observers with a band-limited white noise in the measurements.

Chapter 3

Future Directions of Research

3.1 Fundamental research

3.1.1 Short term perspective

In a short term perspective I will concentrate on the problems announced in the actual research projects, in which I am involved (e.g. AND DIGITSLID, ANR Finite4SoS, CPER DATA, CPET ELSAT2020).

Consistent Discretization of Homogeneous Systems in \mathbb{R}^n

The discretization issues are very important for practical implementation of homogeneous algorithms in a digital control device. The development of discretization schemes for homogeneous systems, which would preserve the convergence properties of the continuous system, is a difficult task. Some of my recent results [59] gives us a possible way for successful solving of this problem in a short term perspective. The proposed scheme is based on a transformation of the original homogeneous model to an equivalent one, which admits consistent implicit or semi-implicit discretization. It has been tested that this approach allows to reject completely the numerical chattering of homogeneous sliding mode controllers and preserve finite-time/fixed-time stability in some systems with a sampled implementation of control laws . Extension of this results to the general case are expected to be provided in a short term perspective. This research direction is proposed by ANR DIGITSLID .

Homogeneous observers for linear systems in \mathbb{B}

Development of a unified approach to the design of finite-time control and estimation algorithms for systems modeled by ODEs, PDEs, time-delay equations, differential inclusions is planned in the context of ANR Finite4SoS. The simplest

way to cover all this models is to deal with abstract evolution equations in a Banach space. The finite-time control design has been presented in the previous chapter. Using the Kalman duality principle for linear system and homogeneity ideas, a universal finite-time observer for evolution systems is planned to be developed in a short term perspective.

Quasi-Homogeneity and Homogeneous Extensions

The homogeneous systems cover rather large class of the nonlinear models. However, the conventional assumptions on the dilation symmetry and the conventional symmetry (homogeneity) relations are still very restrictive. It is rather difficult to use homogeneous arguments for finite-time boundary control of PDEs. For example, an infinite chain of integrators may appear as an approximated model of a boundary controlled heat system. It does not admit a well defined weighted dilation with respect to which the homogeneity property can be established. However, some quasi-homogeneity relations like

$$f(\mathbf{d}_1(s)u) = \mathbf{d}_2(s)f(u), \quad f : \mathcal{D}(f) \subset \mathbb{B} \rightarrow \mathbb{B}, \quad s \in \mathbb{R}, \quad u \in \mathcal{D}(f)$$

where \mathbf{d}_1 and \mathbf{d}_2 are some dilations in \mathbb{B} , are fulfilled. Under additional conditions on the dilations \mathbf{d}_1 and \mathbf{d}_2 , some results about finite-time convergence can be proven similarly to the homogeneous case.

To treat this problem the method of homogeneous extensions introduced in the previous chapter can also be utilized in some cases. In the short term perspective the concepts of quasi-homogeneity and homogeneous extensions is expected developed in the context of ANR Finite4SoS.

3.1.2 Long term perspective

In a long term perspective my research activities in the context of homogeneous systems can be continued in the following directions.

Homogeneous controllers and observers for nonlinear systems in \mathbb{B} . Extension of the homogeneous control and estimation algorithms developed for linear plants in \mathbb{B} to some classes of nonlinear evolution systems are planned for a long term perspective.

Consistent Discretization of Homogeneous Systems in \mathbb{B} . Homogeneity simplifies numerical simulations of the non-linear systems. Indeed, knowing the solutions of some homogeneous systems for initial conditions on a sphere we can easily construct solutions with initial conditions in a whole domain. Due to this symmetry, numerical simulations for some nonlinear infinite dimensional systems can be boosted. The latter is very important, in particular, for control/observation systems based on Navier-Stokes models of the flow, which on-line simulation is currently impossible. The problem of consistent discretization in \mathbb{B} can be also studied in this context (similarly to consistent discretization in \mathbb{R}^n).

Homogeneity in optimal control. The ideas of the dilation symmetry can be useful

in the optimal control theory. For instance, homogeneity of the system will imply a homogeneity of the Hamiltonian and a homogeneity of the Bellman function. This may lead to a simplification of an optimal control design.

Homogeneity in learning control. If it is known (e.g. from physics) that a control plant is homogeneous in a generalized sense, then the learning control design can be essentially using this knowledge. This may imply a homogeneity of a learning function, of the corresponding data utilized for the learning process, etc. This is also an interesting problem to be considered in a long term perspective.

3.2 Applications

Control of flow separation. Active flow control is a very promising research direction in the transportation area. It is aimed at a drag (and, consequently, consumption) reduction of the vehicles (both flying and terrestrial) by means of some actuators (e.g. air-blowers). This task is also important for increase (or on-line control) of the lift force of an airplane under turbulence (e.g. during take off and landing). I develop flow control algorithms in cooperation with colleagues from ONERA, Centrale Lille, The University of Valenciennes, University of Poitiers. The efficiency of the developed control laws is confirmed by successful experimental applications (see, <https://www.youtube.com/watch?v=b5NnAV2qeno>). In a short term perspective, an efficiency of the drag reduction by means of the proposed control scheme is planned to be analyzed at University of Valenciennes during real-life experiments with the car Renault Twingo.

Fast control of flying vehicles. The platform for control experimentation with autonomous vehicles is under construction in Inria Lille in the context of the project ControlHub (see below). In particular, it includes flying vehicles such as Q-Drone of Quanser. Application of the control algorithms proposed in the previous chapter is planned to be done using this platform.

Interconnected systems. The Internet of Things (IoT) is the union of physical devices embedded with electronics, software, sensors, actuators and network interconnectivity allowing these objects to gather and exchange the information. The interconnected systems are expected to be the main model of a control plant in a recent future. The issues of possible application of homogeneity approach to some real interconnected systems are planned to be considered in long term perspective.

3.3 ControlHub Platform

ControlHub is the platform for a rapid prototyping of control algorithms on real physical systems. ControlHub is under construction Inria Lille. I lead this project (<http://controlhub.gforge.inria.fr/>).

The control theory can be considered as a crossing point between engineering (and other sciences such as biology, economy ...) and mathematics. The control

practice deals with a real plant (e.g. distillation column) and a control device (e.g. digital controller), which sends appropriate signals to the plant in order to govern its behavior, while the mathematical control theory considers its mathematical model (usually a dynamical system) with some input (control signals) and output (sensors measurements) parameters. Control design in the mathematical sense is just a synthesis of an appropriate mathematical law. The practical validation of this law cannot be done without a direct interaction with a real system. Apart of a mathematical study, a lot of engineering activities are required for prototyping of control algorithms. ControlHub automatizes the most of them using the modern achievements of computer science and control engineering. It allows the end-user to have a remote access (i.e. using Internet) to real experimental setups in order to validate mathematical control algorithms realized, for example, in MATLAB.

The project ControlHub illustrates my activities related to an innovative research infrastructure for the control systems.

Bibliography

- [1] J. Adamy and A. Flemming. Soft variable-structure controls: a survey. *Automatica*, 40:1821–1844, 2004.
- [2] A.Y. Aleksandrov, A.A. Kosov, and A.V. Platonov. On the asymptotic stability of switched homogeneous systems. *Systems & Control Letters*, 61(1):127–133 127–133 127–133, 2012.
- [3] A. Andreini, A. Bacciotti, and G. Stefani. Global stabilizability of homogeneous vector fields of odd degree. *Systems & Control Letters*, 10:251–256, 1988.
- [4] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous Approximation, Recursive Observer Design, and Output Feedback. *SIAM Journal of Control and Optimization*, 47(4):1814–1850, 2008.
- [5] B. Armstrong-Helouvry. *Control of machines with friction*. Springer, 1991.
- [6] A. Bacciotti and L. Rosier. *Lyapunov Functions and Stability in Control Theory*. Springer, 2001.
- [7] E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov. On homogeneity and its application in sliding mode control. *Journal of The Franklin Institute*, 351(4):1866–1901, 2014.
- [8] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti. Verification of ISS, iISS and IOSS properties applying weighted homogeneity. *System & Control Letters*, 62(12):1159–1167, 2013.
- [9] S. P. Bhat and D. S. Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17:101–127, 2005.
- [10] S.P. Bhat and D.S. Bernstein. Finite time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38(3):751–766, 2000.
- [11] S. Boyd, E. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

- [12] A. Cheremensky and V. Fomin. *Operator Approach to Linear Control Systems*. Springer, B.V., 2013.
- [13] F. L. Chernous'ko, I. M. Ananievski, and S. A. Reshmin. *Control of Nonlinear Dynamical Systems*. Springer, 2008.
- [14] J.-M. Coron, B. d'Andrea Novel, and G. Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, 2007.
- [15] J.-M. Coron, L. Hu, and G. Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *Journal of Functional Analysis*, 271(12):3554–3587, 2016.
- [16] J.-M. Coron and H.-M. Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Archive for Rational Mechanics and Analysis*, 2017 (doi:10.1007/s00205-017-1119-y).
- [17] J.-M. Coron and L. Praly. Adding an integrator for the stabilization problem. *Systems & Control Letters*, 17(2):89–104, 1991.
- [18] S. V. Drakunov, D. B. Izosimov, A. G. Luk'yanov, V. A. Utkin, and V. I. Utkin. The block control design principle I. *Automation and Remote Control*, 51(5):601–608, 1990.
- [19] S. V. Drakunov, D. B. Izosimov, A. G. Luk'yanov, V. A. Utkin, and V. I. Utkin. The block control design principle II. *Automation and Remote Control*, 51(6):737–746, 1990.
- [20] B.K. Driver. *Analysis Tools with Applications*. Springer, 2003.
- [21] D. Efimov and W. Perruquetti. Oscillations conditions in homogenous systems. In *Proc. IFAC NOLCOS Symp.*, pages 1379–1384, 2010.
- [22] G. Falkovich. *Fluid Mechanics (A short course for physicists)*. Cambridge University Press., 2011.
- [23] G. Folland. Subelliptic estimates and function spaces on nilpotent lie groups. *Arkiv for Matematik*, 13(1-2):161–207, 1975.
- [24] A. Fuller. Relay control systems optimized for various performance criteria. In *In Proceedings of the 1st IFAC World Congress*, pages 510–519, 1960.
- [25] L. Grune. Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal of Control and Optimization*, 38(4):1288–1308, 2000.
- [26] W. Hahn. *Stability of Motion*. Springer-Verlag Berlin Heidelberg, New York, 1967.
- [27] V.T. Haimo. Finite time controllers. *SIAM Journal of Control and Optimization*, 24(4):760–770, 1986.

- [28] H. Halkin. Implicit functions and optimization problems without continuous differentiability of the data. *SIAM Journal of Control and Optimization*, 12(2), 1974.
- [29] H. Hermes. Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24(4):731, 1986.
- [30] L. Hu and F. Di Meglio. Finite-time backstepping boundary stabilization of 3x3 hyperbolic systems. In *European Control Conference (ECC)*, pages 67–73, 2015.
- [31] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, N. Y. Inc., 1995.
- [32] M. Kawski. Homogeneous stabilizing feedback laws. *Control Theory and Advanced Technology*, 6(4):497–516, 1990.
- [33] M. Kawski. Families of dilations and asymptotic stability. *Analysis of Controlled Dynamical Systems*, pages 285–294, 1991.
- [34] M. Kawski. Geometric homogeneity and stabilization. In Arthur Krener and David Mayne, editors, *Proc. IFAC Nonlinear Control Symposium*, pages 164–169, Lake Tahoe, CA, 1995.
- [35] H. K. Khalil and L. Praly. High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 24(6):993–1015, 2014.
- [36] V. V. Khomenuk. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Matematika.*, 3(22):157–164, 1961 (in Russian).
- [37] V.I. Korobov. A general approach to synthesis problem. *Doklady Akademii Nauk SSSR*, 248:1051–1063, 1979.
- [38] V.I. Korobov. *Method of Controlability Functions (in Russian)*. NIC, Moscow, 2007.
- [39] M. Krstic and A. Smyshlyaev. *Boundary Control of PDEs: A Course on Backstepping Designs*. SIAM, 2008.
- [40] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993.
- [41] A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5):823–830, 2005.
- [42] A. Levant. Quasi-continuous high-order sliding-mode controllers. *IEEE Transactions on Automatic Control*, 50(11):1812–1816, 2005.
- [43] D. Liberzon. *Switchings in Systems and Control*. Birkhauser, Boston, 2003.
- [44] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, 87(1):52–60, 2018.

- [45] M. Misrikhanov and V. Ryabchenko. Pole placement for controlling a large scale power system. *Automation and Remote Control*, 72(10):2123–2146, 2011.
- [46] H. Nakamura, Y. Yamashita, and H. Nishitani. Smooth lyapunov functions for homogeneous differential inclusions. In *Proceedings of the 41st SICE Annual Conference*, pages 1974–1979, 2002.
- [47] Y. Orlov. Application of Lyapunov method in distributed systems. *Automation and Remote Control*, 44:426–430, 1983.
- [48] Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4):1253–1271, 2005.
- [49] Y Orlov and V Utkin. Unit sliding mode control in infinite-dimensional systems. *Journal of Applied Mathematics and Computer Science*, 8:7–20, 1998.
- [50] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.
- [51] V. Perrollaz and L. Rosier. Finite-time stabilization of 2x2 hyperbolic systems on tree-shaped networks. *SIAM Journal of Control and Optimization*, 52(1):143–163, 2014.
- [52] W. Perruquetti, T. Floquet, and E. Moulay. Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360, 2008.
- [53] A. Pisano, Y. Orlov, and E. Usai. Tracking control of the uncertain heat and wave equation via power-fractional and sliding-mode techniques. *SIAM Journal on Control and Optimization*, 49(2):363–382, 2011.
- [54] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.
- [55] A. Polyakov. Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 2018 (DOI: 10.1002/rnc.4058).
- [56] A. Polyakov, J.-M. Coron, and L. Rosier. On finite-time stabilization of evolution equations: A homogeneous approach. In *Conference on Decision and Control*, pages 3143–3148, 2016.
- [57] A. Polyakov, J.-M. Coron, and L. Rosier. Boundary finite-time feedback design for heat equation. In *IFAC World Congress*, 2017.
- [58] A. Polyakov, J.-M. Coron, and L. Rosier. On homogeneous finite-time control for linear evolution equation in hilbert space. *IEEE Transactions on Automatic Control*, 63(9):3143–3150, 2018.

- [59] A. Polyakov, D. Efimov, and B. Brogliato. Consistent discretization of finite-time stable homogeneous systems. In *15th International Workshop on Variable Structure Systems and Sliding Mode Control*, 2018.
- [60] A. Polyakov, D. Efimov, E. Fridman, and W. Perruquetti. On homogeneous distributed parameters equations. *IEEE Transactions on Automatic Control*, 61(11):3657–3662, 2016.
- [61] A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time stabilization: Implicit Lyapunov function approach. *Automatica*, 51(1):332–340, 2015.
- [62] A. Polyakov, D. Efimov, and W. Perruquetti. Robust stabilization of MIMO systems in finite/fixed time. *International Journal of Robust and Nonlinear Control*, 26(1):69–90, 2016.
- [63] A. Polyakov and L. Fridman. Stability notions and Lyapunov functions for sliding mode control systems. *Journal of The Franklin Institute*, 351(4):1831–1865, 2014.
- [64] L. Praly. Generalized weighted homogeneity and state dependent time scale for linear controllable systems. In *Proc. IEEE Conference on Decision and Control*, pages 4342–4347, San Diego, USA, 1997.
- [65] A. Prasov and H. Khalil. A nonlinear high-gain observer for systems with measurement noise in a feedback control framework. *IEEE Transactions on Automatic Control*, 58(3):569–580, 2013.
- [66] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19:467–473, 1992.
- [67] L. Rosier. Etude de quelques problèmes de stabilisation. *PhD Thesis, Ecole Normale Supérieure de Cachan (France)*, 1993.
- [68] E. Roxin. On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15:273–283, 1966.
- [69] E.P. Ryan. Universal stabilization of a class of nonlinear systems with homogeneous vector fields. *Systems & Control Letters*, 26:177–184, 1995.
- [70] R. Sepulchre and D. Aeyels. Homogeneous Lyapunov Functions and Necessary Conditions for Stabilization. *Mathematics of Control, Signals and Systems*, 9:34–58, 1996.
- [71] R. Sepulchre and D. Aeyels. Stabilizability does not imply homogeneous stabilizability for controllable homogeneous systems. *SIAM Journal of Control and Optimization*, 34(5):1798–1813, 1996.
- [72] E.D. Sontag. A ‘universal’ construction of artstein’s theorem on nonlinear stabilization. *Systems & Control Letters*, 13:117–123, 1989.

- [73] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhauser, 2009.
- [74] V. I. Utkin. *Sliding Modes in Control Optimization*. Springer-Verlag, Berlin, 1992.
- [75] W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*. Springer, 1985.
- [76] K. Zimenko, D. Efimov, A. Polyakov, and W. Perruquetti. A note on delay robustness for homogeneous systems with negative degree. *Automatica*, 79(5):178–184, 2017.
- [77] K. Zimenko, A. Polyakov, and D. Efimov. Stabilization of chain of integrators with arbitrary in finite-time. In *IEEE Conference on Decision and Control*, 2015.
- [78] V. I. Zubov. *Methods of A.M. Lyapunov and Their Applications*. Noordhoff, Leiden, 1964 (Translated from Russian: V.I. Zubov, Metody Lyapunova i ih primeneniye, Leningrad: LGU, 1957).
- [79] V.I. Zubov. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica.*, 1:80–88, 1958 (in Russian).

Appendix

A. Publications

International journals

- IJ1. A. Polyakov, J.-M. Coron, L. Rosier, On Homogeneous Finite-Time Control for Linear Evolution Equation in Hilbert Space, **IEEE Transactions on Automatic Control**, 2018, .
- IJ2. A. Polyakov, Sliding Mode Control Design Using Canonical Homogeneous Norm, **International Journal of Robust and Nonlinear Control**, 2018, .
- IJ3. F. Lopez Ramirez, D. Efimov, A. Polyakov, W. Perruquetti, Fixed-Time Output Stabilization and Fixed-Time Estimation of a Chain of Integrators, **International Journal of Robust and Nonlinear Control**, 2018, .
- IJ4. K. Zimenko, A. Polyakov, D. Efimov, W. Perruquetti, On simple scheme of finite/fixed-time control design, **International Journal of Control**, 2018, .
- IJ5. B.H. Du, A. Polyakov, G. Zheng, Q. Quan, Quadrotor trajectory tracking by using fixed-time differentiator, **International Journal of Control**, 2018, .
- IJ6. D. Efimov, A. Levant, A. Polyakov, W. Perruquetti, Convergence acceleration for observers by gain commutation, **International Journal of Control**, 2018, .
- IJ7. T. Kharkovskaya, D. Efimov, A. Polyakov, J.-P. Richard, Design of interval observers and controls for PDEs using finite-element approximations, **Automatica**, 93(7), 2018, p. 302-310, .

- IJ8. Y. Dvir, A. Levant, D. Efimov, A. Levant, A. Polyakov, W. Perruquetti, Acceleration of finite-time stable homogeneous systems, **International Journal of Robust and Nonlinear Control**, 2018, 28(5), p. 1757-1777 .
- IJ9. G. Zheng, A. Polyakov, A. Levant, Delay estimation via sliding mode for nonlinear time-delay systems, **Automatica**, 2018, 89 (3), p. 266-273, .
- IJ10. F. Lopez Ramirez, A. Polyakov, D. Efimov, W. Perruquetti, Finite-time and fixed-time observer design: Implicit Lyapunov function approach, **Automatica**, 2018, 87, p. 52-60, .
- IJ11. M. Feingesicht, A. Polyakov, F. Kerherve, J.-P. Richard, SISO model-based control of separated flows: Sliding mode and optimal control approaches, **International Journal of Robust and Nonlinear Control**, 2017, .
- IJ12. D. Efimov, A. Levant, A. Polyakov, W. Perruquetti, Supervisory Acceleration of Convergence for Homogeneous Systems, **International Journal of Control**, 2017, .
- IJ13. K. Zimenko, D. Efimov, A. Polyakov, W Perruquetti, A note on delay robustness for homogeneous systems with negative degree, **Automatica**, 2017, 79(5), p. 178-184, .
- IJ14. D. Efimov, A. Polyakov, A. Levant, W. Perruquetti, Realization and Discretization of Asymptotically Stable Homogeneous Systems, **IEEE Transactions on Automatic Control**, 2017, 62(11), p. 5962 - 5969, .
- IJ15. D. Efimov, A. Levant, A. Polyakov, W. Perruquetti, Supervisory Acceleration of Convergence for Homogeneous Systems, **International Journal of Control**, 2017, 27(18), p. 5008-5027 .
- IJ16. A. Polyakov, L. Hetel, Relay Control Design for Robust Stabilization in a Finite-Time, **IEEE Transactions on Automatic Control**, 2017, 62(4), p. 1985-1991
- IJ17. S. Zhuk, A. Polyakov, O. Nakonechnyi, Note on Minimax Sliding Mode Control Design for Linear Systems, **IEEE Transactions on Automatic Control**, 2017, 62(7), p. 3395 - 3400
- IJ18. H. Rios, M. Mera, D Efimov, A. Polyakov, Robust output control for uncertain linear systems: Homogeneous differentiator-based observer approach, **Int. J. of Rob. and Nonlin. Contr.**, 2017, 27(11), p.1895-1914

- IJ19. K. Zimenko, A. Polyakov, D. Efimov, A. Kremlev, Feedback sensitivity functions analysis of finite-time stabilizing control system, **International Journal of Robust and Nonlinear Control**, 2017, 27(15), p. 2475-2491
- IJ20. M. Defoort, G. Demesure, Z. Zuo, A. Polyakov, M. Djemai, Fixed-time stabilisation and consensus of non-holonomic systems, **IET Control Theory & Applications**, 2016, 10(18), p. 2497 - 2505,
- IJ21. D. Efimov, E. Fridman, A. Polyakov, W. Perruquetti, J.-P. Richard, On design of interval observers with sampled measurement, **Systems & Control Letters**, 2016, 96(10), p. 158-164 (*IF*: 2.656)
- IJ22. A. Polyakov, D. Efimov, E. Fridman, W. Perruquetti, Homogenous Distributed Parameters Systems, **IEEE Transactions on Automatic Control**, 2016, 61(11), p. 3657 - 3662
- IJ23. H. Rios, D. Efimov, A. Polyakov, W. Perruquetti, Homogeneous Time-Varying Systems: Robustness Analysis, **IEEE Transactions on Automatic Control**, 2016, 61(12), p. 4075 - 4080,
- IJ24. D. Efimov, E. Fridman, A. Polyakov, W. Perruquetti, J.-P. Richard, Linear interval observers under delayed measurements and delay-dependent positivity, **Automatica**, 2016, 72(10), p. 123-130,
- IJ25. M. Mera, A. Polyakov, W. Perruquetti, Finite-time attractive ellipsoid method: implicit Lyapunov function approach, **International Journal of Control**, 2016, 89(6), p. 1079-1090,
- IJ26. D. Efimov, A. Polyakov, L. Fridman, W. Perruquetti, J.-P. Richard, Delayed sliding mode control, **Automatica**, 2016, 64(2), p. 37-43,
- IJ27. A. Polyakov, D. Efimov, W. Perruquetti, Robust stabilization of MIMO systems in finite/fixed time, **International Journal of Robust and Nonlinear Control**, 2016, 26(1), p. 69-90.
- IJ28. D. Efimov, A. Polyakov, W. Perruquetti, J.-P. Richard, Weighted Homogeneity for Time-Delay Systems: Finite-Time and Independent of Delay Stability, **IEEE Transactions on Automatic Control**, 2016, 61(1), p. 210-215,
- IJ29. A. Polyakov, D. Efimov, W. Perruquetti, J.-P. Richard, Implicit Lyapunov-Krasovski Functionals For Stability Analysis and Control Design of Time-Delay Systems, **IEEE Transactions on Automatic Control**, 2015,

- IJ30. M. Defoort, A. Polyakov, G. Demesure, M. Djemai, K. Veluvolu, Leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics, **IET Control Theory & Applications**, 2015,
- IJ31. A. Polyakov, D. Efimov, W. Perruquetti, Finite-time and Fixed-time Stabilization: Implicit Lyapunov Function Method, **Automatica**, 2015, 51(1), p.332-340,
- IJ32. A. Polyakov, Time-Suboptimal Feedback Design via Linear Matrix Inequalities, *Automation and Remote Control*, 2015, 76 (5), *Special Issue Dedicated to 80th Anniversary of Prof. Boris T. Polyak*,
- IJ33. D. Efimov, A. Polyakov, J.-P. Richard, Interval Observer Design for Estimation and Control of Time-Delay Descriptor Systems, *European Journal of Control*, 2015, 23 (5), pp.26-35,
- IJ34. A. Polyakov, L. Fridman, Stability Notions and Lyapunov Functions of Sliding Mode Systems, **Journal of the Franklin Institute**, 2014, 351(4), p. 1831-1865, .
- IJ35. D. Efimov, A. Polyakov, E. Fridman, W. Perruquetti, J.-P. Richard, Comments on Finite-Time Stability of Time Delay Systems, **Automatica**, 2014, 50(7), p. 1944-1947, .
- IJ36. V. Azhmyakov, A. Polyakov, A. Poznyak, Consistent approximations and variational description of some classes of sliding mode control processes, **Journal of The Franklin Institute**, 2014, 351(4), P. 1964-1981,
- IJ37. E. Bernuau, D. Efimov, W. Perruquetti and A. Polyakov, On Homogeneity and Its Application in Sliding Mode Control, **Journal of the Franklin Institute**, 2014, 351(4), p. 1866-1901,
- IJ38. A. Polyakov, D. Efimov, W. Perruquetti, J.-P. Richard, Output Stabilization of Time-Varying Input Delay Systems Using Interval Observation Technique, **Automatica**, 2013, 49(11), p. 3402-3410,
- IJ39. E. Bernuau, A. Polyakov, D. Efimov and W. Perruquetti, Verification of ISS, iISS and IOSS properties applying weighted homogeneity, **System & Control Letters**, 2013, 62(12), p. 1159-1167, (*IF*: 2.656)
- IJ40. A. Polyakov, Nonlinear feedback design for fixed-time stabilization of Linear Control Systems, **IEEE Transactions on Automatic Control**, 2012, 57(8), p. 2106-2110,

- IJ41. A. Polyakov, A. Poznyak, Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems, *IMA J. of Mathematical Control and Information*, 2012, 29(4), p. 529-550, (IF: 1.034).
- IJ42. A. Polyakov, Minimization of disturbances effects in time delay predictor-based sliding mode control systems, **Journal of The Franklin Institute**, 2012, 349(4), p. 1380-1396,
- IJ43. A. Polyakov, A. Poznyak, Invariant Ellipsoid Method for Minimization of Unmatched Disturbances Effects in Sliding Mode Control, 2011, **Automatica**, 47(7), p. 1450-1454,
- IJ44. A. Poznyak, A. Polyakov, V. Strygin, Analysis of finite-time convergence by the method of Lyapunov functions in systems with second-order sliding modes, *J. of Applied Mathematics and Mechanics*, 2011, 75(3), p. 289-303.
- IJ45. S. Gonzalez-Garcia, A. Polyakov, A. Poznyak, Using the Method of Invariant Ellipsoids for Linear Robust Output Stabilization of Spacecraft, *Automation and Remote Control*, 2011, 72(3), p.540-555,
- IJ46. A. Polyakov, A. Poznyak, Method of Lyapunov Function for Systems with Higher-order Sliding Modes, *Automation and Remote Control*, 2011, 72(5), p.944-963,
- IJ47. A. Polyakov, On Practical Stabilization of the Systems with Relay Delay Control, *Automation and Remote Control*, 2010, 71(11), p. 2331-2344,
- IJ48. A. Polyakov, A. Poznyak, Lyapunov Function Design for Finite-Time Convergence Analysis: "Twisting" Controller for Second Order Sliding Mode Realization, **Automatica**, 2009. 45(2), , p. 444-448,
- IJ49. V. Azmyakov, R. Galvan-Guerra, A. Polyakov, On the Method of Dynamic Programming for Linear-Quadratic Problems of Optimal Control in Hybrid Systems, *Automation and Remote Control*, 2009, 70(5), p. 787-799,
- IJ50. A. Polyakov, A. Poznyak, Reaching Time Estimation for "Super-Twisting" Second Order Sliding Mode Controller via Lyapunov Function Designing, **IEEE Transaction on Automatic Control**, 2009, 54(8), p. 1951-1955,
- IJ51. M. Yefremov, A. Polyakov, V. Strygin, An active stabilization algorithm for a spacecraft with viscoelastic elements under uncertainties, *Journal of Applied Mathematics and Mechanics*, 2006, 70(5), p. 800-811.

- IJ52. M. Yefremov, A. Polyakov, V. Strygin, A new tracking algorithm for certain mechanical systems, *Journal of Applied Mathematics and Mechanics*, 2005, 69(1), p. 27-37.
- IJ53. L. Fridman, V. Strygin, A. Polyakov, Nonlocal Stabilization via Delayed Relay Control Rejecting Uncertainty in a Time Delay, **International Journal of Robust and Nonlinear Control**, 2004, 14(1), p.15-37,
- IJ54. P. Acosta, A. Polyakov, L. Fridman, V. Strygin. Estimation of Amplitude of Oscillations in Sliding Mode Systems Caused by Time Delay, **Asian Journal of Control**, 2004, 6(4), p.513-519 (*IF*: 2.779).
- IJ55. L. Fridman, V. Strygin, A. Polyakov, Stabilization of Oscillations Amplitudes via Relay Delay Control, **International Journal of Control**, 2003, 76(8), p. 770-780,

Reviewed international conferences

- IC1. N. Espitia, A. Polyakov, D. Efimov, W. Perruquetti, On continuous boundary time-varying feedbacks for finite-time stabilization of coupled reaction-diffusion systems, **IEEE Conference on Decision and Control**, 2018.
- IC2. K. Zimenko, A. Polyakov, D. Efimov, On dynamical feedback control design for generalized homogeneous differential inclusions, **IEEE Conference on Decision and Control**, 2018.
- IC3. F. Lopez-Ramirez, D. Efimov, A. Polyakov, W. Perruquetti, On Implicit Finite-Time and Fixed-Time ISS Lyapunov Functions, **IEEE Conference on Decision and Control**, 2018.
- IC4. T. Kharkovskaya, D. Efimov, A. Polyakov, E. Fridman, J.-P. Richard, Interval estimation for second-order delay differential equations with delayed measurements and uncertainties, **IEEE Conf. on Decision and Contr.**, 2018.
- IC5. J. P. Epperlein, O. V. Iftime, S. Zhuk, A. Polyakov, Asymptotic Output-Feedback Stabilization of Linear Evolution Equations with Uncertain Inputs via Equivalent Control Method, **IEEE Conf. on Decision and Control**, 2018.
- IC6. D. Efimov, A. Polyakov, A. Aleksandrov, On State-dependent Discretization of Stable Homogeneous Systems, **IEEE Conference on Decision and Control**, 2018.

- IC7. T. Sanchez, A. Polyakov, L. Hetel, E. Fridman, A switching controller for a class of MIMO bilinear time-delay systems, **IEEE International Conference on the Science of Electrical Engineering**, 2018.
- IC8. A. Polyakov, D. Efimov, B. Brogliato, Consistent Discretization of Finite-time Stable Homogeneous Systems, **Workshop on Variable Structure Systems**, 2018.
- IC9. B. Brogliato; A. Polyakov, D. Efimov, The implicit discretization of the super-twisting sliding-mode control algorithm, **Workshop on Variable Structure Systems**, 2018.
- IC10. T. Sanchez, A. Polyakov, J.-P. Richard, D. Efimov, A robust Sliding Mode Controller for a class of SISO bilinear delayed systems, **Workshop on Variable Structure Systems**, 2018
- IC11. K. Zimenko,A. Polyakov, D. Efimov, Finite-Time and Fixed-Time Stabilization for Integrator Chain of Arbitrary Order, **European Control Conference**, 2018.
- IC12. K. Zimenko,A. Polyakov, D. Efimov, Generalized Feedback Homogenization and Stabilization of Linear MIMO Systems, **European Control Conference**, 2018.
- IC13. F. Lopez-Ramirez, D. Efimov, A. Polyakov, W. Perruquetti, On Necessary and Sufficient Conditions for Fixed-Time Stability of Continuous Autonomous Systems, **European Control Conference**, 2018.
- IC14. K. Zimenko, D. Efimov, A. Polyakov, A. Kremlev, Control of Systems with Arbitrary Bounded Input Delay Using Implicit Lyapunov Function Technique, **European Control Conference**, 2018.
- IC15. A. Polyakov, Quadratic-like Stability of Nonlinear Homogeneous Functions, **IEEE Conference on Decision and Control**, 12-15 December 2017.
- IC16. A. Polyakov, L. Hetel, C. Fiter, Relay Control Design using Attractive Ellipsoids Method, **IEEE Conference on Decision and Control**, 12-15 December 2017.
- IC17. T. Sanchez, D. Efimov, A. Polyakov, J. Moreno, W. Perruquetti, A homogeneity property of a class of discrete-time systems, **IEEE Conference on Decision and Control**, 12-15 December 2017.

- IC18. A. Polyakov, D. Efimov, E. Fridman, W. Perruquetti, J.-P. Richard, On Hyper Exponential Stabilization of Linear State-Delay Systems, **IEEE Conference on Decision and Control**, 12-15 December 2017.
- IC19. A. Polyakov, J.-M. Coron, L. Rosier, On Boundary Finite-Time Feedback Control for Heat Equation, **IFAC World Congress**, 9-14 July 2017.
- IC20. M. Feingesicht, A. Polyakov, F. Kerherve, J.-P. Richard, Sliding Mode Control for Turbulent Flows, **IFAC World Congress**, 9-14 July 2017.
- IC21. M. Feingesicht, A. Polyakov, F. Kerherve, J.-P. Richard, Model-Based Feedforward Optimal Control applied to a Turbulent Separated Flow, **IFAC World Congress**, 9-14 July 2017.
- IC22. T. Kharkovskaya,, D. Efimov, E. Fridman, A. Polyakov, J.-P. Richard, On design of interval observers for parabolic PDEs, **IFAC World Congress**, 9-14 July 2017.
- IC23. D. Efimov, A. Polyakov, A. Levant, W. Perruquetti, Switched gain differentiator with fixed-time convergence, **IFAC World Congress**, 9-14 July 2017.
- IC24. S. Zhuk, A. Polyakov, O. Nakonechnyi, Sliding Mode Control Design for Linear Evolution Equations with Uncertain Measurements and Exogenous Perturbations, **IFAC World Congress**, 9-14 July 2017.
- IC25. A. Polyakov, J.-M. Coron, L. Rosier, On Finite-Time Stabilization of Evolution Equations: A Homogeneous Approach, **IEEE Conference on Decision and Control**, Las Vegas, USA, 12-14 December 2016.
- IC26. D. Efimov, A. Polyakov, A. Levant, W. Perruquetti, Discretization of asymptotically stable homogeneous systems by explicit and implicit Euler methods, **IEEE Conf. on Decision and Control**, Las Vegas, USA, 12-14 December 2016.
- IC27. A. Levant, D. Efimov, A. Polyakov, W. Perruquetti, Stability and Robustness of Homogeneous Differential Inclusions, **IEEE Conference on Decision and Control**, Las Vegas, USA, 12-14 December 2016.
- IC28. D. Efimov, A. Levant, A. Polyakov, W. Perruquetti, On Acceleration of a Class of Asymptotically Stable Homogeneous Systems, **IEEE Conference on Decision and Control**, Las Vegas, USA, 12-14 December 2016.

- IC29. F. Lopez-Ramirez, D. Efimov, A. Polyakov, W. Perruquetti, Fixed-Time Output Stabilization of a Chain of Integrators, **IEEE Conference on Decision and Control**, Las Vegas, USA, 12-14 December 2016.
- IC30. T. Kharkovskaya, D. Efimov, A. Polyakov, J.-P. Richard, Interval observers for PDEs: approximation approach, **NOLCOS**, Monterey, USA, 22-25 August 2016
- IC31. K. Zimenko, D. Efimov, A. Polyakov, W. Perruquetti, Time-delay Robustness Analysis for Systems with Negative Degree of Homogeneity, **NOLCOS**, Monterey, USA, 22-25 August 2016
- IC32. K. Zimenko, A. Polyakov, D. Efimov, A. Kremlev, Frequency Domain Analysis of Control System Based on Implicit Lyapunov Function, **European Control Conference**, Aalborg, Denmark, 29 June-1 July 2016
- IC33. M. Feingesicht, C. Raibaud, A. Polyakov, F. Kerherve, J.-P. Richard, A bilinear input-output model with state-dependent delay for separated flow control, **European Cont. Conf.**, Aalborg, Denmark, 29 June-1 July 2016
- IC34. F. Lopez-Ramirez, A. Polyakov, D. Efimov, W. Perruquetti, Finite-Time and Fixed-Time Observers Design via Implicit Lyapunov Function, **European Control Conference**, Aalborg, Denmark, 29 June-1 July 2016
- IC35. D. Efimov, E. Fridman, A. Polyakov, W. Perruquetti, J.P. Richard, On Design of Sampled-Data Interval Observers, **European Control Conference**, Aalborg, Denmark, 29 June-1 July 2016
- IC36. A. Polyakov, L. Hetel, On Finite-Time Stabilization Via Relay Feedback Control, **IEEE Conference on Decision and Control**, 2015, Osaka, Japan, p. 4098 - 4102
- IC37. A. Polyakov, Y. Orlov, H. Oza, S. Spurgeon, Robust Finite-time Stabilization and Observation of a Planar System Revisited , **IEEE Conference on Decision and Control**, 2015, Osaka, Japan, p. 5689-5694
- IC38. K. Zimenko, A. Polyakov, D. Efimov, Stabilization of Chain of Integrators with Arbitrary Order in Finite-Time , **IEEE Conference on Decision and Control**, 2015, Osaka, Japan, p. 4637 - 4641
- IC39. B. Brogliato, A. Polyakov, Globally Stable Implicit Euler Time-Discretization of a Nonlinear Single-Input Sliding-Mode Control System, **IEEE Conference on Decision and Control**, 2015, Osaka, Japan, p. 5426 - 5431

- IC40. M. Mera, A. Polyakov, W. Perruquetti, G. Zheng, Finite-Time Attractive Ellipsoid Method Using Implicit Lyapunov Functions, **IEEE Conference on Decision and Control**, 2015, Osaka, Japan, p. 6892-6896
- IC41. A. Polyakov, D. Efimov, E. Fridman, W. Perruquetti, Homogeneous Evolution Equation in a Banach Space, **European Control Conference**, 2015, Linz, Austria, p.2469-2474
- IC42. S. Zhuk, A. Polyakov, Output-Based Sliding Mode Control Design for Linear Plants with Multiplicative Disturbances: The Minimax Approach, **European Control Conference**, 2015, Linz, Austria, p. 227-282
- IC43. D. Efimov, A. Polyakov, E. Fridman, W. Perruquetti, J.-P. Richard, Delay-Dependent Positivity: Application to Interval Observers, **European Control Conference**, 2015, Linz, Austria, p. 2079-2083
- IC44. D. Efimov, A. Polyakov, L. Fridman, W. Perruquetti, J.-P. Richard, A Note on Continuous Delayed Sliding Mode Control, **European Control Conference**, 2015, Linz, Austria, p. 2401-2406
- IC45. A. Polyakov, D. Efimov, W. Perruquetti, J.-P. Richard, Implicit Lyapunov-Krasovski Functions for Time-Delay Systems, **IEEE Conference on Decision and Control**, 15-17 December 2014, Los Angeles, USA.
- IC46. A. Polyakov, D. Efimov, W. Perruquetti, J.-P. Richard, Implicit Lyapunov-Krasovski Functionals for time delay systems, **IEEE Conference on Decision and Control**, 15-17 December 2014, Los Angeles, USA, 2014, p. 1191-1196.
- IC47. D. Efimov, A. Polyakov, W. Perruquetti, J.-P. Richard, Stability Analysis for Non-linear Time-Delay Systems Applying Homogeneity, **IEEE Conference on Decision and Control**, 15-17 December 2014, Los Angeles, USA, p. 37-42.
- IC48. A. Polyakov, I. Chairez, A New Homogeneous Quasi-Continuous Second Order Sliding Mode Control, **Latinamerican Congress on Automatic Control**, 14-17 October 2014, Cancun, Mexico.
- IC49. A. Polyakov, Discontinuous Lyapunov Functions for Nonasymptotic Stability Analysis, **IFAC World Congress**, 24-29 August 2014, Cape Town, South Africa, p. 5455-5460.
- IC50. A. Polyakov, J.-P. Richard, Suboptimal LMI-based Solution of Minimum Time Control Problem, **IFAC World Congress**, 24-29 August 2014, Cape Town, South Africa, p. 2475-2480.

- IC51. S. Zhuk, A. Polyakov, On Output-based Sliding Mode Control Design Using Minimax Observer, **International Workshop on Variable Structure Systems**, 29 June-2 July 2014, Nantes, France.
- IC52. M. Kryachkov, A. Polyakov, Graph-based field automata for modeling of sliding mode systems, **1 International Workshop on Variable Structure Systems**, 29 June-2 July 2014, Nantes, France.
- IC53. A. Polyakov, D. Efimov, W. Perruquetti, Sliding Mode Control Design for MIMO Systems: Implicit Lyapunov Function Approach, **European Control Conference**, 24-27 June 2014, Strasburg, France, p. 2612-2617.
- IC54. A. Polyakov, D. Efimov, W. Perruquetti, Homogeneous Differentiator Design using Implicit Lyapunov Function Method, **European Control Conference**, 24-27 June 2014, Strasburg, France, p. 288-293.
- IC55. D. Efimov, A. Polyakov, J.-P. Richard, Interval estimation for systems with time delays and algebraic constraints, **European Control Conference**, 24-27 June 2014, Strasburg, France, p.678-683.
- IC56. S. Govindaswamy, L. Hetel, A. Polyakov, T. Floquet, On relay control for discrete time systems using linear matrix inequalities, **European Control Conference**, 24-27 June 2014, Strasburg, France, p. 2225-2230.
- IC57. A. Polyakov, A. Poznyak, J.-P. Richard, Robust Output Stabilization of Time-Varying Input Delay Systems using Attractive Ellipsoid Method, **IEEE Conference on Decision and Control**, 10-13 December 2013, Florence, Italy.
- IC58. S. Parsegov, A. Polyakov, P.S. Scherbakov, Fixed-Time Consensus Algorithm for Multi-Agent Systems with Integrator Dynamics, **IFAC Workshop on Distributed Estimation and Control in Networked Systems**, 2013, p. 110-115.
- IC59. A. Polyakov, D. Efimov, W. Perruquetti, Finite-time Stabilization Using Implicit Lyapunov Function Technique, **Symposium on Nonlinear Control Systems**, Toulouse, France, 4-6 September 2013, p. 140-145.
- IC60. A. Polyakov, D. Efimov, W. Perruquetti, J.-P. Richard, Interval Observer Approach to Output Stabilization of Time-Varying Input Delay System, **European Control Conference**, Zurich, 17-19 July 2013, p. 4412- 4417.
- IC61. E. Bernuau, D. Efimov, W. Perruquetti, A. Polyakov, On an extension of homogeneity notion for differential inclusions, **European Control Conference**, Zurich, 17-19 July 2013, p. 2204-2209.

- IC62. E. Bernuau, A. Polyakov, D. Efimov, W. Perruquetti, On ISS and iISS properties of homogeneous systems, **European Control Conference**, 17-19 July 2013, Zurich, Switzerland, p. 1705-1710.
- IC63. E. Bernuau, A. Polyakov, D. Efimov, W. Perruquetti, Robustness of finite-time stability property for sliding modes, **Symposium on System Structure and Control**, 4-6 February 2013, Grenoble, France, p. 391-396.
- IC64. S. Parsegov, A. Polyakov, P. Shcherbakov, Nonlinear Fixed-Time Control Protocol for Uniform Allocation of Agents on a Segment, **IEEE Conference on Decision and Control**, 10-13 December 2012, Maui, Hawaii, USA, p. 7732-7737.
- IC65. M. Mera, A. Poznyak, V. Azhmyakov, A. Polyakov, A Robust Dynamic Controller for a Class of Nonlinear Systems with Sample-Data Outputs, **Int. Conf. on Electrical Engineering, Comp. Science and Autom. Control**, 2012
- IC66. A. Polyakov, Fixed-Time Stabilization via Second Order Sliding Mode Control, **Analysis & Design of Hybrid Systems**, 6-8 June 2012, Eindhoven, Netherlands, P. 254 - 258.
- IC67. A. Polyakov, Fixed-Time Stabilization of Linear Systems via Sliding Mode Control, **International Workshop on Variable Structure Systems**, 12-14 January 2012, Mumbai, India, P. 1- 6.
- IC68. A. Polyakov, On Settling Time Function and Stability of Vector Relay Systems, **International Workshop on Variable Structure Systems**, 12-14 January 2012, Mumbai, India, P. 149-154.
- IC69. A. Polyakov, A. Poznyak, The Lyapunov Function Design for the Stability Analysis of the "Italian Version" of the Second Order Sliding Mode Controllers, **IFAC World Congress**, 28 August - 2 September 2011, Milan, Italy.
- IC70. A. Polyakov, Invariant Ellipsoid Method for Time-Delayed Predictor-Based Sliding Mode Control System, **International Workshop on Variable Structure Systems**, 26-28 June 2010, Mexico D.F., P. 95-99.
- IC71. M. Kryachkov, A. Polyakov, V. Strygin, Finite-time stabilization of an integrator chain using only signs of the state variables, **International Workshop on Variable Structure Systems**, 26-28 June 2010, Mexico D.F.
- IC72. S. Gonzalez-Garcia, A. Polyakov, A. Poznyak, Linear feedback spacecraft stabilization using the method of invariant ellipsoids, **Southeastern Symposium on System Theory**, Tullahoma, TN, USA, March 15-17, 2009.

- IC73. S. Gonzalez-Garcia, A. Polyakov, A. Poznyak, Output Linear Controller for a Class of Nonlinear Systems Using the Invariant Ellipsoid Method, **American Control Conference**, St. Louis, Missouri, USA, 10-12 June 2009.
- IC74. A. Polyakov, A. Poznyak, Minimization of the Unmatched Disturbances in the Sliding Mode Control Systems via Invariant Ellipsoid Method, **IEEE Multi Conference on Systems and Control**, 2009
- IC75. A. Polyakov, A. Poznyak, Lyapunov Function Design for Finite-Time Convergence Analysis of "Twisting" and "Super-Twisting" Second Order Sliding Mode Controllers, **IFAC Workshop on Variable Structure Systems**, 2008
- IC76. S. Gonzalez-Garcia, A. Polyakov, A. Poznyak, Output linear feedback for a class of nonlinear systems based on the invariant ellipsoid method, **Int. Conf. on Electr. Engineering, Comp. Science and Autom. Control**, 2008
- IC77. A. Polyakov, Practical Stabilization via Relay Delayed Control, **IEEE Conference on Decision and Control**, 8-11 December 2008, Cancun, Mexico, P. 5306-5311.
- IC78. S. Gonzalez-Garcia, A. Polyakov, A. Poznyak, Robust Stabilization of a spacecraft with flexible elements using invariant ellipsoid technique, **Int. Conf. on Electr. Engineering, Comp. Science and Autom. Control**, 2008
- IC79. L. Fridman, V. Strygin, A. Polyakov, Semiglobal Stabilization via Relay Controller with Uncertain Time Delay, **IEEE Conference on Decision and Control**, Maui, Hawaii, 9-12 December 2003, p.4074-4079.
- IC80. L. Fridman, P. Acosta, A. Polyakov, V. Strygin. Amplitude Estimation of Oscillations in Delayed Relay Systems, **American Control Conference**, Denver, Colorado, USA, 4-6 June 2003, pp. 2676-2681.
- IC81. L. Fridman, V. Strygin, A. Polyakov. Delayed relay nonlocal stabilization suppressing uncertainty in a time delay. **IFAC Symposium on Robust Control Design**, Milan, 25-27 June 2003, pp.389-394.
- IC82. L. Fridman, V. Strygin, A. Polyakov, Nonlocal Stabilization via Delay Relay Control Gain Adaptation, **American Control Conference**, Anchorage, Alaska, 8-10 May 2002, pp. 2221-2226.
- IC83. L. Fridman, V. Strygin, A. Polyakov, Algorithm of Nonlocal Stabilization via Delayed Control Gain Adaptation, **IEEE Workshop on Variable Structure Systems**, Sarajevo, 17-19 July 2002, pp. 47-56.

IC84. L. Fridman, V. Strygin, A. Polyakov, Stabilization of Oscillations Amplitudes via Relay Delay Control, **IEEE Conference on Decision and Control**, Orlando, Florida, 4-7 December 2001, pp.3704-3709.

IC85. L. Fridman, P. Acosta, A. Polyakov, Robust eigenvalue assignment for uncertain delay control systems, **IFAC Workshop on Time Delay Systems**, Santa Fe, New Mexico, USA, 8-10 December 2001, pp.239-244.

IC86. L. Fridman, V. Strygin, A. Polyakov, Relay Control of Oscillations Amplitudes for Systems with Delay, **IFAC Workshop on Time Delay Systems**, Santa Fe, New Mexico, USA, 8-10 December 2001, pp.245-248.

Books and book chapters

B1. A. Poznyak, A. Polyakov, V. Azhmyakov, **Attractive Ellipsoids in Robust Control**, Birkhauser, 2014.

B2. A. Polyakov, Fast Control Systems: Nonlinear Approach, in **New Perspectives and Applications of Modern Control Theory** (Editors: J. B. Clempner and W. Yu). - Springer, 2017. - pp. 287-316.

B3. S. Zhuk, A. Polyakov, Minimax Observer for Sliding Mode Control Design, in **Recent Trends in Sliding Mode Control** (Editors L. Fridman, J.-P. Barbot, F. Plestan). - London, IET, 2016. - pp. 119-130.

B4. E. Bernuau, D. Efimov, W. Perruquetti, A. Polyakov, Homogeneity of Differential Inclusions, in **Recent Trends in Sliding Mode Control** (Editors L. Fridman, J.-P. Barbot, F. Plestan). - London, IET, 2016. - pp. 103-118.

B5. L. Fridman, V. Strygin, A. Polyakov, Semiglobal Stabilization of Linear Uncertain System via Delayed Relay Control, in **Variable Structure Systems : from Principles to Implementation**. - London, 2004. - pp. 377-400.

National journals

NJ1. K. A. Zimenko, A. S. Kremlev, A. E. Polyakov, D. V. Efimov, Robust stabilization of two-rotor MIMO plant, Scientific and Technical Journal of Information Technologies, Mechanics and Optics, Vol.18, N. 3, 2018, p. 392-398

- NJ2. K. A. Zimenko, A. E. Polyakov, Convergence speed increase of a finite-time stable control system, Scientific and Technical Journal of Information Technologies, Mechanics and Optics, Vol. 17 (6), 2017, p. 1018-1024.
- NJ3. K. A. Zimenko, A. E. Polyakov, D. V. Efimov, A. S. Kremlev, Finite-time Stability of System of Series-Connected Integrators, Izvestiya Vysshikh Uchebnykh Zavedeniy. Priborostroenie, 2015, 58 (9), p. 681-686 (in Russian).
- NJ4. S. Parsegov, A. Polyakov, P. Shcherbakov, Nonlinear fixed-time control protocol for uniform allocation of agents on a segment, Doklady Mathematics, Vol. 87(1), 2013, p. 133-136. (Translated from Proc. of the Russian Acad. of Sci.)
- NJ5. V. Strygin, M. Yefremov, A. Polyakov, S. Prosin, The Algorithm of Angular Stabilization of Spacecraft with Flexible Dynamical Elements under Uncertain Conditions, Proceedings of Voronezh State University, No.1, 2006.(in Russian)
- NJ6. A. Polyakov, K.S. Kobychyev, Stabilization of mechanical systems via relay delayed control, Chernozemny anthology of scientific research. Series: Appl. Maths and Comp. Sciences, Voronezh, 2006, Vol. 1(2), pp. 15-26. (in Russian)
- NJ7. V. Strygin, A. Polyakov, M. Kryachkov On some air-to-fuel ratio control algorithms for automobile engines, Proceedings of Voronezh State University, Series: Mathematics and Physics, No.1, 2005.(in Russian)
- NJ8. V. Strygin, L. Fridman, A. Polyakov, Local Stabilization of Relay Systems With Delay, Doklady Mathematics, Vol. 64(1), 2001, p.106-108. (Translated from Proceedings of the Russian Academy of Sciences) (*IF*:0.289)

Reviewed national conferences

- NC1. M. Kryachkov, A. Polyakov, V. Strygin, Stabilization of uncertain system without quantitative information on system state, Control in Technical Systems, Saint-Petersburg, 2010 (in Russian).
- NC2. A. Polyakov, A. Poznyak, Lyapunov Function Design for Finite-Time Convergence Analysis of The Second Order Sliding Mode Controllers, X International Workshop "Stability and Oscillations of Nonlinear Control Systems", 3 - 6 of June, Moscow, 2008 (in Russian).
- NC3. S.A. Prosin, A. Polyakov, V.V. Strygin, Modeling, control and realization of an Aircraft Landing, X International Workshop "Stability and Oscillations of Nonlinear Control Systems", 3-6 June 2008, Moscow (in Russian).
- NC4. M. Yefremov, A. Polyakov, S. Prosin, V. Strygin, An Universal Algorithm of Semiglobal Stabilization of Mechanical Systems with Uncertainties, Proc. of 9-th All-Russian Congress on Theoretical and Applied Mechanics, 22 - 28 August 2006, N. Novgorod, Russia (in Russian).

B. Selected articles

The following my articles are attached below

- *A. Polyakov*, A. Poznyak, Lyapunov Function Design for Finite-Time Convergence Analysis: "Twisting" Controller for Second Order Sliding Mode Realization, **Automatica**, 2009. 45(2), p. 444-448
- *A. Polyakov*, Nonlinear feedback design for fixed-time stabilization of Linear Control Systems, **IEEE Transactions on Automatic Control**, 2012, 57(8), p. 2106-2110
- *A. Polyakov*, D. Efimov, W. Perruquetti, Finite-time and Fixed-time Stabilization: Implicit Lyapunov Function Method, **Automatica**, 2015, 51(1), p.332-340
- *A. Polyakov*, L. Hetel, Relay Control Design for Robust Stabilization in a Finite-Time, **IEEE Transactions on Automatic Control**, 2017, 62(4), p. 1985-1991

Lyapunov Function Design for Finite-Time Convergence Analysis: "Twisting" Controller for Second Order Sliding Mode Realization

Andrei Polyakov and Alex Poznyak*

Automatic Control Department, CINVESTAV-IPN, A.P. 14-740, C.P. 07360 Mexico D.F, e-mail: apolyakov(apoznyak)@ctrl.cinvestav.mx,

Abstract

A generalization of the Zubov's method of a Lyapunov functions design is presented. It is based on the characteristic method application and related to resolving of the first-order partial differential equation of a special type. A successful resolution of this equation guaranties a finite-time convergence for the corresponding dynamics given by an ordinary differential equation with a discontinuous right-hand side. The suggested method is illustrated by its application to the, so-called, "twisting" controller stability analysis. The constructed Lyapunov function as well as its level lines sections is graphically illustrated.

Key words: Lyapunov function; finite-time convergence; characteristic method; high-order sliding mode

1 Introduction

1.1 On the Lyapunov function approach

In the seminal work published by A.M.Lyapunov in the end of the 19-th century there were presented some very simple (but, philosophically, very profound) theorems (after referred to as the *direct Lyapunov's method*) for deciding the stability or instability of an equilibrium point of an ODE. The idea of this approach consists in the generalization of the concept of "energy" and its "power" which usefulness lies in the fact that the decision on stability can be made by investigating the differential equation itself (in, fact, its right-hand side only) but not by finding its exact solution. The considered class of ODE contained basically the so-called regular differential equations with right-hand sides continuous on both variables: time and state. Later several authors [5], [18] considered more wide class of differential equations where right-hand side was discontinuous on its variables. Modern approaches usually treat such class of differential equations as differential inclusions (DI). This class of systems includes as a subclass the so-called *Variable Structure Systems* (VSS') which has attracted a considerable research interest in the last three decades

Email address:

apolyakov(apoznyak)@ctrl.cinvestav.mx (Andrei Polyakov and Alex Poznyak*).

[17], [8], [9], [13]. Although Lyapunov-type methods have been widely used in practice [3], [16], the need for non-smooth Lyapunov functions in particular has been recognized for non-smooth system models [1]. So, to analyze VSS' with sliding modes, where "kinks" form an essential part of dynamics, a non-smooth Lyapunov-like analysis is required [12].

1.2 Motivation

One of the most important features of VSS' systems with sliding modes is a finite-time convergence to a sliding surface (or manifold), or in other words, a *finite-time reaching phase* t_{reach} . The corresponding Lyapunov functions are non-smooth in this case. For example (see [17]), for the *first-order sliding mode* systems, the simplest representative of which is

$$\dot{x}(t) = -r \operatorname{sign}[x(t)], \quad r > 0 \quad (1)$$

with

$$\operatorname{sign}[x] := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \in [-1, 1] & \text{if } x = 0 \end{cases} \quad (2)$$

evidently, the Lyapunov function is $V(x) = |x|$, and hence, its time-derivative (for $x \neq 0$) taken on the tra-

jectories of (1) satisfies

$$\dot{V}(x(t)) = -r$$

that implies $0 \leq V(x(t)) = V(x(0)) - rt$ and defines $t_{reach} = V(x(0))/r$. For the *second-order sliding modes* (see [8]), such as

$$\ddot{x}(t) = -r_1 \text{sign}[x(t)] - r_2 \text{sign}[\dot{x}(t)], \quad r_1 > r_2 > 0 \quad (3)$$

the finite-time convergence analysis and the reaching time estimation have been done by a geometrical setting [10]. Such proof seems to be difficult to extend to multidimensional case. Besides, in [12] to analyze the convergence of $x(t)$ governed by (3) there has been suggested the Lyapunov function approach with the Lyapunov function

$$V(x, \dot{x}) = r_1 |x| + \dot{x}^2/2 \quad (4)$$

which guarantees only asymptotic (non finite-time) convergence since in this case it turns out that $\dot{V}(x(t), \dot{x}(t)) \leq 0$. The modern approaches to the finite-time convergence analysis of high-order sliding mode systems is based on homogeneity principle [2], [11], [16], [6]. However, it *does not allow* to estimate the reaching time. Thus, the *unified approach* to the problems of the finite-time convergence analysis and the reaching time estimation of the high-order sliding mode systems has *never* been presented before.

So, the *motivation* of this study is based on the following remarks:

- the geometrical approach for the convergence analysis and reaching time estimation can not be definitely extended to multidimensional case;
- the Lyapunov functions providing finite-time convergence of the high-order sliding systems have never been designed;
- the finite-time converged Lyapunov function permits to estimate the reaching time even in the presence of a bounded deterministic noise.

1.3 Structure of the paper

In the next section the main ideas of the Lyapunov function design providing a finite-time convergence are discussed. Then, using the suggested approach, the process of the Lyapunov function designing for the so-called "twisting" controller is given. Finally, it is illustrated by a numerical example.

2 The Method of the Lyapunov Function Design with Finite-Time Convergence

2.1 System description

Consider an affine control system given by

$$\begin{cases} \dot{x} = g(x, y) \\ \dot{y} = a(x, y) + b(x, y)u + f(t, x, y) \end{cases} \quad (5)$$

where $x \in \mathbb{R}^k, y \in \mathbb{R}^n$ are components of the state-space vector, $g: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^k, a: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are smooth system vector functions, $u \in \mathbb{R}^m$ is vector of control inputs, $b: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are control-gain matrix, and $f: \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an uncertain measurable but bounded function (in fact, a deterministic noise), i.e.,

$$|f_j(t, x, y)| \leq C_j, \quad \forall x \in \mathbb{R}^k, \forall y \in \mathbb{R}^n, \forall t \geq 0, j = \overline{1, n} \quad (6)$$

Supposing that a stabilizing control u is already designed as

$$u = \bar{u}(x, y) \quad (7)$$

(which, in fact, is admitted to be discontinuous) we have to prove the stability (asymptotic or finite-time) of zero solution $(0, 0)$ of the system (5) with the control (7) using the Lyapunov function technique.

2.2 The generalization of Zubov's Method

Let $V(x, y)$ be a function differentiable almost everywhere. Then its derivative on the trajectories of (5) is estimated as follows

$$\begin{aligned} \dot{V} &= \langle \nabla_x V, g \rangle + \langle \nabla_y V, a + b\bar{u} + f \rangle \\ &\leq \langle \nabla_x V, g \rangle + \langle \nabla_y V, a + b\bar{u} \rangle + \langle |\nabla_y V|, C \rangle \end{aligned} \quad (8)$$

where $\langle \cdot, \cdot \rangle$ means the scalar product in an Euclidian space, $|\nabla_y V| = (|\frac{\partial V}{\partial y_1}|, |\frac{\partial V}{\partial y_2}|, \dots, |\frac{\partial V}{\partial y_n}|)^T$ and $C = (C_1, C_2, \dots, C_n)^T$ is the vector with non-negative components. Denoting

$$h_j(x, y, \gamma_j) := a_j(x, y) + b_j(x, y)\bar{u}(x, y) + \gamma_j, \quad j = \overline{1, n}$$

the inequality (8) might be rewritten (in the component-wise form) as

$$\frac{dV}{dt} \leq \sum_{i=1}^k \frac{\partial V}{\partial x_i} g_i(x, y) + \sum_{j=1}^n \frac{\partial V}{\partial y_j} h_j(x, y, C_j \text{sign}[\frac{\partial V}{\partial y_j}])$$

Let us try to find the Lyapunov function $V(x, y)$ as an *absolute continuous positive definite solution* of the following partial differential equation

$$\sum_{i=1}^k g_i(x, y) \frac{\partial V}{\partial x_i} + \sum_{j=1}^n h_j(x, y, \gamma_j) \frac{\partial V}{\partial y_j} = -qV^p \quad (9)$$

where $\gamma_j = C_j \text{sign}[\frac{\partial V}{\partial y_j}]$ and $q > 0, \rho > 0$ are some positive parameters.

Remark 1 a) In the case $\rho < 1$ we obtain the Lyapunov function with finite time convergence $t_{reach} \leq \frac{1}{q(1-\rho)} [V(x(0), y(0))]^{1-\rho}$.
b) For $\rho \geq 1$ the positive definite solution of (9) implies only an asymptotic convergence

To solve the partial differential equation (9) we need to define precisely the piecewise constant function γ_j , which depends on $\text{sign}[\frac{\partial V}{\partial y_j}]$. Fortunately, since γ_j takes only two values C_j and $-C_j$, we may consider (9) supposing that γ_j is a one of these two constants. Then, obtaining $V(x, y)$ as a solution of (9), one can try define the parameters γ_j in an appropriate way.

The idea of the Lyapunov function design as the solution of the partial differential equation was presented by V.I Zubov [19]. The well-known variable gradient method (see, for example, [7]) is also based on this idea. The presented generalization of the Zubov's method permits the right-hand side of (9) to be depended on the function V too. Only this assumption allows us to find the Lyapunov function with finite-time convergence.

2.3 Method of Characteristics

The solution of (9) can be found using the so-called *Method of Characteristics* [4]. According to this method we can formulate the following lemma.

Lemma 1 *If an absolute continuous positive definite function $V(x, y)$ satisfies the following system of ODE (in the symbolic form)*

$$\begin{aligned} \frac{dx_1}{g_1(x, y)} &= \dots = \frac{dx_k}{g_k(x, y)} = \frac{dy_1}{h_1(x, y, \gamma_1)} = \dots \\ &= \frac{dy_n}{h_n(x, y, \gamma_n)} = \frac{dV}{-qV^\rho} \end{aligned} \quad (10)$$

for $\|x\|^2 + \|y\|^2 > 0$, then the same function $V(x, y)$ is a solution of (9).

PROOF. For $\|x\|^2 + \|y\|^2 > 0$ we have $V(x, y) > 0$, $dx_i = -g_i(x, y) \frac{dV}{qV^\rho}$, $dy_j = -h_j(x, y, \gamma_j) \frac{dV}{qV^\rho}$ and

$$\begin{aligned} dV &= \sum_{i=1}^k \frac{\partial V}{\partial x_i} dx + \sum_{j=1}^n \frac{\partial V}{\partial y_j} dy \\ &= - \left[\sum_{i=1}^k \frac{\partial V}{\partial x_i} g_i(x, y) + \sum_{j=1}^n \frac{\partial V}{\partial y_j} h_j(x, y, \gamma_j) \right] \frac{dV}{qV^\rho} \end{aligned}$$

implies (9).

Suppose that the system of ODE (10) can be solved and its first integrals are

$$\varphi_i(V, x, y, \gamma, q, \rho) = \text{const} := c_i, \quad i = \overline{1, n+k} \quad (11)$$

Since any function of constants is a constant too (in the partial case, zero), the function $V(x, y, \gamma, q, \rho)$ can be found as a solution of nonlinear algebraic equation

$$\Phi(\varphi_1(V, x, y, \gamma, q, \rho), \dots, \varphi_{n+k}(V, x, y, \gamma, q, \rho)) = 0 \quad (12)$$

where $\Phi(\varphi_1, \dots, \varphi_{n+k})$ is an arbitrary function. An explicit analytical solution of the last equation (if it exists) defines the Lyapunov function candidate. However, the function Φ and the parameters γ, q, ρ should be chosen in such a way that the function $V(x, y)$ should be an absolute continuous and positive definite. Only such functions $V(x, y)$ may be considered as a Lyapunov function candidate.

Remark 2 *Obviously, this approach does not provide a formal algorithm for a Lyapunov function design, but only helps to derive a specific form of a Lyapunov function candidate that reduces the problem of a Lyapunov function design to the problem of the correct parameters definition which guarantee the absolute continuity and positive definiteness of the obtained "energetic" function.*

Below we will illustrate the proposed method designing the Lyapunov function with finite time convergence ($0 < \rho < 1$) for the second order sliding system with "twisting" controller.

3 Lyapunov Function Design for Twisting Control Algorithm

3.1 System description and Problem formulation

Consider the controlled system given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(t, x, y) + u(t) \end{cases} \quad (13)$$

where $x, y \in \mathbb{R}$ are the scalar state variables, $f(t, x, y)$ is a measurable but unknown function treated hereafter as an external noise or perturbation, $u \in \mathbb{R}$ is the so-called "twisting" control (see [8])

$$u(t) = -r_1 \text{sign}[x(t)] - r_2 \text{sign}[y(t)] \quad (14)$$

where $r_1, r_2 > 0$ are control parameters, and the operator $\text{sign}[\cdot]$ is defined by (2). The solution of (13) is understood in the Filippov's sense (see [5]).

It is supposed that

$$|f(t, x, y)| \leq C, \quad \forall x, y \in \mathbb{R} \quad \text{and} \quad \forall t \geq 0 \quad (15)$$

with a known constant C .

Now the main problem is to design the Lyapunov function which provides the finite-time convergence of the solutions of (13) to the origin $(0, 0)$ under the control (14) and the assumptions above.

3.2 Partial Differential Equation

For an arbitrary absolute continuous function $V(x, y)$ we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} = y \frac{\partial V}{\partial x} + (u + f(t, x, y)) \frac{\partial V}{\partial y} \leq \\ y \frac{\partial V}{\partial x} + (u + C \operatorname{sign}[\frac{\partial V}{\partial y}]) \frac{\partial V}{\partial y} &= y \frac{\partial V}{\partial x} - \operatorname{sign}[x] \gamma \frac{\partial V}{\partial y} \\ \gamma &= r_1 + r_2 \operatorname{sign}[xy] - \gamma_0, \gamma_0 = C \operatorname{sign}[x \frac{\partial V}{\partial y}] \end{aligned} \quad (16)$$

According to the proposed method the equation (9) becomes to be as

$$y \frac{\partial V}{\partial x} - \operatorname{sign}[x] \gamma \frac{\partial V}{\partial y} = -kV^{1/2} \quad (17)$$

where $k > 0$ is a positive parameter. Supposing that γ_0 is a constant let us find the solution of (17). The corresponding system (10) of the characteristics ODE is

$$\frac{dx}{y} = \frac{dy}{-\operatorname{sign}[x] \gamma} = \frac{dV}{-k\sqrt{V}} \quad (18)$$

Its first integrals are as follows

$$\varphi_1(x, y) = |x| + \frac{y^2}{2\gamma}, \varphi_2(V, y) = \frac{y \operatorname{sign}[x]}{\gamma} - \frac{2\sqrt{V}}{k} \quad (19)$$

Select Φ as

$$\Phi(\varphi_1, \varphi_2) = k_0 \sqrt{\varphi_1} + \varphi_2 = 0 \quad (20)$$

where k_0 is a real parameter. Substitution (19) in to (20) leads to the equation

$$\frac{2\sqrt{V}}{k} = \frac{y \operatorname{sign}[x]}{\gamma} + k_0 \sqrt{|x| + \frac{y^2}{2\gamma}} \quad (21)$$

Obviously, the right-hand side of (21) makes sense for all $|x| > 0$ if and only if $\gamma > 0$, or equivalently, when $r_1 > r_2 + C$. Since the left-hand side of (21) is always non-negative we have

$$k_0 > -\frac{\operatorname{sign}[xy] \sqrt{2/\gamma}}{\sqrt{1+2\gamma|x|/y^2}} \quad (22)$$

So, the function $V(x, y)$ can be found from (21) as

$$V(x, y) = \frac{k^2}{4} \left(\frac{y \operatorname{sign}[x]}{\gamma} + k_0 \sqrt{|x| + \frac{y^2}{2\gamma}} \right)^2 \quad (23)$$

3.3 Removing of Discontinuities

The next step consists in the removing of the discontinuities of the function (23) by special selection of the parameters k and k_0 .

Considering the partial limits of the function $V(x, y)$ when x tends to zero for any fixed y and when y tends to zero for any fixed x , one derive

$$\begin{aligned} \text{if } x \rightarrow 0 \text{ then } V(x, y) &\rightarrow \frac{k^2}{4} \left(\frac{\operatorname{sign}[xy]}{\gamma} + \frac{k_0}{\sqrt{2\gamma}} \right)^2 y^2 \\ \text{if } y \rightarrow 0 \text{ then } V(x, y) &\rightarrow \frac{k^2 k_0^2}{4} |x| \end{aligned}$$

Hence, to eliminate the discontinuities on the lines $x = 0$ and $y = 0$ it is sufficient to resolve the following system

$$\begin{cases} k^2 \left(\frac{\operatorname{sign}[xy]}{\gamma} + \frac{k_0}{\sqrt{2\gamma}} \right)^2 = \bar{k}^2 \\ k^2 k_0^2 = 1 \end{cases}$$

for some positive value \bar{k} . This gives

$$k = \sqrt{\frac{\gamma}{2}} \left| \sqrt{2\gamma\bar{k}} - 1 \right| > 0, k_0 = \sqrt{\frac{2}{\gamma}} \frac{\operatorname{sign}[xy]}{\sqrt{2\gamma\bar{k}-1}} \quad (24)$$

Therefore, the function $V(x, y)$ (23) in view of the continuity property, discussed above, should satisfy

$$V(0, y) = \frac{\bar{k}^2}{4} y^2 \text{ and } V(x, 0) = \frac{|x|}{4} \quad (25)$$

Combining (24) with (22) we can formulate the following simple result.

Lemma 2 *If $r_1 > r_2 + C$ and $r_2 > C$ and the parameter \bar{k} in (24) is selected in a such way that the following inequalities hold*

$$\frac{1}{\sqrt{2(r_1+r_2-C)}} < \bar{k} < \frac{1}{\sqrt{2(r_1-r_2+C)}} \quad (26)$$

then $k_0 > 0$ and the condition (22) holds for all $xy \neq 0$.

PROOF. 1) Consider the case $xy > 0$. Then we need to check the validity of the following inequality $(\sqrt{2\gamma_+ \bar{k}} - 1)^{-1} > -(1 + 2\gamma_+ |x|/y^2)^{-1/2}$ with $\gamma_+ = r_1 + r_1 + C \operatorname{sign}[x \frac{\partial V}{\partial y}]$ which directly results from the inequalities $\bar{k} > (2(r_1 + r_2 - C))^{-0.5} \geq (2\gamma_+)^{-0.5}$.

2) When $xy < 0$ we have to prove $(1 - \sqrt{2\gamma_- \bar{k}})^{-1} > (1 + 2\gamma_- |x|/y^2)^{-0.5}$ with $\gamma_- = r_1 - r_1 + C \operatorname{sign}[x \frac{\partial V}{\partial y}]$ that results from $0 < \bar{k} < (2(r_1 - r_2 + C))^{-0.5} \leq (2\gamma_-)^{-0.5}$.

3) Since $\sqrt{2\gamma_+ \bar{k}} - 1 > 0$ and $\sqrt{2\gamma_- \bar{k}} - 1 < 0$ we have $\text{sign}[xy] = \text{sign}[\sqrt{2\gamma} \bar{k} - 1]$ and $k_0 > 0$ for all $xy \neq 0$.

So, the Lyapunov function (23) becomes

$$V(x, y) = \begin{cases} \frac{k^2}{4} \left(\frac{y \text{sign}[x]}{\gamma} + k_0 \sqrt{|x| + \frac{y^2}{2\gamma}} \right)^2 & \text{if } xy \neq 0 \\ \bar{k}^2 y^2 / 4 & \text{if } x = 0 \\ |x| / 4 & \text{if } y = 0 \end{cases} \quad (27)$$

with k, k_0 are as in (24) and \bar{k} satisfying (26).

3.4 Main Theorem

The rigorous consideration of the partial derivative $\frac{\partial V}{\partial y}$ shows that $\text{sign}[\frac{\partial V}{\partial y}]$ switches on the line $L\sqrt{|x|} \text{sign}[x] + y = 0$ where L is a constant depending on the parameters of the system (13)-(15). However, defining γ as function of $\text{sign}[L\sqrt{|x|} \text{sign}[x] + y]$ one has to definitely remove new discontinuities of the function $V(x, y)$. So, the more preferable definition of γ is

$$\gamma := r_1 + (r_2 - C) \text{sign}[xy] \quad (28)$$

that leads to the following claim.

Theorem 3 *If $r_1 > r_2 + C, r_2 > C$ then the function $V(x, y)$ (27) with γ given by (28) has the properties:*

- 1) $V(x, y)$ is absolute continuous in all space \mathbb{R}^2 and continuously differentiable when $xy \neq 0$;
- 2) the time derivative of $V(x, y)$ on the trajectories of (13)-(15) satisfies

$$\frac{dV}{dt} \leq -k_{\min} \frac{r_1 - r_2 - C}{r_1 - r_2 + C} \sqrt{V} \quad (29)$$

almost always with $k_{\min} := \min(k)$;

- 3) the corresponding guaranteed reaching time is

$$t_{reach} \leq \frac{2(r_1 - r_2 + C)}{k_{\min}(r_1 - r_2 - C)} \sqrt{V(x(0), y(0))} \quad (30)$$

PROOF. 1) Obviously that the function $V(x, y)$ may have discontinuities only on the lines $x = 0$ and $y = 0$. Since the parameters k and k_0 (24) are selected in such a way that discontinuities have to be removed (see (25)), the function $V(x, y)$ is continuous by the accepted construction.

- 2) The partial derivatives of the function $V(x, y)$ are

$$\frac{\partial V}{\partial x} = \frac{k^2}{2} \left(\frac{1}{\sqrt{1+2\gamma|x|/y^2}} \frac{k_0}{\sqrt{2\gamma}} + \frac{k_0^2 \text{sign}[x]}{2} \right)$$

$$\frac{\partial V}{\partial y} = \frac{k^2}{2} \left(\frac{y}{\gamma^2} + \frac{k_0 y^2}{2\gamma} + \frac{k_0 \text{sign}[x](y^2 + \gamma|x|)}{\gamma^2 \sqrt{|x| + y^2/(2\gamma)}} \right)$$

Since γ, k and k_0 switch only on the lines $x = 0$ and $y = 0$, it follows that $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are continuous for all $x \neq 0$ and $y \neq 0$. Moreover, the left and right limits on these lines are finite everywhere in any bounded region.

- 3) Consider then the time derivative of the function $V(x, y)$ on the trajectories of the system (13):

$$\frac{dV}{dt} = k\sqrt{V} \left(-\frac{\bar{\gamma}}{\gamma} + \sqrt{\frac{\gamma}{2}} \frac{k_0 \text{sign}[xy]}{\sqrt{1+2\gamma|x|/y^2}} \left(1 - \frac{\bar{\gamma}}{\gamma} \right) \right)$$

where $\bar{\gamma} = r_1 + r_2 \text{sign}[xy] - f(t, x, y) \text{sign}[x]$, $\gamma = r_1 + r_2 \text{sign}[xy] - C \text{sign}[xy]$ (see (28)). Since, $1 \leq \frac{\bar{\gamma}}{\gamma}$ for $xy > 0$ and $1 \geq \frac{\bar{\gamma}}{\gamma} \geq \frac{r_1 - r_2 - C}{r_1 - r_2 + C}$ for $xy < 0$ the inequality

$$-\frac{\bar{\gamma}}{\gamma} + \sqrt{\frac{\gamma}{2}} \frac{k_0 \text{sign}[xy]}{\sqrt{1+2\gamma|x|/y^2}} \left(1 - \frac{\bar{\gamma}}{\gamma} \right) \leq -\frac{r_1 - r_2 - C}{r_1 - r_2 + C} \quad (31)$$

holds. So, we have $\frac{dV}{dt} \leq -k_{\min} \frac{r_1 - r_2 - C}{r_1 - r_2 + C} \sqrt{V}$ for $x, y \neq 0$.

- 4) To complete the proof it is enough to notice that the system (13) - (15) doesn't have nontrivial solutions with $x(t) \equiv 0$ or $y(t) \equiv 0$, or in other words, the trajectories cross these switching lines avoiding the movements along them (see [12]). Thus, the inequality (29) holds almost always along the trajectory of the system (13) - (15).

Remark 3 *Notice that the Main Theorem conditions $r_1 > r_2 + C, r_2 > C$ coincide with ones in [9].*

Remark 4 *It is not difficult to show that using the designed Lyapunov function (27) the local stability (with a finite-time convergence) of (13)-(14) takes place if the condition (15) holds only in a local neighborhood of the origin D_0 provided that the system starts from within domain $D(c^{\max})$, where $D(c) = \{(x, y)^T \in \mathbb{R}^2 : V(x, y) \leq c\}$ and $c^{\max} = \max_{\forall c: D(c) \subseteq D_0} c$.*

4 Numerical Example

Consider the system (13)-(15) with $r_1 = 1.3, r_2 = 0.7, C = 0.4$. The Lyapunov function (27) for $\bar{k} = 0.9/\sqrt{2}$ is

$$V(x, y) = \begin{cases} \frac{(0.9\sqrt{\gamma}-1)^2}{8\gamma} \left(|y| + \frac{\sqrt{2\gamma|x|+y^2}}{0.9\sqrt{\gamma-1}} \right)^2 & \text{if } xy \neq 0 \\ 0.10125y^2 & \text{if } x = 0 \\ 0.25|x| & \text{if } y = 0 \end{cases}$$

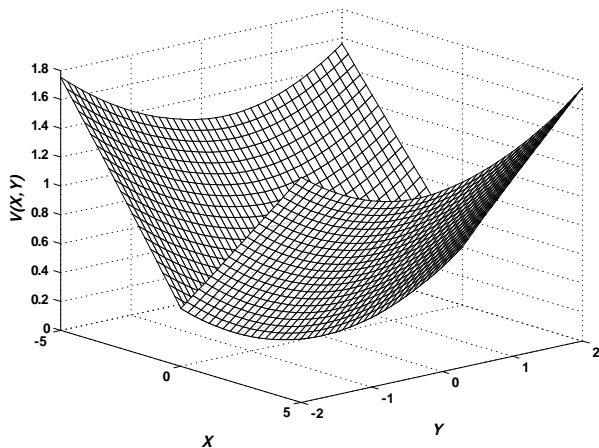


Fig. 1. The Lyapunov function

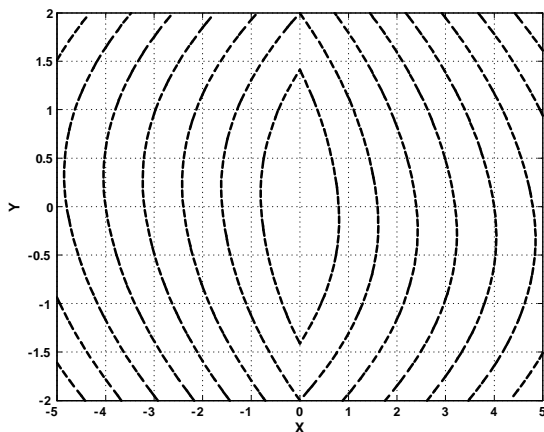


Fig. 2. The level lines of the Lyapunov function

where $\gamma = 1.3 + 0.3 \text{sign}[xy]$ and it is presented in Fig. 1. Fig. 2 shows the level lines of the Lyapunov function.

5 Conclusion

In this paper we present a method of a Lyapunov functions design providing a finite-time convergence of system dynamics to the origin for a class of ODE with discontinuous right-hand side. It is based on Characteristic Method application and related with resolution of the first-order partial differential equation of a special type. Even the suggested approach does not provide a formal algorithm for a Lyapunov function design, it helps to derive a specific form of a Lyapunov function candidate that reduces the problem of a Lyapunov function design to the problem of the correct parameters definition guaranteeing the absolute continuity and positive definiteness of the obtained "energetic" function. The suggested method is successfully illustrated by its application to the "twisting" controller stability analysis.

References

- [1] Bacciotti, A., & Ceragioli F. (1999). Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions, *ESAIM: Control, Optimisation and Calculus of Variations*, 4, 361-376.
- [2] Bacciotti, A., & Rosier, L. (2001). Liapunov functions and stability in control theory, Lecture notes in control and information sciences (Vol. 267). New York: Springer.
- [3] Edwards, C., & Spurgeon, S. (1998). *Sliding mode control: theory and applications*. Taylor&Francis.
- [4] El'sgol'ts, L.E. (1961). *Differential Equations*. International Monographs on Advanced Mathematics and Physics. Hindustan Publisher Company, India, Dheli.
- [5] Filippov, A.F. (1988). *Differential Equations with Discontinuous Right-hand Sides*. Kluwer Academic Publishers, Dordrecht-Boston-London.
- [6] Hong Y., Huang J., Xu Y. (2001). On an output feedback finite-time stabilization problem. *IEEE Trans. Automatic Control*, 46(2), 305-309.
- [7] Khalil H.K.(2002). *Nonlinear systems*, third edition, Prentice Hall, Upper Saddle River, New Jersey.
- [8] Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *Int. J. Control*, 58 (6), 1247-1263.
- [9] Levant, A., & Fridman, L. (2004). Robustness issues of 2-sliding mode control. *Variable Structure Systems: from Principles to Implementations* (edited by A. Sabanovic, L. Fridman, S. Spurgeon), The IEE, London, 131-156.
- [10] Levant, A. (2007). Principles of 2-sliding mode design, *Automatica*, 43 (4), 576-586.
- [11] Orlov, Y. (2005). Finite Time Stability and Robust Control Synthesis of Uncertain Switched Systems. *SIAM J. Control Optim.*, 43(4), pp. 1253-1271
- [12] Orlov, Y. (2006). Extended Invariance Principle and Other Analysis Tools for Variable Structure Systems. *Advances in Variable Structure and Sliding Mode Control* (edited by C. Edwards, E.F. Colet, L. Fridman), Springer, Berlin-Heidelberg-New York, 3-22
- [13] Pan, Y., & Furuta, K. (2006). Variable Structure Control Systems using Sectors for Switching Rule Design. *Advances in Variable Structure and Sliding Mode Control* (edited by C. Edwards, E.F. Colet, L. Fridman), Springer, Berlin-Heidelberg-New York, 89-106.
- [14] Ryan, E. P. (1998). An integral invariance principle for differential inclusions with applications in adaptive control, *SIAM J. Control Optim.*, 36(3), pp. 960-980.
- [15] Sontag, E.D., & Sussman, H.J. (1995) Nonsmooth control-Lyapunov functions. *Proc. 34th IEEE Conf. Decision Control*, USA, New Orleans, 2799-2805.
- [16] Shtessel, Y. B., Shkolnikov, I. A. & Levant, A. (2007). Smooth second-order sliding modes: Missile guidance application. *Automatica*, 43(8), pp. 1470-1476.
- [17] Utkin, V.I. (1992). *Sliding Modes in Control Optimization*. Springer Verlag, Berlin.
- [18] Utkin, V.I., Guldner, J., & Shi, J. (1999). *Sliding Modes in Electromechanical Systems*. Taylor&Francis, London.
- [19] Zubov, V.I. (1964). *Methods of A.M. Lyapunov and their applications*. Groningen: P. Noordho: Limited.

Nonlinear feedback design for fixed-time stabilization of Linear Control Systems

Andrey Polyakov

► **To cite this version:**

Andrey Polyakov. Nonlinear feedback design for fixed-time stabilization of Linear Control Systems. IEEE Transactions on Automatic Control, Institute of Electrical and Electronics Engineers, 2012, 57 (8), pp.2106-2110. <hal-00757561>

HAL Id: hal-00757561

<https://hal.inria.fr/hal-00757561>

Submitted on 27 Nov 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems

Andrey Polyakov

Abstract

Nonlinear control algorithms of two types are presented for uncertain linear plants. Controllers of the first type are stabilizing polynomial feedbacks that allow to adjust a guaranteed convergence time of system trajectories into selected neighborhood of the origin independently on initial conditions. The control design procedure uses block control principles and finite-time attractivity properties of polynomial feedbacks. Controllers of the second type are modifications of the second order sliding mode control algorithms. They provide global finite-time stability of the closed-loop system and allow to adjust a guaranteed settling time independently on initial conditions. Control algorithms are presented for both single-input and multi-input systems. Theoretical results are supported by numerical simulations.

Index Terms

finite-time stability, polynomial feedback, second order sliding mode control

I. INTRODUCTION

Finite-time stability and stabilization problems have often been a subject of research [13], [4], [19], [21]. The control theory provides many systems that exhibit finite-time convergence to the equilibrium. Frequently such systems appear in observation problems when finite-time convergence of the observed states to the real ones is required [3]. The high order sliding mode control algorithms also provide finite-time convergence to the origin [16], [20], [17], [22]. Typically such controllers have mechanical and electromechanical applications [2], [11], [5].

A. Polyakov is with Non-A INRIA - LNE, Parc Scientifique de la Haute Borne 40, avenue Halley Bat.A, Park Plaza 59650 Villeneuve d'Ascq, France, andrey.polyakov@inria.fr

The paper deals with an extension *global finite-time stability* concept that is related to possible predefining of guaranteed convergence(settling) time *independently on initial conditions*. The corresponding property is called in this paper by *fixed-time stability*. Such phenomenon was discovered in [18],[10],[23],[6], where observers with predetermined finite convergence time have been developed. The present paper mostly addresses the control design problem for linear plants providing *fixed-time convergence to the given set*. The developed design procedure requires only controllability of the system, i.e. $\text{rank}[B,AB,\dots,A^{n-1}B]=n$. Some controllers are presented for *fixed-time stabilization of the origin*. These results are restricted by the case $\text{rank}[B,AB]=n$.

The controls considered in this paper basically have polynomial forms. Polynomial state feedback control systems have considerable attention in nonlinear control [8]. This class of control systems appears in models of a wide range of applications such as chemical processes, electronic circuits and mechatronics, biological systems etc. This paper studies a special property of polynomial feedbacks, which is expressed in fixed-time attraction of solutions of the closed-loop system into any selected neighborhood of the origin.

Usually finite-time stability is closely related with homogeneity property of the system. While being asymptotically stable and homogeneous of negative degree, the system is shown to approach the equilibrium point in finite time [16], [20]. The concept of *homogeneity in bi-limit* introduced in [1] generalizes this property providing that an asymptotically stable system is *fixed-time stable* if it is homogeneous of negative degree in 0-limit and homogeneous of positive degree in ∞ -limit. Unfortunately, homogeneous approach does not allow to adjust or even estimate the settling time. To overcome this problem the paper introduces a special modification of the so-called "nested" (terminal) second order sliding mode control algorithm [17] that provides fixed-time stability of the origin and allows to adjust the global settling time of the closed-loop system.

All control algorithms presented in the paper are robust with respect to system disturbances and plant parameters variations in the case when the, so-called, *matching condition* [24] holds, that is, to guarantee successful elimination of system uncertainties or external disturbances they should act in the same subspace as an admissible control.

The paper is organized as follows. The next section discusses finite-time and fixed-time stability concepts. It also extends Lyapunov Function Method to fixed-time stability case. Section 3 states the main problem. Sections 4 and 5 consider control design algorithms for single-input and multi-input systems. Finally, numerical simulations examples and conclusions are presented.

II. FINITE-TIME STABILITY AND SOME FURTHER EXTENSIONS

Consider the following system

$$\dot{x} = g(t, x), \quad x(0) = x_0 \quad (1)$$

where $x \in \mathbb{R}^n$ is the vector of system states, $g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function. The case of discontinuous right-hand sides of the system (1) is not excluded, so solutions of (1) are understood in the sense of Filippov [12]. Assume that the system (1) has zero equilibrium point.

Definition 1 ([4], [20]). *The equilibrium point $x = 0$ of the system (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibria at some finite time moment, i.e. $x(t, x_0) = 0, \forall t \geq T(x_0)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ is the so-called settling-time function.*

The finite-time stability property may demonstrate homogeneous systems with negative degree [4], [20] or the so-called sliding homogeneous systems [16]. For example, any solution of the system $\dot{x} = -x^{-1/3}, x \in \mathbb{R}$ converges to zero equilibrium point in finite time $T(x_0) := \frac{2}{3} \sqrt[3]{|x_0|^2}$.

Definition 2. *The equilibrium point $x = 0$ of the system (1) is said to be globally fixed-time stable if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded by some positive number $T_{\max} > 0$, i.e. $T(x_0) \leq T_{\max}$ for $\forall x_0 \in \mathbb{R}^n$.*

For example, the system $\dot{x} = -x^{1/3} - x^3, x \in \mathbb{R}$ has the fixed-time stable zero equilibrium. Any solution $x(t, x_0)$ of this system converges to the origin in a finite-time and for any $x_0 \in \mathbb{R}$ the equality $x(t, x_0) = 0$ holds for all $t \geq 2.5$.

In the paper [6] the closed-loop control systems that demonstrate dynamics described in Definition 2 was called by "uniformly finite-time convergent". The term "stability" is used here just because this notion is more traditional and natural for describing the referred properties of the system's equilibrium points, but the term "uniformly finite-time stable" can not be used for this purpose, since it has different natural meaning discussed in [20].

The fixed-time stability can be simply extended to the case of attractive set.

Definition 3. *The set M is said to be globally finite-time attractive for the system (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time moment $t = T(x_0)$ and remains there, i.e. $x(t, x_0) \in M, \forall t \geq T(x_0)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ is the settling-time function.*

Definition 4. *The set M is said to be globally fixed-time attractive for the system (1) if it is globally finite-time attractive and the settling-time function $T(x_0)$ is bounded by some positive number $T_{\max} > 0$, i.e. $T(x_0) \leq T_{\max}$ for $x_0 \in \mathbb{R}^n$.*

The property of global finite-time attractivity may demonstrate even some linear systems, but the only nonlinear ones may have fixed-time attractive sets.

Denote by $D^*\varphi(t)$ the upper right-hand derivative of a continuous function $\varphi(t)$, i.e. $D^*\varphi(t) := \sup \lim_{h \rightarrow +0} \frac{\varphi(t+h) - \varphi(t)}{h}$. The following simple result extends the Lyapunov function method providing the background for the fixed-time stability analysis.

Lemma 1. *If there exists a continuous radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ such that 1) $V(x)=0 \Rightarrow x \in M$; 2) any solution $x(t)$ of (1) satisfies the inequality $D^*V(x(t)) \leq -(\alpha V^p(x(t)) + \beta V^q(x(t)))^k$ for some $\alpha, \beta, p, q, k > 0 : pk < 1, qk > 1$ then the set $M \subset \mathbb{R}^n$ is globally fixed-time attractive for the system (1) and $T(x_0) \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)}, \forall x_0 \in \mathbb{R}^n$.*

Proof: Due to 2) we have $D^*V(x(t)) \leq -\alpha^k V^{pk}(x(t))$ if $V(x(t)) \leq 1$ and $D^*V(x(t)) \leq -\beta^k V^{qk}(x(t))$ for $V(x(t)) > 1$. Hence, for any $x(t)$ such that $V(x(0)) > 1$ the last inequality guarantees $V(x(t)) \leq 1$ for $t \geq \frac{1}{\beta^k(qk-1)}$ and for any $x(t)$ such that $V(x(t_0)) \leq 1$ we derive $V(x(t)) = 0$ for $t \geq t_0 + \frac{1}{\alpha^k(1-pk)}$. Therefore, $V(x(t)) = 0$ for all $t \geq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)}$ and any solution $x(t)$ of (1). Applying the condition 1) we finish the proof of this lemma. ■

Replacement of the condition 1) in Lemma 1 by $V(x)=0 \Leftrightarrow x=0$ helps to check invariance of the set M . So, for $M = \{0\}$ Lemma 1 allows to analyze the fixed-time stability of the origin.

III. PROBLEM STATEMENT AND BASIC ASSUMPTIONS

Consider the control system of the form

$$\dot{x} = Ax + Bu + f(t, x) \quad (2)$$

where $x \in \mathbb{R}^n$ is the vector of system states, $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B \in \mathbb{R}^{n \times m}$ is the matrix of control gains, $u \in \mathbb{R}^m$ is the vector of control inputs, and the function $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes systems uncertainties.

We study the system (2) under the following classical *assumptions*: **1.** The pair (A, B) is controllable, i.e. $\text{rank}[B, AB, A^2B, \dots, A^{n-1}B] = n$. **2.** The uncertain function $f(t, x)$ satisfies the so-called matching condition (see, for example, [24]), i.e. $f(t, x) = B\gamma(t, x)$, where $\gamma(t, x)$

is an unknown but bounded function. The function $\gamma(t, x)$ is assumed to be bounded by some known function $\gamma_0(t, x) \geq 0$, i.e. $\|\gamma(t, x)\|_\infty \leq \gamma_0(t, x)$ for $\forall t > 0$ and $\forall x \in \mathbb{R}^n$.

Denote a closed ball of radius $r > 0$ with the center in the origin by B_r , i.e. $B_r := \{x \in \mathbb{R}^n : \|x\|_\infty \leq r\}$, $\|x\|_\infty := \max_{i=1,2,\dots,n} |x_i|$.

This paper address two following problems:

- to design a feedback control $u = u(t, x)$ for the system (2), which provides the fixed-time attractivity property of the given ball B_r for a predefined global settling-time estimate T_{\max} ;
- to design a feedback control $u = u(t, x)$, which guarantees fixed-time stability of the origin of the closed-loop system (2) for a predefined global settling-time estimate T_{\max} .

IV. FIXED-TIME CONTROLLERS FOR SINGLE INPUT SYSTEMS

A. Fixed-time Attractivity

Consider the case $m=1$. Then the linear transformation $y=Gx$, $G=[A^{n-1}b, A^{n-2}b, \dots, Ab, b]^{-1}$ brings (may be after reordering) the initial system (2) to the Brunovsky form

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dots, \quad \dot{y}_n = \sum_{i=1}^n a_i y_i + u(y) + \gamma(t, y) \quad (3)$$

where $a_i \in \mathbb{R}$ and the uncertain term γ is described in the previous section, $\|\gamma(t, y)\|_\infty < \gamma_0(t, y)$.

Introduce the nonlinear coordinate transformation $s = \Phi(y)$ by the following formulas:

$$s_i = y_i + \varphi_i, \quad i=1,2,\dots,n, \quad \varphi_1=0, \quad \varphi_{i+1} = \alpha_i(y_i + \varphi_i) + \beta_i(y_i + \varphi_i)^3 + \sum_{k=1}^i \frac{\partial \varphi_i}{\partial y_k} y_{k+1},$$

where $\alpha_i, \beta_i, i = 1, 2, \dots, n-1$ are some positive numbers. The presented coordinate transformation is smooth and nonsingular. The inverse transformation $y = \Phi^{-1}(s)$ is defined as follows:

$$y_i = s_i + \psi_i, \quad i=1,2,\dots,n, \quad \psi_1=0, \quad \psi_{i+1} = -\alpha_i s_i - \beta_i s_i^3 + \sum_{k=1}^i \frac{\partial \psi_i}{\partial s_k} (-\alpha_k s_k - \beta_k s_k^3 + s_{k+1}).$$

Formal calculations show that the system (3) is equivalent to

$$\dot{s}_1 = -\alpha_1 s_1 - \beta_1 s_1^3 + s_2, \quad \dot{s}_2 = -\alpha_2 s_2 - \beta_2 s_2^3 + s_3, \quad \dots, \quad \dot{s}_n = \sum_{i=1}^n a_i y_i + \sum_{k=1}^{n-1} \frac{\partial \varphi_n}{\partial y_k} y_{k+1} + u + \gamma(t, y) \quad (4)$$

and the feedback control of the form

$$u = -\alpha_n s_n - \beta_n s_n^3 - \sum_{i=1}^n a_i y_i - \sum_{k=1}^{n-1} \frac{\partial \varphi_n}{\partial y_k} y_{k+1}, \quad \text{where } s = \Phi(y) \quad (5)$$

provides fixed-time attractivity property of the closed-loop system (4). The next lemma gives a background for an appropriate adjustment of the control parameters α_i and β_i .

Lemma 2. Let 1) $\varepsilon_i > 0, i = 1, 2, \dots, n$ and $T_{\max} > 0$ are some positive numbers; 2) $\alpha_1 = \bar{\alpha}_1 + \frac{\varepsilon_2}{\varepsilon_1}, \dots, \alpha_{n-1} = \bar{\alpha}_{n-1} + \frac{\varepsilon_n}{\varepsilon_{n-1}}, \alpha_n := \alpha_n(t, y) = \bar{\alpha}_n + \frac{\gamma_0(t, y)}{\varepsilon_n}$, where $\bar{\alpha}_i$ are nonnegative numbers and $\|\gamma(t, y)\|_\infty < \gamma_0(t, y)$; 3) $\beta_i \geq \frac{\bar{\alpha}_i/\varepsilon_i^2}{\exp(2\bar{\alpha}_i \frac{T_{\max}}{n}) - 1}, i = 1, 2, \dots, n$. Then any solution $s(t)$ of the system (4)-(5) satisfies the inequalities $|s_i(t)| \leq \varepsilon_i$ for $\forall t \geq T_{\max}$.

Proof: Denote $V_i(t) = |s_i(t)|, i = 1, 2, \dots, n$. For all $t > 0$ the inequality $D^*V_n(t) \leq -\alpha_n V_n(t) - \beta_n V_n^3(t) + \gamma_0$ holds. Hence, $\dot{V}_n(t) \leq -\bar{\alpha}_n V_n(t) - \beta_n V_n^3(t)$ for $t > 0 : V_n(t) \geq \varepsilon_n$. Studying the last inequality we derive $\frac{\beta_n V_n^2(t)}{\bar{\alpha}_n} + 1 \leq \left(1 - \frac{\beta_n V_n^2(0)}{\beta_n V_n^2(0) + \bar{\alpha}_n} e^{-2\bar{\alpha}_n t}\right)^{-1}$. Hence, $V_n(t) \leq \varepsilon_n$ for all $t \geq T_{\max}/n$. The similar considerations for $i = n - 1$ give $V_i(t) \leq \varepsilon_i$ for all $t \geq (n - i + 1)T_{\max}/n$. The last step ($i = 1$) provides $|s_i(t)| < \varepsilon_i$ for all $t > T_{\max}$. ■

Corollary 1. Selection of the parameters $\varepsilon_i = \varepsilon > 0$ and $\bar{\alpha}_i = 1$ for $i = 1, 2, \dots, n$ under conditions of Lemma 2 implies the inequality $\|s(t)\|_\infty \leq \varepsilon$ for all $t > T_{\max}$, where $s(t)$ is an arbitrary solution of the system (4)-(5) with control parameters $\alpha_i = 2, i = 1, 2, \dots, n - 1; \alpha_n(t, x) = 1 + \gamma_0(t, x)/\varepsilon$ and $\beta_i = \beta := \varepsilon^{-2} (\exp(\frac{2T_{\max}}{n}) - 1)^{-1}, i = 1, 2, \dots, n$.

The presented corollary shows the way for designing of the control, which provides the fixed-time attractivity property of the ball B_ε for the system (4). The radius of the ball ε and the global settling-time T_{\max} can be selected in arbitrary way. Show now that the designed controller will guarantee the same fixed-time attractivity property for the original system (3).

Denote a set of all polynomials of the order k by \mathbb{P}^k .

Corollary 2. If conditions of Corollary 1 hold, then there exist polynomials $p_i \in \mathbb{P}^i, i = 1, 2, \dots, n$ with nonnegative coefficients such that $|y_i(t)| \leq p_i(q)\varepsilon, \forall t > T_{\max}$, where $q = (\exp(\frac{2T_{\max}}{n}) - 1)^{-1}$ and $y(t) = (y_1(t), \dots, y_n(t))^T$ is an arbitrary solution of the system (3), (5).

Proof: Due to Corollary 1 we have that $|s_i| \leq \varepsilon$ for all $t > T_{\max}$. Let $z_i > 0, i = 1, 2, \dots, n$ and $\tilde{\psi}_1=0, \tilde{\psi}_{i+1}=\tilde{\psi}_{i+1}(z_1, z_2, \dots, z_{i+1}):=2z_i+\beta z_i^3+\sum_{k=1}^i \frac{\partial \tilde{\psi}_i}{\partial z_k}(2z_k+\beta z_k^3+z_{k+1})$. It is easy to see that $|\psi_i(s_1, \dots, s_i)| \leq \tilde{\psi}_i(|s_1|, \dots, |s_i|)$. Show now that $\tilde{\psi}_i(\varepsilon, \dots, \varepsilon)=\varepsilon \sum_{j=0}^{i-1} \mu_{ij}(\beta \varepsilon^2)^j$, where μ_{ij} are some nonnegative numbers. Obviously, $\tilde{\psi}_1=0$ and $\tilde{\psi}_2=2z_1+\beta z_1^3$ satisfy the required representation. Assume that it fulfils for $\tilde{\psi}_i$ and show the same property for $\tilde{\psi}_{i+1}$. Due to recursive polynomial structure of $\tilde{\psi}_i$ we conclude that for $z_j=\varepsilon, j = 1, 2, \dots, i, \frac{\partial \tilde{\psi}_i(z_1, \dots, z_i)}{\partial z_k} = \sum_{j=0}^{i-1} \hat{\mu}_{ij}^k(\beta \varepsilon^2)^j$, where $\hat{\mu}_{ij}^k$ are some

nonnegative numbers. Hence, $\tilde{\psi}_{i+1}(\varepsilon, \dots, \varepsilon) = \varepsilon(1 + \beta\varepsilon^2) + \varepsilon \sum_{k=1}^i \sum_{j=0}^{i-1} \hat{\mu}_{ij}^k (\beta\varepsilon^2)^j (2 + \beta\varepsilon^2) = \varepsilon \sum_{j=0}^i \mu_{i+1j} (\beta\varepsilon^2)^j$ and the required property is proven. Finally taking into account $\beta\varepsilon^2 = q$ and $|y_i(t)| \leq |s_i(t)| + |\psi_i(s_1(t), \dots, s_i(t))| \leq p_i(q)\varepsilon$, where $p_i(q) := 1 + \sum_{j=0}^{i-1} \mu_{ij} q^j$, we finish the proof. ■

Remark that the proof of Corollary 2 gives an algorithm of construction of the polynomials $p_i(q) \in \mathbb{P}^i$ introduced in its statement. These polynomials are needed for adjustment of control parameters to guarantee convergence of all solutions of the original system (2) into the required neighborhood of the origin. The explicit form of the corresponding polynomial can be obtained using the recursive formulas for $\tilde{\psi}_i$ calculated in symbolic computation packages such as Mathematica or Symbolic Toolbox of MATLAB. In particular the first three polynomials have the form: $p_1(q) = 1, p_2(q) = 3 + q, p_3(q) = 9 + 12q + 3q^2$.

Taking into account $p_i(q) \leq p_{i+1}(q)$ we may formulate the theorem.

Theorem 1. *If $m=1$ and the control u has the form (5) with parameters $\alpha_i=2, i=1,2,\dots,n-1, \alpha_n(t,x)=1+\gamma_0(t,x)/\varepsilon, \beta_i=\frac{q}{\varepsilon^2}, i=1,2,\dots,n, \varepsilon=r/(\|G^{-1}\|_\infty p_n(q)), q = (\exp(\frac{2T_{\max}}{n}) - 1)^{-1}$, where $r > 0$ and $T_{\max} > 0$ are arbitrary numbers and the polynomial $p_n \in \mathbb{P}^n$ is introduced in Corollary 2, then the ball B_r is the globally fixed-time attractive set of the closed-loop system (2) with the settling-time function bounded by T_{\max} .*

Theorem 1 mostly declares the theoretical possibility for designing of a nonlinear control providing fixed-time attractivity property of the closed-loop linear control system. The practical applications may require more precise adjustment of control parameters basing on Lemma 2.

B. Fixed-time stability

The scheme presented above allows to design the control $u_a(t, x) := u(t, x)$ for the single input case of the system (2), which guarantees fixed-time attractivity property of an arbitrary ball B_r with the center in the origin. The simplest combination of this controller with some control law $u_f(x)$ providing local finite-time stability of the origin for the system (2) gives us a hybrid control algorithm $u_{fx}(t, x) = \begin{cases} u_a(t, x) & \text{for } x \notin B_r \\ u_f(t, x) & \text{for } x \in B_r \end{cases}$, which can afford fixed-time stability of the origin for the system (2). Some design procedures of local finite-time controllers can be found, for example, in [16] or [14]. However, the existed controllers providing local finite-time stability do not have precise algorithms for adjustment of control parameters to predefine the

local settling time estimate (excepting the so-called "twisting" second order ($n = 2$) sliding mode system, see, for example, [22]). To realize the presented hybrid scheme we need to prevent the appearance of switching (chattering) regimes [9] around the boundary of the ball B_r , which may slow down or even prevent the convergence process. Therefore the "solid" (non-hybrid) controller providing *fixed-time stability of the origin* for the system (2) is more preferable. The designing such controller is a difficult problem, which is solved in this paper only for linear plants satisfying *the additional assumption* $\text{rank}[B, AB] = n$. This condition definitely restricts the class of controllable systems, but it still covers a lot of real-life control systems. For example, mechanical systems with maximum numbers of control inputs [5] obviously satisfy this condition. For $m=1$ the presented assumption implies $n=2$.

Introduce the following denotation $z^{[q]} = |z|^q \text{sign}[z]$, $z, q \in \mathbb{R}$, which presents the involution operation without loss of number's sign. Remark, $z^{[1/k]} = z^{1/k}$ and $z^{[k]} = z^k$ for odd integers k .

Theorem 2. *Let $T_{\max} > 0$ be an arbitrary positive number, $m=1$, $n=2$ and the control $u(t, x) = u_{fx}(t, Gx)$, $x \in \mathbb{R}^2$ has the form $u_{fx}(t, y) = -a_1 y_1 - a_2 y_2 - \frac{\alpha_1 + 3\beta_1 y_1^2 + 2\gamma_0(t, y)}{2} \text{sign}[s] - (\alpha_2 s + \beta_2 s^{[3]})^{[\frac{1}{2}]}$, where $s = y_2 + (y_2^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]})^{[0.5]}$, $0.5\alpha_1 = \alpha_2 = 0.5\beta_1 = \beta_2 = 64T_{\max}^{-2}$. Then the origin of the closed-loop system (2) is globally fixed-time stable with the settling-time function bounded by T_{\max} .*

Proof: I. Show that $s = 0$ is a sliding surface of the closed-loop system (2). Consider the Lyapunov function candidate $V(y_1, y_2) := |s(y_1, y_2)|$ and calculate its total upper right-hand Dini derivative along the trajectories of the closed-loop system (3) $D^*V(y_1(t), y_2(t)) = \dot{y}_2(t) \text{sign}[s(t)] + \left(|y_2(t)| \dot{y}_2(t) \text{sign}[s(t)] + \frac{\alpha_1 + 3\beta_1 y_1^2(t)}{2} y_2(t) \text{sign}[s(t)] \right) / \left| y_2^{[2]}(t) + \alpha_1 y_1(t) + \beta_1 y_1^{[3]}(t) \right|^{\frac{1}{2}}$ for $s \neq 0$. Taking into account the equality $(\alpha_2 s + \beta_2 s^{[3]})^{[\frac{1}{2}]} \text{sign}[s] = (\alpha_2 |s| + \beta_2 |s|^3)^{\frac{1}{2}}$ we obtain $\dot{y}_2 \text{sign}[s] = (a_1 y_1 + a_2 y_2 + u_{fx}(y) + \gamma(t, x)) \text{sign}[s] = -\frac{\alpha_1 + 3\beta_1 y_1^2}{2} - (\alpha_2 |s| + \beta_2 |s|^3)^{1/2} - (\gamma_0 - \gamma)$ for $s \neq 0$. Hence, we conclude that $D^*V \leq -(\alpha_2 V + \beta_2 V^3)^{1/2}$ and following the proof of Lemma 1 we guarantee $V(t) := |s(t)| = 0$ for all $t \geq T_s := \frac{2}{\sqrt{\alpha_2}} + \frac{2}{\sqrt{\beta_2}} = \frac{T_{\max}}{2}$.

II. The equality $s=0$ implies $2y_2^{[2]} + \alpha_1 y_1 + \beta_1 y_1^3 = 0$ and the sliding motion equation takes the form $\dot{y}_1 = -\left(\frac{\alpha_1}{2} y_1 + \frac{\beta_1}{2} y_1^3\right)^{[1/2]}$. Using Lemma 1 with the Lyapunov function $V(y_1) = |y_1|$ we show $y_1 = 0$ for all $t \geq \frac{T_{\max}}{2} + \frac{2\sqrt{2}}{\sqrt{\alpha_1}} + \frac{2\sqrt{2}}{\sqrt{\beta_1}} = T_{\max}$. Finally we remark that $y_1 = 0$ and $s = 0$ imply $y_2 = 0$. ■

Remark that for $\beta_1 = 0$ the switching line $s = 0$ coincides with the sliding surface of "nested" controller [17], which provides finite-time stability of the closed-loop system. The polynomial terms $\beta_1 y_1^{[3]}$ and $\beta_2 s^{[3]}$ are introduced to guarantee the fixed-time stability property.

V. FIXED-TIME CONTROLLERS FOR MULTI INPUT SYSTEMS

A. Block Decomposition

Introduce denotations: $\text{rown}(W)$ and $\text{coln}(W)$ are the number of rows and columns of the matrix W , respectively; $\ker(W)$ and $\text{range}(W)$ are the null space and the column space of W ; $\text{null}(W)$ is the matrix with columns defining the orthonormal basis of $\ker(W)$.

To adapt the fixed-time control design scheme developed for the single-input case we need to decompose the original multi input system (2) to a block from [7]. The required coordinate transformation can be constructed using supporting matrices provided by Algorithm 1.

Algorithm 1. **I. Initialization** : $A_0 = A$, $B_0 = B$, $T_0 = I_n$, $k = 0$. **II. Recursion:** While $\text{rank}(B_k) < \text{rown}(A_k)$ do $A_{k+1} = B_k^\perp A_k (B_k^\perp)^T$, $B_{k+1} = B_k^\perp A_k \tilde{B}_k$, $T_{k+1} = \begin{pmatrix} B_k^\perp \\ \tilde{B}_k \end{pmatrix}$, $k=k+1$, where $B_k^\perp := (\text{null}(B_k^T))^T$, $\tilde{B}_k := (\text{null}(B_k^\perp))^T$.

This algorithm can be easily realized in a computational software system such as MATLAB.

Lemma 3. *If the pair (A, B) is controllable then 1) Algorithm 1 terminates after finite number of steps $k \leq n - 1$; 2) the matrices $T_i \in \mathbb{R}^{\text{rown}(B_i) \times \text{rown}(B_i)}$, $i = 1, 2, \dots, k$ are orthogonal; 3) the orthogonal coordinate transformation $y = Gx$*

$$G = \begin{pmatrix} T_k & 0 \\ 0 & I_{w_k} \end{pmatrix} \begin{pmatrix} T_{k-1} & 0 \\ 0 & I_{w_{k-1}} \end{pmatrix} \dots \begin{pmatrix} T_2 & 0 \\ 0 & I_{w_2} \end{pmatrix} T_1, \text{ where } w_i := n - \text{rown}(T_i) \quad (6)$$

reduces the original system (2) to the following block form

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}y_3 \\ \dots \\ \dot{y}_k = A_{k1}y_1 + \dots + A_{kk}y_k + A_{k,k+1}(u + \gamma(t, y)) \end{cases} \quad (7)$$

where $y = (y_1, \dots, y_k)^T$, $y_i \in \mathbb{R}^{n_i}$, $n_i := \text{rank}(B_{k-i})$, $A_{k,k+1} = \tilde{B}_0 B_0$ and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ are blocks of the matrix GAG^T such that $\text{rank}(A_{i,i+1}) = n_i$, $i = 1, 2, \dots, k$

Proof: **I.** Denote $B_0 = B$, $A_0 = A$ and $r_0 = \text{rank}(B_0)$. Controllability of (A_0, B_0) implies $r_0 > 0$. For $r_0 = n$ Algorithm 1 stops after initialization showing that the original system does not require transformation. **II.** Consider the case $r_0 < n$. In this case $\dim(\text{range}(B_0)) = r_0$,

$\dim(\ker(B_0^T))=n-r_0$ and $\mathbb{R}^n=\text{range}(B_0)\oplus\ker(B_0^T)$. Columns of the matrix $\text{null}(B_0^T) \in \mathbb{R}^{n \times (n-r_0)}$ by definition are the orthonormal basis for $\ker(B_0^T) \subset \mathbb{R}^n$, so $\ker(B_0^T)=\text{range}(\text{null}(B_0^T))$. Since $\text{range}(B_0)$ is an orthogonal complement to $\ker(B_0^T)$ then $\text{range}(B_0)=\ker(\text{null}(B_0^T)^T)$ and columns of the matrix $(B_0^\perp)^T=\text{null}(B_0^T)$ together with columns of the matrix $(\tilde{B}_0)^T=\text{null}(\text{null}(B_0^T)^T)$ provide the orthonormal basis in \mathbb{R}^n . So, the block matrix T_1 is square and orthogonal. Since $B_0^T \text{null}(B_0^T)=0$ then the coordinate transformation $\begin{pmatrix} \tilde{y}_1 & \tilde{y}_2 \end{pmatrix}^T = T_1 x$, $\tilde{y}_1 \in \mathbb{R}^{n-r_0}$, $\tilde{y}_2 \in \mathbb{R}^0$ gives

$$\dot{\tilde{y}}_1 = \tilde{A}_{11}\tilde{y}_1 + \tilde{A}_{21}\tilde{y}_2, \quad \dot{\tilde{y}}_2 = \tilde{A}_{21}\tilde{y}_1 + \tilde{A}_{22}\tilde{y}_2 + \tilde{B}_0 B_0(u + \gamma)$$

where $\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}, \tilde{A}_{22}$ are the blocks of the matrix $T_1 A T_1^T$. Obviously that $A_{11} = A_1$ and $A_{12} = B_1$, where matrices A_1 and B_1 are generated by the recursive step of Algorithm 1.

Controllability of the pair (A_0, B_0) implies controllability of (A_1, B_1) [24], so we can repeat the recursive step of Algorithm 1 for A_1 and B_1 . Since for each recursive step we have $\text{rown}(A_{i+1}) = \text{rown}(B_{i+1}) = \text{rown}(A_i) - \text{rank}(B_i)$ and $\text{rank}(B_i) > 0$ then Algorithm 1 terminates in finite steps and the coordinate transformation $y = Gx$ reduces the original system (2) to the block form (7). Finally $\text{rank}(A_{i+1})=\text{rank}(\tilde{B}_{k-i}B_{k-i})=\text{rank}(T_{k-i}B_{k-i})=\text{rank}(B_{k-i}) = n_i$. ■

This paper considers the restricted robust problem statement assuming that all uncertainties and disturbances satisfy the matching condition. The control design for more general case can be done using the robustness analysis of block controllability forms presented in [15].

B. Fixed-time attractivity

Further considerations are presented for the system (2) transformed to the block form (7). Since $\text{rank}(A_{i+1})=\text{rown}(A_{i+1})=n_i$ then $A_{i+1}A_{i+1}^T$ is invertible and $A_{i+1}^+ = A_{i+1}^T(A_{i+1}A_{i+1}^T)^{-1}$ is the right inverse matrix of A_{i+1} . Denote $z^{[p]}=(z_1^{[p]}, \dots, z_k^{[p]})^T$ for $z=(z_1, \dots, z_k)^T \in \mathbb{R}^k$.

By analogy to single input case introduce the nonlinear coordinate transformation $s = \Phi(y)$, $s = (s_1, \dots, s_k)^T$, $s_i \in \mathbb{R}^{n_i}$, $y = (y_1, \dots, y_k)^T$, $y_i \in \mathbb{R}^{n_i}$ by the recurrent formulas: $s_i=y_i + \varphi_i$, $i=1, 2, \dots, k$, $\varphi_1=0$, $\varphi_{i+1}=A_{i+1}^+ \left(\alpha_i(y_i+\varphi_i)+\beta_i(y_i+\varphi_i)^{[3]} + \sum_{j=1}^i A_{ij}y_j + \sum_{r=1}^i \frac{\partial \varphi_i}{\partial y_r} \sum_{j=1}^{r+1} A_{rj}y_j \right)$, where $\alpha_i > 0$, $\beta_i > 0$ and $i = 1, 2, \dots, k - 1$. The presented coordinate transformation is smooth and nonsingular. The inverse transformation $y = \Phi^{-1}(s)$ is defined as follows: $y_i=s_i+\psi_i$, $i=1, 2, \dots, k$, $\psi_1=0$, $\psi_{i+1}=A_{i+1}^+ \left(\sum_{k=1}^i \frac{\partial \psi_i}{\partial s_k} \left(A_{i+1} s_{k+1} - \alpha_k s_k - \beta_k s_k^{[3]} \right) - \alpha_i s_i - \beta_i s_i^{[3]} - \sum_{j=1}^i A_{ij}(s_j + \psi_j) \right)$.

Applying the transformation $s = \Phi(y)$ to the system (7) we obtain

$$\begin{cases} \dot{s}_1 = -\alpha_1 s_1 - \beta_1 s_1^3 + A_{12} s_2 \\ \dot{s}_2 = -\alpha_2 s_2 - \beta_2 s_2^3 + A_{23} s_3 \\ \dots \\ \dot{s}_k = \sum_{i=1}^k A_{ki} y_i + \sum_{i=1}^{k-1} \frac{\partial \varphi_k}{\partial y_i} \sum_{j=1}^{i+1} A_{ij} y_j + A_{k, k+1} (u + \gamma(t, y)) \end{cases} \quad (8)$$

So, the feedback control providing fixed-time attractivity property to the system (8) has the form

$$u = -A_{k, k+1}^+ \left(\alpha_k s_k + \beta_k s_k^3 + \sum_{i=1}^k A_{ki} y_i + \sum_{i=1}^{k-1} \frac{\partial \varphi_k}{\partial y_i} \sum_{j=1}^{i+1} A_{ij} y_j \right), \quad s = \Phi(Gy), \alpha_k, \beta_k > 0 \quad (9)$$

Let ε and T_{max} be positive numbers, $\alpha_i = 1 + \|A_{i, i+1}\|_\infty$, $i=1, 2, \dots, k-1$, $\alpha_k = 1 + \frac{\|A_{k, k+1}\|_\infty \gamma_0(t, y)}{\varepsilon}$ and $\beta_i = \beta := \frac{q}{\varepsilon^2}$, $i=1, 2, \dots, k$, $q = (\exp(\frac{2T_{max}}{n}) - 1)^{-1}$ then repeating the proof of Lemma 2 we can show that any solution $s(t)$ of the system (8)-(9) satisfies the inequality $\|s(t)\|_\infty \leq \varepsilon$, $\forall t \geq T_{max}$.

By analogy with single input case let us show now that there exist polynomials $p_i \in \mathbb{P}^i$, $i = 1, 2, \dots, k$ with nonnegative coefficients such that $\|y_i(t)\|_\infty \leq p_i(q) \varepsilon$ for $\forall t > T_{max}$, where $y(t) = (y_1(t), \dots, y_k(t))^T$ is an arbitrary solution of the system (7), (9).

Denote the vector-modulus and the matrix-modulus operation by $\|\cdot\|$, i.e. $\|v\| = (|v_1|, \dots, |v_r|)^T$ for the vector $v = (v_1, \dots, v_r)^T$ and $\|W\| := \{|w_{ij}|\}$ for the matrix $W = \{w_{ij}\}$. The inequalities $\|v_1\| \preceq \|v_2\|$ and $\|W_1\| \preceq \|W_2\|$ are understood in a component-wise sense.

Let $\tilde{\psi}_1 = 0$, $\tilde{\psi}_{i+1} = \|A_{i, i+1}^+\| \left(\alpha_i z_i + \beta z_i^3 + \sum_{j=1}^i \|A_{ij}\| (z_j + \tilde{\psi}_j) + \sum_{k=1}^i \frac{\partial \tilde{\psi}_i}{\partial z_k} (\alpha_k z_k + \beta z_k^3 + \|A_{i, i+1}\| z_{k+1}) \right)$, where $z_i \in \mathbb{R}_+^{n_i}$, $i=1, 2, \dots, k$. Following the proof of Corollary 2 we can show that $\|\psi_i(s_1, \dots, s_i)\| \preceq \tilde{\psi}_i(\|s_1\|, \dots, \|s_i\|)$, $\tilde{\psi}_i(\varepsilon \bar{e}_{n_1}, \dots, \varepsilon \bar{e}_{n_i}) = \varepsilon \sum_{j=0}^i (\beta \varepsilon^2)^j M_{ij} \bar{e}_{n_j}$, where $\bar{e}_r = (1, \dots, 1)^T \in \mathbb{R}_+^r$ and $M_{ij} \in \mathbb{R}^{n_i \times n_j}$ are matrices with nonnegative elements. Hence it directly follows that $\|y_i\|_\infty \leq \|s_i\|_\infty + \|\psi_i(s_1, \dots, s_i)\|_\infty \leq \varepsilon + \|\tilde{\psi}_i(\varepsilon \bar{e}_{n_1}, \dots, \varepsilon \bar{e}_{n_i})\|_\infty = \varepsilon + \varepsilon \sum_{j=0}^i (\beta \varepsilon^2)^j \|M_{ij}\|_\infty := p_i(q) \varepsilon$.

Similarly to single input case the polynomials $p_i(q)$ can be calculated using some symbolic computation package. For instance, $p_1(q) = 1$ and $p_2(q) = 1 + \|A_{12}^+\|_\infty (1 + \|A_{12}\|_\infty + q + \|A_{11}\|_\infty)$. Therefore, we have just proven the main theorem for the multi input case.

Theorem 3. *If the control u has the form (9) with parameters $\alpha_i = 1 + \|A_{i, i+1}\|_\infty$, $i = 1, 2, \dots, k-1$, $\alpha_k = 1 + \frac{\gamma_0 \|A_{k, k+1}\|_\infty \|G\|_1 p_k(q)}{r}$, $\beta_i = \frac{q \|G\|_1^2 p_k^2(q)}{r^2}$, $i = 1, 2, \dots, k$, $q = (\exp(\frac{2T_{max}}{n}) - 1)^{-1}$, where $r > 0$ and $T_{max} > 0$ are arbitrary positive numbers, then the ball B_r is the globally fixed-time attractive set of the closed-loop system (2) with the settling-time function bounded by T_{max} .*

C. Fixed-time stability

In this subsection we assume $\text{rank}[B, AB]=n$. Then Algorithm 1 terminates after the first recursive step, so the transformation $y = Gx$ brings the original system to the form

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}(u + \gamma) \end{cases} \quad (10)$$

where $y_1 \in \mathbb{R}^{n_1}, y_2 \in \mathbb{R}^{n_2}, n_2 = \text{rank}(B), n_1 = n - n_2, \text{rank}(A_{12}) = n_1 \leq \text{rank}(A_{23}) = n_2$. Denote $A_{12}^\perp = \text{null}(A_{12})^T$.

Theorem 4. *Let the controller $u(t, y), y = (y_1^T, y_2^T)^T \in \mathbb{R}^n$ has the form:*

$$u = A_{23}^+(u_{eq}(y) + u_d(t, y) + u_p(y)), \quad (11)$$

where $u_{eq} = -A_{12}^+((A_{11}^2 + A_{12}A_{21})y_1 + (A_{11}A_{12} + A_{12}A_{22})y_2) - (A_{12}^\perp)^+ A_{12}^\perp(A_{21}y_1 + A_{22}y_2)$,
 $u_d = -\frac{\alpha_1 + 3\beta_1 \|y_1\|_\infty^2 + 2\|A_{12}A_{23}\|_\infty \gamma_0(t, y)}{2} A_{12}^+ \text{sign}[s_1] - \gamma_0(t, y) \|A_{12}^\perp A_{23}\|_\infty (A_{12}^\perp)^+ \text{sign}[s_2]$,
 $u_p = -A_{12}^+(\alpha_2 s_1 + \beta_2 s_1^{[3]})^{[\frac{1}{2}]} - (A_{12}^\perp)^+(\alpha_3 s_2 + \beta_3 s_2^{[3]})^{[\frac{1}{2}]}$,
 $s_1 = A_{11}y_1 + A_{12}y_2 + \left((A_{11}y_1 + A_{12}y_2)^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]} \right)^{[\frac{1}{2}]}$, $s_2 = A_{12}^\perp y_2$

with $0.5\alpha_1 = \alpha_2 = 4\alpha_3 = 0.5\beta_1 = \beta_2 = 4\beta_3 = 64T_{\max}^{-2}$, $T_{\max} > 0$. Then the origin of the closed-loop system (10) is globally fixed-time stable with the settling-time function bounded by T_{\max} .

Proof: First of all remark that $A_{12}(A_{12}^\perp)^+ = 0$ and $A_{12}^\perp A_{12}^+ = 0$. Denote $z = A_{11}y_1 + A_{12}y_2$.

In this case the system (10) can be rewritten in the form

$$\dot{y}_1 = z, \quad \dot{z} = -A_{12}u_d - A_{12}u_q + A_{12}A_{23}\gamma, \quad \dot{s}_2 = -A_{12}^\perp u_d - A_{12}^\perp u_p + A_{12}^\perp A_{23}\gamma$$

and $s_1 = z + \left(z^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]} \right)^{[\frac{1}{2}]}$. Following the proof of Theorem 2 we similarly show that $y_1(t) = 0$ and $z(t) = 0$ for all $t \geq T_{\max}$. Hence $A_{12}y_2(t) = 0$ for all $t \geq T_{\max}$. Taking into account formulas for u_d and u_p we obtain $\dot{s}_2 = -\gamma_0(t, y) \|A_{12}^\perp A_{23}\|_\infty \text{sign}[s_2] - (\alpha_3 s_2 + \beta_3 s_2^{[3]})^{[\frac{1}{2}]} + A_{12}^\perp A_{23}\gamma$. Hence it can be easily shown that $s_2(t) = A_{12}^\perp y_2(t) = 0$ for all $t \geq T_{\max}$. ■

Theorem 4 extends the "nested" second order sliding mode control and the fixed-time controller presented in Theorem 2 to the multi input case.

Remark that the control (11) has the form closed to classical sliding mode controllers [24]. The equivalent control term u_{eq} compensates the inner dynamic of the system, the switching term u_d ensures the sliding mode existence and the new additional term u_p has the form required for fixed-time stability.

For $n_1 = n_2$ we have $\ker(A_{12}) = \emptyset$, so the terms with A_{12}^\perp and $(A_{12}^\perp)^+$ presented in (11) disappear.

VI. NUMERICAL EXAMPLES

Consider as a benchmark example the system (2) with

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 0 & 3 \\ 0 & -1 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -3 \end{pmatrix}, f = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \sin(t), G = \begin{pmatrix} 0.4286 & 0.8571 & 0.2857 \\ -0.8571 & 0.4857 & -0.1714 \\ -0.2857 & -0.1714 & 0.9429 \end{pmatrix}.$$

The transformation $y = Gx$ brings the system to the block from (10) with $y_1 \in \mathbb{R}, y_2 \in \mathbb{R}^2$,

$$A_{11} = -0.5918, A_{23} = \begin{pmatrix} -2.2 & 1.0 \\ -0.4 & -3.0 \end{pmatrix}, A_{21} = \begin{pmatrix} 1.2980 \\ 0.7184 \end{pmatrix}, A_{22} = \begin{pmatrix} 3.0612 & -0.8367 \\ -0.5510 & 2.5306 \end{pmatrix},$$

$$A_{12} = \begin{pmatrix} -0.4449 & 4.9469 \end{pmatrix}. \text{ For this case we have } \gamma(t, x) = (\sin(t), \sin(t))^T \text{ and } \gamma_0(t, x) = 1.$$

A. Example of fixed-time attractivity

Let $T_{max}=6, r=1$. Following Theorem 3 we select the controller $u(t, x)$ in the form (9) with the control parameters $\alpha_1=6.3918, \alpha_2 = 13.333, \beta_1=\beta_2=0.0327$. Fig.1(a) and Fig. 1(b) show evolutions of the system states for $x(0)=(1,0,0)$ and $x(0)=(10,0,0)$. Fig. 1(c) presents the plot for the control magnitude. Numerical experiments show that the predetermined settling time

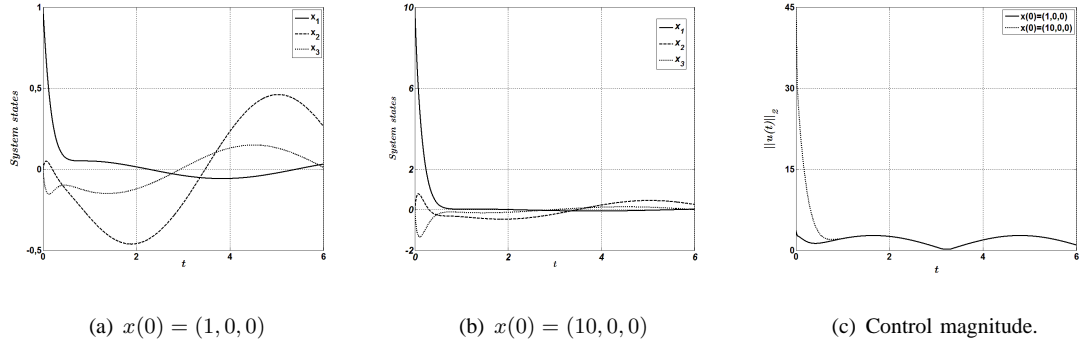


Fig. 1. Simulation results for fixed-time attraction.

estimate $T_{max}=6$ is too conservative. Obtained convergence times to the ball B_1 are less than 1.

B. Example of fixed-time stability

According to Theorem 4 the controller of the form (11) with parameters $\alpha_1=\beta_1=1, \alpha_2=\beta_2=0.5, \alpha_3 = \beta_3 = 0.25$ provides fixed-time stability of the considered system guaranteeing the global settling-time estimate $T_{max}=8$. Simulations results approve declared property (see Fig. 2).

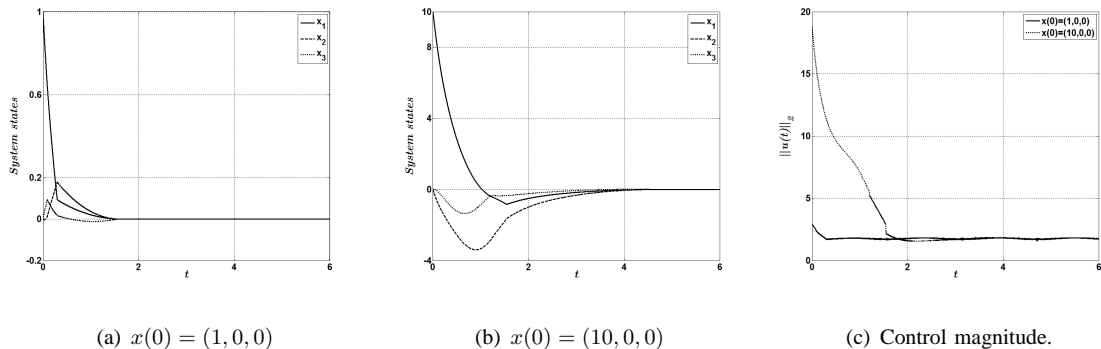


Fig. 2. Simulation results for fixed-time stabilization.

In contrast to polynomial feedback numerical simulations for the fixed-time stabilizing controller demonstrates very small conservatism in settling-time estimate showing that for $x(0) = (10^7, 0, 0)$ the settling-time is equal approximately to 7.2.

VII. CONCLUSIONS

In this paper the new control problems are considered. They consist in designing of the feedback control for linear uncertain plant that guarantees convergence of system trajectories to the origin or into selected neighborhood of the origin in predetermined finite time *independently on initial conditions*. The described property is called in this paper by *fixed-time stability* and *attractivity*, respectively. Controllers that provides fixed-time attractivity to the closed-loop systems are designed in the form of polynomial feedbacks under the classical assumption on controllability of the pair (A, B) . Fixed-time stabilizing controllers presented for more restrictive case $\text{rank}[B, AB] = n$ are modification of the second order sliding mode control algorithm. It is proven that all controllers are robust with respect to matched uncertainties and disturbances.

Numerical experiments approve the declared fixed-time properties for the closed-loop system with the proposed control algorithms. However, they show that for polynomial feedbacks the real convergence time is less than the predetermined. This fact probably points out ineffective usage of the control resource. So, the problem of optimal selection of the control parameters in order to minimize the control energy and magnitude appears. But this is another story.

VIII. ACKNOWLEDGEMENTS

The author wishes to thank professor Boris T. Polyak for helpful discussions and remarks.

REFERENCES

- [1] V. Andrieu, L. Praly, A. Astolfi Homogeneous Approximation, Recursive Observer and Output Feedback, *SIAM Journal of Control and Optimization*, 47(4):1814-1850, 2008.
- [2] G. Bartolini, A. Pisano, E. Punta, E. Usai, A survey of applications of second-order sliding mode control to mechanical systems, *International Journal of Control*, 76(9/10):875-892, 2003.
- [3] F. J. Bejarano & L. M. Fridman, High order sliding mode observer for linear systems with unbounded unknown inputs, *International Journal of Control*, 9:1920-1929, 2010.
- [4] S.P. Bhat, D.S. Bernstein, Finite-time stability of continuous autonomous systems, *SIAM Journal of Control and Optimization*, 38(3):751-766, 2000.
- [5] F.L. Chernousko, I.M. Ananevskii, S.A. Reshmin, *Control of Nonlinear Dynamical Systems: Methods and Applications*, Berlin: Springer-Verlag, 2008.
- [6] E. Cruz-Zavala, J.A. Moreno, L. Fridman, Uniform Second-Order Sliding Mode Observer for Mechanical Systems, *International Workshop on Variable Structure Systems*, 14-19, 2010.
- [7] S.V. Drakunov et al., Block Control Principle I, *Automation and Remote Control*, 51(5):601-609, 1990.
- [8] C. Ebenbauer, F. Allgöwer, Analysis and design of polynomial control systems using dissipation inequalities and sum of squares, *Computers and Chemical Engineering*, 30:1590-1602, 2006.
- [9] D. Efimov, Uniting global and local controllers under acting disturbances, *Automatica*, 42:489-495, 2006.
- [10] R. Engel, G. Kreisselmeier, A Continuous-Time Observer Which Converges in Finite Time, *IEEE Transactions on Automatic Control*, 47:1202-1204, 2002.
- [11] A. Ferrara & L. Giacomini, On multi-input backstepping design with second order sliding modes for a class of uncertain nonlinear systems, *International Journal of Control*, 71(5):767-788, 1998.
- [12] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer Academic Publishers, 1988.
- [13] V. T. Haimo, Finite time controllers. *SIAM Journal of Control and Optimization*, 24(4):760-770, 1986.
- [14] Y. Hong, Z.-P. Jiang, Finite-Time Stabilization of Nonlinear Systems With Parametric and Dynamic Uncertainties *IEEE Transactions on Automatic Control*, 51(12):1950-1956, 2006
- [15] S. A. Krasnova, T. G. Sirotnina, V.A. Utkin, A structural approach to robust control, *Automation and Remote Control*, 72(8):1639-1666, 2011.
- [16] A. Levant, Homogeneity approach to high-order sliding mode design. *Automatica*, 41:823-830, 2005.
- [17] A. Levant, Principles of 2-sliding mode design, *Automatica*, 43:576-586, 2007.
- [18] M. James, Finite Time Observers and Observability, *Conference on Decision and Control*, 770-771, 1990.
- [19] E. Moulay, W. Perruquetti, Finite-time stability and stabilization: state of the art, *Lecture Notes in Control and Information Sciences*, 334:23-41, 2006.
- [20] Y. Orlov, Finite Time Stability and Robust Control Synthesis of Uncertain Switched Systems. *SIAM Journal of Control and Optimization*, 43(4):1253-1271, 2005.
- [21] Y. Orlov, *Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions*. Springer-Verlag, 2009.
- [22] A. Polyakov, A. Poznyak, Lyapunov function design for finite-time convergence analysis: "twisting" controller for second order sliding mode realization. *Automatica*, 45(2):444-448, 2009.
- [23] T. Raff, F. Allgöwer, An Observer that Converges in Finite Time Due to Measurement-based State Updates, *IFAC World Congress*, 2693-2695, 2008.
- [24] V.I. Utkin, *Sliding Modes in Control Optimization*. Springer Verlag, Berlin, 1992.

Finite-Time and Fixed-Time Stabilization: Implicit Lyapunov Function Approach

Andrey Polyakov^{a,b}, Denis Efimov^{a,b}, Wilfrid Perruquetti^{b,a}

^aNon-A INRIA - LNE, Parc Scientifique de la Haute Borne 40, avenue Halley Bat.A, Park Plaza 59650 Villeneuve d'Ascq, France.

^bLAGIS UMR CNRS 8219, Ecole Centrale de Lille, BP 48, Cité Scientifique, 59651 Villeneuve-d'Ascq, France.

Abstract

Theorems on Implicit Lyapunov Functions (ILF) for finite-time and fixed-time stability analysis of nonlinear systems are presented. Based on these results, new nonlinear control laws are designed for robust stabilization of a chain of integrators. High order sliding mode (HOSM) algorithms are obtained as particular cases. Some aspects of digital implementations of the presented algorithms are studied, it is shown that they possess a chattering reduction ability. Theoretical results are supported by numerical simulations.

Key words: Lyapunov Methods, Robust Control, Sliding Mode Control

1 Introduction and related works

Many practical applications require severe time response constraints (for security reasons, or simply to improve productivity). That is why, finite-time stability and stabilization problems have been intensively studied, see [2], [3], [4], [5], [6]. Time constraint may also appear in observation problems when a finite-time convergence of the state estimate to the real values is required: [7], [8], [9], [10], [11].

Let us stress that finite-time stability property is frequently associated with HOSM controls, since these algorithms should ensure finite-time convergence to a sliding manifold [12], [13], [14], [15]. Typically, the associated controllers have mechanical and electromechanical applications [16], [17], [18], [19].

The theoretical background of HOSM control systems is very well developed [12], [20], [21]. However, applications of the existing HOSM control algorithms are complicated, since there are a few constructive algorithms for tuning the HOSM control parameters. Most of them are restricted to the second order sliding mode systems [21], [15], [22], [23].

Fixed-time stability, that demands *boundedness of the settling-time function* for a globally finite-time stable system, was studied in [23], [22], [24]. This property was originally discovered in the context of homogeneous systems [25]. Fixed-time stability looks promising if a controller (observer) has to be designed in order to provide some required control (observation) precision in *a given time* and *independently of initial conditions*.

The main tool for analysis of finite-time and fixed-time stability is the Lyapunov function method (see, for example, [4], [5], [24]), which is lacking for constructive design in the nonlinear case. This paper deals with ILF method [26], which relies on Lyapunov functions defined, implicitly, as solutions to an algebraic equation. Stability analysis does not require solving this equation, since the implicit function theorem [27] helps to check all stability conditions directly from the implicit formulation. The similar approach was presented in [28] for control design and called the controllability function method (see, also [29]).

This paper addresses the problem of a control design for the robust finite-time and fixed-time stabilization of a chain of integrators. The ILF method is used to design the control laws together with Lyapunov functions for closed-loop systems. This method allows us to analyze robustness of the closed-loop system and to design *a high order sliding mode control algorithm*, which rejects bounded matched exogenous disturbances. Finite-time and fixed-time stability conditions were obtained in the form of Linear Matrix Inequalities (LMI). They provide *simple constructive schemes for*

* The preliminary version of this paper was presented at IFAC Symposium on Nonlinear Control Systems 2013 [1]. This work is supported by ANR Grant CHASLIM (ANR-11-BS03-0007)

Email addresses: andrey.polyakov@inria.fr (Andrey Polyakov), denis.efimov@inria.fr (Denis Efimov), wilfrid.perruetti@inria.fr (Wilfrid Perruquetti).

tuning the control parameters in order to predefine the required convergence time and/or to guarantee stability and robustness with respect to disturbances of a given magnitude.

Through the paper the following notation will be used:

- \mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$;
- $\left. \frac{dV}{dt} \right|_{(\cdot)}$ is the time derivative of a function V along the solution of a differential equation numbered as (\cdot) ;
- $\|\cdot\|$ is the Euclidian norm in \mathbb{R}^n ;
- $\text{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with the elements λ_i ;
- a continuous function $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K} if it is monotone increasing and $\sigma(h) \rightarrow 0^+$ as $h \rightarrow 0^+$;
- for a symmetric matrix $P = P^T$ the minimal and maximal eigenvalues are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$,
- $\text{int}(\Omega)$ is the interior of the set $\Omega \subseteq \mathbb{R}^n$.

2 Problem statement

The paper deals with finite-time and fixed-time stabilization problems for the disturbed chain of integrators. Note that a control design scheme developed for such systems usually admits extension to feedback linearizable nonlinear multi-input multi-output systems [30]. The problem statement presented below is also typical for high-order sliding mode control design [12].

Consider a linear single input system of the following form

$$\dot{x}(t) = Ax(t) + bu(t) + d(t, x(t)), \quad t > 0, \quad (1)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix},$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and the function $d : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes the system uncertainties and disturbances. The whole state vector x is assumed to be measured. Let the function d be measurable locally bounded uniformly in time, i.e. $\sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^n : \|x\| < \delta} \|d(t, x)\| < \infty$ for any $\delta > 0$. Both the control function u and the function d are admitted to be discontinuous with respect to x . For example, the function d may describe unknown dry friction of a mechanical model. The analysis of such systems requires a special mathematical framework. In this paper we use Filippov theory [31].

The goal of the paper is to develop control laws such that the origin of the closed-loop system (1) will be globally asymptotically stable and all its trajectories will reach the origin in a finite time or in the fixed time $T_{\max} \in \mathbb{R}_+$. In addition, the control algorithms to be developed should have effective schemes for tuning the control parameters and assigning of the settling time.

The control design is based of ILF approach to finite-time and fixed-time stability analysis, which is developed in the next section.

3 Preliminaries

3.1 Finite-time and fixed-time stability

Consider the system defined by

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0, \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field locally bounded uniformly in time. If f is a locally measurable function that is discontinuous with respect to the state variable x then a solution of the Cauchy problem (2) is understood in the sense of Filippov [31], namely, as an absolutely continuous function satisfying the differential inclusion

$$\dot{x}(t) \in K[f](t, x(t))$$

for almost all $t \in [0, t^*)$, where $t^* \in \mathbb{R}_+$ or $t^* = +\infty$. The set-valued function $K[f] : \mathbb{R}_+ \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined for any fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ as follows

$$K[f](t, x) = \bigcap_{\varepsilon > 0} \bigcap_{N: m(N)=0} \text{co} f(t, B(x, \varepsilon) \setminus N),$$

where $\text{co}(M)$ defines the convex closure of the set $M \subset \mathbb{R}^n$, $B(x, \varepsilon)$ is the ball with the center at $x \in \mathbb{R}^n$ and the radius $\varepsilon \in \mathbb{R}_+$, the equality $m(N) = 0$ means that the Lebesgue measure of the set $N \subset \mathbb{R}^n$ is zero.

Let the origin be an equilibrium of (2), i.e. $0 \in K[f](t, 0)$. The system (2) may have non-unique solutions and may admit both weak and strong stability (see, for example, [31]). This paper deals only with the *strong stability* properties, which ask for stable behavior of all solutions of the system (2). The next definition of uniform finite-time stability is just a compact representation of Definition 2.5 from [13].

Definition 1 *The origin of system (2) is said to be globally uniformly finite-time stable if it is globally uniformly asymptotically stable (see, for example, [13]) and there exists a locally bounded function $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, such that $x(t, x_0) = 0$ for all $t \geq T(x_0)$, where $x(\cdot, x_0)$ is an arbitrary solution of the Cauchy problem (2). The function T is called the settling-time function.*

Asymptotic stability of the time-independent (autonomous) system always implies its uniform asymptotic stability (see, for example, [32]). For finite-time stable systems this is not true in general case (see, for example, [24]), since Definition 1 additionally asks a uniformity of the settling time with respect to initial conditions.

The origin of system $\dot{x}(t) = -|x(t)|^{0.5} \text{sign}[x(t)]$, $x \in \mathbb{R}$ is globally uniformly finite-time stable, since its settling-time function T is locally bounded: $T(x_0) = 2\sqrt{|x_0|}$.

Definition 2 ([23]) The origin of system (2) is said to be globally **fixed-time stable** if it is globally uniformly finite-time stable and the settling-time function T is **globally bounded**, i.e. $\exists T_{\max} \in \mathbb{R}_+$ such that $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

The presented definition just asks more: strong uniformity of finite-time stability with respect to initial condition. The origin of system $\dot{x}(t) = -(|x|^{0.5}(t) + |x|^{1.5}(t)) \text{sign}(x(t))$, $x \in \mathbb{R}$, is globally fixed-time stable, since its settling time function $T(x_0) = 2 \arctan(\sqrt{|x_0|})$ is bounded by $\pi \approx 3.14$.

The uniformity of finite-time and fixed-time stability with respect to system disturbances can also be analyzed. For instance, finite-time stability, which is uniform (in some sense) with respect to both initial conditions and system disturbances, was called equiuniform finite-time stability [13].

3.2 Implicit Lyapunov Function Method

This subsection introduces some stability theorems further used for control design. They refine the known results about global uniform asymptotic, finite-time and fixed-time stability of differential inclusions to the case of implicit definition of Lyapunov function. The next theorem extends Theorem 2 from [26].

Theorem 3 If there exists a continuous function

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \\ (V, x) \rightarrow Q(V, x)$$

satisfying the conditions

C1) Q is continuously differentiable outside the origin;
C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+$ such that $Q(V, x) = 0$;

C3) let $\Omega = \{(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, x) = 0\}$ and $\lim_{x \rightarrow 0} V = 0^+$, $\lim_{V \rightarrow 0^+} \|x\| = 0$, $\lim_{\|x\| \rightarrow \infty} V = +\infty$;
 $(V, x) \in \Omega$

C4) $\frac{\partial Q(V, x)}{\partial V} < 0$ for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$;

C5) $\sup_{t \in \mathbb{R}_+, y \in K[f](t, x)} \frac{\partial Q(V, x)}{\partial x} y < 0$ for all $(V, x) \in \Omega$;

then the origin of (2) is globally uniformly asymptotically stable.

Proof. The conditions C1), C2), C4) and the implicit function theorem [27] imply that the equation $Q(V, x) = 0$ implicitly defines a unique function $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ such that $Q(V(x), x) = 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The function V is continuously differentiable outside the origin and $\frac{\partial V}{\partial x} = - \left[\frac{\partial Q(V, x)}{\partial V} \right]^{-1} \frac{\partial Q(V, x)}{\partial x}$ for $Q(V, x) = 0$, $x \neq 0$. Due to the condition C3) the function V can be continuously prolonged at the origin by setting $V(0) = 0$. In addition, it is radially unbounded and positive definite. Denote

$$Z(V, x) = \left[\frac{\partial Q}{\partial V} \right]^{-1} \sup_{t \in \mathbb{R}_+, y \in K[f](t, x)} \frac{\partial Q(V, x)}{\partial x} y \text{ and } W(x) =$$

$Z(V(x), x)$. The conditions C4) and C5) imply $W(x) > 0$ for

$x \neq 0$. Let $x(t, x_0)$ be a solution of the system (2) with initial condition $x(0, x_0) = x_0$ then the function $V(x(t, x_0))$ is differentiable for almost all t such that $x(t, x_0) \neq 0$ and $\frac{d}{dt} V(x(t, x_0)) \leq -W(x(t, x_0))$. Finally, we finish the proof using Theorem 4.1 from [33] and, for example, Lemmas 4, 6 from [24]. ■

Evidently, the conditions of Theorem 3 mainly repeat (in the implicit form) the requirements of the classical theorem on global asymptotic stability (see, for example, [33]). Indeed, Condition C1) asks for smoothness of the Lyapunov function. Condition C2) and the first two limits from Condition C3) provide its positive definiteness. The last limit from Condition C3) implies radial unboundedness of the Lyapunov function. Condition C5) guarantees the negative definiteness of the total derivative of the Lyapunov function calculated along trajectories of the system (2). The only specific condition is C4), which is imposed by implicit function theorem (see, for example, [27]). This condition is required in order to guarantee that the Lyapunov function is (uniquely) well-defined by the equation $Q(V, x) = 0$.

Theorem 4 If there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies the conditions C1)-C4) of Theorem 3 and the condition

C5bis) there exist $c > 0$ and $0 < \mu \leq 1$ such that $\sup_{t \in \mathbb{R}_+, y \in K[f](t, x)} \frac{\partial Q(V, x)}{\partial x} y \leq cV^{1-\mu} \frac{\partial Q(V, x)}{\partial V}$ for $(V, x) \in \Omega$,

then the origin of the system (2) is globally uniformly finite-time stable and $T(x_0) \leq \frac{V_0^\mu}{c\mu}$, where $Q(V_0, x_0) = 0$.

Proof. Theorem 3 implies global uniform asymptotic stability of the origin of (2). The uniform finite-time stability of the origin follows from the differential inequality

$$\frac{dV(x(t, x_0))}{dt} \leq -cV^{1-\mu}(x(t, x_0)),$$

which, due to the condition C5bis), holds for almost all t such that $x(t, x_0) \neq 0$. For a detailed analysis of the obtained differential inequality see, for example, [4], [5], [24]. ■

Theorem 5 Let there exist two functions Q_1 and Q_2 satisfying the conditions C1)-C4) of Theorem 3 and the conditions C6) $Q_1(1, x) = Q_2(1, x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$;

C7) there exist $c_1 > 0$ and $0 < \mu < 1$ such that the inequality $\sup_{t \in \mathbb{R}_+, y \in K[f](t, x)} \frac{\partial Q_1(V, x)}{\partial x} y \leq c_1 V^{1-\mu} \frac{\partial Q_1(V, x)}{\partial V}$,

holds for all $V \in (0, 1]$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_1(V, x) = 0$;

C8) there exist $c_2 > 0$ and $\nu > 0$ such that the inequality $\sup_{t \in \mathbb{R}_+, y \in K[f](t, x)} \frac{\partial Q_2(V, x)}{\partial x} y \leq c_2 V^{1+\nu} \frac{\partial Q_2(V, x)}{\partial V}$,

holds for all $V \geq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_2(V, x) = 0$, then the system (2) is globally fixed-time stable with the settling-time estimate $T(x_0) \leq \frac{1}{c_1 \mu} + \frac{1}{c_2 \nu}$.

Proof. Let the two functions V_1 and V_2 be defined by the equations $Q_1(V, x) = 0$ and $Q_2(V, x) = 0$ (see, the proof of Theorem 3). Consider the sets $\Sigma_1 = \{x \in \mathbb{R}^n : V_1(x) > 1\}$,

$\Sigma_2 = \{x \in \mathbb{R}^n : V_2(x) > 1\}$ and prove that $\Sigma_1 = \Sigma_2$. Suppose the contrary, i.e. $\exists z \in \mathbb{R}^n$ such that $z \in \Sigma_1$ and $z \notin \Sigma_2$. On the one hand, $Q_1(V_1, z) = 0$ implies $V_1 > 1$ and $Q_1(1, z) > Q_1(V_1, z) = 0$ due to Condition C4). On the other hand, $Q_2(V_2, z) = 0$ implies $V_2 \leq 1$ and $Q_2(1, z) \leq Q_2(V_2, z) = 0$. The contradiction follows from Condition C6).

Therefore, due to C6) and C4) the function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the equality

$$V(x) = \begin{cases} V_1(x) & \text{for } V_1(x) < 1, \\ V_2(x) & \text{for } V_2(x) > 1, \\ 1 & \text{for } V_1(x) = V_2(x) = 1, \end{cases}$$

is positive definite, continuous in \mathbb{R}^n and continuously differentiable for $x \notin \{0\} \cup \{x \in \mathbb{R}^n : V(x) = 1\}$. The function V is Lipschitz continuous outside the origin and has the following Clarke's gradient [34]:

$$\nabla_C V(x) = \xi \nabla V_1(x) + (1 - \xi) \nabla V_2(x), \quad x \in \mathbb{R}^n,$$

where $\xi = 1$ for $0 < V_1(x) < 1$, $\xi = 0$ for $V_2(x) > 1$, $\xi = [0, 1]$ for $V_1(x) = V_2(x) = 1$ and ∇V_i is the gradient of the function V_i , $i = 1, 2$. Hence, due to conditions C7) and C8), the inequality

$$\frac{dV(x(t, x_0))}{dt} \leq \begin{cases} -c_1 V^{1-\mu}(x(t, x_0)) & \text{for } V(x(t, x_0)) < 1, \\ -c_2 V^{1+\nu}(x(t, x_0)) & \text{for } V(x(t, x_0)) > 1, \\ -\min\{c_1, c_2\} & \text{for } V(x(t, x_0)) = 1, \end{cases}$$

holds for almost all t such that $x(t, x_0) \neq 0$, where $x(t, x_0)$ is a solution of the system (2) with the initial condition $x(0) = x_0$. This implies the fixed-time stability of the origin of the system (2) with the estimate of settling-time function given above. Please see [23] or [24] for more details. ■

4 Control design using Implicit Lyapunov Function Method

4.1 Finite-Time Stabilization

Introduce the implicit Lyapunov function candidate

$$Q(V, x) := x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \quad (3)$$

where $V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n \times n}$, $P > 0$, $D_r(\lambda) := \text{diag}\{\lambda^{r_i}\}$ is the matrix with $r = (r_1, \dots, r_n)^T \in \mathbb{R}_+^n$ and $r_i = 1 + (n - i)\mu$, $0 < \mu \leq 1$. Denote $H_\mu := \text{diag}\{-r_i\}_{i=1}^n$.

The function Q is an implicit analog of the quadratic Lyapunov function. Indeed, for $\mu = 0$ the equality $Q(V, x) = 0$ gives $V(x) = \sqrt{x^T P x}$. For $\mu = 1$ it coincides with the implicit Lyapunov function considered in [26].

Theorem 6 (Finite-time Stabilization) *Let 1) $\mu \in (0, 1]$, $\alpha, \beta, \gamma \in \mathbb{R}_+$ such that $\alpha > \beta$ and the system of matrix*

inequalities

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X + \beta I_n \leq 0, \\ -\gamma X \leq XH_\mu + H_\mu X < 0, \quad X > 0 \end{cases} \quad (4)$$

is feasible for some $X = X^T \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^{1 \times n}$;

2) the control u has the form

$$u(V, x) = V^{1-\mu} k D_r(V^{-1}) x, \quad (5)$$

where $k := yX^{-1}$, $V \in \mathbb{R}_+$ is such that $Q(V, x) = 0$ and Q is defined by (3) with $P = X^{-1}$;

3) the function $d(t, x)$ satisfy the inequality

$$\beta^2 V^{-2\mu} \geq \sup_{t \in \mathbb{R}_+} d^T(t, x) D_r^2(V^{-1}) d(t, x), \quad (6)$$

for all $V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ such that $Q(V, x) = 0$;

then the closed-loop system (1) is globally uniformly finite-time stable and $T(x) \leq \frac{\gamma V^\mu}{\mu(\alpha - \beta)}$, where $Q(V, x) = 0$.

Proof. It can be shown that the following inequalities

$\frac{\lambda_{\min}(P)\|x\|^2}{\max\{V^{2+2(n-1)\mu}, V^2\}} \leq Q(V, x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{\min\{V^{2+2(n-1)\mu}, V^2\}}$ hold for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. The function (3) satisfies conditions C1)-C3) of Theorem 3. The condition C4) of Theorem 3 also holds, since $0 > \frac{\partial Q}{\partial V} = V^{-1} x^T D_r(V^{-1})(H_\mu P + P H_\mu) D_r(V^{-1}) x \geq -\gamma V^{-1}$.

Taking into account that $D_r(V^{-1}) A D_r^{-1}(V^{-1}) = V^{-\mu} A$ and $D_r(V^{-1}) b u(x) = V^{-\mu} b k D_r(V^{-1}) x$ we obtain

$$\frac{\partial Q}{\partial x}(Ax + bu(x) + d(t, x)) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^T W \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta^{-1} V^\mu z_2^T z_2 - \alpha V^{-\mu} z_1^T P z_1,$$

where $z_1 = D_r(V^{-1}) x$, $z_2 = D_r(V^{-1}) d$, the matrix

$$W := \begin{pmatrix} \frac{1}{V^\mu} (P(A+bk) + (A+bk)^T P + \alpha P) & P \\ P & -\frac{V^\mu}{\beta} I_n \end{pmatrix}$$

is negative semidefinite due to (4) and the Schur complement. Since $z_1^T P z_1 = 1$ and $z_2^T z_2 \leq \beta^2 V^{-2\mu}$ by (6) then the condition C5bis) of Theorem 4 holds for $c = (\alpha - \beta)/\gamma$. ■

Let us make remarks about the presented control scheme:

- The practical implementation of the control (5) requires to find the solution $V(x)$ of the equation $Q(V, x) = 0$, which can be solved numerically and on-line using the current value of the state vector. A simple numerical scheme that can be utilized for this purpose is presented in Section 5.
- If $n = 2$ and $\mu = 1$ the function $V(x)$ can be found analytically. Indeed, the equation $Q(V, x) = 0$ becomes $V^4 - p_{22} x_2^2 V^2 - 2p_{12} x_1 x_2 V - p_{11} x_1^2 = 0$, where $\{p_{ij}\}$ are elements of the matrix $P > 0$ and

$(x_1, x_2)^T = x \in \mathbb{R}^2$. The roots of the obtained quartic (with respect to V) equation can be found using, for example, Ferrari formulas. Due to Theorem 6 the equation has a unique positive root for any $(x_1, x_2)^T \in \mathbb{R}^2$.

Denote $z(x_1, x_2) = \frac{2p_{22}x_2^2 - C_1(x_1, x_2) - C_2(x_1, x_2)}{3}$, where $C_i(x_1, x_2) = \sqrt[3]{\frac{(-1)^i \sqrt{\Delta_1^2(x_1, x_2) - 4\Delta_0^3(x_1, x_2)} + \Delta_1(x_1, x_2)}{2}}$, $\Delta_0(x_1, x_2) = p_{22}^2 x_2^4 - 12p_{11} x_1^2$ and $\Delta_1(x_1, x_2) = p_{22}^3 x_2^6 - 180p_{12}^2 x_1^2 x_2^2$. The roots of the quartic equation coincide with roots of two quadratic equations [35]: $V^2 + (-1)^i \sqrt{z(x_1, x_2)}V + r_i(x_1, x_2) = 0$, $i = 1, 2$, where $r_i(x_1, x_2) = \frac{p_{22}x_2^2 + z(x_1, x_2)}{2} + (-1)^i \frac{p_{12}x_1x_2}{\sqrt{z(x_1, x_2)}}$.

The explicit representations of roots are rather cumbersome. They are omitted in order to save the space.

- The implicit restriction (6) to the system disturbances and uncertainties takes an explicit form when $\mu = 1$ and the matching condition [14] holds, i.e. $d_i(t, x) \equiv 0$ for $i = 1, \dots, n-1$ and $d = (d_1, \dots, d_n)^T$. In this case the condition (6) becomes $|d_n(t, x)| \leq \beta$ and the term βI_n in (4) can be replaced with βE , where the matrix $E = \{e_{ij}\} \in \mathbb{R}^{n \times n}$ has only one nonzero element: $e_{n,n} = 1$. So, the finite-time control (5) designed for $\mu = 1$ rejects the bounded disturbances realized HOSM algorithm. The HOSM version of (5) has a discontinuity only at zero similarly to the quasi-continuous HOSM algorithm [20].
- For $\mu = 1$ the control (5) is globally bounded. Indeed, the equality $x^T D_r(V^{-1}) P D_r(V^{-1}) x = 1$ implies $\|D_r(V^{-1})x\|^2 \leq \frac{1}{\lambda_{\min}(P)}$ and

$$\|u(x)\| \leq \|k\| \cdot \|D_r(V^{-1})x\| \leq \frac{\|k\|}{\sqrt{\lambda_{\min}(P)}}.$$

Hence the condition $\|u(x)\| \leq u_0$, $u_0 \in \mathbb{R}_+$ can be formulated as additional LMI:

$$\begin{pmatrix} X & y^T \\ y & u_0^2 \end{pmatrix} \geq 0. \quad (7)$$

Indeed, the inequality $u^2 = x^T D_r(V^{-1}) k^T k D_r(V^{-1}) x \leq u_0^2 = u_0^2 x^T D_r(V^{-1}) P D_r(V^{-1}) x$ holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ such that $Q(V, x) = 0$. Hence $k^T k \leq u_0^2 P$ and Schur complement gives (7) for $X = P^{-1}$, $y = kX$.

- The advantage of the control design scheme presented in Theorem 6 is related to simplicity of tuning the control parameters, which is based on LMI technique. The parameters γ and α are introduced in (4) in order to tune the settling time of the closed-loop system.

Corollary 7 *Let the conditions of Theorem 6 hold and the control law is modified as $u(x) = u(V_0, x)$, where $V_0 \in \mathbb{R}_+$ is an arbitrary fixed number, then the ellipsoid*

$$\Pi(V_0) = \left\{ x \in \mathbb{R}^n : x^T (D_r(V_0) X D_r(V_0))^{-1} x \leq 1 \right\} \quad (8)$$

is strictly positively invariant set of the closed-loop system (1), i.e. $x(t_0) \in \Pi(V_0) \Rightarrow x(t) \in \text{int } \Pi(V_0), t > t_0$, where

$x(\cdot)$ is any trajectory of the closed-loop system (1) and $t_0 \geq 0$ is an arbitrary instant of time.

Proof. Rewrite the matrix inequality (4) in the form: $PA + A^T P + Pbk + k^T b^T P + \alpha P + \beta P^2 \leq 0$. Hence we derive $D_r(V_0^{-1})(PA + A^T P + Pbk + k^T b^T P + \alpha P + \beta P^2)D_r(V_0^{-1}) \leq 0$. Denoting $P_0 = D_r(V_0^{-1})P D_r(V_0^{-1}) > 0$ and taking into account $D_r^{-1}(V_0^{-1})A D_r(V_0^{-1}) = V_0^\mu A$, $D_r^{-1}(V_0^{-1})b = V_0 b$ we obtain the following matrix inequality:

$$P_0 A + A^T P_0 + P_0 b k_0 + k_0^T b^T P_0 + \frac{\alpha P_0 + \beta P_0 D_r^2(V_0) P_0}{V_0^\mu} \leq 0,$$

where $k_0 = V_0^{-\mu} k D_r(V_0^{-1})$, $\alpha > \beta > 0$ and $P_0 > 0$. This means that the matrix $A + bk_0$ is Hurwitz, i.e. $u(x) = u(V_0, x) = k_0 x$ is a stabilizing linear feedback control for the system (1) with the Lyapunov function $\tilde{V}(x) = x^T P_0 x$. Note that using Schur Complement the obtained matrix inequality can be transformed into the form $W_0 \leq 0$, where

$$W_0 = \begin{pmatrix} P_0(A+bk_0) + (A+bk_0)^T P_0 + \frac{\alpha}{V_0^\mu} P_0 & P_0 \\ P_0 & -\frac{V_0^\mu D_r^2(V_0^{-1})}{\beta} \end{pmatrix}.$$

Taking into account the inequality (6) for $V = V_0$ (or equivalently $\tilde{V} = 1$) we derive

$$\begin{aligned} \frac{d}{dt} \tilde{V} \Big|_{(1)} &= x^T (P_0 A + A^T P_0 + P_0 b k_0 + k_0^T b^T P_0) x + 2d^T P_0 x \\ &= \begin{pmatrix} x \\ d \end{pmatrix}^T W_0 \begin{pmatrix} x \\ d \end{pmatrix} + \frac{V_0^\mu}{\beta} d^T D_r^2(V_0^{-1}) d - \frac{\alpha \tilde{V}}{V_0^\mu} \\ &\leq \frac{\beta - \alpha}{V_0^\mu} < 0 \quad \text{for } \tilde{V}(x) = 1. \end{aligned}$$

The obtained inequality implies that the ellipsoid (8) is strictly positively invariant. ■

The next corollary proves stability of the sampled-time realization for the ILF-based control algorithm (5).

Corollary 8 *Let $\{t_i\}_{i=0}^\infty$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Let all conditions of Theorem 6 hold and the control u is applied as follows: $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i > 0 : Q(V_i, x(t_i)) = 0$. Then the origin of the system (1) is globally asymptotically stable.*

Proof. Let $V(x)$ be a positive definite function implicitly defined by the equation $Q(V, x) = 0$ and $x(t)$ be a trajectory of the closed-loop system (1) with the sampled control application described above. Let us prove that the sequence $\{V(x(t_i))\}_{i=1}^\infty$ is monotone decreasing and tends to 0. This, obviously, implies convergence of $x(t)$ to the origin.

Consider the time interval $[t_i, t_{i+1})$ and the function $\tilde{V}_i(x) = x^T P_i x$, where $P_i = D_r(V^{-1}(x(t_i))) P D_r(V^{-1}(x(t_i))) >$

0. The switching control $u(x) = u_i(x)$ on this interval takes the form $u_i(x) = u(V(x(t_i)), x) = k_i x$, where $k_i := V^{1-\mu}(x(t_i))kD_r(V^{-1}(x(t_i)))$. Repeating the proof of Corollary 7 we show $\tilde{V}_i(x(t_i)) > \tilde{V}_i(x(t))$ for all $t \in (t_i, t_{i+1})$. Since $V(x)$ is such that $Q(V(x), x) = 0$ then for all $t \in (t_i, t_{i+1})$

$$\begin{aligned} Q(V(x(t_i)), x(t)) &= \\ x^T(t)D_r(V^{-1}(x(t_i)))PD_r(V^{-1}(x(t_i)))x(t) - 1 &= \\ \tilde{V}_i(x(t)) - 1 < \tilde{V}_i(x(t_i)) - 1 = Q(V(t_i), x(t_i)) &= \\ 0 = Q(V(x(t)), x(t)). \end{aligned}$$

For any given $x \in \mathbb{R}^n \setminus \{0\}$ the function $Q(V, x)$ is monotone decreasing for all $V \in \mathbb{R}^+$ (see Condition C4) of Theorem 3 and the proof of Theorem 6). Then the obtained inequality implies $V(x(t)) < V(x(t_i))$, $\forall t \in (t_i, t_{i+1}]$. Moreover $V(x(t)) \leq V(x(0))$ for all $t > 0$, i.e. *the origin of the closed-loop system (1) is Lyapunov stable*. The proven properties of closed-loop system also imply that $\|x(t)\|$ and $\|d(t, x(t))\|$ are bounded by some constants for all $t > 0$.

Since the function $V(x)$ is positive definite then the monotone decreasing sequence $\{V(x(t_i))\}_{i=1}^{\infty}$ converge to some limit. Let us show now that this limit is zero. Suppose the contrary, i.e. $\lim_{i \rightarrow \infty} V(x(t_i)) = V_* > 0$ or equivalently $\forall \varepsilon > 0 \exists N = N(\varepsilon) : 0 \leq V(x(t_i)) - V_* < \varepsilon, \forall i \geq N$. Let us represent the sampled control law u in the form

$$u(V(x(t_i)), x) = u(V_*, x) + \Delta_i x,$$

$$\Delta_i = V^{1-\mu}(x(t_i))kD_r(V^{-1}(x(t_i))) - V_*^{1-\mu}kD_r(V_*^{-1}).$$

Since the control function $u = u(V, x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^n$ and then there exists $\gamma \in \mathcal{K}$ (possibly depended on μ, k, V_*, n) such that $\|\Delta_i\| \leq \gamma(\varepsilon)$ for all $i \geq N(\varepsilon)$. This means that for $t > t_{N(\varepsilon)}$ the closed-loop system (1) can be presented in the form

$$\dot{x} = (A + bk_*)x + b\Delta_i x + d(t, x) \quad (9)$$

where $k_* = V_*^{1-\mu}kD_r(V_*^{-1})$. Consider the Lyapunov function candidate $\tilde{V}(x) = x^T P_* x$, where $P_* = D_r(V_*^{-1})PD_r(V_*^{-1})$. Under the assumptions made above we have $\tilde{V}(x(t)) \geq 1$ for all $t > 0$ and repeating the proof of Corollary 7 we derive

$$\begin{aligned} \left. \frac{d\tilde{V}(x(t))}{dt} \right|_{(1)} &\leq \frac{V_*^\mu d^T D_r^2(V_*^{-1})d}{\beta} - \frac{\alpha \tilde{V}(x(t))}{V_*^\mu} + 2x^T(t)P_* b \Delta_i x(t) \\ &\leq \frac{-\alpha + \beta}{V_*^\mu} + 2x^T(t)P_* b \Delta_i x(t) \\ &\quad + \frac{V_*^\mu (d^T D_r^2(V_*^{-1})d - \beta^2 V_*^{-2\mu})}{\beta}. \end{aligned}$$

Since $\|x(t)\|$ and $\|d(t, x(t))\|$ are bounded by some constants for all $t > 0$ then for sufficiently small $\varepsilon > 0$ we have $\left. \frac{d\tilde{V}(x(t))}{dt} \right|_{(1)} \leq \frac{-\alpha + \beta}{2V_*^\mu} < 0$ for all $t > t_{N(\varepsilon)}$. This contradicts

the condition $V(x(t)) \geq V_*$ for all $t > 0$. Consequently, $\lim_{i \rightarrow \infty} V(x(t_i)) = 0$. Hence, we derive the asymptotic stability of the closed-loop system (1). ■

The proven corollary shows that the sampled-time realization of the developed "implicit" control scheme preserves the asymptotic stability to the origin of the closed-loop system (1) *independently on the sampling period*. Between two switching instants the unperturbed system is *linear*, so analysis of its discrete-time version can be studied using discretization schemes of linear systems (see, e.g. [36]).

4.2 Fixed-time Stabilization

Consider now two functions:

$$Q_1(V, x) = x^T D_{r_1}(V^{-1})PD_{r_1}(V^{-1})x - 1,$$

$$Q_2(V, x) = x^T D_{r_2}(V^{-1})PD_{r_2}(V^{-1})x - 1,$$

where $P \in \mathbb{R}^{n \times n}$, $P = P^T > 0$,

$$\begin{aligned} r_1 &= (1 + (n-1)\mu, 1 + (n-2)\mu, \dots, 1 + \mu, 1)^T \in \mathbb{R}^n, \\ r_2 &= (1, 1 + \nu, \dots, 1 + (n-2)\nu, 1 + (n-1)\nu)^T \in \mathbb{R}^n, \\ 0 &< \mu \leq 1 \text{ and } \nu \in \mathbb{R}_+. \text{ Denote } H_\nu = \text{diag}\{-\nu\}_{i=1}^n. \end{aligned}$$

Theorem 9 (Fixed-time stabilization) *If 1) the system of matrix inequalities*

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X + \beta I_n \leq 0, \\ -\gamma_1 X \leq XH_\mu + H_\mu X < 0, \\ -\gamma_2 X \leq XH_\nu + H_\nu X < 0, \\ X > 0, \end{cases} \quad (10)$$

is feasible for some $X = X^T \in \mathbb{R}^{n \times n}$, $y \in \mathbb{R}^{1 \times n}$ and numbers $\mu \in (0, 1]$, $\nu, \alpha, \beta, \gamma_1, \gamma_2 \in \mathbb{R}_+$ such that $\alpha > \beta$;

2) the control u has the form

$$u(V, x) = \begin{cases} V^{1-\mu}kD_{r_1}(V^{-1})x & \text{for } x^T P x < 1, \\ V^{1+\nu}kD_{r_2}(V^{-1})x & \text{for } x^T P x \geq 1, \end{cases} \quad (11)$$

where $k = yX^{-1}$, $P = X^{-1}$ and V defined by

$$V = V(x) \text{ such that } \begin{cases} Q_1(V, x) = 0 & \text{for } x^T P x < 1, \\ Q_2(V, x) = 0 & \text{for } x^T P x \geq 1; \end{cases}$$

3) the disturbance function d satisfy

$$\beta^2 V^{-2\mu} \geq \sup_{t \in \mathbb{R}_+} d^T(t, x)D_{r_1}^2(V^{-1})d(t, x) \text{ if } x^T P x < 1,$$

$$\beta^2 V^{2\nu} \geq \sup_{t \in \mathbb{R}_+} d^T(t, x)D_{r_2}^2(V^{-1})d(t, x) \text{ if } x^T P x \geq 1;$$

then the closed-loop system (1) is fixed-time stable with the settling-time estimate: $T(x) \leq (\gamma_1/\mu + \gamma_2/\nu)(\alpha - \beta)^{-1}$.

Proof. The functions $Q_1(V, x)$ and $Q_2(V, x)$ satisfy the conditions C1)-C4) of Theorem 3. Since $Q_1(1, x) = Q_2(1, x)$, then the condition C6) of Theorem 5 also holds.

Repeating the proof of Theorem 6 it can be shown that $\frac{\partial Q_1}{\partial x}(Ax+bu(x)+d(t,x)) \leq \frac{\alpha-\beta}{\gamma_1} V^{1-\mu} \frac{\partial Q_1}{\partial V}$ for $x^T Px \leq 1$, $\frac{\partial Q_2}{\partial x}(Ax+bu(x)+d(t,x)) \leq \frac{\alpha-\beta}{\gamma_2} V^{1+\nu} \frac{\partial Q_2}{\partial V}$ for $x^T Px \geq 1$. Applying Theorem 5 we finish the proof. ■

If $\mu = 1$, $|d_n(t,x)| \leq \beta$ and the matching condition holds then Theorem 9 provides a HOSM control with fixed-time reaching phase.

Let us call the controls (5) and (11) by *the finite-time ILF control* and *the fixed-time ILF control*, respectively. The ILF control with $\mu = 1$ we call *the HOSM ILF control*.

It is worth to stress that Corollaries 7 and 8 stay true for the case of fixed-time control (11) application. We do not prove the finite-time and fixed-time stability properties for the sampled-time realization of the ILF controls. However, the motion of the sampled system will be close to the original one if the sampling period is sufficiently small.

5 Aspects of practical implementation

Corollaries 7 and 8 give some remarks on possible implementation of the developed control scheme. A detailed study of sampled-time and discrete-time versions of the presented control algorithms goes beyond the scope of this paper providing the subject for a future research. In this section we provide just some general ideas to be used for ILF control implementation.

The control scheme (5) can be realized in digital control devices, which allow us to solve the equation $Q(V,x) = 0$ numerically and on-line for any given $x \in \mathbb{R}^n \setminus \{0\}$. Rather simple numerical procedures can be utilized for this purpose. The function $Q(V,x)$ satisfy the properties C1)-C4) of Theorem 3. So, for each fixed $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ the function $\bar{Q}(V) = Q(V,\bar{x})$ is monotone decreasing and has the unique zero on the interval $(0, +\infty)$. In this case we may use, for example, the bisection method in order to solve the scalar equation $\bar{Q}(V) = 0$.

Let the control $u(V,x)$ be given by (5) and the parameter V may change its value at some time instants $t_0 = 0, t_i > 0, i = 1, 2, \dots$. Namely, let the control signal $u(t)$ be defined as $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$ and $V_i \in \mathbb{R}_+$. Recall that $u(V_i, x)$ is a linear stabilizing feedback for any $V_i \in \mathbb{R}_+$ (see, the proof of Corollary 7). Denote $x_i = x(t_i)$. If $V_i = V(x_i)$ then stability of the sampling control scheme follows from Corollary 8.

The simplest algorithm for finding the switching control parameter V_i is given below.

Algorithm 10 (Implementation of the ILF control)

INITIALIZATION: $a = V_{\min}; V_0 > V_{\min}; b = V_0;$
STEP :

```

If  $x_i^T D_r(b^{-1}) P D_r(b^{-1}) x_i > 1$  then  $a = b; b = 2b;$ 
elseif  $x_i^T D_r(a^{-1}) P D_r(a^{-1}) x_i < 1$  then
     $b = a; a = \max\{\frac{a}{2}, V_{\min}\};$ 
else  $c = \frac{a+b}{2};$ 
    If  $x_i^T D_r(c^{-1}) P D_r(c^{-1}) x_i < 1$  then  $b = c;$ 
    else  $a = \max\{V_{\min}, c\};$ 
endif;
endif;
 $V_i = b,$ 

```

where $V_{\min} \in \mathbb{R}_+$ is a minimal admissible value of V_i .

INITIALIZATION defines a linear feedback $u(V_0, x)$ on the first sampling interval $[t_0, t_1)$ by means of selection of the value $V_0 \in \mathbb{R}_+$. STEP of the algorithm is applied at each sampling time instances t_i in order to define the value V_i (and the linear feedback $u(V_i, x)$) for the next interval $[t_i, t_{i+1})$. Let $x_i \in \mathbb{R}^n$ is a current state vector. If at the sampling instant t_i STEP of the algorithm is repeated many times for the same x_i (for example, there exists a loop containing STEP) then Algorithm 10 realizes: 1) a localization of the unique positive root of the equation $Q(V, x_i) = 0$, i.e. $V(x_i) \in [a, b]$; 2) improvement of the obtained localization by means of the bisection method, i.e. $|b - a| \rightarrow 0$. Such an application of Algorithm 10 allows us to calculate $V(x_i)$ with a high precision, however, it requests a high computational capability of a control device.

Therefore, it is more reasonable to realize STEP of Algorithm 10 just once or a few (2 or 3) times at each sampling instant. Some additional considerations are needed in order to show that the algorithm will work properly in this case.

Denote $\Pi(V_i) = \{x \in \mathbb{R}^n : x^T D_r(V_i^{-1}) P D_r(V_i^{-1}) x \leq 1\}$. If the set of initial conditions is known then the value $V_0 \in \mathbb{R}_+$ can be always selected such that $x_0 \in \Pi(V_0)$. Otherwise, applying Algorithm 10 for sufficiently small sampling period there always exists a finite number $i^* \in \{0, 1, 2, \dots\}$ such that $x_{i^*} \in \Pi(V_{i^*})$. Corollary 7 implies that the ellipsoid $\Pi(V_i)$ is *strictly* positively invariant for the closed-loop system (1) with the feedback law $u(x) = u(V_i, x)$ of the form (5). Monotonicity condition $\frac{\partial Q(V,x)}{\partial V} < 0$ implies that $\Pi(V') \subset \Pi(V'')$ for $V' < V''$. In order to guarantee stability of the sampled-time realization of the developed control, on each STEP of Algorithm 10 the *upper estimate* (i.e. $V_i = b$) is selected in order to design the linear feedback $u(V_i, x)$ for the next interval $[t_i, t_{i+1})$. Such selection ensures $V_{i+1} \in (0, V_i]$ if $x_i \in \Pi(V_i)$ and the sequence $\{V_i\}_{i=i^*}^\infty$ generated by Algorithm 10 is non-increasing. The bisection procedure will operate until $x(t) \notin \Pi(V_{\min})$. So, it can be shown that $V_i \rightarrow V_{\min}$ as $i \rightarrow \infty$.

The parameter V_{\min} defines the lower possible value of V and the "minimal" attractive invariant set $\Pi(V_{\min})$ for the closed-loop system. This parameter cannot be selected arbitrary small due to finite numerical precision of digital devices. Fixed-time control application can be realized with small changes of Algorithm 10.

Remark 11 An additional advantage of the developed control scheme is related to the possible reduction of the chattering effect for HOSM ILF control application. Indeed, the HOSM control of the form (5) with $\mu = 1$ has the unique discontinuity point $x = 0$. According Algorithm 10, near discontinuity point the feedback law is defined as follows: $u(x) = kD(V_{\min})x$, $\forall x \in \Pi(V_{\min})$, where $V_{\min} \in \mathbb{R}_+$, i.e. we always have a linear continuous control inside the ellipsoid $\Pi(V_{\min})$. Such a modification of the control law obviously follows the classical idea of the chattering reduction developed for the first order sliding mode algorithms [14]. Namely, for practical realization of the sliding mode control, the discontinuous feedback law can be replaced with a high-gain linear feedback if the system state is close to the switching manifold [14].

6 Numerical examples

6.1 HOSM ILF control

Consider the system (1) for $n = 4$, $d_1(t, x) = d_2(t, x) = d_3(t, x) = 0$ and $|d_4(t, x)| \leq \beta = 0.2$. We take $d_4(t, x) = 0.1 \sin(t) + 0.1 \cos(x_4)$ and $x(0) = (2, 0, 0, 0)^T$.

The numerical solution of ODE for the closed-loop system has been obtained using the explicit Euler method with a fixed step size $h \in \mathbb{R}_+$. In order to show the effectiveness of the developed control scheme with respect to the chattering reduction, we select quite a large step size $h = 0.1$.

The Fig. 1(a) shows the simulation results for the closed-loop system with the HOSM ILF control (5) ($\mu = 1$) that is restricted by $|u(x)| \leq 1$ (see, (7)) and applied by the scheme presented in Algorithm 10 for $V_{\min} = 0.1$ and $h = 0.1$.

The parameters of the HOSM ILF control (5) were selected by means of solving the LMI system (4), (7) for $\mu = 1$ and $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 10$, $u_0 = 1$:

$$P = \begin{pmatrix} 0.0058 & 0.0318 & 0.0644 & 0.0517 \\ 0.0318 & 0.1983 & 0.4351 & 0.3798 \\ 0.0644 & 0.4351 & 1.0798 & 1.0393 \\ 0.0517 & 0.3798 & 1.0393 & 1.3857 \end{pmatrix},$$

$$k = (-0.0330, -0.2522, -0.7823, -1.1386).$$

In order to compare the obtained results with some existing HOSM controller, the simulations also have been done for the fourth order sliding mode (4-SM) nested controller [12] of the form

$$u = -\text{sign} \left[x_4 + 0.5 (|x_1|^3 + |x_2|^4 + |x_3|^6)^{\frac{1}{12}} \text{sign}(\phi) \right],$$

$$\phi = x_3 + \frac{1}{4} (|x_1|^3 + |x_2|^4)^{\frac{1}{6}} \text{sign} \left[x_2 + 0.15 |x_1|^{\frac{3}{4}} \text{sign}[x_1] \right].$$

The effective procedures for parameters adjustment of high

order sliding mode control algorithms are not developed yet for $n \geq 3$. So, the parameters of this controller have been selected manually. The simulations of the 4-SM nested control algorithm were initially done for the small sampling period: $h = 10^{-3}$. They showed good performances of this control: fast convergence rate and rejection of matched bounded disturbances. The chattering effect appears in the system, when the sampling period becomes larger. The simulation results with $h = 0.1$ for the 4-SM nested control are depicted on the Fig. 1(b). The Fig. 1(c) shows the control inputs generated by two considered algorithms.

Note that the comparison of the sliding mode control laws for the sufficiently large sampling period $h = 0.1$ is motivated by practical reasons. For instance, the HOSM control is frequently demonstrated on the control problems for mobile robotics (see, for instance, [12]). However, the computational restrictions of autonomous mobile systems does not allow us to spend too much computational power for the generation of control actions. In practice, the reasonable sampling period can be 0.05 – 0.1 second [37].

6.2 Fixed-time ILF control

Define the fixed-time control u in the form (11) with $n = 3$, $\mu = 0.5$, $\nu = 0.1$, where the matrix

$$P = \begin{pmatrix} 5.0575 & 4.5868 & 1.4558 \\ 4.5868 & 7.4884 & 2.2003 \\ 1.4558 & 2.2003 & 1.7959 \end{pmatrix}$$

and the vector $k = (-2.9319, -5.6235, -2.6998)$ are obtained from the LMI (10) with $\beta = 0$, $\alpha = 1$, $\gamma_1 = 4.5$, $\gamma_2 = 4.5$. In this case the settling time estimate provided by Theorem 9 gives $T(x_0) \leq 54$.

The numerical simulation has been done using the Euler method with a fixed step size $h = 0.01$. The control has been applied using Algorithm 10 with $V_{\min} = 0.001$. The simulation results are presented on Fig. 2 for $x_0 = (1, 0, 0)^T$ and $x_0 = (9, 0, 0)^T$. They show a "week" dependence of the convergence time on the initial conditions. Definitely, it requires a rather high magnitude of the control.

7 Conclusion

The paper develops the Implicit Lyapunov Function method for finite-time and fixed-time stability analysis. Using this theoretical framework new algorithms of finite-time and fixed-time stabilization for a chain of integrators and high order sliding mode control design are developed. The obtained control schemes have the following **advantages**:

- The control design algorithms are constructive. The schemes for tuning the control parameters have LMI representations.
- For $\mu = 1$ the algorithms provide high order sliding mode control rejecting matched bounded disturbances.

- The digital implementation of the developed control scheme admits the effective chattering reduction by means of tuning the parameter V_{\min} .
- The fixed-time ILF controllers allow us to prescribe the convergence time independently of the initial condition.

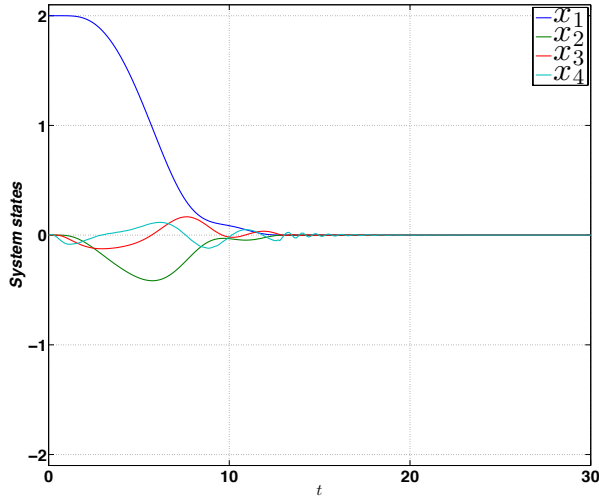
and **disadvantages**:

- The algorithms are applicable only in digital controllers.
- The practical realization of the developed control schemes asks for additional computational power of the digital control device, which is required for on-line computation of the ILF value at the current state.

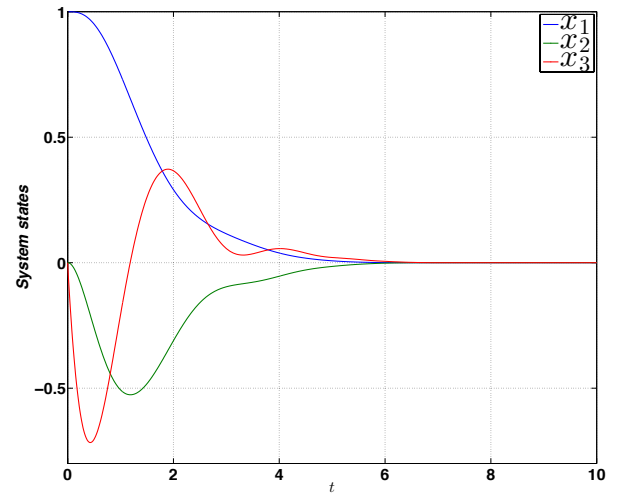
The developed Implicit Lyapunov Function method for finite-time and fixed-time stability analysis is promising to tackle many other problems such as fast observation and estimation, development of fast adaptation algorithms or fast consensus protocols, etc.

References

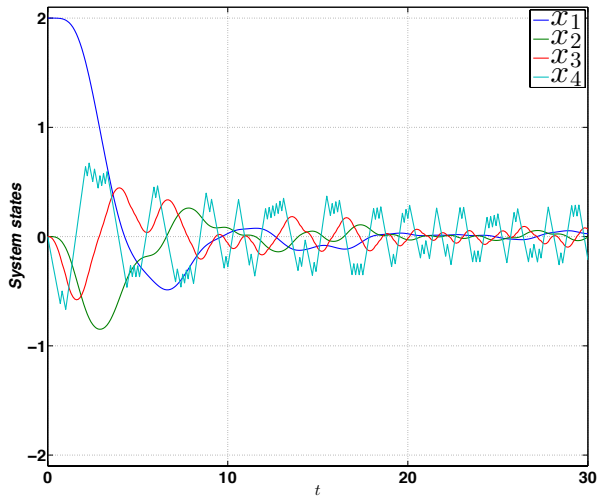
- [1] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time stabilization using implicit lyapunov function technique," in *9th Symposium on Nonlinear Control Systems*, 2013, pp. 140–145.
- [2] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15(3), pp. 273–283, 1966.
- [3] V. Haimo, "Finite time controllers," *SIAM Journal of Control and Optimization*, vol. 24(4), pp. 760–770, 1986.
- [4] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal of Control and Optimization*, vol. 38(3), pp. 751–766, 2000.
- [5] E. Moulay and W. Perruquetti, "Finite-time stability and stabilization: State of the art," *Lecture Notes in Control and Information Sciences*, vol. 334, pp. 23–41, 2006.
- [6] Y. Orlov, *Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions*. Springer-Verlag, 2009.
- [7] R. Engel and G. Kreisselmeier, "A continuous-time observer which converges in finite time," *IEEE Transactions on Automatic Control*, vol. 47, pp. 1202–1204, 2002.
- [8] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: Application to secure communication," *IEEE Transactions on Aut.*, vol. 53(1), pp. 356–360, 2008.
- [9] T. Menard, E. Moulay, and W. Perruquetti, "A global high-gain finite-time observer," *IEEE Transactions on Automatic Control*, vol. 55(6), pp. 1500–1506, 2010.
- [10] F. J. Bejarano and L. M. Fridman, "High order sliding mode observer for linear systems with unbounded unknown inputs," *International Journal of Control*, vol. 9, pp. 1920–1929, 2010.
- [11] Y. Shen, Y. Huang, and J. Gu, "Global finite-time observers for lipschitz nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 56(2), pp. 418–424, 2011.
- [12] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, pp. 823–830, 2005.
- [13] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43(4), pp. 1253–1271, 2005.
- [14] V. Utkin, J. Guldner, and J. Shi, *Sliding Mode Control in Electro-Mechanical Systems*. CRC Press., 2009.
- [15] A. Polyakov and A. Poznyak, "Lyapunov function design for finite-time convergence analysis: "twisting" controller for second-order sliding mode realization," *Automatica*, vol. 45, pp. 444–448, 2009.
- [16] G. Bartolini, A. Pisano, E. Punta, and E. Usai, "A survey of applications of second-order sliding mode control to mechanical systems," *International Journal of Control*, vol. 76(9/10), pp. 875–892, 2003.
- [17] A. Ferrara and L. Giacomini, "On multi-input backstepping design with second order sliding modes for a class of uncertain nonlinear systems," *International Journal of Control*, vol. 71(5), pp. 767–788, 1998.
- [18] F. Chernousko, I. Anan'evskii, and S. Reshmin, *Control of nonlinear dynamical systems: methods and applications*. Springer-Verlag, 2008.
- [19] F. Dinuzzo and A. Ferrara, "Higher order sliding mode controllers with optimal reaching," *IEEE Transactions on Automatic Control*, vol. 54(9), pp. 2126–2136, 2009.
- [20] A. Levant, "Quasi-continuous high-order sliding mode controllers," *IEEE Transactions on Automatic Control*, vol. 50(11), pp. 1812–1816, 2005.
- [21] —, "Construction principles of 2-sliding mode design," *Automatica*, vol. 43(4), pp. 576–586, 2007.
- [22] E. Cruz-Zavala, J. Moreno, and L. Fridman, "Uniform robust exact differentiator," *IEEE Transactions on Automatic Control*, vol. 56(11), pp. 2727–2733, 2011.
- [23] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57(8), pp. 2106–2110, 2012.
- [24] A. Polyakov and L. Fridman, "Stability notions and Lyapunov functions for sliding mode control systems," *Journal of The Franklin Institute*, vol. 351, no. 4, pp. 1831–1865, 2014.
- [25] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer and output feedback," *SIAM Journal of Control and Optimization*, vol. 47(4), pp. 1814–1850, 2008.
- [26] J. Adamy and A. Flemming, "Soft variable-structure controls: a survey," *Automatica*, vol. 40, pp. 1821–1844, 2004.
- [27] R. Courant and F. John, *Introduction to calculus and analysis (Vol. II/1)*. New York: Springer, 2000.
- [28] V. Korobov, "A general approach to synthesis problem," *Doklady Akademii Nauk SSSR*, vol. 248, pp. 1051–1063, 1979.
- [29] I. Anan'evskii, "Control design for dynamic systems using Lyapunov function method," *Contemporary Mathematics. Fundamental Directions*, vol. 42, pp. 23–29, 2011 (in Russian).
- [30] A. Isidori, *Nonlinear Control Systems*. Springer, 1995.
- [31] A. Filippov, *Differential equations with discontinuous right-hand sides*. Kluwer, Dordrecht, 1988.
- [32] F. H. Clarke, Y. S. Ledyaev, and R. Stern, "Asymptotic stability and smooth Lyapunov functions," *Journal of differential Equations*, vol. 149, pp. 69–114, 1998.
- [33] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*, 2nd ed., B. Springer, Ed. Springer, 2005.
- [34] F. Clarke, *Optimization and Nonsmooth Analysis*. SIAM, Philadelphia, 1990.
- [35] W. Carpenter, "On the solution of the real quartic," *Mathematics Magazine*, vol. 39, no. 1, pp. 28–30, 1966.
- [36] R. Dorf and R. Bishop, *Modern Control Systems*. Prentice Hall, 2008.
- [37] M. Guerra, D. Efimov, G. Zheng, and W. Perruquetti, "Finite-time supervisory stabilization for a class of nonholonomic mobile robots under input disturbances," in *Proc. 19th IFAC World Congress*, 2014, pp. 4868–4872.



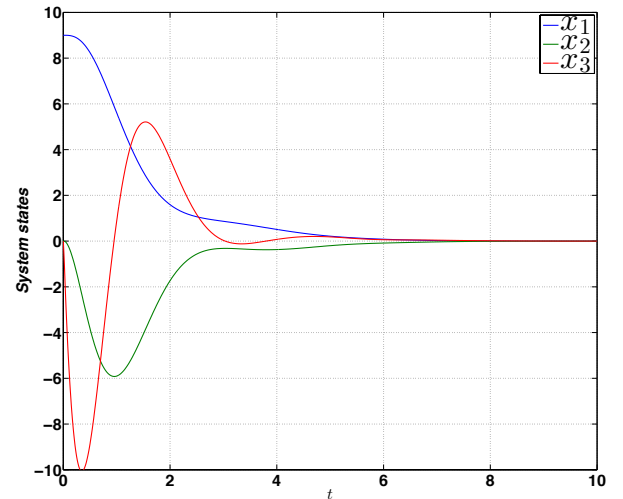
(a) The HOSM ILF control.



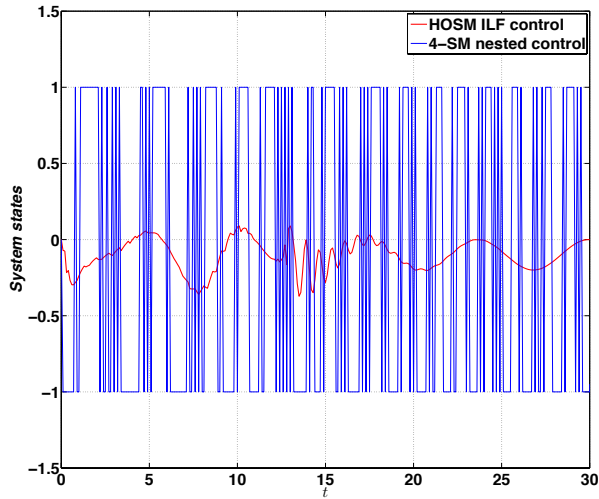
(a) $x_0 = (1, 0, 0)^T$



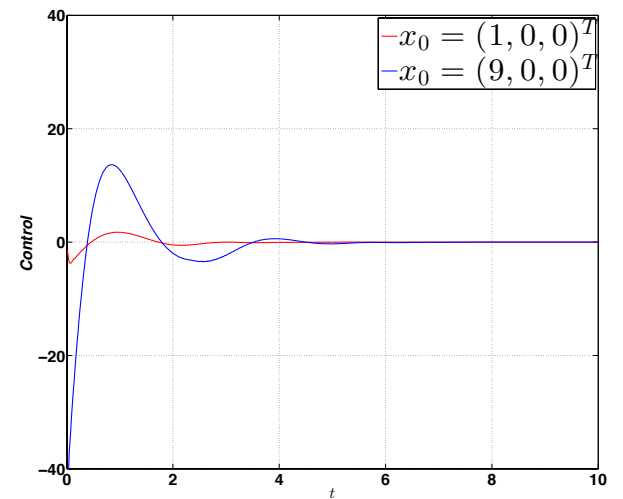
(b) The 4-SM nested control [12].



(b) $x_0 = (9, 0, 0)^T$



(c) Comparison of control inputs.



(c) Control inputs.

Fig. 1. The simulation results for the HOSM ILF control ($V_{\min} = 0.1$) and the 4-SM nested control.

Fig. 2. The simulation results for the fixed-time ILF control.

Relay Control Design for Robust Stabilization in a Finite-Time

Andrey Polyakov and Laurentiu Hetel

Abstract—The problem of robust finite-time stabilization of perturbed multi-input linear system by means of generalized relay feedback is considered. A new control design procedure, which combines convex embedding technique with Implicit Lyapunov Function (ILF) method, is developed. The sufficient conditions for both local and global finite-time stabilization are provided. The issues of practical implementation of the obtained implicit relay feedback are discussed. Our theoretical result is supported by numerical simulation for a Buck converter.

I. INTRODUCTION

Theory of relay automatic control systems has a long outstanding history. Relay feedbacks appeared in the early technological developments of the 19th century. However, the systematic theoretical study of relay control methods was provided in 1950s [1], [2]. The modern frequency domain approach to analysis and design of the relay systems can be found in [3].

When the sliding mode control methodology [4] was invented, it suggested to utilize a proper fast relay switching strategy in order to maintain the motion of the control system on a prescribed surface in the state space. Indeed, the classical example of the sliding mode system has the form of relay feedback: $\dot{x}(t) = -\text{sign}[x(t)]$, $t > 0$, $x(0) = x_0 \in \mathbb{R}$, where the sign function is defined as follows: $\text{sign}[\rho] = 1$ if $\rho > 0$ and $\text{sign}[\rho] = -1$ if $\rho < 0$. Any trajectory of this system reaches the state $x = 0$ in a finite time and remains thereafter. In fact, finite-time stability frequently accompanies the relay and sliding mode feedback systems [4], [5], [6], [7], [8]. The main application domains of sliding mode approach are electrical and electro-mechanical systems [4], [7]. Historically, being invented in 1960s the sliding mode is one of the first robust control approaches. Indeed, the sliding mode control is nonsensitive with respect to the so-called matched disturbances [9], [4].

Nowadays, the relay control systems can be found in different application domains. For instance, relay actuators (switches) are used in power converters [10] for control of output voltage but the relay (pulsed) jets are utilized in fluid dynamics for control of separated (turbulent) flows [11], [12].

The modern theoretical framework of hybrid dynamical systems [13], [14], [15] includes relay feedbacks as a particular case of switched affine systems [16], [17], [18], [19]. To the best of our knowledge, the finite-time stabilization problems have never been studied in the context of switched affine systems.

This research is supported by ANR ROCC-SYS (ANR-14-CE27-0008). A. Polyakov (andrey.polyakov@inria.fr) is with Non-A team-project, Inria Lille-Nord Europe, 40, avenue Halley, 59650 Villeneuve d'Ascq, France. L. Hetel (laurentiu.hetel@ec-lille.fr) and A. Polyakov are with CRISStAL UMR CNRS 9189, 59650 Villeneuve d'Ascq, France.

The present paper addresses the *robust finite-time stabilization of linear multi-input system using the so-called generalized relay control*. The generalized relay assumes a fixed set of possible values for the whole vector of control inputs [20]. This means, in particular, that the inputs cannot be switched independently. Such type of relay is motivated by practical demands of power electronics [21], where, for example, some switch cannot be turned on while another one has the same state. Such restrictions in general case do not allow us to apply directly conventional schemes of relay sliding mode control design [4], [6]. In order to tackle this challenge, the technical note uses the convex embedding procedure. Recently, the ideas of convex embedding have been applied in order to design a locally exponentially stabilizing relay switching law based on the existence of a stabilizing linear feedback [20]. Here based on Implicit Lyapunov Function (ILF) method [22], [23], [24], [25] the convex embedding procedure allows us

- to design a finite-time stabilizing feedback for the case of generalized relays;
- to formulate stability conditions and restrictions to control parameters in terms of linear matrix inequalities;
- to obtain an estimate of both matched and mismatched uncertainties and disturbances, which can be rejected by the generalized relay feedback;
- to derive sufficient conditions for global finite-time stabilization and to estimate an attraction domain in the local case;
- to provide a settling-time estimate of the closed-loop system.

One more challenge tackled in the paper is the development of a feedback control rejecting state-dependent discontinuous disturbances. This allows us, for example, to take into account an unknown dry friction in mechanical or electro-mechanical models.

A preliminary version of this paper appeared in [26]. The key differences with respect to [26] are as follows: in this technical note we design a *robust* relay feedback control for *perturbed* multi-input system, present the proofs to all claims (*Appendix*), provide the global finite-time stabilization conditions (*Proposition 7*) and study a possible application of the proposed relay control to buck converter (*Section VI*).

The paper is organized as follows. The section II discusses problem statement and basic assumptions. After that preliminaries are considered. Next, the main result is presented. Finally, numerical simulation example and concluding remarks are given. All proofs are presented in Appendix.

Notation: \mathbb{R} is the set of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; $\|x\|$ denotes the Euclidian norm of the vector

$x \in \mathbb{R}^n$; $\text{Range}(B)$ is the column space of the matrix $B \in \mathbb{R}^{n \times m}$; $\text{diag}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix with elements λ_i ; the order relation $P > 0$ ($< 0, \geq 0, \leq 0$) for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive (negative) definite (semidefinite); if $P > 0$ then the matrix $P^{1/2} := B$ is such that $B^2 = P$; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote maximum and minimum eigenvalues of the symmetric matrix $P \in \mathbb{R}^{n \times n}$; $\text{co}(U)$ is the convex closure of the set $U \subset \mathbb{R}^n$; $\text{int}\{U\}$ denotes the interior of the set U ; $\mathcal{B}(r) = \{u \in \mathbb{R}^m : \|u\| \leq r\}$ is the ball of the radius ε in \mathbb{R}^m .

II. PROBLEM STATEMENT

Let us consider a model of a control system described by the ordinary differential equation (ODE):

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t, x(t), u(t)), \quad t \in \mathbb{R}_+, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the vector of control inputs, $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B \in \mathbb{R}^{n \times m}$ is the matrix of control gains and the locally measurable function $d : \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}^n$ describes the exogenous disturbances and parametric uncertainties.

It is assumed that the matrices A and B are known, $\text{rank}(B) = m \leq n$ and the pair (A, B) is controllable; the whole state vector x can be measured and utilized for control purposes. The control input u is assumed to be a generalized relay, i.e. it can take values from a given discrete set:

$$u(t) \in \mathcal{U} := \{v_1, v_2, \dots, v_N\}, \quad v_i \in \mathbb{R}^m, \quad t \in \mathbb{R}_+, \quad (2)$$

where N is a natural number. In addition, the assumption

$$0 \in \int \{\text{co}(\mathcal{U})\} \subset \mathbb{R}^m \quad (3)$$

is possessed in order to guarantee the existence of the locally stabilizing relay control (see, [20] for the details). As we will see further, this configuration includes as a particular case the classical sliding control generated by sign functions. This control configuration may also be related to the simplex method in [27], [28] and to the stabilization of switched affine systems [17], [18]. Filippov theory of differential equations with discontinuous right-hand sides [29] is utilized below in order to take into account the discontinuity of the control law and possible discontinuity of the disturbances.

The control aims are

- to stabilize the origin of the system (1) in a finite time,
- to describe a class of uncertainties d , which can be rejected by relay feedback control,
- to specify a set of admissible initial conditions (i.e. the domain of finite-time attraction).

Following the ideas of [20] the relay stabilizing control law can be designed in two steps. Initially, some (possibly continuous) stabilizing feedback should be selected. For this purpose the method of the Implicit Lyapunov Functions (ILF) is utilized [22], [24], [25], [30]. Next, a proper convex embedding procedure [20] is applied in order to construct the relay switching law in the form

$$u(t) \in u_r(t, x(t)) = \underset{v \in \mathcal{U}}{\text{argmin}} \Gamma^T(t, x(t))v, \quad (4)$$

where $\Gamma : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$ is a continuous (outside the origin) nonlinear function to be defined. The inclusion in (4) indicates that argmin is not unique in general case. In particular, if $m = 1$ and $\mathcal{U} = \{-1, 1\}$ then $u_r(t, x) = -\overline{\text{sign}}[\Gamma(t, x)]$ similarly to the sliding mode control [4], where

$$\overline{\text{sign}}[\rho] = \begin{cases} 1 & \text{if } \rho > 0, \\ -1 & \text{if } \rho < 0, \\ \{-1, 1\} & \text{if } \rho = 0. \end{cases}$$

Note that in order to define the control input according to the formula (4) we just need to select the minimum of $\Gamma^T(t, x(t))v$ over finite set values $v \in \mathcal{U}$. This operation does not need applying any finite or infinite dimensional optimization procedure. Just N scalar products $\Gamma^T(t, x(t))v_i$, $i = 1, \dots, N$ must be calculated at each instant of time.

III. PRELIMINARIES

A. Finite-Time Stability

Let us consider the differential inclusion [29] of the form

$$\dot{x}(t) \in F(t, x), \quad t \in \mathbb{R}_+, \quad (5)$$

where $x \in \mathbb{R}^n$ is the state vector, $F : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$ is a multivalued map that is convex-valued, compact-valued and upper semi-continuous. It is well known [29] that an absolute continuous function $x : (a, b) \subset \mathbb{R}^n$ is called Caratheodory solution of (5) if it satisfies the differential inclusion (5) almost everywhere on the time interval (a, b) .

Let the origin be an equilibrium point of the system (5), i.e. $0 \in F(t, 0)$. Only strong uniform stability properties of the system (5) are studied in this paper, so the corresponding words "strong uniform" will be omitted below for shortness and simplicity of the presentation.

Definition 1 ([31], [32], [5]): The origin of system (5) is said to be **finite-time stable** if it is asymptotically stable and finite-time attractive, i.e. for any initial condition $x(0) = x_0 \in \mathcal{M} \setminus \{0\}$ there exists $T(x_0) \in \mathbb{R}_+$ such that $x(t) = 0$ for all $t \geq T(x_0)$, where \mathcal{M} is a neighborhood of the origin and T is called the *settling-time function* of the system (5). If $\mathcal{M} = \mathbb{R}^n$ then the origin is globally finite-time stable.

According to [29] the discontinuous system (1), (4) can also be treated as differential inclusion (5) properly constructed by means of the so-called Filippov regularization procedure. Below we deal only with Filippov solutions of the control system (1), (4).

B. Implicit Lyapunov Function Method

The next theorem is utilized below in order to design the feedback law.

Theorem 2: [25] If there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies the conditions

- C1) Q is continuously differentiable in $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$;
- C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exist $V \in \mathbb{R}_+$ such that $Q(V, x) = 0$;
- C3) let $\Omega = \{(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, x) = 0\}$ and

$$\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty;$$

C4) the inequality $\frac{\partial Q(V,x)}{\partial V} < 0$ holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$;

C5) there exist $c \in \mathbb{R}_+$ and $\mu \in (0, 1]$ such that $\sup_{t \in \mathbb{R}_+, y \in F(t,x)} \frac{\partial Q(V,x)}{\partial x} y \leq cV^{1-\mu} \frac{\partial Q(V,x)}{\partial V}$, $(V, x) \in \Omega$;

then the origin of system (5) is globally finite time stable with the following settling time estimate: $T(x_0) \leq \frac{V_0^\mu}{c\mu}$, where $V_0 \in \mathbb{R}_+$: $Q(V_0, x_0) = 0$.

Theorem 2 provides the sufficient conditions of finite-time stability for implicit definition of Lyapunov function. The conditions C1)-C4) guarantee existence and uniqueness of a continuously differentiable (outside the origin) positive definite radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, which is implicitly defined by the equation $Q(V, x) = 0$. The implicit function theorem [33] gives $\frac{\partial V}{\partial x} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial x}$. Due to C4) and C5) the estimate $\dot{V} \leq \sup_{t \in \mathbb{R}_+, y \in F(t,x)} \frac{\partial V}{\partial x} y \leq -cV^{1-\mu}$ implies the finite-time stability of the origin of (5) if $\mu \in (0, 1]$. We refer the reader to [31], [23], [32], [5], [6], [25] for more details about finite-time stability and implicit Lyapunov function method.

Corollary 3: If the conditions C1)-C4) of Theorem 2 are fulfilled then the set

$$\varepsilon(V_0) = \{z \in \mathbb{R}^n : Q(V_0, z) \leq 0\} \quad (6)$$

is the V_0 -level set $\{s \in \mathbb{R}^n : V(s) \leq V_0\}$ of the positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ implicitly defined by the equation $Q(V, s) = 0$.

Proof: Indeed, if $\tilde{s} \in \mathbb{R}^n$ is such that $V(\tilde{s}) = \alpha$, where $\alpha \in \mathbb{R}_+$ then $Q(\alpha, \tilde{s}) = 0$, i.e. $\tilde{s} \in \varepsilon(\alpha)$. The condition C4) implies that $Q(V', \tilde{s}) < 0$ (i.e. $\tilde{s} \in \varepsilon(V')$) for any $V' > \alpha$ and $Q(V'', \tilde{s}) > 0$ (i.e. $\tilde{s} \notin \varepsilon(V'')$) for any $V'' < \alpha$. ■

Corollary 3 allows us to adapt Theorem 2 to local finite-time stability analysis possessing the condition C5) locally, i.e. $0 < V < \bar{V}$ and $x \in \varepsilon(\bar{V})$ for some given $\bar{V} \in \mathbb{R}_+$. The level set $\varepsilon(\bar{V})$ specifies the finite-time attraction domain \mathcal{M} in this case (see, Definition 1).

IV. IMPLICIT RELAY FEEDBACK LAW

A. Block Decomposition

Let us initially decompose the original multi-input system (1) to a block form [34]. In order to make the paper self-contained the block decomposition procedure studied in [35], [30] is very briefly discussed in Appendix. It constructs the non-singular coordinate transformation

$$s = \Theta x \quad (7)$$

reducing the original system (1) to the block form

$$\dot{s}(t) = \tilde{A}s(t) + \tilde{B}u(t) + \tilde{d}(t, s(t), u(t)), \quad (8)$$

where $\tilde{d}(t, s, u) = \Theta d(t, \Theta^{-1}s, u) + K_{lin}s$ and $K_{lin} \in \mathbb{R}^{n \times n}$,

$$\tilde{A} = \begin{pmatrix} 0 & A_{12} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & A_{k-1,k} \\ 0 & \dots & \dots & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ \dots \\ 0 \\ A_{kk+1} \end{pmatrix} \in \mathbb{R}^{n \times n_k}, \quad (9)$$

the matrices $A_{i,i+1} \in \mathbb{R}^{n_i \times n_{i+1}}$ are of full row rank, $j = 1, 2, \dots, k$, $n_1 + \dots + n_k = n$, $n_k = m$ and $A_{kk+1} \in \mathbb{R}^{m \times m}$ is a nonsingular matrix, where k is the number of blocks.

The function \tilde{d} may be discontinuous. Following Filippov regularization procedure [29] we construct the compact and convex-valued upper semi-continuous map $\Delta : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$:

$$\Delta(t, s) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} co\left(\tilde{d}(t, s + \mathcal{B}(\delta) \setminus N, co(\mathcal{U}))\right), \quad (10)$$

which collects all possible perturbations of the system (1).

B. Relay Feedback Design

Let us introduce the ILF function [30] of the form

$$Q(V, s) = s^T D_r(V^{-1}) P D_r(V^{-1}) s - 1, \quad (11)$$

where $s = (s_1, \dots, s_k)^T$, $s_i \in \mathbb{R}^{n_i}$, $V \in \mathbb{R}_+$, $D_r(\lambda)$ is the so-called homogeneous dilation matrix [30]

$$D_r(\lambda) = \begin{pmatrix} \lambda^{r_1} I_{n_1} & 0 & \dots & 0 \\ 0 & \lambda^{r_2} I_{n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda^{r_k} I_{n_k} \end{pmatrix}, \quad (12)$$

where $\lambda \in \mathbb{R}_+$ is a non-negative scaling parameter, $r_i = 1 + (k-i)\mu$, $i = 1, 2, \dots, k$ are scaling weights with $0 < \mu \leq 1$, and $P \in \mathbb{R}^{n \times n}$, $P > 0$. Denote $H_\mu := \text{diag}\{r_i I_{n_i}\}_{i=1}^k$.

Theorem 4: Let $\mu \in (0, 1)$, $\alpha > 0$, $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n_k \times n}$ satisfy the system of matrix inequalities:

$$\begin{aligned} \tilde{A}X + X\tilde{A}^T + \tilde{B}Y + Y^T\tilde{B}^T + \alpha(H_\mu X + XH_\mu) + R &\leq 0, \\ XH_\mu + H_\mu X &> 0, \quad X > 0, \end{aligned} \quad (13)$$

then there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ implicitly defined by the equation $Q(V, s) = 0$ with Q is given by (11) for $P = X^{-1}$. Let $\beta \in (0, \alpha)$, $V_{\max} \in \mathbb{R}_+ \cup \{+\infty\}$ and the inequality

$$\begin{aligned} \delta^T D_r(V^{-1}) R^{-1} D_r(V^{-1}) \delta &\leq \\ \beta V^{-2\mu} s^T D_r(V^{-1}) (H_\mu P + PH_\mu) D_r(V^{-1}) s, \end{aligned} \quad (14)$$

holds for $\delta \in \Delta(t, s)$, $V \in (0, V_{\max})$, $s \in \mathbb{R}^n : Q(V, s) = 0$ and for almost all $t \in \mathbb{R}_+$, then the control (4) with

$$\Gamma^T(t, x) = x^T \Theta^T D_r\left(\frac{1}{V(\Theta x)}\right) P D_r\left(\frac{1}{V(\Theta x)}\right) \tilde{B}, \quad (15)$$

locally stabilizes the origin of the system (1) in a finite time and the settling-time function T can be estimated as follows

$$T(x_0) \leq \frac{V_0^\mu}{(\alpha - \beta)\mu}, \quad \forall x_0 \in \mathbb{R}^n : \Theta x_0 \in \varepsilon(\bar{V}), \quad (16)$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, \Theta x_0) = 0$ and $\varepsilon(\bar{V})$ is the finite-time attraction domain (6) with a positive $\bar{V} \in \mathbb{R}_+ : \bar{V} \leq V_{\max}$.

All proofs are given in Appendix, where the estimate of the finite-time attraction domain is also provided.

The system of matrix inequalities (13) can be easily solved using LMI toolbox of MATLAB or, for example, SeDuMi solver. The solution of (13) also can be constructed analytically using the proof (see, [30]) of the next proposition.

Proposition 5 ([30]): If the pair (A, B) is controllable then the system of matrix inequalities (13) is feasible for any $\mu \in \mathbb{R}_+$.

Obviously, if $\tilde{d} \equiv 0$ then $\Delta \equiv \{0\}$ and (14) holds for any R . The matrix R is introduced in (13) in order to characterize disturbance functions d , which do not affect the finite-time stability property of the closed-loop system. The formula (14) gives the implicit restrictions to the system disturbances. In order to provide the explicit ones let us introduce the matrices E_i , $i = 1, 2, \dots, k$ by the following formula

$$E_i = \begin{pmatrix} 0_{n_1} & \dots & 0_{n_1 \times n_i} & \dots & 0_{n_1 \times n_k} \\ \dots & \dots & \dots & \dots & \dots \\ 0_{n_i \times n_1} & \dots & I_{n_i} & \dots & 0_{n_i \times n_k} \\ \dots & \dots & \dots & \dots & \dots \\ 0_{n_k \times n_1} & \dots & 0_{n_k \times n_i} & \dots & 0_{n_k} \end{pmatrix}. \quad (17)$$

Note that if $E_i \tilde{d} \neq 0$ just for some $i = i_1, i_2, \dots, i_p$ then Theorem 4 stays true even when the term R in the LMI (13) is replaced with $(E_{i_1} + E_{i_2} + \dots + E_{i_p})R(E_{i_1} + E_{i_2} + \dots + E_{i_p})$.

Proposition 6: Let $X \in \mathbb{R}^{n \times n}$ be a solution of the LMI system (13) with $R = \text{diag}\{\beta_1 I_{n_1}, \beta_2 I_{n_2}, \dots, \beta_k I_{n_k}\}$, $\beta_i \in \mathbb{R}_+$: $\beta = \beta_1 + \dots + \beta_k < \alpha$ and $P = X^{-1}$. If

$$\delta^T E_i \delta \leq \beta_i^2 \gamma \begin{cases} (\lambda_{\min}(P) s^T s)^{1+(k-i-1)\mu} & \text{if } s^T P s \leq 1, \\ (\lambda_{\min}(P) s^T s)^{\frac{1+(k-i-1)\mu}{1+(k-1)\mu}} & \text{if } s^T P s > 1, \end{cases} \quad (18)$$

for $\delta \in \Delta(t, s)$, $\gamma = \lambda_{\min}(P^{-\frac{1}{2}}(PH_\mu + H_\mu P)P^{-\frac{1}{2}})$ and $i = 1, 2, \dots, k$ then the inequality (14) holds for any $V \in \mathbb{R}_+$: $Q(V, s) = 0$.

In the view of Corollary 3 the inequality $V \leq 1$ is equivalent to $s^T P s \leq 1$, so the restriction (18) can be used locally for $V_{\max} = 1$ (see, (14)).

The class of the disturbances which can be rejected by the proposed control law is essentially depended on the parameter μ . If $\mu = 1$ then, obviously, condition (18) implies $\delta^T E_k \delta \leq \beta_k^2 \gamma = \text{const}$ for any $s \in \mathbb{R}^n$, i.e. the constructed relay feedback rejects the so-called bounded matched disturbances [4]. The mismatched disturbances $E_i \delta$, $i = 1, 2, \dots, k-1$ should vanish as $s \rightarrow 0$ and admit polynomial estimates close to the origin (see (18)).

Proposition 7: Theorem 4 stays true for $\mu = 1$ if the matrix inequality (13) is supplied with the additional condition

$$\begin{pmatrix} X & Y^T \\ Y & u_0^2 I_m \end{pmatrix} \geq 0, \quad (19)$$

where $0 < u_0 \leq u_{\max} = \sup_{r \in \mathbb{R}_+ : \mathcal{B}(r) \subset c_0\{U\}} r$. Moreover, the closed-loop system (1), (4), (15) is globally finite-time stable if $V_{\max} = +\infty$.

This proposition provides sufficient condition for global robust finite-time stability of the closed-loop relay system (1), (4), (15). In particular, it is globally finite-time stable if (13) and (19) holds and \tilde{d} satisfies (18) for $\mu = 1$.

V. PRACTICAL IMPLEMENTATION

In order to realize the control algorithm (4), (15) in practice we need to know V . In some cases the function V can be calculated analytically [36]. The function V can also be approximated numerically on a grid constructed in the finite-time attraction domain $\varepsilon(\bar{V})$. Finally, the relay control

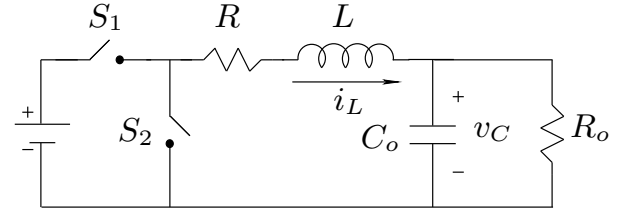


Fig. 1. Scheme of Buck converter

law (4) can be applied by means of on-line estimation of V . The following corollary may be utilized for this purpose.

Corollary 8: If 1) the conditions of Theorem 4 hold; 2) $\{t_i\}_{i=0}^{+\infty}$ is an arbitrary sequence of time instances such that $0 = t_0 < t_1 < t_2 < \dots$ and $\lim_{i \rightarrow +\infty} t_i = +\infty$; 3) the relay control u_r has the form (4) with the sampled computation of the switching function $\Gamma(t, x) = \tilde{\Gamma}_i(\Theta x)$ for $t \in [t_i, t_{i+1})$,

$$\tilde{\Gamma}_i^T(s) = s^T D_r (V_i^{-1}) P D_r (V_i^{-1}) \tilde{B}, \quad Q(V_i, s(t_i)) = 0.$$

Then the system (1), (4) is globally asymptotically stable.

The corollary shows that asymptotic stability of the closed loop system is maintained even if the switching function Γ can be updated only at some isolated instants of time. The estimation of the switching parameter V_i can be obtained using the simple bisection method. We refer the reader to [25], [30] for more details about practical implementation of the control algorithms based on implicit Lyapunov function.

VI. EXAMPLE: CONTROL OF BUCK CONVERTER

Let us consider the model of Buck converter (Fig. 1) studied in [10]: $x = (v_C \ i_L)^T$, $x \in \mathbb{R}^2$, $\mathcal{U} = \{0, v_{in}\}$,

$$A = \begin{pmatrix} -1/(R_o C_o) & 1/C_o \\ -1/L & -R_c/L \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1/L \end{pmatrix},$$

where i_L - inductor current, v_C - capacitor voltage, $v_{in} = 100$ V, $R = 2$ Ω , $L = 500$ μ H, $C_o = 470$ μ F and $R_o = 50$ Ω . In the active mode, the switches S_1 and S_2 operate as follows: if S_1 is opened then S_2 is closed and vice versa.

Let us apply the coordinate transformation:

$$s = \Theta(x - x^*), \quad \Theta = \begin{pmatrix} 1 & 0 \\ -1/R_o & 1 \end{pmatrix}, \quad x^* = \begin{pmatrix} v_{out} \\ v_{out}/R_o \end{pmatrix},$$

where $v_{out} \in (0, v_{in}/(1 + R_c/R_o))$ is the required output voltage. We derive the following relay control system

$$\dot{s}(t) = \begin{pmatrix} 0 & 1/C_o \\ 0 & 0 \end{pmatrix} s + B \hat{u} + \tilde{d}(s),$$

where $\hat{u} \in \{v_1, v_2\}$,

$$\begin{aligned} v_1 &= -v_{out}(1 + R_c/R_o), \\ v_2 &= v_{in} - v_{out}(1 + R_c/R_o) \end{aligned}$$

and $\tilde{d}(s) = -(1/L \ R_c/L - 1/(R_o C_o))s$. The obtained system satisfies the assumption (3) and all conditions of Theorem 4. The system of matrix inequalities (13), (19) gives

$$P = \begin{pmatrix} 0.1768 & 0.1172 \\ 0.1172 & 0.1059 \end{pmatrix}, K = \begin{pmatrix} -0.0960 & -0.0707 \end{pmatrix}$$

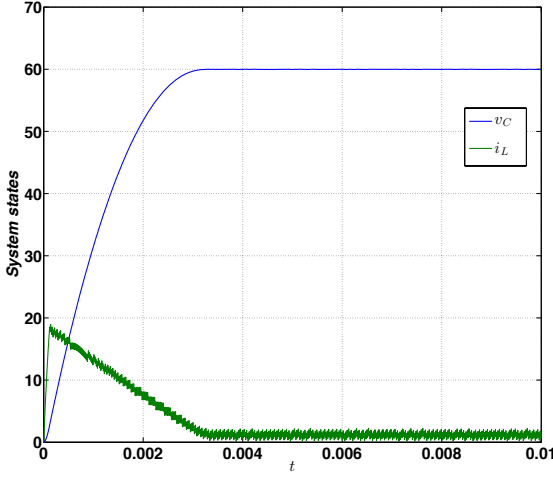


Fig. 2. Evolution of the system state

for the matrix $R = 10^5 I_2 \in \mathbb{R}^{2 \times 2}$, for parameters $\mu = \alpha = 1$ and $u_0 = \min(|v_1|, v_2)$, $v_{out} = 60$. The switching surface Γ is defined by (15) and the dilation matrix D_r has the weights $r_1 = 2$ and $r_2 = 1$.

The Fig. 2 shows the simulation results of the control application for $x(0) = (0, 0)^T$ and the sampling period 10^{-5} . During the simulations the value of the implicit Lyapunov function has been calculated on-line using bisection method.

It was expectable that relay finite-time stabilizing algorithm should be a sort of (high order) sliding mode control. This fact is indirectly confirmed by simulations, since the chattering phenomenon appear. Fortunately, the oscillations are observed only for the current i_L . They are not destructive, since i_K is the fast variable of the system.

VII. CONCLUSION

The paper presents relay feedback control algorithm for stabilization of linear multi-input system provided non-asymptotic transitions. The control design procedure combines the ILF method and convex embedding technique. This approach allows us to develop a simple procedure for implicit switching surface design using LMIs. The algorithm of practical implementation of the obtained implicit relay feedback is presented and tested on the numerical example. Its sampled-time analysis is considered as the subject for future research.

VIII. APPENDIX

A. Transformation to Block Form

Let us denote by $\text{rown}(W)$ the number of rows of a matrix W and by $\text{null}(W)$ the matrix that has the columns defining an orthonormal basis of the null space of a matrix W .

Let the orthogonal matrices T_i be defined by the following simple algorithm:

Initialization : $A_0 = A$, $B_0 = B$, $T_0 = I_n$, $k = 0$.

Loop: While $\text{rank}(B_k) < \text{rown}(A_k)$ do

$$A_{k+1} = B_k^\perp A_k (B_k^\perp)^T, \quad B_{k+1} = B_k^\perp A_k \hat{B}_k,$$

$$T_{k+1} = \begin{pmatrix} B_k^\perp \\ \hat{B}_k \end{pmatrix}, \quad k = k + 1,$$

where $B_k^\perp = (\text{null}(B_k^T))^T$, $\hat{B}_k = (\text{null}(B_k^\perp))^T$.

It was proven (see, e.g. [35]) that

$$GAG^T = \begin{pmatrix} A_{11} & A_{12} & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{k-1,1} & A_{k-1,2} & \dots & A_{k-1,k-1} & A_{k-1,k} \\ A_{k1} & A_{k2} & \dots & A_{kk-1} & A_{kk} \end{pmatrix}, \quad (20)$$

$$\tilde{B} = GB = \begin{pmatrix} 0 & 0 & \dots & 0 & A_{k,k+1}^T \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}^T,$$

$$G = \begin{pmatrix} T_k & 0 \\ 0 & I_{w_k} \end{pmatrix} \begin{pmatrix} T_{k-1} & 0 \\ 0 & I_{w_{k-1}} \end{pmatrix} \dots \begin{pmatrix} T_2 & 0 \\ 0 & I_{w_2} \end{pmatrix} T_1, \quad (21)$$

where $w_i := n - \text{rown}(T_i)$, $A_{kk+1} = \hat{B}_0 B_0$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $n_i := \text{rank}(B_{k-i})$, $i, j = 1, 2, \dots, k$ and $\text{rank}(A_{i,i+1}) = n_i$.

Recall that the B has full column rank ($\text{rank}(B) = m$). Consequently, A_{kk+1} is square and nonsingular. Since $\text{rank}(A_{i,i+1}) = n_i = \text{rown}(A_{i,i+1})$ then $A_{i,i+1} A_{i,i+1}^T$ is invertible and $A_{i,i+1}^+ = A_{i,i+1}^T (A_{i,i+1} A_{i,i+1}^T)^{-1}$ is the right inverse matrix of $A_{i,i+1}$. Introduce the linear coordinate transformation $s = \Phi y$, $s = (s_1, \dots, s_k)^T$, $s_i \in \mathbb{R}^{n_i}$, $y = (y_1, \dots, y_k)^T$, $y_i \in \mathbb{R}^{n_i}$ by the formulas:

$$s_i = y_i + \varphi_i, \quad i = 1, 2, \dots, k, \quad \varphi_1 = 0,$$

$$\varphi_{i+1} = A_{i,i+1}^+ \left(\sum_{j=1}^i A_{ij} y_j + \sum_{r=1}^i \frac{\partial \varphi_i}{\partial y_r} \sum_{j=1}^{r+1} A_{rj} y_j \right). \quad (22)$$

The presented coordinate transformation is linear and nonsingular. The inverse transformation $y = \Phi^{-1} s$ is defined as follows: $y_i = s_i + \psi_i$, $i = 1, 2, \dots, k$, $\psi_1 = 0$, $\psi_{i+1} = A_{i,i+1}^+ \left(\sum_{k=1}^i \frac{\partial \psi_i}{\partial s_k} A_{ii+1} s_{k+1} - \sum_{j=1}^i A_{ij} (s_j + \psi_j) \right)$.

Applying the transformation $s = \Theta x$ with $\Theta = \Phi G$ to (1)

we derive $\dot{s} = \begin{pmatrix} 0 & A_{12} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & A_{k-1,k} \\ \tilde{A}_{k1} & \dots & \dots & \tilde{A}_{kk} \end{pmatrix} s + \tilde{B}u + \Theta d$, that is equivalent to (8) with $K_{lin} = \tilde{B} A_{k,k+1}^{-1} (\tilde{A}_{k1} \dots \tilde{A}_{kk})$.

B. Proof of Theorem 4

The proof is divided into several steps. First, based on implicit Lyapunov function method we design a continuous feedback ensuring global finite-time stabilization. Next, we provide an estimation of the domain in which the continuous feedback may be re-configured as a relay using convex embedding procedure. We continue with the Filippov regularization of the closed-loop relay system. At last, we show the local finite-time stability of the closed-loop system when Filippov solutions are being considered. The construction of the switching surface Γ is the most specific part of proof, since, usually [20], it is depended on the Lyapunov function, which does not have an explicit representation in our case.

1) *The continuous feedback law providing finite-time stabilization:* In the proof we use the a result presented in [30]. Let $\mu \in (0, 1)$, $\alpha > 0$, $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n_k \times n}$ satisfy the system of matrix inequalities (13). Let $\Theta \in \mathbb{R}^{n \times n}$ be given by (7), $K := YX^{-1}$ and the positive definite function

$V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $Q(V(s), s) = 0$ with Q defined by (11) and $P := X^{-1}$. It is known [30] that the function V is smooth outside the origin and the continuous feedback law

$$u_{con}(x(t)) \stackrel{def}{=} \tilde{u}(\Theta x(t)), \quad (23)$$

$$\tilde{u}(s) = V^{1-\mu}(s) K D_r(V^{-1}(s)) s, \quad (24)$$

ensures the inequality

$$\frac{\partial V}{\partial s} (\tilde{A}s + \tilde{B}u_{con}(s) + \delta) \leq -(\alpha - \beta)V^{1-\mu}(s), \quad (25)$$

for any δ satisfying (14). Hence, for $V_{\max} = \infty$ the origin of the closed-loop system (1) is globally finite-time stable and the settling-time function is bounded as follows

$$T(x_0) = \frac{V_0^\mu}{\mu(\alpha - \beta)}, \quad (26)$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, \Phi G x_0) = 0$. The function V is the Lyapunov function of the closed-loop system [30], which is defined implicitly by means of equation $Q(V, s) = 0$. If $V_{\max} < +\infty$ then the given feedback law guarantees local finite-time stabilization for $x_0 \in \varepsilon(V_{\max})$, where $\varepsilon(V_{\max})$ is the level set of the Lyapunov function V (see, Corollary 3).

2) *Estimation of attraction domain:* As follows we show how the continuous feedback (23), (24) can be re-configured as a relay feedback using convex optimization arguments. The representation of the continuous feedback (23), (24) as a convex combination of the relay vectors $v_i, i = 1, \dots, N$,

$$u_{con}(s) = \sum_{i=1}^N \alpha_i(s) v_i, \quad \alpha_i(s) \geq 0, \quad \sum_{i=1}^N \alpha_i(s) = 1 \quad (27)$$

can be applied locally for this purpose. In contrast to the feedback law (23), (24) the relay control (4) is always bounded. Therefore, the representation (27) holds locally in the domain defined as follows

$$\mathcal{C}_U = \{z \in \mathbb{R}^n : u_{con}(z) \in co(\mathcal{U})\}. \quad (28)$$

Since u_{con} is the continuous function [30] and $u_{con}(0) = 0$, then \mathcal{C}_U is a nonempty compact set. Moreover, due to (3) the inclusion $0 \in \text{int}\{\mathcal{C}_U\}$ holds. Let $\bar{V} \in \mathbb{R}_+$ be defined as

$$\bar{V} = \min \left\{ V_{\max}, \sup_{V \in \mathbb{R}_+ : \varepsilon(V) \subset \mathcal{C}_U} V \right\},$$

where $\bar{V} > 0$ due to $0 \in \text{int}\{\mathcal{C}_U\}$. Below we show that $\varepsilon(\bar{V})$ is the finite-time attraction domain of the system (1),(4),(15).

3) *Filippov regularization of the closed-loop relay system:* Since the closed-loop system (1), (4), (15) has a discontinuous right-hand side, classical solutions may not exist. In order to describe the system behavior, we considered solution $x(t)$ in the sense of Filippov and construct the differential inclusion

$$\dot{x}(t) \in F(t, x(t)),$$

$$F(t, x) = \left\{ Ax + Bco \left(\underset{v \in \mathcal{U}}{\text{argmin}} \Gamma^T(t, x)v \right) + \Theta^{-1} \Delta(t, \Theta x) \right\}.$$

Obviously, the obtained inclusion contains all Filippov solutions of the closed-loop system (1), (4), (15). For its finite-time stability analysis Theorem 2 can be utilized.

4) Finite-time stability of the closed-loop relay system:

Let us show that the function the implicit Lyapunov function (11) satisfying (25) is also Lyapunov function for the system (1) closed by the relay feedback (4), (15). It is sufficient to show that

$$\frac{\partial V}{\partial s} (\tilde{A}s + \delta) + \sup_{z \in co \left(\underset{v \in \mathcal{U}}{\text{argmin}} \tilde{\Gamma}^T(s)v \right)} \frac{\partial V}{\partial s} \tilde{B}z \leq -(1 - \beta)V^{1-\mu}(s)$$

for all $\delta \in \Delta(t, s)$ and all $s \in \mathbb{R}^n \setminus \{0\} : s \in \varepsilon(\bar{V})$, where

$$\tilde{\Gamma}(s) = \left(s^T D_r(V^{-1}(s)) P D_r(V^{-1}(s)) \tilde{B} \right)^T. \quad (29)$$

Obviously, $\Gamma(t, x) = \tilde{\Gamma}(\Theta x)$ (see, the formula (15)). Recall that the implicit function theorem [33] gives $\frac{\partial V}{\partial s} = - \left[\frac{\partial Q}{\partial V} \right]^{-1} \frac{\partial Q}{\partial x}$. So, we derive the following representation:

$$\frac{\partial V}{\partial s} = 2V(s) \frac{s^T D_r(V^{-1}(s)) P D_r(V^{-1}(s))}{s^T D_r(V^{-1}(s)) (H_\mu P + P H_\mu) D_r(V^{-1}(s)) s},$$

where $H_\mu P + P H_\mu > 0$ (see, (13)). Hence, we conclude that the relation

$$\underset{v \in \mathcal{U}}{\text{argmin}} \tilde{\Gamma}^T(s)v = \underset{v \in \mathcal{U}}{\text{argmin}} \frac{\partial V}{\partial s} \tilde{B}v$$

holds for any $s \in \mathbb{R}^n \setminus \{0\}$. Moreover, the inequality $\frac{\partial V}{\partial s} \tilde{B}z \leq \frac{\partial V}{\partial s} \tilde{B}v_i$, holds for any $v_i \in \mathcal{U}$ and any $z \in \underset{v \in \mathcal{U}}{\text{argmin}} \tilde{\Gamma}^T(s)v$. Using the standard convexity arguments, it can be easily shown that this inequality also holds for any $z \in co \left(\underset{v \in \mathcal{U}}{\text{argmin}} \tilde{\Gamma}^T(s)v \right)$. Therefore, we derive

$$\frac{\partial V}{\partial s} \tilde{B}z = \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial V}{\partial s} \tilde{B}z \leq \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial V}{\partial s} \tilde{B}v_i$$

$$\forall \tilde{\alpha}_i \geq 0 : \sum_{i=1}^N \tilde{\alpha}_i = 1 \text{ and } \forall z \in co \left(\underset{v \in \mathcal{U}}{\text{argmin}} \tilde{\Gamma}^T(s)v \right).$$

Finally, taking into account the representation (27) of the control law (23), (24) and the equality (25) we derive with $\tilde{\alpha}_i = \alpha_i(s), i = 1, \dots, N$, the representation

$$\frac{\partial V}{\partial s} (\tilde{A}s + \delta) + \sum_{i=1}^N \alpha_i(s) \frac{\partial V}{\partial s} \tilde{B}v_i \leq -(\alpha - \beta)V^{1-\mu},$$

$\sum_{i=1}^N \alpha_i(s) = 1, \alpha_i(s) \geq 0$ for any $\delta \in \Delta(t, s)$ and any $s \in \varepsilon(\bar{V})$. The obtained representation completes the proof.

C. Proof of Corollary 8

In [30] it was proven that for any fixed $V_i \in \mathbb{R}_+$ the control (23), (24) with $V = V_i$ is the linear stabilizing feedback for the system (8) and the corresponding quadratic Lyapunov function V_i is defined as follows $V_i(s) = s^T P_i s$, where $P_i := D_r(V_i^{-1}) P D_r(V_i^{-1}) > 0$. Hence, the ellipsoid $\varepsilon(V_i)$ defined in Corollary 3 is strictly positively invariant set.

Corollary 9 ([30]): If 1) the conditions (13) of Theorem 4 hold; 2) $\{t_i\}_{i=0}^{+\infty}$ is an arbitrary sequence of time instances such that $0 = t_0 < t_1 < t_2 < \dots$ and $\lim_{i \rightarrow +\infty} t_i = +\infty$; 3) the control has the form (23) with $\tilde{u}(s) = \tilde{u}_{V_i}(s)$ on each time interval $[t_i, t_{i+1})$, where $\tilde{u}_{V_i}(s)$ is defined by (24) and $V_i \in R_+ : Q(V_i, s(t_i)) = 0$. Then the closed-loop system (1) is globally asymptotically stable.

The convex embedding technique for design relay feedback law using existing linear stabilizing feedback is studied in [20]. Applying this technique to the switching linear feedback described in this corollary (see the condition 3)) we finish the proof of Corollary 8. Note that the ellipsoids $\varepsilon(V_i)$ are also strictly positively invariant for the relay feedback.

D. Proof of Proposition 6

For proof of this proposition we use the ideas presented in [30]. Obviously, $\gamma I_n \leq P^{1/2} H_\mu P^{-1/2} + P^{-1/2} H_\mu P^{1/2}$ or equivalently $\gamma P \leq P H_\mu + H_\mu P$. On the one hand, we have $\gamma = \gamma s^T D_r (V^{-1}) P D_r (V^{-1}) s \leq s^T D_r (V^{-1}) (P H_\mu + H_\mu P) D_r (V^{-1}) s$, where $(V, s) \in \mathbb{R}_+ \times \mathbb{R}^n$ such that $Q(V, s) = 0$. On the other hand,

$$1 = s^T D_r (V^{-1}) P D_r (V^{-1}) s \geq \begin{cases} \lambda_{\min}(P) V^{-2} s^T s & \text{for } s^T P s \leq 1, \\ \lambda_{\min}(P) V^{-2-2(k-1)\mu} s^T s & \text{for } s^T P s > 1. \end{cases}$$

Hence, we derive $\delta^T D_r (V^{-1}) R^{-1} D_r (V^{-1}) \delta = \sum_{i=1}^k \beta_i^{-1} V^{-2-2(k-i)\mu} \delta^T E_i \delta \leq \frac{\gamma}{\sqrt{2\mu}} (\beta_1 + \dots + \beta_k) \leq \beta V^{-2\mu} s^T D_r (V^{-1}) (P H_\mu + H_\mu P) D_r (V^{-1}) s$, i.e. the inequality (14) holds.

E. Proof of Proposition 7

If $\mu = 1$ then the control function \tilde{u}_v is *continuous outside the origin* and *bounded* for all $x \in \mathbb{R}^n$. Indeed, since $s^T D_r (V^{-1}) P D_r (V^{-1}) s = 1 \rightarrow \|D_r (V^{-1}) s\|^2 \leq \frac{1}{\lambda_{\min}(P)}$ and $\tilde{u}^2(s) \leq s D_r (V^{-1}) K^T K D_r (V^{-1}) s \leq \frac{\lambda_{\max}(K^T K)}{\lambda_{\min}(P)}$.

Hence, in order to restrict globally the control magnitude by $\|\tilde{u}(s)\| \leq u_0$ the inequality $K^T K \leq u_0^2 P$ should be added to (13). Therefore, the inequality (19) guarantees that \mathcal{C}_U is nonempty and all steps of the proof of Theorem 4 can be repeated for $\mu = 1$. Moreover, $\{s \in \mathbb{R}^n : \|u_{con}(s)\| \leq u_{\max}\} \subset \mathcal{C}_U$, so $\bar{V} = \min\{V_{\max}, +\infty\} = V_{\max}$.

REFERENCES

- [1] I. Flugge-Lotz, *Discontinuous automatic control*. Princeton University Press, 1953.
- [2] Y. Z. Tsyppkin, *Relay Control Systems*. Moscow: Nauka, 1984.
- [3] I. Boiko, *Discontinuous Control Systems: Frequency-Domain Analysis and Design*. Springer, 2009.
- [4] V. Utkin, J. Guldner, and J. Shi, *Sliding Mode Control in Electro-Mechanical Systems*. CRC Press., 2009.
- [5] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43(4), pp. 1253–1271, 2005.
- [6] A. Levant, "Principles of 2-sliding mode design," *Automatica*, vol. 43, pp. 576–586, 2007.
- [7] G. Bartolini, A. Pisano, E. Punta, and E. Usai, "A survey of applications of second-order sliding mode control to mechanical systems," *International Journal of Control*, vol. 76(9/10), pp. 875–892, 2003.
- [8] A. Polyakov and A. Poznyak, "Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems," *IMA Journal of Mathematical Control and Information*, vol. 29(4), pp. 529–550, 2012.
- [9] B. Drazhenovic, "The invariance conditions in variable structure systems," *Automatica*, vol. 5, no. 3, pp. 287–295, 1969.
- [10] G. S. Deaecto, J. C. Geromel, F. S. Garcia, and J. A. Pomilio, "Switched affine systems control design with application to dc-dc converters," *IET Control Theory & Applications*, vol. 4, no. 4, pp. 1201–1210, 2010.
- [11] S. L. Brunton and B. R. Noack, "Closed-loop turbulence control: Progress and challenges," *Applied Mechanics Reviews*, vol. 67, no. 5, p. 050801, 2015.
- [12] M. Feingensicht, C. Raibaudo, A. Polyakov, F. Kerhervé, and J.-P. Richard, "A bilinear input-output model with state-dependent delay for separated flow control," in *European Control Conference 2016*, Aalborg, 2016.
- [13] A. Van Der Schaft and H. Schumacher, *An introduction to hybrid dynamical systems*. Springer, 2000.
- [14] D. Liberzon, *Switching in Systems and Control*. Birkhauser, 2003.
- [15] G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten, "A nonconservative lmi condition for stability of switched systems with guaranteed dwell time," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1297–1302, 2012.
- [16] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *Invited paper for SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.
- [17] P. Bolzern and W. Spinelli, "Quadratic stabilization of a switched affine system about a nonequilibrium point," in *Proceeding of the 2004 American Control Conference*, Boston, Massachusetts, USA, 2004.
- [18] G. Deaecto, J. Geromel, F. Garcia, and J. Pomilio, "Switched affine systems control design with application to dc-dc converters," *Control Theory Applications, IET*, vol. 4, no. 7, pp. 1201–1210, July 2010.
- [19] X. Xu and G. Zhai, "Practical stability and stabilization of hybrid and switched systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1897–1903, 2005.
- [20] L. Hetel, E. Fridman, and T. Floquet, "Variable structure control with generalized relays: A simple convex optimization approach," *IEEE Transactions on Automatic Control*, vol. 60(2), pp. 497–502, 2015.
- [21] S. Bacha, I. Munteanu, and A. Bratcu, *Power Electronic Converters Modeling and Control: with Case Studies*. Springer, 2014.
- [22] V. Korobov, "A solution of the problem of synthesis using a controllability function," *Doklady Akademii Nauk SSSR*, vol. 248, pp. 1051–1063, 1979.
- [23] —, "A general approach to the solution of the bounded control synthesis problem in a controllability problem," *Math. USSR Sbornik*, vol. 37, no. 4, pp. 535–557, 1979.
- [24] J. Adamy and A. Flemming, "Soft variable-structure controls: a survey," *Automatica*, vol. 40, pp. 1821–1844, 2004.
- [25] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: Implicit lyapunov function approach," *Automatica*, vol. 51, pp. 332–340, 2015.
- [26] A. Polyakov and L. Hetel, "On finite-time stabilization via relay control," in *Conference on Decision and Control*, 2015, pp. 4098–4102.
- [27] G. Bartolini, E. Punta, and T. Zolezzi, "Simplex sliding mode control of multi-input systems with chattering reduction and mono-directional actuators," *Automatica*, vol. 47, pp. 2433–2437, 2011.
- [28] S. V. Bajda and D. B. Izosimov, "Vector method of design of sliding motion and simplex algorithms," *Automation and Remote Control*, vol. 46, pp. 830–837, 1985.
- [29] A. Filippov, *Differential equations with discontinuous right-hand sides*. Kluwer, Dordrecht, 1988.
- [30] A. Polyakov, D. Efimov, and W. Perruquetti, "Robust stabilization of mimo systems in finite/fixed time," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 1, pp. 69–90, 2016.
- [31] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15(3), pp. 273–283, 1966.
- [32] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal of Control and Optimization*, vol. 38(3), pp. 751–766, 2000.
- [33] R. Courant and F. John, *Introduction to calculus and analysis (Vol. II/1)*. New York: Springer, 2000.
- [34] S. Drakunov, D. Izosimov, A. Lukyanov, V. Utkin, and V. Utkin, "Block control principle I," *Automation and Remote Control*, vol. 51(5), pp. 601–609, 1990.
- [35] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57(8), pp. 2106–2110, 2012.
- [36] A. Polyakov and I. Chairez, "A new homogeneous quasi-continuous second order sliding mode control," in *Latinamerican Congress on Automatic Control, Cancun, Mexico*, 2014.