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Implicit Lyapunov methods for analysis and synthesis of superexponentially stable systems

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CENTRALE LILLE

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d'obtenir le grade de

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Par

Artem NEKHOROSHIKH

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**Méthodes implicites de Lyapunov pour l'analyse et la synthèse
de systèmes superexponentiellement stables**

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SUMMARY

Implicit Lyapunov methods for analysis and synthesis of superexponentially stable systems

One of the most important performance indices for an automatic control system is the speed of response which refers to the time taken by the system to respond to the given input or external disturbance. Achieving fast response is a challenging engineering problem, for the solution of which various control design methods are developed. For example, in the simplest case, the settling time of a control system can be reduced by appropriately increasing the feedback gains of a linear static controller. However, increasing the speed of response in such a way leads to transient oscillations of considerable amplitude or even to the loss of stability in the case of time-delay systems. An alternative way of achieving the required performance is to design a nonlinear control system. Compared to their linear counterparts, nonlinear controllers allow one not only to significantly accelerate the speed of response but also to guarantee the finite-time decay of the transients. However, due to the complexity of the stability analysis of nonlinear systems, algorithms for calculating the nonlinear controller (observer) parameters either do not exist at all or are applicable only for low-order systems. Therefore, the objective of the research was to develop a simple and constructive way of nonlinear control design. To this end, the Implicit Lyapunov method, which is based on the study of a Lyapunov function implicitly defined by some nonlinear algebraic equation, was chosen as the main tool for stability analysis in the thesis. Due to the implicit formulation, sufficient stability conditions for nonlinear control systems can be presented in the form of linear matrix inequalities, which can be numerically checked very efficiently using appropriate mathematical software. As a result, the calculation of the controller (observer) parameters, which ensure the required performance of the closed-loop system, is significantly simplified. To demonstrate the advantages and capabilities of the Implicit Lyapunov method, several problems related to superexponential (hyperexponential and finite/fixed-time) stabilization and state estimation of dynamical systems have been solved in the thesis.

Firstly, a Razumikhin-like method has been proposed for hyperexponential and fixed-time stability analysis of retarded time-delay systems. Differently from the original Lyapunov-Razumikhin method, the proposed approach allows one not only to study the stability of a time-delay system but also to estimate the speed at which trajectories of the system converge to the equilibrium point. However, due to the complexity of formulating Razumikhin-like sufficient conditions for hyperexponential and fixed-time stability by means of a single function, in the proposed method, stability analysis is carried out in two steps using a different Lyapunov-Razumikhin function for each of them. First, it is proven that any trajectory of the system enters a specified closed region centered at the origin in finite time and never leaves it again. Then, the second function is used to show that once the trajectories are within the specified region, they will converge to the origin hyperexponentially or in fixed time, respectively. Furthermore, to make the proposed method more suitable for the nonlinear control design, Implicit

Lyapunov-Razumikhin theorems have also been formulated. The advantage of the implicit formulation has been illustrated by solving the problems of hyperexponential and fixed-time stabilization of a special subclass of time-delay systems. It has been shown that, under some nonrestrictive assumptions, both problems can be easily solved by using the same nonlinear controller that stabilizes the corresponding delay-free system in fixed time. Applying the developed Implicit Lyapunov-Razumikhin method for stability analysis of the closed-loop system, the tuning of the nonlinear controller parameters, which guarantee superexponential stabilization with the required speed, was reduced to verification of linear matrix inequalities. The obtained theoretical results have been supported by numerical simulation of the designed control system for different initial conditions and time delays.

Secondly, the notion of practical fixed-time input-to-state stability has been introduced for neutral time-delay systems with external bounded disturbances and characterized by the Lyapunov-Krasovskii method, which has been formulated both explicitly and implicitly. Based on the obtained theoretical results, an alternative way of robust output practical fixed-time stabilization of linear systems in the controllable canonical form has been proposed. To this end, the state vector was first approximated by means of the finite difference method, i.e., based on the past values of the output signal. Differently from the observer-based approaches, the finite-difference approximation scheme does not require solving additional differential equations in real time, which simplifies its practical implementation. Then, a nonlinear controller was designed to practically stabilize the system in fixed time. To achieve fast stabilization, the nonlinear degree of the feedback is dynamically changed depending on how far from the origin trajectories of the closed-loop system are. To apply the developed Lyapunov-Krasovskii method, it has been shown that the closed-loop system has a neutral time-delay representation due to the special integral relation between the state and its finite-difference approximation. Using the formulated Implicit Lyapunov-Krasovskii theorem, sufficient stability conditions for the designed nonlinear control system were presented in the form of linear matrix inequalities, solutions of which are used for the calculation of the controller parameters. Furthermore, the impact of the artificially induced time delay on the stabilization accuracy has also been quantitatively studied. Finally, it has been theoretically proven and numerically illustrated that, both in the disturbance-free and disturbed cases, the proposed nonlinear controller stabilizes the considered system in the vicinity of the origin much faster than its linear counterpart.

Thirdly, the problem of robust output finite-time stabilization of linear systems under state constraints has been addressed. Geometrically, the considered class of state constraints represents a closed region (hyperoctahedron, hypersphere or hypercube) centered at the origin, within which trajectories of the closed-loop system must remain. To solve the problem, a nonlinear Luenberger-like observer was first designed using the Implicit Lyapunov method in order to reconstruct the state vector in finite time. Then, a continuous control law was proposed, which is linear when trajectories of the closed-loop system risk violating the state constraints, and nonlinear otherwise. The linear feedback was chosen to make the closed-loop system matrix Hurwitz and diagonally dominant. Due to this, trajectories of the closed-loop system not only exponentially converge to the origin but also do not leave the specified region. Once the trajectories reach the switching surface, the nonlinear feedback is used to accelerate the convergence rate, namely, to stabilize the system in finite time. However, differently from the finite-time observer, the practical implementation of the finite-time controller requires the online computation of the Implicit Lyapunov function. Since the analytical solution of the corresponding nonlinear

equation cannot be found in the general case, the bisection method was used to numerically calculate the Implicit Lyapunov function. Compared to the existing methods of stabilization of dynamical systems under state constraints based on control barrier functions and barrier Lyapunov functions, the tuning of the proposed nonlinear control system is extremely simple: the observer and controller parameters are found from the solutions of linear matrix inequalities and equations. Numerical simulation of the designed control system has shown that, for sufficiently small external disturbances, the linear system is stabilized in finite time without violating the state constraints.

Therefore, the thesis has demonstrated how various problems related to superexponential stabilization and state estimation of dynamical systems can be effectively solved using the Implicit Lyapunov method and its modifications. Compared to existing methods of nonlinear control design, the main advantage of the developed approach is that all parameters of a nonlinear controller (observer) can be numerically calculated by solving linear matrix inequalities and equations. Such a simple computation method provides a constructive way for tuning the control parameters to achieve the required speed of response and/or robustness with respect to external bounded disturbances. However, the automatic control systems designed using the implicit Lyapunov method are not without drawbacks. On the one hand, the practical implementation of the controller generally requires the online computation of the Implicit Lyapunov function by means of numerical root-finding methods (e.g., the bisection method), which increases the computational complexity of the control algorithm. On the other hand, despite the relatively simple implementation of the finite-time observer, the calculation of its parameters becomes more complicated due to the need to check a parameterized matrix inequality (e.g., using the iterative algorithm proposed in the thesis). Thus, one of the possible directions for future research could be modifying the structure of the controller and the observer in order to simplify their implementation and tuning, respectively. Finally, in addition to improving the proposed control schemes, it is also essential to evaluate their performance in practice and compare them with existing approaches (e.g., PID or model predictive control).

Keywords: Implicit Lyapunov method, finite-time stability, fixed-time stability, hyperexponential stability, time-delay systems, input-to-state stability

RÉSUMÉ

Méthodes implicites de Lyapunov pour l'analyse et la synthèse de systèmes superexponentiellement stables

L'un des indices de performance les plus importants pour un système de contrôle automatique est la vitesse de réponse, qui correspond au temps mis par le système pour répondre à une entrée donnée ou à une perturbation externe. L'obtention d'une réponse rapide est un problème d'ingénierie difficile, pour la solution duquel diverses méthodes de conception d'une commande automatique sont développées. Par exemple, dans le cas le plus simple, le temps d'établissement d'un système de contrôle peut être réduit en augmentant de manière appropriée les gains de rétroaction d'un contrôleur statique linéaire. Cependant, l'augmentation de la vitesse de réponse de cette manière conduit à des oscillations transitoires d'une amplitude considérable, voire à la perte de stabilité dans le cas de systèmes à retard. Une autre façon d'obtenir les performances requises consiste à concevoir un système de contrôle non linéaire. Par rapport à leurs homologues linéaires, les contrôleurs non linéaires permettent non seulement d'accélérer considérablement la vitesse de réponse mais aussi de garantir la décroissance en temps fini des transitoires. Cependant, en raison de la complexité de l'analyse de stabilité des systèmes non linéaires, les algorithmes de calcul des paramètres d'un contrôleur (observateur) non linéaire n'existent pas du tout ou ne sont applicables que pour les systèmes d'ordre inférieur. Par conséquent, l'objectif de la recherche était de développer une méthode simple et constructive de conception d'une commande automatique non linéaire. À cette fin, la méthode implicite de Lyapunov, qui est basée sur l'étude d'une fonction de Lyapunov définie implicitement par une certaine équation algébrique non linéaire, a été choisie comme outil principal pour l'analyse de stabilité dans la thèse. Grâce à la formulation implicite, les conditions de stabilité suffisantes pour les systèmes de contrôle non linéaires peuvent être présentées sous la forme d'inégalités matricielles linéaires, qui peuvent être vérifiées numériquement de manière très efficace en utilisant un logiciel mathématique approprié. Par conséquent, le calcul des paramètres du contrôleur (observateur), qui assurent la performance requise du système en boucle fermée, est considérablement simplifié. Pour démontrer les avantages et les capacités de la méthode implicite de Lyapunov, plusieurs problèmes liés à la stabilisation et à l'estimation d'état superexponentielle (hyperexponentielle et en temps fini/fixe) des systèmes dynamiques ont été résolus dans la thèse.

Premièrement, une méthode de type Razumikhin a été proposée pour l'analyse de la stabilité hyperexponentielle et en temps fixe des systèmes à retard. Contrairement à la méthode originale de Lyapunov-Razumikhin, l'approche proposée permet non seulement d'étudier la stabilité d'un système à retard mais aussi d'estimer la vitesse à laquelle les trajectoires du système convergent vers le point d'équilibre. Cependant, en raison de la complexité de la formulation des conditions suffisantes de type Razumikhin pour la stabilité hyperexponentielle et en temps fixe au moyen d'une seule fonction, dans la méthode proposée, l'analyse de la stabilité est effectuée en deux étapes en utilisant une fonction de

Lyapunov-Razumikhin différente pour chacune d'elles. Tout d'abord, il est prouvé que toute trajectoire du système entre dans une région fermée spécifiée centrée sur l'origine en temps fini et ne la quitte plus jamais. Ensuite, la deuxième fonction est utilisée pour montrer qu'une fois que les trajectoires sont dans la région spécifiée, elles convergent vers l'origine de manière hyperexponentielle ou en temps fixe, respectivement. En outre, pour rendre la méthode proposée plus adaptée à la conception d'une commande automatique non linéaire, les théorèmes implicites de Lyapunov-Razumikhin ont également été formulés. L'avantage de la formulation implicite a été illustré en résolvant les problèmes de stabilisation hyperexponentielle et en temps fixe d'une sous-classe spéciale de systèmes à retard. Il a été démontré que, sous certaines hypothèses non restrictives, les deux problèmes peuvent être facilement résolus en utilisant le même contrôleur non linéaire qui stabilise le système sans retard correspondant en temps fixe. En appliquant la méthode implicite de Lyapunov-Razumikhin développée pour l'analyse de stabilité du système en boucle fermée, le réglage des paramètres du contrôleur non linéaire, qui garantit la stabilisation superexponentielle avec la vitesse requise, a été réduit à la vérification d'inégalités matricielles linéaires. Les résultats théoriques obtenus ont été étayés par la simulation numérique du système de contrôle conçu pour différentes conditions initiales et différents retards.

Deuxièmement, la notion de stabilité entrée-état pratique en temps fixe a été introduite pour les systèmes à retard de type neutre avec des perturbations externes bornées et caractérisées par la méthode de Lyapunov-Krasovskii, qui a été formulée à la fois explicitement et implicitement. Sur la base des résultats théoriques obtenus, une méthode alternative de stabilisation en temps fixe, robuste et pratique, des systèmes linéaires sous la forme canonique commandable a été proposée. À cette fin, le vecteur d'état a d'abord été approximé au moyen de la méthode des différences finies, c'est-à-dire sur la base des valeurs passées du signal de sortie. Contrairement aux approches basées sur les observateurs, le schéma d'approximation par différences finies ne nécessite pas la résolution d'équations différentielles supplémentaires en temps réel, ce qui simplifie sa mise en œuvre pratique. Ensuite, un contrôleur non linéaire a été conçu pour stabiliser pratiquement le système en temps fixe. Pour obtenir une stabilisation rapide, le degré non linéaire de la rétroaction est modifié dynamiquement en fonction de la distance par rapport à l'origine des trajectoires du système en boucle fermée. Pour appliquer la méthode de Lyapunov-Krasovskii développée, il a été démontré que le système en boucle fermée est un système à retard de type neutre en raison de la relation intégrale spéciale entre l'état et son approximation en différence finie. En utilisant le théorème implicite de Lyapunov-Krasovskii formulé, des conditions de stabilité suffisantes pour le système de contrôle non linéaire conçu ont été présentées sous la forme d'inégalités matricielles linéaires, dont les solutions sont utilisées pour le calcul des paramètres du contrôleur. De plus, l'impact du retard induit artificiellement sur la précision de la stabilisation a également été étudié quantitativement. Enfin, il a été prouvé théoriquement et illustré numériquement que, tant dans les cas sans perturbation que dans les cas perturbés, le contrôleur non linéaire proposé stabilise le système considéré au voisinage de l'origine beaucoup plus rapidement que son homologue linéaire.

Troisièmement, le problème de la stabilisation robuste en temps fini de la sortie de systèmes linéaires sous contraintes d'état a été abordé. Géométriquement, la classe de contraintes d'état considérée représente une région fermée (hyperoctaèdre, hypersphère ou hypercube) centrée à l'origine, dans laquelle les trajectoires du système en boucle fermée doivent rester. Pour résoudre le problème, un

observateur non linéaire de type Luenberger a d'abord été conçu en utilisant la méthode implicite de Lyapunov afin de reconstruire le vecteur d'état en temps fini. Ensuite, une loi de commande continue a été proposée, qui est linéaire lorsque les trajectoires du système en boucle fermée risquent de violer les contraintes d'état, et non linéaire dans le cas contraire. La rétroaction linéaire a été choisie pour que la matrice du système en boucle fermée soit de Hurwitz et à diagonale dominante. De ce fait, les trajectoires du système en boucle fermée convergent non seulement de manière exponentielle vers l'origine mais ne quittent pas non plus la région spécifiée. Une fois que les trajectoires atteignent la surface de commutation, la rétroaction non linéaire est utilisée pour accélérer le taux de convergence, c'est-à-dire pour stabiliser le système en temps fini. Cependant, contrairement à l'observateur en temps fini, la mise en œuvre pratique du contrôleur en temps fini nécessite le calcul en ligne de la fonction implicite de Lyapunov. Comme la solution analytique de l'équation non linéaire correspondante ne peut être trouvée dans le cas général, la méthode de la bisection a été utilisée pour calculer numériquement la fonction implicite de Lyapunov. Par rapport aux méthodes existantes de stabilisation des systèmes dynamiques sous contraintes d'état basées sur les fonctions de barrière de contrôle et les fonctions de barrière de Lyapunov, le réglage du système de contrôle non linéaire proposé est extrêmement simple : les paramètres de l'observateur et du contrôleur sont trouvés à partir des solutions des inégalités et des équations matricielles linéaires. La simulation numérique du système de contrôle conçu a montré que, pour des perturbations externes suffisamment petites, le système linéaire est stabilisé en temps fini sans violer les contraintes d'état.

Par conséquent, la thèse a démontré comment divers problèmes liés à la stabilisation et à l'estimation d'état superexponentielle des systèmes dynamiques peuvent être résolus efficacement en utilisant la méthode implicite de Lyapunov et ses modifications. Par rapport aux méthodes existantes de conception d'une commande automatique non linéaire, le principal avantage de l'approche développée est que tous les paramètres d'un contrôleur (observateur) non linéaire peuvent être calculés numériquement en résolvant des inégalités et des équations matricielles linéaires. Une méthode de calcul aussi simple offre un moyen constructif de régler les paramètres de commande afin d'obtenir la vitesse de réponse requise et/ou la robustesse par rapport à des perturbations externes bornées. Cependant, les systèmes de contrôle automatique conçus à l'aide de la méthode implicite de Lyapunov ne sont pas sans inconvénients. D'une part, la mise en œuvre pratique du contrôleur nécessite généralement le calcul en ligne de la fonction implicite de Lyapunov au moyen de méthodes numériques de recherche d'un zéro d'une fonction (par exemple, de la méthode de la bisection), ce qui augmente la complexité de calcul de l'algorithme de contrôle. D'autre part, malgré la mise en œuvre relativement simple de l'observateur en temps fini, le calcul de ses paramètres devient plus compliqué en raison de la nécessité de vérifier une inégalité matricielle paramétrée (par exemple, en utilisant l'algorithme itératif proposé dans la thèse). Ainsi, une des directions possibles pour les recherches futures pourrait être de modifier la structure du contrôleur et de l'observateur afin de simplifier leur implémentation et leur réglage, respectivement. Enfin, en plus d'améliorer les schémas de contrôle proposés, il est également essentiel d'évaluer leurs performances dans la pratique et de les comparer aux approches existantes (par exemple, au contrôle PID ou à la commande prédictive).

Mots-clefs : méthode implicite de Lyapunov, stabilité en temps fini, stabilité en temps fixe, stabilité hyperexponentielle, systèmes à retard, stabilité entrée-état

РЕФЕРАТ

Неявные методы Ляпунова для анализа и синтеза суперэкспоненциально устойчивых систем

Одним из важнейших показателей качества системы автоматического управления является ее быстродействие, характеризующее скорость реагирования системы на появление задающих и возмущающих воздействий. Повышение быстродействия является сложной инженерной задачей, для решения которой разрабатываются различные методы синтеза систем автоматического управления. Например, в простейшем случае длительность переходных процессов в системе управления может быть сокращена путем надлежащего увеличения коэффициентов обратной связи линейного статического регулятора. Однако повышение быстродействия системы подобным образом приводит к возникновению колебаний значительной амплитуды при переходных процессах или даже к потере устойчивости при наличии запаздывания в системе. Альтернативный способ обеспечения требуемых динамических показателей качества заключается в проектировании нелинейной системы управления. По сравнению с линейными аналогами, нелинейные регуляторы позволяют не только существенно увеличить скорость протекания переходных процессов в системе, но и гарантировать их затухание за конечный промежуток времени. Однако ввиду сложности анализа устойчивости нелинейных систем, алгоритмы расчета параметров нелинейного регулятора (наблюдателя) либо не существуют вовсе, либо ограничиваются применением для систем невысокого порядка. В связи с этим, целью диссертационного исследования являлась разработка простого и конструктивного способа синтеза нелинейных систем управления. Для этого в работе в качестве основного инструмента анализа устойчивости был выбран неявный метод Ляпунова, основанный на исследовании функции Ляпунова, неявно заданной некоторым нелинейным алгебраическим уравнением. Благодаря неявной формулировке достаточные условия устойчивости нелинейных систем управления могут быть представлены в виде линейных матричных неравенств, численная проверка которых может быть осуществлена достаточно эффективно при помощи соответствующего математического программного обеспечения. В результате, расчет параметров регулятора (наблюдателя), обеспечивающих требуемое быстродействие системы, существенно упрощается. Для демонстрации преимуществ и возможностей неявного метода Ляпунова в диссертационной работе было решено несколько задач, связанных с суперэкспоненциальной (гиперэкспоненциальной и за конечное/фиксированное время) стабилизацией и оценкой состояния динамических систем.

Во-первых, был предложен метод типа Разумихина для анализа гиперэкспоненциальной устойчивости и устойчивости за фиксированное время систем с запаздыванием. В отличие от оригинального метода Ляпунова-Разумихина, предложенный подход позволяет не только исследовать устойчивость системы с запаздыванием, но также и оценить скорость, с которой траектории системы сходятся к положению равновесия. Однако из-за сложности формулиров-

ки достаточных условий гиперэкспоненциальной устойчивости и устойчивости за фиксированное время типа Разумихина с помощью одной функции, в предложенном методе анализ устойчивости проводится в два этапа, для каждого из которых используются своя функция Ляпунова-Разумихина. Сначала доказываем, что любая траектория системы попадает в заданную замкнутую область с центром в начале координат за конечное время и больше никогда ее не покидает. Затем вторая функция используется для того, чтобы показать, что оказавшиеся в заданной области траектории сходятся к началу координат гиперэкспоненциально или за фиксированное время соответственно. Кроме того, чтобы сделать предложенный метод более пригодным для синтеза нелинейных систем управления, были также сформулированы неявные теоремы Ляпунова-Разумихина. Преимущество неявной формулировки было проиллюстрировано на примере решения задач гиперэкспоненциальной стабилизации и стабилизации за фиксированное время специального подкласса систем с запаздыванием. Было показано, что при некоторых неограниченных допущениях, обе задачи могут быть легко решены с помощью того же нелинейного регулятора, что стабилизирует соответствующую систему без запаздывания за фиксированное время. Применяя разработанный неявный метод Ляпунова-Разумихина для анализа устойчивости замкнутой системы, настройка параметров нелинейного регулятора, гарантирующих суперэкспоненциальную стабилизацию с требуемой скоростью, была сведена к проверке линейных матричных неравенств. Полученные теоретические результаты были подтверждены численным моделированием разработанной системы управления при различных начальных условиях и величинах запаздывания.

Во-вторых, было введено понятие практической устойчивости «вход-состояние» за фиксированное время для систем с запаздыванием нейтрального типа, подверженных воздействию внешних ограниченных возмущений, а также охарактеризовано с помощью метода Ляпунова-Красовского, сформулированного как в явном, так и в неявном виде. На основе полученных теоретических результатов был предложен альтернативный способ робастной практической стабилизации линейных систем, заданных в канонической управляемой форме, по выходу за фиксированное время. Для этого сначала вектор состояния был приближенно оценен с помощью метода конечных разностей, то есть на основе значений выходного сигнала в предыдущие моменты времени. В отличие от подходов, основанных на использовании наблюдателя, конечно-разностная аппроксимация вектора состояния не требует решения дополнительных дифференциальных уравнений в реальном времени, что упрощает ее практическую реализацию. Затем был разработан нелинейный регулятор для практической стабилизации системы за фиксированное время. Повышение скорости стабилизации было обеспечено за счет динамического изменения показателя нелинейности обратной связи в зависимости от степени удаления траекторий замкнутой системы от начала координат. Для того чтобы применить разработанный метод Ляпунова-Красовского, было показано, что замкнутая система является системой с запаздыванием нейтрального типа благодаря специальному интегральному соотношению между вектором состояния и его конечно-разностной аппроксимацией. Используя сформулированную неявную теорему Ляпунова-Красовского, достаточные условия устойчивости разработанной нелинейной системы управления были представлены в виде линейных матричных неравенств, решения которых используются для расчета параметров регулятора. Кроме того, было также количественно исследовано влияние искусственно вводимого запаздывания на точность

стабилизации. Наконец, было теоретически доказано и численно проиллюстрировано, что и при отсутствии, и при наличии внешних возмущений предложенный нелинейный регулятор стабилизирует рассматриваемую систему в окрестности начала координат гораздо быстрее, чем его линейный аналог.

В-третьих, была рассмотрена проблема робастной стабилизации линейных систем по выходу за конечное время при наличии фазовых ограничений. Геометрически рассматриваемый класс фазовых ограничений представляет собой замкнутую область (гипероктаэдр, гиперсферу или гиперкуб) с центром в начале координат, внутри которой должны оставаться траектории замкнутой системы. Для того чтобы решить поставленную задачу, сначала с помощью неявного метода Ляпунова был спроектирован нелинейный наблюдатель типа Люенбергера для восстановления вектора состояния за конечное время. Затем был предложен непрерывный закон управления, который является линейным, когда траектории замкнутой системы находятся вблизи границ области допустимых значений вектора состояния, и нелинейным в противном случае. Линейная обратная связь была выбрана таким образом, чтобы матрица замкнутой системы была гурвицевой и обладала свойством диагонального преобладания. Благодаря этому траектории замкнутой системы не только экспоненциально сходятся к началу координат, но также при этом не покидают пределы заданной области допустимых значений. Как только траектории достигают поверхности переключения, нелинейный закон управления используется для повышения скорости стабилизации, а именно, для стабилизации системы за конечное время. Однако в отличие от конечно-временного наблюдателя, практическая реализация конечно-временного регулятора требует вычисления неявной функции Ляпунова в реальном времени. Поскольку аналитическое решение соответствующего нелинейного уравнения в общем случае не может быть найдено, для приближенного вычисления неявной функции Ляпунова был использован метод бисекции. По сравнению с существующими методами стабилизации динамических систем при наличии фазовых ограничений, основанными на использовании барьерных функций управления и барьерных функций Ляпунова, настройка предложенной нелинейной системы управления предельно проста: параметры наблюдателя и регулятора находятся из решений линейных матричных неравенств и уравнений. Численное моделирование спроектированной системы управления показало, что при достаточно малых внешних возмущениях линейная система стабилизируется за конечное время с соблюдением фазовых ограничений.

Таким образом, в диссертационной работе продемонстрировано, как различные задачи, связанные с суперэкспоненциальной стабилизацией и оценкой состояния динамических систем, могут быть эффективно решены с помощью неявного метода Ляпунова и его модификаций. По сравнению с существующими методами проектирования нелинейных систем управления, основным преимуществом разработанного подхода является то, что все параметры нелинейного регулятора (наблюдателя) могут быть рассчитаны путем численного решения линейных матричных неравенств и уравнений. Такой простой метод расчета обеспечивает конструктивный способ настройки параметров нелинейной системы управления для обеспечения требуемого быстродействия и/или робастности по отношению ко внешним ограниченными возмущениям. Однако системы автоматического управления, спроектированные с использованием неявного метода Ляпунова, не лишены недостатков. С одной стороны, практическая реализация регулятора в общем случае требует вычисления неявной функции Ляпунова в реальном времени с помощью

численных методов решения уравнений (например, метода бисекции), что ведет к увеличению вычислительной сложности алгоритма управления. С другой стороны, несмотря на относительно простую реализацию конечно-временного наблюдателя, расчет его параметров усложняется необходимостью проверки параметризованного матричного неравенства (например, с помощью предложенного в диссертационной работе итерационного алгоритма). Следовательно, одним из возможных направлений будущих исследований является структурная модификация регулятора и наблюдателя с целью упрощения их реализации и настройки, соответственно. Наконец, помимо совершенствования разработанных систем управления, важно также оценить их работоспособность на практике и сравнить с существующими подходами (например, ПИД-регулированием или управлением с прогнозирующими моделями).

Ключевые слова: неявный метод Ляпунова, устойчивость за конечное время, устойчивость за фиксированное время, гиперэкспоненциальная устойчивость, системы с запаздыванием, устойчивость «вход-состояние»

NOTATION AND ABBREVIATIONS

Notation

SETS AND SPACES

\mathbb{R} the set of real numbers

\mathbb{R}_+ the set of non-negative numbers

\mathbb{R}_+^* the set of positive numbers

\mathbb{N} the set of natural numbers

$\overline{1, n}$ a series of natural numbers up to n , i.e. $1, \dots, n$

\mathbb{R}^n the Euclidean space equipped with the p -norm, $p \in \{1, 2, \infty\}$, defined for any column vector x with elements $x_i \in \mathbb{R}$, $i = \overline{1, n}$, as

$$\|x\|_p := \begin{cases} \sum_{i=1}^n |x_i| & \text{if } p = 1, \\ \sqrt{\sum_{i=1}^n |x_i|^2} & \text{if } p = 2, \\ \max_{i=\overline{1, n}} |x_i| & \text{if } p = \infty \end{cases}$$

$\mathbb{R}^{n \times m}$ the space of real $n \times m$ matrices equipped with the induced p -norm, $p \in \{1, 2, \infty\}$, defined for any matrix A with elements $a_{ij} \in \mathbb{R}$, $i = \overline{1, n}$, $j = \overline{1, m}$, as

$$\|A\|_p := \begin{cases} \max_{i=\overline{1, n}} \sum_{j=1}^m |a_{ij}| & \text{if } p = 1, \\ \sigma_{\max}(A) & \text{if } p = 2, \\ \max_{j=\overline{1, m}} \sum_{i=1}^n |a_{ij}| & \text{if } p = \infty, \end{cases}$$

where $\sigma_{\max}(A)$ is the largest singular value of A

$\mathbb{S}^{n \times n}$ the space of symmetric real $n \times n$ matrices

$C(\Omega, \mathbb{R}^n)$ the space of continuous functions $\phi : \mathbb{R} \supseteq \Omega \rightarrow \mathbb{R}^n$ equipped with the norm

$$\|\phi\|_C := \max_{\tau \in \Omega} \|\phi(\tau)\|_2$$

$C^m(\Omega, \mathbb{R}^n)$ the space of m times continuously differentiable functions $\mathbb{R} \supseteq \Omega \rightarrow \mathbb{R}^n$

$L_p^\infty(\Omega, \mathbb{R}^n)$ the space of Lebesgue measurable essentially bounded functions $\phi : \mathbb{R} \supseteq \Omega \rightarrow \mathbb{R}^n$ equipped with the norm

$$\|\phi\|_{L_p^\infty} := \text{ess sup}_{\tau \in \Omega} \|\phi(\tau)\|_p < +\infty$$

$L^2(\Omega, \mathbb{R}^n)$ the space of Lebesgue square integrable functions $\phi : \mathbb{R} \supseteq \Omega \rightarrow \mathbb{R}^n$ equipped with the norm

$$\|\phi\|_{L^2} := \sqrt{\int_{\Omega} \|\phi(\tau)\|_2^2 ds} < +\infty$$

$W(\Omega, \mathbb{R}^n)$ the space of absolutely continuous functions $\phi : \mathbb{R} \supseteq \Omega \rightarrow \mathbb{R}^n$ with Lebesgue square integrable first-order derivatives ϕ' equipped with the norm

$$\|\phi\|_W := \|\phi\|_C + \|\phi'\|_{L^2},$$

$W_0(\Omega, \mathbb{R}^n)$ the subspace of $W(\Omega, \mathbb{R}^n)$ defined as $W_0 := \{\phi \in W : \phi(0) = 0\}$

$\mathcal{B}_{\mathcal{X}}(\varrho), \mathcal{B}_{\mathcal{X}}[\varrho]$ open and closed balls of radius $\varrho > 0$ centered at the origin in the normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ defined, respectively, as

$$\mathcal{B}_{\mathcal{X}}(\varrho) := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} < \varrho\} \quad \text{and} \quad \mathcal{B}_{\mathcal{X}}[\varrho] := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq \varrho\}$$

$\text{co}(\Omega)$ the convex hull of a set $\Omega \in \mathbb{R}^n$

COMPARISON FUNCTIONS

$\mathcal{K}(\mathcal{GK})$ the class of functions $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

- 1) $w(0) = 0$ (resp. $w(s) = 0$ for all $s \in [0, s_0]$, where $s_0 > 0$);
- 2) $w(s)$ is strictly increasing (resp. on $[s_0, +\infty)$)

\mathcal{K}_{∞} the subclass of \mathcal{K} defined as $\mathcal{K}_{\infty} := \{w \in \mathcal{K} : \lim_{s \rightarrow +\infty} w(s) = +\infty\}$

\mathcal{IK}_{∞} the class of functions $q : \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ such that:

- 1) q is continuous on $\mathbb{R}_+^* \times \mathbb{R}_+^*$;
- 2) for any $s \in \mathbb{R}_+^*$ there exists $\sigma \in \mathbb{R}_+^*$ such that $q(\sigma, s) = 0$;
- 3) for any fixed $s \in \mathbb{R}_+^*$, the function $q(\cdot, s)$ is strictly decreasing on \mathbb{R}_+^* ;
- 4) for any fixed $\sigma \in \mathbb{R}_+^*$, the function $q(\sigma, \cdot)$ is strictly increasing on \mathbb{R}_+^* ;
- 5) $\lim_{s \rightarrow 0^+} \sigma = 0$, $\lim_{\sigma \rightarrow 0^+} s = 0$ and $\lim_{s \rightarrow +\infty} \sigma = +\infty$ for all pairs $(\sigma, s) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ such that $q(\sigma, s) = 0$

$\mathcal{KL}(\mathcal{GKL})$ the class of functions $v : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

- 1) for any fixed $t \in \mathbb{R}_+$, the function $v(\cdot, t)$ belongs to the class \mathcal{K} (resp. \mathcal{GK});
- 2) for any fixed $s \in \mathbb{R}_+$, the function $v(s, \cdot)$ is decreasing and $v(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ (resp. $t \rightarrow T(s)$ for some finite $T(s) \geq 0$ and $v(s, t) = 0$ for all $t \geq T(s)$)

VECTORS AND MATRICES

x^\top, A^\top the transpose of a vector x and a matrix A , respectively

$\text{col}\{a_1, \dots, a_n\},$
 $\text{col}\{a_i\}_{i=1}^n$ a column vector with elements $a_i \in \mathbb{R}^{m_i}, i = \overline{1, n}$

$\text{row}\{a_1^\top, \dots, a_n^\top\},$
 $\text{row}\{a_i^\top\}_{i=1}^n$ a row vector with elements $a_i \in \mathbb{R}^{m_i}, i = \overline{1, n}$

$\text{diag}\{A_1, \dots, A_n\},$
 $\text{diag}\{A_i\}_{i=1}^n$ a block diagonal matrix with blocks $A_i \in \mathbb{R}^{m_i \times m_i}, i = \overline{1, n}$

I_n	the $n \times n$ identity matrix
$O_{n \times m}$	an $n \times m$ zero matrix
$\lambda_i(A)$	the i -th eigenvalue of a real matrix A
$\lambda_{\min}(P), \lambda_{\max}(P)$	the smallest and largest eigenvalues of a symmetric real matrix P , respectively
$P \succ 0$ ($P \prec 0$)	a symmetric real matrix P is positive (negative) definite
$P \succcurlyeq 0$ ($P \preccurlyeq 0$)	a symmetric real matrix P is positive (negative) semidefinite
$\ x\ _P$	the weighted Euclidean norm defined for any $x \in \mathbb{R}^n$ and a positive definite matrix $P \in \mathbb{S}^{n \times n}$ as

$$\|x\|_P := \sqrt{x^\top P x}$$

$\exp(A)$	the matrix exponential of $A \in \mathbb{R}^{n \times n}$ defined as
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$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where $A^0 = I_n$

$\text{rank}(A)$	the rank of a matrix A
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*	the corresponding element of a block symmetric matrix
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OPERATORS

$f \circ g$	the composition of mappings f and g
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$\nabla V(x)$	the gradient of a differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined for any vector $x = \text{col}\{x_i\}_{i=1}^n$ as
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$$\nabla V(x) := \text{col} \left\{ \frac{\partial V}{\partial x_i} \right\}_{i=1}^n$$

$\text{sgn}(x)$	the signum function defined for any $x \in \mathbb{R}$ as
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$$\text{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases}$$

$\text{SGN}(x)$	the set-valued signum function defined for any $x \in \mathbb{R}$ as
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$$\text{SGN}(x) := \begin{cases} \{1\} & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0 \end{cases}$$

$[x]^a$	the signed power defined for any $x \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$ as
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$$[x]^a := \text{sgn}(x)|x|^a$$

$\text{sat}(x, b)$	the saturation function defined for any $x \in \mathbb{R}$ and $b \in \mathbb{R}_+^*$ as
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$$\text{sat}(x, b) := \begin{cases} b & \text{if } x > b, \\ x & \text{if } x \in [-b, b], \\ -b & \text{if } x < -b \end{cases}$$

Abbreviations

- a.e.** almost everywhere
- ILF** Implicit Lyapunov function
- ILKF** Implicit Lyapunov-Krasovskii functional
- ILRF** Implicit Lyapunov-Razumikhin function
- ISS** input-to-state stability (*also* input-to-state stable)
- LMI** linear matrix inequality
- resp.** respectively

CHAPTER

1

INTRODUCTION

In many control scenarios, a failure to perform a required task within a limited period of time leads to fatal consequences. For instance, in order to significantly reduce the cost of space launches, modern spacecraft companies develop and manufacture reusable rocket stages that can be recovered and reflown after successful vertical landings. Obviously, such technology requires a rocket stage to be stabilized in a vertical position and its velocity reduced to zero before the touchdown since otherwise the stage may tip over or crash down. Similarly, to avoid a possible collision between two separate free-flying space vehicles during mating operations (e.g., when a cargo spacecraft is docking with a space station), the speed of both spacecraft must be perfectly matched at the same time they meet in the orbit. Another scenario where the speed of response plays a key role is the interception of attacking missiles. Clearly, anti-missile defense systems must detect, track and destroy a hostile missile before it hits its target.

These and many other examples demonstrate the significant demand for reliable automatic systems which guarantee that the control objective is achieved without violating the time constraints. However, the design of such control systems is a challenging engineering problem, for the solution of which various theoretical methods are developed. In the dissertation, we propose an alternative way of control design that, differently from the existing approaches, provides a simple and constructive algorithm for calculating the parameters of a control system. The main idea of the proposed solution is to prove stability of the closed-loop system using the Implicit Lyapunov method, the advantages and capabilities of which are described in detail below. Also in this chapter, we explain what superexponential stable systems are and why they are studied in the thesis.

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1.1 Decay rates of dynamical systems

Consider a nonlinear system in the form:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad (1.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function such that $f(0) = 0$.

In order to compare different decay rates, we assume that for all $x_0 \in \mathbb{R}^n$ solutions $x(t, x_0)$ of the system (1.1) can be majorized as follows:

$$\|x(t, x_0)\|_2 \leq \varphi(\|x_0\|_2, t), \quad \forall t \geq 0, \quad (1.2)$$

where φ is some function of class \mathcal{KL} or \mathcal{GKL} . Note that the estimate (1.2) implies global asymptotic stability of the system (1.1) at the origin. Moreover, since the equilibrium point is unique, then we can say that the whole system is stable. Then we can introduce the following indicator function:

$$\Phi(x_0, \alpha) := \lim_{t \rightarrow +\infty} e^{\alpha t} \varphi(\|x_0\|_2, t), \quad (1.3)$$

where $\alpha > 0$ is a parameter used to characterize the decay rate of the function $\varphi(\|x_0\|_2, t)$ for any fixed $x_0 \in \mathbb{R}^n$. Note that a similar indicator function was used by A. M. Lyapunov in his seminal work [51] to define the so-called characteristic number of a continuous function.

Following [12], we will present the classification of systems in the form (1.1) based on the value of the indicator function $\Phi(x_0, \alpha)$.

Definition 1.1. *Let the estimate (1.2) hold. Then the system (1.1) is called globally:*

- **exponentially stable** with a decay rate $\alpha > 0$ if $\Phi(x_0, \alpha) = \psi(\|x_0\|_2) \in \mathcal{K}$ for all $x_0 \in \mathbb{R}^n$;
- **subexponentially stable** if $\Phi(x_0, \alpha) = +\infty$ for all $x_0 \in \mathbb{R}^n$ and any $\alpha > 0$;
- **superexponentially stable** if $\Phi(x_0, \alpha) = 0$ for all $x_0 \in \mathbb{R}^n$ and any $\alpha > 0$.

In other words, sub/superexponential stability means that solutions of the system (1.1) converge to the equilibrium slower/faster than exponentially with any decay rate $\alpha > 0$. Let us illustrate that by some scalar examples.

Example 1.1. Consider a scalar differential equation in the form

$$\dot{x}(t) = -\alpha [x(t)]^{1+\mu}, \quad (1.4)$$

where $\alpha > 0$ and $\mu \in (-1, +\infty)$.

We begin with a well-known fact that system (1.4) is exponentially stable with a decay rate α if $\mu = 0$. Indeed, substituting $|x(t, x_0)| = |x_0|e^{-\alpha t}$ in formula (1.3) yields $\Phi(x_0, \alpha) = |x_0|$. Graphically, exponential stability means that the tangent lines to $\ln |x(t, x_0)|$ have a constant slope for all $t \geq 0$ (see Figures 1.1–1.2).

On the other hand, it is easy to check that for $\mu > 0$ solutions of system (1.4) satisfy the following relation

$$|x(t, x_0)| = \frac{|x_0|}{(1 + \alpha\mu|x_0|^\mu t)^{1/\mu}}.$$

Since, in this case, $x(t)$ decreases polynomially, then $\Phi(x_0, \alpha) = +\infty$ for all $x_0 \in \mathbb{R}$ and any $\alpha > 0$, which implies subexponential stability of system (1.4). Graphically, it means that the tangent line to $\ln |x(t, x_0)|$ becomes parallel to the t -axis as t tends to $+\infty$ (see Figure 1.1a). However, it is worth mentioning that since $|x(t, x_0)| \leq (\alpha\mu t)^{-1/\mu}$ then for any $x_0 \in \mathbb{R}$ solutions of the system reach the set $\mathcal{B}_2[\varrho_0]$, where $\varrho_0 > 0$, in fixed time $T(\varrho_0) := \varrho_0^{-\mu}/(\alpha\mu)$. Following [19], we will call this property as **nearly fixed-time** stability of the origin.

Finally, if $\mu \in (-1, 0)$ then one gets

$$|x(t, x_0)| = \begin{cases} (|x_0|^{|\mu|} - \alpha|\mu|t)^{1/|\mu|}, & \text{if } t \in [0, T(x_0)], \\ 0, & \text{if } t \geq T(x_0), \end{cases}$$

where $T(x_0) := |x_0|^{|\mu|}/(\alpha|\mu|)$ is called the settling time. In other words, $x(t)$ converges to the origin in **finite time** $T(x_0)$, which implies superexponential stability of system (1.4). Graphically, it means that the tangent line to $\ln |x(t, x_0)|$ becomes normal to the t -axis as t tends to $T(x_0)$ (see Figure 1.1b). ■

Example 1.2. Consider a scalar differential equation in the form

$$\dot{x}(t) = -0.5\alpha([\lceil x(t) \rceil]^{1-\mu} + \lceil x(t) \rceil^{1+\mu}), \quad (1.5)$$

where $\mu \in (0, 1)$. Introducing the coordinate transformation $y := \arctan |x|^\mu$, solutions of system (1.5) can be easily found:

$$|x(t, x_0)| = \begin{cases} (\tan(\arctan(|x_0|^\mu) - 0.5\alpha\mu t))^{1/\mu}, & \text{if } t \in [0, T(x_0)], \\ 0, & \text{if } t \geq T(x_0), \end{cases}$$

where $T(x_0) := 2 \arctan(|x_0|^\mu)/(\alpha\mu)$. It is worth mentioning that, differently from finite-time stability, the settling time $T(x_0)$ of system (1.5) is uniformly bounded (**fixed**) for all $x_0 \in \mathbb{R}$, i.e., $\sup_{x_0 \in \mathbb{R}} T(x_0) = T_0 := \pi/(\alpha\mu)$ (see Figure 1.2a). ■

Example 1.3. Consider a scalar differential equation in the form

$$\dot{x}(t) = \begin{cases} -\alpha(1 + |\ln |x(t)||)x(t), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (1.6)$$

Note that the right-side of (1.6) is continuous at $x = 0$ since $\lim_{|x| \rightarrow 0^+} (1 + |\ln |x||)|x| = \lim_{|x| \rightarrow 0^+} (1 + \ln |x|^{-1})/|x|^{-1} = 0$. One can check that solutions of (1.6) are given as follows:

- for $|x_0| \leq 1$

$$|x(t, x_0)| = e^{1 - (1 - \ln |x_0|)e^{\alpha t}};$$

- for $|x_0| > 1$

$$|x(t, x_0)| = \begin{cases} e^{(1 + \ln |x_0|)e^{-\alpha t} - 1}, & \text{if } t \in [0, T(x_0)], \\ e^{1 - e^{\alpha(t - T(x_0))}}, & \text{if } t \geq T(x_0), \end{cases}$$

where $T(x_0) := \ln(1 + \ln |x_0|)/\alpha$. Clearly, $\Phi(x_0, \alpha) = 0$ for all $x_0 \in \mathbb{R}$ and any $\alpha > 0$. Thus, system (1.6)

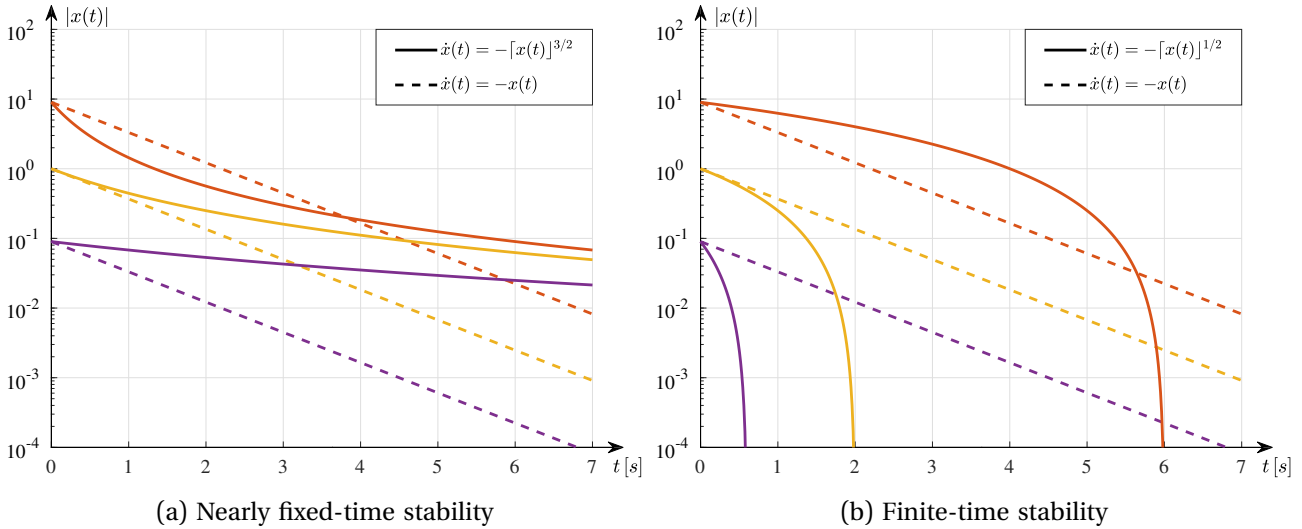


Figure 1.1: Examples of subexponentially (a) and superexponentially (b) stable systems

is superexponentially stable. Since the decay rate itself changes exponentially, following [64], we will call such type of convergence as *hyperexponential* one (see Figure 1.2b). ■

The above examples demonstrate an interesting feature of superexponentially stable systems: the system state may reach the origin in finite time. However, in many practical cases, it is sufficient to guarantee finite-time convergence only to some small neighborhood of the origin. For example, this requirement is easily achieved for linear controllable systems if the control law is chosen in the form of linear state feedback. Then the convergence time can be tuned using the pole placement method. But this strategy has limited use due to the so-called “peaking phenomenon” when reducing of the convergence time leads to significant overshoots during the initial phase of the stabilization [37]. Moreover, increasing of the feedback gains may result in robustness degradation (note that time-delay systems can even become unstable in this case). Since such a transient behavior is unacceptable in practice, then a nonlinear approach may be another possible way for convergence acceleration.

Furthermore, finite-time convergence can drastically simplify control design when only a part of the system state is available for measurements. In this case, one has to estimate the remaining part of the state, for example, by introducing an additional dynamical system called the observer [50]. Differently from linear systems, the controller and the observer cannot be designed independently since the closed-loop system might be unstable. But we can get a nonlinear analogue of the separation principle [38, 21] if the state vector is reconstructed in finite time. The only additional requirement is to ensure that the system performance has not deteriorated by that time.

Finally, let us say some words about the robustness properties of superexponentially stable systems. To this end, consider again systems (1.4)–(1.6), but in the presence of matched parametric uncertainties and external disturbances:

$$\dot{x}(t) = -\alpha K(x(t))x(t) + \beta x(t) + d(t), \quad (1.7)$$

where $K : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ is a state dependent gain defined as $K(x) = K_1(x, \mu) := |x|^\mu$, $K(x) = K_2(x) := 0.5(|x|^{-\mu} + |x|^\mu)$ and $K(x) = K_3(x) := 1 + |\ln |x||$, respectively; $\beta > 0$ is an unknown parameter; $d(t) \in \mathbb{R}$ is external essentially bounded disturbance. If $K(x) = K_1(x, 0) = 1$, then system (1.7) is linear. It is a well-known fact that, in this case, solutions of the system remain bounded only if

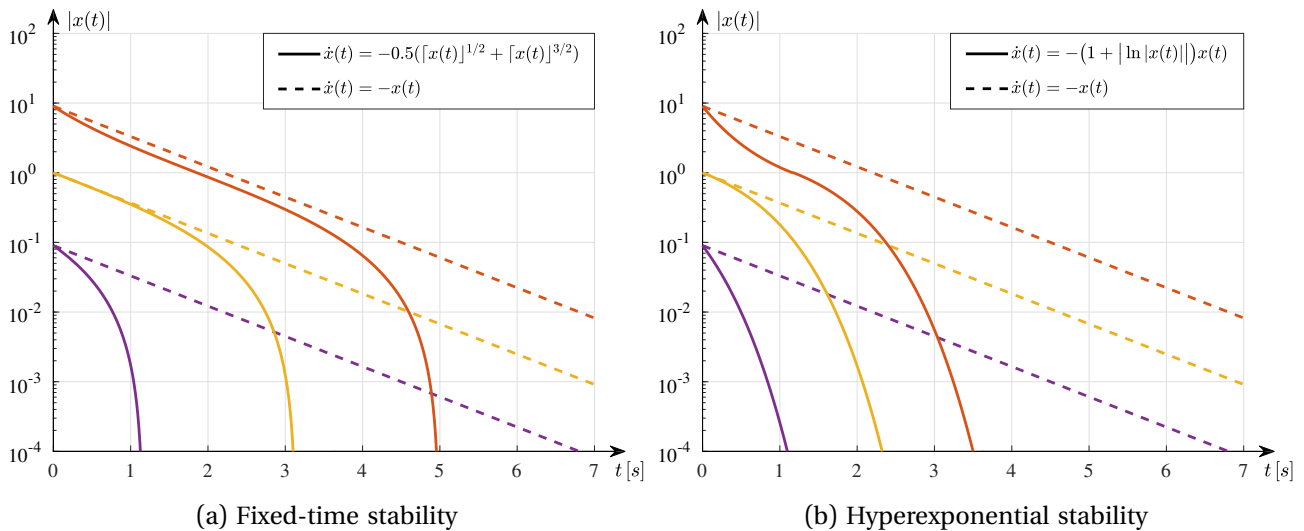


Figure 1.2: Examples of superexponentially stable systems

$\beta < \alpha$. However, this requirement may be not guaranteed if there is no information about parameter β . Contrarily, if $K(x) = K_2(x)$ or $K(x) = K_3(x)$, then, for any $\beta > 0$, system (1.7) is stable with respect to some set $\mathcal{A} \subset \mathbb{R}$. Therefore, superexponentially stable systems are robust to matched parametric uncertainties. Moreover, since $K_2(x) \geq 1$ and $K_3(x) \geq 1$ for any $x \in \mathbb{R}$, then matched external disturbances are better rejected than in the linear case.

Therefore, consideration of superexponentially stable systems is intriguing not only for theory, but also for practice. In the next section, we will recall the main results related to analysis and synthesis of superexponentially stable systems.

1.2 Superexponential stability: brief history and state of the art

First, note that it is difficult, if not impossible, to consider superexponentially stable systems in general. Instead, in the thesis, we will focus only on a few subclasses whose definition is given below.

Definition 1.2 [59, 64, 68]. *The system (1.1) is called globally:*

- **finite-time stable** if it is Lyapunov stable and for any $x_0 \in \mathbb{R}^n$ there exists $0 \leq T(x_0) < +\infty$ such that $x(t, x_0) = 0$ for all $t \geq T(x_0)$. The functional $T(x_0) := \inf\{T \geq 0 : x(t, x_0) = 0 \forall t \geq T\}$ is called **the settling time of the system (1.1)**;
- **fixed-time stable** if it is finite-time stable and there exists $T_0 < +\infty$ such that $\sup_{x_0 \in \mathbb{R}^n} T(x_0) \leq T_0$;
- **hyperexponentially stable** with a decay rate $\vartheta > 0$ if it is Lyapunov stable and there exists $\psi \in \mathcal{K}$ such that $\|x(t, x_0)\|_2 \leq \psi(\|x_0\|_2)e^{-e^{\vartheta t}}$ for all $t \geq 0$ and any $x_0 \in \mathbb{R}^n$.

Now let us discuss each type of stability in more detail.

It appears that finite-time convergence was first studied in control theory as a time-optimization problem [1]. Typically, this problem is solved by using the so-called bang-bang control when the control signal switches abruptly between two extreme values. However, practical implementation of such a discontinuous control strategy may lead to unwanted side effects (e.g., chattering [25]). Moreover,

stability analysis of systems with discontinuous dynamics is rather complicated [65]. These facts stimulated the search for continuous finite-time controllers and, at the same time, the development of methods for the corresponding stability analysis. Apparently, the first step was taken by Emilio Roxin who systematically introduced the notion of finite-time stability in [68] and then proposed to characterize it by means of Lyapunov functions in [69]. Later the Lyapunov method was refined (see the next section) and applied to construct a continuous finite-time controller for the double integrator [31, 5]. Then it was shown in [6] that stability analysis can be simplified for the so-called homogeneous systems [81]. In fact, since then, most finite-time control and state estimation algorithms have been developed using these two approaches. It is worth mentioning that there exists another popular method of finite-time stabilization, namely, sliding mode control that forces a system to “slide” along a desired trajectory [77]. However, due to discontinuity of the control signal that alters the dynamics of the system, sliding mode control will not be considered in the thesis.

The notion of fixed-time stability, which is closely related to the concept of finite-time stability, was introduced in [59]. The main feature of fixed-time stable systems is that their trajectories reach the equilibrium point in finite time, no matter how far from the origin they start (see Example 1.2). Clearly, such a property is very important in finite-time stabilization (state estimation) when little is known about the initial conditions [62, 48, 49]. Moreover, recently it was shown that the settling time can be chosen irrespective of the initial conditions, i.e., $T(x_0) = Const$ for any $x_0 \in \mathbb{R}^n$ [73, 34]. However, this property requires analysis of time-varying systems, which is beyond the scope of this thesis.

Note that, in some situations, finite/fixed-time stability may be difficult to achieve. For example, this is typical for time-delay systems, for which $x(t) = 0$ does not necessarily implies $\dot{x}(t) = 0$ due to the presence of time delay [52, 23]. Although in this case the equilibrium point cannot be reached in finite time, it is still possible to significantly accelerate the decay rate. To this end, one can consider the concept of hyperexponential stability [64], when the decay rate grows exponentially in time (see Example 1.3).

1.3 Lyapunov functions method

In modern control theory, the Lyapunov functions method is de facto the main tool for stability analysis of nonlinear differential equations. The main advantage of the method is that it allows one to check stability of an equilibrium point without solving a differential equation. This is achieved by introducing a special function and studying its evolution in time. Moreover, the Lyapunov functions method is not only used for qualitative analysis (stability/instability), but it also provides important quantitative estimates (e.g., on the domain of attraction or the rate of convergence).

Nevertheless, to apply the method, one has to find a suitable Lyapunov function candidate. Unfortunately, there is no systematic way for this and the choice is usually based on the intuition of the designer. However, in some cases, it is possible to give some recommendations (see [40]). For example, in the simplest case of linear systems, a Lyapunov function has the quadratic form. In case of mechanical or electrical systems, it seems natural to use the energy function as an appropriate candidate.

Furthermore, the search for a Lyapunov function becomes harder as more severe restrictions are imposed on the convergence rate of solutions to an equilibrium point. To show this, let us recall the Lyapunov theorems on exponential and finite-time stability.

Theorem 1.1 [7]. *If there exists a continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that:*

C_{e1}) V is continuously differentiable outside the origin;

C_{e2}) there exist functions $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that for all $\chi \in \mathbb{R}^n$:

$$\sigma_1(\|\chi\|_2) \leq V(\chi) \leq \sigma_2(\|\chi\|_2);$$

C_{e3}) there exist constants $\mu \in (-1, 0]$ and $\alpha > 0$ such that for all $x(t) \in \mathbb{R}^n \setminus \{0\}$ satisfying (1.1):

$$\dot{V}(x(t)) \leq -\alpha V^{1+\mu}(x(t)),$$

where $\dot{V}(x(t)) := \frac{\partial V(\chi)}{\partial \chi} \Big|_{\chi=x(t)} f(x(t))$, then the origin of the system (1.1) is

- exponentially stable with the decay rate α if $\mu = 0$;
- finite-time stable with the settling time $T(x_0) \leq V_0^{|\mu|} / (\alpha|\mu|)$, where $V_0 := V(x_0)$, otherwise.

Remark 1.1. *It is worth mentioning that the condition C_{e1}) on continuous differentiability of a function V is chosen to make Theorem 1.1 consistent with its implicit counterpart (see Theorem 1.2). In fact, the result of Theorem 1.1 is also valid for a locally Lipschitz continuous function V if $\dot{V}(x(t))$ denotes the upper Dini derivative.*

Note that, similar to the scalar system (1.4), the type of stability is completely determined by the parameter μ . It may seem that if one found a Lyapunov function satisfying condition C_{e3}) for $\mu = 0$, then, by continuity, it can be easily modified to fulfill condition C_{e3}) for $\mu \neq 0$. However, as it is shown in the next example, such a modification is not so obvious.

Example 1.4. Consider a chain of $n \geq 2$ integrators:

$$\dot{x}(t) = A_0 x(t) + B_0 u(x(t)), \quad x(0) = x_0, \quad (1.8)$$

where $u(x(t)) \in \mathbb{R}$ is a control input to be designed, system matrices $A_0 \in \mathbb{R}^{n \times n}$ and $B_0 \in \mathbb{R}^n$ are of the form:

$$A_0 := \begin{bmatrix} O_{(n-1) \times 1} & I_{n-1} \\ 0 & O_{1 \times (n-1)} \end{bmatrix}, \quad B_0 := \begin{bmatrix} O_{(n-1) \times 1} \\ 1 \end{bmatrix}.$$

Let us find a control law $u(t)$ exponentially stabilizing system (1.8) using Theorem 1.1. To this end, choose a Lyapunov function candidate $V(\chi) := \sqrt{\chi^\top P \chi}$, where $P \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix. Clearly, for this function conditions C_{e1}) and C_{e2}) hold with $\sigma_1(\|\chi\|_2) := \sqrt{\lambda_{\min}(P)} \|\chi\|_2$ and $\sigma_2(\|\chi\|_2) := \sqrt{\lambda_{\max}(P)} \|\chi\|_2$. Then taking into account the definition of the function $\dot{V}(x(t))$ and dropping argument t , condition C_{e3}) requires to find $u(x)$ such that:

$$\dot{V}(x) = \frac{x^\top P (A_0 x + B_0 u(x))}{\sqrt{x^\top P x}} \leq -\alpha \sqrt{x^\top P x}$$

for given $\alpha > 0$. Apparently, the simplest choice of the control input would be a linear state feedback $u(x) = Kx$ since in this case it suffices to find gains $K \in \mathbb{R}^{1 \times n}$ for which the matrix inequality $(A_0 + B_0 K)^\top P + P(A_0 + B_0 K) \preceq -2\alpha P$ holds. Note that, for any $\alpha > 0$, matrix K can always be found due to the controllability of the pair (A_0, B_0) .

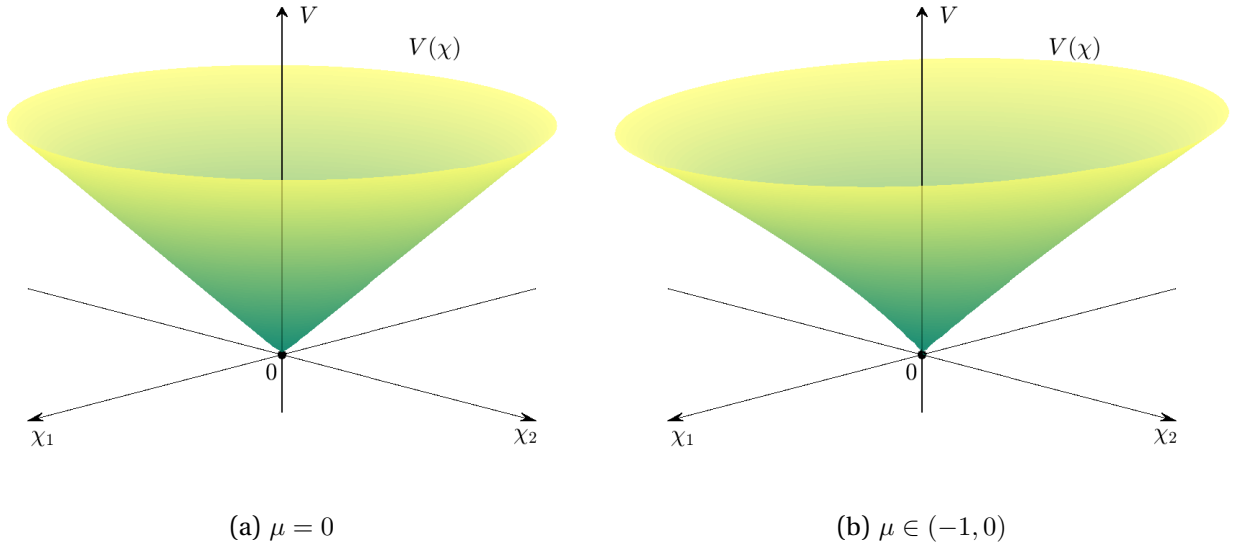


Figure 1.3: Locus of the equation $Q(V, \chi) = 0$ for $n = 2$

Contrarily, for the case of finite-time stabilization, it is unclear how to select a Lyapunov function candidate. Note that several attempts have been made to solve this problem (e.g., [8, 36]) and the corresponding problem of finite-time observer design (e.g., [2, 55]). However, either the proposed solutions are restricted to low-order systems or they only guarantee existence of small enough $|\mu|$ for which the controller/observer is finite-time stable. ■

Motivated by this simple, but yet important example, it was suggested in [62] to reformulate Theorem 1.1 in the implicit way. As it will be shown later, such an approach drastically simplifies control design of superexponentially stable systems. The next theorem presents the fundamental result that will be extensively used in this work.

Theorem 1.2 [62]. *If there exists a continuous function $Q : \mathbb{R}_+^* \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that:*

$C_i1)$ *Q is continuously differentiable outside the origin;*

$C_i2)$ *for any $\chi \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+^*$ such that $Q(V, \chi) = 0$;*

$C_i3)$ *$\frac{\partial Q(V, \chi)}{\partial V} < 0$ for all $V \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$;*

$C_i4)$ *there exist functions $q_1, q_2 \in \mathcal{IK}_\infty$ such that for all $V \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$:*

$$q_1(V, \|\chi\|_2) \leq Q(V, \chi) \leq q_2(V, \|\chi\|_2);$$

$C_i5)$ *there exist constants $\mu \in (-1, 0]$ and $\alpha > 0$ such that for all $x(t) \in \mathbb{R}^n \setminus \{0\}$ satisfying (1.1):*

$$(V, x(t)) \in \Omega := \{(s, \varphi) \in \mathbb{R}_+^* \times \mathbb{R}^n \setminus \{0\} : Q(s, \varphi) = 0\} \implies F(V, x(t)) \leq -\alpha V^{1+\mu},$$

where $F(V, x(t)) := -\left(\frac{\partial Q(V, x(t))}{\partial V}\right)^{-1} \frac{\partial Q(V, \chi)}{\partial \chi} \Big|_{\chi=x(t)} f(x(t))$, then the origin of the system (1.1) is

- exponentially stable with the decay rate α if $\mu = 0$;
- finite-time stable with the settling time $T(x_0) \leq V_0^{|\mu|} / (\alpha|\mu|)$, where V_0 is the solution of the equation $Q(V_0, x_0) = 0$, otherwise.

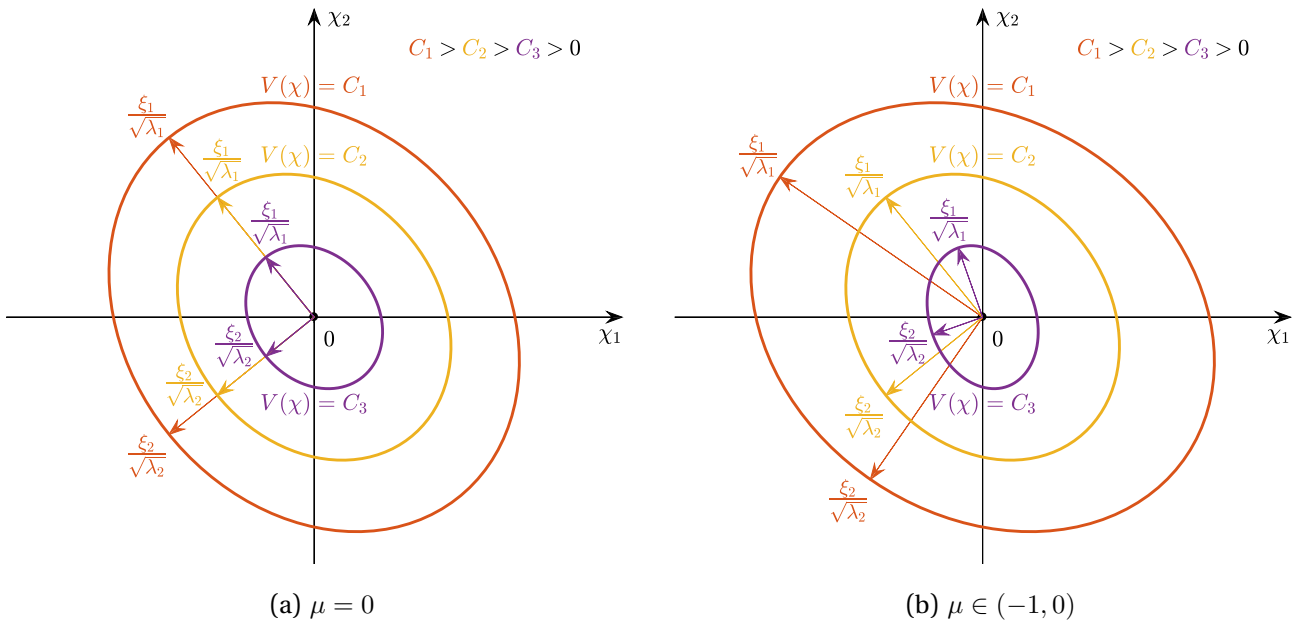


Figure 1.4: Loci of the equation $Q(V, \chi) = 0$ for different values of V and $n = 2$

Let us explain the conditions of Theorem 1.2 in more detail.

Firstly, conditions C_i1 – C_i3) together with the Implicit Function Theorem A.1 (see Appendix A) imply existence of the unique function $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+^*$ implicitly defined by the equation $Q(V, \chi) = 0$ for all $\chi \in \mathbb{R}^n \setminus \{0\}$. Moreover, the function $V(\chi)$ is continuously differentiable outside the origin and $\frac{\partial V(\chi)}{\partial \chi} = -\left(\frac{\partial Q(V, \chi)}{\partial V}\right)^{-1} \frac{\partial Q(V, \chi)}{\partial \chi}$.

Secondly, due to the properties of \mathcal{IK}_∞ functions, it is possible to show [64] that for any $q \in \mathcal{IK}_\infty$ there exists $\sigma \in \mathcal{K}_\infty$ such that $q(\sigma(s), s) = 0$ for all $s \in \mathbb{R}_+^*$ and $\sigma(0) = 0$. Then it follows from condition C_i4) that $q_1(V, \|\chi\|_2) \leq 0 = q_1(\sigma_1(\|\chi\|_2), \|\chi\|_2)$ and $q_2(V, \|\chi\|_2) \geq 0 = q_2(\sigma_2(\|\chi\|_2), \|\chi\|_2)$ for all $V \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$ such that $Q(V, \chi) = 0$. Taking into account that function $\tilde{q}(\cdot) := q(\cdot, s)$ is strictly decreasing for any fixed $s \in \mathbb{R}_+^*$ and setting $V(0) = 0$, we get the same inequalities as in condition C_e2).

Finally, note that, for all $(V, x(t)) \in \Omega$, the function $F(V, x(t))$ can be rewritten as $F(V, x(t)) = \frac{\partial V(\chi)}{\partial \chi} \Big|_{\chi=x(t)} f(x(t))$, which coincides with the definition of the function $\dot{V}(x(t))$.

Therefore, Theorem 1.2 implicitly repeats the conditions of Theorem 1.1. To better understand the idea of the Implicit Lyapunov functions method and to demonstrate its efficiency, we consider again the problem formulated in Example 1.4.

Example 1.4 (Continuation). Introduce an Implicit Lyapunov function (ILF) candidate in the form

$$Q(V, \chi) := \chi^\top \Lambda(V^{-1}) P \Lambda(V^{-1}) \chi - 1, \quad (1.9)$$

where $\Lambda(V^{-1}) := \text{diag}\{V^{-r_i}\}_{i=1}^n$ is the dilation matrix with weights $r_i := 1 - (n - i)\mu$, $i = \overline{1, n}$. Let us say some words about geometrical features of the ILF (1.9). One can see in Figure 1.3 that the locus of the equation $Q(V, \chi) = 0$ is an elliptic cone which, in general ($\mu \neq 0$), is twisted around the V -axis. This fact can be explained by taking into account that the equation $Q(V, \chi) = 0$, where V is fixed, defines an ellipsoid $\chi^\top \tilde{P}(V^{-1}) \chi = 1$, where $\tilde{P}(V^{-1}) := \Lambda(V^{-1}) P \Lambda(V^{-1})$ (see Figure 1.4). Recall that the eigenvectors ξ_i of $\tilde{P}(V^{-1})$ are the principal axes of the ellipsoid, and the eigenvalues

λ_i of $\tilde{P}(V^{-1})$ are the reciprocals of the squares of the semi-axes¹. Therefore, for different values of V , the principal axes of the ellipsoids have different orientation in space. On the other hand, if $\mu = 0$, i.e., $\Lambda(V^{-1}) = V^{-1}I_n$, then the equation $Q(V, \chi) = 0$ can be solved explicitly with respect to V as $V(\chi) = \sqrt{\chi^\top P \chi}$. Clearly, in this case, all the ellipsoids share the same principal axes.

Now we will show that conditions $C_i1)$ – $C_i5)$ hold for ILF (1.9). First, note that the partial derivatives of $Q(V, \chi)$ given as

$$\begin{aligned}\frac{\partial Q(V, \chi)}{\partial V} &= -V^{-1}\chi^\top \Lambda(V^{-1})(PG + GP)\Lambda(V^{-1})\chi, \\ \frac{\partial Q(V, \chi)}{\partial \chi} &= 2\chi^\top \Lambda(V^{-1})P\Lambda(V^{-1}),\end{aligned}$$

where $G := \text{diag}\{r_i\}_{i=1}^n$, are continuous for all $V \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n$. Moreover, condition $C_i3)$ holds if $PG + GP \succ 0$. Note that this linear matrix inequality (LMI) is always feasible for some $P \succ 0$, since matrix $-G$ is Hurwitz. If, additionally, $\mu = 0$, then $G = I_n$ and the LMI reads as $P \succ 0$.

On the other hand, it is easy to see that ILF (1.9) satisfies the following inequalities:

$$q_1(V, \|\chi\|_2) := \frac{\lambda_{\min}(P)\|\chi\|_2^2}{\max\{V^{2r_1}, V^{2r_n}\}} - 1 \leq Q(V, \chi) \leq \frac{\lambda_{\max}(P)\|\chi\|_2^2}{\min\{V^{2r_1}, V^{2r_n}\}} - 1 := q_2(V, \|\chi\|_2). \quad (1.10)$$

Note that $q_1(V, \|\chi\|_2) > 0$ for all $V < \tilde{\sigma}_1(\lambda_{\min}(P)\|\chi\|_2^2)$, where $\tilde{\sigma}_1(s) := \min\{s^{1/(2r_1)}, s^{1/(2r_n)}\}$. Analogously, $q_2(V, \|\chi\|_2) < 0$ for all $V > \tilde{\sigma}_2(\lambda_{\max}(P)\|\chi\|_2^2)$, where $\tilde{\sigma}_2(s) := \max\{s^{1/(2r_1)}, s^{1/(2r_n)}\}$. Therefore, for any $\chi \in \mathbb{R}^n \setminus \{0\}$, it is possible to choose $V^+ > 0$ and $V^- > V^+$ such that $Q(V^+, \chi) > 0$ and $Q(V^-, \chi) < 0$. Since $Q(V, \chi)$ is continuous, then there exists at least one $V \in (V^+, V^-)$ such that $Q(V, \chi) = 0$.

Finally, by the definition of the function $F(V, x)$ we have:

$$F(V, x) = V \frac{2x^\top \Lambda(V^{-1})P\Lambda(V^{-1})(A_0x + B_0u(x))}{x^\top \Lambda(V^{-1})(PG + GP)\Lambda(V^{-1})x}.$$

Taking into account that $\Lambda(V^{-1})A_0 = V^\mu A_0 \Lambda(V^{-1})$ and $\Lambda(V^{-1})B_0 = V^{-1}B_0$, condition $C_i5)$ is rewritten as follows:

$$F(V, x) = V^{1+\mu} \frac{2x^\top \Lambda(V^{-1})P(A_0\Lambda(V^{-1})x + V^{-1-\mu}B_0u(x))}{x^\top \Lambda(V^{-1})(PG + GP)\Lambda(V^{-1})x} \leq -\alpha V^{1+\mu}, \quad \forall (V, x) \in \Omega.$$

To fulfill this requirement, one can simply choose $u(x)$ in the form:

$$u(x) := \begin{cases} V^{1+\mu}K\Lambda(V^{-1})x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad (1.11)$$

where V is the solution of the equation $Q(V, x) = 0$, and $K \in \mathbb{R}^{1 \times n}$ is a matrix chosen such that $(A_0 + B_0K)^\top P + P(A_0 + B_0K) \preceq -\alpha(PG + GP)$. The control law (1.11) can be seen as an extension of the conventional linear feedback $u(x) = Kx$ with state-dependent gains. It is worth stressing that the function (1.11) is continuous and locally bounded for all $x \in \mathbb{R}^n$. Indeed, it follows from (1.9) and

¹Since any symmetric matrix $\tilde{P} \in \mathbb{S}^{n \times n}$ can be decomposed as $\tilde{P} = MDM^{-1} = MDM^\top$, where $M := \text{col}\{\xi_i\}_{i=1}^n$ and $D := \text{diag}\{\lambda_i\}_{i=1}^n$.

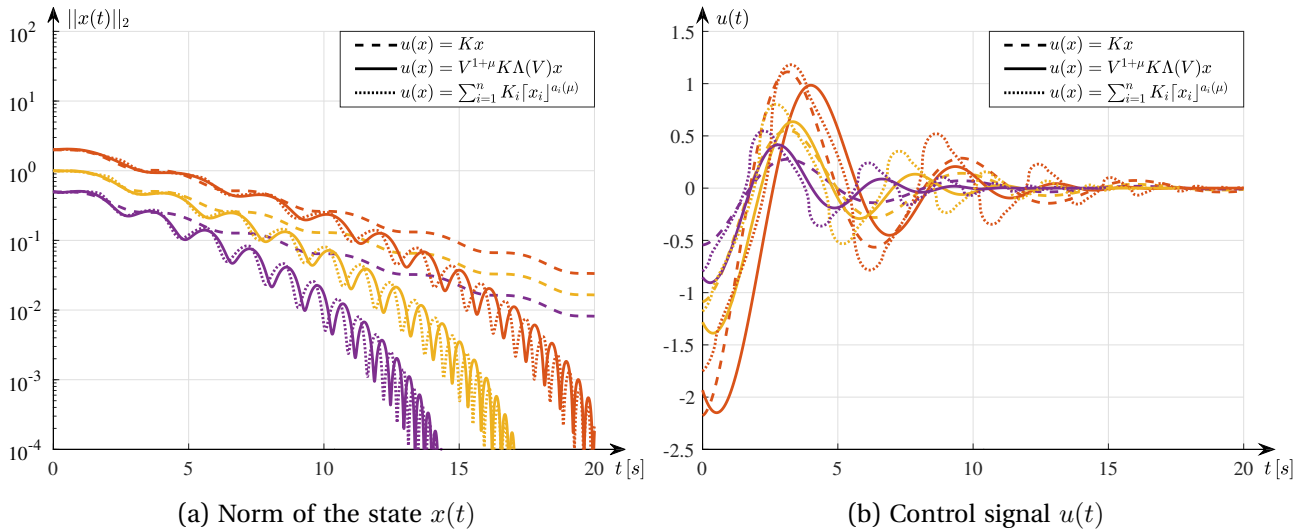


Figure 1.5: Exponential and finite-time stabilization of system (1.8) using control laws (1.11) and (1.12)

(1.10) that $\|\Lambda(V^{-1})x\|_2 \leq \lambda_{\min}^{-1/2}(P)$ and $V \leq \tilde{\sigma}_2(\lambda_{\max}(P)\|x\|_2^2)$, respectively, for all $(V, x) \in \Omega$. Since $\mu > -1$, then $|u(x)| \leq \beta(\|x\|)$, where $\beta \in \mathcal{K}_\infty$, which implies continuity and boundedness of (1.11). Note that a similar “implicit” finite-time controller was also studied by V. I. Korobov in [44], where $V(x)$ was called the controllability function.

One can see that the practical implementation of (1.11) requires online solving the equation $Q(V, x) = 0$ with respect to V . Due to monotonicity of the function $Q(V, x)$ for any fixed $x \in \mathbb{R}^n \setminus \{0\}$, this can be done, for example, using the bisection method (see Appendix D for more details). However, it was shown in [48] that under additional restrictions on parameters μ , P and K given in the form of LMIs the control law (1.11) can be substituted by its “explicit” counterpart

$$u(x) := \sum_{i=1}^n K_i [x_i]^{a_i(\mu)}, \quad (1.12a)$$

$$a_i(\mu) := \frac{1 + \mu}{1 - (n - i)\mu}, \quad i = \overline{1, n}. \quad (1.12b)$$

where $K_i < 0$, $i = \overline{1, n}$, are feedback gains. One can see in Figure 1.5 that such a modification of the control law (1.11) does not affect the performance of the closed-loop system. Since $a_i \in (0, 1)$, $i = \overline{1, n}$, for any $\mu \in (-1, 0)$, then similarly to (1.11), the function (1.12a) is continuous and locally bounded for all $x \in \mathbb{R}^n$.

Finally, note that both control schemes (1.11) and (1.12) coincide with the linear controller $u(x) = Kx$ if $\mu = 0$. Indeed, in this case, $V^{1+\mu}K\Lambda(V^{-1})x = V^{1+\mu}KV^{-1}I_n x = Kx$ and $[x_i]^{a_i(\mu)} = x_i$. This is not a surprise since, as we have said it before, the equation $Q(V, \chi) = 0$ implicitly defines the Lyapunov function $V(\chi) = \sqrt{\chi^\top P \chi}$ that we used for the exponential stabilization. ■

Therefore, differently from the conventional Lyapunov method, the stability conditions of Theorem 1.2 written with respect to the system (1.8) and the ILF (1.9) admit the LMI representation even for $\mu \neq 0$. Moreover, the obtained LMIs have a **universal** form since the same ILF is used for any $n \geq 2$ and $\mu \in (-1, 0]$. This means that the tuning process of the controller parameters can be easily programmed using appropriate mathematical software.

Note that the Implicit Lyapunov method also can be successfully applied for observer design [49], robustness analysis [80], stabilization of multi-input systems [63]. Furthermore, it is possible to formulate an implicit counterpart of the Lyapunov-Krasovskii functionals method for stability analysis of time-delay systems [64].

1.4 Research problem and contributions of the thesis

In the previous section, it has been shown that the Implicit Lyapunov method drastically simplifies design of superexponentially stable systems: the controller parameters can be found as solutions of LMIs. Inspired by this observation, in this thesis, we further develop the Implicit Lyapunov method in order to provide alternative and simpler solutions to some existing problems of control theory.

1.4.1 Theoretical contribution

The theoretical contribution of the thesis is related to the development of the Lyapunov method for analysis and synthesis of time-delay systems with the superexponential decay rate. Note that the developed methods are formulated both explicitly and implicitly.

First, an alternative way of stability analysis and control design of hyperexponentially/fixed-time stable retarded time-delay systems is formulated in Chapter 2. Differently from the Implicit Lyapunov-Krasovskii approach [64], the proposed modification of the Lyapunov-Razumikhin method allows one to carry out stability analysis by means of a Lyapunov function rather than a functional. For example, one can initially choose the same Lyapunov function as for the analysis of the corresponding delay-free system.

Then in Chapter 3, the Implicit Lyapunov-Krasovskii method is extended for input-to-state stability (ISS)² analysis of practically fixed-time stable neutral time-delay systems. Therefore, compared to [62], a wider class of time-delay systems is considered and robustness properties with respect to external bounded disturbances are studied. Note that differently from [20], where the same problem was addressed, the proposed approach does not require homogeneity of the system.

1.4.2 Practical contribution

Based on the Implicit Lyapunov method, two alternative control schemes for output superexponential stabilization of linear systems in the presence of external disturbances and measurement noises are proposed in the thesis.

For example, in Chapter 3, the problem of practical fixed-time stabilization by using artificial delays is addressed. Differently from the observer-based design [48], it is suggested to approximate the state vector by the finite differences. As a result, the proposed controller has a quite simple structure and therefore can be easily implemented in practice. Furthermore, due to the developed Implicit Lyapunov Krasovskii method, the problem can be effectively solved for systems of arbitrarily dynamic order.

Next, the problem of finite-time stabilization under state constraints was studied in Chapter 4. It is shown that for particular classes of state constraints, when trajectories of a system must stay in some hyperoctahedron, hypersphere or hypercube centered at the origin, the control design admits a relatively simple solution. In this case, it is suggested to define a continuous control law, which is linear

²Depending on the context, ISS also stands for *input-to-state stable*

when trajectories of the system risk violating the state constraints, and nonlinear otherwise. While the linear controller guarantees exponential stabilization of the system under the state constraints, the nonlinear controller accelerates the rate of convergence to the equilibrium point.

1.5 List of publications and structure of the thesis

1.5.1 Main publications

The main results of the research were presented at three international conferences and accepted for publication in two peer-reviewed journals, namely:

Chapter 2 is based on:

- 1) A. Nekhoroshikh, D. Efimov, A. Polyakov, W. Perruquetti, and I. Furtat. “Hyperexponential and fixed-time stability of time-delay systems: Lyapunov-Razumikhin method”. In: *IEEE Transactions on Automatic Control* (2022), pp. 1–8 (*in print*).
- 2) A. N. Nekhoroshikh, D. Efimov, A. Polyakov, W. Perruquetti, and I. B. Furtat. “On finite-time stabilization of a class of nonlinear time-delay systems: Implicit Lyapunov-Razumikhin approach”. In: *2020 59th IEEE Conference on Decision and Control (CDC)*. 2020, pp. 1975–1980.

Chapter 3 is based on:

- 3) A. Nekhoroshikh, D. Efimov, E. Fridman, W. Perruquetti, I. Furtat, and A. Polyakov. “Practical fixed-time ISS of neutral time-delay systems with application to stabilization by using delays”. In: *Automatica* 143 (2022), p. 110455.
- 4) A. N. Nekhoroshikh, D. Efimov, A. Polyakov, W. Perruquetti, I. B. Furtat, and E. Fridman. “On output-based accelerated stabilization of a chain of integrators: Implicit Lyapunov-Krasovskii functional approach”. In: *IFAC-PapersOnLine* 53.2 (2020), pp. 5982–5987.

Chapter 4 is based on:

- 5) A. N. Nekhoroshikh, D. Efimov, A. Polyakov, W. Perruquetti, and I. B. Furtat. “Finite-time stabilization under state constraints”. In: *2021 60th IEEE Conference on Decision and Control (CDC)*. 2021, pp. 4682–4687.

1.5.2 Other publications

Here is a list of the publications which were also prepared during the PhD research, but not included in the thesis.

- 6) A. N. Nekhoroshikh, D. Efimov, A. Polyakov, W. Perruquetti, and I. B. Furtat. “On simple robust finite-time observer design”. In: *2022 61st IEEE Conference on Decision and Control (CDC)*. 2022 (*accepted*).
- 7) I. Furtat, A. Nekhoroshikh, and P. Gushchin. “Synchronization of multi-machine power systems under disturbances and measurement errors”. In: *International Journal of Adaptive Control and Signal Processing* 36.6 (2022), pp. 1272–1284.

- 8) I. B. Furtat, A. N. Nekhoroshikh, and P. A. Gushchin. “Robust Stabilization of Linear Plants in the Presence of Disturbances and High-Frequency Measurement Noise”. In: *Automation and Remote Control* 82.7 (2021), pp. 1248–1261.
- 9) I. Furtat, P. Gushchin, and A. Nekhoroshikh. “Divergence Conditions for Stability Study of Autonomous Nonlinear Systems”. In: *IFAC-PapersOnLine* 53.2 (2020), pp. 6317–6320.
- 10) I. Furtat, P. Gushchin, A. Nekhoroshikh, and A. Peregudin. “Output Control of Linear Time-invariant Systems Under Input and Output Disturbances”. In: *IFAC-PapersOnLine* 53.2 (2020), pp. 4534–4539.
- 11) I. Furtat, A. Nekhoroshikh, and P. Gushchin. “Robust stabilization of linear plants in presence of disturbances and high-frequency measurement noises”. In: *Large-Scale Systems Control* 86 (2020), pp. 32–54 (*in Russian*).
- 12) I. Furtat, A. Nekhoroshikh, P. Gushchin, and Yu. Chugina. “Synchronization of the electric power network in the conditions of high-frequency measurement noises”. In: *Mekhatronika, Avtomatizatsiya, Upravlenie* 21.10 (2020), pp. 584–594 (*in Russian*).
- 13) P. Gushchin, I. Furtat, A. Nekhoroshikh, Ju. Chugina, S. Vrazhevsky, and M. Tarasov. “Control of distillation column under perturbations: a case study”. In: *Cybernetics and Physics* 9.4 (2020), pp. 182–186.
- 14) I. B. Furtat, P. A. Gushchin, A. N. Nekhoroshikh, S. A. Vrazhevsky, and Ju. V. Chugina. “Modified Backstepping Algorithm for Plants under Mismatched Disturbances and Varying Time-Delay”. In: *2019 6th International Conference on Control, Decision and Information Technologies (CoDIT)*. IEEE, 2019, pp. 604–609.
- 15) I. B. Furtat, P. A. Gushchin, A. N. Nekhoroshikh, S. A. Vrazhevsky, M. S. Tarasov, and Ju. V. Chugina. “Algorithms for Prediction of Smooth Bounded Signals”. In: *2019 6th International Conference on Control, Decision and Information Technologies (CoDIT)*. IEEE, 2019, pp. 581–586.
- 16) I. B. Furtat and A. N. Nekhoroshikh. “Modified backstepping algorithm and its application to control of distillation column”. In: *Mekhatronika, Avtomatizatsiya, Upravlenie* 20.2 (2019), pp. 90–96.
- 17) I. B. Furtat and A. N. Nekhoroshikh. “Investigation of electric generator robust algorithm under measurement noises”. In: *Cybernetics and Physics* 7.4 (2018), pp. 204–209.

**HYPEREXPONENTIAL AND FIXED-TIME
STABILITY OF TIME-DELAY SYSTEMS:
IMPLICIT LYAPUNOV-RAZUMIKHIN METHOD**

In this chapter, a Razumikhin-like method is proposed for hyperexponential and fixed-time stability analysis of retarded time-delay systems. Differently from the original Lyapunov-Razumikhin method, the proposed approach allows one not only to study the stability of a time-delay system but also to estimate the speed at which trajectories of the system converge to the equilibrium point. Furthermore, to make the developed method more suitable for the nonlinear control design, Implicit Lyapunov-Razumikhin theorems are also formulated. The advantage of the implicit formulation is illustrated by solving the problems of hyperexponential and fixed-time stabilization of a special subclass of time-delay systems. It is shown that, under some nonrestrictive assumptions, both problems can be easily solved by using the same nonlinear controller that stabilizes the corresponding delay-free system in fixed time. Applying the developed Implicit Lyapunov-Razumikhin method for stability analysis of the closed-loop system, the tuning of the nonlinear controller parameters, which guarantee superexponential stabilization with the required speed, is reduced to verification of linear matrix inequalities. The obtained theoretical results are supported by numerical simulation of the designed control system for different initial conditions and time delays.

Outline of the current chapter

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2.1 Introduction

Stability analysis of time-delay systems is usually made by means of either Lyapunov-Krasovskii [45] or Lyapunov-Razumikhin [67] methods. The idea of the former approach consists in shifting the Lyapunov analysis to the functional space which the state of a time-delay system belongs to. Thus, using this framework, it is possible not only to prove (asymptotic) stability [33, 42] but also to estimate how fast the system can be stabilized at the equilibrium point: (hyper)exponentially [26, 64] or in finite/(nearly) fixed time [52, 19]. Besides, such a method provides both sufficient and necessary stability conditions for time-delay systems. Nevertheless, generally the choice of a Lyapunov-Krasovskii functional is difficult. Furthermore, it is even harder to find a functional satisfying the restrictive conditions of the theorem on finite-time stability [52, 23].

Contrarily, the latter approach allows a conventional Lyapunov function to be used for the stability analysis [30]. For example, a Lyapunov function for the corresponding delay-free case can be chosen initially. However, such a paradigm does not provide necessary stability conditions or any quantitative estimates on the system convergence. The latter is the major drawback of Lyapunov-Razumikhin method since in many applications the settling time is one of the main performance criteria. Yet recent extensions of this framework made it possible to analyze exponential [47] and finite-/nearly fixed-time stability [19]. It is worth stressing that exponential stability of time-delay systems can also be proven without introduction of Lyapunov functionals by checking Halanay's inequality [32]. Based on this idea, in [71] global hyperexponential stability of a time-delay system with unbounded time-varying coefficients was investigated.

Notwithstanding, to the best of our knowledge, Razumikhin-like methods of hyperexponential and fixed-time stability analysis have not been proposed yet. Note that both types of convergence are superior to any exponential. It means that the transient time can be significantly reduced. For example, it can be also done by monotonous increasing feedback gains [24]. However, such a method makes the stability margin of time-delay systems drastically small. Another approach consists in introducing nonlinear feedback with comparatively small gains. Unfortunately, control design of nonlinear systems is a complicated task even for linear plants, e.g., a chain of integrators [2]. To overcome this difficulty the so-called Implicit Lyapunov method has been proposed for delay-free [62, 48] and time-delay systems [64]. This approach provides a constructive way for the control synthesis: it allows the controller parameters to be found as the solution of LMIs.

Therefore, the main objective of this chapter is to formulate both explicit and implicit Razumikhin-like theorems on hyperexponential and fixed-time stability of time-delay systems. Compared to the existing approaches, the contribution is as follows:

- 1) Development of an alternative way of stability analysis and control design of hyperexponentially/fixed-time stable time-delay systems: Lyapunov-Razumikhin method instead of Lyapunov-Krasovskii one [64]. It will be shown how the problem of hyperexponential stabilization studied in [64] can be successfully solved using the proposed method. Moreover, differently from [64], the closed-loop system is *globally* stable.

- 2) Extension of the results presented in [19] to the case of hyperexponential and fixed-time stability analysis and stabilization of retarded time-delay systems. Due to the complexity of formulating Razumikhin-like sufficient conditions by means of a single function, in the proposed method, the stability analysis is carried out using two Lyapunov-Razumikhin functions instead.

2.2 Definitions of stability

Consider a retarded time-delay system of the form [33]

$$\begin{cases} \dot{x}(t) = f(x_t), & t > 0, \\ x(\tau) = x_0(\tau), & \tau \in [-h, 0], \end{cases} \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state, $x_t \in C([-h, 0], \mathbb{R}^n)$ is the functional state defined for any time delay $h > 0$ as $x_t(\tau) := x(t + \tau)$ with $\tau \in [-h, 0]$, $x_0 \in C([-h, 0], \mathbb{R}^n)$ is the initial value function, $f : C([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is a continuous operator. Assume that the origin is an equilibrium point of (2.1), i.e., $f(0) = 0$.

Let us recall the definitions of (hyper)exponential, finite/(nearly) fixed-time stability for (2.1). For the conventional definition of Lyapunov stability the reader is referred to [26]. Denote a solution of the system (2.1) by $x(t, x_0)$.

Definition 2.1 [19, 33, 52, 64]. *The origin of (2.1) is said to be **globally***

- **exponentially stable** with a decay rate $\vartheta > 0$ if it is Lyapunov stable and there exists $\psi \in \mathcal{K}$ such that $\|x(t, x_0)\|_2 \leq \psi(\|x_0\|_C) e^{-\vartheta t}$ for all $t \geq 0$ and any $x_0 \in C([-h, 0], \mathbb{R}^n)$;
- **hyperexponentially stable** with a decay rate $\vartheta > 0$ if it is Lyapunov stable and there exists $\psi \in \mathcal{K}$ such that $\|x(t, x_0)\|_2 \leq \psi(\|x_0\|_C) e^{-e^{\vartheta t}}$ for all $t \geq 0$ and any $x_0 \in C([-h, 0], \mathbb{R}^n)$;
- **finite-time stable** if it is Lyapunov stable and for any $x_0 \in C([-h, 0], \mathbb{R}^n)$ there exists $0 \leq T(x_0) < +\infty$ such that $x(t, x_0) = 0$ for all $t \geq T(x_0)$. The functional $T(x_0) := \inf\{T \geq 0 : x(t, x_0) = 0, \forall t \geq T\}$ is called **the settling time** of the system (2.1);
- **nearly fixed-time stable** if it is Lyapunov stable and for any $\varrho_0 > 0$ there exists $0 < T(\varrho_0) < +\infty$ such that $\|x(t, x_0)\|_2 \leq \varrho_0$ for all $t \geq T(\varrho_0)$ and any $x_0 \in C([-h, 0], \mathbb{R}^n)$;
- **fixed-time stable** if it is finite-time stable and $\sup_{x_0 \in C([-h, 0], \mathbb{R}^n)} T(x_0) \leq T_0 < +\infty$, i.e., **the settling time is uniformly bounded**.

It is worth mentioning that the definition of the exponential stability differs from the conventional one, where $\psi(\|x_0\|_C) = k\|x_0\|_C$ with $k \geq 1$ (e.g., see [26]). However, in order to be consistent with the classification of the delay-free systems given in Definition 1.1, we will assume only that ψ is a class- \mathcal{K} function.

2.3 Explicit Lyapunov-Razumikhin theorems

Now we are ready to formulate Razumikhin-like theorems on hyperexponential and fixed-time stability of the system (2.1). Since the formulation of sufficient stability conditions by means of a single Lyapunov-Razumikhin function is difficult or even impossible in this case, we will carry out the stability analysis of the system (2.1) in two steps using a different Lyapunov-Razumikhin function for each of them. First, it will be proven that any trajectory of the system enters a specified closed region centered at the origin in finite time and never leaves it again. Then, the second function will be used to show that once the trajectories are within the specified region, they will converge to the origin hyperexponentially or in fixed time, respectively.

2.3.1 Hyperexponential stability

Theorem 2.1. *If there exist continuous functions $V_k : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $k = 1, 2$, such that:*

C_e1) V_k are continuously differentiable³ outside the origin;

C_e2) for some $\sigma_{1,k}, \sigma_{2,k} \in \mathcal{K}_\infty$ and all $\chi \in \mathbb{R}^n$

$$\sigma_{1,k}(\|\chi\|_2) \leq V_k(\chi) \leq \sigma_{2,k}(\|\chi\|_2);$$

C_e3) there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\|\chi\|_2 \leq \sigma_{1,1}(\|\chi\|_2)$ for all $c_1\|\chi\|_2 \leq 1$ and $c_2\|\chi\|_2 \leq \sigma_{1,2}(\|\chi\|_2)$ for all $c_2\|\chi\|_2 > 1$;

C_e4) $V_1(\chi) \leq 1$ for all $\chi \in \mathbb{R}^n$ such that $V_2(\chi) \leq 1$;

C_e5) for some $\nu > 1$, $\beta > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:

$$(a) \quad V_2(x_t(0)) > 1 \text{ and } \max_{\tau \in [-h, 0]} V_2(x_t(\tau)) \leq V_2^\nu(x_t(0))e^{\nu-1} \\ \implies \dot{V}_2(x_t(0)) \leq -\beta(1 + \ln V_2(x_t(0)))V_2(x_t(0)),$$

$$(b) \quad 0 < V_1(x_t(0)) \leq 1 \text{ and } \max_{\tau \in [-h, 0]} V_1^\nu(x_t(\tau)) \leq V_1(x_t(0))e^{\nu-1} \\ \implies \dot{V}_1(x_t(0)) \leq -\beta(1 - \ln V_1(x_t(0)))V_1(x_t(0)),$$

where $\dot{V}_k(x_t(0)) := \frac{\partial V_k(\chi)}{\partial \chi} \Big|_{\chi=x_t(0)} f(x_t)$, $k = 1, 2$, then system (2.1) is globally hyperexponentially stable at the origin with the decay rate ϑ given by

$$\vartheta = \min \left\{ \frac{\ln \nu}{h}, \beta \right\}. \quad (2.2)$$

Proof. Following the ideas of [19], the objective of the proof is to show that the Lyapunov-Razumikhin functions $\bar{V}_k(t) := V_k(x_t(0)) = V_k(x(t))$, $k = 1, 2$ decrease hyperexponentially along solutions of the time-delay system (2.1) for all $\bar{V}_2(t) \geq 1$ and $\bar{V}_1(t) \in [0, 1]$, respectively. To this end, we will study the time evolution of the functions $\bar{V}_k(t)$ both when the relations between $\max_{\tau \in [-h, 0]} \bar{V}_k(t + \tau)$ and $\bar{V}_k(t)$ given in condition C_e5) hold and when they do not. Depending on the initial value of $\bar{V}_2(t)$, we will consider two cases: $\bar{V}_2(0) > 1$ and $\bar{V}_2(0) \leq 1$.

Case 1: $\bar{V}_2(0) > 1$

I. Condition C_e5a) check. Without loss of generality, first assume that

$$\max_{\tau \in [-h, 0]} \bar{V}_2(t + \tau) > \bar{V}_2^\nu(t)e^{\nu-1}, \quad \forall t \in [0, t_1], \quad (2.3)$$

where $t_1 \in [0, +\infty)$ is the moment of time such that either condition C_e5a) starts to hold or $\bar{V}_2(t_1) = 1$. It is worth stressing that the inequality (2.3) implies $\max_{\tau \in [-h, 0]} \bar{V}_2(t + \tau) \neq \bar{V}_2(t)$ for all $t \in [0, t_1]$ since $\nu > 1$. Introduce $\theta_{t',2} := \min\{\theta \in [-h, 0) : \bar{V}_2(t' + \theta) = \max_{\tau \in [-h, 0]} \bar{V}_2(t' + \tau)\}$ for all $t' \in [0, t]$. As a result, it follows from (2.3) that

$$\ln(e\bar{V}_2(t)) < \nu^{-1} \ln(e\bar{V}_2(t + \theta_{t,2})) = e^{-\ln \nu} \ln(e\bar{V}_2(t + \theta_{t,2})).$$

³See Remark 1.1 on page 27

This estimate is successively applied until the sum $t + \theta_{t,2} + \theta_{t+\theta_{t,2},2} + \dots$ belongs to the interval $[-h, 0]$. Clearly, this process will stop after a *finite* number of iterations $m_2 \geq t/h$ since $\theta_{t,2}, \theta_{t+\theta_{t,2},2}, \dots < 0$. Taking into account (2.2), it yields:

$$\ln(e\bar{V}_2(t)) < e^{-(t/h)\ln\nu} \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)) \leq e^{-\vartheta t} \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)), \quad \forall t \in [0, t_1].$$

Evidently, it follows that $t_1 < T' := \vartheta^{-1} \ln(\max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau))) + h < +\infty$.

Now suppose that condition C_{e5a} holds for $t \in [t_1, t_2]$, where $t_2 \in [t_1, T')$ is the moment of time such that either $\max_{\tau \in [-h, 0]} \bar{V}_2(t_2 + \tau) > \bar{V}_2^\nu(t_2)e^{\nu-1}$ or $\bar{V}_2(t_2) = 1$. Then applying the Comparison Lemma A.1 (see Appendix A), we deduce that

$$\ln(e\bar{V}_2(t)) \leq e^{-\beta(t-t_1)} \ln(e\bar{V}_2(t_1)) \leq e^{-\vartheta(t-t_1)} \ln(e\bar{V}_2(t_1)), \quad \forall t \in [t_1, t_2].$$

Summarizing both cases, we get

$$\ln(e\bar{V}_2(t)) < e^{-\vartheta t} \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)), \quad \forall t \in [0, t_2]. \quad (2.4)$$

Repeating the same steps, it can be shown that estimate (2.4) holds for all $t \in [0, T)$, where $T \in [t_2, T')$ such that $\bar{V}_2(T) = 1$, and $\bar{V}_2(t) \leq 1$ for all $t \geq T$. Moreover, let us prove that for all $t \in [0, T)$ the following inequality holds:

$$e^{-\vartheta t} \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)) \leq 1 - e^{\vartheta t} + \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)).$$

Indeed, since the function $y(t) := p(e^{-\vartheta t} - 1) + e^{\vartheta t} - 1$, where $p := \max_{\tau \in [-h, 0]} \ln(e\bar{V}_2(\tau)) > 1$, is convex with respect to time t , then it is sufficient to check that $y(0)$ and $y(T)$ are nonpositive. Obviously, $y(0) = p(1 - 1) + 1 - 1 = 0$. On the other hand, $1 = \ln(e\bar{V}_2(T)) < pe^{-\vartheta T}$ due to (2.4) and, therefore, $y(T) = p(e^{-\vartheta T} - 1) + e^{\vartheta T} - 1 = -(pe^{-\vartheta T} - 1)(e^{\vartheta T} - 1) < 0$.

Hence, taking into account C_{e2} and C_{e3} , one can see that

$$c_2 \|x(t, x_0)\|_2 \leq \bar{V}_2(t) < e^{1-e^{\vartheta t}} \max_{\tau \in [-h, 0]} \bar{V}_2(\tau), \quad \forall t \in [0, T). \quad (2.5)$$

II. Condition C_{e5b} check. Since $\bar{V}_2(t) \leq 1$ for all $t \geq T$, then it follows from C_{e4} that $\bar{V}_1(t) \leq 1$ for all $t \geq T$. Assume that

$$\max_{\tau \in [-h, 0]} \bar{V}_1^\nu(t + \tau) > \bar{V}_1(t)e^{\nu-1}, \quad \forall t \in [T, t_3], \quad (2.6)$$

where $t_3 \geq T$ (possibly *infinite*) is the moment of time such that condition C_{e5b} starts to hold or $\bar{V}_1(t_3) = 0$. It is worth pointing out that (2.6) implies $\max_{\tau \in [-h, 0]} \bar{V}_1(t + \tau) \neq \bar{V}_1(t)$ for all $t \in [T, t_3]$ since $\bar{V}_1(t) \leq 1$ for all $t \geq T$. Introduce $\theta_{t',1} := \min\{\theta \in [-h, 0) : \bar{V}_1(t' + \theta) = \max_{\tau \in [-h, 0]} \bar{V}_1(t' + \tau)\}$ for all $t' \in [T, t]$. Thus,

$$\ln(\bar{V}_1(t)/e) < \nu \ln(\bar{V}_1(t + \theta_{t,1})/e) = e^{\ln\nu} \ln(\bar{V}_1(t + \theta_{t,1})/e).$$

This estimate is successively applied until the sum $t + \theta_{t,1} + \theta_{t+\theta_{t,1},1} + \dots$ belongs to the interval $[T, T + h]$. Clearly, this process will stop after a *finite* number of iterations $m_1 \geq (T - t + h)/h$ since

$\theta_{t,1}, \theta_{t+\theta_{t,1},1}, \dots < 0$. Eventually, it yields:

$$\ln(\bar{V}_1(t)/e) < e^{(t-T+h)/h \ln v} \max_{\tau \in [0,h]} \ln(\bar{V}_1(T+\tau)/e) \leq e^{\vartheta(t-T+h)} \max_{\tau \in [0,h]} \ln(\bar{V}_1(T+\tau)/e), \quad \forall t \in [T, t_3].$$

Now, suppose that condition $C_e5b)$ holds for $t \in [t_3, t_4)$, where $t_4 \geq t_3$ (again possibly *infinite*) is the moment of time such that $\max_{\tau \in [-h,0]} \bar{V}_1^\nu(t_4+\tau) > \bar{V}_1(t_4)e^{\nu-1}$ or $\bar{V}_1(t_4) = 0$. Then applying the Comparison Lemma A.1 (see Appendix A), we deduce that

$$\ln(\bar{V}_1(t)/e) \leq e^{\beta(t-T-t_3)} \ln(\bar{V}_1(T+t_3)/e) \leq e^{\vartheta(t-T-t_3)} \ln(\bar{V}_1(T+t_3)/e), \quad \forall t \in [t_3, t_4).$$

Summarizing both cases, for all $t \in [T, t_4)$ we get

$$\ln(\bar{V}_1(t)/e) < e^{\vartheta(t-T+h)} \max_{\tau \in [0,h]} \ln(\bar{V}_1(T+\tau)/e), \quad \forall t \in [T, t_4). \quad (2.7)$$

Again, repeating the same steps, it can be shown that estimate (2.7) holds for all $t \geq T$. Moreover, since $\max_{\tau \in [0,h]} \bar{V}_1(T+\tau) \leq 1$ for all $t \geq T$, then

$$e^{\vartheta(t-T+h)} \max_{\tau \in [0,h]} \ln(\bar{V}_1(T+\tau)/e) \leq 1 - e^{\vartheta(t-T)} + \max_{\tau \in [0,h]} \ln(\bar{V}_1(T+\tau)/e).$$

Thus, taking into account $C_e2)$ and $C_e3)$, one can see

$$c_1 \|x(t, x_0)\|_2 \leq \bar{V}_1(t) < e^{1-e^{\vartheta(t-T)}} \max_{\tau \in [0,h]} \bar{V}_1(T+\tau), \quad \forall t \geq T. \quad (2.8)$$

Finally, combining (2.5) and (2.8) and taking into account condition $C_e2)$, we conclude that system (2.1) is globally hyperexponentially stable at the origin with the decay rate ϑ and function $\psi(\|x_0\|_C) = \sigma_{2,2}(\|x_0\|_C)e / \min\{c_1, c_2\}$.

Case 2: $\bar{V}_2(0) \leq 1$

If $\bar{V}_2(0) \leq 1$ then $\bar{V}_1(0) \leq 1$ due to condition $C_e4)$. Therefore, all the results obtained in sections II and III of Case 1 remain valid with $T = 0$ and $\psi(\|x_0\|_C) = \sigma_{2,1}(\|x_0\|_C)e/c_1$. \square

Example 2.1. Let us show how Theorem 2.1 can be used for the stability analysis of the scalar time-delay system

$$\dot{x}(t) = -2ex^{1/3}(t) - 2ex^3(t) + x(t-h).$$

Choose $V_1(\chi) = \chi^{2/3}$ and $V_2(\chi) = \chi^2$, for which conditions $C_e1)$ – $C_e4)$ hold with $\sigma_{1,1}(s) = \sigma_{2,1}(s) = s^{2/3}$, $\sigma_{1,2}(s) = \sigma_{2,2}(s) = s^2$ and $c_1 = c_2 = 1$. Let us prove that condition $C_e5)$ is fulfilled for $\nu = 3$ and $\beta = 2/3$. Indeed, one can see that if $x_t^2(-h) = V_2(x_t(-h)) \leq V_2^3(x_t(0))e^{3-1} = x_t^6(0)e^2$ then $\dot{V}_2(x_t(0)) \leq -2eV_2^2(x_t(0)) \leq -2e(1 + \ln V_2(x_t(0)))V_2(x_t(0))$. Analogously, $x_t^2(-h) = V_1^3(x_t(-h)) \leq V_1(x_t(0))e^{3-1} = x_t^{2/3}(0)e^2$ implies $\dot{V}_1(x_t(0)) \leq -\frac{2}{3}e = -\frac{2}{3}eV_1^{-1}(x_t(0))V_1(x_t(0)) \leq -\frac{2}{3}e(1 + \ln V_1^{-1}(x_t(0)))V_1(x_t(0))$. As a result, the system is globally hyperexponentially stable with the decay rate $\vartheta = \min\{\frac{\ln 3}{h}, \frac{2}{3}e\}$. \blacksquare

Repeating the steps given in the proof of Theorem 2.1, it can be shown (see [19]) that global exponential stability of the time-delay system (2.1) at the origin can be studied by checking a Lyapunov-Razumikhin condition similar to the one given in [47]. Moreover, in this case, the stability analysis can be done using a single Lyapunov-Razumikhin function.

Corollary 2.1. *If there exists a continuous function $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that conditions $C_e1)$, $C_e2)$ hold with $k = 0$ and, furthermore:*

$C_e^*3)$ *there is a constant $c_0 > 0$ such that $c_0\|\chi\|_2 \leq \sigma_{1,0}(\|\chi\|_2)$ for all $\chi \in \mathbb{R}^n$;*

$C_e^*4)$ *for some $\kappa > 1$, $\beta > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:*

$$V_0(x_t(0)) > 0 \text{ and } \max_{\tau \in [-h, 0]} V_0(x_t(\tau)) \leq \kappa V_0(x_t(0)) \implies \dot{V}_0(x_t(0)) \leq -\beta V_0(x_t(0)),$$

where $\dot{V}_0(x_t(0)) := \frac{\partial V_0(\chi)}{\partial \chi} \Big|_{\chi=x_t(0)} f(x_t)$, then system (2.1) is globally exponentially stable at the origin with the decay rate ϑ given by

$$\vartheta = \min \left\{ \frac{\ln \kappa}{h}, \beta \right\}. \quad (2.9)$$

Example 2.2. Using Corollary 2.1, it is easy to verify that the following scalar time-delay system

$$\dot{x}(t) = -2\kappa x(t) + x(t-h),$$

where $\kappa > 1$, is globally exponentially stable with the decay rate $\vartheta = \min\{\frac{\ln \kappa}{h}, \kappa\}$. Indeed, choosing a Lyapunov-Razumikhin function $V_0(\chi) = |\chi|$, one can see that conditions $C_e1)$, $C_e2)$ and $C_e^*3)$ hold with $\sigma_{1,0}(s) = \sigma_{2,0}(s) = s$ and $c_0 = 1$. On the other hand, the fulfillment of condition $C_e^*4)$ follows from the fact that $\dot{V}_0(x_t(0)) \leq -\kappa V_0(x_t(0))$ for all $|x_t(-h)| = V_0(x_t(-h)) \leq \kappa V_0(x_t(0)) = \kappa|x_t(0)|$. ■

2.3.2 Fixed-time stability

One can see that the time-delay system given in Example 2.1 cannot be stabilized in fixed (or even finite) time since $\dot{x}(t) \neq 0$ when $x(t) = 0$. Therefore, to guarantee fixed-time stability of time-delay systems, more severe restrictions must be imposed on the right-hand side of (2.1).

Theorem 2.2. *If there exist two continuous functions $V_k : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $k = 1, 2$, such that conditions $C_e1)$, $C_e2)$ and $C_e4)$ hold and, furthermore,*

$C_e^*5)$ *for some $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\alpha > 0$, $\rho > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:*

$$(a) \quad V_2(x_t(0)) > 1 \text{ and } \left(\max_{\tau \in [-h, 0]} V_2(x_t(\tau)) \right)^{-\mu_2} + \rho \geq V_2^{-\mu_2}(x_t(0)) \\ \implies \dot{V}_2(x_t) \leq -\alpha V_2^{1+\mu_2}(x_t(0)),$$

$$(b) \quad 0 < V_1(x_t(0)) \leq 1 \text{ and } \left(\max_{\tau \in [-h, 0]} V_1(x_t(\tau)) \right)^{-\mu_1} \leq V_1^{-\mu_1}(x_t(0)) + \rho \\ \implies \dot{V}_1(x_t) \leq -\alpha V_1^{1+\mu_1}(x_t(0)),$$

then system (2.1) is globally fixed-time stable at the origin with the settling time T_0 given by

$$T_0 = \frac{1}{\min\{\alpha\mu_2, \rho/h\}} + \frac{1}{\min\{-\alpha\mu_1, \rho/h\}} \quad (2.10)$$

Proof. The proof is based on the fact that system (2.1) is nearly fixed-time stable with respect to the set $\mathcal{A} := \{x \in \mathbb{R}^n : V_2(x) \leq 1\}$ due to $C_e^*5a)$ (see Theorem 4 of [19]) and finite-time stable with the domain

of attractivity $\mathcal{X} := \{x \in \mathbb{R}^n : V_1(x) \leq 1\}$ due to $C_e^*5b)$ (see Theorem 3 of [19]). Since condition $C_e4)$ implies $\mathcal{A} \subseteq \mathcal{X}$, then the system is fixed-time stable. Moreover, the following estimates hold

$$\begin{aligned}\bar{V}_2(t) &< \left(\max_{\tau \in [-h, 0]} \bar{V}_2^{-\mu_2}(\tau) + \min\{\alpha\mu_2, \rho/h\}t \right)^{-1/\mu_2}, & \forall t \in [0, T_2), \\ \bar{V}_1(t) &< \left(\max_{\tau \in [0, h]} \bar{V}_1^{-\mu_1}(T + \tau) - \min\{-\alpha\mu_1, \rho/h\}(t - T) \right)^{-1/\mu_1}, & \forall t \in [T_2, T_2 + T_1), \\ \bar{V}_1(t) &= 0, & \forall t \geq T_2 + T_1,\end{aligned}$$

where $T_2 \in [0, \frac{1}{\min\{\alpha\mu_2, \rho/h\}}]$ is the moment of time such that $\bar{V}_2(T + \tau) = 1$, and $T_1 \in [0, \frac{1}{\min\{-\alpha\mu_1, \rho/h\}}]$. Denoting $T_0 = T_2 + T_1$, we get (2.10). \square

Example 2.3. Let us consider a scalar *bilinear* time-delay system

$$\dot{x}(t) = -2[x(t)]^{1/2} - 2[x(t)]^2 + x(t) \text{sat}(x(t-h), b),$$

where $b \in (4, +\infty)$.

It is worth stressing that due to the boundedness of the delay term, it is possible to carry out the delay-free stability analysis by letting $\text{sat}(x(t-h), b) = b$. However, such a method results in a very conservative result since parameter b can be large enough. For example, the corresponding delay-free system is only locally finite-time stable at the origin if $x(0) \in \mathcal{X} := \{x \in \mathbb{R} : |x|^{-1/2} + |x| > b/2, |x| \leq 1\}$, and nearly fixed-time stable with respect to the set $\mathcal{A} := \{x \in \mathbb{R} : |x|^{-1/2} + |x| \leq b/2\}$ otherwise. Therefore, the delay-free analysis does not allow us to prove the fixed-time stability of the original time-delay system.

Now we will show how the stability analysis can be carried out using Theorem 2.2. To this end, we choose the same Lyapunov-Razumikhin functions as in Example 2.1, i.e., $V_1(\chi) = \chi^{2/3}$ and $V_2(\chi) = \chi^2$, for which conditions $C_e1)$, $C_e2)$ and $C_e4)$ are satisfied. Let us prove that condition $C_e^*5)$ holds for $\mu_1 = -3/4$, $\mu_2 = 1/2$, $\alpha = 2/3$ and $\rho \in (0, 2/(2b + b^{5/2})]$. First, note that $\dot{V}_2(x_t) \leq -2|x_t(0)|^3$ for all $|x_t(0)| \geq b$. On the other hand, one can see that if $|x_t(-h)| = V_2^{1/2}(x_t(-h)) \leq (V_2^{-1/2}(x_t(0)) - \rho)^{-1} = (|x_t(0)|^{-1} - \rho)^{-1}$ then $\dot{V}_2(x_t) \leq -2x_t^2(0)(2|x_t(0)|^{-1/2} + 2|x_t(0)| - (|x_t(0)|^{-1} - \rho)^{-1})$. Since $|x_t(0)|^{-1} - \rho \geq (2|x_t(0)|^{-1/2} + |x_t(0)|)^{-1}$ for all $x_t(0) \in (1, b)$ if $\rho \leq 2/(2b + b^{5/2})$, then $\dot{V}_2(x_t) \leq -2|x_t(0)|^3 = -2V_2^{1+\mu_2}(x_t(0))$ for all $V_2(x_t(0)) > 1$. Clearly, if the delay term is unbounded, i.e., $b = +\infty$, then $\rho = 0$ and condition $C_e^*5a)$ cannot be fulfilled. Similarly, $|x_t(-h)| = V_1^{3/2}(x_t(-h)) \leq (V_1^{3/4}(x_t(0)) + \rho)^2 = (|x_t(0)|^{1/2} + \rho)^2$ implies $\dot{V}_1(x_t) \leq -\frac{2}{3}x_t^{2/3}(0)(2|x_t(0)|^{-1/2} + 2|x_t(0)| - (|x_t(0)|^{1/2} + \rho)^2)$. Taking into account that $(|x_t(0)|^{1/2} + \rho)^2 \leq |x_t(0)|^{-1/2} + 2|x_t(0)|$ for all $x_t(0) \in (0, 1]$ and $\rho \in (0, 2/(2b + b^{5/2})]$, it follows that $\dot{V}_1(x_t) \leq -\frac{2}{3}|x_t(0)|^{1/6} = -\frac{2}{3}V_1^{1+\mu_1}(x_t(0))$ for all $V_1(x_t(0)) \in (0, 1]$. As a result, the system is globally fixed-time stable with the settling time $T_0 = \frac{1}{\min\{1/3, \rho/h\}} + \frac{1}{\min\{1/2, \rho/h\}}$. \blacksquare

2.4 Implicit Lyapunov-Razumikhin theorems

In this section, implicit analogues of Theorems 2.1 and 2.2 as well as Corollary 2.1 are introduced.

Theorem 2.3. *If there exist two continuous functions $Q_k : \mathbb{R}_+^* \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that:*

$C_i1)$ Q_k are continuously differentiable outside the origin;

$C_i2)$ for any $\chi \in \mathbb{R}^n \setminus \{0\}$ there exists $V_k \in \mathbb{R}_+^*$ such that $Q_k(V_k, \chi) = 0$;

$C_i3)$ $\frac{\partial Q_k(V_k, \chi)}{\partial V_k} < 0$ for all $V_k \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$;

$C_i4)$ there exist $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$ such that for all $V_k \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$:

$$q_{1,k}(V_k, \|\chi\|_2) \leq Q_k(V_k, \chi) \leq q_{2,k}(V_k, \|\chi\|_2);$$

$C_i5)$ there are constants $c_1 > 0$ and $c_2 > 0$ such that $q_{1,1}(c_1\|\chi\|_2, \|\chi\|_2) \geq 0$ for all $c_1\|\chi\|_2 \leq 1$ and $q_{1,2}(c_2\|\chi\|_2, \|\chi\|_2) \geq 0$ for all $c_2\|\chi\|_2 > 1$;

$C_i6)$ $Q_1(1, \chi) = Q_2(1, \chi)$ for all $\chi \in \mathbb{R}^n$;

$C_i7)$ for some $\nu > 1, \beta > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:

$$(a) \quad (V_2, x_t) \in \Omega_2 := \{(s, \phi) \in (1, +\infty) \times C([-h, 0], \mathbb{R}^n) : Q_2(s, \phi(0)) = 0, \\ \max_{\tau \in [-h, 0]} Q_2(s^\nu e^{\nu-1}, \phi(\tau)) \leq 0\} \implies F_2(V_2, x_t) \leq -\beta(1 + \ln V_2)V_2,$$

$$(b) \quad (V_1, x_t) \in \Omega_1 := \{(s, \phi) \in (0, 1] \times C([-h, 0], \mathbb{R}^n) : Q_1(s, \phi(0)) = 0, \\ \max_{\tau \in [-h, 0]} Q_1(s^{1/\nu} e^{1-1/\nu}, \phi(\tau)) \leq 0\} \implies F_1(V_1, x_t) \leq -\beta(1 - \ln V_1)V_1,$$

where $F_k(V_k, x_t) := -\left(\frac{\partial Q_k(V_k, x_t(0))}{\partial V_k}\right)^{-1} \frac{\partial Q_k(V_k, \chi)}{\partial \chi} \Big|_{\chi=x_t(0)} f(x_t)$, then system (2.1) is globally hyperexponentially stable at the origin with the decay rate ϑ given by (2.2).

Proof. In order to prove Theorem 2.3, let us show that there exist two unique functions $V_k : \mathbb{R}^n \rightarrow \mathbb{R}_+$ that satisfy conditions of Theorem 2.1. Firstly, it was shown in Chapter 1 (see Theorem 1.2) that conditions $C_i1)$ – $C_i3)$ imply existence of unique functions $V_k(\chi)$ defined by equality $Q_k(V_k(\chi), \chi) = 0$ with $V_k(0) = 0$.

Secondly, it follows from properties of \mathcal{IK}_∞ functions that there exist $\sigma_{1,k}, \sigma_{2,k} \in \mathcal{K}_\infty$ such that $q_{1,k}(\sigma_{1,k}(\|\chi\|_2), \|\chi\|_2) = q_{2,k}(\sigma_{2,k}(\|\chi\|_2), \|\chi\|_2) = 0 = Q_k(V_k(\chi), \chi)$ for all $\chi \in \mathbb{R}^n \setminus \{0\}$. Therefore, $C_i4) \iff C_e2)$ and $C_i5) \iff C_e3)$.

Thirdly, for any given $\chi \in \mathbb{R}^n$ such that $V_2(\chi) \leq 1$ we have $Q_1(1, \chi) = Q_2(1, \chi) \leq Q_2(V_2(\chi), \chi) = 0 = Q_1(V_1(\chi), \chi)$ due to $C_i3)$. Then $V_1(\chi) \leq 1$ and, hence, $C_i6) \implies C_e4)$.

Finally, conditions $C_e5)$ and $C_i7)$ are equivalent. Indeed, take any pair (V_k, x_t) from Ω_k . Since functions $s \mapsto Q_k(s, x_t(\tau))$, $\tau \in [-h, 0]$, are monotonically decreasing due to condition $C_i3)$, for any $\tilde{s} \in \mathbb{R}_+^*$ we have:

$$\max_{\tau \in [-h, 0]} Q_k(\tilde{s}, x_t(\tau)) \leq 0 = Q_k(V_k(x_t(\tau)), x_t(\tau)) \iff \max_{\tau \in [-h, 0]} V_k(x_t(\tau)) \leq \tilde{s}.$$

On the other hand, due to the Implicit Function Theorem A.1 (see Appendix A) for all $(V_k, x_t) \in \Omega_k$, the function $F_k(V_k, x_t)$ can be rewritten as $F_k(V_k, x_t) = \frac{\partial V_k(\chi)}{\partial \chi} \Big|_{\chi=x_t(0)} f(x_t)$, which coincides with the definition of the function $\dot{V}_k(x_t(0))$. Taking these facts into account, we finish the proof. \square

Corollary 2.2. *If there exists a continuous function $Q_0 : \mathbb{R}_+^* \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that conditions $C_i1)$ – $C_i4)$ hold with $k = 0$ and, furthermore:*

$C_i^*5)$ there is a constant $c_0 > 0$ such that $q_{1,0}(c_0\|\chi\|_2, \|\chi\|_2) \geq 0$ for all $\chi \in \mathbb{R}_+$;

$C_i^*6)$ for some $\kappa > 1$, $\beta > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:

$$(V_0, x_t) \in \Omega_0 := \left\{ (s, \phi) \in \mathbb{R}_+^* \times C([-h, 0], \mathbb{R}^n) : Q_0(s, \phi(0)) = 0, \right. \\ \left. \max_{\tau \in [-h, 0]} Q_0(\kappa s, \phi(\tau)) \leq 0 \right\} \implies F_0(V_0, x_t) \leq -\beta V_0,$$

where $F_0(V_0, x_t) := -\left(\frac{\partial Q_0(V_0, x_t(0))}{\partial V_0}\right)^{-1} \frac{\partial Q_0(V_0, x)}{\partial x} \Big|_{x=x_t(0)} f(x_t)$, then system (2.1) is globally exponentially stable at the origin with the decay rate ϑ given by (2.9).

Theorem 2.4. *If there exist two continuous functions $Q_k : \mathbb{R}_+^* \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that conditions $C_i1)$ - $C_i4)$ and $C_i6)$ hold and, furthermore:*

$C_i^*7)$ for some $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\alpha > 0$, $\rho > 0$ and all $x_t \in C([-h, 0], \mathbb{R}^n)$ satisfying (2.1) we have:

$$(a) \quad (V_2, x_t) \in \Omega_2^* := \left\{ (s, \phi) \in (1, +\infty) \times C([-h, 0], \mathbb{R}^n) : Q_2(s, \phi(0)) = 0, \right. \\ \left. \max_{\tau \in [-h, 0]} Q_2((\max\{0, s^{-\mu_2} - \rho\})^{-1/\mu_2}, \phi(\tau)) \leq 0 \right\} \implies F_2(V_2, x_t) \leq -\alpha V_2^{1+\mu_2},$$

$$(b) \quad (V_1, x_t) \in \Omega_1^* := \left\{ (s, \phi) \in (0, 1] \times C([-h, 0], \mathbb{R}^n) : Q_1(s, \phi(0)) = 0, \right. \\ \left. \max_{\tau \in [-h, 0]} Q_1((s^{-\mu_1} + \rho)^{-1/\mu_1}, \phi(\tau)) \leq 0 \right\} \implies F_1(V_1, x_t) \leq -\alpha V_1^{1+\mu_1},$$

then system (2.1) is globally fixed-time stable at the origin with the settling time T_0 given by (2.10).

Theorem 2.4 can be proven similarly to Theorem 2.3.

2.5 Examples

2.5.1 Problem formulation

Let us consider a subclass of (2.1) with $n \geq 2$:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 \xi(x_t) + B_0 u(x(t)), & t > 0, \\ x(\tau) = x_0(\tau), & \tau \in [-h, 0], \end{cases} \quad (2.11)$$

where $u(x(t)) \in \mathbb{R}$ is a control input to be designed, $A_1 = \text{diag}\{A_{1,i}\}_{i=1}^n$ is a known matrix, system matrices $A_0 \in \mathbb{R}^{n \times n}$ and $B_0 \in \mathbb{R}^n$ are of the form:

$$A_0 := \begin{bmatrix} O_{(n-1) \times 1} & I_{n-1} \\ 0 & O_{1 \times (n-1)} \end{bmatrix}, \quad B_0 := \begin{bmatrix} O_{(n-1) \times 1} \\ 1 \end{bmatrix},$$

$\xi(x_t) = \text{col}\{\xi_i(x_{t,i})\}_{i=1}^n$ is a vector-valued function the components $\xi_i(x_{t,i})$, $i = \overline{1, n}$, of which satisfy the following conditions:

– in case of hyperexponential stabilization [64]:

$$|\xi_i(x_{t,i})| \leq |x_i(t-h)|; \quad (2.12a)$$

– in case of fixed-time stabilization:

$$|\xi_i(x_{t,i})| \leq |\text{sat}(x_i(t), 1)| |\text{sat}(x_i(t-h), b)|, \quad (2.12b)$$

where $b \in (\max\{1, \max_i\{1/A_{1,i}\}\}, +\infty)$.

Note that many dynamical processes, such as vehicular traffic flow [26], regenerative chatter in metal cutting [30] or population dynamics [26], can be modeled by system (2.11) with conditions (2.12) fulfilled.

Remark 2.1. *It is worth mentioning that system (2.11) could be finite-time stable if $\dot{x}(t)|_{x(t)=0} = 0$ (see [23, 52]). Since we are looking for a feedback $u(x(t))$ such that $u(0) = 0$, then the delay term $A_1\xi(x_t)$ has to be identically zero whenever $x(t) = 0$. Obviously, condition (2.12a) does not guarantee this and function $\xi(x_t)$ has to satisfy more severe restrictions as, for example, given in (2.12b). However, constraint (2.12b) is not very restrictive since it holds for many bounded functions (e.g., sine and different sigmoids) with a unit slope. Moreover, the amplitude of the time-delay term can be significantly large.*

2.5.2 Control design

For both hyperexponential and fixed-time stabilization we will define a continuous control law as follows [48]:

$$u(x) := \sum_{i=1}^n K_i [x_i]^{a_i(\tilde{\mu}(\|x\|_P))}, \quad (2.13a)$$

$$a_i(\tilde{\mu}) := \frac{1 + \tilde{\mu}}{1 - (n - i)\tilde{\mu}}, \quad i = \overline{1, n}, \quad (2.13b)$$

$$\tilde{\mu}(\|x\|_P) := \begin{cases} \mu_2, & \text{if } \|x\|_P \geq \Delta, \\ \mu_1, & \text{if } \|x\|_P \leq 1, \\ \frac{\mu_2 - \mu_1}{\Delta - 1} \|x\|_P + \frac{\Delta\mu_1 - \mu_2}{\Delta - 1}, & \text{otherwise,} \end{cases} \quad (2.13c)$$

where $K_i < 0$, $i = \overline{1, n}$, $0 < P \in \mathbb{S}^{n \times n}$, $\mu_1 \in (-1, 0)$, $\mu_2 \in (0, \frac{1}{n-1})$ and $\Delta > 1$ are controller parameters to be selected.

Note that if $\mu_1 = \mu_2 = 0$ then $a_i(\tilde{\mu}) \equiv 1$ for all $i = \overline{1, n}$. As a result, the control law (2.13) is reduced to the conventional linear feedback

$$u(x) = \sum_{i=1}^n K_i x_i = Kx, \quad (2.14)$$

where $K := \text{row}\{K_i\}_{i=1}^n$.

2.5.3 Controller parameters tuning algorithm

In this subsection, we will present the restrictions on the choice of the controller parameters such that the sufficient conditions for global hyperexponential (fixed-time) stability given in Theorem 2.3 (2.4) are satisfied (the proof can be found in Appendices 2.A and 2.B). To this end, we will introduce Implicit Lyapunov-Razumikhin function (ILRF) candidates $Q_k(V_k, \chi)$, $k = 1, 2$, [62]

$$Q_k(V_k, \chi) := \chi^\top \Lambda_k(V_k^{-1}) P \Lambda_k(V_k^{-1}) \chi - 1, \quad (2.15)$$

where $\Lambda_k(\lambda) := \text{diag}\{\lambda^{r_{k,i}}\}_{i=1}^n$ is the dilation matrix with weights

$$r_{k,i}(\mu_k) := 1 - (n - i)\mu_k, \quad i = \overline{1, n}. \quad (2.16)$$

Proposition 2.1. *If there exist $\mu \in (0, \frac{1}{n-1})$, $\delta > 1$, $\eta > 0$ and $\alpha \in (0, \eta/2]$ such that LMIs:*

$$0 \prec XG_k + G_kX \preceq \gamma X, \quad (2.17a)$$

$$\begin{bmatrix} R & \|\zeta\|_2 B_0 Y \\ \|\zeta\|_2 Y^\top B_0^\top & -\alpha^2 \gamma X \end{bmatrix} \preceq 0, \quad (2.17b)$$

$$2\alpha I_n \preceq X \preceq \eta I_n, \quad 2\eta I_n \preceq Z, \quad (2.17c)$$

where $G_k := \text{diag}\{r_{k,i}\}_{i=1}^n$, $R := A_0 X + B_0 Y + X A_0^\top + Y^\top B_0^\top + 2\alpha \gamma X + \frac{\varrho^2}{\alpha \gamma} A_1 Z A_1^\top$,

$$\zeta = \text{col}\{\zeta_i\}_{i=1}^n, \quad \zeta_i := \max \left\{ \bar{g} \left(\sqrt{\eta}, \frac{1 + \mu_1}{r_{1,i}} \right), \bar{g} \left(\sqrt{\eta}, \frac{1 + \mu_2}{r_{2,i}} \right) \right\} + \sqrt{\eta} (1 - \delta^{p_i}),$$

$$p_i := \frac{r_{2,i}}{r_{1,i}} (1 + \mu_1) - (1 + \mu_2), \quad \gamma := 2(1 - n\mu_1), \quad \varrho = e^{-\mu_1}, \quad \mu_1 := -\frac{\mu}{1 - (n-2)\mu}, \quad \mu_2 := \mu,$$

with functions $r_{k,i}(\mu_k)$ and $\bar{g}(\bar{s}, \bar{\epsilon})$ defined in (2.16) and Lemma A.2, respectively, are feasible for some $X \in \mathbb{S}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and $Z \in \mathbb{S}^{n \times n}$, then the closed-loop system (2.11), (2.12a), (2.13) with parameters

$$K = YX^{-1}, \quad P = X^{-1}, \quad \mu_1 = -\frac{\mu}{1 - (n-2)\mu}, \quad \mu_2 = \mu, \quad \Delta = \sqrt{\lambda_{\min}(X^{1/2} \Lambda_2(\delta) X^{-1} \Lambda_2(\delta) X^{1/2})}$$

is **globally hyperexponentially stable** with the decay rate ϑ given by (2.2), where $\nu = 1 + \mu$ and $\beta = \alpha\mu$.

Proposition 2.2. *If there exist $\mu \in (0, \frac{1}{n-1})$, $\delta > 1$, $\eta > 0$, $\alpha \in (0, \eta/2]$ and $\rho \in (0, \bar{\rho}]$, where*

$$\bar{\rho} := \begin{cases} \bar{\rho}_2, & \text{if } \sqrt{\eta} \leq 1, \\ \bar{\rho}_1, & \text{if } \sqrt{\eta} \geq b\sqrt{n}, \\ \min\{\bar{\rho}_1, \bar{\rho}_2\}, & \text{otherwise,} \end{cases}$$

$$\bar{\rho}_1 := \frac{\sqrt{\eta}^{\mu_1^2/r_{1,1}} - 1}{\sqrt{\eta}^{-\mu_1/r_{1,1}} - 1}, \quad \bar{\rho}_2 := \frac{1 - (b\sqrt{n/\eta})^{-\mu_2^2/r_{2,2}}}{(b\sqrt{n/\eta})^{\mu_2/r_{2,2}} - (b\sqrt{n/\eta})^{-\mu_2^2/r_{2,2}}},$$

such that LMIs (2.17) with $\varrho = (1 - \rho)^{-(1+\mu)/\mu}$ are feasible for some $X \in \mathbb{S}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and $Z \in \mathbb{S}^{n \times n}$, then the closed-loop system (2.11), (2.12b), (2.13) with parameters

$$K = YX^{-1}, \quad P = X^{-1}, \quad \mu_1 = -\frac{\mu}{1 - (n-2)\mu}, \quad \mu_2 = \mu, \quad \Delta = \sqrt{\lambda_{\min}(X^{1/2} \Lambda_2(\delta) X^{-1} \Lambda_2(\delta) X^{1/2})}$$

is **globally fixed-time stable** with the settling time T_0 given by (2.10).

Note that Propositions 2.1 and 2.2 provide a simple algorithm for tuning the parameters of the nonlinear controller 2.13: given the expressions (2.2) and (2.10), one has to maximize the decay rate ϑ or to minimize the settling time T_0 , respectively, subject to the LMI constraints (2.17).

Furthermore, using Corollary 2.2, we can show (see the proof in Appendix 2.C) that if the proposed nonlinear controller stabilizes the system (2.11), (2.12) hyperexponentially or in fixed time then its linear counterpart (2.14) can be used for the exponential stabilization.

Corollary 2.3. *If Proposition 2.1 (or 2.2) holds, then the closed-loop system (2.11), (2.12), (2.14) with the same feedback gains K is **globally exponentially stable** with the decay rate ϑ given by (2.9), where $\kappa = \varrho\sqrt{2}$ and $\beta = \alpha\gamma/2$.*

2.6 Numerical simulation

Let us illustrate the theoretical results obtained in the previous section by numerically simulating system (2.11), (2.12) governed either by feedback (2.13) or its linear counterpart (2.14). For both cases consider $n = 3$, $a_i = 0.02$, $i = \overline{1,3}$, and a constant initial function x_0 , i.e., $x_0(\tau) = Const$ for any $\tau \in [-h, 0]$. The numerical simulation has been done in MATLAB Simulink by using the explicit Euler method with a state-dependent step [22]. The basic and minimum discretization steps, the maximum number of iterations and the homogeneous norm have been defined as $\Delta t_0 = 10^{-2}$, $\Delta t_{\min} = 10^{-4}$, $N_{\max} = 3 \cdot 10^5$ and $\|x\|_{hom} := (\sum_{i=1}^n |x_i|^{r_1(\tilde{\mu})/r_i(\tilde{\mu})})^{1/r_1(\tilde{\mu})}$, where $r_i(\tilde{\mu}) := (1 - (n - i)\tilde{\mu})$, respectively.

2.6.1 Hyperexponential vs. exponential stabilization

Let us choose parameters $\mu = 0.054$, $\delta = 1.1$, $\eta = 1.79$ and $\alpha = 0.153$ such that LMIs (2.17) are feasible and the decay rate $\vartheta = 0.0083$ [1/s] is maximal for $h = 1$ [s]. Thus, solving LMIs (2.17), we find the following parameters of the controller (2.13):

$$K = \begin{bmatrix} -1.92 & -4.12 & -2.92 \end{bmatrix}, \quad P = \begin{bmatrix} 1.27 & 1.04 & 0.50 \\ 1.04 & 2.32 & 0.69 \\ 0.50 & 0.69 & 0.92 \end{bmatrix}, \quad \mu_1 = -0.057, \quad \mu_2 = 0.05, \quad \Delta = 1.09.$$

Besides, Corollary 2.3 implies that the system (2.11), (2.12a), (2.14) is globally exponentially stable.

The norm of the trajectories $\|x(t)\|_2$ of (2.11), (2.12a) for different initial conditions and time delays are depicted in Fig. 2.1 in the logarithmic scale. Solid and dashed lines correspond to the nonlinear controller (2.13) and its linear counterpart (2.14), respectively. Clearly, in both cases, the hyperexponentially stable system converges faster to the origin than the exponentially stable one. Moreover, the numerical simulation confirms the dependence of the decay rate ϑ on the value of time delay h given by formulas (2.2) and (2.9): ϑ is inversely proportional to h .

2.6.2 Fixed-time vs. exponential stabilization

As it has been highlighted in Remark 2.1, for fixed-time stabilization, condition (2.12a) has to be replaced with (2.12b), where parameter b can be significantly large. To this end, let us set $b = 10^{20}$. We choose parameters $\mu = 0.05$, $\delta = 1.1$, $\eta = 1.91$, $\alpha = 0.161$ and $\rho = 0.011 = \bar{\rho}$ such that LMIs (2.17) are feasible and the settling time $T_0 = 243$ [s] is minimal for $h = 1$ [s]. Therefore, solving LMIs (2.17), we find the following parameters of the controller (2.13):

$$K = \begin{bmatrix} -1.98 & -4.20 & -2.99 \end{bmatrix}, \quad P = \begin{bmatrix} 1.20 & 1.00 & 0.48 \\ 1.00 & 2.19 & 0.66 \\ 0.48 & 0.66 & 0.87 \end{bmatrix}, \quad \mu_1 = -0.053, \quad \mu_2 = 0.05, \quad \Delta = 1.09.$$

Moreover, Corollary 2.3 implies that the system (2.11), (2.12b), (2.14) is globally exponentially stable.

The norm of the trajectories $\|x(t)\|_2$ of (2.11), (2.12b) for different initial conditions are depicted in Fig. 2.2 in the logarithmic scale. One can see that the fixed-time stable system converges faster to the origin than the exponentially stable one. It is worth mentioning that numerical simulation of the system (2.11), (2.12b), (2.13) has shown the same settling time T_0 for different values of time delay h .

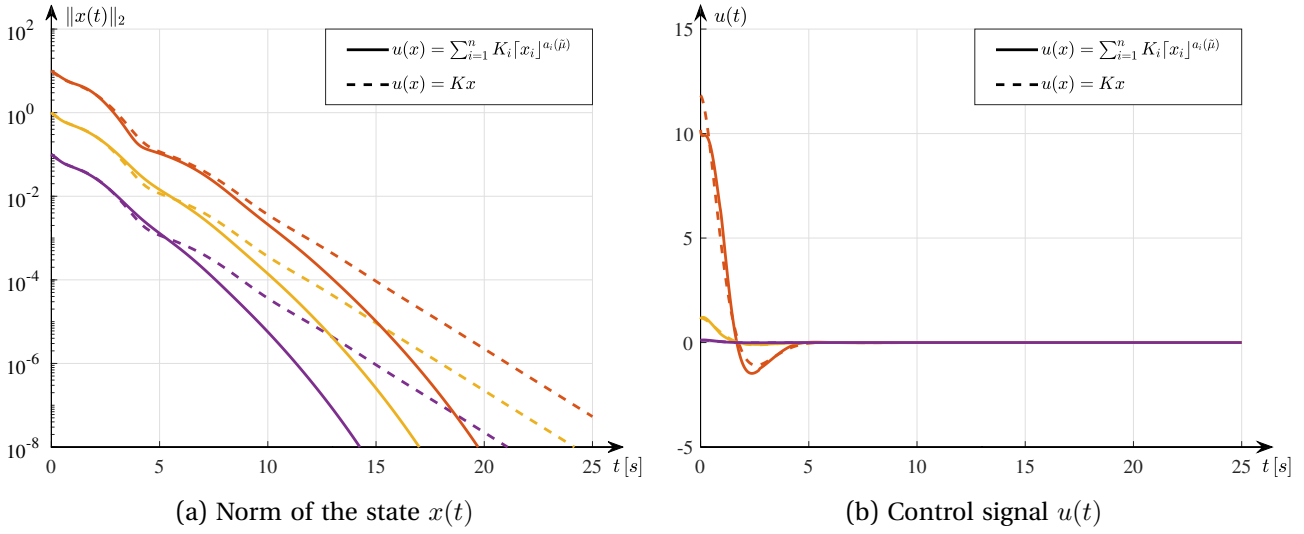
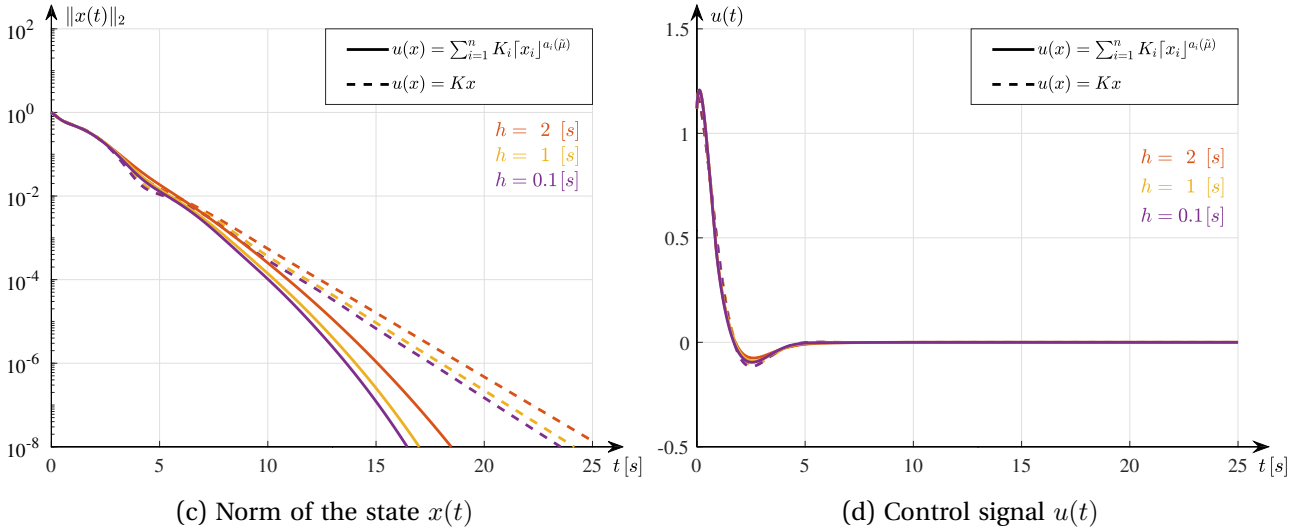
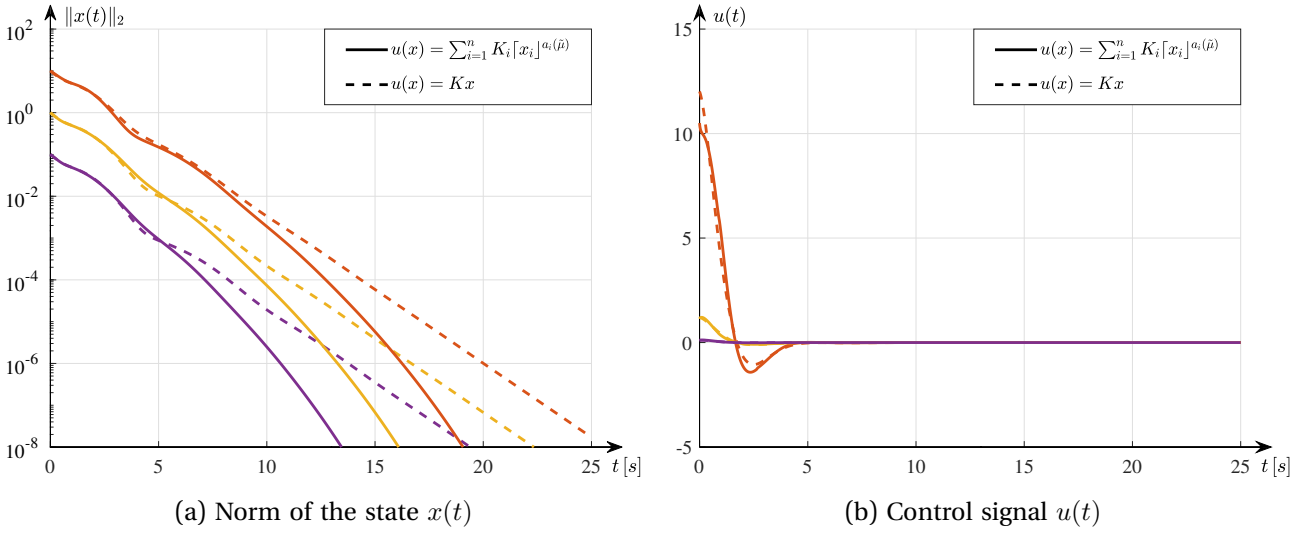
Case 1: Different initial functions ($h = 1$ [s])Case 2: Different time delays ($\|x_0\|_C = 1$)

Figure 2.1: Transients in the closed-loop system (2.11), (2.12a), (2.13)

2.7 Conclusion

In this chapter, a Razumikhin-like method has been proposed for hyperexponential and fixed-time stability analysis of retarded time-delay systems. Differently from the original Lyapunov-Razumikhin method, the proposed approach allows one not only to study the stability of a time-delay system but also to estimate the speed at which trajectories of the system converge to the equilibrium point. However, due to the complexity of formulating Razumikhin-like sufficient conditions for hyperexponential and fixed-time stability by means of a single function, in the proposed method, stability analysis is carried out in two steps using a different Lyapunov-Razumikhin function for each of them. First, it is proven that any trajectory of the system enters a specified closed region centered at the origin in finite time and never leaves it again. Then, the second function is used to show that once the trajectories are within the specified region, they will converge to the origin hyperexponentially or in fixed time, respectively.

Case 1: Different initial functions ($h = 1$ [s])



Case 2: Different time delays ($\|x_0\|_C = 1$)

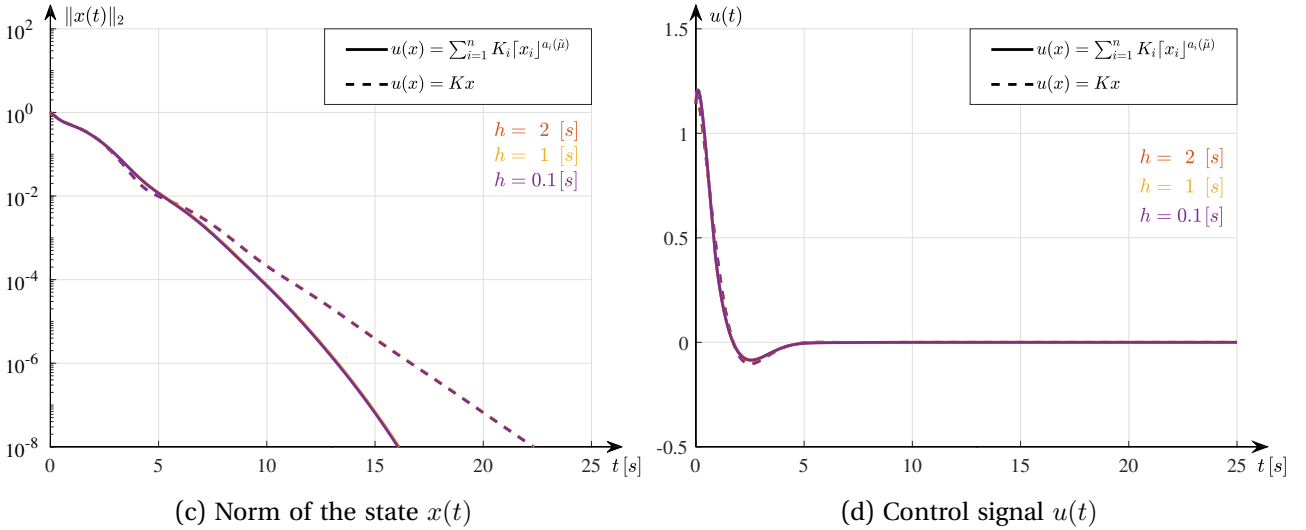


Figure 2.2: Transients in the closed-loop system (2.11), (2.12b), (2.13)

Furthermore, to make the proposed method more suitable for the nonlinear control design, Implicit Lyapunov-Razumikhin theorems have also been formulated. The advantage of the implicit formulation has been illustrated by solving the problems of hyperexponential and fixed-time stabilization of a special subclass of time-delay systems. It has been shown that, under some nonrestrictive assumptions, both problems can be easily solved by using the same nonlinear controller that stabilizes the corresponding delay-free system in fixed time. Applying the developed Implicit Lyapunov-Razumikhin method for stability analysis of the closed-loop system, the tuning of the nonlinear controller parameters, which guarantee superexponential stabilization with the required speed, was reduced to verification of linear matrix inequalities. The obtained theoretical results have been supported by numerical simulation of the designed control system for different initial conditions and time delays. The robustness analysis of the closed-loop system with respect to external disturbances, such as state perturbations and measurement noises, is left for future research.

Appendices

2.A Proof of Proposition 2.1

Let us show that ILRFs (2.15) satisfy all the conditions of Theorem 2.3 with respect to the system (2.11), (2.12a), (2.13).

A.I. Clearly, $Q_1(1, \chi) = Q_2(1, \chi)$ for all $\chi \in \mathbb{R}^n$. It can be shown that the following inequalities

$$q_{1,k}(V_k, \|\chi\|_2) := \frac{\lambda_{\min}(P)\|\chi\|_2^2}{\max\{V_k^{2r_{k,1}}, V_k^{2r_{k,n}}\}} - 1 \leq Q_k(V_k, \chi) \leq \frac{\lambda_{\max}(P)\|\chi\|_2^2}{\min\{V_k^{2r_{k,1}}, V_k^{2r_{k,n}}\}} - 1 := q_{2,k}(V_k, \|\chi\|_2)$$

hold for all $V_k \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n$. Therefore, the function (2.15) satisfies conditions $C_i1)$, $C_i2)$ and $C_i4)$. Moreover, there exist $c_1 = c_2 = \sqrt{\lambda_{\min}(P)}$ such that $q_{1,1}(c_1\|\chi\|_2, \|\chi\|_2) \geq 0$ for all $c_1\|\chi\|_2 \leq 1$ and $q_{1,2}(c_2\|\chi\|_2, \|\chi\|_2) \geq 0$ for all $c_2\|\chi\|_2 > 1$.

A.II. One can see that LMI (2.17a) implies that condition $C_i3)$ holds for all $V_k \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$ since by definition

$$\frac{\partial Q_k(V_k, \chi)}{\partial V_k} := -V_k^{-1}\chi^\top \Lambda_k(V_k^{-1})(G_k P + P G_k)\Lambda_k(V_k^{-1})\chi.$$

A.III. Firstly, let us introduce a new variable $z_k := \Lambda_k(V_k^{-1})x_t(0)$. Secondly, note that $\Lambda_k(V_k^{-1})A_0 = V_k^{\mu_k}A_0\Lambda_k(V_k^{-1})$, $\Lambda_k(V_k^{-1})A_1 = A_1\Lambda_k(V_k^{-1})$ and $\Lambda_k(V_k^{-1})B_0 = V_k^{-1}B_0$. Then adding and subtracting $2V_k^{\mu_k}z_k^\top P B_0 K^\top z_k$, we obtain:

$$F_k(V_k, x_t) = 2V_k \frac{V_k^{\mu_k}z_k^\top P(A_0 + B_0 K)z_k + z_k^\top P A_1 \zeta_{h,k} + V_k^{\mu_k}z_k^\top P B_0 K \zeta_{\mu,k}}{z_k^\top (G_k P + P G_k)z_k},$$

where $\zeta_{h,k} := \Lambda_k(V_k^{-1})\xi(x_t)$ and $\zeta_{\mu,k} := V_k^{-1-\mu_k} \text{col}\{[x_{t,i}(0)]^{a_i(\bar{\mu}(\|x_t(0)\|_P))}\}_{i=1}^n - z_k$.

Denote $\varsigma_k := \text{col}\{z_k, V_k^{-\mu_k}\zeta_{h,k}/\varrho, \zeta_{\mu,k}/\|\zeta\|_2\}$. Adding and subtracting the terms $2\alpha\gamma V_k^{\mu_k}\|z_k\|_P^2$, $\frac{\alpha\gamma}{\varrho^2}V_k^{-\mu_k}\|\zeta_{h,k}\|_S^2$ and $\frac{\alpha^2\gamma}{\|\zeta\|_2^2}V_k^{\mu_k}\|\zeta_{\mu,k}\|_P^2$, we get:

$$F_k(V_k, x_t) = V_k^{1+\mu_k} \frac{\varsigma_k^\top \Psi \varsigma_k - 2\alpha\gamma\|z_k\|_P^2 + \frac{\alpha\gamma}{\varrho^2}V_k^{-2\mu_k}\|\zeta_{h,k}\|_S^2 + \frac{\alpha^2\gamma}{\|\zeta\|_2^2}\|\zeta_{\mu,k}\|_P^2}{z_k^\top (G_k P + P G_k)z_k},$$

where matrix Ψ is defined as follows:

$$\Psi := \begin{bmatrix} P(R - \frac{\varrho^2}{\alpha\gamma}A_1 Z A_1^\top)P & \varrho P A_1 & \|\zeta\|_2 P B_0 K \\ \varrho A_1^\top P & -\alpha\gamma S & O_{n \times n} \\ \|\zeta\|_2 K^\top B_0^\top P & O_{n \times n} & -\alpha^2\gamma P \end{bmatrix}.$$

Let $Z = S^{-1}$, then $\Psi \preceq 0$ due to (2.17b) and the Schur complement (see Lemma A.4 in Appendix A). Moreover, LMIs (2.17c) imply that $2\eta\|\cdot\|_S^2 \leq \|\cdot\|_2^2$ and $2\alpha\|\cdot\|_P^2 \leq \|\cdot\|_2^2$. Hence,

$$F_k(V_k, x_t) \leq -V_k^{1+\mu_k} \frac{\alpha\gamma + \frac{\alpha\gamma}{2\eta\varrho^2}V_k^{-2\mu_k}(\eta\varrho^2 V_k^{2\mu_k} - \|\zeta_{h,k}\|_2^2) + \frac{\alpha\gamma}{2\|\zeta\|_2^2}(\|\zeta\|_2^2 - \|\zeta_{\mu,k}\|_2^2)}{z_k^\top (G_k P + P G_k)z_k}.$$

Taking into account that (2.17a) implies $z_k^\top (G_k P + P G_k)z_k \leq \gamma$ for all $(V_k, x_t) \in \Omega_k$, one can see that

$$F_k(V_k, x_t) \leq -\alpha V_k^{1+\mu_k} \leq -\alpha\mu_k \ln(e^{1/\mu_k} V_k) V_k$$

if $\|\zeta_{h,k}\|_2 \leq \sqrt{\eta}\varrho V_k^{\mu_k}$ and $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$ for all $(V_k, x_t) \in \Omega_k$. Finally, since $\mu_2 \leq |\mu_1| < 1$, then $e^{1/\mu_2} > e$ and $e^{1/\mu_1} < e^{-1}$, hence, condition $C_i7)$ holds with $\beta = \alpha\mu_2$.

A.IV. Proof of $\|\zeta_{h,k}\|_2 \leq \sqrt{\eta}\varrho V_k^{\mu_k}$ in Ω_k

For all $(V_k, x_t) \in \Omega_k$ we have:

$$\begin{aligned}\sqrt{\eta} &\geq \|\Lambda_1(V_1^{-1/\nu} e^{1/\nu-1})x_t(-h)\|_2 \geq (V_1/e)^{\frac{r_{1,1}(\nu-1)}{\nu}} \|z_1\|_2, \\ \sqrt{\eta} &\geq \|\Lambda_2(V_2^{-\nu} e^{1-\nu})x_t(-h)\|_2 \geq (V_2e)^{1-\nu} \|z_2\|_2.\end{aligned}$$

Taking into account that $r_{1,1}\mu_2 = -\mu_1(1 + \mu_2)$ and $|\mu_1| \geq \mu_2$, we deduce that the estimate indeed holds with $\nu = 1 + \mu$.

A.V. Proof of $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$ in Ω_k

Adding and subtracting the term $V_k^{r_{k,i}a_i(\tilde{\mu})-1-\mu_k} z_{k,i}$ to $\zeta_{\mu,k,i}$, we get

$$|\zeta_{\mu,k,i}| \leq V_k^{r_{k,i}a_i(\tilde{\mu})-1-\mu_k} \left| |z_{k,i}|^{a_i(\tilde{\mu})} - |z_{k,i}| \right| + |z_{k,i}| \left| V_k^{r_{k,i}a_i(\tilde{\mu})-1-\mu_k} - 1 \right|$$

For the following analysis let us show that $V_1 \leq 1$ and $V_2 \geq \delta$ imply $\|x\|_P \leq 1$ and $\|x\|_P \geq \Delta$, respectively. Indeed, if $V_1 \leq 1$, then $Q_1(V_1, x) = 0 \geq Q_1(1, x) = x^\top \Lambda_1(1)P\Lambda_1(1)x - 1 = x^\top Px - 1$. Similarly, since $x^\top Px = x^\top \Lambda_2(\delta^{-1})P^{1/2}[P^{-1/2}\Lambda_2(\delta)P\Lambda_2(\delta)P^{-1/2}]P^{1/2}\Lambda_2(\delta^{-1})x \geq \Delta^2 x^\top \Lambda_2(\delta^{-1})P\Lambda_2(\delta^{-1})x$, for all $V_2 \geq \delta$ we have $Q_2(V_2, x) = 0 \leq Q_2(\delta, x) = x^\top \Lambda_2(\delta^{-1})P\Lambda_2(\delta^{-1})x - 1 \leq \Delta^{-2}x^\top Px - 1$.

Now, applying Lemma A.2 given in Appendix A with $s = |z_{k,i}|$, $\bar{s} = \sqrt{\eta}$ (since $|z_{k,i}| \leq \sqrt{\eta}$) and $\epsilon = a_i(\tilde{\mu})$, the term $\left| |z_{k,i}|^{a_i(\tilde{\mu})} - |z_{k,i}| \right|$ can be bounded as follows:

$$\left| |z_{k,i}|^{a_i(\tilde{\mu})} - |z_{k,i}| \right| \leq \max \left\{ \bar{g} \left(\sqrt{\eta}, \frac{1 + \mu_1}{r_{1,i}} \right), \bar{g} \left(\sqrt{\eta}, \frac{1 + \mu_2}{r_{2,i}} \right) \right\}.$$

Secondly, it is clear that $V_k^{r_{k,i}a_i(\tilde{\mu})-1-\mu_k} = 1$ for all $V_1 \leq 1$ and $V_2 \geq \delta$. On the other hand, it is easy to show that $r_{2,i}a_i(\tilde{\mu}) - 1 - \mu_2 \leq 0$ for all $i = \overline{1, n}$. Moreover, since $\partial(r_{2,i}a_i(\tilde{\mu}) - 1 - \mu_2)/\partial\tilde{\mu} > 0$, then $\min_{\tilde{\mu} \in [\mu_1, \mu_2]} (r_{2,i}a_i(\tilde{\mu}) - 1 - \mu_2) = r_{2,i}a_i(\mu_1) - 1 - \mu_2 = p_i$. Therefore, we have $\delta^{p_i} \leq V_2^{r_{2,i}a_i(\tilde{\mu})-1-\mu_2} \leq 1$ and, finally, $|\zeta_{\mu,k,i}| \leq \zeta_i$. \square

2.B Proof of Proposition 2.2

Repeating the steps given in the proof of Proposition 2.1, one can see that conditions $C_i1)$ - $C_i4)$ and $C_i6)$ are satisfied and, furthermore, condition $C_i7)$ holds if $\|\zeta_{h,k}\|_2 \leq \sqrt{\eta}(1 - \rho)^{-(1+\mu)/\mu} V_k^{\mu_k}$ and $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$ for all $(V_k, x_t) \in \Omega_k^*$. Since the latter is fulfilled (see the proof of Proposition 2.1), then we have to prove that the former holds. Recall that $|\zeta_{h,k,i}| \leq V_k^{-r_{k,i}} |\text{sat}(x_{t,i}(0), 1)| |\text{sat}(x_{t,i}(-h), b)|$.

For all $(V_1, x_t) \in \Omega_1^*$ we have:

$$\begin{aligned}\eta &\geq \|\Lambda_1(V_1^{-1})x_t(0)\|_2^2 \geq V_1^{-2r_{1,i}} x_{t,i}^2(0), \\ \eta &\geq \|\Lambda_1((V_1^{-\mu_1} + \rho)^{1/\mu_1})x_t(-h)\|_2^2 = \sum_{i=1}^n (V_1^{-\mu_1} + \rho)^{2r_{1,i}/\mu_1} x_{t,i}^2(-h).\end{aligned}$$

Taking into account that $|\text{sat}(x_{t,i}(0), 1)| = \min\{|x_{t,i}(0)|, 1\}$ and $|\text{sat}(x_{t,i}(-h), b)| \leq |x_{t,i}(-h)|$, we need to show that the following estimate holds for all $V_1 \in (0, 1]$ and $i = \overline{1, n}$:

$$V_1^{-r_{1,i}} \min\{\sqrt{\eta}V_1^{r_{1,i}}, 1\} \leq (1 - \rho)^{-(1+\mu)/\mu} (V_1^{-\mu_1} + \rho)^{r_{1,i}/\mu_1} V_1^{\mu_1}. \quad (2.18)$$

Since $V_1^{-r_{1,i}} \min\{\sqrt{\eta}V_1^{r_{1,i}}, 1\} \leq \sqrt{\eta}$ for all $V_1 \in (0, \bar{V}_1]$, where $\bar{V}_1 := \min\{1, \sqrt{\eta}^{-1/r_{1,1}}\}$, and the function $V_1 \mapsto (V_1^{-\mu_1} + \rho)^{r_{1,i}/\mu_1} V_1^{\mu_1}$ is strictly decreasing, then we can consider only the interval $V_1 \in [\bar{V}_1, 1]$. In this case, inequality (2.18) is rewritten as follows:

$$(1 + \rho V_1^{\mu_1})^{-r_{1,i}/\mu_1} \leq (1 - \rho)^{-(1+\mu)/\mu} V_1^{\mu_1}. \quad (2.19)$$

Note that the function $V_1^{\mu_1} \mapsto (1 + \rho V_1^{\mu_1})^{-r_{1,i}/\mu_1}$ is convex and strictly increasing for all $V_1 \in [\bar{V}_1, 1]$. Thus, it suffices to check inequality (2.19) only at the boundary points, i.e., $(1 + \rho \bar{V}_1^{\mu_1})^{-r_{1,i}/\mu_1} \leq (1 - \rho)^{-(1+\mu)/\mu} \bar{V}_1^{\mu_1}$ and $(1 + \rho)^{-r_{1,i}/\mu_1} \leq (1 - \rho)^{-(1+\mu)/\mu}$. Taking into account that $r_{1,i} \leq r_{1,1} = -\mu_1 \frac{1+\mu}{\mu}$ for all $i = \overline{1, n}$, one can verify that both inequalities hold for all $\rho > 0$ such that $\rho(\bar{V}_1^{\mu_1} - 1) \leq (\bar{V}_1^{-\mu_1^{2/r_{1,1}}} - 1)$.

For $k = 2$, we consider two intervals: $V_2 \in (1, \bar{V}_2)$ and $V_2 \geq \bar{V}_2$, where $\bar{V}_2 := \max\{1, (b\sqrt{n/\eta})^{1/r_{2,2}}\}$. Since $|\text{sat}(x_{t,i}(0), 1)| \leq 1$ and $|\text{sat}(x_{t,i}(-h), b)| \leq b$, then for all $V_2 \geq \bar{V}_2$ we have to show that $V_2^{-r_{2,i}} b \leq \sqrt{\eta/n}(1 - \rho)^{-(1+\mu)/\mu} V_2^{\mu_2}$. Clearly, it suffices to check the inequality only for $V_2 = \bar{V}_2$, i.e., $b\sqrt{n/\eta} \leq (1 - \rho)^{-(1+\mu)/\mu} \bar{V}_2^{r_{2,i} + \mu_2}$. Taking into account that $\bar{V}_2^{r_{2,i} + \mu_2} \geq \bar{V}_2^{r_{1,2} + \mu_2} = b\sqrt{n/\eta}$ for all $i = \overline{1, n}$ and that $(1 - \rho)^{-(1+\mu)/\mu} > 1$, one can see that the inequality indeed holds.

On the other hand, for all $(V_2, x_t) \in \Omega_2^*$ such that $V_2 \in (1, \bar{V}_2)$ we have:

$$\eta \geq \|\Lambda_2((V_2^{-\mu_2} - \rho)^{1/\mu_2} x_t(-h))\|_2^2 = \sum_{i=1}^n V_2^{-2r_{2,i}} (1 - \rho V_2^{\mu_2})^{2r_{2,i}/\mu_2} x_{t,i}^2(-h).$$

Since $|\text{sat}(x_{t,i}(-h), b)| \leq |x_{t,i}(-h)|$, we need to prove that the following estimate holds for all $V_2 \in (1, \bar{V}_2)$ and $i = \overline{1, n}$:

$$(1 - \rho V_2^{\mu_2})^{-r_{2,i}/\mu_2} \leq (1 - \rho)^{-(1+\mu)/\mu} V_2^{\mu_2}. \quad (2.20)$$

Note that the function $V_2^{\mu_2} \mapsto (1 - \rho V_2^{\mu_2})^{-r_{2,i}/\mu_2}$ is convex and strictly increasing for all $V_2 \in (1, \bar{V}_2]$. Thus, it suffices to check inequality (2.20) only at the boundary points, i.e., $(1 - \rho)^{-r_{2,i}/\mu_2} \leq (1 - \rho)^{-(1+\mu)/\mu}$ and $(1 - \rho \bar{V}_2^{\mu_2})^{-r_{2,i}/\mu_2} \leq (1 - \rho)^{-(1+\mu)/\mu} \bar{V}_2^{\mu_2}$. Taking into account that $r_{2,i} \leq r_{n,2} = 1 = \mu_2/\mu$ for all $i = \overline{1, n}$ and that $(1 - \rho)^{-1} > 1$, it is not difficult to verify that both inequalities hold for all $\rho > 0$ such that $\rho(\bar{V}_2^{\mu_2} - \bar{V}_2^{-\mu_2^2}) \leq (1 - \bar{V}_2^{-\mu_2^2})$. \square

2.C Proof of Corollary 2.3

Introduce an ILRF in the form $Q_0(V_0, \chi) := V_0^{-2} \chi^\top P \chi - 1$. One can see that for such an ILRF conditions $C_i 1) - C_i 3)$ and $C_i^* 5)$ hold with $q_{1,0}(V_0, \|\chi\|_2) = V_0^{-2} \lambda_{\min}(P) \|\chi\|_2^2 - 1$, $q_{2,0}(V_0, \|\chi\|_2) = V_0^{-2} \lambda_{\max}(P) \|\chi\|_2^2 - 1$ and $c_0 = \sqrt{\lambda_{\min}(P)}$. Moreover, $\partial Q_0(V_0, \chi)/\partial V_0 := -2V_0^{-3} \chi^\top P \chi < 0$ for all $V_0 \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$. Taking into account that for (2.14) the term $\zeta_{\mu,k}$ equals to zero (since $\mu = 0$), it follows from the proof of Proposition 2.1 that

$$F_0(V_0, x_t) \leq V_0 \frac{-2\alpha\gamma + \frac{\alpha\gamma}{\varrho^2} V_0^{-2} \|\xi(x_t)\|_S^2}{2V_0^{-2} x_t^\top(0) P x_t(0)} = -V_0 \frac{\alpha\gamma + \frac{\alpha\gamma}{\varrho^2} (\varrho^2 - V_0^{-2} \|\xi(x_t)\|_S^2)}{2} \leq -\beta V_0,$$

where $\beta = \alpha\gamma/2$, if $\|\xi(x_t)\|_S \leq \varrho V_0$ for all $(V_0, x_t) \in \Omega_0$. Indeed, by definition of the set Ω_0 we have $Q_0(\kappa V_0, x_t(-h)) \leq 0 \Leftrightarrow \|x_t(-h)\|_P \leq \kappa V_0$. And since (2.17c) implies $\sqrt{2\eta} \|\xi(x_t)\|_S \leq \|\xi(x_t)\|_2 \leq \|x_t(-h)\|_2 \leq \sqrt{\eta} \|x_t(-h)\|_P$, we conclude the proof with $\kappa = \varrho\sqrt{2} > 1$. \square

CHAPTER

3

PRACTICAL FIXED-TIME INPUT-TO-STATE STABILITY OF NEUTRAL TIME-DELAY SYSTEMS: IMPLICIT LYAPUNOV-KRASOVSKII METHOD

In this chapter, the notion of practical fixed-time input-to-state stability is introduced for neutral time-delay systems with external bounded disturbances and characterized by the Lyapunov-Krasovskii method, which is formulated both explicitly and implicitly. Based on the obtained theoretical results, an alternative way of robust output practical fixed-time stabilization of linear systems in the controllable canonical form is proposed. Differently from the observer-based approaches, the state vector is approximated by means of the finite difference method, i.e., based on the past values of the output signal. As a result, due to the special integral relation between the state and its finite-difference approximation, the closed-loop system has a neutral time-delay representation. Applying the developed Implicit Lyapunov-Krasovskii method, sufficient stability conditions for the designed nonlinear control system are presented in the form of linear matrix inequalities, solutions of which are used for the calculation of the controller parameters. Furthermore, the impact of the artificially induced time delay on the stabilization accuracy is also quantitatively studied. Finally, it is theoretically proven and numerically illustrated that, both in the disturbance-free and disturbed cases, the proposed nonlinear controller stabilizes the considered system in the vicinity of the origin much faster than its linear counterpart.

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3.1 Introduction

Stabilization of dynamical systems with a faster than exponential rate of convergence has become one of the main trends in modern control theory [48, 62]. Frequently, such an approach allows systems to be stabilized at the origin in a finite time. For example, for homogeneous autonomous systems, a special class of nonlinear ones, the type of convergence is defined by their degree of homogeneity [3]. For perturbed systems this concept can be extended to non-asymptotic ISS [35] when the steady-state error is upper bounded by the norm of external disturbance. In [4] robustness of homogeneous systems with respect to bounded exogenous disturbances was studied.

However, finite-time stabilization is hard to obtain for time-delay systems [23, 52]. For instance, to ensure such a property the delays have to diminish proportionally to the norm of the state vector and vanish at the origin, or time-delay terms have to be multiplied by the instantaneous state vector. But in many applications it is sufficient to stabilize a system in finite time only in the vicinity of the origin, the radius of which depends on the time delay and external disturbances, and following [20] such a problem is investigated in this work. In [20] the homogeneity theory was extended to neutral type systems and it was shown how the convergence can be accelerated by selecting a non-zero degree of homogeneity. Nevertheless, it is worth mentioning that for linear systems any stable set is reachable in a finite time also and the settling time can be reduced by feedback gains increasing. But differently from the delay-free case, this approach has limited use for time-delay systems: for any given delay h sufficiently large gains make the closed-loop system unstable, which motivated [20].

Stability analysis of time-delay systems could be done by using different methods [26, 30, 33, 42]. For example, similar to linear time-invariant systems, one may check if the characteristic polynomial is Hurwitz or not [41]. However, such an approach is difficult to use for the synthesis of control systems with delays since the polynomials are transcendental in this case. Another conventional tools are Lyapunov-Krasovskii [45] or Lyapunov-Razumikhin [67] methods. They impose restrictions on the time derivative of an auxiliary functional or function, respectively, with respect to the differential equation of the system. Being well-developed for analysis, both of them do not provide a constructive way for control design in the nonlinear case. On the contrary, their implicit extensions are free of such a drawback: all stability conditions can be checked directly by analyzing some algebraic equations, which implicitly define Lyapunov functionals (functions) [64]. Moreover, the controller parameters can be obtained by solving a system of LMIs.

The goal of the work, presented in this chapter, is to extend the exponential ISS concept for neutral time-delay systems to its fixed-time analog. Both, Lyapunov-Krasovskii theorem and its implicit counterpart, are introduced. Then the proposed approach is applied for static nonlinear output-feedback stabilization of a delay-free linear system in the controllable canonical form with parametric uncertainties, bounded state perturbations and measurement noises. To this end, the unmeasured states are approximated by the finite differences method [10, 27, 28], i.e., using the past values of the output signal. In [70] it was shown that in this case closed-loop system has a neutral time-delay representation. Moreover, since no observers/predictors are introduced, the control law is static, which essentially simplifies its practical implementation. Differently from [20], in this work

- 1) the homogeneity is not used to prove superexponential stability,
- 2) the designed control system is practically *fixed-time* stable,
- 3) feedback gains are explicitly calculated.

3.2 Input-to-state stability of neutral time-delay systems

Consider a functional differential equation of neutral type with external disturbance:

$$\begin{cases} \dot{x}(t) = f(x_t, \dot{x}_t, d(t)), & t > 0, \\ x(\tau) = x_0(\tau), & \tau \in [-h, 0], \end{cases} \quad (3.1)$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state; $x_t \in W([-h, 0], \mathbb{R}^n)$ is the functional state defined for any time delay $h > 0$ as $x_t(\tau) := x(t + \tau)$ with $\tau \in [-h, 0]$, $\dot{x}_t \in L^2([-h, 0], \mathbb{R}^n)$; $x_0 \in W([-h, 0], \mathbb{R}^n)$ is the initial value function; $d(t) \in \mathbb{R}^m$ is the external disturbance, $d \in L_2^\infty(\mathbb{R}_+, \mathbb{R}^m)$. A continuous operator $f : W([-h, 0], \mathbb{R}^n) \times L^2([-h, 0], \mathbb{R}^n) \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is Lipschitz in the second variable with a constant smaller than one, ensuring forward uniqueness and existence of the system solutions at least locally in time [43]. Assume that the origin is an equilibrium point of the system (3.1), i.e., $f(0, 0, 0) = 0$. A solution of the system (3.1) is denoted by $x(t, x_0, d) \in \mathbb{R}^n$ or $x_t(x_0, d) \in W([-h, 0], \mathbb{R}^n)$.

Following [35], we present the concept of the practical fixed-time ISS stability of neutral time-delay systems with external inputs.

Definition 3.1. *The system (3.1) is called practically locally fixed-time ISS, if there exist a constant $\varrho_0 \geq 0$ and functions $w \in \mathcal{K}$, $v \in \mathcal{GKL}$ with the settling time $T_0 := \sup_{s < \varrho_x} T(s) < +\infty$ such that:*

$$\|x(t, x_0, d)\|_2 \leq \varrho_0 + v(\|x_0\|_W, t) + w(\|d\|_{L_2^\infty}), \quad \forall t \geq 0, \quad (3.2)$$

for all $x_0 \in \mathcal{B}_W(\varrho_x)$ and $d \in \mathcal{B}_{L_2^\infty}(\varrho_d)$.

If $\varrho_0 = 0$, then system (3.1) is called locally fixed-time ISS. If additionally $\varrho_x = \varrho_d = +\infty$, then system (3.1) is called fixed-time ISS.

The following theorem provides sufficient conditions to check practical local fixed-time ISS property (3.2) by using Lyapunov-Krasovskii functionals.

Theorem 3.1. *If there exist two continuous functionals $V_k : W([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}_+$, $k = 1, 2$, such that:*

C_e1) V_k are continuously Fréchet differentiable⁴ outside the origin;

C_e2) for some $\sigma_{1,k}, \sigma_{2,k} \in \mathcal{K}_\infty$ and all $\chi \in \mathbb{R}^n$

$$\sigma_{1,k}(\|\chi(0)\|_2) \leq V_k(\chi) \leq \sigma_{2,k}(\|\chi\|_W);$$

C_e3) $V_1(\chi) \leq 1$ for all $\chi \in W([-h, 0], \mathbb{R}^n)$ such that $V_2(\chi) \leq 1$;

C_e4) for some $\bar{\varrho}_0 \in [0, 1)$, $\bar{\varrho}_x > 1$, $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\alpha > 0$, $\bar{w} \in \mathcal{K}$ and all $x_t \in W([-h, 0], \mathbb{R}^n)$ satisfying (3.1) we have:

$$(a) \quad \max\{1, \bar{w}(\|d\|_{L_2^\infty})\} < V_2(x_t) < \bar{\varrho}_x \implies \dot{V}_2(x_t) \leq -\alpha V_2^{1+\mu_2}(x_t),$$

$$(b) \quad \max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\} < V_1(x_t) \leq 1 \implies \dot{V}_1(x_t) \leq -\alpha V_1^{1+\mu_1}(x_t),$$

⁴The definition of the Fréchet derivative can be found in Appendix B

where $\dot{V}_k(x_t) := DV_k(x_t)\dot{x}_t$, then the system (3.1) is practically locally fixed-time ISS (3.2) with ϱ_0 , ϱ_x , ϱ_d , $w(s)$ and $v(s, t)$ given by

$$\begin{aligned} \varrho_0 &= \tilde{\sigma}_{1,1}(\bar{\varrho}_0), \quad \varrho_x = \tilde{\sigma}_{2,2}(\bar{\varrho}_x), \quad \varrho_d = \tilde{w}(\bar{\varrho}_x), \quad w(s) = \begin{cases} \tilde{\sigma}_{1,1}(\bar{w}(s)), & \text{if } \bar{w}(s) < 1, \\ \tilde{\sigma}_{1,2}(\bar{w}(s)), & \text{if } \bar{w}(s) \geq 1, \end{cases} \\ v(s, t) &= \begin{cases} \tilde{\sigma}_{1,2}((\mu_2\alpha(t - T_2(s)) + 1)^{-1/\mu_2}), & t \in [0, T_2(s)), \\ \tilde{\sigma}_{1,1}((\mu_1\alpha(t - T_2(s) - T_1(s)))^{-1/\mu_1}), & t \in [T_2(s), T_2(s) + T_1(s)), \\ 0, & t \geq T_2(s) + T_1(s), \end{cases} \end{aligned} \quad (3.3)$$

where functions \tilde{w} , $\tilde{\sigma}_{1,1}$, $\tilde{\sigma}_{1,2}$ and $\tilde{\sigma}_{2,2}$ are inverse of \bar{w} , $\sigma_{1,1}$, $\sigma_{1,2}$ and $\sigma_{2,2}$, respectively, and

$$T_1(s) := \max \left\{ 0, \frac{\min\{1, \sigma_{2,1}^{-\mu_1}(s)\}}{|\mu_1|\alpha} \right\} \leq \frac{1}{|\mu_1|\alpha}, \quad T_2(s) := \max \left\{ 0, \frac{1 - \sigma_{2,2}^{-\mu_2}(s)}{\mu_2\alpha} \right\} \leq \frac{1}{\mu_2\alpha}.$$

Proof. If $\bar{w}(\|d\|_{L_2^\infty}) < 1$, then applying the Comparison Lemma A.1 (see Appendix A) to the function $\bar{V}_2(t) := V_2(x_t)$ from C_e4a) on interval $t \in [0, T_2)$, where $T_2 = \inf\{t \geq 0 : \bar{V}_2(t) \leq 1\}$, we get $\bar{V}_2(t) \leq (\mu_2\alpha t + \bar{V}_2^{-\mu_2}(0))^{-1/\mu_2}$. Obviously, $T_2 \leq (1 - \bar{V}_2^{-\mu_2}(0))/(\mu_2\alpha)$. Hence, if $\sigma_{2,2}(\|x_0\|_W) \leq 1$, then C_e2) implies $\bar{V}_2(t) \leq 1$ and $T_2 = 0$. Otherwise, $\|x(t)\|_2 \leq \tilde{\sigma}_{1,2}(\bar{V}_2(t))$ for $t \in [0, T_2)$ and $\bar{V}_2(0) \leq \sigma_{2,2}(\|x_0\|_W)$ due to C_e2). On the other hand, if $\bar{w}(\|d\|_{L_2^\infty}) \geq 1$, then there exists a moment of time $T_2' \in [0, T_2)$ such that $\bar{V}_2(t) \leq \bar{w}(\|d\|_{L_2^\infty})$ for $t \geq T_2'$. Thus, one can conclude that $\|x(t)\|_2 \leq v_2(\|x_0\|_W, t) + w(\|d\|_{L_2^\infty})$ for all $t \in [0, T_2)$. Moreover, $\bar{V}_2(0) < \bar{\varrho}_x$ if $\|x_0\|_W < \tilde{\sigma}_{2,2}(\bar{\varrho}_x)$.

If $\bar{w}(\|d\|_{L_2^\infty}) < 1$, then condition C_e3) implies $\bar{V}_1(t) := V_1(x_t) \leq 1$ for $t \geq T_2$. Assume first that $\max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\} = 0$. Applying the Comparison Lemma A.1 to the function $\bar{V}_1(t)$ from C_e4b) on interval $t \in [T_2, T_2 + T_1)$, where $T_2 + T_1 = \inf\{t \geq 0 : \bar{V}_1(t) = 0\}$, we get $\bar{V}_1(t) \leq (\mu_1\alpha(t - T_2) + \bar{V}_1^{-\mu_1}(T_2))^{-1/\mu_1}$. It is clear that $T_1 \leq \bar{V}_1^{-\mu_1}(T_2)/(-\mu_1\alpha)$, where $\bar{V}_1(T_2) \leq 1$ if $T_2 > 0$ or $\bar{V}_1(T_2) \leq \sigma_{2,1}(\|x_0\|_W)$ if $T_2 = 0$. Hence, $\|x(t)\|_2 \leq \tilde{\sigma}_{1,1}(\bar{V}_1(t)) \leq v_1(\|x_0\|_W, t)$ for $t \in [T_2, T_2 + T_1)$ and $\|x(t)\|_2 = 0$ for $t \geq T_2 + T_1$ due to C_e2). Now assume that $0 < \max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\} < 1$. Then there exists a moment of time $T_1' \in [T_2, T_2 + T_1)$ such that $\bar{V}_1(t) \leq \max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\}$ for $t \geq T_1'$. Thus, $\|x(t)\|_2 \leq v_1(\|x_0\|_W, t) + \varrho_0 + w(\|d\|_{L_2^\infty})$ for all $t \geq T_2$. \square

One can see that conditions C_e4a) and C_e4b) in general are hard to check, especially in a control design scenario. As it has been shown in [64], this problem can be overcome by defining functionals V_k in Theorem 3.1 implicitly.

Theorem 3.2. *If there exist two continuous functionals $Q_k : \mathbb{R}_+^* \times W([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$, $k = 1, 2$ such that:*

C_i1) $Q_k(V_k, \chi)$ are continuously Fréchet differentiable on $\mathbb{R}_+^* \times W([-h, 0], \mathbb{R}^n)$;

C_i2) for any $\chi \in W([-h, 0], \mathbb{R}^n)$ there exists $V_k \in \mathbb{R}_+^*$ such that $Q_k(V_k, \chi) = 0$;

C_i3) $\frac{\partial Q_k(V_k, \chi)}{\partial V_k} < 0$ for all $V_k \in \mathbb{R}_+^*$ and $\chi \in W([-h, 0], \mathbb{R}^n) \setminus \{0\}$;

C_i4) for some $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$ and all $V_k \in \mathbb{R}_+^*$

$$\begin{aligned} q_{1,k}(V_k, \|\chi(0)\|_2) &\leq Q_k(V_k, \chi), \quad \forall \chi \in W([-h, 0], \mathbb{R}^n) \setminus W_0([-h, 0], \mathbb{R}^n), \\ Q_k(V_k, \chi) &\leq q_{2,k}(V_k, \|\chi\|_W), \quad \forall \chi \in W([-h, 0], \mathbb{R}^n) \setminus \{0\}; \end{aligned}$$

$C_i5)$ $Q_1(1, \chi) = Q_2(1, \chi)$ for all $\chi \in W([-h, 0], \mathbb{R}^n)$;

$C_i6)$ for some $\bar{\varrho}_0 \in [0, 1)$, $\bar{\varrho}_x > 1$, $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\alpha > 0$, $\bar{w} \in \mathcal{K}$ and all $x_t \in W([-h, 0], \mathbb{R}^n)$ satisfying (3.1) we have:

$$(a) \quad (V_2, x_t) \in \Omega_2 := \{(s, \phi) \in (\max\{1, \bar{w}(\|d\|_{L_2^\infty})\}, \bar{\varrho}_x) \times W([-h, 0], \mathbb{R}^n) : Q_2(s, \phi) = 0\} \\ \implies F_2(V_2, x_t) \leq -\alpha V_2^{1+\mu_2},$$

$$(b) \quad (V_1, x_t) \in \Omega_1 := \{(s, \phi) \in (\max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\}, 1] \times W([-h, 0], \mathbb{R}^n) : Q_1(s, \phi) = 0\} \\ \implies F_1(V_1, x_t) \leq -\alpha V_1^{1+\mu_1},$$

where $F_k(V_k, x_t) := -D_{V_k}^{-1}Q_k(V_k, x_t)D_\chi Q_k(V_k, x_t)\dot{x}_t$, then the system (3.1) is practically locally fixed-time ISS (3.2) with $\varrho_0, \varrho_x, \varrho_d, w(s)$ and $v(s, t)$ given by (3.3), where functions $\sigma_{i,k}(s)$ implicitly defined by equations $q_{i,k}(\sigma_{i,k}(s), s) = 0$, respectively, $i, k = 1, 2$.

Proof. For the sake of brevity, denote the space $W([-h, 0], \mathbb{R}^n)$ simply as W . In order to prove the theorem it is sufficient to show that there exist functionals $V_k : W \rightarrow \mathbb{R}_+$, satisfying conditions of Theorem 3.1. Indeed, $C_i2)$ and $C_i3)$ guarantee existence of unique functionals $V_k : W \setminus \{0\} \rightarrow \mathbb{R}_+^*$ such that $Q_k(V_k(\chi), \chi) = 0$ for any $\chi \in W \setminus \{0\}$. Moreover, the Implicit Function Theorem B.2 (see Appendix B) and condition $C_i1)$ guarantee that functionals V_k are continuously Fréchet differentiable on $W \setminus \{0\}$ and $DV_k(\chi) = -D_{V_k}^{-1}Q_k(V_k, \chi) \circ D_\chi Q_k(V_k, \chi)$.

From $C_i4)$ it follows that $q_{1,k}(V_k(\chi), \|\chi(0)\|_2) \leq Q_k(V_k(\chi), \chi) = 0 = q_{1,k}(\sigma_{1,k}(\|\chi(0)\|_2), \|\chi(0)\|_2)$ for all $\chi \in W \setminus W_0$ and $q_{2,k}(\sigma_{2,k}(\|\chi\|_W), \|\chi\|_W) = 0 = Q_k(V_k(\chi), \chi) \leq q_{2,k}(V_k(\chi), \|\chi\|_W)$ for all $\chi \in W \setminus \{0\}$. Due to the properties of \mathcal{IK}_∞ functions, the obtained inequalities imply $\sigma_{1,k}(\|\chi(0)\|_2) \leq V_k(\chi)$ for all $\chi \in W \setminus W_0$ and $V_k(\chi) \leq \sigma_{2,k}(\|\chi\|_W)$ for all $\chi \in W \setminus \{0\}$. Thus, the functionals $V_k(\chi)$ can be extended by continuity to W as $V_k(0) = 0$. Taking into account that $0 = \sigma_{1,k}(\|\chi(0)\|_2) < V_k(\chi)$ for all $\chi \in W_0 \setminus \{0\}$, we finally derive condition $C_e2)$.

Conditions $C_i5)$ and $C_i3)$ guarantee that $C_e3)$ holds. Indeed, if $V_1(\chi) \leq 1$, then $Q_2(1, \chi) = Q_1(1, \chi) \leq Q_1(V_1(\chi), \chi) = 0 = Q_2(V_2(\chi), \chi)$ and, consequently, $V_2(\chi) \leq 1$.

Finally, note that, for all $(V_k, x_t) \in \Omega_k$, the functional $F_k(V_k, x_t)$ can be rewritten as $F_k(V_k, x_t) = DV_k(x_t)\dot{x}_t$, which coincides with the definition of the function $\dot{V}_k(x_t)$. \square

Despite the seeming complexity of conditions $C_i6a)$ and $C_i6b)$, in the next section it will be shown that for linear systems in the controllable canonical form they can be represented in the form of linear matrix inequalities which are easily checked using appropriate mathematical software.

3.3 Nonlinear delay-induced control

In this section, we will demonstrate how the problem of robust practical fixed-time stabilization of linear systems in the controllable canonical form can be effectively solved using the Implicit Lyapunov-Krasovskii method. Differently from the observer-based approaches, the state vector will be approximated by means of the finite difference method, i.e., based on the past values of the output signal. Due to the special integral relation between the state and its finite-difference approximation, the closed-loop system has a neutral time-delay representation and therefore the developed method can be applied for the control design.

3.3.1 Problem statement

Consider a linear system in the controllable canonical form with a relative degree $n \geq 2$, matched parametric uncertainties, state perturbation and measurement noise:

$$\begin{cases} \dot{x}(t) = A_0x(t) + B_0(u(t) + d_x(t) + c^\top x(t)), \\ y(t) = C_0x(t) + d_y(t), \end{cases} \quad (3.4)$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}$ is the control input; $y(t) \in \mathbb{R}$ is the measured output; $d_x(t) \in \mathbb{R}$ is the state perturbation; $d_y(t) \in \mathbb{R}$ is the measurement noise; $d = \text{col}\{d_x, d_y\} \in \mathcal{B}_{L_2^\infty}(\varrho_d)$; $c \in \mathbb{R}^{n \times 1}$ is the vector of unknown coefficients such that $\|c\|_2^2 \leq \epsilon$; system matrices $A_0 \in \mathbb{R}^{n \times n}$, $B_0 \in \mathbb{R}^{n \times 1}$ and $C_0 \in \mathbb{R}^{1 \times n}$ are of the form

$$A_0 = \begin{bmatrix} O_{(n-1) \times 1} & I_{n-1} \\ 0 & O_{1 \times (n-1)} \end{bmatrix}, \quad B_0 = \begin{bmatrix} O_{(n-1) \times 1} \\ 1 \end{bmatrix}, \quad C_0 = [1 \quad O_{1 \times (n-1)}].$$

Note that all linear single-input single-output controllable systems with the relative degree n can be rewritten in the canonical form (3.4) by applying a linear coordinate transformation. Moreover, for many nonlinear systems, such as a pendulum ($n = 2$), a magnetic suspension system ($n = 3$) or a single link manipulator with flexible joints and negligible damping ($n = 4$), there is a change of variables that transforms them into the form (3.4) [40].

The goal is to design a static output-feedback control practically stabilizing the system (3.4) with the rate of convergence faster than exponential.

3.3.2 Control design

Inspired by [48], we will define a nonlinear control law in the following form:

$$u(\hat{y}) := \sum_{j=1}^n K_j [\hat{y}_j]^{a_j(\tilde{\mu}(\|\hat{y}\|_2))}, \quad (3.5a)$$

$$a_j(\tilde{\mu}) := \frac{1}{1 - (n+1-j)\tilde{\mu}}, \quad j = \overline{1, n}, \quad (3.5b)$$

$$\tilde{\mu}(\|\hat{y}\|_2) := \begin{cases} \mu_2, & \text{if } \|\hat{y}\|_2 \geq \Delta, \\ \mu_1, & \text{if } \|\hat{y}\|_2 \leq 1, \\ \frac{\mu_2 - \mu_1}{\Delta - 1} \|\hat{y}\|_2 + \frac{\Delta\mu_1 - \mu_2}{\Delta - 1}, & \text{otherwise,} \end{cases} \quad (3.5c)$$

where $K_j < 0$, $j = \overline{1, n}$, $\mu_1 \in (-1, 0)$, $\mu_2 \in (0, 1/n)$ and $\Delta > 1$ are controllers parameters to be selected; $\hat{y} = \text{col}\{\hat{y}_j\}_{j=1}^n \in \mathbb{R}^n$, $\hat{y}_1(t) := y(t)$, $\hat{y}_{i+1}(t)$ is the approximation of the i -th output derivative $y^{(i)}(t)$, $i = \overline{1, n-1}$.

Instead of introducing a state observer, we approximate the output derivatives by the finite differences $\hat{y}_{i+1}(t) \approx y^{(i)}(t)$, $i = \overline{1, n-1}$:

$$\hat{y}_{i+1}(t) := \frac{\hat{y}_i(t) - \hat{y}_i(t-h)}{h} = \frac{1}{h^i} \sum_{s=0}^i (-1)^s \frac{i!}{s!(i-s)!} y(t-sh), \quad (3.6)$$

where $h > 0$ is a time delay. Since the value of $y(t-sh)$ is undefined for $t \in [0, sh)$, then we set it equal to $y(0)$.

Selection of approximation (3.6) follows from the well-known fact: if $h \rightarrow 0$ then $\hat{y}_{i+1}(t) \rightarrow y^{(i)}(t)$. It is worth noting that the proposed scheme is similar to a high-gain observer [40], since only for sufficiently small delays $h > 0$ derivative estimates $\hat{y}_{i+1}(t)$ can be used in stabilizing feedback [27, 28]. Differently from the conventional observer-based control, the finite-difference approximation scheme (3.6) does not require solving additional differential equations in real time, which simplifies its practical implementation.

Nevertheless, to apply Theorem 3.2, we have to present $\hat{y}_{i+1}(t)$ in a different form. First, for each $i = \overline{1, n-1}$, define variables $\hat{x}_{i+1}(t)$ and $\hat{d}_{y,i+1}(t)$ as follows:

$$\hat{x}_{i+1}(t) := \frac{\hat{x}_i(t) - \hat{x}_i(t-h)}{h}, \quad (3.7a)$$

$$\hat{d}_{y,i+1}(t) := \frac{\hat{d}_{y,i}(t) - \hat{d}_{y,i}(t-h)}{h}, \quad (3.7b)$$

where $\hat{x}_1(t) := x_1(t)$ and $\hat{d}_{y,1}(t) := d_y(t)$. Here $\hat{x}_{i+1}(t)$ represents the approximation of $x_{i+1}(t)$, and $\hat{d}_{y,i+1}(t)$ is a new disturbance. One can see that (3.6) and (3.7) imply $\hat{y}_{i+1}(t) = \hat{x}_{i+1}(t) + \hat{d}_{y,i+1}(t)$. To proceed further, we need to recall the following result.

Proposition 3.1 [70]. *If $x(t) \in C^i(\Omega, \mathbb{R})$ and $x^{(i)}(t)$ is absolutely continuous, $i \in \mathbb{N}$, then the following relation holds:*

$$\varepsilon_i(t) := \hat{x}_{i+1}(t) - x^{(i)}(t) = - \int_{t-ih}^t \varphi_i\left(\frac{t-s}{h}\right) x^{(i+1)}(s) ds, \quad (3.8)$$

where $\varphi_1(\xi) := 1 - \xi$ and for $i \in \mathbb{N} \setminus \{1\}$:

$$\varphi_i(\xi) := \begin{cases} \int_0^\xi \varphi_{i-1}(\lambda) d\lambda + 1 - \xi, & \xi \in [0, 1], \\ \int_{\xi-1}^\xi \varphi_{i-1}(\lambda) d\lambda, & \xi \in (1, i-1), \\ \int_{\xi-1}^{i-1} \varphi_{i-1}(\lambda) d\lambda, & \xi \in [i-1, i]. \end{cases} \quad (3.9)$$

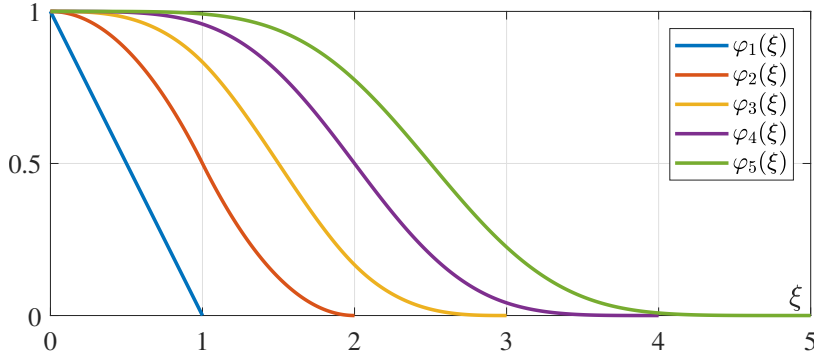
Since $x_1(t) \in C^n(\mathbb{R}_+, \mathbb{R})$ and $x_1^{(n)}(t)$ is absolutely continuous, it follows from (3.6)–(3.8) that $\hat{y}_{i+1}(t) = x_{i+1}(t) + \varepsilon_i(t) + \hat{d}_{y,i+1}(t)$. Therefore, the closed-loop system (3.4), (3.5) is in the form (3.1) and Theorem 3.2 can be applied. To this end, introduce two Implicit Lyapunov-Krasovskii functional (ILKF) candidates $Q_k(V_k, \chi)$, $k = 1, 2$, by the equality:

$$Q_k(V_k, \chi) := -1 + \|\Lambda_k(V_k^{-1})\chi(0)\|_P^2 + \sum_{i=1}^{n-1} \frac{i}{2S_i} V_k^{-2r_{k,i+2} + \mu_k} \int_{-ih}^0 \psi_i\left(\frac{-\tau}{h}\right) \dot{\chi}_{i+1}^2(\tau) d\tau, \quad (3.10)$$

where $0 \prec P \in \mathbb{S}^{n \times n}$, $S_i > 0$, $i = \overline{1, n-1}$, $\psi_i(\xi) := \int_\xi^i \varphi_i(\lambda) d\lambda$, $\Lambda_k(\lambda) := \text{diag}\{\lambda^{r_{k,j}}\}_{j=1}^n$ is the dilation matrix with weights

$$r_{k,j}(\mu_k) := 1 - (n+1-j)\mu_k, \quad j = \overline{1, n}. \quad (3.11)$$

Note that in the linear setting ($\mu_k = 0$), equation $Q_k(V_k, \chi) = 0$ defines a Lyapunov-Krasovskii functional $V_k(\chi) = \sqrt{\|\chi(0)\|_P^2 + \sum_{i=1}^{n-1} \frac{i}{2S_i} \int_{-ih}^0 \psi_i\left(\frac{-\tau}{h}\right) \dot{\chi}_{i+1}^2(\tau) d\tau}$.

Figure 3.1: Plots of $\varphi_i(\xi)$ for $i = \overline{1, 5}$

i	ω_i
1	2
2	2.1577
3	2.3282
4	2.4614
5	2.5680

Table 3.1: Values of ω_i for $i = \overline{1, 5}$

For the following Lyapunov-Krasovskii analysis we will need some characteristics of the functions $\varphi_i(\xi)$ (see Fig. 3.1) and their integrals $\psi_i(\xi)$ (see the proof in Appendix 3.A).

Proposition 3.2. *The functions $\varphi_i(\xi)$ defined in (3.9) and their integrals $\psi_i(\xi)$ possess the following properties:*

- P1) $\varphi'_i(\xi) < 0$ on $\xi \in (0, i)$;
- P2) $0 \leq \varphi_i(\xi) \leq 1$ for all $\xi \in [0, i]$;
- P3) $\varphi_i(\xi) + \varphi_i(i - \xi) = 1$ for all $\xi \in [0, i]$;
- P4) $\varphi''_i(\xi) < 0$ on $\xi \in (0, i/2)$ and $\varphi''_i(\xi) > 0$ on $\xi \in (i/2, i)$ for $i \geq 2$;
- P5) $\psi_i(0) = i/2$ and $\psi_i(i) = 0$;
- P6) $\psi_i(\xi) \leq (i/2)\varphi_i(\xi)$ for all $\xi \in [0, i]$;
- P7) for all $i \in \mathbb{N}$ the following integral is well-defined:

$$\omega_i := \int_0^i \frac{\varphi_i^2(\xi)}{\psi_i(\xi)} d\xi. \quad (3.12)$$

Remark 3.1. *It is worth mentioning that parameters ω_i are independent of time delay $h > 0$ and, thus, can be calculated in advance. For example, direct computation of ω_1 gives a quite simple result: $\omega_1 = 2$. Other values of ω_i can be found by numerical integration (see Table 3.1).*

Now we are ready to present the restrictions on the choice of the controller parameters given in the form of LMIs such that Theorem 3.2 holds for ILKFs (3.10) with respect to the system (3.4), (3.5) (see the proof in Appendix 3.B).

Proposition 3.3. *Given $\epsilon > 0$, let there exist $\mu \in (0, 1/n)$, $h > 0$, $\delta > 0$ such that the system of LMIs:*

$$XG_k + G_kX \succ 0, \quad (3.13a)$$

$$\max\{\|\zeta\|_2, \delta\}I_n \preccurlyeq X \preccurlyeq I_n/2, \quad (3.13b)$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & Y \\ * & \Psi_{22} & \Psi_{12}^\top B_0 \\ * & * & -\frac{4S_{n-1}}{(n-1)^2} \end{bmatrix} \preccurlyeq 0, \quad \begin{bmatrix} Z & X \\ * & \Upsilon_1 \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} \Upsilon_2 & \Upsilon_2 \rho \\ * & X \rho \end{bmatrix} \succcurlyeq 0, \quad (3.13c)$$

$$\begin{aligned}
\text{where } G_k &:= \text{diag}\{r_{k,j}\}_{j=1}^n, & \Psi_{11} &:= XA_0^\top + YB_0^\top + A_0X + B_0Y^\top + Z + \frac{2}{n-1}X, \\
\Xi_1 &:= \text{diag}\{b_2, b_3, 4S_i/i^2\}_{i=1}^{n-2}, & \Psi_{12} &:= [B_0Y^\top\sqrt{\rho}, B_0Y^\top\sqrt{\|\zeta\|_2}, B_0Y^\top\sqrt{\delta}, B_0, B_0\sqrt{\epsilon}], \\
\Xi_2 &:= \frac{1}{n-1}\text{diag}\{b_4, S_i\}_{i=1}^{n-1}, & \Psi_{22} &:= -\frac{1}{n-1}\text{diag}\{(1-\rho)X, \frac{1}{2}X, \frac{1}{2}X, \frac{1}{4}b_1, \frac{1}{2}\}, \\
\zeta &:= \text{col}\{\zeta_j\}_{j=1}^n, & \zeta_j &:= \max\left\{\bar{g}\left(\Delta, \frac{1}{r_{1,j}}\right), \bar{g}\left(\Delta, \frac{1}{r_{2,j}}\right)\right\} + \Delta^{1/(1+\mu)}\bar{g}\left(\Delta^{1/r_{2,j}}, \frac{r_{2,j}}{r_{1,j}}\right), \\
\rho &:= h^{1-\sqrt{\mu}}, & \mu_1 &:= -\mu, & \mu_2 &:= \mu, & \Delta &:= \sqrt{1+\delta},
\end{aligned}$$

with functions $r_{k,j}(\mu_k)$ and $\bar{g}(\bar{s}, \bar{\epsilon})$ defined in (3.11) and Lemma A.2, respectively, is feasible for some $b_l > 0$, $0 < S_i \leq i/(4\omega_i\rho)$, $X \in \mathbb{S}^{n \times n}$, $Y \in \mathbb{R}^{n \times 1}$, $Z \in \mathbb{S}^{n \times n}$, where ω_i defined in (3.12), $i = \overline{1, n-1}$, $j = \overline{1, n}$, $k = 1, 2$, $l = \overline{1, 4}$.

Then the closed-loop system (3.4), (3.5) with parameters

$$K := \text{row}\{K_j\}_{j=1}^n = X^{-1}Y, \quad \mu_1 = -\mu, \quad \mu_2 = \mu, \quad \Delta = \sqrt{1+\delta}$$

is practically locally fixed-time ISS (3.2) with ϱ_x , ϱ_d , T_0 , ϱ_0 and $w(s)$ given by

$$\begin{aligned}
\varrho_x &= \frac{h^{-r_{2,1}/\sqrt{\mu}}}{\sqrt{\max\{\|\zeta\|_2^{-1}, h\sqrt{\mu}S_0\}}}, & \varrho_d &= \frac{h^{-r_{2,1}/\sqrt{\mu}}}{\eta}, & w(s) &= \frac{1}{\sqrt{2}} \begin{cases} (\eta s)^{r_{1,n}/r_{1,1}}, & \text{if } \eta s < 1, \\ (\eta s)^{r_{2,n}/r_{2,1}}, & \text{if } \eta s > 1, \end{cases} \\
T_0 &= \frac{1}{(n-1)\mu\gamma}, & \varrho_0 &= \frac{h^{r_{1,n}/\sqrt{\mu}}}{\sqrt{2}},
\end{aligned}$$

where $\gamma := \max\{\lambda_{\max}\{X^{1/2}G_kX^{-1/2} + X^{-1/2}G_kX^{1/2}\}, 2 + (2n-3)\mu\}$, $S_0 := \max_{i=\overline{1, n-1}} i^2/(4S_i)$ and $\eta := \sqrt{\max\{b_1, \frac{(2/h)^{2n-1}}{(2/h)^2-1}/\delta^2\}}$.

Let us give some comments on the choice of tuning parameters. Firstly, LMIs (3.13) are always feasible provided ϵ , μ , h and δ are sufficiently small. Obviously, this is true for $\epsilon = \mu = h = \delta = 0$. Indeed, taking into account that in this case $\|\zeta\|_2 = \rho = 0$ and $G_k = I_n$, one can see that LMIs (3.13) hold for some $0 \prec X \preceq I_n/2$, $Y, Z \succeq 0$ and sufficiently large b_l, S_i . Clearly, LMIs (3.13) remain feasible for some positive nonzero ϵ, μ, h and δ since $r_{k,j}, \zeta_j$ and ρ are continuous functions of μ, h and δ .

Secondly, it follows from Proposition 3.3 that the settling time T_0 is inversely proportional to parameter μ . Thus, the best strategy of parameter tuning consists in maximizing μ , for which LMIs (3.13) are feasible for given ϵ . On the other hand, note that $\varrho_x = \varrho_x(h)$, $\varrho_d = \varrho_d(h)$ and $\varrho_0 = \varrho_0(h)$ are the functions of the time delay h for the fixed nonlinear degree μ . Obviously, ϱ_x (ϱ_d) and ϱ_0 can be enlarged and decreased, respectively, by reducing time delay h . Moreover, in the limit case, i.e., $h \rightarrow 0^+$, the closed-loop system is globally fixed-time ISS since $\varrho_x(\varrho_d) \rightarrow +\infty$ and $\varrho_0 \rightarrow 0^+$. As a result, we get the same property as in [48] where the state $x(t)$ was reconstructed using an observer.

However, in practice, time delay h cannot be chosen arbitrarily small due to related implementation problems. For example, note that, similar to high-gain observers, approximation (3.6) is sensitive to high-frequency measurement noises [39]. In order to show this, let us assume that $d_y(t)$ is a Lipschitz continuous function of time, i.e., there exists a positive constant L such that $|d_y(t_1) - d_y(t_2)| \leq L\|d_y\|_{L_2^\infty}|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$. Taking into account (3.7b), it can be shown that in this case $\eta = \sqrt{\max\{b_1, (1 + L^2 \frac{(2/h)^{2(n-1)} - 1}{(2/h)^2 - 1})/\delta^2\}}$, which coincides with the one given in Proposition 3.3 if $hL = 2$. Thus, the slower the measurement noise d_y changes (the smaller L), the smaller the steady-state error. Nevertheless, the problem of making approximation (3.6) more robust to high-frequency measurement noises (e.g., by introducing low-pass filters [29, 39]) is out of the scope of this work.

Now let us show what the main advantage of the proposed control law is (3.5) compared to its linear analog ($\mu = 0$) with the same feedback gains K .

Proposition 3.4. *Let the conditions of Proposition 3.3 be fulfilled. Then there are $h^* \in (0, h]$ and $\varrho_x^* \in (0, \varrho_x(h^*)]$ such that for all $x_0 \in \mathcal{B}_W(\varrho_x) \setminus \mathcal{B}_W(\varrho_x^*)$ and $d \in \mathcal{B}_{L_2^\infty}(\varrho_d)$ the system (3.4), (3.5) with time delay h^* converges faster to the set $\mathcal{A} := \{x \in \mathbb{R}^n : \|x(t, x_0, d)\|_2 \leq \varrho_0 + w(\|d\|_{L_2^\infty})\}$ than its linear counterpart ($\mu = 0$).*

The proof of Proposition 3.4 is given in Appendix 3.C.

In other words, for sufficiently large initial conditions or sufficiently small external disturbances, the proposed control system always converges faster to the vicinity of the origin than its linear analog.

3.4 Numerical simulation

Let $n = 3$ and $\epsilon = 0.05$. Choose $\mu = 0.01$, $h = 0.02$ and $\delta = 10^{-3}$. Then solving LMIs (3.13), we find parameters of the controller (3.5)

$$K = \begin{bmatrix} -3.19 & -6.08 & -4.20 \end{bmatrix}, \quad \mu_1 = -0.01, \quad \mu_2 = 0.01, \quad \Delta = 1.0005$$

and the following coefficients:

$$\varrho_0 = 4.9 \cdot 10^{-18}, \quad \varrho_x = 6.15 \cdot 10^{15}, \quad \varrho_d = 3.02 \cdot 10^9, \quad \eta = 10^7.$$

For further comparison we set $c = 0.125 \cdot \text{col}\{1, 1, 1\}$ such that $\|c\|_2^2 = 0.047 < \epsilon$. The numerical simulation of the closed-loop system (3.1), (3.5) has been done in MATLAB Simulink by using the explicit Euler method with a state-dependent step [22]. The basic and minimum discretization steps, the maximum number of iterations and the homogeneous norm have been defined as $\Delta t_0 = 10^{-2}$, $\Delta t_{\min} = 10^{-4}$, $N_{\max} = 2 \cdot 10^4$ and $\|x\|_{hom} := (\sum_{j=1}^n |x_j|^{a_j(\tilde{\mu})/a_1(\tilde{\mu})})^{a_1(\tilde{\mu})}$, respectively. Initial conditions have been chosen as $x_0(\tau) = 10^{2-i} \cdot \text{col}\{0.6, 0, -0.8\}$, $i = \overline{1, 3}$ for all $\tau \in [-0.04, 0]$.

First, we will show that the proposed control scheme (3.4), (3.5) is indeed practically fixed-time stable. To this end, we will compare it with its linear analog ($\mu = 0$) when $\|d\|_{L_2^\infty} = 0$. The results of the simulation depicted in Figure 3.2 in the logarithmic scale, where solid lines corresponds to the proposed control law (3.5) and dashed ones represent its linear counterpart ($\mu = 0$). The results illustrate Proposition 3.4: the solutions of the nonlinear system (3.4), (3.5) converge faster than its linear analog. However, the superiority of the proposed control over its linear counterpart is not so evident due to the smallness of μ . Recall that this parameter should be chosen as large as possible to ensure the feasibility of LMIs (3.13). Since our Lyapunov analysis is rather conservative, one might expect that the closed-loop system (3.1), (3.5) remain fixed-time stable even for larger μ . To demonstrate this, we chose $\mu = 0.1$ and kept other controller parameters the same. The results of this numerical comparison are depicted in Figure 3.2. Clearly, the proposed control significantly does outperform the linear one.

Now we compare performance of the proposed control system (3.4), (3.5) with its linear counterpart in the presence of the state perturbation $d_x(t) = 0.5 \cos(t)$ and the measurement noise $d_y(t) = 0.01 \sin(10t)$. As a result, $w(\|d\|_{L_2^\infty}) = (\eta\|d\|_{L_2^\infty})^{r_2, n/r_2, 1} = 7 \cdot 10^6$. The norm of the solutions $x(t, x_0, d)$ is depicted in Figure 3.3 in the logarithmic scale, where the initial conditions x_0 are chosen the same

as for the disturbance-free case. Again the obtained results go with Proposition 3.4. As well as in the disturbance-free case, for larger values of μ the difference between nonlinear and linear approaches becomes clearer (see Fig. 3.3).

3.5 Conclusion

In this chapter, the notion of practical fixed-time input-to-state stability has been introduced for neutral time-delay systems with external bounded disturbances and characterized by the Lyapunov-Krasovskii method, which has been formulated both explicitly and implicitly. Based on the obtained theoretical results, an alternative way of robust output practical fixed-time stabilization of linear systems in the controllable canonical form has been proposed. To this end, the state vector was first approximated by means of the finite difference method, i.e., based on the past values of the output signal. Differently from the observer-based approaches, the finite-difference approximation scheme does not require solving additional differential equations in real time, which simplifies its practical implementation. Then, a nonlinear controller was designed to practically stabilize the system in fixed time. To achieve fast stabilization, the nonlinear degree of the feedback is dynamically changed depending on how far from the origin trajectories of the closed-loop system are. To apply the developed Lyapunov-Krasovskii method, it has been shown that the closed-loop system has a neutral time-delay representation due to the special integral relation between the state and its finite-difference approximation. Using the formulated Implicit Lyapunov-Krasovskii theorem, sufficient stability conditions for the designed nonlinear control system were presented in the form of linear matrix inequalities, solutions of which are used for the calculation of the controller parameters. Furthermore, the impact of the artificially induced time delay on the stabilization accuracy has also been quantitatively studied. Finally, it was theoretically proven and numerically illustrated that, both in the disturbance-free and disturbed cases, the proposed nonlinear controller stabilizes the considered system in the vicinity of the origin much faster than its linear counterpart. Robustification of the proposed control scheme with respect to high-frequency measurement noises (e.g., by prefiltering the output signal $y(t)$ before using it for the finite-difference approximation of the state) is left for future research.

Appendices

3.A Proof of Proposition 3.2

P1)–P2) First, it is clear to see that $\varphi'_1(\xi) = -1 < 0$ for all $\xi \in [0, 1]$. Differentiating (3.9) with respect to ξ , we obtain:

$$\varphi'_i(\xi) := \begin{cases} \varphi_{i-1}(\xi) - 1, & \xi \in [0, 1], \\ \varphi_{i-1}(\xi) - \varphi_{i-1}(\xi - 1), & \xi \in (1, i - 1), \\ -\varphi_{i-1}(\xi - 1), & \xi \in [i - 1, i]. \end{cases} \quad (3.14)$$

Obviously, using induction, one can prove that $\varphi'_i(\xi) < 0$ on $\xi \in (0, i)$ for $i \geq 2$. Indeed, if $\varphi'_{i-1}(\xi) < 0$ on $\xi \in (0, i-1)$, then $\varphi_{i-1}(\xi)$ is strictly decreasing. Then taking into account that (3.9) implies $\varphi_i(0) = 1$ and $\varphi_i(i) = 0$, we finish the proof.

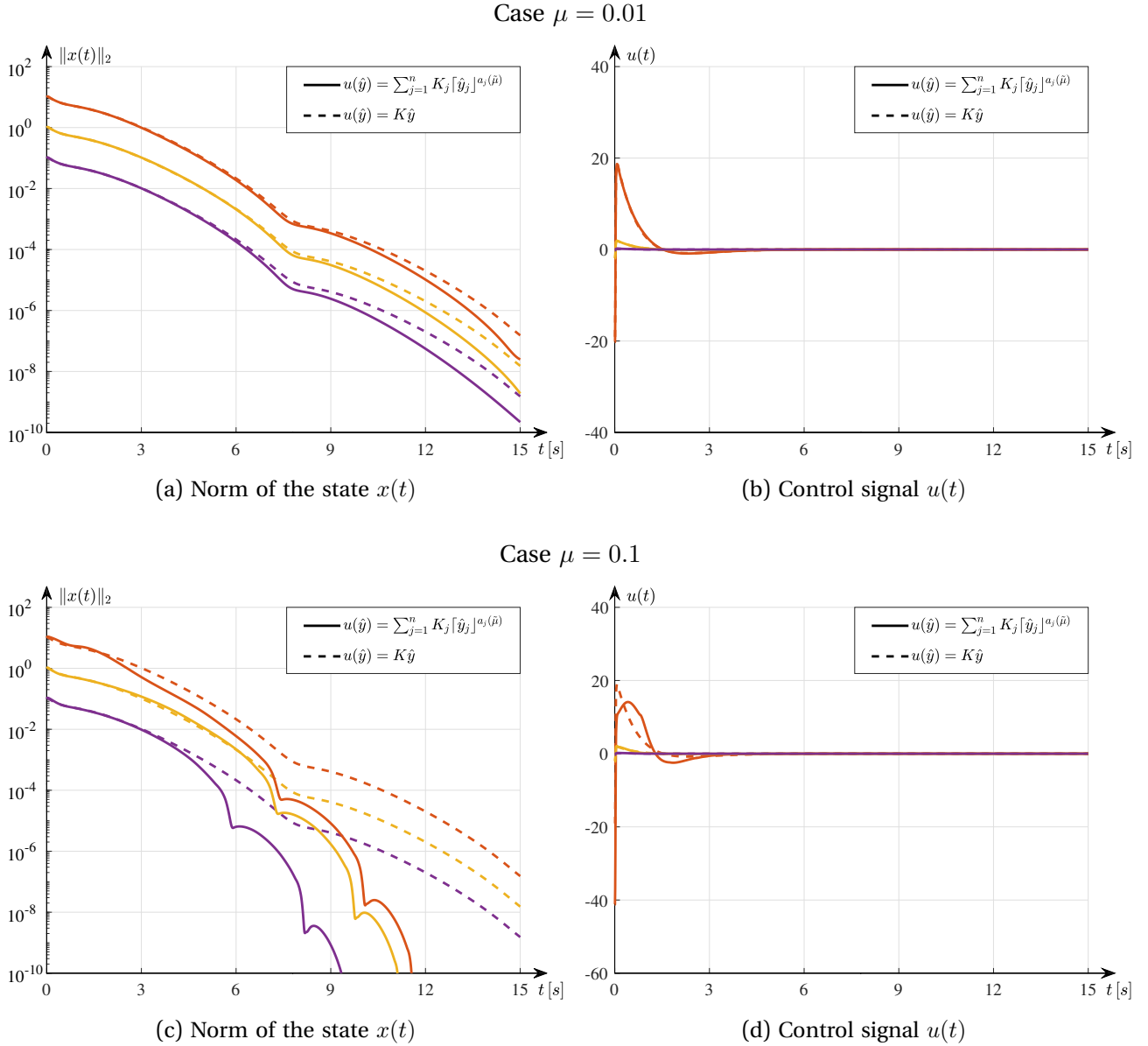


Figure 3.2: Transients in the closed-loop system (3.4), (3.5), (3.6) for $d = 0$

P3) Property 4) from Proposition 2 in [70] postulates that $\bar{\varphi}_i(h\xi) + \bar{\varphi}_i(h(i - \xi)) = 1$, where functions $\bar{\varphi}_i(h\xi)$ are such that $\bar{\varphi}_i(h\xi) = \varphi_i(\xi)$. Thus, $\varphi_i(\xi) + \varphi_i(i - \xi) = 1$.

P4) Differentiating (3.14) with respect to ξ , we get:

$$\varphi_i''(\xi) := \begin{cases} \varphi_{i-1}'(\xi), & \xi \in [0, 1], \\ \varphi_{i-1}'(\xi) - \varphi_{i-1}'(\xi - 1), & \xi \in (1, i - 1), \\ -\varphi_{i-1}'(\xi - 1), & \xi \in [i - 1, i]. \end{cases}$$

For $i = 2$ it is obvious that $\varphi_2''(\xi) < 0$ on $\xi \in [0, 1]$ and $\varphi_2''(\xi) > 0$ on $\xi \in (1, 2]$, since $\varphi_1'(\xi) = -1 < 0$ for all $\xi \in [0, 1]$. Moreover, $\varphi_2''(\xi)$ is strictly decreasing and strictly increasing on corresponding intervals.

Applying property P1) for $i > 2$, it is sufficient to prove by using induction that function $\varphi_i''(\xi)$ has the unique zero at $\xi_{0i} = i/2$. Indeed, $\varphi_{i-1}''((i - 1)/2) = 0$ implies that $\varphi_{i-1}'(\xi)$ is strictly decreasing on $\xi \in (0, (i - 1)/2)$ and strictly increasing on $\xi \in ((i - 1)/2, i - 1)$. Therefore, $\varphi_i''(\xi)$ has the only one

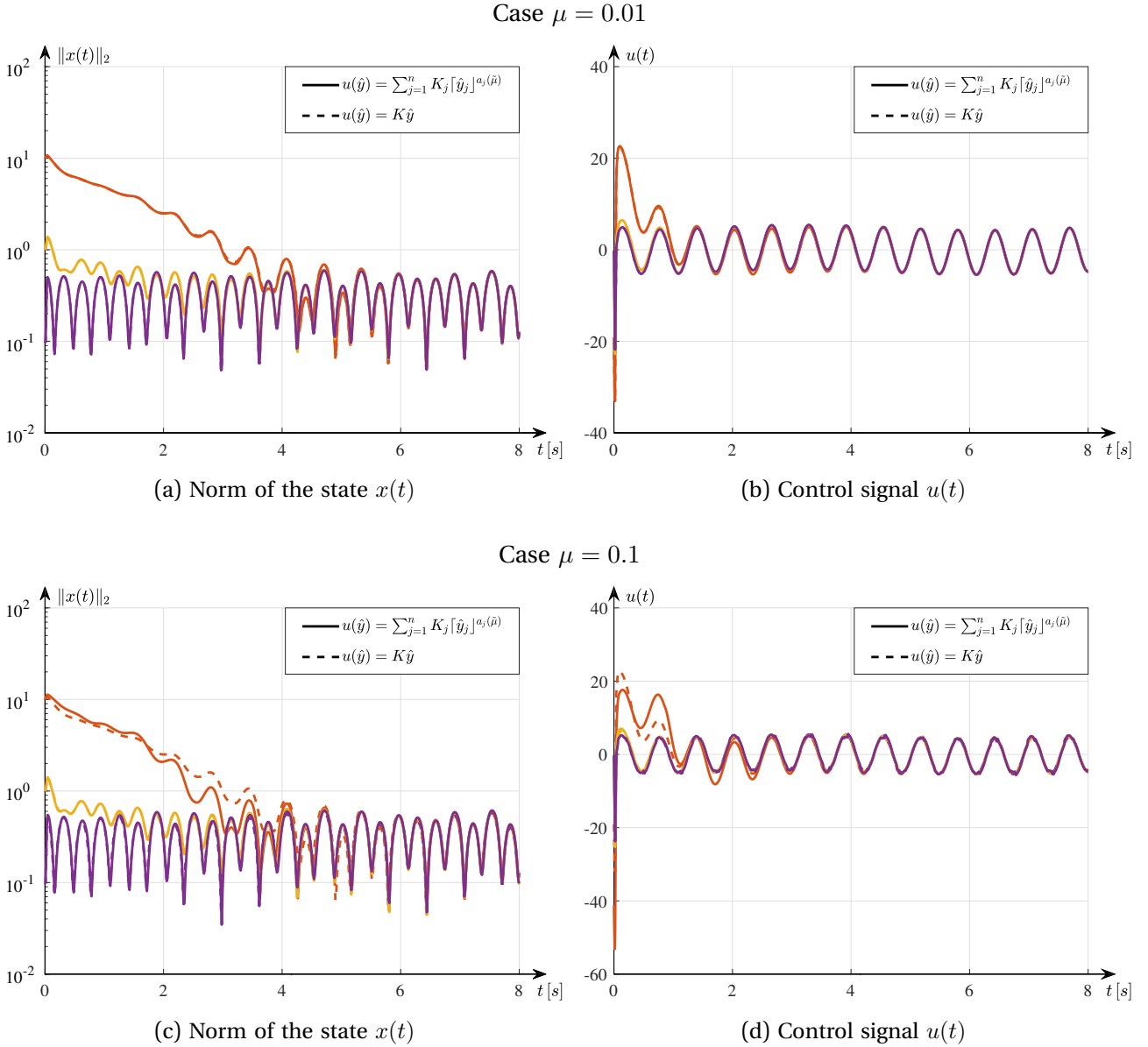


Figure 3.3: Transients in the closed-loop system (3.4), (3.5), (3.6) for $d \neq 0$

zero on $\xi \in (0, i)$. Let us show that $\xi_{0i} = i/2$. Using (3.14), the condition $\varphi_i''(i/2) = \varphi_{i-1}'(i/2) - \varphi_{i-1}'(i/2 - 1) = 0$ can be equivalently rewritten as

$$\begin{aligned} 2\varphi_1\left(\frac{1}{2}\right) &= 1 && \text{for } i = 3, \\ 2\varphi_{i-2}\left(\frac{i-2}{2}\right) &= \varphi_{i-2}\left(\frac{i}{2}\right) + \varphi_{i-2}\left(\frac{i-4}{2}\right) && \text{for } i \geq 4, \end{aligned}$$

since $3/2 \in [1, 2]$ and $(3/2 - 1) \in [0, 1]$ for $i = 3$, $i/2$ and $(i/2 - 1) \in [1, i - 1]$ for $i \geq 4$. Applying property P3), one can see that these relations hold and, therefore, $\xi_{0i} = i/2$ is the unique inflection point of $\varphi_i(\xi)$ for $i > 2$.

P5) It is obvious that $\psi_i(i) = 0$. Then using the change of variable $\tilde{\lambda} = i - \lambda$ and property P3), we get

$$\psi_i(0) = \int_0^{i/2} \varphi_i(\lambda) d\lambda + \int_{i/2}^i \varphi_i(\tilde{\lambda}) d\tilde{\lambda} = \int_0^{i/2} (\varphi_i(\lambda) + \varphi_i(i - \lambda)) d\lambda = \frac{i}{2}.$$

P6) Note that function $\psi_i(\xi)$ could be rewritten as follows:

$$\psi_i(\xi) = \begin{cases} \frac{i}{2} - \int_0^\xi \varphi_i(\lambda) d\lambda, & \xi \in [0, i/2], \\ \int_\xi^i \varphi_i(\lambda) d\lambda, & \xi \in [i/2, i]. \end{cases}$$

Taking into account property P4), integral terms can be estimated by the area of a trapezoid from below and a triangle from above, respectively:

$$\psi_i(\xi) \leq \begin{cases} \frac{i}{2} - \frac{\xi(1 + \varphi_i(\xi))}{2}, & \xi \in [0, i/2], \\ \frac{(i - \xi)\varphi_i(\xi)}{2}, & \xi \in [i/2, i]. \end{cases}$$

Since $i - \xi \leq i\varphi_i(\xi)$ for $[0, i/2]$ and $i - \xi \leq i$ for all $\xi \in [0, i]$, we conclude the proof.

P7) Since function $\tilde{\varphi}_i(\xi) := \psi_i^{-1}(\xi)\varphi_i^2(\xi)$ is continuous on $\xi \in [0, i]$, it is sufficient to prove that $\tilde{\varphi}_i(i + 0^-) < +\infty$. Indeed, applying L'Hôpital's rule, we get $\tilde{\varphi}_i(i + 0^-) = -2\varphi_i'(i)$. From (3.14) it follows that $\varphi_i'(i) = \varphi_{i-1}(i - 1) = 0$. Therefore, $\omega_i = \int_0^i \tilde{\varphi}_i(\xi) d\xi$ is well-defined and function $\tilde{\varphi}_i(\xi)$ can be prolonged to $\xi = i$ by defining $\tilde{\varphi}_i(i) = 0$. \square

3.B Proof of Proposition 3.3

Let us show that all the conditions of Theorem 3.2 are fulfilled for ILKFs (3.10) with respect to the system (3.4)–(3.6). For the sake of brevity, denote the space $W([- (n - 1)h, 0], \mathbb{R}^n)$ simply as W .

B.I. Verification of conditions $C_i1)$ – $C_i5)$

Obviously, the functionals $Q_k(V_k, \chi)$ defined by equation (3.10) satisfy condition $C_i5)$. On the other hand, for any $(V_k, \chi) \in \mathbb{R}_+^* \times W$, both partial Fréchet derivatives $D_{V_k}Q_k(V_k, \chi)$ and $D_\chi Q_k(V_k, \chi)$ exist and given as follows:

$$\begin{aligned} D_{V_k}Q_k(V_k, \chi)\tilde{V}_k &:= -(\tilde{V}_k/V_k)\left(\chi^\top(0)\Lambda_k(V_k^{-1})(G_kP + PG_k)\Lambda_k(V_k^{-1})\chi(0)\right. \\ &\quad \left.- \sum_{i=1}^{n-1} \frac{i(2r_{k,i+2} - \mu_k)}{2S_i} V_k^{-2r_{k,i+2} + \mu_k} \int_{-ih}^0 \psi_i\left(\frac{-\tau}{h}\right) \dot{\chi}_{i+1}^2(\tau) d\tau\right), \\ D_\chi Q_k(V_k, \chi)\tilde{\chi} &:= 2\chi^\top(0)\Lambda_k(V_k^{-1})P\Lambda_k(V_k^{-1})\tilde{\chi}(0) \\ &\quad + 2 \sum_{i=1}^{n-1} \frac{i}{2S_i} V_k^{-2r_{k,i+2} + \mu_k} \int_{-ih}^0 \psi_i\left(\frac{-\tau}{h}\right) \dot{\chi}_{i+1}(\tau) \left(\frac{d}{d\tau} \tilde{\chi}_{i+1}(\tau)\right) d\tau, \end{aligned}$$

where $\tilde{V}_k \in \mathbb{R}_+^*$ and $\tilde{\chi} \in W$. Clearly, the mappings $(V_k, \chi) \mapsto D_{V_k}Q_k(V_k, \chi)$ and $(V_k, \chi) \mapsto D_\chi Q_k(V_k, \chi)$ are continuous on $\mathbb{R}_+^* \times W$. Therefore, condition $C_i1)$ holds. Furthermore, since $G_kP + PG_k \succ 0$ due to (3.13a) and $2r_{k,i+2} > \mu_k$ for all $i = \overline{1, n-1}$, then $D_{V_k}Q_k(V_k, \chi) < 0$ for all $V_k \in \mathbb{R}_+^*$ and $\chi \in W \setminus \{0\}$. Taking into account that $\partial Q_k(V_k, \chi)/\partial V_k := D_{V_k}Q_k(V_k, \chi)$, we conclude that condition $C_i3)$ holds, too.

Since $P \succ 0$ and $\psi_i(\xi) \leq i/2$, then the estimates

$$\frac{\lambda_{\min}(P)\|\chi(0)\|_2^2}{\max\{V_k^{2-2\mu_k}, V_k^{2-2n\mu_k}\}} \leq Q_k(V_k, \chi) + 1 \leq \frac{\lambda_{\max}(P)\|\chi(0)\|_2^2 + \sum_{i=1}^{n-1} \frac{i^2}{4S_i} V_k^{-\mu_k} \int_{-(n-1)h}^0 |\dot{\chi}_{i+1}(\tau)|^2 d\tau}{\min\{V_k^{2-2\mu_k}, V_k^{2-2n\mu_k}\}}$$

hold for all $V_k \in \mathbb{R}_+^*$ and $\chi \in W$. As a result, it is easy to see that for any $\chi \in W$ there exists $V_k \in \mathbb{R}_+^*$ such that $Q_k(V_k, \chi) = 0$. Taking into account (3.13b), introduce functions $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$ by the following formulas:

$$q_{1,k}(\sigma, s) = \frac{2s^2}{\max\{\sigma^{2-2\mu_k}, \sigma^{2-2n\mu_k}\}} - 1, \quad q_{2,k}(\sigma, s) = \frac{\max\left\{\frac{1}{\max\{\|\zeta\|_2, \delta\}}, S_0\sigma^{-\mu_k}\right\}s^2}{\min\{\sigma^{2-2\mu_k}, \sigma^{2-2n\mu_k}\}} - 1,$$

where $\sigma, s \in \mathbb{R}_+^*$. The obtained estimates guarantee that $q_{1,k}(V_k, \|\chi(0)\|_2) \leq Q(V_k, \chi) \leq q_{2,k}(V_k, \|\chi\|_W)$ for all $V_k \in \mathbb{R}_+^*$ and $\chi \in W$. \square

B.II. Verification of condition C_i6)

First we calculate $D_\chi Q_k(V_k, x_t)\dot{x}_t$. By the definition of the partial Fréchet derivative $D_\chi Q_k(V_k, \chi)$, we have the following:

$$\begin{aligned} D_\chi Q_k(V_k, x_t)\dot{x}_t &= 2x_t^\top(0)\Lambda_k(V_k^{-1})P\Lambda_k(V_k^{-1})\left(A_0x(t) + B_0(u(t) + d_x(t) + c^\top x(t))\right) \\ &\quad + \sum_{i=1}^{n-1} \frac{i}{2S_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{-ih}^0 \psi_i\left(\frac{-\tau}{h}\right) \left(\frac{d}{d\tau} \dot{x}_{t,i+1}^2(\tau)\right) d\tau, \end{aligned}$$

since $\dot{x}_t(0) = \dot{x}(t)$ and $2\dot{x}_{t,i+1}(\tau)\left(\frac{d}{d\tau} \dot{x}_{t,i+1}(\tau)\right) = \frac{d}{d\tau} \dot{x}_{t,i+1}^2(\tau)$. Then applying the integration by parts formula and using property P5), we get

$$\begin{aligned} D_\chi Q_k(V_k, x_t)\dot{x}_t &= \underbrace{2x_t^\top(0)\Lambda_k(V_k^{-1})P\Lambda_k(V_k^{-1})\left(A_0x(t) + B_0(u(t) + d_x(t) + c^\top x(t))\right)}_{R_{1,k}} \\ &\quad + \underbrace{\sum_{i=1}^{n-1} \frac{i^2}{4S_i} V_k^{-2r_{k,i+2}+\mu_k} \dot{x}_{t,i+1}^2(0)}_{R_{2,k}} - \underbrace{\sum_{i=1}^{n-1} \frac{i}{2hS_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{-ih}^0 \varphi_i\left(\frac{-\tau}{h}\right) \dot{x}_{t,i+1}^2(\tau) d\tau}_{R_{3,k}}. \end{aligned} \quad (3.15)$$

Note that $\Lambda_k(V_k^{-1})A_0 = V_k^{\mu_k} A_0\Lambda_k(V_k^{-1})$ and $\Lambda_k(V_k^{-1})B_0 = V_k^{-r_{k,n}} B_0 = V_k^{-1+\mu_k} B_0$. Then taking into account that $\hat{y}_{i+1}(t) = \hat{x}_{i+1}(t) + \hat{d}_{y,i+1}(t) = x_{i+1}(t) + \varepsilon_i(t) + \hat{d}_{y,i+1}(t)$, $i = \overline{1, n-1}$ due to (3.6)–(3.8), the term $R_{1,k}$ could be rewritten as follows:

$$R_{1,k} = 2V_k^{\mu_k} z_k^\top P \left((A_0 + B_0K)z_k + B_0K(\zeta_{h,k} + \zeta_{\mu,k} + \zeta_{d_y,k}) + B_0V_k^{-1}(d_x + c^\top \Lambda_k(V_k)z_k) \right),$$

where $z_k := \Lambda_k(V_k^{-1})x_t(0)$, $\zeta_{h,k} := \Lambda_k(V_k^{-1})\text{col}\{0, \varepsilon_1, \dots, \varepsilon_{n-1}\}$, $\zeta_{\mu,k} := V_k^{-1}\text{col}\{[\hat{y}_j]^{a_j(\bar{\mu})}\}_{j=1}^n - \Lambda_k(V_k^{-1})\hat{y}$ and $\zeta_{d_y,k} := \Lambda_k(V_k^{-1})\text{col}\{\hat{d}_{y,j}\}_{j=1}^n$.

Since $b_2, b_3 > 0$, then the term $R_{2,k}$ admits the following estimate:

$$R_{2,k} \leq V_k^{\mu_k} z_k^\top \Xi_1^{-1} z_k + V_k^{\mu_k} \frac{(n-1)^2}{4S_{n-1}} (V_k^{-1} \dot{x}_{t,n}(0))^2 = V_k^{\mu_k} z_k^\top \Xi_1^{-1} z_k + V_k^{\mu_k} \varsigma_k^\top \Theta^\top \frac{(n-1)^2}{4S_{n-1}} \Theta \varsigma_k,$$

where $\Theta := [Y, B_0^\top \Psi_{12}]$ and $\varsigma_k := \text{col}\left\{Pz_k, \frac{P\zeta_{h,k}}{\sqrt{\rho}}, \frac{P\zeta_{\mu,k}}{\sqrt{\|\zeta\|_2}}, \frac{P\zeta_{d_y,k}}{\sqrt{\delta}}, \frac{d_x}{V_k}, \frac{c^\top \Lambda_k(V_k)z_k}{V_k\sqrt{\varepsilon}}\right\}$.

Using property P6) and applying Lemma A.3 with $\vartheta = \varphi_i$, $\phi = \dot{x}_{t,i+1}(0)$, $\varpi = 1$, the term $R_{3,k}$ can be upper-bounded in the following manner:

$$R_{3,k} = \rho R_{3,k} + (1-\rho)R_{3,k} \leq -\frac{2\rho}{(n-1)h} (1 - z_k^\top Pz_k) - \frac{1-\rho}{(n-1)h^2} V_k^{-\mu_k} \zeta_{h,k}^\top \Xi_2^{-1} \zeta_{h,k}.$$

Adding and subtracting the corresponding terms to (3.15) in order to construct a quadratic form with respect to the vector ς_k and matrix $\Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix}$, we obtain

$$\begin{aligned} D_{\mathcal{X}}Q_k(V_k, x_t)\dot{x}_t + V_k^{\mu_k} \frac{1}{2(n-1)} &\leq V_k^{\mu_k} \varsigma_k^{\top} \left(\Psi + \Theta^{\top} \frac{(n-1)^2}{4S_{n-1}} \Theta \right) \varsigma_k + V_k^{\mu_k} z_k^{\top} \left(\Xi_1^{-1} - PZP \right) z_k \\ &- \frac{2}{n-1} (1 - z_k^{\top} P z_k) \left(\frac{\rho}{h} - V_k^{\mu_k} \right) - V_k^{-\mu_k} \frac{1-\rho}{(n-1)\rho^2} \zeta_{h,k}^{\top} \left(\frac{\rho^2}{h^2} \Xi_2^{-1} - V_k^{2\mu_k} \rho P \right) \zeta_{h,k} \\ &+ \frac{V_k^{\mu_k}}{4(n-1)} \left(b_1 (V_k^{-1} d_x)^2 + \frac{2}{\epsilon} (V_k^{-1} c^{\top} \Lambda_k (V_k) z_k)^2 - 2 \right) + \frac{V_k^{\mu_k}}{2(n-1)} \left(\frac{\|\zeta_{d_y,k}\|_P^2}{\delta} + \frac{\|\zeta_{\mu,k}\|_P^2}{\|\zeta\|_2} - 2 \right). \end{aligned} \quad (3.16)$$

Let us show that the right-hand side of the inequality (3.16) is nonpositive for all $(V_k, x_t) \in \Omega_k$ such that $V_1 \in (\max\{\bar{\varrho}_0, \bar{w}(\|d\|_{L_2^\infty})\}, 1]$ and $V_2 \in (\max\{1, \bar{w}(\|d\|_{L_2^\infty})\}, \bar{\varrho}_x)$.

Firstly, using the Schur complement (see Lemma A.4 in Appendix A), LMIs (3.13c) yield $\Psi + \Theta^{\top} \frac{(n-1)^2}{4S_{n-1}} \Theta \preceq 0$, $PZP - \Xi_1^{-1} \succcurlyeq 0$ and $\Xi_2^{-1} - \rho P \succcurlyeq 0$, respectively. Therefore, the first two terms are not positive. Moreover, the third and fourth terms are negative if $\rho/h > V_k^{\mu_k}$, i.e. if $\bar{\varrho}_0 = \bar{\varrho}_x^{-1} = h^{1/\sqrt{\mu}}$.

Secondly, note that $V_k^{-1} \Lambda_k(V_k) \Lambda_k(V_k) V_k^{-1} = \text{diag}\{V_k^{2(r_{k,j}-1)}\}_{j=1}^n \prec I_n$ for all $V_1 \leq 1$ and $V_2 > 1$ since $r_{1,1} > r_{1,n} > 1$ and $r_{2,1} < r_{2,n} < 1$. Hence, $(V_k^{-1} c^{\top} \Lambda_k(V_k) z_k)^2 \leq \epsilon z_k^{\top} z_k \leq \epsilon/2$ for all $(V_k, x_t) \in \Omega_k$ due to (3.13b). Moreover, one can see that $b_1 (V_k^{-1} d_x)^2 \leq \eta^2 (V_k^{-1} d)^2 \leq 1$ if $\bar{w}(s) \geq \eta s$. Thus, the fifth term is also negative.

Thirdly, taking into account (3.7b), it can be shown that $|\hat{d}_{y,j}| \leq (2/h)^{j-1} \|d_y\|_{L_2^\infty}$ for $j = \overline{1, n}$. As a result, $\|\zeta_{d_y,k}\|_2^2 \leq \frac{(2/h)^{2n-1}}{(2/h)^{2-1}} \|d\|_{L_2^\infty}^2 \max\{V_k^{-2r_{k,1}}, V_k^{-2r_{k,n}}\}$. Then assuming that $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$ (see the proof below), it is clear that the sixth term in (3.16) is negative if $\bar{w}(s) \geq \max\{(\eta s)^{1/r_{1,1}}, (\eta s)^{1/r_{2,1}}\}$.

Finally, since $P^{-1/2} G_k P^{1/2} + P^{1/2} G_k P^{-1/2} \preceq \gamma I_n$ implies $G_k P + P G_k \preceq \gamma P$ and, on the other hand, $\max_{i=\overline{1, i-1}, k=1,2} \{2r_{k,i+2} - \mu_k\} = 2r_{1,3} - \mu_1 = 2 + (2n-3)\mu$, then $-\partial Q_k(V_k, x_t)/\partial V_k \leq \gamma V_k^{-1}$ for all $(V_k, x_t) \in \Omega_k$. Therefore, one can conclude that conditions C_i6a and C_i6b hold with $\bar{w}(s) = \max\{(\eta s)^{1/r_{1,1}}, (\eta s)^{1/r_{2,1}}\}$, $\bar{\varrho}_0 = \bar{\varrho}_x^{-1} = h^{1/\sqrt{\mu}}$ and $\alpha_k^{-1} = 2(n-1)\gamma$. Taking into account the formulas of $q_{1,k}$ and $q_{2,k}$, $k = 1, 2$ parameters $\varrho_0, \varrho_x, \varrho_d, T_0$ and function $w \in \mathcal{K}$ can be easily calculated using formulas (3.3). \square

B.III. Proof of the estimate $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$ in Ω_k

Adding and subtracting the terms $V_k^{-1} |\hat{y}_j|^{1/r_{k,j}}$, the disturbance $\|\zeta_{\mu,k}\|_2^2$ can be rewritten in the following form:

$$\|\zeta_{\mu,k}\|_2^2 = \sum_{j=1}^n \left((V_k^{-1} |\hat{y}_j|^{1/r_{k,j}} - |V_k^{-r_{k,j}} \hat{y}_j|) + V_k^{-1} (|\hat{y}_j|^{a_j(\bar{\mu})} - |\hat{y}_j|^{1/r_{k,j}}) \right)^2. \quad (3.17)$$

First, applying Lemma A.3 given in Appendix A with $\vartheta = \varphi_i$, $\phi = \dot{x}_{t,i+1}(0)$, $\varpi = \psi_i/\varphi_i$ to (3.10) for all $(V_k, x_t) \in \Omega_k$ such that $V_k^{-\mu_k} \rho/h > 1$, we deduce that:

$$z_k^{\top} P z_k + \sum_{i=1}^{n-1} \frac{i}{2\rho\omega_i S_i} (V_k^{-r_{k,i+1}} \varepsilon_i)^2 \leq 1.$$

Taking into account that $P \succcurlyeq 2I_n$ due to (3.13b) and, on the other hand, $4\omega_i \rho S_i \leq i$, it follows that

$$2V_k^{-2r_{k,1}} x_{t,1}^2(0) + 2 \sum_{i=1}^{n-1} V_k^{-2r_{k,i+1}} (x_{t,i+1}^2(0) + \varepsilon_i^2) \leq 1.$$

Then $\|\Lambda_k(V_k^{-1})\hat{y}\|_2^2 \leq 1 + 2\|\zeta_{d_y,k}\|_2^2 \leq 1 + \delta = \Delta^2$, what implies that $\|\hat{y}\|_2 \leq \Delta \max\{V_k^{r_{k,1}}, V_k^{r_{k,n}}\}$ and $|V_k^{-r_{k,j}}\hat{y}_j| \leq \Delta$. Thus, applying Lemma A.2 given in Appendix A with $s = |V_k^{-r_{k,j}}\hat{y}_j|$, $\bar{s} = \Delta$ and $\epsilon = 1/r_{k,j}$, the first term in (3.17) can be bounded as follows:

$$|V_k^{-r_{k,j}}\hat{y}_j|^{1/r_{k,j}} - |V_k^{-r_{k,j}}\hat{y}_j| \leq \bar{g}\left(\Delta, \frac{1}{r_{k,j}}\right). \quad (3.18)$$

Since $V_1 \leq \Delta^{1/(\mu_1-1)}$ implies $\|\hat{y}\|_2 \leq 1$, we deduce that $r_{1,j}a_j(\tilde{\mu}) = 1$ for all $V_1 \leq \Delta^{1/(\mu_1-1)}$. On the other hand, if $|\hat{y}_j| \geq \Delta$ for all $j = \overline{1, n}$, then $\|\hat{y}\|_2 \geq \Delta$ and $r_{2,j}a_j(\tilde{\mu}) = 1$. Thus, for all $V_1 \in (\Delta^{1/(\mu_1-1)}, 1]$ and $V_2 > 1$, the second term in (3.17) could be estimated as:

$$V_k^{-1}(|\hat{y}_j|^{a_j(\tilde{\mu})} - |\hat{y}_j|^{1/r_{k,j}}) \leq \Delta^{1/(1-\mu_1)} \max_{|\hat{y}| \in [0, \Delta]} \left| |\hat{y}_j|^{1/r_{1,j}} - |\hat{y}_j|^{1/r_{2,j}} \right| = \Delta^{1/(1-\mu_1)} \bar{g}\left(\Delta^{1/r_{2,j}}, \frac{r_{2,j}}{r_{1,j}}\right).$$

Taking into account (3.18), one can finally conclude that $\|\zeta_{\mu,k}\|_2 \leq \|\zeta\|_2$. \square

3.C Proof of Proposition 3.4

It is a well-known fact [70] that the system (3.4), (3.5) with $\mu = 0$ is exponentially ISS with a decay rate $\beta \in (0, \beta_0)$, where $\beta_0 > 0$ is the decay rate of the corresponding state-feedback control, i.e. for all $x_0 \in W([-h, 0], \mathbb{R}^n)$ and $d \in L_2^\infty(\mathbb{R}_+, \mathbb{R}^2)$ there exists a constant $c_0 > 0$ and a function $\tilde{w} \in \mathcal{K}$ such that

$$\|x(t, x_0, d)\|_2 \leq c_0 \|x_0\|_W e^{-\beta t} + \tilde{w}(\|d\|_{L_2^\infty}), \quad \forall t \geq 0.$$

Define by T_0^* the moment of time when the system (3.4), (3.5) with time delay h^* and $\mu = 0$ reaches the set \mathcal{A} , i.e. $T_0^* = \inf\{t \geq 0 : \|x(t, x_0, d)\|_2 \leq \varrho_0 + w(\|d\|_{L_2^\infty})\}$. Obviously, $T_0^* \geq \max\{0, \beta_0^{-1} \ln(c_0 \|x_0\|_W / (\varrho_0 + w(\varrho_d)))\}$ if $\varrho_0 + w(\|d\|_{L_2^\infty}) \geq \tilde{w}(\|d\|_{L_2^\infty})$. Otherwise, the set \mathcal{A} is unreachable. Therefore, it is easy to see that $T_0 \leq T_0^*$ if

$$\|x_0\|_W \geq \varrho_x^* := e^{\frac{\beta_0}{(n-1)\mu\gamma}} (\varrho_0 + w(\varrho_d)) / c_0.$$

Clearly, there is a small enough h^* such that $\varrho_x \geq \varrho_x^*$. \square

CHAPTER**4**

**OUTPUT FINITE-TIME STABILIZATION
UNDER STATE CONSTRAINTS:
IMPLICIT LYAPUNOV METHOD**

In this chapter, the problem of robust output finite-time stabilization of linear systems under state constraints is addressed. Geometrically, the considered class of state constraints represents a closed region (hyperoctahedron, hypersphere or hypercube) centered at the origin, within which trajectories of the closed-loop system must remain. The problem is solved in two steps. First, a nonlinear Luenberger-like observer is designed using the Implicit Lyapunov method in order to reconstruct the state vector in finite time. Then, a continuous control law is proposed, which is linear when trajectories of the closed-loop system risk violating the state constraints, and nonlinear otherwise. It is shown that while the linear controller guarantees exponential stabilization of the system under the state constraints, the nonlinear one accelerates the rate of convergence to the equilibrium point. Compared to the existing methods of stabilization of dynamical systems under state constraints based on control barrier functions and barrier Lyapunov functions, the tuning of the proposed nonlinear control system is extremely simple: the observer and controller parameters are found from the solutions of linear matrix inequalities and equations. Numerical simulation of the designed control system shows that, for sufficiently small external disturbances, the linear system is stabilized in finite time without violating the state constraints.

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4.1 Introduction

In practice, physical limitations and operational requirements impose severe restrictions on the functioning of dynamical systems. For example, state constraints require that trajectories of a system always remain in some bounded region of the state space. Furthermore, in many applications, a control objective must be achieved within a prescribed finite-time interval. Clearly, neglecting such constraints at the control design stage may lead to performance degradation, system failures or safety risks.

Although numerous approaches have been developed to address these problems, the most popular ones are Lyapunov-like methods based on *control barrier functions* (CBFs) [66, 78] or *barrier Lyapunov functions* (BLFs) [53, 76]. The former method consists in introducing a continuously differentiable function 1) that is nonpositive for all admissible values of the state vector and 2) the time derivative of which does not increase along trajectories of the system. However, this approach has limited use in practice due to the restrictive assumption on the smoothness of CBFs. On the other hand, the latter method involves the construction of a Lyapunov function that grows to infinity whenever trajectories of the system risk violating the state constraints. Though a wider class of state constraints can be considered compared to the CBF-based method, it is clear that BLFs are essentially nonlinear. Thus, the stability analysis becomes a difficult problem even for linear systems.

Nevertheless, for particular classes of state constraints, the control design for linear systems admits a relatively simple solution. For example, if trajectories of a system must stay in some hyperoctahedron, hypersphere or hypercube [57, 56] centered at the origin, then a control law can be chosen in the form of the conventional linear feedback. As a result, the control design can be easily formulated as a linear optimization problem [58]. However, in this case, the controllability of the system is not sufficient to solve the problem and some additional restrictions, such as the diagonal dominance of the closed-loop system matrix [79]), must be imposed.

On the other hand, it is clear that finite-time stabilization cannot be achieved by means of a linear controller, and thus another, nonlinear regulator has to be designed. For example, this can be effectively done by using the Implicit Lyapunov method [61]. Due to the implicit formulation, stability analysis is reduced to solving of linear matrix inequalities and equations that can be easily done using appropriate mathematical software. Moreover, due to the structure of the obtained nonlinear controller, it is possible to design a continuous linear-nonlinear control law.

Therefore, the main objective of the work, presented in this chapter, is to provide an alternative way of robust output finite-time stabilization of linear systems under a particular class of state constraints, which geometrically represents a closed region (hyperoctahedron, hypersphere or hypercube) centered at the origin, within which trajectories of the closed-loop system must remain. To solve the problem, we will first design a nonlinear Luenberger-like observer in order to reconstruct the system state in finite time. Then, based on the obtained state estimate, we will propose a continuous switching control law, which is linear when trajectories of the closed-loop system risk violating the state constraints, and nonlinear otherwise. It will be shown that while the linear controller guarantees exponential stabilization of the system under the state constraints, the nonlinear one accelerates the rate of convergence to the equilibrium point. Furthermore, the closed-loop system is also ISS with respect to external disturbances. Another advantage of the proposed control scheme compared to [53, 76] is a simple tuning algorithm: the observer and controller parameters are obtained as solutions of linear matrix equations and inequalities.

4.2 Problem formulation

Consider a linear system of the following form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + d_x(t), & x(0) = x_0, \\ y(t) = Cx(t) + d_y(t), \end{cases} \quad (4.1)$$

where $x(t) \in \mathbb{R}^n$ is the state; $x_0 \in \mathbb{R}^n$ is the initial value of the state; $u(t) \in \mathbb{R}^m$ is the control input, $m < n$; $y(t) \in \mathbb{R}^k$ is the output, $k \leq n$; $d_x(t) \in \mathbb{R}^n$ is the state perturbation, $d_y(t) \in \mathbb{R}^k$ is the measurement noise, $d = \text{col}\{d_x, d_y\} \in L_p^\infty(\mathbb{R}_+, \mathbb{R}^{n+k})$. Assume that system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{k \times n}$ are such that the pairs (A, B) and (A, C) are controllable and observable, respectively, $\text{rank}(B) = m$ and $\text{rank}(C) = k$.

The goal is to design continuous output feedback $u(y)$ such that for all $x_0 \in \mathcal{B}_p[\varrho_x]$ and $d \in \mathcal{B}_{L_p^\infty}[\varrho_d]$, where $p \in \{1, 2, \infty\}$, solutions of the closed-loop system $x(t, x_0, d)$ satisfy the estimates

$$\|x(t, x_0, d)\|_p \leq \varrho_x, \quad \forall t \geq 0, \quad (4.2a)$$

$$\|x(t, x_0, d)\|_p \leq v(\|x_0\|_p, t) + w(\|d\|_{L_p^\infty}), \quad \forall t \geq 0, \quad (4.2b)$$

where $v \in \mathcal{GKL}$ and $w \in \mathcal{K}_\infty$. The first condition represents the state constraints that can be formulated as follows: for sufficiently small external disturbances d , system trajectories $x(t, x_0, d)$ starting inside the closed ball $\mathcal{B}_p[\varrho_x]$ never leave it. The second condition guarantees that the system is stabilized at the origin in finite time in the absence of external disturbances or in its vicinity otherwise.

We will solve the problem in two steps. First, a nonlinear observer will be designed to reconstruct the system state in finite time. Then, we will synthesize a switching, linear-nonlinear controller ensuring fulfillment of the estimates (4.2). To this end, we will use two theorems on ISS of dynamical systems presented in the next section.

4.3 Input-to-state stability of dynamical systems

Consider a nonlinear system in the form:

$$\dot{x}(t) = f(x(t), d(t)), \quad x(0) = x_0, \quad (4.3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^n$ is the external disturbance, $d \in L_p^\infty(\mathbb{R}_+, \mathbb{R}^n)$, function $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous in x and d . A solution to system (4.3) with initial conditions $x_0 \in \mathbb{R}^n$ is denoted as $x(t, x_0, d)$.

For finite-time ISS analysis of (4.3), the following Implicit Lyapunov theorem can be used (see the explicit version in [35]).

Theorem 4.1. *If there exists a continuous function $Q : \mathbb{R}_+^* \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that:*

C_i1) Q is continuously differentiable outside the origin;

C_i2) for any $\chi \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+^$ such that $Q(V, \chi) = 0$;*

C_i3) $\frac{\partial Q(V, \chi)}{\partial V} < 0$ for all $V \in \mathbb{R}_+^$ and $\chi \in \mathbb{R}^n \setminus \{0\}$;*

$C_i4)$ there exist $q_1, q_2 \in \mathcal{IK}_\infty$ such that for all $V \in \mathbb{R}_+^*$ and $\chi \in \mathbb{R}^n \setminus \{0\}$:

$$q_1(V, \|\chi\|_2) \leq Q(V, \chi) \leq q_2(V, \|\chi\|_2);$$

$C_i5)$ for some $\mu > 0$, $\alpha > 0$, $\bar{w} \in \mathcal{K}$ and all $x(t) \in \mathbb{R}^n$ satisfying (4.3) we have:

$$(V, x(t)) \in \Omega := \{(s, \varphi) \in (\bar{w}(\|d\|_{L_2^\infty}), +\infty) \times \mathbb{R}^n : Q(s, \varphi) = 0\} \implies F(V, x(t)) \leq -\alpha V^{1-\mu},$$

where $F(V, x(t)) := -\left(\frac{\partial Q(V, x(t))}{\partial V}\right)^{-1} \frac{\partial Q(V, \chi)}{\partial \chi} \Big|_{\chi=x(t)} f(x(t), d(t))$, then the system (4.3) is finite-time ISS, i.e., the following estimate holds:

$$\|x(t, x_0, d)\|_2 \leq v(\|x_0\|_2, t) + w(\|d\|_{L_2^\infty}), \quad \forall t \geq 0,$$

where $v \in \mathcal{GKL}$ and $w \in \mathcal{K}_\infty$.

On the other hand, it is well known that a linear time-invariant system

$$\dot{x}(t) = Dx(t) + d(t), \quad x(0) = x_0, \quad (4.4)$$

where $D \in \mathbb{R}^{n \times n}$ is the system matrix, is ISS if and only if it is globally asymptotically (exponentially) stable in the input-free case ($d \equiv 0$), i.e., D is Hurwitz [74]. However, the latter property does not necessarily imply that the L_1 , L_2 or L_∞ norm of the state would decrease monotonically if $d \equiv 0$. In the next theorem, we present additional constraints on the system matrix D under which a better estimate on the solutions of (4.4) with respect to the specified norms can be obtained.

Theorem 4.2. For given $p \in \{1, 2, \infty\}$ solutions of the system (4.4) admit the estimate

$$\|x(t, x_0, d)\|_p \leq e^{-\sigma_p t} \|x_0\|_p + \frac{1 - e^{-\sigma_p t}}{\sigma_p} \|d\|_{L_p^\infty}, \quad \forall t \geq 0, \quad (4.5)$$

where $\sigma_p > 0$, for all $x_0 \in \mathbb{R}^n$ and $d \in L_p^\infty(\mathbb{R}_+, \mathbb{R}^n)$, if for

$p = 1$: matrix D is strictly column diagonally dominant, i.e.

$$\sigma_1 = \sigma_1(D) := -\max_{j=1, n} \left\{ D_{jj} + \sum_{i \neq j} |D_{ij}| \right\} > 0; \quad (4.6)$$

$p = 2$: the symmetric part of D is negative definite, i.e.

$$\sigma_2 = \sigma_2(D) := -\lambda_{\max} \left(\frac{D + D^\top}{2} \right) > 0; \quad (4.7)$$

$p = \infty$: matrix D is strictly row diagonally dominant, i.e.

$$\sigma_\infty = \sigma_\infty(D) := -\max_{i=1, n} \left\{ D_{ii} + \sum_{j \neq i} |D_{ij}| \right\} > 0. \quad (4.8)$$

Proof. For each $p \in \{1, 2, \infty\}$, introduce a Lipschitz continuous Lyapunov function $V_p(x) := \|x\|_p$ and calculate its time derivative $\dot{V}_p(x)$ along the solutions of the system (4.4) using Theorem C.1 given in Appendix C.

p = 1: Taking into account (C.2) and (4.6), one can get:

$$\begin{aligned}\dot{V}_1(x) &\stackrel{\text{a.e.}}{=} \sum_{i=1}^n \text{SGN}(x_i) \left(\sum_{j=1}^n D_{ij} x_j + d_i \right) = \sum_{j=1}^n \left(\sum_{i=1}^n \text{SGN}(x_i) D_{ij} \right) x_j + \sum_{i=1}^n \text{SGN}(x_i) d_i \\ &\leq \sum_{j=1}^n \left(D_{jj} + \sum_{i \neq j} |D_{ij}| \right) |x_j| + \sum_{i=1}^n |d_i| \leq -\sigma_1 \sum_{j=1}^n |x_j| + \|d\|_{L_1^\infty} = -\sigma_1 V_1(x) + \|d\|_{L_1^\infty}.\end{aligned}$$

p = 2: From the Cauchy–Schwarz inequality and (4.7) it follows that:

$$\dot{V}_2(x) = \frac{x^\top (D + D^\top)x + 2x^\top d}{2\|x\|_2} \leq -\sigma_2 \|x\|_2 + \|d\|_{L_2^\infty} = -\sigma_2 V_2(x) + \|d\|_{L_2^\infty}.$$

p = ∞: Taking into account (C.3) and (4.8), one can get:

$$\begin{aligned}\dot{V}_\infty(x) &\stackrel{\text{a.e.}}{=} \sum_{i \in I(x)} c_i \text{sgn}(x_i) \left(\sum_{j=1}^n D_{ij} x_j + d_i \right) = \sum_{i \in I(x)} c_i \left(D_{ii} |x_i| + \sum_{j \neq i} \text{sgn}(x_i) D_{ij} x_j \right) + \sum_{i \in I(x)} c_i \text{sgn}(x_i) d_i \\ &\leq \sum_{i \in I(x)} c_i \left(D_{ii} + \sum_{j \neq i} |D_{ij}| \right) \|x\|_\infty + \sum_{i \in I(x)} c_i |d_i| \leq (-\sigma_\infty \|x\|_\infty + \|d\|_{L_\infty^\infty}) \sum_{i \in I(x)} c_i = -\sigma_\infty V_\infty(x) + \|d\|_{L_\infty^\infty}.\end{aligned}$$

Since $\dot{V}_p(x(t)) \leq -\sigma_p V_p(x(t)) + \|d\|_{L_p^\infty}$ almost everywhere, then the Comparison Lemma A.1 implies $V_p(x(t)) \leq e^{-\sigma_p t} (V_p(x_0) - \|d\|_{L_p^\infty} / \sigma_p) + \|d\|_{L_p^\infty} / \sigma_p$. Therefore, the estimate (4.5) holds. \square

Remark 4.1. Obviously, Theorem 4.2 can be generalized to the case of weighted norms $\|Mx\|_p$, where $M \in \mathbb{R}^{n \times n}$ is an invertible matrix. By introducing new coordinates $\tilde{x} := Mx$, system (4.4) can be rewritten as follows:

$$\dot{\tilde{x}}(t) = \tilde{D}\tilde{x}(t) + \tilde{d}(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad (4.9)$$

where $\tilde{D} := MDM^{-1}$, $\tilde{d}(t) := Md(t)$ and $\tilde{x}_0 := Mx_0$.

Therefore, Theorem 4.2 remains valid for system (4.9) if x , d and D are replaced by \tilde{x} , \tilde{d} and \tilde{D} , respectively. For example, estimate (4.5) holds for the weighted Euclidean norm $\|x\|_P$, i.e., $M = P^{1/2}$, if matrix $P^{1/2}DP^{1/2} + P^{-1/2}D^\top P^{1/2}$ is negative definite. Clearly, this condition is fulfilled if $PD + D^\top P \prec 0$. On the other hand, if $M = \text{diag}\{m_i\}_{i=1}^n$, $m_i > 0$, then $\tilde{D}_{ii} = D_{ii}$ and $\tilde{D}_{ij} = \frac{m_i}{m_j} D_{ij}$, $i \neq j$, in (4.6) and (4.8). Moreover, if matrix D is Metzler ($D_{ij} \geq 0$, $i \neq j$), then conditions (4.6) and (4.8) are satisfied if $m^\top D < 0$ and $Dm < 0$, respectively, for $m = [m_1, \dots, m_n]^\top$.

Remark 4.2. It is worth stressing that for more sophisticatedly defined norms similar analysis can be done by using the method of vector Lyapunov functions [9, 46]. However, in this work we will focus only on the L_1 , L_2 and L_∞ norms that are mostly used in practice.

Compared to the general formulation of the ISS property for linear time-invariant systems, the main advantage of the estimate (4.5) is lack of the “peaking phenomenon” related to nonzero initial conditions. Due to this improvement, the following important observation can be made.

Corollary 4.1. Let conditions of Theorem 4.2 be satisfied, then for all $x_0 \in \mathcal{B}_p[\varrho_x]$ and $d \in \mathcal{B}_{L_p^\infty}[\varrho_d]$, where $\varrho_d \leq \varrho_x \sigma_p$, solutions $x(t, x_0, d)$ of the system (4.4) remain in the set $\mathcal{B}_p[\varrho_x]$ for all $t \geq 0$.

Therefore, if the control input $u(t)$ in (4.1) is chosen as a linear feedback such that the closed-loop system matrix satisfies one of the conditions (4.6)–(4.8) then the state constraints (4.2a) are not violated for sufficiently small external disturbances.

4.4 Observer design

Let $\hat{x}(t) \in \mathbb{R}^n$ be the estimate of the state vector $x(t)$ reconstructed by the following observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + g(C\hat{x}(t) - y(t)), \quad \hat{x}(0) = \hat{x}_0, \quad (4.10)$$

where the injection term $g(\vartheta)$ to be defined. Then the estimation error $\varepsilon := \hat{x} - x$ satisfies the following differential equation:

$$\dot{\varepsilon}(t) = A\varepsilon(t) + g(C\varepsilon(t) - d_y(t)) - d_x(t), \quad \varepsilon(0) = \varepsilon_0, \quad (4.11)$$

where $\varepsilon_0 := \hat{x}_0 - x_0$. A solution of (4.11) is denoted as $\varepsilon(t, \varepsilon_0, d)$.

Let us apply Theorem 4.1 to design a nonlinear injection term $g(\vartheta)$ such that the system (4.11) is finite-time ISS. To this end, introduce an ILF candidate $Q_c(V_c, \chi)$ as follows [60]:

$$Q_o(V_o, \chi) := \chi^\top \Lambda_o(V_o^{-1}) P_o \Lambda_o(V_o^{-1}) \chi - 1, \quad (4.12)$$

where $0 \prec P_o \in \mathbb{S}^{n \times n}$, $\Lambda_o(\lambda) := \exp\{G_o \ln \lambda\}$ is the dilation matrix with an anti-Hurwitz⁵ matrix $G_o \in \mathbb{R}^{n \times n}$ called the dilation generator [54].

Proposition 4.1. *Let matrices $X_o \in \mathbb{R}^{n \times n}$ and $L_o \in \mathbb{R}^{n \times k}$ be a solution of the matrix equations*

$$AX_o = (X_o + I_n)(A + L_o C), \quad (4.13a)$$

$$CX_o = O_{k \times n}. \quad (4.13b)$$

If for adjustable $\mu \in (-1, 0)$ and $\alpha_o > 0$ the system of matrix inequalities

$$(A + L_o C)^\top P_o + P_o(A + L_o C) + C^\top Y_o^\top + Y_o C + 8\alpha_o P_o \preceq 0, \quad (4.14a)$$

$$P_o \succ C^\top C, \quad P_o \succ 0, \quad (4.14b)$$

$$\Xi_\mu(\xi) P_o^{-1} Y_o Y_o^\top P_o^{-1} \Xi_\mu^\top(\xi) \preceq \alpha_o^2 P_o^{-1}, \quad \forall \xi \in [0, \bar{\xi}], \quad (4.14c)$$

$$P_o G_o + G_o^\top P_o \succ 2|\mu| P_o, \quad (4.14d)$$

where $G_o := \mu X_o + I_n$, $\Xi_\mu(\xi) := \xi(\exp\{\mu(X_o + I_n) \ln \xi\} - I_n)$ and $\bar{\xi} := 1 + \sqrt{\|d\|_{L_2^\infty}}$, is feasible for some $P_o \in \mathbb{S}^{n \times n}$ and $Y_o \in \mathbb{R}^{n \times k}$, then system (4.11) with the continuous injection term defined as

$$g(\vartheta) := \begin{cases} L_o \vartheta + \|\vartheta\|_2^{\mu-1} \Lambda_o(\|\vartheta\|_2) L \vartheta, & \text{if } \vartheta \neq 0, \\ 0, & \text{if } \vartheta = 0, \end{cases} \quad (4.15)$$

where $L := P_o^{-1} Y_o$, is finite-time ISS:

$$\|\varepsilon(t, \varepsilon_0, d)\|_{P_o} \leq v_o(\|\varepsilon_0\|_{P_o}, t) + w_o(\|d\|_{L_2^\infty}), \quad \forall t \geq 0,$$

with functions $v_o \in \mathcal{GKL}$ and $w_o \in \mathcal{K}_\infty$ given by

$$v_o(s, t) := \begin{cases} \left(\frac{\alpha_o |\mu|}{\gamma_o} (T'_o(s) - t) + 1 \right)^{\gamma_o/|\mu|}, & \text{if } t \in [0, T'_o(s)], \\ \left(\frac{\alpha_o |\mu|}{\gamma_o} (T_o(s) - t) \right)^{\beta_o/|\mu|}, & \text{if } t \in [T'_o(s), T_o(s)], \\ 0, & \text{if } t \in [T_o(s), +\infty), \end{cases}$$

$$w_o(s) := \max \{w_1(s), w_2(\eta_o \|L\|_2 s), w_3(\eta_o (1 + \|L_o\|_2) s)\},$$

⁵A square matrix G is called anti-Hurwitz if $-G$ is Hurwitz.

where $\gamma_o := 0.5\lambda_{\max}(P_o^{1/2}G_oP_o^{-1/2} + P_o^{-1/2}G_o^\top P_o^{1/2})$, $\beta_o := 0.5\lambda_{\min}(P_o^{1/2}G_oP_o^{-1/2} + P_o^{-1/2}G_o^\top P_o^{1/2})$,
 $\eta_o := \frac{\sqrt{\lambda_{\max}(P_o)}}{\alpha_o}$, $T_o'(s) := \frac{\gamma_o \max\{0, s^{|\mu|/\beta_o} - 1\}}{\alpha_o|\mu|}$, $T_o(s) := \frac{\gamma_o \min\{1, s^{|\mu|/\gamma_o}\}}{\alpha_o|\mu|} + T_o'(s)$,
 $w_1(s) := \max\{s^{\gamma_o/2}, s^{\beta_o/2}\}$, $w_2(s) := \max\{s^{\gamma_o}, s^{\beta_o}\}$, $w_3(s) := \max\{s^{\beta_o/(\gamma_o+\mu)}, s^{\gamma_o/(\beta_o+\mu)}\}$.

The proof of Proposition 4.1 is given in Appendix 4.B.

It is worth mentioning that matrix equations (4.13) always have a solution since the pair (A, C) is observable (see Lemma 4.1 in Appendix 4.A). Furthermore, since matrix $X_o + I_n$ is invertible, then matrix equations (4.13) can be transformed into a linear form by introducing a new decision variable $Z_o := (X_o + I_n)L_o$.

On the other hand, the system of matrix inequalities (4.14) is always feasible for sufficiently small $|\mu|$. Indeed, for any given $\alpha_o > 0$, observability of the pair (A, C) implies feasibility of LMI (4.14a) for some $P_o \succ 0$ and Y_o . Note that conditions (4.14b) is not restrictive since it can always be fulfilled by properly scaling matrices $P_o \succ 0$ and Y_o . Obviously, for the obtained P_o there exists sufficiently small $|\mu| \neq 0$ such that LMI (4.14d) holds. Finally, since $\|\xi^{-1}\Xi_\mu(\xi)\|_2 = \|\sum_{i=1}^{\infty} \frac{\mu^i (X_o + I_n)^i \ln^i \xi}{i!}\|_2 \leq \sum_{i=1}^{\infty} \frac{|\mu|^i \|X_o + I_n\|_2^i \ln^i \xi}{i!}$, then $\|\Xi_\mu(\xi)\|_2 \leq \xi(\xi^{-|\mu|\|X_o + I_n\|_2} - 1)$ for $\xi \in [0, 1]$ and $\|\Xi_\mu(\xi)\|_2 \leq \bar{\xi} \sum_{i=1}^{\infty} \frac{|\mu|^i \|X_o + I_n\|_2^i \ln^i \bar{\xi}}{i!}$ for $\xi \in [1, \bar{\xi}]$. It follows from the proof of Proposition 11.1 given in [60] that $\sup_{\xi \in [0, 1]} \|\Xi_\mu(\xi)\|_2 \leq |\mu| \|X_o + I_n\|_2$ for all $|\mu| \|X_o + I_n\|_2 < 1$. Hence, $\sup_{\xi \in [0, \bar{\xi}]} \|\Xi_\mu(\xi)\|_2 \rightarrow 0$ as $|\mu| \rightarrow 0$, what implies feasibility of the matrix inequality (4.14c) for small $|\mu|$.

However, note that smaller values of $|\mu|$ lead to slower state estimation (see the definition of T_o). In the limit case $\mu = 0$, the state $x(t)$ cannot be reconstructed in finite time since the proposed observer (4.10), (4.15) becomes linear with the infinite settling time $T_o = +\infty$. Therefore, to accelerate the convergence rate as much as possible, one has to find the largest $|\mu|$ for which matrix inequalities (4.14) are feasible.

In order to apply Proposition 4.1, one must solve the system of parameterized nonlinear matrix inequalities (4.14) with respect to matrices P_o and Y_o . By fixing parameter $\xi \in [0, \bar{\xi}]$ and assuming that $P_o^{-1}Y_o Y_o^\top P_o^{-1} \preccurlyeq cP_o^{-1}$, where $c > 0$ is some constant, the parameterized matrix inequality (4.14c) can be expressed in the linear form. However, the obtained system of LMIs still must be checked for any $\xi \in [0, \bar{\xi}]$. Nonetheless, due to the smoothness of the matrix-valued function $\Xi_\mu(\xi)$ with respect to the parameter ξ , it suffices to check the LMIs only for a finite number of ξ from the interval $[0, \bar{\xi}]$. Taking this into account, the next proposition provides a simple LMI-based algorithm for verification of the parameterized matrix inequality (4.14c).

Proposition 4.2. *The parameterized matrix inequality (4.14c) holds if for adjustable $\mu \in (-1, 0)$, $\alpha_o > 0$ and $N \in \mathbb{N}$ the system of LMIs*

$$\begin{bmatrix} P_o & \sqrt{|\mu|}Y_o \\ \sqrt{|\mu|}Y_o^\top & I_k \end{bmatrix} \succcurlyeq 0, \quad (4.16a)$$

$$\frac{1}{|\mu|} \Xi_\mu^\top(\xi_i) P_o \Xi_\mu(\xi_i) \preccurlyeq \left(\alpha_o^2 - \frac{2i-1}{2N^2} \bar{\xi}^2 \right) P_o, \quad i = \overline{1, N}, \quad (4.16b)$$

$$\begin{bmatrix} P_o(\mu X_o + I_n) + (\mu X_o^\top + I_n)P_o + 2\mu P_o & \sqrt{|\mu|}P_o(X_o + I_n) \\ \sqrt{|\mu|}(X_o^\top + I_n)P_o & P_o \end{bmatrix} \succcurlyeq 0, \quad (4.16c)$$

where $\xi_i := (i/N)\bar{\xi}$, is feasible for some $0 \prec P_o \in \mathbb{S}^{n \times n}$ and $Y_o \in \mathbb{R}^{n \times k}$.

Proof. Since $P_o^{-1} \succcurlyeq |\mu| P_o^{-1} Y Y^\top P_o^{-1}$ due to (4.16a), then condition (4.14c) holds if $\Xi_\mu(\xi) P_o^{-1} \Xi_\mu^\top(\xi) \preccurlyeq |\mu| \alpha_o^2 P_o^{-1}$. Using twice the Schur complement (see Lemma A.4 in Appendix A), the latter can be rewritten as $\Xi_\mu^\top(\xi) P_o \Xi_\mu(\xi) \preccurlyeq |\mu| \alpha_o^2 P_o$. Note that

$$\frac{d}{d\xi} \Xi_\mu(\xi) = \xi^{-1} \Xi_\mu(\xi) + \mu(X_o + I_n) \exp\{\mu(X_o + I_n) \ln \xi\} = \xi^{-1} (I_n + \mu(X_o + I_n)) \Xi_\mu(\xi) + \mu(X_o + I_n).$$

Then for $W(\xi, z) := z^\top \Xi_\mu^\top(\xi) P_o \Xi_\mu(\xi) z$, where $z \in \mathbb{R}^n$ is an arbitrary vector, we have

$$\begin{aligned} \frac{\partial}{\partial \xi} W(\xi, z) &= z^\top \left(\left(\frac{d}{d\xi} \Xi_\mu(\xi) \right)^\top P_o \Xi_\mu(\xi) + \Xi_\mu^\top(\xi) P_o \frac{d}{d\xi} \Xi_\mu(\xi) \right) z \\ &= z^\top \begin{bmatrix} \Xi_\mu(\xi) \\ I_n \end{bmatrix}^\top \begin{bmatrix} \xi^{-1} \left(P_o(\mu X_o + I_n) + (\mu X_o^\top + I_n) P_o + 2\mu P_o \right) & |\mu| P_o(X_o + I_n) \\ |\mu|(X_o^\top + I_n) P_o & O_{n \times n} \end{bmatrix} \begin{bmatrix} \Xi_\mu(\xi) \\ I_n \end{bmatrix} z. \end{aligned}$$

Condition (4.16c) implies that $\frac{\partial}{\partial \xi} W(\xi, z) \geq -|\mu| \xi z^\top P_o z$ and $W(\xi, z) \leq W(\xi_i, z) + |\mu| \frac{\xi_i^2 - \xi^2}{2} z^\top P_o z$ for any $\xi \in [\xi_i - \frac{\bar{\xi}}{N}, \xi_i]$, $i = 1, N$. Taking into account that $\xi_i^2 - \xi^2 \leq \frac{2i-1}{N^2} \bar{\xi}^2$, inequality (4.16b) yields $W(\xi_i, z) + |\mu| \frac{\xi_i^2 - \xi^2}{2} z^\top P_o z \leq |\mu| \alpha_o^2 z^\top P_o z$. Therefore, condition (4.14c) indeed holds. \square

Clearly, conditions (4.16a) and (4.16c) are fulfilled for sufficiently small $|\mu|$. Furthermore, since $\sup_{\xi \in [0, \bar{\xi}]} \|\Xi_\mu(\xi)\|_2 / \sqrt{|\mu|} \rightarrow 0$ as $|\mu| \rightarrow 0$, then there exists sufficiently large $N > (\bar{\xi}/\alpha_o)^2$ such that LMI (4.16b) is feasible.

4.5 Output-feedback stabilization

In this section, we will show how to design linear and nonlinear controllers for the system (4.1) to respect the state constraints (4.2a) and to guarantee the finite-time ISS property (4.2b), respectively.

4.5.1 Linear feedback

Let us begin with a simple linear feedback

$$u = u_0(\hat{x}) := K \hat{x}, \quad (4.17)$$

where $K \in \mathbb{R}^{m \times n}$ are controller gains to be chosen. Clearly, the closed-loop system (4.1), (4.10), (4.17) is in the form (4.4) with $D = A + BK$ and $d = d_x + BK\varepsilon(\varepsilon_0, d)$. Note that Corollary 4.1 implies that the state constraints (4.2a) are fulfilled if $\|d\|_{L^\infty} \leq \varrho_x \sigma_p(D) = \varrho_x \sigma_p(A + BK)$. The next proposition presents the way of selecting matrix K such that this inequality holds.

For all $\varrho_\varepsilon \geq 0$ and $\varrho_d \geq 0$, define the function $\zeta(\varrho_\varepsilon, \varrho_d) := \sqrt{\frac{n}{\lambda_{\min}(P_o)}} (\tilde{v}_o(\varrho_\varepsilon) + w_o(\sqrt{n}\varrho_d))$, where $\tilde{v}_o(s) := v_o(s\sqrt{n\lambda_{\max}(P_o)}, 0) = \max\{(s\sqrt{n\lambda_{\max}(P_o)})^{\gamma_o/\beta_o}, (s\sqrt{n\lambda_{\max}(P_o)})^{\beta_o/\gamma_o}\}$.

Proposition 4.3. *Let $x_0 \in \mathcal{B}_p[\varrho_x]$, $\varepsilon_0 \in \mathcal{B}_p[\varrho_\varepsilon]$ and $d \in \mathcal{B}_{L^\infty}[\varrho_d]$. Then solutions $x(t, x_0, \varepsilon_0, d)$ of the system (4.1), (4.10), (4.17) remain in the set $\mathcal{B}_p[\varrho_x]$ for all $t \geq 0$ if for*

p = 1: the system of LMIs

$$A_{jj} + \sum_{q=1}^m B_{jq} K_{qj} + \sum_{i \neq j} \varphi_{ij} \leq -\sigma, \quad j = \overline{1, n}, \quad (4.18a)$$

$$\zeta(\varrho_\varepsilon, \varrho_d) \sum_{s=1}^n \psi_{sr} \leq \varrho_x \sigma - \varrho_d, \quad r = \overline{1, n}, \quad (4.18b)$$

$$-\varphi_{ij} \leq A_{ij} + \sum_{q=1}^m B_{iq}K_{qj} \leq \varphi_{ij}, \quad i, j = \overline{1, n}, i \neq j, \quad (4.18c)$$

$$-\psi_{sr} \leq \sum_{q=1}^m B_{sq}K_{qr} \leq \psi_{sr}, \quad r, s = \overline{1, n}, \quad (4.18d)$$

is feasible for some $K \in \mathbb{R}^{m \times n}$, $\sigma > 0$, $\varphi_{ij} > 0$, $i, j = \overline{1, n}$, $i \neq j$, and $\psi_{sr} > 0$, $r, s = \overline{1, n}$;
p = 2: the system of LMIs

$$(A + BK) + (A + BK)^\top \preceq -2\sigma I_n, \quad (4.19a)$$

$$\begin{bmatrix} (\varrho_x \sigma - \varrho_d) I_n & \zeta(\varrho_\varepsilon, \varrho_d) (BK)^\top \\ \zeta(\varrho_\varepsilon, \varrho_d) BK & (\varrho_x \sigma - \varrho_d) I_n \end{bmatrix} \succeq 0 \quad (4.19b)$$

is feasible for some $K \in \mathbb{R}^{m \times n}$ and $\sigma > 0$;

p = ∞: LMIs (4.18c), (4.18d) and

$$A_{ii} + \sum_{q=1}^m B_{iq}K_{qi} + \sum_{j \neq i} \varphi_{ij} \leq -\sigma, \quad i = \overline{1, n}, \quad (4.20a)$$

$$\zeta(\varrho_\varepsilon, \varrho_d) \sum_{r=1}^n \psi_{sr} \leq \varrho_x \sigma - \varrho_d, \quad s = \overline{1, n}, \quad (4.20b)$$

are feasible for some $K \in \mathbb{R}^{m \times n}$, $\sigma > 0$, $\varphi_{ij} > 0$, $i, j = \overline{1, n}$, $i \neq j$, and $\psi_{sr} > 0$, $r, s = \overline{1, n}$.

Proof. Taking into account that $v_o(\|\varepsilon_0\|_{P_o}, t) \leq v_o(\|\varepsilon_0\|_{P_o}, 0) \leq v_o(\sqrt{n\lambda_{\max}(P_o)}\|\varepsilon_0\|_p, 0) = \tilde{v}_o(\|\varepsilon_0\|_p)$ and $\|d\|_{L_p^\infty} \leq \varrho_d + \|BK\|_p \|\varepsilon\|_{L_p^\infty}$, it suffices to show that $\varrho_d + \|BK\|_p \zeta(\varrho_\varepsilon, \varrho_d) \leq \varrho_x \sigma_p(A + BK)$. For the sake of brevity, denote $\zeta = \zeta(\varrho_\varepsilon, \varrho_d)$.

First note that $|D_{ij}| \leq \varphi_{ij}$ and $|(BK)_{sr}| \leq \psi_{sr}$ due to LMIs (4.18c) and (4.18d), respectively. Then taking into account that $\|BK\|_1 := \max_{r=\overline{1, n}} \sum_{s=1}^n |(BK)_{sr}|$ and $\|BK\|_\infty := \max_{s=\overline{1, n}} \sum_{r=1}^n |(BK)_{sr}|$, it follows from LMIs (4.18a), (4.18b) and (4.20a), (4.20b) that $\varrho_d + \zeta \|BK\|_1 \leq \varrho_x \sigma \leq \varrho_x \sigma_1(D)$ and $\varrho_d + \zeta \|BK\|_\infty \leq \varrho_x \sigma \leq \varrho_x \sigma_\infty(D)$, respectively. Analogously, since $\|BK\|_2 := \sqrt{\lambda_{\max}((BK)^\top BK)}$, then LMIs (4.19) imply $\varrho_d + \zeta \|BK\|_2 \leq \varrho_x \sigma \leq \varrho_x \sigma_2(D)$. \square

Note that, for given ϱ_x , feasibility of LMIs (4.18)–(4.20) requires not only the smallness of the external disturbances d_x and d_y but also the closeness of the initial conditions \hat{x}_0 and x_0 . As a result, the initial state of the system (4.1) must be known with the required accuracy ϱ_ε in order to properly set the initial state of the observer (4.10).

4.5.2 Nonlinear feedback

Obviously, it is impossible to stabilize the system (4.1) in finite time, i.e., to fulfill the estimate (4.2b), by applying linear control (4.17). Therefore, a nonlinear controller must be designed. To this end, we will use Theorem 4.1 with an ILF candidate $Q_c(V_c, \chi)$:

$$Q_c(V_c, \chi) := \chi^\top \Lambda_c(V_c^{-1}) P_c \Lambda_c(V_c^{-1}) \chi - 1, \quad (4.21)$$

where $0 \prec P_c \in \mathbb{S}^{n \times n}$, $\Lambda_c(\lambda) := \exp\{G_c \ln \lambda\}$ is the dilation matrix with the anti-Hurwitz dilation generator $G_c \in \mathbb{R}^{n \times n}$.

Proposition 4.4. *Let matrices $X_c \in \mathbb{R}^{n \times n}$ and $K_0 \in \mathbb{R}^{m \times n}$ be a solution of the matrix equations*

$$(A + BK_0)(X_c - I_n) = X_c A, \quad (4.22a)$$

$$X_c B = O_{n \times m}. \quad (4.22b)$$

If for adjustable $\nu \in (-1, 0)$ and $\alpha_c > 0$ the system of matrix inequalities

$$(A + BK)^\top P_c + P_c(A + BK) + 4\alpha_c P_c \preceq 0, \quad (4.23a)$$

$$P_c G_c + G_c^\top P_c \succ 2|\nu|P_c, \quad (4.23b)$$

where $G_c = \nu X_c + I_n$, is feasible for some $0 \prec P_c \in \mathbb{S}^{n \times n}$ and $K \in \mathbb{R}^{m \times n}$, then the system (4.10) governed by the continuous, locally bounded control

$$u = u_\nu(\hat{x}) := \begin{cases} K_0 \hat{x} + V_c^{1+\nu}(K - K_0)\Lambda_c(V_c^{-1})\hat{x}, & \text{if } \hat{x} \neq 0, \\ 0, & \text{if } \hat{x} = 0, \end{cases} \quad (4.24)$$

where $V_c \in \mathbb{R}_+^*$ is the solution of the equation $Q_c(V_c, \hat{x}) = 0$, is finite-time ISS:

$$\|\hat{x}(t, \hat{x}_0, g(C\varepsilon - d_y))\|_{P_c} \leq v_c(\|\hat{x}_0\|_{P_c}, t) + w_c(\|g(C\varepsilon - d_y)\|_{L_\infty}), \quad \forall t \geq 0,$$

with functions $v_c \in \mathcal{GKL}$ and $w_c \in \mathcal{K}_\infty$ given by

$$v_c(s, t) := \begin{cases} \left(\frac{\alpha_c |\nu|}{\gamma_c} (T'_c(s) - t) + 1\right)^{\gamma_c/|\nu|}, & \text{if } t \in [0, T'_c(s)], \\ \left(\frac{\alpha_c |\nu|}{\gamma_c} (T_c(s) - t)\right)^{\beta_c/|\nu|}, & \text{if } t \in [T'_c(s), T_c(s)], \\ 0, & \text{if } t \in [T_c(s), +\infty), \end{cases}$$

$$w_c(s) := \max\{(\eta_c s)^{\beta_c/(\gamma_c + \nu)}, (\eta_c s)^{\gamma_c/(\beta_c + \nu)}\},$$

where $\gamma_c := 0.5\lambda_{\max}(P_c^{1/2}G_cP_c^{-1/2} + P_c^{-1/2}G_c^\top P_c^{1/2})$, $\beta_c := 0.5\lambda_{\min}(P_c^{1/2}G_cP_c^{-1/2} + P_c^{-1/2}G_c^\top P_c^{1/2})$,

$$T'_c(s) := \frac{\gamma_c \max\{0, s^{|\nu|/\beta_c} - 1\}}{\alpha_c |\nu|}, \quad T_c(s) := \frac{\gamma_c \min\{1, s^{|\nu|/\gamma_c}\}}{\alpha_c |\nu|} + T'_c(s), \quad \eta_c := \frac{\sqrt{\lambda_{\max}(P_c)}}{\alpha_c}.$$

The proof of Proposition 4.4 is given in Appendix 4.C.

It is worth mentioning that matrix equations (4.22) always have a solution since the pair (A, B) is controllable (see Lemma 4.1 in Appendix 4.A). Furthermore, since matrix $X_c - I_n$ is invertible, then matrix equations (4.22) can be transformed into a linear form by introducing a new decision variable $Z_c := K_0(X_c - I_n)$.

On the other hand, the system of matrix inequalities (4.23) is always feasible for sufficiently small $|\nu|$. Indeed, for any given $\alpha_c > 0$, controllability of the pair (A, B) implies feasibility of the matrix inequality (4.23a) for some $P_c \succ 0$ and K . Obviously, for obtained P_c there exists sufficiently small $|\nu| \neq 0$ such that LMI (4.23b) is fulfilled. Note that in the sequel matrix K will be chosen the same as the one calculated for the linear controller (4.17) to guarantee continuity of the control law (4.25) on the switching surface. As a result, the matrix inequality (4.23a) becomes linear with respect to matrix P_c . It is worth stressing that the obtained LMI remains feasible for sufficiently small $\alpha_c > 0$.

However, note that smaller values of $|\nu|$ leads to slower stabilization (see the definition of T_c). In the limit case $\nu = 0$, the system (4.10) cannot be stabilized at the origin in finite time since the proposed controller (4.24) becomes linear with the infinite settling time $T_c = +\infty$. Therefore, to accelerate the convergence rate as much as possible, one has to find the largest $|\nu|$ for which matrix inequalities (4.23) are feasible.

4.5.3 Switching feedback

In order to stabilize system (4.1) in finite time without violating the state constraints, let us introduce the following switching control law:

$$u = u_s(\hat{x}) := \begin{cases} u_0(\hat{x}), & \text{if } \|\hat{x}\|_{P_c} > 1, \\ u_\nu(\hat{x}), & \text{otherwise,} \end{cases} \quad (4.25)$$

where $u_0(\hat{x})$ and $u_\nu(\hat{x})$ are defined in (4.17) and (4.24), respectively. Note that the control signal (4.25) is continuous if K is the same matrix in $u_0(\hat{x})$ and $u_\nu(\hat{x})$. Indeed, it follows from (4.21) that $V_c = 1$ for all $\hat{x} \in \mathbb{R}^n$ such that $\|\hat{x}\|_{P_c} = 1$, and, therefore, $u_\nu(\hat{x}) = K_0\hat{x} + (K - K_0)\hat{x} = Kx = u_0(\hat{x})$.

Let us now formulate the main result of this chapter.

Proposition 4.5. *Let matrices $X_c \in \mathbb{R}^{n \times n}$ and $K_0 \in \mathbb{R}^{m \times n}$ be a solution of the matrix equations (4.22), and matrix K be the solution of the LMIs (4.18)–(4.20) for given $\zeta(\varrho_\varepsilon, \varrho_d)$. If for adjustable $\nu \in (-1, 0)$ and $\alpha > 0$ the LMIs (4.23) are feasible for some $0 \prec P_c \in \mathbb{S}^{n \times n}$ such that*

$$(\varrho_x - \zeta(\varrho_\varepsilon, \varrho_d))^2 P_c \succcurlyeq nI_n, \quad (4.26)$$

then for all $x_0 \in \mathcal{B}_p[\varrho_x]$, $\varepsilon_0 \in \mathcal{B}_p[\varrho_\varepsilon]$ and $d \in \mathcal{B}_{L_p^\infty}[\varrho_d]$ solutions of the closed-loop system (4.1), (4.10), (4.25) satisfy the estimates (4.2).

Proof. I. State constraints (4.2a)

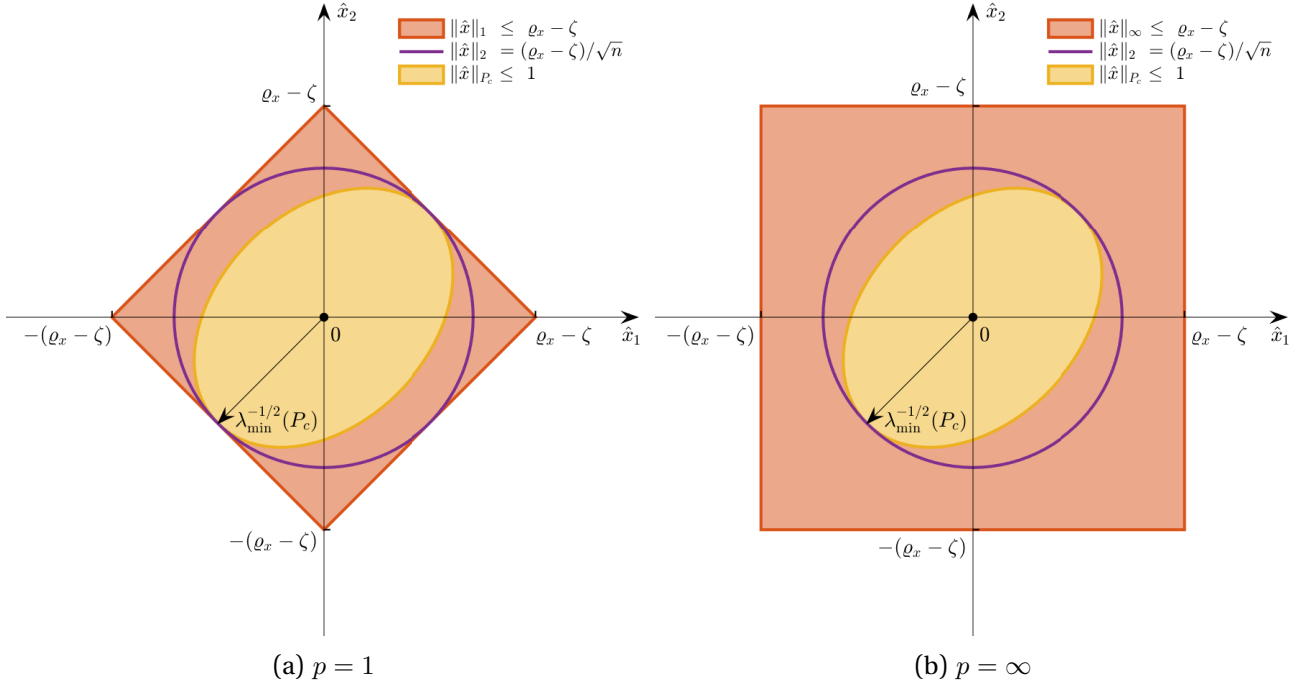
Since K is a solution of the LMIs (4.18)–(4.20), then it follows from Proposition 4.3 that $x(t) \in \mathcal{B}_p[\varrho_x]$ if $\|\hat{x}(t)\|_{P_c} > 1$. Therefore, it is necessary to prove that solutions $x(t)$ remain in the set $\mathcal{B}_p[\varrho_x]$ if $\|\hat{x}(t)\|_{P_c} \leq 1$ as well. Obviously, this requirement is met if $\|\hat{x}(t)\|_{P_c} \leq 1$ implies $\|\hat{x}(t)\|_p + \|\varepsilon(t)\|_p \leq \varrho_x$. Taking into account that $\|\cdot\|_p \leq \sqrt{n}\|\cdot\|_2 \leq \sqrt{n/\lambda_{\min}(P_c)}\|\cdot\|_{P_c}$ and $\|\varepsilon(t)\|_p \leq \zeta(\varrho_\varepsilon, \varrho_d)$, it is clear that the required inequality holds due to (4.26). In other words, condition (4.26) guarantees that the switching surface $\mathcal{S} := \{\hat{x} \in \mathbb{R}^n : \|\hat{x}(t)\|_{P_c} = 1\}$ is inside the closed ball $\mathcal{B}_p[\varrho_x - \zeta]$ (see Figure 4.1). It is worth mentioning that condition (4.26) is not restrictive, since it does not affect the feasibility of LMIs (4.23).

II. Finite-time ISS property (4.2b)

Note that the closed-loop system (4.1), (4.25) is finite-time ISS if so are the observer (4.10), (4.25) and the system (4.11). Since Proposition 4.1 implies finite-time ISS of (4.11), we have to prove that this property also holds for the observer. To this end, consider a Lipschitz continuous Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as follows

$$V(\hat{x}) = \|\hat{x}\|_s := \begin{cases} \|\hat{x}\|_{P_c}, & \text{if } \|\hat{x}\|_{P_c} \geq 1, \\ V_c(\hat{x}), & \text{if } \|\hat{x}\|_{P_c} < 1, \end{cases} \quad (4.27)$$

where $V_c(\hat{x})$ is the solution of the equation $Q_c(V_c, \hat{x}) = 0$.

Figure 4.1: Geometric meaning of condition (4.26) for $n = 2$

Using formula (C.1) from Appendix C to calculate the generalized gradient of $V(\hat{x})$, we get

$$\nabla_C V(\hat{x}) = \begin{cases} \nabla \|\hat{x}\|_{P_c}, & \text{if } \|\hat{x}\|_{P_c} > 1, \\ c\nabla \|\hat{x}\|_{P_c} + (1-c)\nabla V_c(\hat{x}), & \text{if } \|\hat{x}\|_{P_c} = 1, \\ \nabla V_c(\hat{x}), & \text{if } \|\hat{x}\|_{P_c} \in (0, 1), \end{cases}$$

where $c \in [0, 1]$. Clearly, to prove the ISS property, it suffices to consider two cases: $\|\hat{x}\|_{P_c} \geq 1$ and $\|\hat{x}\|_{P_c} \in (0, 1]$. Denote $\vartheta := C\varepsilon - d_y$.

For $\|\hat{x}\|_{P_c} \geq 1$, we have

$$\dot{V}(\hat{x}) = \frac{\hat{x}^\top (P_c(A+BK) + (A+BK)^\top P_c)\hat{x} + 2\hat{x}^\top P_c g(\vartheta)}{2\hat{x}^\top P_c \hat{x}}.$$

Taking into account that $\hat{x}^\top P_c g(\vartheta) \leq \|\hat{x}\|_{P_c} \|g(\vartheta)\|_{P_c}$, it follows from (4.23a) that $\dot{V}(\hat{x}) \leq -2\alpha_c \|\hat{x}\|_{P_c} - \|g(\vartheta)\|_{P_c}$. Therefore, $\dot{V}(\hat{x}) \leq -\alpha_c \|\hat{x}\|_{P_c} = -\alpha_c V(\hat{x})$ for all $\|\hat{x}\|_{P_c} \geq \eta \|g(\vartheta)\|_{L^\infty}$.

On the other hand, it follows from the proof of Proposition 4.4 (see Appendix 4.C) that for $\|\hat{x}\|_{P_c} \in (0, 1]$ such that $\|\hat{x}\|_{P_c} \geq (\eta_c \|g(\vartheta)\|_{L^\infty})^{\beta_c/(\gamma_c+\nu)}$, we have $\dot{V}(\hat{x}) \leq -(\alpha_c/\gamma_c)V^{1+\nu}$. Finally, applying the Comparison Lemma A.1, we conclude that system (4.10), (4.25) is finite-time ISS:

$$\|\hat{x}(t, \hat{x}_0, g(C\varepsilon - d_y))\|_{P_c} \leq \tilde{v}_c(\|\hat{x}_0\|_{P_c}, t) + \tilde{w}_c(\|g(C\varepsilon - d_y)\|_{L^\infty})$$

with the functions $\tilde{w}_c \in \mathcal{GKL}$ and $\tilde{v}_c \in \mathcal{K}_\infty$ given by

$$\tilde{v}_c(s, t) := \begin{cases} e^{(\tilde{T}'_c(s)-t)/2}, & \text{if } t \in [0, \tilde{T}'_c(s)], \\ \left(\frac{\alpha_c |\nu|}{\gamma_c} (\tilde{T}'_c(s) - t)\right)^{\beta_c/|\nu|}, & \text{if } t \in [\tilde{T}'_c(s), \tilde{T}_c(s)], \\ 0, & \text{if } t \in [\tilde{T}_c(s), +\infty), \end{cases}$$

$$\tilde{w}_c(s) := \begin{cases} \eta_c s, & \text{if } \eta_c s > 1, \\ (\eta_c s)^{\beta_c/(\gamma_c+\nu)}, & \text{if } \eta_c s \leq 1, \end{cases}$$

where parameters $\alpha_c, \beta_c, \eta_c$ are given in Proposition 4.4, and

$$\tilde{T}'_c(s) := \max\{0, 2 \ln s\}, \quad \tilde{T}_c(s) := \frac{\gamma_c \min\{1, s^{|\nu|/\gamma_c}\}}{\alpha_c |\nu|} + \tilde{T}'_c(s).$$

The proof is complete. \square

4.6 Example

Let matrices A, B and C be as follows:

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -3 & 1 & 2 \\ 2 & -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix}.$$

Note that matrix A is not Hurwitz since its eigenvalues are $\lambda_1(A) = -2$ and $\lambda_{2,3}(A) = \pm i$, where i is the imaginary unit.

Let $\varrho_x = 1$. To guarantee feasibility of LMIs (4.18)–(4.20), we set $\varrho_d = 10^{-4}$ and $\varrho_\varepsilon = 3 \cdot 10^{-3}$.

4.6.1 Calculation of the observer parameters

For the given observable pair (A, C) , matrix equations (4.13) have the unique solution

$$X_o = \begin{bmatrix} 1.02 & -0.27 & -0.51 \\ 1.04 & 0.63 & -0.65 \\ -2.56 & 0.23 & 1.35 \end{bmatrix}, \quad L_0 = \begin{bmatrix} 0.78 \\ -0.24 \\ 0.56 \end{bmatrix}.$$

Choose $\mu = -0.07$ and $\alpha_o = 0.27$. Then solving LMIs (4.14) and (4.16), we find

$$P_o = \begin{bmatrix} 7.04 & 3.34 & 2.24 \\ 3.34 & 3.93 & 3.44 \\ 2.24 & 3.44 & 5.07 \end{bmatrix}, \quad L = P_o^{-1} Y_o = \begin{bmatrix} -1.30 \\ 3.19 \\ -2.33 \end{bmatrix}.$$

4.6.2 Calculation of the controller parameters

For given ϱ_ε and ϱ_d , we have $\zeta(\varrho_\varepsilon, \varrho_d) = 0.13$. Then from the solutions of LMIs (4.18)–(4.20) we find a matrix K for each $p \in \{1, 2, \infty\}$:

$$K|_{p=1} = \begin{bmatrix} -1.25 & 1.50 & 1 \end{bmatrix}, \quad K|_{p=2} = \begin{bmatrix} -1.23 & 1.51 & 0.41 \end{bmatrix}, \quad K|_{p=\infty} = \begin{bmatrix} -1.88 & 2 & 0.54 \end{bmatrix}.$$

For the given controllable pair (A, B) , matrix equations (4.22) have the unique solution

$$X_c = \begin{bmatrix} 1.33 & 0.67 & 0 \\ -3.00 & -1.00 & 1 \\ -1.56 & -2.44 & -3.33 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 5.56 & 2.44 & 1.33 \end{bmatrix}.$$

Choose $\nu = -0.07$ and $\alpha_c = 0.5$. Then solving LMIs (4.23) and (4.26), we find

$$P_c|_{p=1} = \begin{bmatrix} 40.52 & -2.24 & 18.54 \\ -2.24 & 33.49 & -4.54 \\ 18.54 & -4.54 & 28.13 \end{bmatrix}, \quad P_c|_{p=2} = \begin{bmatrix} 26.54 & -0.44 & 2.01 \\ -0.44 & 23.80 & 0.91 \\ 2.01 & 0.91 & 30.85 \end{bmatrix}, \quad P_c|_{p=\infty} = \begin{bmatrix} 23.95 & 5.46 & 0.11 \\ 5.46 & 21.52 & 1.02 \\ 0.11 & 1.02 & 35.03 \end{bmatrix}.$$

4.6.3 Numerical simulation

The numerical simulation of the closed-loop system (4.1), (4.10), (4.25) has been done in MATLAB Simulink by using the explicit Euler method with a state-dependent step [22]. The basic and minimum discretization steps, the maximum number of iterations and the homogeneous norm have been defined as $\Delta t_0 = 10^{-2}$, $\Delta t_{\min} = 10^{-4}$, $N_{\max} = 2 \cdot 10^4$ and $\|x\|_{hom} := \|x\|_s$, where the function $\|\cdot\|_s$ is defined in (4.27), respectively. Initial states of the system (4.1) and the observer (4.10), and the external disturbances have been chosen as follows:

$$x_0 = \varrho_x \frac{\text{col}\{(-1)^i, (-1)^j, 1\}}{\|\text{col}\{1, 1, 1\}\|_p}, \quad i, j = 1, 2, \quad \hat{x}_0 = \left(1 - \frac{\varrho_\varepsilon}{\varrho_x}\right)x_0,$$

$$d_x(t) = \varrho_d \frac{\text{col}\{\sin(5t), \cos(t), \frac{1}{2}(1 + \sin(2t))\}}{\|\text{col}\{1, 1, 1, 1\}\|_p}, \quad d_y(t) = \varrho_d \frac{\frac{1}{2}(1 + \cos(10t))}{\|\text{col}\{1, 1, 1, 1\}\|_p}.$$

Such a choice implies that $\|x_0\|_p = \varrho_x$, $\|\varepsilon_0\|_p := \|\hat{x}_0 - x_0\|_p = \varrho_\varepsilon$ and $\|d\|_{L_p^\infty} := \|\text{col}\{d_x, d_y\}\|_{L_p^\infty} = \varrho_d$.

The simulation results for each $p \in \{1, 2, \infty\}$ are depicted in Figures 4.2–4.4, where solid and dashed lines correspond to the case of switching (4.25) and linear (4.17) feedback, respectively. One can see that, in both cases, trajectories of the closed-loop system remain in the region specified by the state constraints (4.2a). On the other hand, due to the nonlinear component (4.24), the proposed controller (4.25) allows one to stabilize the system (4.1) faster than exponentially.

4.7 Conclusion

In this chapter, the problem of robust output finite-time stabilization of linear systems under state constraints has been addressed. Geometrically, the considered class of state constraints represents a closed region (hyperoctahedron, hypersphere or hypercube) centered at the origin, within which trajectories of the closed-loop system must remain. To solve the problem, a nonlinear Luenberger-like observer was first designed using the Implicit Lyapunov method in order to reconstruct the state vector in finite time. Then, a continuous control law was proposed, which is linear when trajectories of the closed-loop system risk violating the state constraints, and nonlinear otherwise. The linear feedback was chosen to make the closed-loop system matrix Hurwitz and diagonally dominant. Due to this, trajectories of the closed-loop system not only exponentially converge to the origin but also do not leave the specified region. Once the trajectories reach the switching surface, the nonlinear feedback is used to accelerate the convergence rate, namely, to stabilize the system in finite time. However, differently from the finite-time observer, the practical implementation of the finite-time controller requires the online computation of the Implicit Lyapunov function. Since the analytical solution of the corresponding nonlinear equation cannot be found in the general case, the bisection method was used to numerically calculate the Implicit Lyapunov function. Compared to the existing methods of stabilization of dynamical systems under state constraints based on control barrier functions [66, 78] and barrier Lyapunov functions [53, 76], the tuning of the proposed nonlinear control system is extremely simple: the observer and controller parameters are found from the solutions of linear matrix inequalities and equations. Numerical simulation of the designed control system has shown that, for sufficiently small external disturbances, the linear system is stabilized in finite time without violating the state constraints.

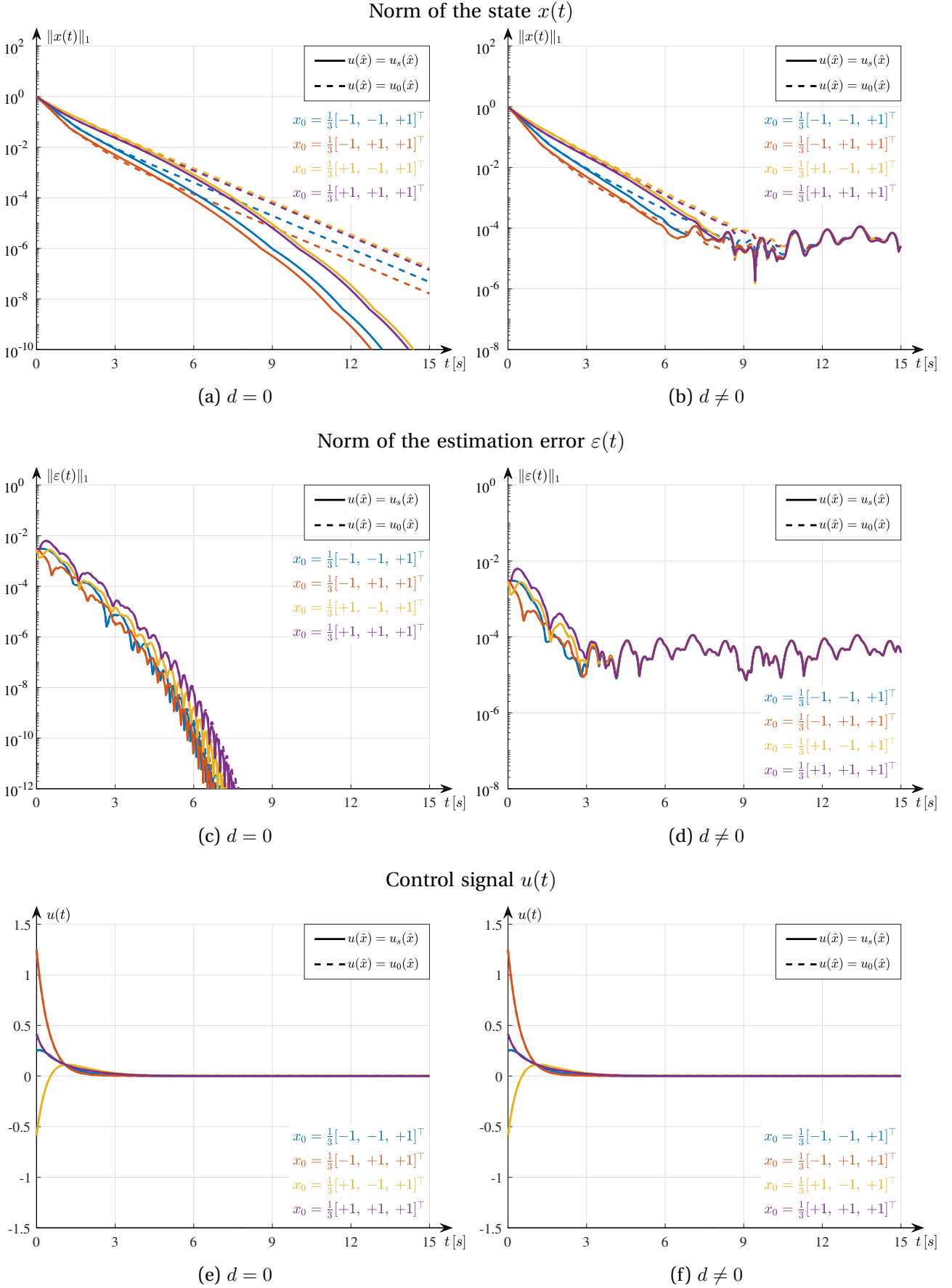


Figure 4.2: Transients in the closed-loop system (4.1), (4.10), (4.25) for $p = 1$

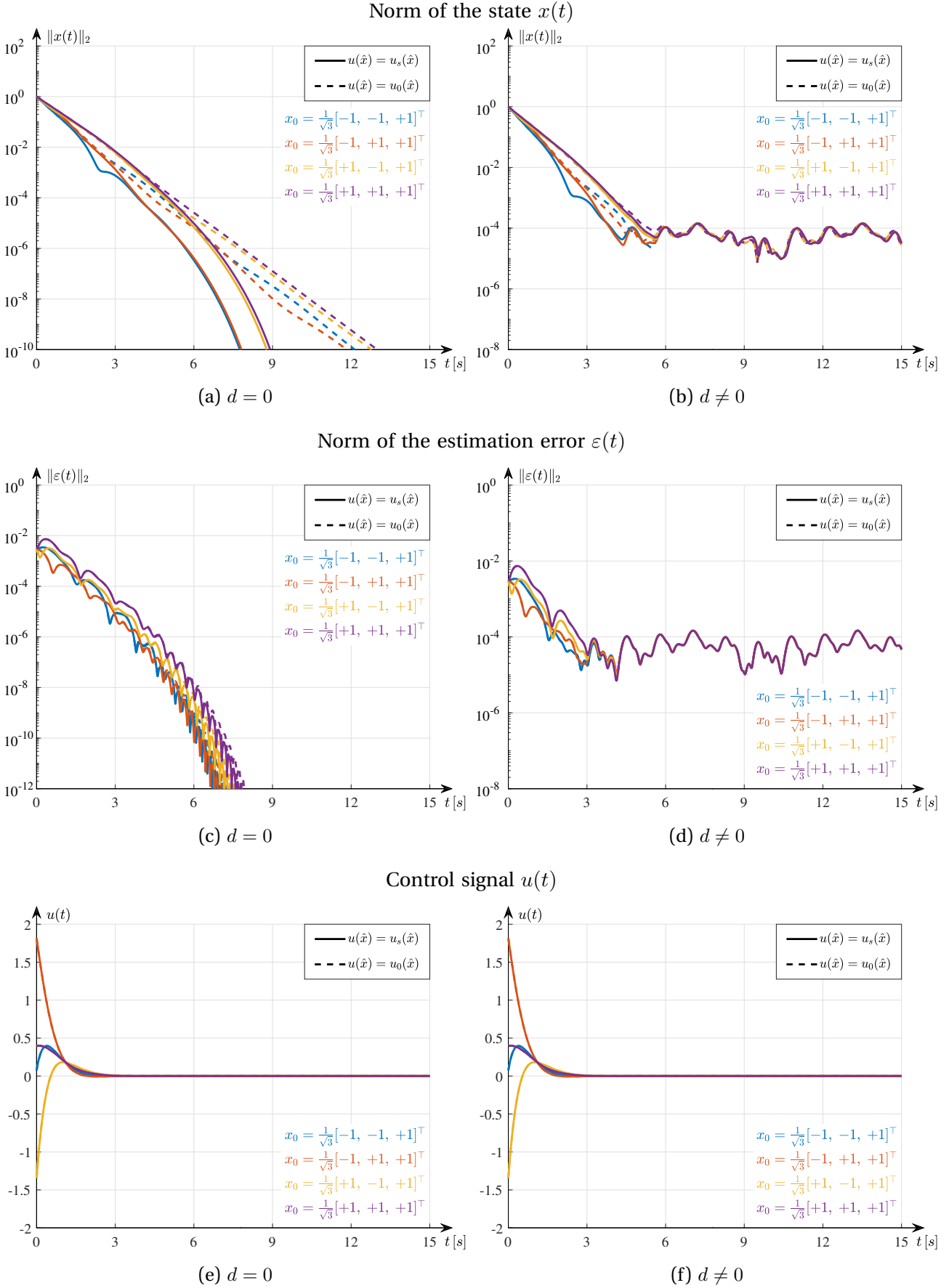


Figure 4.3: Transients in the closed-loop system (4.1), (4.10), (4.25) for $p = 2$

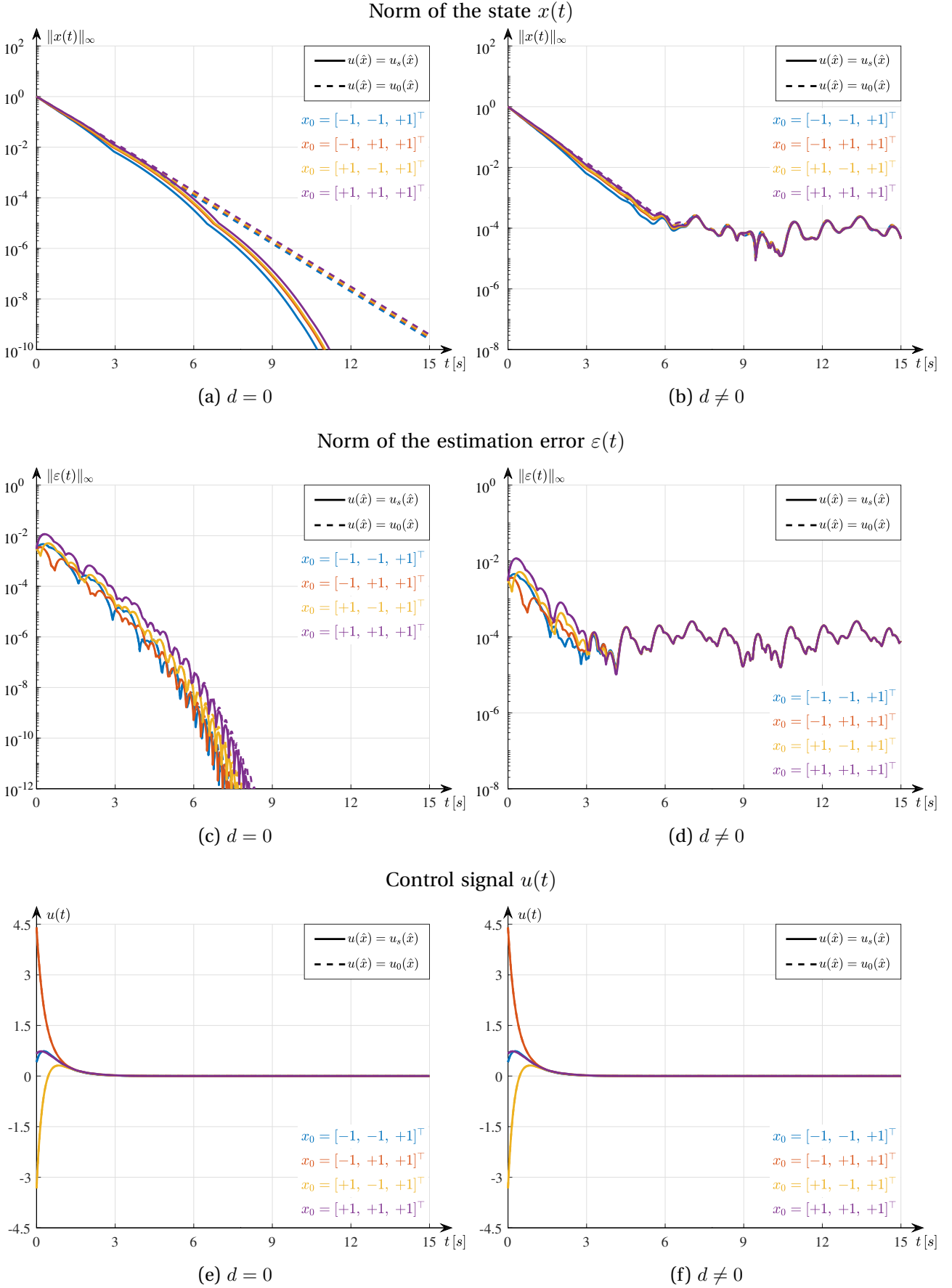


Figure 4.4: Transients in the closed-loop system (4.1), (4.10), (4.25) for $p = \infty$

Appendices

4.A On feasibility of matrix equations (4.13) and (4.22)

Lemma 4.1. *If matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ ($\text{rank}(B) = m$) are such that the pair (A, B) is controllable, then matrix equations (4.22) always have a solution.*

Similarly, if matrices $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{k \times n}$ ($\text{rank}(C) = k$) are such that the pair (A, C) is observable, then matrix equations (4.13) always have a solution.

Proof. Note that equations (4.13) can be rewritten in the form (4.22) by denoting $\tilde{A} = A^\top$, $\tilde{B} = C^\top$, $\tilde{K}_0 = L_0^\top$ and $\tilde{X}_c = -X_o^\top$. Since observability of the pair (A, C) implies controllability of the pair (A^\top, C^\top) , then feasibility of equations (4.13) and uniqueness of their solutions follow from the same properties of equations (4.22).

Applying the block decomposition given in [63], it can be shown that there exists a nonsingular transformation matrix $\Psi \in \mathbb{R}^{n \times n}$ such that $A = \Psi^{-1}(\bar{A} + \bar{B}S)\Psi$ and $B = \Psi^{-1}\bar{B}$ with $S \in \mathbb{R}^{m \times n}$,

$$\bar{A} = \begin{bmatrix} O_{n_1 \times n_1} & \bar{A}_{12} & O_{n_1 \times n_3} & \cdots & O_{n_1 \times n_k} \\ O_{n_2 \times n_1} & O_{n_2 \times n_2} & \bar{A}_{23} & \cdots & O_{n_2 \times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n_{k-1} \times n_1} & O_{n_{k-1} \times n_2} & O_{n_{k-1} \times n_3} & \cdots & \bar{A}_{(k-1)k} \\ O_{n_k \times n_1} & O_{n_k \times n_2} & O_{n_k \times n_3} & \cdots & O_{n_k \times n_k} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} O_{n_1 \times m} \\ O_{n_2 \times m} \\ \vdots \\ O_{n_{k-1} \times m} \\ I_m \end{bmatrix},$$

where k is the number of blocks, $n_1 + \dots + n_k = n$, $n_k = m$, matrices $\bar{A}_{(i-1)i} \in \mathbb{R}^{n_{i-1} \times n_i}$, $i = \overline{2, k}$, have full row rank.

Therefore, equations (4.22) can be rewritten as follows:

$$\begin{aligned} \Psi^{-1}(\bar{A} + \bar{B}(S + \bar{K}))(\bar{X} - I_n)\Psi &= \Psi^{-1}\bar{X}(\bar{A} + \bar{B}S)\Psi, \\ \Psi^{-1}\bar{X}\bar{B} &= O_{n \times m}, \end{aligned} \quad (4.28)$$

where matrix $\bar{X} := \Psi X_c \Psi^{-1}$ and $\bar{K} := K_0 \Psi^{-1}$ consist of blocks $\bar{X}_{sr} \in \mathbb{R}^{n_s \times n_r}$ and $\bar{K}_s \in \mathbb{R}^{n_m \times n_s}$, $s, r = \overline{1, k}$, respectively.

Taking into account that $\Psi^{-1}\bar{X}\bar{B}S\Psi = O_{n \times n}$, one can check that equations (4.28) have a solution for any Ψ if and only if:

$$\begin{cases} \bar{A}_{(i-1)i}\bar{X}_{i1} = O_{n_{i-1} \times n_1}, \\ \bar{A}_{(i-1)i}\bar{X}_{ii} = (\bar{X}_{(i-1)(i-1)} + I_{n_{i-1}})\bar{A}_{(i-1)i}, \\ \bar{A}_{(i-1)i}\bar{X}_{ij} = \bar{X}_{(i-1)(j-1)}\bar{A}_{(j-1)j}, \\ \bar{X}_{sk} = O_{n_s \times n_k}, \\ \sum_{s=1}^k (S_s + \bar{K}_s)\bar{X}_{s1} = S_1 + \bar{K}_1, \\ \sum_{s=1}^k (S_s + \bar{K}_s)\bar{X}_{sj} = S_j + \bar{K}_j + \bar{X}_{k(j-1)}\bar{A}_{(j-1)j}, \end{cases} \quad (4.29)$$

$$\begin{cases} \sum_{s=1}^k (S_s + \bar{K}_s)\bar{X}_{s1} = S_1 + \bar{K}_1, \\ \sum_{s=1}^k (S_s + \bar{K}_s)\bar{X}_{sj} = S_j + \bar{K}_j + \bar{X}_{k(j-1)}\bar{A}_{(j-1)j}, \end{cases} \quad (4.30)$$

where $i, j = \overline{2, k}$, $j \neq i$.

Since $\text{rank}(\bar{A}_{(i-1)i}) = n_{i-1} \leq n_i$, it can be shown that the solution to matrix equations (4.29) is given by the following formulas:

$$\begin{aligned}\bar{X}_{rr} &= -(k-r)I_{n_r}, \\ \bar{X}_{1j} &= O_{n_1 \times n_j}, \\ \bar{X}_{i1} &= R_i W_{i1}, \\ \bar{X}_{ij} &= \begin{cases} O_{n_i \times n_j}, & \text{if } i < j, \\ R_i W_{ij} + \bar{A}_{(i-1)i}^+ \bar{X}_{(i-1)(j-1)} \bar{A}_{(j-1)j}, & \text{if } i > j, \end{cases}\end{aligned}$$

where $W_{iq} \in \mathbb{R}^{n_i \times n_q}$, $q = \overline{1, i-1}$, are arbitrary matrices, $\bar{A}_{(i-1)i}^+ := \bar{A}_{(i-1)i}^\top (\bar{A}_{(i-1)i} \bar{A}_{(i-1)i}^\top)^{-1}$ is the right pseudoinverse matrix of $\bar{A}_{(i-1)i}$ and $R_i := I_{n_i} - \bar{A}_{(i-1)i}^+ \bar{A}_{(i-1)i}$.

Taking into account that $\bar{X}_{sj} = O_{n_s \times n_j}$ for all $s < j$ and $\bar{X}_{jj} = -(k-j)I_{n_j}$, one can see that the solution to equations (4.30) is given by:

$$\begin{aligned}\bar{K}_1 &= -S_1 + k^{-1} \sum_{s=2}^k (S_s + \bar{K}_s) \bar{X}_{sj}, \\ \bar{K}_j &= -S_j - \frac{\bar{X}_{k(j-1)} \bar{A}_{(j-1)j} - \sum_{s=j+1}^k (S_s + \bar{K}_s) \bar{X}_{sj}}{1+k-j}.\end{aligned}$$

In a general case, \bar{X} is a lower triangular matrix, where all the entries below the main diagonal depend on R_i and W_{iq} . Obviously, matrix \bar{X} always can be chosen in the diagonal form with blocks $\bar{X}_{rr} = -(k-r)I_{n_r}$, $r = \overline{1, k}$.

One can see that system (4.29), (4.30) has a unique solution if and only if $R_i = O_{n_i \times n_i}$ for all $i = \overline{2, k}$. It means that all matrices $\bar{A}_{(i-1)i}$ must have full column rank, i.e., $\text{rank}(\bar{A}_{(i-1)i}) = n_i$. Since $\text{rank}(\bar{A}_{(i-1)i}) = n_{i-1}$, then this holds if and only if $n_1 = \dots = n_k = m$. For example, this is always true for $m = 1$ since $1 \leq n_1 \leq \dots \leq n_k = m$. \square

4.B Proof of Proposition 4.1

To simplify readability, we will drop the subscript o and the argument t .

I. Continuity of the function $g(\vartheta)$

Clearly, it suffices to show that $g(\vartheta)$ is continuous at $\vartheta = 0$. Note that for all $\vartheta \neq 0$ the function $g(\vartheta)$ can be rewritten as follows:

$$g(\vartheta) = L_0 \vartheta + \exp\{I_n + \mu(X + I_n) \ln \|\vartheta\|_2\} L \frac{\vartheta}{\|\vartheta\|}.$$

Since matrix $I_n + \mu(X + I_n)$ is anti-Hurwitz due to (4.14d), then $g(\vartheta) \rightarrow 0$ as $\vartheta \rightarrow 0$ and, thus, $g(\vartheta)$ is continuous at $\vartheta = 0$.

II. Verification of conditions $C_i1)$ – $C_i4)$

Clearly, the ILF (4.12) is continuously differentiable outside the origin. Furthermore, note that for any $\chi \in \mathbb{R}^n$, the following estimates hold:

$$\begin{aligned}V^{-\beta} \|\chi\|_P &\leq \|\Lambda(V^{-1})\chi\|_P \leq V^{-\gamma} \|\chi\|_P, & \text{if } V \leq 1, \\ V^{-\gamma} \|\chi\|_P &\leq \|\Lambda(V^{-1})\chi\|_P \leq V^{-\beta} \|\chi\|_P, & \text{if } V \geq 1.\end{aligned}\tag{4.31}$$

Indeed, since $\frac{\partial}{\partial V} \|\Lambda(V^{-1})\chi\|_P^2 = -V^{-1} \chi^\top \Lambda(V^{-1}) (PG + G^\top P) \Lambda(V^{-1}) \chi$, then it follows from (4.14d)

that $2\beta\|\Lambda(V^{-1})\chi\|_P^2 \leq -V\frac{\partial}{\partial V}\|\Lambda(V^{-1})\chi\|_P^2 \leq 2\gamma\|\Lambda(V^{-1})\chi\|_P^2$, where $0 < \beta \leq \gamma$. The obtained differential inequalities imply estimates (4.31). Thus, for any $\chi \in \mathbb{R}^n$, there exists $V \in \mathbb{R}_+^*$ such that $\|\Lambda(V^{-1})\chi\|_P = 1 \Leftrightarrow Q(V, \chi) = 0$. Taking into account (4.31), functions $q_1, q_2 \in \mathcal{IK}_\infty$ can be introduced as follows:

$$q_1 := \frac{\sqrt{\lambda_{\min}(P)}\|\chi\|_2}{\max\{V^\beta, V^\gamma\}} - 1, \quad q_2 := \frac{\sqrt{\lambda_{\max}(P)}\|\chi\|_2}{\min\{V^\beta, V^\gamma\}} - 1.$$

III. Verification of condition C_i5)

By the definition of the function $F(V, \varepsilon)$, we have:

$$F(V, \varepsilon) = V \frac{2\varepsilon^\top \Lambda(V^{-1})P\Lambda(V^{-1})(A\varepsilon + g(C\varepsilon - d_y) - d_x)}{\varepsilon^\top \Lambda(V^{-1})(PG + G^\top P)\Lambda(V^{-1})\varepsilon}.$$

Note that equation (4.13a) implies $(A + L_0C)X = (X + I_n)(A + LC_0)$. Therefore, $(A + L_0C)(G - \mu I_n) = G(A + L_0C)$ and, by induction, $(A + L_0C)(G - \mu I_n)^k = G^k(A + L_0C)$ for all $k \in \mathbb{N}$. Similarly, it follows from (4.13b) that $CG = C(\mu X + I_n) = C$ and, by induction, $CG^k = C$ for all $k \in \mathbb{N}$. Taking into account the definition of the matrix exponential, the following is true for any $\lambda \in \mathbb{R}_+^*$:

$$\begin{aligned} \Lambda(\lambda)(A + L_0C) &= \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} G^k(A + LC_0) = \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} (A + L_0C)(G - \mu I_n)^k \\ &= (A + L_0C) \exp\{(G - \mu I_n) \ln \lambda\} = (A + L_0C)\Lambda(\lambda)\lambda^{-\mu}, \\ C\Lambda(\lambda) &= \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} CG^k = \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} C = \lambda C. \end{aligned}$$

Denote $z := \Lambda(V^{-1})\varepsilon$ and $\xi := V^{-1}\|C\varepsilon - d_y\|_2 = \|Cz - V^{-1}d_y\|_2$. Therefore

$$\begin{aligned} F(V, \varepsilon) &= V^{1+\mu} \frac{2z^\top P((A + L_0C)z + \xi^{\mu-1}\Lambda(\xi)L(Cz - V^{-1}d_y) - V^{-\mu}\Lambda(V^{-1})(d_x + L_0d_y))}{z^\top (PG + G^\top P)z} \\ &= V^{1+\mu} \frac{2z^\top P((A + (L_0 + L)C)z + \xi^{-1}\Xi_\mu(\xi)L(Cz - V^{-1}d_y) - V^{-1}Ld_y - V^{-\mu}\Lambda(V^{-1})d_{\bar{x}})}{z^\top (PG + G^\top P)z}, \end{aligned}$$

where $d_{\bar{x}} := d_x + L_0d_y$. Since $2a^\top b \leq c\|a\|_2^2 + c^{-1}\|b\|_2^2$ for any $a, b \in \mathbb{R}^n$ and $c > 0$, taking into account (4.14a), one can get

$$\begin{aligned} F(V, \varepsilon) &\leq V^{1+\mu} \frac{-8\alpha z^\top Pz + \alpha\xi^{-2}\|Cz - V^{-1}d_y\|_2^2 + \alpha^{-1}\|L^\top \Xi_\mu^\top(\xi)Pz\|_2^2}{z^\top (PG + G^\top P)z} \\ &\quad + V^{1+\mu} \frac{2\alpha z^\top Pz + \alpha^{-1}(\|V^{-\mu}\Lambda(V^{-1})d_{\bar{x}}\|_P^2 + \|V^{-1}Ld_y\|_P^2)}{z^\top (PG + G^\top P)z}. \end{aligned}$$

Firstly, since $\|Cz\|_2^2 \leq z^\top Pz = 1$ due to (4.14b) and $V^{-1} \leq \max\{\|\varepsilon\|_P^{-1/\gamma}, \|\varepsilon\|_P^{-1/\beta}\}$ for all $(V, \varepsilon) \in \Omega$ due to (4.31), then $\xi \leq 1 + \max\{\|\varepsilon\|_P^{-1/\gamma}, \|\varepsilon\|_P^{-1/\beta}\}\|d\|_{L_2^\infty}$. As a result, condition (4.14c) implies $\|L^\top \Xi_\mu^\top(\xi)Pz\|_2^2 < \alpha^2 z^\top Pz = (\alpha)^2$ for all $\|\varepsilon\|_P \geq w_1(\|d\|_{L_2^\infty})$. Analogously, if $\|\varepsilon\|_P \geq w_2(\eta_o\|L\|_2\|d\|_{L_2^\infty})$, then $V^{-1}\|Ld_y\|_P \leq \alpha$.

Secondly, it follows from the estimate (4.31) that $\|V^{-\mu}\Lambda(V^{-1})d_{\bar{x}}\|_P \leq \max\{V^{-\gamma-\mu}, V^{-\beta-\mu}\}\|d_{\bar{x}}\|_P \leq \max\{\|\varepsilon\|_P^{-(\gamma+\mu)/\beta}, \|\varepsilon\|_P^{-(\beta+\mu)/\gamma}\}\|d_{\bar{x}}\|_P$. Furthermore, since $PG + G^\top P \succcurlyeq 2\beta P$, then LMI (4.14d) guarantees that $\beta > |\mu|$. Thus, $\|V^{-\mu}\Lambda(V^{-1})d_{\bar{x}}\|_P \leq \alpha$ for all $\|\varepsilon\|_P \geq w_3(\eta_o(1 + \|L_0\|_2)\|d\|_{L_2^\infty})$.

Taking into account that $z^\top (PG + G^\top P)z \leq 2\gamma z^\top Pz = 2\gamma$, one can see that $F(V, \varepsilon) \leq -(\alpha/\gamma)V^{1+\mu}$ for all $\|\varepsilon\|_P \geq w(\|d\|_{L_2^\infty})$. Finally, applying the Comparison Lemma A.1, we conclude the proof. \square

4.C Proof of Proposition 4.4

To simplify readability, we will drop the subscript c and the argument t .

I. Continuity and local boundedness of the control law $u_\nu(\hat{x})$

Clearly, it suffices to show that $u_\nu(\hat{x})$ is continuous at the origin. Since $V^{1+\nu}\|\Lambda(V^{-1})\hat{x}\|_P \leq \max\{\|\hat{x}\|_P^{(1+\nu)/\gamma}, \|\hat{x}\|_P^{(1+\nu)/\beta}\}$, then $\|u_\nu(\hat{x})\|_P \leq \|K_0\hat{x}\|_P + \max\{\|\hat{x}\|_P^{(1+\nu)/\gamma}, \|\hat{x}\|_P^{(1+\nu)/\beta}\}\|K - K_0\|_P$. Thus, $u_\nu(0) = 0$.

II. Verification of conditions $C_i1)$ – $C_i5)$

Since ILFs (4.21) and (4.12) have the same structure, then conditions $C_i1)$ – $C_i5)$ can be verified similarly (see subsection II in 4.B). Moreover, the estimates (4.31) are valid for ILF (4.21), too. Therefore, we need to check only condition $C_i5)$. By the definition of the function $F(V, \varepsilon)$, we have:

$$F(V, \hat{x}) = V \frac{2\hat{x}^\top \Lambda(V^{-1})P\Lambda(V^{-1})(A\hat{x} + BK_0\hat{x} + BV^{1+\nu}(K - K_0)\Lambda(V^{-1})\hat{x} + g(C\varepsilon - d_y))}{\hat{x}^\top \Lambda(V^{-1})(PG + G^\top P)\Lambda(V^{-1})\hat{x}}.$$

Note that equation (4.22a) implies $X(A + BK_0) = (A + BK_0)(X - I_n)$. Therefore, $G(A + BK_0) = (A + BK_0)(G - \nu I_n)$ and, by induction, $G^k(A + BK_0) = (A + BK_0)(G - \nu I_n)^k$ for all $k \in \mathbb{N}$. Similarly, it follows from (4.22b) that $GB = (\mu X + I_n)B = B$ and, by induction, $G^k B = B$ for all $k \in \mathbb{N}$. Taking into account the definition of the matrix exponential, the following is true for any $\lambda \in \mathbb{R}_+^*$:

$$\begin{aligned} \Lambda(\lambda)(A + BK_0) &= \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} G^k (A + BK_0) = \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} (A + BK_0)(G - \nu I_n)^k \\ &= (A + BK_0) \exp\{(G - \nu I_n) \ln \lambda\} = (A + BK_0)\Lambda(\lambda)\lambda^{-\nu}, \\ \Lambda(\lambda)B &= \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} G^k B = \sum_{k=0}^{\infty} \frac{\ln^k \lambda}{k!} B = \lambda B. \end{aligned}$$

Introduce a new variable $z := \Lambda(V^{-1})\hat{x}$. Then

$$\begin{aligned} F(V, \hat{x}) &= V^{1+\nu} \frac{z^\top (P(A + BK) + (A + BK)^\top P)z + 2V^{-\nu} z^\top P\Lambda(V^{-1})g(C\varepsilon - d_y)}{z^\top (PG + G^\top P)z} \\ &\leq V^{1+\nu} \frac{z^\top (P(A + BK) + (A + BK)^\top P)z + \alpha z^\top Pz + \alpha^{-1} \|V^{-\nu}\Lambda(V^{-1})g(C\varepsilon - d_y)\|_P^2}{z^\top (PG + G^\top P)z}. \end{aligned}$$

Note that $\|V^{-\nu}\Lambda(V^{-1})g(\vartheta)\|_P \leq \max\{\|\hat{x}\|_P^{-(\gamma+\nu)/\beta}, \|\hat{x}\|_P^{-(\beta+\nu)/\gamma}\}\|g(\vartheta)\|_P$ due to (4.31). Then taking into account that $z^\top (PG + G^\top P)z \leq 2\gamma z^\top Pz = 2\gamma$, the matrix inequality (4.23a) implies $F(V, \hat{x}) \leq -(\alpha/\gamma)V^{1+\nu}$ for all $\|\hat{x}\|_P \geq w(\|g(C\varepsilon - d_y)\|_{L_2^\infty})$. Finally, applying the Comparison Lemma A.1, we conclude the proof. \square

5.1 Concluding remarks

The objective of the thesis was to provide a simple and constructive method for design of nonlinear controllers (observers) that must stabilize dynamical systems (reconstruct their state) faster than exponentially, i.e., superexponentially. To this end, the Lyapunov theorems have been formulated with respect to implicitly defined Lyapunov functions (functionals). Due to this, sufficient stability conditions can be represented in the form of linear matrix inequalities and equations, which can be numerically solved very efficiently using appropriate mathematical software. To demonstrate the capabilities of the developed method, several theoretical and practical problems have been solved in the thesis.

In Chapter 2, a Razumikhin-like method has been proposed for hyperexponential and fixed-time stability analysis of retarded time-delay systems. Differently from the original Lyapunov-Razumikhin method, the proposed approach allows one not only to study the stability of a time-delay system but also to estimate the speed at which trajectories of the system converge to the equilibrium point. Furthermore, to make the developed method more suitable for the nonlinear control design, Implicit Lyapunov-Razumikhin theorems have also been formulated. The advantage of the implicit formulation has been illustrated by solving the problems of hyperexponential and fixed-time stabilization of a special subclass of time-delay systems. It has been shown that, under some nonrestrictive assumptions, both problems can be easily solved by using the same nonlinear controller that stabilizes the corresponding delay-free system in fixed time. Applying the developed Implicit Lyapunov-Razumikhin method for stability analysis of the closed-loop system, the tuning of the nonlinear controller parameters, which guarantee superexponential stabilization with the required speed, was reduced to verification of linear matrix inequalities. The obtained theoretical results have been supported by numerical simulation of the designed control system for different initial conditions and time delays.

In Chapter 3, the notion of practical fixed-time input-to-state stability has been introduced for neutral time-delay systems with external bounded disturbances and characterized by the Lyapunov-Krasovskii method, which has been formulated both explicitly and implicitly. Based on the obtained theoretical results, an alternative way of robust output practical fixed-time stabilization of linear systems in the controllable canonical form has been proposed. Differently from the observer-based approaches, the state vector was approximated by means of the finite difference method, i.e., based on the past values of the output signal. As a result, due to the special integral relation between the state and its finite-difference approximation, the closed-loop system has a neutral time-delay representation. Applying the developed Implicit Lyapunov-Krasovskii method, sufficient stability conditions for the

designed nonlinear control system were presented in the form of linear matrix inequalities, solutions of which are used for the calculation of the controller parameters. Furthermore, the impact of the artificially induced time delay on the stabilization accuracy has also been quantitatively studied. Finally, it has been theoretically proven and numerically illustrated that, both in the disturbance-free and disturbed cases, the proposed nonlinear controller stabilizes the considered system in the vicinity of the origin much faster than its linear counterpart.

In Chapter 4, the problem of robust output finite-time stabilization of linear systems under state constraints has been addressed. Geometrically, the considered class of state constraints represents a closed region (hyperoctahedron, hypersphere or hypercube) centered at the origin, within which trajectories of the closed-loop system must remain. The problem has been solved in two steps. First, a nonlinear Luenberger-like observer was designed using the Implicit Lyapunov method in order to reconstruct the state vector in finite time. Then, a continuous control law was proposed, which is linear when trajectories of the closed-loop system risk violating the state constraints, and nonlinear otherwise. It has been shown that while the linear controller guarantees exponential stabilization of the system under the state constraints, the nonlinear one accelerates the rate of convergence to the equilibrium point. Compared to the existing methods of stabilization of dynamical systems under state constraints based on control barrier functions and barrier Lyapunov functions, the tuning of the proposed nonlinear control system is extremely simple: the observer and controller parameters are found from the solutions of linear matrix inequalities and equations. Numerical simulation of the designed control system has shown that, for sufficiently small external disturbances, the linear system is stabilized in finite time without violating the state constraints.

The obtained theoretical and numerical results have demonstrated that the Implicit Lyapunov method can be effectively applied to solve various problems related to superexponential stabilization and state estimation of dynamical systems. The developed approach provides a constructive algorithm for calculating the nonlinear controller (observer) parameters to achieve the required speed of response and/or robustness with respect to external bounded disturbances.

5.2 Future work

Possible directions for future research might be the further development of the Implicit Lyapunov method as well as the improvement of the proposed control systems, in particular:

- robustness analysis of superexponentially stable time-delay systems with respect to external bounded disturbances by means of the Lyapunov-Razumikhin method presented in Chapter 2;
- robustification of the control scheme designed in Chapter 3 with respect to high-frequency measurement noises by prefiltering the output signal before using it for the finite-difference approximation;
- structural modification of the finite-time observer from Chapter 4 to avoid checking parameterized matrix inequalities and as a result to simplify the tuning of its parameters;
- refinement of the obtained LMIs in order to find such controller parameters under which the closed-loop system stabilizes as fast as possible.

In addition to improving the proposed control schemes, it is also essential to evaluate their performance in practice and compare them with existing approaches (e.g., PID or model predictive control).

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APPENDIX

A

AUXILIARY THEOREMS AND LEMMAS

Theorem A.1 (Implicit function theorem for Euclidean spaces [17]). *If $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}$, is a continuous function such that:*

- 1) $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$;
- 2) $f(x, y)$ is continuously differentiable in a neighborhood of the point (x_0, y_0) ;
- 3) $\frac{\partial f(x_0, y)}{\partial y} \Big|_{y=y_0} \neq 0$,

then there exists a unique function $g : \mathcal{X}_0 \rightarrow \mathcal{Y}$ defined in a neighborhood $\mathcal{X}_0 \in \mathcal{X}$ of x_0 such that $f(x, g(x)) = 0$ for all $x \in \mathcal{X}_0$. Moreover, g is continuously differentiable on \mathcal{X}_0 and its derivative is given by

$$\frac{\partial g(x)}{\partial x} = - \left(\frac{\partial f(x, y)}{\partial y} \right)^{-1} \frac{\partial f(x, y)}{\partial x}.$$

Lemma A.1 (Comparison lemma [75]). *Consider the scalar differential equation*

$$\dot{u}(t) = f(t, u(t)), \quad u(t_0) = u_0, \tag{A.1}$$

where the function $f : (a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, i.e.,

- *for each fixed t , the function $f(t, u)$ is continuous in u ;*
- *for each fixed u , the function $f(t, u)$ is measurable in t ;*
- *there exists a Lebesgue integrable function $m : (a, b) \rightarrow \mathbb{R}_+$ such that $|f(t, u)| \leq m(t)$.*

Let $u(t)$ be the right-hand maximum solution of (A.1) with initial conditions $(t_0, u_0) \in (a, b) \times \mathbb{R}$ defined on $[t_0, b)$. Let $v(t)$ be an absolutely continuous function defined on $[t_0, b)$.

If $v(t)$ satisfies the initial inequality $v(t_0) \leq u_0$ and the differential inequality $\dot{v}(t) \leq f(t, v(t))$ almost everywhere on $t \in (t_0, b)$, then $v(t) \leq u(t)$ for all $[t_0, b)$.

Lemma A.2 [48]. *The function $g : \mathbb{R}_+ \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+$, defined as $g(s, \epsilon) := |s^\epsilon - s|$, for any $\bar{s} \in \mathbb{R}_+^*$ and $(\bar{\epsilon}_1, \bar{\epsilon}_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ admits the following estimate:*

$$\max_{s \in [0, \bar{s}], \epsilon \in [\bar{\epsilon}_1, \bar{\epsilon}_2]} g(s, \epsilon) = \max\{\bar{g}(\bar{s}, \bar{\epsilon}_1), \bar{g}(\bar{s}, \bar{\epsilon}_2)\},$$

where

$$\bar{g}(\bar{s}, \bar{\epsilon}) := \begin{cases} 0, & \text{if } \bar{\epsilon} = 1, \\ g(\bar{s}, \bar{\epsilon}), & \text{if } \bar{\epsilon} \neq 1 \text{ and } \bar{s} \leq \hat{s}, \\ \max(g(\hat{s}, \bar{\epsilon}), g(\bar{s}, \bar{\epsilon})), & \text{if } \bar{\epsilon} \neq 1 \text{ and } \bar{s} > \hat{s} \end{cases}$$

with $\hat{s} := \bar{\epsilon}^{1/(1-\bar{\epsilon})}$.

Lemma A.3 (Jensen's inequality [72]). *Let functions $\phi : [a, b] \rightarrow \mathbb{R}$ and $\varpi, \vartheta : [a, b] \rightarrow [0, +\infty)$ be such that integration concerned is well-defined. Then:*

$$\left(\int_a^b \vartheta(s)\phi(s)ds \right)^2 \leq \int_a^b \frac{\vartheta(s)}{\varpi(s)} ds \int_a^b \varpi(s)\vartheta(s)\phi^2(s)ds.$$

Lemma A.4 (Schur complement [11]). *Let X be a symmetric real matrix partitioned as*

$$X = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix},$$

where A and C are square matrices. Then the following properties hold:

- 1) *If A is invertible, then $X \succ 0$ if and only if $A \succ 0$ and $C - B^\top A^{-1}B \succ 0$;*
- 2) *If C is invertible, then $X \succ 0$ if and only if $C \succ 0$ and $A - BC^{-1}B^\top \succ 0$;*
- 3) *If $A \succ 0$, then $X \succcurlyeq 0$ if and only if $C - B^\top A^{-1}B \succcurlyeq 0$;*
- 4) *If $C \succ 0$, then $X \succcurlyeq 0$ if and only if $A - BC^{-1}B^\top \succcurlyeq 0$.*

APPENDIX

B

ELEMENTS OF DIFFERENTIAL CALCULUS IN BANACH SPACES

In this appendix, we recall some elements of the differential calculus in Banach spaces which are used in Chapter 2 for the analysis of functional differential equations. For a thorough treatment and detailed references on this subject, the reader may refer, for example, to works [13], [14] and [18]. Throughout the text, let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ and $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ be some Banach spaces with open subsets \mathcal{X}' , \mathcal{Y}' and \mathcal{Z}' , respectively.

B.1 Fréchet derivative

The next definition extends the notion of the total derivative in Banach spaces.

Definition B.1 [14]. *A mapping $f : \mathcal{X}' \rightarrow \mathcal{Y}$, $x \mapsto f(x)$, is said to be Fréchet differentiable at $x_0 \in \mathcal{X}'$ if there exists a bounded linear operator $Df(x_0) : \mathcal{X} \rightarrow \mathcal{Y}$ such that*

$$\frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_{\mathcal{Y}}}{\|x - x_0\|_{\mathcal{X}}} \rightarrow 0 \quad \text{as } \|x - x_0\|_{\mathcal{X}} \rightarrow 0.$$

Furthermore, the operator $Df(x_0)$ is called the Fréchet derivative of f at x_0 .

Note that for finite-dimensional spaces, when $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^m$, if the Fréchet (total) derivative exists at x_0 then it is given by the Jacobian matrix of f at x_0 .

Definition B.2 [13]. *A mapping $f : \mathcal{X}' \rightarrow \mathcal{Y}$, $x \mapsto f(x)$, is said to be Fréchet differentiable in \mathcal{X}' if it is Fréchet differentiable at each $x_0 \in \mathcal{X}'$.*

Furthermore, if the derived mapping $x \mapsto Df(x)$ is continuous in \mathcal{X}' , then f is continuously Fréchet differentiable in \mathcal{X}' .

For a function of several variables the notion of the partial Fréchet derivatives also can be introduced. For the sake of brevity, we consider only the case of functions of two variables.

Definition B.3 [18]. *A mapping $f : \mathcal{X}' \times \mathcal{Y}' \rightarrow \mathcal{Z}$, $(x, y) \mapsto f(x, y)$, is said to be Fréchet differentiable at $(x_0, y_0) \in \mathcal{X}' \times \mathcal{Y}'$ with respect to the variable x (resp. y) if the mapping $g(x) := f(x, y_0)$ (resp. $g(y) := f(x_0, y)$) is Fréchet differentiable at x_0 (resp. y_0).*

The Fréchet derivative of g at x_0 (resp. y_0) is called the partial Fréchet derivative of f at (x_0, y_0) with respect to the variable x (resp. y) and denoted as $D_x f(x_0, y_0)$ (resp. $D_y f(x_0, y_0)$).

Clearly, the partial Fréchet derivative $D_x f(x, y)$ (resp. $D_y f(x, y)$) coincides with the conventional one $\frac{\partial f(x, y)}{\partial x}$ (resp. $\frac{\partial f(x, y)}{\partial y}$) if $\mathcal{Z} = \mathbb{R}$ and $\mathcal{X} = \mathbb{R}$ (resp. $\mathcal{Y} = \mathbb{R}$).

Definition B.4 [18]. A mapping $f : \mathcal{X}' \times \mathcal{Y}' \rightarrow \mathcal{Z}$, $(x, y) \mapsto f(x, y)$, is said to be continuously Fréchet differentiable in $\mathcal{X}' \times \mathcal{Y}'$ if

- 1) it is Fréchet differentiable at each $(x_0, y_0) \in \mathcal{X}' \times \mathcal{Y}'$ with respect to the variables x and y ;
- 2) the derived mappings $(x, y) \mapsto D_x f(x, y)$ and $(x, y) \mapsto D_y f(x, y)$ are continuous in $\mathcal{X}' \times \mathcal{Y}'$.

Note that if only the first condition given in Definition B.4 is fulfilled then f is not necessarily Fréchet differentiable in $\mathcal{X}' \times \mathcal{Y}'$.

B.2 The chain rule and the implicit function theorem

In this section, the generalization of the chain rule and the implicit function theorem (see Theorem A.1 in Appendix A) are given.

Theorem B.1 (The chain rule [18]). If a mapping $f : \mathcal{X}' \rightarrow \mathcal{Y}'$, $x \mapsto f(x)$, is Fréchet differentiable at $x_0 \in \mathcal{X}'$ and a mapping $g : \mathcal{Y}' \rightarrow \mathcal{Z}$, $y \mapsto g(y)$, is Fréchet differentiable at $y_0 := f(x_0)$, then the composite mapping $h := g \circ f$ is Fréchet differentiable at x_0 and

$$Dh(x_0) = Dg(y_0) \circ Df(x_0).$$

Theorem B.2 (Implicit function theorem for Banach spaces [18]). If $f : \mathcal{X}' \times \mathcal{Y}' \rightarrow \mathcal{Z}'$ is a continuous mapping such that:

- 1) $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in \mathcal{X}' \times \mathcal{Y}'$;
- 2) $f(x, y)$ is continuously Fréchet differentiable in a neighborhood of the point (x_0, y_0) ;
- 3) the partial Fréchet derivative $D_y f(x_0, y_0)$ is an invertible operator,

then there exists a unique mapping $g : \mathcal{X}_0 \rightarrow \mathcal{Y}$ defined in a neighborhood $\mathcal{X}_0 \subset \mathcal{X}'$ of x_0 such that $f(x, g(x)) = 0$ for all $x \in \mathcal{X}_0$. Moreover, g is continuously Fréchet differentiable in \mathcal{X}_0 and its Fréchet derivative is given by

$$Dg(x) = -D_y^{-1} f(x, y) \circ D_x f(x, y).$$

APPENDIX

C

ELEMENTS OF NONSMOOTH ANALYSIS

In this appendix, we recall the definition of the generalized gradients and some elements of their calculus which are used in Chapter 4 for the analysis of nonsmooth Lyapunov functions. For a thorough treatment and detailed references on this subject, the reader may refer, for example, to the works [15] and [16]. Throughout the text, $e_i \in \mathbb{R}^n$ denotes the standard-basis vector with a 1 in the i -th coordinate and 0's elsewhere.

C.1 Clarke generalized gradient

The next definition generalizes the notion of the gradient of Lipschitz continuous functions which are differentiable almost everywhere (a.e.) due to Rademacher theorem.

Definition C.1 [15]. *Let $x_0 \in \mathbb{R}^n$, and let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz continuous near x_0 . Let Ω be any subset of zero measure in \mathbb{R}^n , and let Ω_V be the set of points in \mathbb{R}^n at which V fails to be differentiable. Then the generalized gradient $\nabla_C V(x_0)$ of V at x_0 is given by*

$$\nabla_C V(x_0) := \text{co}\{\lim \nabla V(x_k) : x_k \rightarrow x_0, x_k \notin \Omega, x_k \notin \Omega_V\}. \quad (\text{C.1})$$

The meaning of formula (C.1) is the following: consider any sequence $\{x_k\}$ of vectors $x_k \in \mathbb{R}^n$ converging to x_0 while avoiding both Ω and points at which V is not differentiable, and such the sequence $\{\nabla V(x_k)\}$ converges; then the convex hull of all such limits is $\nabla_C V(x_0)$.

In the next two lemmas, the generalized gradient is calculated for the functions $V(x) = \|x\|_1$ and $V(x) = \|x\|_\infty$ which are studied in Chapter 4.

Lemma C.1. *If $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by $V(x) = \|x\|_1 := \sum_{i=1}^n |x_i|$ then*

$$\nabla_C V(x) = \sum_{i=1}^n \text{SGN}(x_i) e_i, \quad (\text{C.2})$$

where $\text{SGN}(x_i)$ is the set-valued signum function.

Proof. For each $x_0 \in \mathbb{R}^n$, define the set of indices $I(x_0) := \{i = \overline{1, n} : x_{0,i} = 0\}$. Note that the function $V(x) = \|x\|_1$ fails to be differentiable at x_0 if the set $I(x_0)$ is nonempty. Let $\{x_k\}$ be a sequence converging to x_0 such that $(-1)^{s_i} x_{k,i} > 0$ for all $i \in I(x_0)$, where $s_1, \dots, s_i \in \{1, 2\}$. Then the function $V(x)$ is differentiable at each x_k and $\lim_{x_k \rightarrow x_0} \nabla V(x_k) = \sum_{i \in I(x_0)} (-1)^{s_i} e_i + \sum_{i \notin I(x_0)} \text{sgn}(x_{0,i}) e_i$. Constructing sequences $\{x_k\}$ in a similar way for every possible combination of s_1, \dots, s_i , and calculating

the convex hull of all the limits, we get

$$\nabla_C V(x_0) = \left\{ \sum_{i \in I(x_0)} c_i e_i + \sum_{i \notin I(x_0)} \operatorname{sgn}(x_{0,i}) e_i : c_i \in [-1, 1] \right\}.$$

Taking into account the definition of $\operatorname{SGN}(\cdot)$ and $I(x_0)$, the latter can be rewritten in the form (C.2). \square

Lemma C.2. *If $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by $V(x) = \|x\|_\infty := \max_{i=\overline{1,n}} |x_i|$ then*

$$\nabla_C V(x) = \left\{ \sum_{i \in I(x)} c_i \operatorname{sgn}(x_i) e_i : c_i \in [0, 1], \sum_{i \in I(x)} c_i = 1 \right\}, \quad (\text{C.3})$$

where $I(x) := \{i = \overline{1,n} : |x_i| = V(x)\}$.

Proof. Note that the function $V(x) = \|x\|_\infty$ is differentiable at x_0 only if the set $I(x_0)$ is a singleton. Choose $i \in I(x_0)$ and let $\{x_k\}$ be a sequence converging to $x_0 \in \mathbb{R}^n$ such that $|x_{k,i}| > |x_{k,j}|$ for all $j = \overline{1,n}, i \neq j$. Then $V(x_k) = |x_{k,i}|$ and $\lim_{x_k \rightarrow x_0} \nabla(x_k) = \operatorname{sgn}(x_{0,i}) e_i$. Constructing sequences $\{x_k\}$ in a similar way for every $i \in I(x_0)$ and calculating the convex hull of all the limits, we get (C.3). \square

C.2 The chain rule

The following theorem presents the chain rule for the Clarke generalized gradient.

Theorem C.1 [65]. *Let a Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined in an open nonempty set $\mathcal{X} \subseteq \mathbb{R}^n$ and an absolutely continuous function $x : \mathbb{R} \rightarrow \mathbb{R}^n$ be defined in \mathcal{T} such that $x(t) \in \mathcal{X}$ for $t \in \mathcal{T}$. Then there exists a function $\xi : \mathbb{R} \rightarrow \mathbb{R}^n$ defined in \mathcal{T} such that $\xi(t) \in \nabla_C V(x(t))$ and*

$$\dot{V}(x(t)) \stackrel{\text{a.e.}}{=} \xi(t)^\top \dot{x}(t).$$

APPENDIX

D

CALCULATION OF IMPLICITLY DEFINED LYAPUNOV FUNCTIONS

In order to implement control schemes (1.11) and (4.24), it is required to solve a nonlinear equation $Q(V, x) = 0$ for any given $x \in \mathbb{R} \setminus \{0\}$. Due to the properties of Implicit Lyapunov functions, the function $\bar{Q}(V) := Q(V, x)$ is monotonically decreasing with the unique zero in \mathbb{R}_+^* . Therefore, the scalar equation $\bar{Q}(V) = 0$ can be solved using, for example, the bisection method. The next algorithm [62] demonstrates how to calculate $V(t_i)$ at time instants $t_i > 0$, $i \in \mathbb{N}$.

Algorithm D.1 Calculation of $V(t_i)$, $i \in \mathbb{N}$

Require: $x(t_i)$, $V(t_{i-1})$, $V_{\min} > 0$, $V_0 > V_{\min}$, $N \geq 1$

```
1:  $\chi \leftarrow x(t_i)$ 
2:  $a \leftarrow V_{\min}$ 
3: if  $i = 1$  then
4:    $b \leftarrow V_0$ 
5: else
6:    $b \leftarrow V(t_{i-1})$ 
7: end if
8:  $j \leftarrow 1$ 
9: while  $j \leq N$  do
10:  if  $Q(b, \chi) > 1$  then           //  $V > b$ 
11:     $a \leftarrow b$ 
12:     $b \leftarrow 2b$ 
13:  else if  $Q(a, \chi) < 1$  then     //  $V < a$ 
14:     $b \leftarrow a$ 
15:     $a \leftarrow \max\{a/2, V_{\min}\}$ 
16:  else                             //  $V \in [a, b]$ 
17:     $c \leftarrow (a + b)/2$ 
18:    if  $Q(c, \chi) > 1$  then       //  $V \in (c, b]$ 
19:       $a \leftarrow c$ 
20:    else                             //  $V \in [a, c)$ 
21:       $b \leftarrow c$ 
22:    end if
23:  end if
24:   $j \leftarrow j + 1$ 
25: end while
26:  $V(t_i) \leftarrow b$ 
```

Here V_{\min} is the minimal admissible value of $V(t_i)$, V_0 is the value of $V(t_0)$, N is the number of iterations.

Titre en français : Méthodes implicites de Lyapunov pour l'analyse et la synthèse de systèmes super-exponentiellement stables

Résumé en français

L'un des indices de performance les plus importants pour un système de contrôle automatique est la vitesse de réponse, qui correspond au temps mis par le système pour répondre à une entrée donnée ou à une perturbation externe. La conception de systèmes de contrôle fiables à action rapide est un problème d'ingénierie difficile, pour la solution duquel diverses méthodes théoriques sont développées. Cependant, en raison de la complexité de l'analyse de stabilité requise, aucune des approches existantes ne fournit un algorithme simple et constructif pour calculer les paramètres d'un système de contrôle. Par conséquent, l'objectif de la recherche était de développer une méthode alternative de conception d'une commande automatique qui fournirait un tel algorithme. À cette fin, la méthode implicite de Lyapunov, qui est basée sur l'étude d'une fonction de Lyapunov définie implicitement par une certaine équation algébrique non linéaire, a été choisie comme outil principal pour l'analyse de stabilité dans la thèse. Grâce à la formulation implicite, les conditions de stabilité suffisantes pour les systèmes de contrôle peuvent être présentées sous la forme d'inégalités matricielles linéaires qui peuvent être vérifiées numériquement de manière très efficace en utilisant un logiciel mathématique approprié. Par conséquent, le calcul des paramètres de commande, qui assurent la performance et la robustesse souhaitées du système en boucle fermée, est considérablement simplifié. Pour démontrer les avantages et les capacités de la méthode implicite de Lyapunov, plusieurs problèmes liés à la stabilisation et à l'estimation d'état superexponentielle (hyperexponentielle et en temps fini/fixe) des systèmes dynamiques ont été résolus dans la thèse.

Mots-clefs : méthode implicite de Lyapunov, stabilité en temps fini, stabilité en temps fixe, stabilité hyperexponentielle, systèmes à retard, stabilité entrée-état

Title in English: Implicit Lyapunov methods for analysis and synthesis of superexponentially stable systems

Summary in English

One of the most important performance indices for an automatic control system is the speed of response which refers to the time taken by the system to respond to the given input or external disturbance. The design of reliable fast-acting control systems is a challenging engineering problem, for the solution of which various theoretical methods are developed. However, due to the complexity of the required stability analysis, none of the existing approaches provides a simple and constructive algorithm for calculating the parameters of a control system. Therefore, the objective of the research was to develop an alternative way of control design that would provide such an algorithm. To this end, the Implicit Lyapunov method, that is based on the study of a Lyapunov function implicitly defined by some nonlinear algebraic equation, was chosen as the main tool for stability analysis in the thesis. Due to the implicit formulation, sufficient stability conditions for control systems can be presented in the form of linear matrix inequalities that can be numerically checked very efficiently using appropriate mathematical software. As a result, the calculation of the control parameters, which ensure the desired performance and robustness of the closed-loop system, is significantly simplified. To demonstrate the advantages and capabilities of the Implicit Lyapunov method, several problems related to superexponential (hyperexponential and finite/fixed-time) stabilization and state estimation of dynamical systems have been solved in the thesis.

Keywords: Implicit Lyapunov method, finite-time stability, fixed-time stability, hyperexponential stability, time-delay systems, input-to-state stability