



**HAL**  
open science

# Analytical Methods for Solving Nonlinear Models and Variational Problems

Pascal Alain Dkengne Sielenou

► **To cite this version:**

Pascal Alain Dkengne Sielenou. Analytical Methods for Solving Nonlinear Models and Variational Problems. Analysis of PDEs [math.AP]. Université d'Abomey-Calavi (Bénin), 2011. English. NNT: . tel-03562935

**HAL Id: tel-03562935**

**<https://inria.hal.science/tel-03562935>**

Submitted on 9 Feb 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



**HAL**  
open science

# Analytical Methods for Solving Nonlinear Models and Variational Problems

Mahouton Norbert Hounkonnou

► **To cite this version:**

Mahouton Norbert Hounkonnou. Analytical Methods for Solving Nonlinear Models and Variational Problems. Analysis of PDEs [math.AP]. Université d'Abomey-Calavi (Bénin), 2011. English. tel-03562895

**HAL Id: tel-03562895**

**<https://hal.inria.fr/tel-03562895>**

Submitted on 9 Feb 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

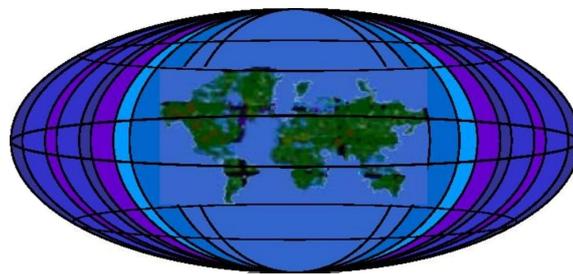
L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



United Nations  
Educational, Scientific and  
Cultural Organization



# Analytical Methods for Solving Nonlinear Models and Variational Problems



International Chair of Mathematical Physics and Applications  
(ICMPA-UNESCO Chair)  
University of Abomey-Calavi  
072 B.P. : 50 Cotonou, Rep. of Benin.

**Ph.D. Thesis in Mathematics**

By

**Pascal Alain Dkengne Sielenou**

Supervisor :

**Prof. Mahouton Norbert Hounkonnou**

**ICMPA Publishing © 2011**



University of Abomey-Calavi (UAC), BENIN  
Faculty of Sciences and Technics (FAST)  
Chaire Internationale en Physique Mathématique et Applications  
(CIPMA - Chaire UNESCO)

N° 07/Ph.D./CIPMA/FAST/UAC/2011.

# Analytical Methods for Solving Nonlinear Models and Variational Problems

**Ph.D. Thesis**

Presented for obtaining the degree of  
**Doctor in sciences of University of Abomey-Calavi, Benin**  
Option : **Mathematics**

by

**Pascal Alain Dkengne Sielenou**

Cotonou, Rep. of Benin, November 2<sup>nd</sup>, 2011

**Jury :**

**President :**

Prof. Kalyan Bidhan SINHA

(Jawaharlal Nehru Center for Advanced Scientific Research, Jakkur-India)

**Referees :**

Prof. Mahouton Norbert HOUNKONNOU (Université d'Abomey-Calavi, Benin)

Prof. Jean-Pierre ANTOINE (Université Catholique de Louvain, Belgique)

Prof. Kinvi KANGNI (Université de Cocody, Côte d'Ivoire)

Prof. Rébecca R. WALO (Université de Kinshasa, République Démocratique du Congo)

**Examiners :**

Prof. Koffi ASSIAMOUA (Université de Lomé, Togo)

Prof. Gabriel BISSANGA (Université Marien Ngouabi, Congo)

**Supervisor :**

Prof. Mahouton Norbert HOUNKONNOU (Université d'Abomey-Calavi, Benin)

**ANALYTICAL METHODS FOR SOLVING  
NONLINEAR MODELS  
AND  
VARIATIONAL PROBLEMS**

Pascal Alain Dkengne Sielenou

University of Abomey-Calavi, International Chair in Mathematical  
Physics and Applications (ICMPA–UNESCO Chair)

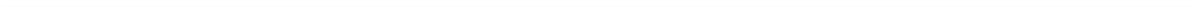
072 B.P. 50 Cotonou, Republic of Benin

Ph.D. Thesis in Mathematics



# Dedication

TO MY FAMILY





# Acknowledgments

I could never have achieved this research program without the support and encouragement of a lot of people.

Many thanks to my advisor, Prof. Mahouton Norbert Hounkonnou, for the numerous hours he has spent guiding me through this project. With his insight and considerable knowledge, he has constantly been able to suggest improvements which often led to new discoveries and better methods. Furthermore, he has revised and improved my written English. I owe you so much.

Thanks also to Dr. Elvis Benzo Ngompe Nkouankam who was at the origin of my collaboration with Prof. M. N. Hounkonnou.

I am very grateful to Dr. Mahaman Kabir. During the first year of my thesis, his patient guidance and helpful suggestions have played a significant role in the fact this work has resulted in a Ph.D. thesis.

I would like to express my thanks to my colleague of ICMIPA Mr. Bukweli Désiré. I have learned so much from you during my stay.

I extend my appreciation to the ICMIPA administrators and secretaries: Dr. Enzivi Baloitcha, Mrs. Allagnon, Mr. Soglonou and Mr. Changoniyi to solve my problems with formalities.

I am also indebted towards the teachers of mathematics of the university of Dschang and Yaoundé for the good quality of the lessons that they ensured me.

This work is partially supported by the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the Office of External Activities (OEA)-Prj-15. The ICMIPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France. I am very grateful to these institutions for their financial support.

Of course, greatest gratitude goes to my parents, my brothers, my sisters and my friends Patricia Djath, Nathacha Djath, Sandrine Djakouré, René Fotsing, Yaogan Mensah, Ousmane Samary Dine, Isiaka, Aremua and Charky Kengne for their prayers, patience, encouragement, sacrifice, help and support.

Finally, my deepest and greatest thanks to God Almighty for every think good in my life, and also to allow me to make my dreams true. I pray that I might have the incredible opportunity and responsibility to continue and use my work through out my life.

---



# Abstract

This work substantially deals with our contribution to the theory of nonlinear systems of partial differential equations and variational calculus.

Direct methods and symmetry analysis for investigation of analytical exact or approximate solutions of linear as well as nonlinear models are performed. We start by presenting the factorization method for linear and nonlinear partial differential equations as well as for their systems and give necessary and sufficient conditions of factorization in the particular case of second order equations. Then, we provide with three classes of factorizable second order linear ordinary differential equations together with their solutions. We come up with the generalization of the technique of finding exact solutions to nonlinear differential equations as an expansion of well-known functions. Known previous results are recovered as particular cases. The invariance of some differential equations with respect to certain symmetry transformation groups is a property which plays a considerable role in the study of these equations. The procedure to find symmetry groups and to construct group invariant solutions for systems of differential equations is described in this document with application to the family of fifth order KdV equations. It is sometimes difficult to handle exact solutions for differential equations. In this case, analytical approximate solutions can be computed by known powerful procedures, among which the so-called Adomian decomposition method plays a significant role. We make a detailed description of the latter method and perform its extension to the class of under determined systems of nonlinear differential equations. An illustration to the famous Ginzburg-Landau system is probed.

For partial differential equations admitting Lie symmetry groups, one can also construct their conserved quantities. Conservation laws play an important role in science. Another aim of this thesis is to provide an overview on this topic and develop new methods for constructing conservation laws using Lie group theory. The derivation of conservation laws for invariant variational problems is based on Noether theorem. This approach allows one to obtain new conservation laws even for equations without Lagrangians. A formal Lagrangian can be introduced and used for computing nonlocal conservation laws. For self-adjoint or quasi-self-adjoint equations, nonlocal conservation laws can be transformed into local ones. We extend this approach, only developed in the existing literature for determined systems, to under determined systems of partial differential equations possessing symmetry groups. An alternative direct algebraic method of constructing, for nonlinear evolution partial differential equations, conservation laws that may depend not only on dependent variables and its derivatives but also explicitly on independent variables is also proposed. This procedure starts by attributing to the differential equation a scaling symmetry group and then judiciously exploits the Euler-Lagrange and the homotopy operators to compute conserved quantities (conserved density and its related flux). Several examples on constructing conservation laws for some well-known equations are provided.

Differential equations also appear in the study of optimization problems defined by a variational integral. Classical solutions to minimization problems in the calculus of variations are prescribed by boundary value problems involving certain types of differential equations, known as the associated Euler-Lagrange equations. The mathematical techniques that have been developed to handle such optimization problems are fundamental in many areas of mathematics,

---

physics, engineering, and other applications. We present the basic mathematical analysis of nonlinear minimization principles on function spaces. The solutions to the Euler-Lagrange equations may also include (local) maxima, as well as other non-extremum critical functions. Our investigation of the second variation of a functional leads to the construction of an associated matrix. We show that the second order necessary and sufficient conditions that a function should satisfy to be either a minimum or a maximum are, in general, characterized by the properties of this matrix.

Serious efforts are made to make this book self-contained so that it is readable to many graduate students and researchers who are not specialists in these topics.

# Résumé

Ce travail présente substantiellement notre contribution à la théorie des équations aux dérivées partielles non linéaires et à la théorie du calcul des variations.

Des méthodes directes et d'analyse des symétries pour l'étude des solutions analytiques exactes ou approchées des modèles non linéaires ont été élaborées. De manière plus précise, nous avons développé une méthode de factorisation des systèmes d'équations aux dérivées partielles non linéaires et donné les conditions nécessaires et suffisantes de factorisation dans le cas particulier des systèmes du second ordre. Ensuite, nous avons fourni, ensemble avec leurs solutions, trois classes d'équations différentielles ordinaires linéaires factorisables du second ordre. Une généralisation de la technique de construction des solutions des équations aux dérivées partielles non linéaires comme une expansion de fonctions bien connues a été aussi faite. La procédure pour calculer les groupes de symétrie des systèmes d'équations aux dérivées partielles et pour construire des solutions qui sont invariantes par ces groupes a été décrite dans ce document. Des solutions analytiques approchées ont été calculées par la méthode de décomposition d'Adomian. Une description détaillée de cette dernière méthode et son extension à la classe des systèmes d'équations différentielles sous déterminées ont été présentées et illustrées par le système de Ginzburg-Landau. Cette thèse présente aussi des méthodes de construction des lois de conservation utilisant la théorie des groupes de Lie. Cette nouvelle approche adoptée permet l'obtention des lois de conservation même pour des équations différentielles sans Lagrangien. Une méthode algébrique alternative de construction directe, pour des équations aux dérivées partielles nonlinéaires évolutives, des lois de conservations a été proposée. Nous avons construit à titre d'exemples plusieurs lois de conservations pour des équations bien connues. Nous avons enfin montré que les conditions nécessaires et suffisantes du second ordre qu'une fonction critique doit satisfaire pour être soit un minimum, soit un maximum sont, en générale, caractérisées par les propriétés d'une matrice de type Hessienne.

---



# Abbreviations and Notations

## Abbreviations

DE: Differential Equation  
 ODE: Ordinary Differential Equation  
 PDE: Partial Differential Equation  
 NLPDE: Nonlinear Partial Differential Equation  
 KdV equation: Korteweg-de Vries equation

## Notations

Consider  $X$ , an  $n$ -dimensional independent variable space, and  $U$ , an  $m$ -dimensional dependent variable space. Let  $x = (x^1, \dots, x^n) \in X$  and  $u = (u^1, \dots, u^m) \in U$ . We define the space  $U^{(s)}$ ,  $s \in \mathbb{N}$  as:

$$U^{(s)} := \left\{ u^{(s)} : u^{(s)} = \bigotimes_{\alpha=1}^m \left( \bigotimes_{k=0}^s u_{(k)}^\alpha \right) \right\}, \quad (0.0.1)$$

where  $u_{(k)}^\alpha$  is the

$$p_k = \binom{n+k-1}{k} \text{-tuple} \quad (0.0.2)$$

of all distinct  $k$ -order partial derivatives of  $u^\alpha$ . The  $u_{(k)}^\alpha$  vector components are recursively obtained as follows:

- i)  $u_{(0)}^\alpha = u^\alpha$  and  $u_{(1)}^\alpha = (u_{x^1}^\alpha, u_{x^2}^\alpha, \dots, u_{x^n}^\alpha)$ .
- ii) Assume that  $u_{(k)}^\alpha$  is known.

- Form the tuples  $\hat{u}_{(k+1)}^\alpha(l)$  as follows

$$\hat{u}_{(k+1)}^\alpha(l) = \left( \frac{\partial}{\partial x^1} u_{(k)}^\alpha[l], \frac{\partial}{\partial x^2} u_{(k)}^\alpha[l], \dots, \frac{\partial}{\partial x^n} u_{(k)}^\alpha[l] \right), \quad l = 1, 2, \dots, p_k;$$

where  $u_{(k)}^\alpha[l]$  is the  $l$ -th component of the vector  $u_{(k)}^\alpha$ .

- Form, by iteration, the tuples  $\tilde{u}_{(k+1)}^\alpha(l)$  following the scheme  $\tilde{u}_{(k+1)}^\alpha(1) = \hat{u}_{(k+1)}^\alpha(1)$  and for  $l = 2, 3, \dots, p_k$ , the vector  $\tilde{u}_{(k+1)}^\alpha(l)$  is nothing but the tuple  $\hat{u}_{(k+1)}^\alpha(l)$  in which all components already present in  $\tilde{u}_{(k+1)}^\alpha(i)$ ,  $i = 1, 2, \dots, l-1$  are excluded.
- Finally, form the vector

$$u_{(k+1)}^\alpha = \left( \tilde{u}_{(k+1)}^\alpha(1), \tilde{u}_{(k+1)}^\alpha(2), \dots, \tilde{u}_{(k+1)}^\alpha(p_k) \right).$$

As a matter of clarity, let us immediately illustrate this construction by the following.

---

**Example 0.0.1** • For  $n = 2$ ,  $x = (x^1, x^2)$  and we have:

$$u_{(1)}^\alpha = (u_{x^1}^\alpha, u_{x^2}^\alpha),$$

$$\widehat{u}_{(2)}^\alpha(1) = \left( \frac{\partial}{\partial x^1} u_{(1)}^\alpha[1], \frac{\partial}{\partial x^2} u_{(1)}^\alpha[1] \right) = (u_{2x^1}^\alpha, u_{x^1 x^2}^\alpha),$$

$$\widehat{u}_{(2)}^\alpha(2) = \left( \frac{\partial}{\partial x^1} u_{(1)}^\alpha[2], \frac{\partial}{\partial x^2} u_{(1)}^\alpha[2] \right) = (u_{x^2 x^1}^\alpha, u_{2x^2}^\alpha),$$

$$\widetilde{u}_{(2)}^\alpha(1) = \widehat{u}_{(2)}^\alpha(1) = (u_{2x^1}^\alpha, u_{x^1 x^2}^\alpha), \quad \widetilde{u}_{(2)}^\alpha(2) = (\check{u}_{x^2 x^1}^\alpha, u_{2x^2}^\alpha) = (u_{2x^2}^\alpha),$$

$$u_{(2)}^\alpha = \left( \widetilde{u}_{(2)}^\alpha(1), \widetilde{u}_{(2)}^\alpha(2) \right) = (u_{2x^1}^\alpha, u_{x^1 x^2}^\alpha, u_{2x^2}^\alpha).$$

• For  $n = 3$ ,  $x = (x^1, x^2, x^3)$  and the same scheme leads to

$$u_{(2)}^\alpha = (u_{2x^1}^\alpha, u_{x^1 x^2}^\alpha, u_{x^1 x^3}^\alpha, u_{2x^2}^\alpha, u_{x^2 x^3}^\alpha, u_{2x^3}^\alpha),$$

$$u_{(3)}^\alpha = (u_{3x^1}^\alpha, u_{2x^1 x^2}^\alpha, u_{2x^1 x^3}^\alpha, u_{x^1 2x^2}^\alpha, u_{x^1 x^2 x^3}^\alpha, u_{x^1 2x^3}^\alpha, u_{3x^2}^\alpha, u_{2x^2 x^3}^\alpha, u_{x^2 2x^3}^\alpha, u_{3x^3}^\alpha),$$

for  $k = 2$  and  $k = 3$ , respectively.

An element  $u^{(s)}$ , in the space  $U^{(s)}$ , is the

$$q_s = m(1 + p_1 + p_2 + \cdots + p_s) = m \binom{n+s}{s} \text{-tuple} \quad (0.0.3)$$

defined by

$$u^{(s)} = (u_{(0)}^1, u_{(1)}^1, \cdots, u_{(s)}^1, u_{(0)}^2, u_{(1)}^2, \cdots, u_{(s)}^2, \cdots, u_{(0)}^m, u_{(1)}^m, \cdots, u_{(s)}^m). \quad (0.0.4)$$

We denote by  $X \times U^{(s)}$ , the total space whose coordinates are denoted by  $(x, u^{(s)})$ , encompassing the independent variables  $x$  and the dependent variables with their derivatives up to order  $s$ , globally denoted by  $u^{(s)}$ .

In the sequel, a  $q_s$ -uple  $u^{(s)}$  is referred to (0.0.4), whereas the integers  $p_k$  and  $q_s$  are defined by (0.0.2) and (0.0.3), respectively.

The notations which follow are valid only in the first chapter. We denote again by  $u_{(k)}^\alpha$  the  $p_k = n^k$ -tuple of all  $k$ -order partial derivatives of  $u^\alpha$  distinct or not. The  $u_{(k)}^\alpha$  vector components are recursively obtained as follows:

i)  $u_{(0)}^\alpha = u^\alpha$  and  $u_{(1)}^\alpha = (u_{x^1}^\alpha, u_{x^2}^\alpha, \cdots, u_{x^n}^\alpha)$ .

ii) Assume that  $u_{(k)}^\alpha$  is known.

– Form the tuples  $\widehat{u}_{(k+1)}^\alpha(l)$  as follows

$$\widehat{u}_{(k+1)}^\alpha(l) = \left( \frac{\partial}{\partial x^1} u_{(k)}^\alpha[l], \frac{\partial}{\partial x^2} u_{(k)}^\alpha[l], \cdots, \frac{\partial}{\partial x^n} u_{(k)}^\alpha[l] \right), \quad l = 1, 2, \cdots, p_k;$$

where  $u_{(k)}^\alpha[l]$  is the  $l$ -th component of the vector  $u_{(k)}^\alpha$ .

– Finally, form the vector

$$u_{(k+1)}^\alpha = \left( \widehat{u}_{(k+1)}^\alpha(1), \widehat{u}_{(k+1)}^\alpha(2), \dots, \widehat{u}_{(k+1)}^\alpha(p_k) \right).$$

Define differential operators  $D_{k,h}$  whose action on a regular function  $u$  is

$$D_{k,h} u = u_{(k)}[h]. \quad (0.0.5)$$

Note that the differential operators  $D_{k,h}$  satisfy the following properties:

- (i)  $D_{0,1} u = u$  (identity),
- (ii)  $D_{1,h} D_{k',h'} u = D_{k'+1,n(h'-1)+h} u$  (composition rule),
- (iii)  $D_{k,h} u = D_{1,h} D_{k-1,1} u$ ,  $k \geq 1$  (decomposition rule).

**Remark 0.0.1** Operators  $D_{k,h}$  allow the simplification of the writing of certain differential operators. For example, the operator

$$\mathcal{T} = \sum_{l_1+l_2+\dots+l_n=0}^s \frac{\partial^{l_1+l_2+\dots+l_n}}{(\partial x^1)^{l_1} (\partial x^2)^{l_2} \dots (\partial x^n)^{l_n}}$$

can be shortly expressed as

$$\mathcal{T} = \sum_{k=0}^s \sum_{h=1}^{p_k} D_{k,h}.$$

It is clearly seen that there is a considerable reduction of the summation indices.



# Introduction

This dissertation deals with different algebraic aspects of NLPDEs and variational calculus. We discuss here the motivations and the problem, and sketch the outlines of our main achievements.

- *How important are NLPDEs in our life?*

Nonlinear partial differential equations are widely used to describe important phenomena and processes in various areas starting from mathematical sciences to life sciences. For instance, the Burger-Fisher equation [102] is used to describe the structure of shock waves, the traffic flow, the acoustic transmission and the process of interaction between diffusion and reaction. The Kompaneets equation [58], also known as the photon diffusion equation, describes an interaction of free electrons and electromagnetic radiation, specifically, an interaction of a low-energy homogeneous photon gas with a rarefied electron gas via the Compton scattering. The Novikov equation [11] is used to model dispersive shallow wave propagation. The Ginzburg-Landau (GL) equations [85, 86] models the low-temperature superconductivity while the Korteweg-de Vries (KdV) equations [102] serve to describe a variety of nonlinear phenomena, including ion acoustic waves in plasmas and shallow water waves. Differential equations can be also derived from the transformations performed on equations modelling given systems. Thus, many important equations, pertaining to physical and technical applications, are reducible to Helmholtz equation [25] if time dependence is separated. This property extends to equations generally describing quite the propagation of waves like the diffusion equation, the wave equation, the damped wave equation, the transmission line equation and the vector wave equation. The Helmholtz and Laplace equations are expressible, using a separation of variables in appropriate coordinate systems, into linear ordinary differential equations which belong to the class of Bôcher equations [76]. For instance, Heine and Wangerin equations [76, 112], which are special cases of Bôcher equations, appear when the Laplace's equation is solved in the bi-cyclide and flat-ring cyclide coordinate systems, respectively.

- *What about the issue of analytical methods for solving NLPDEs?*

In recent years, the search for exact solutions of differential equations is very challenging in mathematics, but their usefulness in the proper understanding of qualitative features of phenomena and processes in various areas of natural science merits to get down to such an investigation. Indeed, exact solutions can be used to verify the consistency and estimate errors of various numerical, asymptotic and approximate analytical methods. Unfortunately, there does not always exist a method adapted for the resolution of any type of differential equations. Very often, one tries to reduce the equations in order to make easier its resolution. But such a reduction often requires the knowledge of suitable transformations or changes of variables. The latter usually give rise to another problem the issue of which is not always favorable. It is well known that the lowest the order of a differential equation is, the greatest is the possibility of finding suitable analytical methods for its resolution. Therefore, when a differential equation cannot be

---

directly analytically solved, it is meaningful to transform it into an equation of lower order. In practice, it is generally difficult to perform such a reduction. In [46], we have addressed classes of  $n$ -order, ( $n \geq 2$ ), nonlinear ordinary differential equations (ODEs) reducible to lower order ones, using the well known method of parameter variation usually applicable to linear ODEs. Known classical equations such as the Bernoulli, Emden-Fowler, homogenous and parabolic cylinder equations are recovered in illustrated relevant examples.

A simple approach for the reduction of a differential equation into a lower order one consists in seeking a factorization, if there exists, of the differential operator associated with it. Note that for the particular case of second order linear ordinary differential equations of Schrödinger or Sturm-Liouville type, the factorization of the associated differential operators also allows to obtain partially or completely their spectrum, under certain assumptions of integrability [31, 43, 62, 98]. Note also that in recent years, there has been much interest devoted to the problem of factorization of differential equations, especially based on linear [16, 20, 99] and nonlinear [19, 20, 21] ordinary differential equations. Although effective, the used methods are rather restrictive in their applications.

- *What have we achieved in the algebraic theory of NLPDEs factorization?*

In the first two chapters, we have described the principle of factorization of nonlinear partial differential equations and their systems and given the related necessary and sufficient conditions. These results extend our previous works [47] on the factorization of nonlinear ordinary differential equations and systems. Remark also that the Beke's method [16] and van Hoiej's methods [99] for factorization of ordinary linear differential operators are also restrictive in their application. Unfortunately, none of these methods of factorization does give indication on what type of linear ordinary differential equations is factorizable or not. In other words, the existing factorization methods and symbolic computation codes do not tell us what kinds of linear equations are factorizable. More specifically, they do not answer to the question: given a second order linear differential equation, does it admit a factorizable form? We have partially fulfill such a lack in the third chapter by proceeding to a systematic classification of factorizable second order linear ordinary differential equations with polynomial coefficients whose degrees satisfy some particular relations, using an algebraic method [44, 54, 89, 90] of differential operator decomposition into a product of lower order differential operators.

- *What have we achieved in the algebraic theory of analytical methods for solving NLPDEs?*

A variety of methods, e.g. the exponential function method [38], the hyperbolic tangent function method [77], the Jacobi elliptic function expansion method [74], the sine-cosine function method [17], the  $F$ -expansion method [72], the projective Riccati equation expansion method [5], the  $\frac{G'}{G}$ -expansion method [18], the  $(\frac{G'}{G}, \frac{1}{G})$ -expansion method [73] and their numerous extensions [4, 6, 23, 32, 33, 34, 65, 75, 100, 101, 103, 109, 110, 111], etc. are used to obtain exact solutions for nonlinear differential equations. The common issue to all these methods is the rational form of the final solution. The rationality here is with respect to either trigonometric functions, or hyperbolic functions, or elliptic functions, or exponential functions, or other given functions. Furthermore, the scope of these methods is restricted to polynomial differential equations. We describe in the

---

fourth chapter the general principle from which all the above mentioned methods derive for the integration of autonomous NLPDEs.

- *What is our contribution relative to the use of Lie symmetry methods and their generalizations?*

Suppose we have to solve a differential equation. How can we proceed? First, we will certainly check whether the differential equation belongs to a class we know how to treat or whether it can be transformed into a member of such a class. If unsuccessful, we may look up a textbook or a collection of solutions. If that does not help and the problem is of some importance, we will try more elaborate transformation methods and ansätze to find a solution. But it may sometimes happen that we have to leave the problem unsolved. However, in the majority of the cases where exact solutions of a differential equation can be found, the underlying property is a symmetry of that equation. Indeed, this was the state of the art around the middle of the nineteenth century, when Sophus Lie made the profound and far-reaching discovery that many special methods were, in fact, special cases of general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation at once unified and significantly extended the available integration techniques. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group elements. Alternatively, one can use symmetry groups to perform some reductions in the system. The reduction here is understood in the sense that the system is transformed into an equivalent one, where either the order or the number of independent variables are reduced. Thus, the solutions to the original system can be recovered from those of the reduced system when the latter is integrable. In the fifth chapter, we describe the process to find one-parameter Lie symmetry groups of transformations for differential equations [37, 81]. Then we show how to use these symmetries for the integration of differential equations. In [45], we have applied this method to find exact solutions of the fifth order KdV equations.

- *What have we achieved regarding the Adomian decomposition method?*

In most cases, analytic solutions of differential equations are very difficult to achieve. Usually, one uses analytic approaches that linearize the system by assuming that the nonlinearities are relatively insignificant. Such procedures change the actual problem to make it tractable by the conventional methods. This changes drastically the solution of the real problem. In the other side, the numerical methods are based on discretization techniques and require intensive computer time to solve the problem; they also permit to obtain only approximate solutions for finite values of independent variables, overlooking some important phenomena. The above drawbacks of linearization and numerical methods justify the need to search for an alternative techniques to solve the nonlinear differential equations. Such techniques encompass the homotopy perturbation method,

the iteration variational method [39, 40, 41, 102] and the Adomian decomposition method [7, 8, 9]. The Adomian decomposition method consists of splitting the given equation into linear and nonlinear parts, identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian polynomials, and finding the successive terms of the series solution by recurrence relation. It was shown in [10] that the Adomian decomposition method is a special case of the homotopy analysis method. In addition the criteria for the convergence of the homotopy analysis method was shown to be the same criteria for the convergence of the Adomian decomposition method and that one has a variety of choices for the linear operator and therefore a variety of choices for the initial estimation to start the Adomian decomposition iteration process.

It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Cherruault examined the convergence of Adomian method in [27]. In addition, Cherruault and Adomian presented a new proof of convergence of the method in [28]. For more details about the proofs presented to discuss the rapid convergence, the reader is advised to see the references mentioned above and the references therein. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. It was also shown by many authors that the series obtained by evaluating few terms gives an approximation of high degree of accuracy if compared with other numerical techniques [102]. However, although Adomian decomposition method has been used intensively to solve nonlinear problems, in our best knowledge of the literature, it is not developed to handle under determined systems of nonlinear partial differential equations. The sixth, seventh and eighth chapters are devoted to the generalization of this method to classes of under determined systems.

- *How have we contributed to answer to the question of how to associate a conservation law with every infinitesimal generator of symmetries of arbitrary PDEs?*

The existence of symmetry groups to differential equations also enables the computation of its conserved quantities. Conservation laws play a vital role in the study of PDEs in the search for qualitative properties such as integrability, stability, existence of global solutions and the linearizability conditions [78, 92, 94, 104, 105]. Their usefulness has considerably increased since the work by Jacobi [61] in 1884, who showed a connection between conserved quantities and symmetries of the equations of a particle's motion in classical mechanics. Klein [66] has obtained similar result for the equations of the general relativity and predicted that a connection between conservation laws and symmetries could be found for any differential equation obtained from a variational principle. Noether [80] has showed that the conservation laws were associated with the invariance of variational integrals with respect to continuous transformation groups. She obtained the sufficient condition for existence of conservation laws. In 1921, Bessel-Hagen [22] applied Noether's theorem with the so-called "divergence" condition to the Maxwell equations

and calculated their conservation laws. In 1951, Hill [42] presented the explicit formula in terms of variations for conservation laws in the case of a first order Lagrangian. Ibragimov [55] proved the generalized version of Noether's theorem and conservation laws related to the invariance of the extremal values of variational integrals. He derived the necessary and sufficient condition for the existence of conservation laws and gave the explicit expressions in the case of a Lagrangian of any order. Recently, some relevant results on conservation laws have emerged. See for instance the works by Wolf [106], Anco [13, 14], Poole [83], Ibragimov [56, 57], Naz [79] and Khamitova [64]. Despite all this progress, there remains an important question: How to associate a conservation law with every infinitesimal generator of symmetries of arbitrary PDEs? Ibragimov [57] achieved this goal for any system of PDEs where the number of equations is equal to the number of dependent variables. In the ninth chapter, we extend Ibragimov's result to under determined system of DEs.

- *What about the construction of time-space dependent conservation laws for nonlinear evolution PDEs?*

The literature prospers in various approaches to construct local conservation laws. For instance, we can cite the direct method [13, 14, 70], the Noether method [12, 80], the characteristic method and the variational derivative method [81, 97, 106]. All these approaches allow, in most cases, to compute polynomial conservation laws that only depend on dependent variables and its derivatives, and not explicitly on independent variables. In the tenth chapter, an alternative study of time-space dependent conservation laws for some classes of nonlinear evolution PDEs is performed by judiciously exploiting known algebraic methods [83, 107]. It generalizes, to any dimensional space, a previous algorithmic scheme [108] elaborated to construct spatio-temporal dependent conservation laws for  $n$ -order  $(1 + 1)$ -dimensional Korteweg de Vries (KdV) equations.

- *What have we done in the field of variational calculus?*

The differential equations are closely related to the calculus of variations in the sense that solving extremal problems for a functional can pass by the resolution of DEs. The calculus of variations encompasses a very broad range of mathematical applications. The methods of variational analysis can be applied to an enormous variety of physical systems, whose equilibrium configurations inevitably minimize or maximize a suitable functional which typically represents the potential energy of the system. The critical functions are characterized as solutions to a system of PDEs, known as the Euler-Lagrange equations associated with the variational principle. Each solution to the problem specified by the Euler-Lagrange equations subject to appropriate boundary conditions is thus a candidate for extremums of the functional defining the variational problem. In many applications, the Euler-Lagrange boundary value problem suffices to single out the physically relevant solutions, and one does not need to press onto the considerably more difficult second variation. In general, the solutions to the Euler-Lagrange boundary value problem are critical functions for the functional defining the variational problem, and hence include all (smooth) local and global extremums. The determination of which solutions are genuine minima or maxima requires further analysis of the positivity properties of the second variation. But, as stated by Olver in [82], a complete analysis

of the positive definiteness of the second variation of multi-dimensional variational problems is quite complicated, and still awaits a completely satisfactory resolution! This is thus a reason for which, in the literature, second order conditions of extremums are customary established only for functional whose Lagrangian involves dependent variables together with at most their first order derivatives [15, 35, 36, 84, 82]. In the eleventh chapter, we make a brief reviews of relevant known results in variational calculus that we are going to extend. We start the last part by dealing with variational problems without constraints or with constraints of holonomic type and give some first order extremum necessary conditions for these problems. Then, we finish with some satisfactory expressions of the second order extremum conditions for a functional whose Lagrangian also depends on the higher order derivatives of the dependent variables.

- *How is the dissertation partitioned?*

This document is organized into four parts. The first part contains some methods to solve nonlinear models such as the factorization scheme, the function expansion principle and the use of symmetries to find group-invariant solutions. In the second part, we describe roughly the construction of analytical approximate solutions to any systems of nonlinear partial differential equations using the Adomian decomposition technique. The third part is devoted to group analysis of differential equations in which we show how to use symmetry group of differential equations for the computation of conserved quantities. The fourth part provides a detailed description on the first and second order extremum conditions for any functional defined by a variational integral.

# List of Figures

6.1	Function $\phi$ , approximate solution of the Kompaneets equation, versus $x$ and $t$ . . . . .	83
6.2	Function $\phi$ , approximate solution of the Novikov equation, versus $x$ and $t$ . . . . .	85
7.1	Approximate solution of the Ginzburg-Landau system (7.2.11) versus $X = x + y + z$ and $t$ . . .	93
8.1	Approximate solution of the Ginzburg-Landau system (8.2.16) versus $X = x + y + z$ and $t$ . . .	104





# Contents

Dedication	i
Acknowledgments	iii
Abstract	v
Résumé	vii
Abbreviations and Notations	ix
Introduction	xiii
List of Figures	xix
Contents	xix
<b>I Reductions and Exact Solutions of Nonlinear Models</b>	<b>1</b>
<b>1 Factorization of Linear Differential Equations and Systems</b>	<b>3</b>
1.1 Factorization of linear differential equations . . . . .	3
1.1.1 General setting . . . . .	3
1.1.2 Case of second order linear ODEs . . . . .	4
1.1.3 Case of second order linear PDEs . . . . .	5
1.2 Factorization of systems of linear differential equations . . . . .	7
1.2.1 General considerations . . . . .	7
1.2.2 Case of systems of second order linear ODEs . . . . .	8
1.2.3 Case of systems of second order linear PDEs . . . . .	10
<b>2 Factorization of Nonlinear Differential Equations and Systems</b>	<b>13</b>
2.1 Factorizations of nonlinear differential equations . . . . .	13
2.1.1 General setting and results . . . . .	13
2.1.2 Case of second order nonlinear ODEs . . . . .	14
2.1.3 Case of second order nonlinear PDEs . . . . .	14
2.2 Factorizations of systems of nonlinear differential equations . . . . .	16
2.2.1 Theoretical considerations and principles . . . . .	16
2.2.2 Case of systems of second order nonlinear ODEs . . . . .	16
2.2.3 Case of systems of second order nonlinear PDEs . . . . .	18
<b>3 Three Factorizable Classes of 2<sup>nd</sup> Order Linear ODEs</b>	<b>23</b>
3.1 First lass of factorizable 2 <sup>nd</sup> order linear ODEs . . . . .	25
3.2 Second class of factorizable 2 <sup>nd</sup> order linear ODEs . . . . .	30
3.3 Third class of factorizable 2 <sup>nd</sup> order linear ODEs . . . . .	35

---

<b>4</b>	<b>Function Expansion Methods to Solve Autonomous NLPDEs</b>	<b>41</b>
4.1	Single function expansion method . . . . .	41
4.2	Double function expansion method . . . . .	43
4.3	Triple function expansion method . . . . .	44
4.4	Application to some relevant NLPDEs in physics . . . . .	46
<b>5</b>	<b>Group Invariant Solutions of Differential Equations</b>	<b>53</b>
5.1	One-parameter groups of transformations . . . . .	53
5.2	Invariance criteria . . . . .	57
5.3	Determination of Lie group of differential equations . . . . .	61
5.4	Change of variables . . . . .	63
5.5	The use of symmetries to ordinary differential equations . . . . .	67
5.6	The use of symmetries to partial differential equations . . . . .	71
<b>II</b>	<b>Approximate Analytical Solutions of Nonlinear Models</b>	<b>75</b>
<b>6</b>	<b>Adomian Method for PDEs</b>	<b>79</b>
6.1	Brief description of the Adomian method . . . . .	79
6.2	Theoretical considerations . . . . .	80
6.3	Application to Kompaneets equation . . . . .	81
6.4	Application to Novikov equation . . . . .	84
<b>7</b>	<b>Adomian Method for Determined Systems of PDEs</b>	<b>87</b>
7.1	Theoretical considerations . . . . .	87
7.2	Application to Ginzburg-Landau equations . . . . .	88
<b>8</b>	<b>Adomian Method for Under Determined Systems of PDEs</b>	<b>95</b>
8.1	Theoretical considerations . . . . .	95
8.2	Application to Ginzburg-Landau equations . . . . .	98
<b>III</b>	<b>Symmetry Groups and Conservation Laws of PDEs</b>	<b>105</b>
<b>9</b>	<b>Symmetry Groups, Noether Theorem and Conservation Laws</b>	<b>107</b>
9.1	Basic definitions and Noether theorem . . . . .	107
9.2	Conservation laws for determined systems of NLPDEs . . . . .	109
9.3	Conservation laws for under determined systems of NLPDEs . . . . .	111
<b>10</b>	<b>Scaling Symmetries, Homotopy Operators and Conservation Laws</b>	<b>115</b>
10.1	Basic definitions and theorems . . . . .	115
10.2	Homotopy operator formula . . . . .	117
10.3	Conservation laws of evolution NLPDEs . . . . .	118
10.3.1	Determination of a scaling symmetry group . . . . .	119
10.3.2	Construction of a density and its associated flux . . . . .	120
10.3.3	Application to a fifth order KdV equation . . . . .	120
10.4	Time-space dependent conservation laws of some evolution NLPDEs . . . . .	122
10.4.1	Case of a fifth order KdV equation . . . . .	123

---

10.4.2	Case of the $(2 + 1)$ -dimensional Zakharov-Kuznetsov equation . . . . .	123
10.4.3	Case of the $(3 + 1)$ -dimensional Zakharov-Kuznetsov equation . . . . .	125
<b>IV</b>	<b>Extremum Conditions for Variational Problems</b>	<b>129</b>
<b>11</b>	<b>First Order Extremum Conditions</b>	<b>133</b>
11.1	Some known first order extremum conditions . . . . .	133
11.2	First variation and extremum conditions for unconstrained problems . . . . .	134
11.3	First variation and extremum conditions for constrained problems . . . . .	137
<b>12</b>	<b>Second Order Extremum Conditions</b>	<b>143</b>
12.1	Some known second order extremum conditions . . . . .	143
12.2	Second variation and conditions for local extremums . . . . .	145
12.3	Applications to some concrete examples . . . . .	156
	<b>Conclusion</b>	<b>163</b>
	<b>Bibliography</b>	<b>165</b>



Part I

Reductions and Exact Solutions of  
Nonlinear Models

---



# Factorization of Linear Differential Equations and Systems

---

In this chapter, an algebraic approach for factorizing linear partial differential equations (PDEs) and systems of PDEs is provided. In the particular case of second order linear PDEs and systems of PDEs, necessary and sufficient conditions of factorization are given. The main results of this chapter have been published [51].

## 1.1 Factorization of linear differential equations

In this section, we develop an algebraic method of factorization applicable to linear differential operators (LDOs) and to systems of LDOs.

### 1.1.1 General setting

The general setting of the factorization problem for LDOs is developed. Necessary and sufficient conditions are derived for the factorization of second order linear ordinary and partial differential operators with two independent variables.

Let  $s \geq 2$  be a positive integer and  $\Lambda$  be an open subset of  $\mathbb{R}^n$ . Let

$$\mathcal{P}(s) = \sum_{k=0}^s \sum_{h=1}^{p_k} g_{k,h}(x) D_{k,h} \quad (1.1.1)$$

a linear differential operator of order  $s$ , where  $g_{k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ . The operator  $\mathcal{P}(s)$  acts on a function  $u \in \mathcal{C}^s(\Lambda, \mathbb{R})$  as follows

$$\mathcal{P}(s) u = \sum_{k=0}^s \sum_{h=1}^{p_k} g_{k,h}(x) D_{k,h} u. \quad (1.1.2)$$

The method of factorization consists in seeking a decomposition of the differential operator (1.1.1) in the following form

$$\mathcal{P}(s) = \prod_{i=1}^l \mathcal{Q}_i(s_i) \quad (1.1.3)$$

with  $\sum_{i=1}^l s_i = s$  and

$$\mathcal{Q}_i(s_i) = \sum_{k=0}^{s_i} \sum_{h=1}^{p_k} b_{i,k,h}(x) D_{k,h}, \quad (1.1.4)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $b_{i,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_j}(\Lambda, \mathbb{R})$ ,  $i = 2, 3, \dots, l$ .

---

**Proposition 1.1.1** Let  $\mathcal{P}(s)$  be an operator which can be decomposed into the form (1.1.3). If the function  $u_0$  satisfies

$$\mathcal{Q}_l(s_l)u_0 = 0, \quad (1.1.5)$$

and  $u_1, \dots, u_{l-1}$  are solutions of the system

$$\prod_{k=l-j+1}^l \mathcal{Q}_k(s_k)u_j = v_j, \quad j = 1, 2, \dots, l-1, \quad (1.1.6)$$

where  $v_j, j = 1, 2, \dots, l-1$ , are solutions of

$$\prod_{i=1}^{l-j} \mathcal{Q}_i(s_i)v_j = 0, \quad (1.1.7)$$

then  $u_0, u_1, \dots, u_{l-1}$  are  $l$  particular solutions of the equation  $\mathcal{P}(s)u = 0$ .

**Proof.** Let  $u_0$  and  $u_j, j = 1, 2, \dots, l-1$  be solutions of (1.1.5) and (1.1.6), respectively. Then

$$\mathcal{P}(s)u_0 = \left( \prod_{i=1}^{l-1} \mathcal{Q}_i(s_i) \right) \mathcal{Q}_l(s_l)u_0 = 0,$$

and for  $j = 1, 2, \dots, l-1$ ,

$$\begin{aligned} \mathcal{P}(s)u_j &= \left( \prod_{i=1}^{l-j} \mathcal{Q}_i(s_i) \right) \left( \prod_{k=l-j+1}^l \mathcal{Q}_k(s_k) \right) u_j \\ &= \prod_{i=1}^{l-j} \mathcal{Q}_i(s_i)v_j = 0, \end{aligned}$$

where the use of (1.1.6) and (1.1.7) has been made. ■

Expanding (1.1.3) leads to the relations between unknown functions  $b_{i,k,h}$  of the differential operators  $\mathcal{Q}_i(s_i)$  and the known functions  $g_{k,h}$  of the original differential operator  $\mathcal{P}(s)$ . Without loss of generality and as matter of clarity, this study will be concentrated to second order equations, the generalization being straightforward.

### 1.1.2 Case of second order linear ODEs

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}$  such that  $\Lambda_0 \subset \Lambda$ . Consider the second order linear ordinary differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^2 \sum_{h=1}^{p_k} g_{k,h}(x)D_{k,h} \\ &= g_{0,1}(x)D_{0,1} + g_{1,1}(x)D_{1,1} + g_{2,1}(x)D_{2,1}, \end{aligned} \quad (1.1.8)$$

where  $g_{k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $x = x^1$ . Write  $\mathcal{P}(2)$  in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1) \cdot \mathcal{Q}_2(1) \\ &= \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{1,k,h}(x)D_{k,h} \right] \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{2,k,h}(x)D_{k,h} \right] \\ &= [b_{1,0,1}(x)D_{0,1} + b_{1,1,1}(x)D_{1,1}] [b_{2,0,1}(x)D_{0,1} + b_{2,1,1}(x)D_{1,1}], \end{aligned} \quad (1.1.9)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $b_{2,k,h} \in \mathcal{C}^1(\Lambda, \mathbb{R})$ .

Let  $u \in \mathcal{C}^2(\Lambda_0, \mathbb{R})$ . We have

$$\mathcal{P}(2)u = g_{0,1}(x)u + g_{1,1}(x)u_x + g_{2,1}(x)u_{2x} \quad (1.1.10)$$

and after expansion

$$\begin{aligned} \mathcal{P}(2)u &= [b_{1,0,1}(x)D_{0,1} + b_{1,1,1}(x)D_{1,1}] [b_{2,0,1}(x)D_{0,1} + b_{2,1,1}(x)D_{1,1}]u \\ &= b_{1,1,1}b_{2,1,1}u_{2x} + [b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,1,1})]u_x \\ &+ [b_{1,0,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,0,1})]u. \end{aligned} \quad (1.1.11)$$

Identifying (1.1.10) with (1.1.11) yields

**Proposition 1.1.2** *A necessary and sufficient condition to the differential operator  $\mathcal{P}(2)$  defined by (1.1.8) to be decomposed into the form (1.1.9) is that:*

$$g_{2,1} = b_{1,1,1}b_{2,1,1}, \quad (1.1.12)$$

$$g_{1,1} = b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,1,1}), \quad (1.1.13)$$

$$g_{0,1} = b_{1,0,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,0,1}). \quad (1.1.14)$$

Propose an approach to solve system (1.1.12)-(1.1.14). Assume that  $g_{2,1}$  does not vanish on  $\Lambda$ . Thus, it is always possible to find two nonzero functions on  $\Lambda$  namely,  $b_{1,1,1}$  and  $b_{2,1,1}$  which satisfy (1.1.12). Substituting  $X = b_{1,0,1}$  and  $Y = b_{2,0,1}$  in (1.1.13) yields

$$X = \frac{1}{b_{2,1,1}} [g_{1,1} - b_{1,1,1}D_{1,1}(b_{2,1,1}) - b_{1,1,1}Y]. \quad (1.1.15)$$

The substitution of (1.1.15) into (1.1.14) implies that the decomposition (1.1.9) is strongly related to the existence of a solution to the following Riccati equation in  $Y$

$$D_{1,1}(Y) - \frac{b_{1,1,1}}{g_{2,1}}Y^2 + \frac{g_{1,1} - b_{1,1,1}D_{1,1}(b_{2,1,1})}{g_{2,1}}Y - \frac{g_{0,1}}{b_{1,1,1}} = 0. \quad (1.1.16)$$

### 1.1.3 Case of second order linear PDEs

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}^2$  such that  $\Lambda_0 \subset \Lambda$ . Consider the second order linear partial differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^2 \sum_{h=1}^{p_k} g_{k,h}(x)D_{k,h} \\ &= g_{0,1}(x)D_{0,1} + g_{1,1}(x)D_{1,1} + g_{1,2}(x)D_{1,2} \\ &+ g_{2,1}(x)D_{2,1} + g_{2,2}(x)D_{2,2} + g_{2,3}(x)D_{2,3} + g_{2,4}(x)D_{2,4}, \end{aligned} \quad (1.1.17)$$

where  $g_{k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $x = (x^1, x^2)$ . Write  $\mathcal{P}(2)$  in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1) \cdot \mathcal{Q}_2(1) \\ &= \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{1,k,h}(x)D_{k,h} \right] \left[ \sum_{k=0}^2 \sum_{h=1}^{p_k} b_{2,k,h}(x)D_{k,h} \right] \\ &= [b_{1,0,1}(x)D_{0,1} + b_{1,1,1}(x)D_{1,1} + b_{1,1,2}(x)D_{1,2}] \\ &\times [b_{2,0,1}(x)D_{0,1} + b_{2,1,1}(x)D_{1,1} + b_{2,1,2}(x)D_{1,2}], \end{aligned} \quad (1.1.18)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $b_{2,k,h} \in \mathcal{C}^1(\Lambda, \mathbb{R})$ .

Let  $u \in \mathcal{C}^2(\Lambda_0, \mathbb{R})$ . We have

$$\begin{aligned} \mathcal{P}(2)u &= g_{0,1}u + g_{1,1}u_{x^1} + g_{1,2}u_{x^2} \\ &+ g_{2,1}u_{2x^1} + (g_{2,2} + g_{2,3})u_{x^1x^2} + g_{2,4}u_{2x^2} \end{aligned} \quad (1.1.19)$$

and after expansion

$$\begin{aligned} \mathcal{P}(2)u &= [b_{1,0,1}(x)D_{0,1} + b_{1,1,1}(x)D_{1,1} + b_{1,1,2}(x)D_{1,2}] \\ &\times [b_{2,0,1}(x)D_{0,1} + b_{2,1,1}(x)D_{1,1} + b_{2,1,2}(x)D_{1,2}]u \\ &= [b_{1,0,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,0,1}) + b_{1,1,2}D_{1,2}(b_{2,0,1})]u \\ &+ [b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,1,1}) + b_{1,1,2}D_{1,2}(b_{2,1,1})]u_{x^1} \\ &+ [b_{1,0,1}b_{2,1,2} + b_{1,1,2}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,1,2}) + b_{1,1,2}D_{1,2}(b_{2,1,2})]u_{x^2} \\ &+ b_{1,1,1}b_{2,1,1}u_{2x^1} + [b_{1,1,2}b_{2,1,1} + b_{1,1,1}b_{2,1,2}]u_{x^1x^2} + b_{1,1,2}b_{2,1,2}u_{2x^2}. \end{aligned} \quad (1.1.20)$$

Identifying (1.1.19) with (1.1.20) yields

**Proposition 1.1.3** *A necessary and sufficient condition to the differential operator  $\mathcal{P}(2)$  defined by (1.1.17) to be decomposed into the form (1.1.18) is that:*

$$g_{2,1} = b_{1,1,1}b_{2,1,1}, \quad (1.1.21)$$

$$g_{2,2} + g_{2,3} = b_{1,1,2}b_{2,1,1} + b_{1,1,1}b_{2,1,2}, \quad (1.1.22)$$

$$g_{2,4} = b_{1,1,2}b_{2,1,2}, \quad (1.1.23)$$

$$g_{1,1} = b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + \mathcal{L}(b_{2,1,1}), \quad (1.1.24)$$

$$g_{1,2} = b_{1,0,1}b_{2,1,2} + b_{1,1,2}b_{2,0,1} + \mathcal{L}(b_{2,1,2}), \quad (1.1.25)$$

$$g_{0,1} = b_{1,0,1}b_{2,0,1} + \mathcal{L}(b_{2,0,2}), \quad (1.1.26)$$

where  $\mathcal{L} = b_{1,1,1}D_{1,1} + b_{1,1,2}D_{1,2}$ .

Propose an approach to solve system (1.1.21)-(1.1.26). Assume that at least one of the functions  $g_{2,1}$  and  $g_{2,4}$  does not vanish on  $\Lambda$ , says  $g_{2,1}$ . It is always possible to find two nonzero functions on  $\Lambda$ , namely  $b_{1,1,1}$  and  $b_{2,1,1}$  which satisfy (1.1.21). Substituting  $X_1 = b_{1,1,2}$  and  $X_2 = b_{2,1,2}$  into (1.1.22) yields

$$X_1 = \frac{1}{b_{2,1,1}}(g_{2,2} + g_{2,3} - b_{1,1,1}X_2). \quad (1.1.27)$$

The substitution of (1.1.27) into (1.1.23) shows that  $b_{2,1,2}$  is a solution of the second degree algebraic equation

$$\frac{b_{1,1,1}}{g_{2,1}}X_2^2 - \frac{g_{2,2} + g_{2,3}}{g_{2,1}}X_2 + \frac{g_{2,4}}{b_{1,1,1}} = 0. \quad (1.1.28)$$

The discriminant of equation (1.1.28) is

$$\Delta = (g_{2,2} + g_{2,3})^2 - 4g_{2,1}g_{2,4} = (b_{1,1,2}b_{2,1,1} - b_{1,1,1}b_{2,1,2})^2 \geq 0. \quad (1.1.29)$$

If  $\Delta > 0$ , then the substitution of  $Y = b_{1,0,1}$  and  $Z = b_{2,0,1}$  into (1.1.24) and (1.1.25) implies that the decomposition (1.1.18) is possible if the unique solution to the following algebraic system in  $Y$  and  $Z$

$$\begin{aligned} g_{1,1} - \mathcal{L}(b_{2,1,1}) &= b_{2,1,1}Y + b_{1,1,1}Z \\ g_{1,2} - \mathcal{L}(b_{2,1,2}) &= b_{2,1,2}Y + b_{1,1,2}Z \end{aligned} \quad (1.1.30)$$

satisfying (1.1.26). Indeed, the determinant of the system (1.1.30) is

$$b_{1,1,2}b_{2,1,1} - b_{1,1,1}b_{2,1,2} = \pm\sqrt{\Delta} \neq 0.$$

If  $\Delta = 0$ , then the substitution of  $Y = b_{1,0,1}$  and  $Z = b_{2,0,1}$  into (1.1.24) yields

$$Y = \frac{1}{b_{2,1,1}} [g_{1,1} - \mathcal{L}(b_{2,1,1}) - b_{1,1,1}Z]. \tag{1.1.31}$$

Then, the substitution of (1.1.31) into (1.1.26) implies that the decomposition (1.1.18) is strongly related to the existence of a solution to the following first order quasi-linear partial differential equation in  $Z$

$$\mathcal{L}(Z) - \frac{b_{1,1,1}}{b_{2,1,1}} Z^2 + \frac{g_{1,1} - \mathcal{L}(b_{2,1,1})}{b_{2,1,1}} Z - g_{0,1} = 0 \tag{1.1.32}$$

which satisfies (1.1.25).

## 1.2 Factorization of systems of linear differential equations

The previous analysis is now made for systems of linear differential equations.

### 1.2.1 General considerations

Let  $\Lambda$  be an open subset of  $\mathbb{R}^n$ . Examine now the factorization process for systems of  $s$ -th order, ( $s \geq 2$ ), linear differential equations with  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m \geq 2$  dependent variables  $u = {}^t(u^1, \dots, u^m)$ ,  $u = u(x)$  whose associated matrix operator,  $\mathcal{M}(s)$ , is of the form

$$\mathcal{M}(s) = [\mathcal{R}_{p,q}(s_{p,q})]_{1 \leq p,q \leq m} \tag{1.2.33}$$

with  $\mathcal{R}_{p,q}(s_{p,q})$  are  $s_{p,q}$ -th order linear differential operators

$$\mathcal{R}_{p,q}(s_{p,q}) = \sum_{k=0}^{s_{p,q}} \sum_{h=1}^{p_k} f_{p,q,k,h}(x) D_{k,h}, \tag{1.2.34}$$

where  $f_{p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ ,  $s_{p,q} = s - 1 + \delta_{p,q}$ ,  $\delta_{p,p} = 1$  and  $\delta_{p,q} = 0$  if  $p \neq q$ .

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}^n$  such that  $\Lambda_0 \subset \Lambda$ . The matrix operator  $\mathcal{M}(s)$  acts on a vector valued function  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^s(\Lambda_0, \mathbb{R}^m)$  as follows

$$\mathcal{M}(s) u = [\mathcal{R}_{p,q}(s_{p,q})]_{1 \leq p,q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q}(s_{p,q}) u^q \right]_{1 \leq p \leq m}.$$

The method of factorization consists in seeking a decomposition of the matrix  $\mathcal{M}(s)$  under the following form

$$\mathcal{M}(s) = \prod_{i=1}^l \mathcal{N}_i(s_i) \tag{1.2.35}$$

where

$$\mathcal{N}_i(s_i) = [\mathcal{T}_{i,p,q}(s_{i,p,q})]_{1 \leq p,q \leq m} \tag{1.2.36}$$

and

$$\mathcal{T}_{i,p,q}(s_{i,p,q}) = \sum_{k=0}^{s_{i,p,q}} \sum_{h=1}^{p_k} a_{i,p,q,k,h}(x) D_{k,h}, \quad (1.2.37)$$

with  $\sum_{i=1}^l s_i = s$ ,  $s_{i,p,q} = s_i - 1 + \delta_{p,q}$ ,  $a_{1,p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $a_{i,p,q,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_{i,p,q}}(\Lambda, \mathbb{R})$ ,  $i = 2, 3, \dots, l$ .

**Proposition 1.2.1** *Let  $\mathcal{M}(s)$  be a matrix of differential operators defined by (1.2.33) which can be decomposed into the form (1.2.35). If the function  $u_0 = {}^t(u_0^1, \dots, u_0^m)$  satisfies*

$$\mathcal{N}_l(s_l)u_0 = 0, \quad (1.2.38)$$

and  $u_j = {}^t(u_j^1, \dots, u_j^m)$ ,  $j = 1, 2, \dots, l - 1$  are solutions of the system

$$\prod_{k=l-j+1}^l \mathcal{N}_k(s_k)u_j = v_j, \quad j = 1, 2, \dots, l - 1, \quad (1.2.39)$$

where  $v_j = {}^t(v_j^1, \dots, v_j^m)$ ,  $j = 1, 2, \dots, l - 1$ , are solutions of

$$\prod_{i=1}^{l-j} \mathcal{N}_i(s_i)v_j = 0, \quad (1.2.40)$$

then  $u_0, u_1, \dots, u_{l-1}$  are  $l$  particular solutions of the equation  $\mathcal{M}(s)u = 0$ .

**Proof.** The proof is similar to that of the Proposition 5.2.1. ■

Expanding (1.2.35) leads to the relations between the unknown functions  $a_{i,p,q,k,h}$  of  $\mathcal{N}_i(s_i)$  and the known functions  $f_{p,q,k,h}$  of  $\mathcal{M}(s)$ .

As matter of clarity, in the sequel we explicitly derive necessary and sufficient conditions for the factorization of systems of second order linear ordinary and partial differential operators with two independent variables.

### 1.2.2 Case of systems of second order linear ODEs

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}$  such that  $\Lambda_0 \subset \Lambda$ . Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m}, \quad (1.2.41)$$

where

$$\mathcal{R}_{p,p} = \sum_{k=0}^2 \sum_{h=1}^{p_k} f_{p,p,k,h}(x) D_{k,h} = f_{p,p,0,1} D_{0,1} + f_{p,p,1,1} D_{1,1} + f_{p,p,2,1} D_{2,1} \quad (1.2.42)$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} = \sum_{k=0}^1 \sum_{h=1}^{p_k} f_{p,q,k,h}(x) D_{k,h} = f_{p,q,0,1} D_{0,1} + f_{p,q,1,1} D_{1,1} \quad (1.2.43)$$

with  $f_{p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ ,  $x = x^1$ . Write  $\mathcal{M}(2)$  in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \quad (1.2.44)$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1 \leq p,q \leq m} \quad (1.2.45)$$

with

$$\mathcal{T}_{i,p,p} = \sum_{k=0}^1 \sum_{h=1}^{p_k} a_{i,p,p,k,h}(x) D_{k,h} = a_{i,p,p,0,1} D_{0,1} + a_{i,p,p,1,1} D_{1,1} \quad (1.2.46)$$

and for  $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1} D_{0,1}, \quad (1.2.47)$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda, \mathbb{R})$ .

Let  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^2(\Lambda_0, \mathbb{R})$ . We have

$$\mathcal{M}(2) u = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q} u^q \right]_{1 \leq p \leq m} \quad (1.2.48)$$

where

$$\mathcal{R}_{p,p} u^p = f_{p,p,0,1} u^p + f_{p,p,1,1} u_x^p + f_{p,p,2,1} u_{2x}^p$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} u^q = f_{p,q,0,1} u^q + f_{p,q,1,1} u_x^q.$$

On the other hand, after expansion of (1.2.44), we have

$$\mathcal{M}(2) u = [\tilde{\mathcal{R}}_{p,q}]_{1 \leq p,q \leq m} u = \left[ \sum_{q=1}^m \tilde{\mathcal{R}}_{p,q} u^q \right]_{1 \leq p \leq m}, \quad (1.2.49)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_{p,p} u^p &= a_{1,p,p,1,1} a_{2,p,p,1,1} u_{2x}^p \\ &+ [a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,1,1})] u_x^p \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,0,1}) \right] u^p \end{aligned}$$

and for  $p \neq q$

$$\begin{aligned} \tilde{\mathcal{R}}_{p,q} u^q &= [a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}] u_x^q \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,q,0,1}) \right] u^q. \end{aligned}$$

Identifying (1.2.48) with (1.2.49) yields

**Proposition 1.2.2** *A necessary and sufficient condition to the differential operator  $\mathcal{M}(2)$  defined by (1.2.41) to be decomposed into the form (1.2.44) is that:*

$$f_{p,p,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,0,1}), \quad (1.2.50)$$

$$f_{p,p,1,1} = a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,1,1}), \quad (1.2.51)$$

$$f_{p,p,2,1} = a_{1,p,p,1,1} a_{2,p,p,1,1} \quad (1.2.52)$$

and for  $p \neq q$

$$f_{p,q,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,q,0,1}), \quad (1.2.53)$$

$$f_{p,q,1,1} = a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}. \quad (1.2.54)$$

### 1.2.3 Case of systems of second order linear PDEs

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}^2$  such that  $\Lambda_0 \subset \Lambda$ . Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m}, \quad (1.2.55)$$

where

$$\begin{aligned} \mathcal{R}_{p,p} &= \sum_{k=0}^2 \sum_{h=1}^{p_k} f_{p,p,k,h}(x) D_{k,h} \\ &= f_{p,p,0,1} D_{0,1} + f_{p,p,1,1} D_{1,1} + f_{p,p,1,2} D_{1,2} \\ &\quad + f_{p,p,2,1} D_{2,1} + f_{p,p,2,2} D_{2,2} + f_{p,p,2,3} D_{2,3} + f_{p,p,2,4} D_{2,4} \end{aligned} \quad (1.2.56)$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} = \sum_{k=0}^1 \sum_{h=1}^{p_k} f_{p,q,k,h}(x) D_{k,h} = f_{p,q,0,1} D_{0,1} + f_{p,q,1,1} D_{1,1} + f_{p,q,1,2} D_{1,2} \quad (1.2.57)$$

with  $f_{p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ ,  $x = (x^1, x^2)$ . Write  $\mathcal{M}(2)$  in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \quad (1.2.58)$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1 \leq p,q \leq m} \quad (1.2.59)$$

with

$$\mathcal{T}_{i,p,p} = \sum_{k=0}^1 \sum_{h=1}^{p_k} a_{i,p,p,k,h}(x) D_{k,h} = a_{i,p,p,0,1} D_{0,1} + a_{i,p,p,1,1} D_{1,1} + a_{i,p,p,1,2} D_{1,2} \quad (1.2.60)$$

and for  $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1} D_{0,1}, \quad (1.2.61)$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$  and  $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda, \mathbb{R})$ .

Let  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^2(\Lambda_0, \mathbb{R})$ . We have

$$\mathcal{M}(2) u = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q} u^q \right]_{1 \leq p \leq m}, \quad (1.2.62)$$

where

$$\begin{aligned} \mathcal{R}_{p,p} u^p &= f_{p,p,0,1} u^p + f_{p,p,1,1} u_{x^1}^p + f_{p,p,1,2} u_{x^2}^p \\ &\quad + f_{p,p,2,1} u_{2x^1}^p + (f_{p,p,2,2} + f_{p,p,2,3}) u_{x^1 x^2}^p + f_{p,p,2,4} u_{2x^2}^p \end{aligned}$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} u^q = f_{p,q,0,1} u^q + f_{p,q,1,1} u_{x^1}^q + f_{p,q,1,2} u_{x^2}^q.$$

On the other hand, after expansion of (1.2.58), we have

$$\mathcal{M}(2) u = \left[ \tilde{\mathcal{R}}_{p,q} \right]_{1 \leq p, q \leq m} u = \left[ \sum_{q=1}^m \tilde{\mathcal{R}}_{p,q} u^q \right]_{1 \leq p \leq m}, \quad (1.2.63)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_{p,p} u^p &= a_{1,p,p,1,1} a_{2,p,p,1,1} u_{2x^1}^p + a_{1,p,p,1,2} a_{2,p,p,1,2} u_{2x^2}^p \\ &+ (a_{1,p,p,1,2} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,1,2}) u_{x^1 x^2}^p \\ &+ [a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,1})] u_{x^1}^p \\ &+ [a_{1,p,p,0,1} a_{2,p,p,1,2} + a_{1,p,p,1,2} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2})] u_{x^2}^p \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}) \right] u^p \end{aligned}$$

and for  $p \neq q$

$$\begin{aligned} \tilde{\mathcal{R}}_{p,q} u^q &= [a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}] u_{x^1}^q \\ &+ [a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2}] u_{x^2}^q \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}) \right] u^q, \end{aligned}$$

where  $\mathcal{L}_p = a_{1,p,p,1,1} D_{1,1} + a_{1,p,p,1,2} D_{1,2}$ . Identifying (1.2.62) with (1.2.63) yields

**Proposition 1.2.3** *A necessary and sufficient condition to the differential operator  $\mathcal{M}(2)$  defined by (1.2.55) to be decomposed into the form (1.2.58) is that:*

$$f_{p,p,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}), \quad (1.2.64)$$

$$f_{p,p,1,1} = a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,1}), \quad (1.2.65)$$

$$f_{p,p,1,2} = a_{1,p,p,0,1} a_{2,p,p,1,2} + a_{1,p,p,1,2} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2}), \quad (1.2.66)$$

$$f_{p,p,2,1} = a_{1,p,p,1,1} a_{2,p,p,1,1}, \quad (1.2.67)$$

$$f_{p,p,2,2} + f_{p,p,2,3} = a_{1,p,p,1,2} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,1,2}, \quad (1.2.68)$$

$$f_{p,p,2,4} = a_{1,p,p,1,2} a_{2,p,p,1,2} \quad (1.2.69)$$

and for  $p \neq q$

$$f_{p,q,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}), \quad (1.2.70)$$

$$f_{p,q,1,1} = a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}, \quad (1.2.71)$$

$$f_{p,q,1,2} = a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2}. \quad (1.2.72)$$



# Factorization of Nonlinear Differential Equations and Systems

---

In this chapter, an algebraic approach for factorizing nonlinear partial differential equations (PDEs) and systems of PDEs is provided. In the particular case of second order nonlinear PDEs and systems of PDEs, necessary and sufficient conditions of factorization are given. The main results of this chapter have been published [51].

## 2.1 Factorizations of nonlinear differential equations

We start with general considerations and then deduce the main results on conditions of factorization.

### 2.1.1 General setting and results

Let  $s \geq 2$  be a positive integer,  $\Lambda$  be an open subset of  $\mathbb{R}^n$  and  $\Omega$  and open subset of  $\mathbb{R}$ . Let

$$\mathcal{P}(s) = \sum_{k=0}^s \sum_{h=1}^{p_k} g_{k,h}(x, \cdot) D_{k,h} \quad (2.1.1)$$

be a nonlinear differential operator of order  $s$ , where  $g_{k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ . The operator  $\mathcal{P}(s)$  acts on a function  $u \in \mathcal{C}^s(\Lambda, \Omega)$  as follows

$$\mathcal{P}(s) u = \sum_{k=0}^s \sum_{h=1}^{p_k} g_{k,h}(x, u) D_{k,h} u. \quad (2.1.2)$$

The method of factorization consists in seeking a decomposition of the differential operator (2.1.1) in the following form

$$\mathcal{P}(s) = \prod_{i=1}^l \mathcal{Q}_i(s_i) \quad (2.1.3)$$

with  $\sum_{i=1}^l s_i = s$  and

$$\mathcal{Q}_i(s_i) = \sum_{k=0}^{s_i} \sum_{h=1}^{p_k} b_{i,k,h}(x, \cdot) D_{k,h}, \quad (2.1.4)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $b_{i,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_j}(\Lambda \times \Omega, \mathbb{R})$ ,  $i = 2, 3, \dots, l$ .

Expanding (2.1.3) leads to the relations between unknown functions  $b_{i,k,h}$  of the differential operators  $\mathcal{Q}_i(s_i)$  and the known functions  $g_{k,h}$  of the original differential operator  $\mathcal{P}(s)$ .

---

### 2.1.2 Case of second order nonlinear ODEs

Let  $\Omega, \Lambda$  and  $\Lambda_0$  be three open subsets of  $\mathbb{R}$  such that  $\Lambda_0 \subset \Lambda$ . Consider the second order nonlinear ordinary differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^2 \sum_{h=1}^{p_k} g_{k,h}(x, \cdot) D_{k,h} \\ &= g_{0,1}(x, \cdot) D_{0,1} + g_{1,1}(x, \cdot) D_{1,1} + g_{2,1}(x, \cdot) D_{2,1}, \end{aligned} \quad (2.1.5)$$

where  $g_{k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $x = x^1$ . Write  $\mathcal{P}(2)$  in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1) \cdot \mathcal{Q}_2(1) \\ &= \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{1,k,h}(x, \cdot) D_{k,h} \right] \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{2,k,h}(x, \cdot) D_{k,h} \right] \\ &= [b_{1,0,1}(x, \cdot) D_{0,1} + b_{1,1,1}(x, \cdot) D_{1,1}] [b_{2,0,1}(x, \cdot) D_{0,1} + b_{2,1,1}(x, \cdot) D_{1,1}], \end{aligned} \quad (2.1.6)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $b_{2,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$ .

Let  $u \in \mathcal{C}^2(\Lambda_0, \Omega)$ . We have

$$\mathcal{P}(2) u = g_{0,1}(x, u) u + g_{1,1}(x, u) u_x + g_{2,1}(x, u) u_{2x} \quad (2.1.7)$$

and after expansion

$$\begin{aligned} \mathcal{P}(2) u &= [b_{1,0,1}(x, \cdot) D_{0,1} + b_{1,1,1}(x, \cdot) D_{1,1}] [b_{2,0,1}(x, \cdot) D_{0,1} + b_{2,1,1}(x, \cdot) D_{1,1}] u \\ &= [b_{1,0,1} b_{2,1,1} + b_{1,1,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,1,1}) + b_{1,1,1} D_{1,2}(b_{2,0,1}) u] u_x \\ &+ b_{1,1,1} D_{1,2}(b_{2,1,1}) u_x^2 + [b_{1,0,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,0,1})] u + b_{1,1,1} b_{2,1,1} u_{2x}. \end{aligned} \quad (2.1.8)$$

Identifying (2.1.7) with (2.1.8) yields

**Proposition 2.1.1** *A necessary and sufficient condition to the differential operator  $\mathcal{P}(2)$  defined by (2.1.5) to be decomposed into the form (2.1.6) is that:*

$$g_{2,1} = b_{1,1,1} b_{2,1,1}, \quad (2.1.9)$$

$$g_{1,1} = b_{1,0,1} b_{2,1,1} + b_{1,1,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,1,1}) + b_{1,1,1} D_{1,2}(b_{2,0,1}) u, \quad (2.1.10)$$

$$0 = b_{1,1,1} D_{1,2}(b_{2,1,1}), \quad (2.1.11)$$

$$g_{0,1} = b_{1,0,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,0,1}). \quad (2.1.12)$$

### 2.1.3 Case of second order nonlinear PDEs

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}^2$  such that  $\Lambda_0 \subset \Lambda$ . Let  $\Omega$  and open subset of  $\mathbb{R}$ . Consider the second order nonlinear partial differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^2 \sum_{h=1}^{p_k} g_{k,h}(x, \cdot) D_{k,h} \\ &= g_{0,1}(x, \cdot) D_{0,1} + g_{1,1}(x, \cdot) D_{1,1} + g_{1,2}(x, \cdot) D_{1,2} + g_{2,1}(x, \cdot) D_{2,1} \\ &+ g_{2,2}(x, \cdot) D_{2,2} + g_{2,3}(x, \cdot) D_{2,3} + g_{2,4}(x, \cdot) D_{2,4}, \end{aligned} \quad (2.1.13)$$

where  $g_{k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $x = (x^1, x^2)$ . Write  $\mathcal{P}(2)$  in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1) \cdot \mathcal{Q}_2(1) \\ &= \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{1,k,h}(x, \cdot) D_{k,h} \right] \left[ \sum_{k=0}^1 \sum_{h=1}^{p_k} b_{2,k,h}(x, \cdot) D_{k,h} \right] \\ &= [b_{1,0,1}(x, \cdot) D_{0,1} + b_{1,1,1}(x, \cdot) D_{1,1} + b_{1,1,2}(x, \cdot) D_{1,2}] \\ &\quad \times [b_{2,0,1}(x, \cdot) D_{0,1} + b_{2,1,1}(x, \cdot) D_{1,1} + b_{2,1,2}(x, \cdot) D_{1,2}], \end{aligned} \quad (2.1.14)$$

where  $b_{1,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $b_{2,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$ .

Let  $u \in \mathcal{C}^2(\Lambda_0, \Omega)$ . We have

$$\begin{aligned} \mathcal{P}(2) u &= g_{0,1}(x, u) u + g_{1,1}(x, u) u_{x^1} + g_{1,2}(x, u) u_{x^2} + g_{2,1}(x, u) u_{2x^1} \\ &\quad + (g_{2,2}(x, u) + g_{2,3}(x, u)) u_{x^1 x^2} + g_{2,4}(x, u) u_{2x^2} \end{aligned} \quad (2.1.15)$$

and after expansion

$$\begin{aligned} \mathcal{P}(2) u &= [b_{1,0,1}(x, \cdot) D_{0,1} + b_{1,1,1}(x, \cdot) D_{1,1} + b_{1,1,2}(x, \cdot) D_{1,2}] \\ &\quad \times [b_{2,0,1}(x, \cdot) D_{0,1} + b_{2,1,1}(x, \cdot) D_{1,1} + b_{2,1,2}(x, \cdot) D_{1,2}] u \\ &= [b_{1,0,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,0,1}) + b_{1,1,2} D_{1,2}(b_{2,0,1})] u \\ &\quad + [b_{1,0,1} b_{2,1,1} + b_{1,1,1} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,1,1}) + b_{1,1,2} D_{1,2}(b_{2,1,1}) \\ &\quad + b_{1,1,1} D_{1,3}(b_{2,0,1})] u u_{x^1} + [b_{1,0,1} b_{2,1,2} + b_{1,1,2} b_{2,0,1} + b_{1,1,1} D_{1,1}(b_{2,1,2}) \\ &\quad + b_{1,1,2} D_{1,2}(b_{2,1,2}) + b_{1,1,2} D_{1,3}(b_{2,0,1})] u u_{x^2} + b_{1,1,1} D_{1,3}(b_{2,1,1}) u_{x^1}^2 \\ &\quad + [b_{1,1,1} D_{1,3}(b_{2,1,2}) + b_{1,1,2} D_{1,3}(b_{2,1,1})] u_{x^1} u_{x^2} + b_{1,1,2} D_{1,3}(b_{2,1,2}) u_{x^2}^2 \\ &\quad + b_{1,1,1} b_{2,1,1} u_{2x^1} + [b_{1,1,2} b_{2,1,1} + b_{1,1,1} b_{2,1,2}] u_{x^1 x^2} + b_{1,1,2} b_{2,1,2} u_{2x^2}. \end{aligned} \quad (2.1.16)$$

Identifying (2.1.15) with (2.1.16) yields

**Proposition 2.1.2** *A necessary and sufficient condition to the differential operator  $\mathcal{P}(2)$  defined by (2.1.13) to be decomposed into the form (2.1.14) is that:*

$$g_{2,1} = b_{1,1,1} b_{2,1,1}, \quad (2.1.17)$$

$$g_{2,2} + g_{2,3} = b_{1,1,2} b_{2,1,1} + b_{1,1,1} b_{2,1,2}, \quad (2.1.18)$$

$$g_{2,4} = b_{1,1,2} b_{2,1,2}, \quad (2.1.19)$$

$$g_{1,1} = b_{1,0,1} b_{2,1,1} + b_{1,1,1} b_{2,0,1} + \mathcal{L}(b_{2,1,1}) + b_{1,1,1} D_{1,3}(b_{2,0,1}) u, \quad (2.1.20)$$

$$g_{1,2} = b_{1,0,1} b_{2,1,2} + b_{1,1,2} b_{2,0,1} + \mathcal{L}(b_{2,1,2}) + b_{1,1,2} D_{1,3}(b_{2,0,1}) u, \quad (2.1.21)$$

$$g_{0,1} = b_{1,0,1} b_{2,0,1} + \mathcal{L}(b_{2,0,2}), \quad (2.1.22)$$

$$0 = b_{1,1,2} D_{1,3}(b_{2,1,2}), \quad (2.1.23)$$

$$0 = b_{1,1,1} D_{1,3}(b_{2,1,1}), \quad (2.1.24)$$

$$0 = b_{1,1,1} D_{1,3}(b_{2,1,2}) + b_{1,1,2} D_{1,3}(b_{2,1,1}), \quad (2.1.25)$$

where  $\mathcal{L} = b_{1,1,1} D_{1,1} + b_{1,1,2} D_{1,2}$ .

## 2.2 Factorizations of systems of nonlinear differential equations

### 2.2.1 Theoretical considerations and principles

Let  $\Lambda$  be an open subset of  $\mathbb{R}^n$  and  $\Omega$ , an open subset of  $\mathbb{R}^m$ . Examine now the factorization process for systems of  $s$ -th order, ( $s \geq 2$ ), nonlinear differential equations with  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m \geq 2$  dependent variables  $u = {}^t(u^1, \dots, u^m)$ ,  $u = u(x)$  whose associated matrix operator,  $\mathcal{M}(s)$ , is of the form

$$\mathcal{M}(s) = [\mathcal{R}_{p,q}(s_{p,q})]_{1 \leq p,q \leq m} \quad (2.2.26)$$

with  $\mathcal{R}_{p,q}(s_{p,q})$  are  $s_{p,q}$ -th order nonlinear differential operators

$$\mathcal{R}_{p,q}(s_{p,q}) = \sum_{k=0}^{s_{p,q}} \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h}, \quad (2.2.27)$$

where  $f_{p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ ,  $s_{p,q} = s - 1 + \delta_{p,q}$ ,  $\delta_{p,p} = 1$  and  $\delta_{p,q} = 0$  if  $p \neq q$ .

Let  $\Lambda$  and  $\Lambda_0$  be two open subsets of  $\mathbb{R}^n$  such that  $\Lambda_0 \subset \Lambda$ . The matrix operator  $\mathcal{M}(s)$  acts on a vector valued function  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^s(\Lambda_0, \Omega)$  as follows

$$\mathcal{M}(s) u = [\mathcal{R}_{p,q}(s_{p,q})]_{1 \leq p,q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q}(s_{p,q}) u^q \right]_{1 \leq p \leq m},$$

with

$$\mathcal{R}_{p,q}(s_{p,q}) u^q = \sum_{k=0}^{s_{p,q}} \sum_{h=1}^{p_k} f_{p,q,k,h}(x, u^1, \dots, u^m) D_{k,h} u^q. \quad (2.2.28)$$

The method of factorization consists in seeking a decomposition of the matrix  $\mathcal{M}(s)$  under the following form

$$\mathcal{M}(s) = \prod_{i=1}^l \mathcal{N}_i(s_i) \quad (2.2.29)$$

where

$$\mathcal{N}_i(s_i) = [\mathcal{T}_{i,p,q}(s_{i,p,q})]_{1 \leq p,q \leq m} \quad (2.2.30)$$

and

$$\mathcal{T}_{i,p,q}(s_{i,p,q}) = \sum_{k=0}^{s_{i,p,q}} \sum_{h=1}^{p_k} a_{i,p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h}, \quad (2.2.31)$$

with  $\sum_{i=1}^l s_i = s$ ,  $s_{i,p,q} = s_i - 1 + \delta_{p,q}$ ,  $a_{1,p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $a_{i,p,q,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_{i,p,q}}(\Lambda \times \Omega, \mathbb{R})$ ,  $i = 2, 3, \dots, l$ .

Expanding (2.2.29) leads to the relations between the unknown functions  $a_{i,p,q,k,h}$  of  $\mathcal{N}_i(s_i)$  and the known functions  $f_{p,q,k,h}$  of  $\mathcal{M}(s)$ .

### 2.2.2 Case of systems of second order nonlinear ODEs

Let  $\Lambda$ ,  $\Lambda_0$  be two open subsets of  $\mathbb{R}$  such that  $\Lambda_0 \subset \Lambda$ , and  $\Omega$  an open subset of  $\mathbb{R}^m$ . Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m}, \quad (2.2.32)$$

where

$$\begin{aligned} \mathcal{R}_{p,p} &= \sum_{k=0}^2 \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\ &= f_{p,p,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + f_{p,p,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} + f_{p,p,2,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{2,1} \end{aligned} \quad (2.2.33)$$

and for  $p \neq q$

$$\begin{aligned} \mathcal{R}_{p,q} &= \sum_{k=0}^1 \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\ &= f_{p,q,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + f_{p,q,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} \end{aligned} \quad (2.2.34)$$

with  $f_{p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ ,  $x = x^1$ . Write  $\mathcal{M}(2)$  in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \quad (2.2.35)$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1 \leq p, q \leq m} \quad (2.2.36)$$

with

$$\begin{aligned} \mathcal{T}_{i,p,p} &= \sum_{k=0}^1 \sum_{h=1}^{p_k} a_{i,p,p,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\ &= a_{i,p,p,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + a_{i,p,p,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} \end{aligned} \quad (2.2.37)$$

and for  $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1}, \quad (2.2.38)$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$ .

Let  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^2(\Lambda_0, \Omega)$ . We have

$$\mathcal{M}(2) u = [\mathcal{R}_{p,q}]_{1 \leq p, q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q} u^q \right]_{1 \leq p \leq m} \quad (2.2.39)$$

where

$$\mathcal{R}_{p,p} u^p = f_{p,p,0,1}(x, u) u^p + f_{p,p,1,1}(x, u) u_x^p + f_{p,p,2,1}(x, u) u_{2x}^p$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} u^q = f_{p,q,0,1}(x, u) u^q + f_{p,q,1,1}(x, u) u_x^q.$$

On the other hand, after expansion of (2.2.35), we have

$$\mathcal{M}(2) u = \left[ \tilde{\mathcal{R}}_{p,q} \right]_{1 \leq p, q \leq m} u = \left[ \sum_{q=1}^m \tilde{\mathcal{R}}_{p,q} u^q \right]_{1 \leq p \leq m}, \quad (2.2.40)$$

where

$$\begin{aligned}
 \tilde{\mathcal{R}}_{p,p} u^p &= a_{1,p,p,1,1} a_{2,p,p,1,1} u_{2x}^p + a_{1,p,p,1,1} \sum_{\tilde{h}=1}^m D_{1,\tilde{h}+1}(a_{2,p,p,1,1}) u_x^{\tilde{h}} u_x^p \\
 &+ a_{1,p,p,1,1} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq p}}^m D_{1,\tilde{h}+1}(a_{2,p,p,0,1}) u_x^{\tilde{h}} u^p + [a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} \\
 &+ a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,1,1}) + a_{1,p,p,1,1} D_{1,p+1}(a_{2,p,p,0,1}) u^p] u_x^p \\
 &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,0,1}) \right] u^p
 \end{aligned}$$

and for  $p \neq q$

$$\begin{aligned}
 \tilde{\mathcal{R}}_{p,q} u^q &= [a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1} + a_{1,p,p,1,1} D_{1,q+1}(a_{2,p,q,0,1}) u^q] u_x^q \\
 &+ a_{1,p,p,1,1} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq q}}^m D_{1,\tilde{h}+1}(a_{2,p,q,0,1}) u_x^{\tilde{h}} u^q \\
 &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,q,0,1}) \right] u^q.
 \end{aligned}$$

Identifying (2.2.39) with (2.2.40) yields

**Proposition 2.2.1** *A necessary and sufficient condition to the differential operator  $\mathcal{M}(2)$  defined by (2.2.32) to be decomposed into the form (2.2.35) is that:*

$$f_{p,p,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,0,1}), \quad (2.2.41)$$

$$\begin{aligned}
 f_{p,p,1,1} &= a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} \\
 &+ a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,1,1}) + a_{1,p,p,1,1} D_{1,p+1}(a_{2,p,p,0,1}) u^p, \quad (2.2.42)
 \end{aligned}$$

$$f_{p,p,2,1} = a_{1,p,p,1,1} a_{2,p,p,1,1}, \quad (2.2.43)$$

$$0 = D_{1,\tilde{h}+1}(a_{2,p,p,0,1}), \quad \tilde{h} \in \{1, 2, \dots, m\} \setminus \{p\}, \quad (2.2.44)$$

$$0 = D_{1,\tilde{h}+1}(a_{2,p,p,1,1}), \quad \tilde{h} = 1, 2, \dots, m \quad (2.2.45)$$

and for  $p \neq q$

$$f_{p,q,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,q,0,1}), \quad (2.2.46)$$

$$f_{p,q,1,1} = a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1} + a_{1,p,p,1,1} D_{1,q+1}(a_{2,p,q,0,1}) u^q, \quad (2.2.47)$$

$$0 = D_{1,\tilde{h}+1}(a_{2,p,q,0,1}), \quad \tilde{h} \in \{1, 2, \dots, m\} \setminus \{q\}. \quad (2.2.48)$$

### 2.2.3 Case of systems of second order nonlinear PDEs

Let  $\Lambda, \Lambda_0$  be two open subsets of  $\mathbb{R}^2$  such that  $\Lambda_0 \subset \Lambda$ , and  $\Omega$  an open subset of  $\mathbb{R}^m$ . Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1 \leq p, q \leq m}, \quad (2.2.49)$$

where

$$\begin{aligned}
 \mathcal{R}_{p,p} &= \sum_{k=0}^2 \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\
 &= f_{p,p,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + f_{p,p,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} + f_{p,p,1,2}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,2} \\
 &+ f_{p,p,2,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{2,1} + f_{p,p,2,2}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{2,2} \\
 &+ f_{p,p,2,3}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{2,3} + f_{p,p,2,4}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{2,4}
 \end{aligned} \tag{2.2.50}$$

and for  $p \neq q$

$$\begin{aligned}
 \mathcal{R}_{p,q} &= \sum_{k=0}^1 \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\
 &= f_{p,q,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + f_{p,q,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} + f_{p,q,1,2}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,2}
 \end{aligned} \tag{2.2.51}$$

with  $f_{p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ ,  $x = (x^1, x^2)$ . Write  $\mathcal{M}(2)$  in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \tag{2.2.52}$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1 \leq p, q \leq m} \tag{2.2.53}$$

with

$$\begin{aligned}
 \mathcal{T}_{i,p,p} &= \sum_{k=0}^1 \sum_{h=1}^{p_k} a_{i,p,p,k,h}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{k,h} \\
 &= a_{i,p,p,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1} + a_{i,p,p,1,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,1} + a_{i,p,p,1,2}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{1,2}
 \end{aligned} \tag{2.2.54}$$

and for  $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}(x, \underbrace{\cdot, \dots, \cdot}_{m\text{-entries}}) D_{0,1}, \tag{2.2.55}$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$  and  $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$ .

Let  $u = {}^t(u^1, \dots, u^m) \in \mathcal{C}^2(\Lambda_0, \Omega)$ . We have

$$\mathcal{M}(2) u = [\mathcal{R}_{p,q}]_{1 \leq p, q \leq m} u = \left[ \sum_{q=1}^m \mathcal{R}_{p,q} u^q \right]_{1 \leq p \leq m}, \tag{2.2.56}$$

where

$$\begin{aligned}
 \mathcal{R}_{p,p} u^p &= f_{p,p,0,1} u^p + f_{p,p,1,1} u_{x^1}^p + f_{p,p,1,2} u_{x^2}^p \\
 &+ f_{p,p,2,1} u_{2x^1}^p + (f_{p,p,2,2} + f_{p,p,2,3}) u_{x^1 x^2}^p + f_{p,p,2,4} u_{2x^2}^p
 \end{aligned} \tag{2.2.57}$$

and for  $p \neq q$

$$\mathcal{R}_{p,q} u^q = f_{p,q,0,1} u^q + f_{p,q,1,1} u_{x^1}^q + f_{p,q,1,2} u_{x^2}^q.$$

On the other hand, after expansion of (2.2.52), we have

$$\mathcal{M}(2) u = \left[ \tilde{\mathcal{R}}_{p,q} \right]_{1 \leq p, q \leq m} u = \left[ \sum_{q=1}^m \tilde{\mathcal{R}}_{p,q} u^q \right]_{1 \leq p \leq m}, \quad (2.2.58)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_{p,p} u^p &= a_{1,p,p,1,1} a_{2,p,p,1,1} u_{2x^1}^p + (a_{1,p,p,1,2} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,1,2}) u_{x^1 x^2}^p \\ &+ a_{1,p,p,1,2} a_{2,p,p,1,2} u_{2x^2}^p [a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} \\ &+ \mathcal{L}_p(a_{2,p,p,1,1}) + a_{1,p,p,1,1} D_{1,p+2}(a_{2,p,p,0,1}) u^p] u_{x^1}^p + [a_{1,p,p,0,1} a_{2,p,p,1,2} \\ &+ a_{1,p,p,1,2} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2}) + a_{1,p,p,1,2} D_{1,p+2}(a_{2,p,p,0,1}) u^p] u_{x^2}^p \\ &+ a_{1,p,p,1,1} \sum_{\tilde{h}=1}^m D_{1,\tilde{h}+2}(a_{2,p,p,1,1}) u_{x^1}^{\tilde{h}} u_{x^1}^p + a_{1,p,p,1,1} \sum_{\tilde{h}=1}^m D_{1,\tilde{h}+2}(a_{2,p,p,1,2}) u_{x^1}^{\tilde{h}} u_{x^2}^p \\ &+ a_{1,p,p,1,2} \sum_{\tilde{h}=1}^m D_{1,\tilde{h}+2}(a_{2,p,p,1,1}) u_{x^2}^{\tilde{h}} u_{x^1}^p + a_{1,p,p,1,2} \sum_{\tilde{h}=1}^m D_{1,\tilde{h}+2}(a_{2,p,p,1,2}) u_{x^2}^{\tilde{h}} u_{x^2}^p \\ &+ a_{1,p,p,1,1} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq p}}^m D_{1,\tilde{h}+2}(a_{2,p,p,0,1}) u_{x^1}^{\tilde{h}} u^p + a_{1,p,p,1,2} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq p}}^m D_{1,\tilde{h}+2}(a_{2,p,p,0,1}) u_{x^2}^{\tilde{h}} u^p \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}) \right] u^p \end{aligned}$$

and for  $p \neq q$

$$\begin{aligned} \tilde{\mathcal{R}}_{p,q} u^q &= [a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1} + a_{1,p,p,1,1} D_{1,q+2}(a_{2,p,q,0,1}) u^q] u_{x^1}^q \\ &+ [a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2} + a_{1,p,p,1,2} D_{1,q+2}(a_{2,p,q,0,1}) u^q] u_{x^2}^q \\ &+ a_{1,p,p,1,1} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq q}}^m D_{1,\tilde{h}+2}(a_{2,p,q,0,1}) u_{x^1}^{\tilde{h}} u^q + a_{1,p,p,1,2} \sum_{\substack{\tilde{h}=1 \\ \tilde{h} \neq q}}^m D_{1,\tilde{h}+2}(a_{2,p,q,0,1}) u_{x^2}^{\tilde{h}} u^q \\ &+ \left[ \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}) \right] u^q, \end{aligned}$$

where  $\mathcal{L}_p = a_{1,p,p,1,1} D_{1,1} + a_{1,p,p,1,2} D_{1,2}$ . Identifying (2.2.56) with (2.2.58) yields

**Proposition 2.2.2** *A necessary and sufficient condition to the differential operator  $\mathcal{M}(2)$  de-*

finned by (2.2.49) to be decomposed into the form (2.2.52) is that:

$$f_{p,p,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}), \quad (2.2.59)$$

$$f_{p,p,1,1} = a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,1}) + a_{1,p,p,1,1} D_{1,p+2}(a_{2,p,p,0,1}) u^p, \quad (2.2.60)$$

$$f_{p,p,1,2} = a_{1,p,p,0,1} a_{2,p,p,1,2} + a_{1,p,p,1,2} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2}) + a_{1,p,p,1,2} D_{1,p+2}(a_{2,p,p,0,1}) u^p, \quad (2.2.61)$$

$$f_{p,p,2,1} = a_{1,p,p,1,1} a_{2,p,p,1,1}, \quad (2.2.62)$$

$$f_{p,p,2,2} + f_{p,p,2,3} = a_{1,p,p,1,2} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,1,2}, \quad (2.2.63)$$

$$f_{p,p,2,4} = a_{1,p,p,1,2} a_{2,p,p,1,2}, \quad (2.2.64)$$

$$0 = D_{1,\tilde{h}+2}(a_{2,p,p,0,1}), \quad \tilde{h} \in \{1, 2, \dots, m\} \setminus \{p\}, \quad (2.2.65)$$

$$0 = D_{1,\tilde{h}+2}(a_{2,p,p,1,1}), \quad \tilde{h} = 1, 2, \dots, m, \quad (2.2.66)$$

$$0 = D_{1,\tilde{h}+2}(a_{2,p,p,1,2}), \quad \tilde{h} = 1, 2, \dots, m \quad (2.2.67)$$

and for  $p \neq q$

$$f_{p,q,0,1} = \sum_{l=1}^m a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}), \quad (2.2.68)$$

$$f_{p,q,1,1} = a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1} + a_{1,p,p,1,1} D_{1,q+2}(a_{2,p,q,0,1}) u^q, \quad (2.2.69)$$

$$f_{p,q,1,2} = a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2} + a_{1,p,p,1,2} D_{1,q+2}(a_{2,p,q,0,1}) u^q, \quad (2.2.70)$$

$$0 = D_{1,\tilde{h}+2}(a_{2,p,q,0,1}), \quad \tilde{h} \in \{1, 2, \dots, m\} \setminus \{q\}. \quad (2.2.71)$$



# Three Factorizable Classes of 2<sup>nd</sup> Order Linear ODEs

---

This chapter addresses necessary and sufficient factorizability conditions for classes of second order ODEs characterized by the degrees of their corresponding polynomial functions coefficients. A pure algebraic method is used to solve a system of linear algebraic equations whose solutions satisfy a compatibility criterion and generate two first order differential operators factorizing the considered second order differential operator. Concrete examples are probed, including special cases of Böcher ODEs like Heun, Wangerin and Heine's differential equations. The main results of this chapter have been published [49].

Our study is restricted here to the second order linear differential operator

$$\mathcal{P}(2, \mathcal{D}) = g_2(x)\mathcal{D}^2 + g_1(x)\mathcal{D} + g_0(x). \quad (3.0.1)$$

Provided the factorized form

$$\mathcal{P}(2, \mathcal{D}) = \mathcal{Q}_1(1, \mathcal{D})\mathcal{Q}_2(1, \mathcal{D}) = (\mathcal{L}_{11}\mathcal{D} + \mathcal{L}_{10})(\mathcal{L}_{21}\mathcal{D} + \mathcal{L}_{20}), \quad (3.0.2)$$

the functions  $\mathcal{L}_{ij}$  satisfy the following algebraic and differential equations (3.0.3)-(3.0.5):

$$\mathcal{L}_{11}\mathcal{L}_{21} = g_2, \quad (3.0.3)$$

$$\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20} = g_1, \quad (3.0.4)$$

$$\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x = g_0. \quad (3.0.5)$$

Finally, two particular solutions  $u_0$  and  $u_1$  of the equation associated with the operator  $\mathcal{P}(2, \mathcal{D})$  defined by (3.0.1) can be obtained by solving the following differential equations:

$$\mathcal{Q}_2(1, \mathcal{D})u_0(x) := \mathcal{L}_{21}(x)u_0'(x) + \mathcal{L}_{20}(x)u_0(x) = 0, \quad (3.0.6)$$

$$\mathcal{Q}_1(1, \mathcal{D})v_1(x) := \mathcal{L}_{11}(x)v_1'(x) + \mathcal{L}_{10}(x)v_1(x) = 0, \quad (3.0.7)$$

$$\mathcal{Q}_2(1, \mathcal{D})u_1(x) := \mathcal{L}_{21}(x)u_1'(x) + \mathcal{L}_{20}(x)u_1(x) = v_1(x). \quad (3.0.8)$$

Every first order right factor of (3.0.2) leads to a hyperexponential solution [67],  $u_0$ , of the differential equation associated with (3.0.1) which can be written in terms of exponential functions. Another solution,  $u_1$ , of the same equation is obtained with the functions  $u_0$  and  $v_1$ , solutions of (3.0.6) and (3.0.7), respectively, as follows:

$$u_1(x) = u_0(x) \int \frac{v_1(x)}{u_0(x) \mathcal{L}_{21}(x)} dx.$$


---

Now, we probe various classes of factorizable second order linear ODEs with rational coefficients. Dealing with the second order linear differential operator (3.0.1), where, for analysis convenience, we define

$$g_2(x) := P_p(x) = \sum_{i=1}^p \sigma_i x^i, \quad g_1(x) := Q_q(x) = \sum_{j=1}^q \gamma_j x^j, \quad (3.0.9)$$

$$g_0(x) := R_r(x) = \sum_{l=1}^r \rho_l x^l, \quad p, q, r \in \mathbb{N}, \quad \sigma_i, \gamma_j, \rho_l \in \mathbb{R}, \quad (3.0.10)$$

one can deduce from (3.0.3)-(3.0.5) the following three types of second order linear ODEs:

$$P_{k+1}(x) u''(x) + Q_{k+1}(x) u'(x) + R_k(x) u(x) = 0, \quad k \in \mathbb{N}; \quad (3.0.11)$$

$$P_k(x) u''(x) + Q_{k+h+1}(x) u'(x) + R_{k+h}(x) u(x) = 0, \quad (k, h) \in \mathbb{N}^* \times \mathbb{N}; \quad (3.0.12)$$

$$P_{k+2}(x) u''(x) + Q_{k+1}(x) u'(x) + R_k(x) u(x) = 0, \quad k \in \mathbb{N}. \quad (3.0.13)$$

Depending on the relations between the degrees  $p, q, r$  of the polynomial functions  $g_i$ , ( $i = 0, 1, 2$ ), these ODEs can be factorized into the form (3.0.2). They are worth something as they contain a large class of relevant second order linear ODEs of mathematical physics such as the equations of Heun, Heine and Wangerin, which will be treated in the sequel.

Recall that, by the fundamental theorem of algebra, the polynomial  $P_p$  can be put in the form:  $P_p(x) = a_p \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}$ , where  $a_p, \lambda_i$  are complex numbers such that  $\lambda_i \neq \lambda_j$  for  $i \neq j$  and  $a_p \neq 0$ ;  $p_0, m_i$  are positive integers such that  $p_0 \leq p$  and  $\sum_{i=1}^{p_0} m_i = p$ . In what follows, without loss of generality, we set  $a_p = 1$ . Besides, using the Euclidean division and the partial fraction expansion theorem in the set of rational functions with complex coefficients  $\mathbb{C}[X]$ ,

$$\frac{\sum_{j=0}^q \gamma_j x^j}{\prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}} = E(x) + \sum_{i=1}^{p_0} \sum_{j=1}^{m_i} \frac{\mu_{i,j}}{(x - \lambda_i)^j}, \quad (3.0.14)$$

where  $\mu_{i,j}$  are complex numbers;  $E(x)$  is a nonzero polynomial of degree  $q - p$  if  $q \geq p$  and  $E(x) = 0$  if  $q < p$ . There results that equation (3.0.1) together with (3.0.9) and (3.0.10) can be transformed into the following canonical form:

$$u''(x) + \left( E(x) + \sum_{i=1}^{p_0} \sum_{j=1}^{m_i} \frac{\mu_{i,j}}{(x - \lambda_i)^j} \right) u'(x) + \frac{\sum_{l=0}^r \rho_l x^l}{\prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}} u(x) = 0. \quad (3.0.15)$$

**Remark 3.0.1** *Böcher equations*

$$u''(x) + \left( \sum_{j=1}^n \frac{\epsilon_j}{x - \lambda_j} \right) u'(x) + \frac{\sum_{l=0}^r \rho_l x^l}{\prod_{i=1}^n (x - \lambda_i)^{m_i}} u(x) = 0, \quad (3.0.16)$$

where  $n, r, m_i \in \mathbb{N}$ ,  $\epsilon_j, a_i, \rho_l \in \mathbb{C}$ ,  $\lambda_i \neq \lambda_j$  for  $i \neq j$ , are particular cases of (3.0.15) with  $E(x) = 0$ .

### 3.1 First lass of factorizable 2<sup>nd</sup> order linear ODEs

In this section, we investigate the classes of factorizable second order linear ODEs of the type

$$P_{k+1}(x) u''(x) + Q_{k+1}(x) u'(x) + R_k(x) u(x) = 0, \quad k \in \mathbb{N} \quad (3.1.17)$$

explicitly written as

$$\left( \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i} \right) u''(x) + \left( \sum_{j=0}^{k+1} \gamma_j x^j \right) u'(x) + \left( \sum_{l=0}^k \rho_l x^l \right) u(x) = 0, \quad (3.1.18)$$

where  $\sum_{i=1}^{p_0} m_i = k + 1$ , or, equivalently, in the canonical form: ( $E_0 \neq 0$ )

$$u''(x) + \left( E_0 + \sum_{i=1}^{p_0} \sum_{j=1}^{m_i} \frac{\mu_{i,j}}{(x - \lambda_i)^j} \right) u'(x) + \frac{\sum_{l=0}^k \rho_l x^l}{\prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}} u(x) = 0. \quad (3.1.19)$$

**Proposition 3.1.1 (Necessary condition for the factorization of (3.1.17))** *Let equation (3.1.17) be factorizable into the form (3.0.2). Then, the degrees of the polynomials  $\mathcal{L}_{ij}$  satisfy the following relations:*

$$\deg \mathcal{L}_{11} + \deg \mathcal{L}_{21} = k + 1 \quad \text{and} \quad (3.1.20)$$

$$\begin{cases} \deg \mathcal{L}_{10} = p \\ \deg \mathcal{L}_{20} = k - p, \end{cases} \quad 0 \leq p \leq k \quad \text{or} \quad (3.1.21)$$

$$\begin{cases} \deg \mathcal{L}_{20} = k + 1 - p, & 1 \leq p \leq k + 1 \\ \deg \mathcal{L}_{10} = j, & 0 \leq j \leq p - 1, \end{cases} \quad (3.1.22)$$

where  $p = \deg \mathcal{L}_{11}$ .

**Proof.** The system (3.0.3)-(3.0.5) becomes:

$$\mathcal{L}_{11} \mathcal{L}_{21} = P_{k+1}, \quad (3.1.23)$$

$$\mathcal{L}_{10} \mathcal{L}_{21} + \mathcal{L}_{11} (\mathcal{L}_{21})_x + \mathcal{L}_{11} \mathcal{L}_{20} = Q_{k+1}, \quad (3.1.24)$$

$$\mathcal{L}_{10} \mathcal{L}_{20} + \mathcal{L}_{11} (\mathcal{L}_{20})_x = R_k. \quad (3.1.25)$$

The identification of both sides of the equation (3.1.23) yields:

$$\deg (\mathcal{L}_{11} \mathcal{L}_{21}) = \deg (P_{k+1})$$

which implies

$$\deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{21}) = k + 1. \quad (3.1.26)$$

Since  $p = \deg \mathcal{L}_{11}$ , we have from the relation (3.1.26):

$$\deg (\mathcal{L}_{21}) = k + 1 - p. \quad (3.1.27)$$

From the equation (3.1.24), we can write:

$$\deg (\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20}) = \deg (Q_{k+1})$$

giving

$$\begin{aligned} k + 1 &= \max \{ \deg (\mathcal{L}_{10}\mathcal{L}_{21}), \deg (\mathcal{L}_{11}(\mathcal{L}_{21})_x), \deg (\mathcal{L}_{11}\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{21}), \deg (\mathcal{L}_{11}) + \deg ((\mathcal{L}_{21})_x), \\ &\quad \deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{21}), \deg (\mathcal{L}_{11}) + [\deg (\mathcal{L}_{21}) - 1], \\ &\quad \deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{20}) \}. \end{aligned} \quad (3.1.28)$$

The substitution of (3.1.27) into (3.1.28) gives:

$$\begin{aligned} k + 1 &= \max \{ \deg (\mathcal{L}_{10}) + k + 1 - p, p + [(k + 1 - p) - 1], p + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k + 1 - p, k, p + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k + 1 - p, p + \deg (\mathcal{L}_{20}) \} \equiv m_1. \end{aligned}$$

Besides, the identification of both sides of the equation (3.1.25) allows to write:

$$\deg (\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x) = \deg (R_k)$$

or equivalently

$$\begin{aligned} k &= \max \{ \deg (\mathcal{L}_{10}\mathcal{L}_{20}), \deg (\mathcal{L}_{11}(\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{20}), \deg (\mathcal{L}_{11}) + \deg ((\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{20}), p + [\deg (\mathcal{L}_{20}) - 1] \} \equiv m_2. \end{aligned} \quad (3.1.29)$$

- If  $m_1 = \deg (\mathcal{L}_{10}) + k + 1 - p$  then  $\deg (\mathcal{L}_{10}) = p$  and

$$m_2 = \max \{ p + \deg (\mathcal{L}_{20}), p + \deg (\mathcal{L}_{20}) - 1 \} = p + \deg (\mathcal{L}_{20})$$

which gives, taking into account (3.1.29),  $\deg (\mathcal{L}_{20}) = k - p$ .

- If  $m_1 = p + \deg (\mathcal{L}_{20})$  then  $\deg (\mathcal{L}_{20}) = k + 1 - p$  and

$$\begin{aligned} m_2 &= \max \{ \deg (\mathcal{L}_{10}) + k + 1 - p, p + [(k + 1 - p) - 1] \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k + 1 - p, k \} \end{aligned}$$

which gives, taking into account (3.1.29),

$$\deg (\mathcal{L}_{10}) + k + 1 - p \leq k, \text{ i.e. } \deg (\mathcal{L}_{10}) = j, \quad 0 \leq j \leq p - 1.$$

■

The polynomials  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  are characterized by  $(j + 1) + (k + 1 - p + 1) = k + j - p + 3$  constants,  $0 \leq j \leq p - 1$ . The results of the Proposition 3.1.2 are determined in the case where  $j = p - 1$  because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (3.1.24) only. After substitution of polynomials  $\mathcal{L}_{11}$ ,  $\mathcal{L}_{10}$ ,  $\mathcal{L}_{21}$ ,  $\mathcal{L}_{20}$  determined by equations (3.1.23) and (3.1.24) into the equation (3.1.25), a simple identification of coefficients gives a set of relations expressing the  $\rho_l$  as functions of the constants  $\lambda_i$ ,  $E_0$  and  $\mu_{i,j}$ . These relations can be easily computed using a symbolic computational software, for instance Maple. The two following situations are worthy of attention:

- (i) the first order equation associated with the left factor of (3.0.2) admits the solution  $v_1$  given by:

$$v_1(x) = e^{-E_0 x} \prod_{n=1}^q (x - \lambda_{i_n})^{-\mu_{i_n,1}} \exp \left( \sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right), \quad (3.1.30)$$

if  $1 \leq p \leq k + 1$ , while the first order equation of the right factor of (3.0.2) admits the solution  $u_0$  :

$$u_0(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp \left( \sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i} \right) \quad (3.1.31)$$

which is a particular solution of equation (3.1.19).

- (ii) the first order equation associated with the left factor of (3.0.2) admits the solution  $v_1$  given by:

$$v_1(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{-\mu_{i_n,1}} \exp \left( \sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right) \quad (3.1.32)$$

while the first order equation corresponding to the right factor of (3.0.2) generates the solution  $u_0$  given by:

$$u_0(x) = e^{-E_0 x} \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp \left( \sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i} \right), \quad (3.1.33)$$

which is a particular solution of equation (3.1.19).

Here  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  
 $m_{i_l}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

**Proposition 3.1.2 (Sufficient condition for the factorization of (3.1.19))** Consider the equation (3.1.19) and assume that the polynomial

$$R_k(x) = \sum_{l=0}^k \rho_l x^l \quad (3.1.34)$$

satisfies the relation

$$R_k(x) = \mathcal{L}_{10}(x)\mathcal{L}_{20}(x) + \mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \quad (3.1.35)$$

with

$$\begin{cases} \mathcal{L}_{11}(x) = 1 & \text{if } p = 0 \\ \mathcal{L}_{11}(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}} & \text{if } 1 \leq p \leq k + 1, \end{cases} \quad (3.1.36)$$

and  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  explicitly given by one of the two following situations:

(i)

$$\mathcal{L}_{10}(x) = E_0 \quad \text{if } p = 0, \quad (3.1.37)$$

$$\begin{aligned} \mathcal{L}_{10}(x) &= \sum_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}-1} \left[ \mu_{i_n,1} + \sum_{j=1}^{m_{i_n}-1} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right] \prod_{\substack{l=1 \\ l \neq n}}^q (x - \lambda_{i_l})^{m_{i_l}} \\ &+ E_0 \prod_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}} \quad \text{if } 1 \leq p \leq k+1; \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-1} [(\mu_{j_n,1} - m_{j_n}) \\ &+ \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q} (x - \lambda_{j_l})^{m_{j_l}}; \end{aligned} \quad (3.1.38)$$

(ii)

$$\mathcal{L}_{10}(x) = \sum_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}-1} \left[ \mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1} \frac{\mu_{i_n,i+1}}{(x - \lambda_{i_n})^i} \right] \prod_{\substack{l=1 \\ l \neq n}}^q (x - \lambda_{i_l})^{m_{i_l}};$$

$$\begin{aligned} \mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-1} [(\mu_{j_n,1} - m_{j_n}) \\ &+ \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q} (x - \lambda_{j_l})^{m_{j_l}} \\ &+ E_0 \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}} \quad \text{if } 1 \leq p \leq k+1, \end{aligned}$$

where  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  
 $m_{i_l}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

Then, the second order differential operator governing the equation (3.1.19) can be written in the form (3.0.2) where

$$\mathcal{L}_{21}(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}} \quad (3.1.39)$$

is such that

$$\mathcal{L}_{11}(x) \mathcal{L}_{21}(x) = \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}. \quad (3.1.40)$$

**Proof.** Given the expressions of  $\mathcal{L}_{11}$  and  $\mathcal{L}_{21}$  from Proposition 3.1.2, then  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  can be explicitly determined using (3.0.6) and (3.0.7) as follows:

$$\mathcal{L}_{10}(x) = -\mathcal{L}_{11}(x) \frac{v_1'(x)}{v_1(x)} \tag{3.1.41}$$

$$\mathcal{L}_{20}(x) = -\mathcal{L}_{21}(x) \frac{u_0'(x)}{u_0(x)}. \tag{3.1.42}$$

■

**Example 3.1.1** Consider the confluent Heun equation [44, 91]

$$u''(x) + \left( E_0 + \frac{\mu_{1,1}}{x - \lambda_1} + \frac{\mu_{2,1}}{x - \lambda_2} \right) u'(x) + \frac{\rho_0 + \rho_1 x}{(x - \lambda_1)(x - \lambda_2)} u(x) = 0, \tag{3.1.43}$$

where  $E_0, \lambda_1, \lambda_2, \mu_{1,1}, \mu_{2,1}, \rho_0, \rho_1$  are constants such that  $E_0 \neq 0$  and  $\lambda_1 \neq \lambda_2$ . We distinguish here the following three formal factorisable classes:

(i) First class,  $\rho_1 = E_0\mu_{1,1} + E_0\mu_{2,1} - 2E_0, \quad \rho_0 = \mu_{1,1} + \mu_{2,1} - 2 - E_0\mu_{1,1}\lambda_2 - E_0\mu_{2,1}\lambda_1 + E_0\lambda_1 + E_0\lambda_2 :$

$$\begin{aligned} \mathcal{L}_{11}(x) &= 1, & \mathcal{L}_{21}(x) &= (x - \lambda_1)(x - \lambda_2), \\ \mathcal{L}_{10}(x) &= h_0, & \mathcal{L}_{20}(x) &= k_0 + k_1 x, \end{aligned}$$

$$h_0 = E_0, \quad k_0 = -\mu_{1,1}\lambda_2 - \mu_{2,1}\lambda_1 + \lambda_1 + \lambda_2, \quad k_1 = \mu_{1,1} + \mu_{2,1} - 2.$$

Two particular solutions emerge, given by

$$\begin{aligned} u_0(x) &= (x - \lambda_1)^{1-\mu_{1,1}} (x - \lambda_2)^{1-\mu_{2,1}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_1)^{\mu_{1,1}-2} (x - \lambda_2)^{\mu_{2,1}-2} e^{-E_0 x} dx. \end{aligned}$$

(ii) Second class,  $\rho_0 = -E_0\lambda_1 - E_0\mu_{1,1}\lambda_2 + \mu_{1,1}\mu_{2,1} - \mu_{1,1}, \quad \rho_1 = E_0 + E_0\mu_{1,1} :$

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x - \lambda_1), & \mathcal{L}_{21}(x) &= (x - \lambda_2), \\ \mathcal{L}_{10}(x) &= h_0, & \mathcal{L}_{20}(x) &= k_0 + k_1 x, \end{aligned}$$

$$h_0 = \mu_{1,1}, \quad k_0 = -E_0\lambda_2 + \mu_{2,1} - 1, \quad k_1 = E_0.$$

There exist the following two particular solutions:

$$\begin{aligned} u_0(x) &= (x - \lambda_2)^{1-\mu_{2,1}} e^{-E_0 x}, \\ u_1(x) &= u_0(x) \int (x - \lambda_2)^{\mu_{2,1}-2} (x - \lambda_1)^{-\mu_{1,1}} e^{E_0 x} dx. \end{aligned}$$

(iii) Third class,  $\rho_0 = E_0\lambda_1 - E_0\mu_{2,1}\lambda_1 - \mu_{1,1} + \mu_{1,1}\mu_{2,1}, \quad \rho_1 = -E_0 + E_0\mu_{2,1} :$

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x - \lambda_1), & \mathcal{L}_{21}(x) &= (x - \lambda_2), \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, & \mathcal{L}_{20}(x) &= k_0, \end{aligned}$$

$$h_0 = -E_0\lambda_1 + \mu_{1,1}, \quad k_0 = -1 + \mu_{2,1}, \quad h_1 = E_0.$$

Two particular solutions of the corresponding equation (3.1.43) are given by

$$\begin{aligned} u_0(x) &= (x - \lambda_2)^{1-\mu_{2,1}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_2)^{\mu_{2,1}-2} (x - \lambda_1)^{-\mu_{1,1}} e^{-E_0x} dx. \end{aligned}$$

**Example 3.1.2** Consider the following second order linear ODE

$$u''(x) + \left( E_0 + \frac{\mu_{1,1}}{x-\lambda_1} + \frac{\mu_{1,2}}{(x-\lambda_1)^2} + \frac{\mu_{1,3}}{(x-\lambda_1)^3} \right) u'(x) + \frac{\rho_0 + \rho_1 x + \rho_2 x^2}{(x-\lambda_1)^3} u(x) = 0, \quad (3.1.44)$$

where  $E_0, \lambda_1, \mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \rho_0, \rho_1, \rho_2$  are constants such that  $E_0 \neq 0$ . When  $\mu_{1,3} = 0$ , (3.1.44) is reduced to the double confluent Heun equation [44, 91]. Then, the equation (3.1.44) admits a unique formal factorizable class characterized by:

$$\begin{aligned} \rho_0 &= -2\mu_{1,1}\lambda_1 + 6\lambda_1 + \mu_{1,2} - 3E_0\lambda_1^2 + E_0\mu_{1,1}\lambda_1^2 + E_0\mu_{1,3} - E_0\mu_{1,2}\lambda_1, \\ \rho_1 &= -6 + 2\mu_{1,1} - 2E_0\mu_{1,1}\lambda_1 + 6E_0\lambda_1 + E_0\mu_{1,2}, \\ \rho_2 &= -3E_0 + E_0\mu_{1,1}; \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{11}(x) &= 1, & \mathcal{L}_{21}(x) &= (x - \lambda_1)^3, \\ \mathcal{L}_{10}(x) &= h_0, & \mathcal{L}_{20}(x) &= k_0 + k_1 x + k_2 x^2, \end{aligned}$$

$$\begin{aligned} k_1 &= -2\mu_{1,1}\lambda_1 + 6\lambda_1 + \mu_{1,2}, & h_0 &= E_0, \\ k_0 &= -3\lambda_1^2 + \mu_{1,1}\lambda_1^2 + \mu_{1,3} - \mu_{1,2}\lambda_1, & k_2 &= -3 + \mu_{1,1}. \end{aligned}$$

Two particular solutions of the related equation (3.1.44) are given by

$$\begin{aligned} u_0(x) &= (x - \lambda_1)^{3-\mu_{1,1}} e^{\frac{\mu_{1,2}}{x-\lambda_1} + \frac{1}{2} \frac{\mu_{1,3}}{(x-\lambda_1)^2}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_1)^{\mu_{1,1}-6} e^{-\frac{\mu_{1,2}}{x-\lambda_1} - \frac{1}{2} \frac{\mu_{1,3}}{(x-\lambda_1)^2}} e^{-E_0x} dx. \end{aligned}$$

## 3.2 Second class of factorizable 2<sup>nd</sup> order linear ODEs

In this section, we examine the classes of factorizable second order linear ODEs of the type

$$P_k(x) u''(x) + Q_{k+h+1}(x) u'(x) + R_{k+h}(x) u(x) = 0, \quad (k, h) \in \mathbb{N}^* \times \mathbb{N} \quad (3.2.45)$$

explicitly written as

$$\left( \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i} \right) u''(x) + \left( \sum_{j=1}^{k+h+1} \gamma_j x^j \right) u'(x) + \left( \sum_{l=1}^{k+h} \rho_l x^l \right) u(x) = 0, \quad (3.2.46)$$

where  $\sum_{i=1}^{p_0} m_i = k$ , or, equivalently, in the canonical form:  $b_{h+1} \neq 0$

$$u''(x) + \left( \sum_{j=0}^{h+1} b_j x^j + \sum_{i=1}^{p_0} \sum_{j=1}^{m_i} \frac{\mu_{i,j}}{(x - \lambda_i)^j} \right) u'(x) + \frac{\sum_{l=0}^{k+h} \rho_l x^l}{\prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}} u(x) = 0. \quad (3.2.47)$$

**Proposition 3.2.1** (Necessary condition for the factorization of (3.2.45)) *Let equation (3.2.45) be decomposable into the form (3.0.2). Then, the degrees of polynomials  $\mathcal{L}_{ij}$  satisfy the following relations:*

$$\deg \mathcal{L}_{11} + \deg \mathcal{L}_{21} = k \quad \text{and} \tag{3.2.48}$$

$$\begin{cases} \deg \mathcal{L}_{10} = h + p + 1 \\ \deg \mathcal{L}_{20} = k - p - 1, \quad 0 \leq p \leq k - 1 \quad \text{or} \end{cases} \tag{3.2.49}$$

$$\begin{cases} \deg \mathcal{L}_{20} = k + h + 1 - p, & 1 \leq p \leq k \\ \deg \mathcal{L}_{10} = j, & 0 \leq j \leq p - 1, \end{cases} \tag{3.2.50}$$

where  $p = \deg \mathcal{L}_{11}$ .

**Proof.** The system (3.0.3)-(3.0.5) becomes:

$$\mathcal{L}_{11}\mathcal{L}_{21} = P_k, \tag{3.2.51}$$

$$\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20} = Q_{k+h+1}, \tag{3.2.52}$$

$$\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x = R_{k+h}. \tag{3.2.53}$$

The identification of both sides of the equation (3.2.51) yields:

$$\deg (\mathcal{L}_{11} \mathcal{L}_{21}) = \deg (P_k)$$

which implies

$$\deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{21}) = k. \tag{3.2.54}$$

Since  $p = \deg \mathcal{L}_{11}$  we have from the relation (3.2.54):

$$\deg (\mathcal{L}_{21}) = k - p. \tag{3.2.55}$$

From the equation (3.2.52), we can deduce:

$$\deg (\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20}) = \deg (Q_{k+h+1})$$

which implies

$$\begin{aligned} k + h + 1 &= \max \{ \deg (\mathcal{L}_{10}\mathcal{L}_{21}), \deg (\mathcal{L}_{11}(\mathcal{L}_{21})_x), \deg (\mathcal{L}_{11}\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{21}), \deg (\mathcal{L}_{11}) + \deg ((\mathcal{L}_{21})_x), \\ &\quad \deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{21}), \deg (\mathcal{L}_{11}) + [\deg (\mathcal{L}_{21}) - 1], \\ &\quad \deg (\mathcal{L}_{11}) + \deg (\mathcal{L}_{20}) \}. \end{aligned} \tag{3.2.56}$$

The substitution of (3.2.55) into (3.2.56) gives:

$$\begin{aligned} k + h + 1 &= \max \{ \deg (\mathcal{L}_{10}) + k - p, p + [(k - p) - 1], p + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k - p, k - 1, p + \deg (\mathcal{L}_{20}) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k - p, p + \deg (\mathcal{L}_{20}) \} \equiv m_1. \end{aligned}$$

Besides, the identification of both sides of the equation (3.2.53) yields:

$$\deg (\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x) = \deg (R_{k+h})$$

which implies

$$\begin{aligned} k + h &= \max \{ \deg (\mathcal{L}_{10}\mathcal{L}_{20}), \deg (\mathcal{L}_{11}(\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{20}), \deg (\mathcal{L}_{11}) + \deg ((\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + \deg (\mathcal{L}_{20}), p + \deg [(\mathcal{L}_{20}) - 1] \} \equiv m_2. \end{aligned}$$

- If  $m_1 = \deg (\mathcal{L}_{10}) + k - p$  then  $\deg (\mathcal{L}_{10}) = h + p + 1$  and

$$\begin{aligned} m_2 &= \max \{ h + p + 1 + \deg (\mathcal{L}_{20}), p + \deg (\mathcal{L}_{20}) - 1 \} \\ &= h + p + 1 + \deg (\mathcal{L}_{20}) \end{aligned}$$

which gives, taking into account (3.2.47),  $\deg (\mathcal{L}_{20}) = k - p - 1$ .

- If  $m_1 = p + \deg (\mathcal{L}_{20})$  then  $\deg (\mathcal{L}_{20}) = k + h + 1 - p$  and

$$\begin{aligned} m_2 &= \max \{ \deg (\mathcal{L}_{10}) + k + h + 1 - p, p + [(k + h + 1 - p) - 1] \} \\ &= \max \{ \deg (\mathcal{L}_{10}) + k + h + 1 - p, k + h \} \end{aligned}$$

which implies, taking into account (3.2.47),

$$\deg (\mathcal{L}_{10}) + k + h + 1 - p \leq k + h, \text{ i.e. } \deg (\mathcal{L}_{10}) = j, \quad 0 \leq j \leq p - 1.$$

■

The polynomials  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  are characterized by  $(j+1)+(k+h+1-p+1) = k+h+j-p+3$  constants,  $0 \leq j \leq p-1$ . The results of the Proposition 3.2.2 are determined in the case where  $j = p-1$  because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (3.2.52) only. After substitution of polynomials  $\mathcal{L}_{11}$ ,  $\mathcal{L}_{10}$ ,  $\mathcal{L}_{21}$ ,  $\mathcal{L}_{20}$  determined by equations (3.2.51) and (3.2.52) into the equation (3.2.53), a simple identification of coefficients gives a set of relations expressing the  $\rho_l$  as functions of the constants  $\lambda_i$ ,  $b_j$  and  $\mu_{i,j}$ . As in the previous case, these relations can be also easily computed using a symbolic computational software, for instance Maple. There follow two possibilities:

- (i) the corresponding first order left factor of (3.0.2) admits the solution  $v_1$  given by:

$$v_1(x) = e^{-\left(\sum_{j=0}^{h+1} \frac{b_j}{j+1} x^{j+1}\right)} \quad \text{if } p = 0,$$

$$v_1(x) = e^{-\left(\sum_{j=0}^{h+1} \frac{b_j}{j+1} x^{j+1}\right)} \prod_{n=1}^q (x - \lambda_{i_n})^{-\mu_{i_n,1}} \exp \left( \sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n, j+1}}{(x - \lambda_{i_n})^j} \right)$$

if  $1 \leq p \leq k$ ,

while the first order right factor of (3.0.2) admits the solution  $u_0$  given by:

$$u_0(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp \left( \sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n, i+1}}{(x - \lambda_{j_n})^i} \right)$$

which is a particular solution of equation (3.2.47).

(ii) the corresponding first order left factor of (3.0.2) admits the solution  $v_1$  given by:

$$v_1(x) = 1 \quad \text{if } p = 0,$$

$$v_1(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{-\mu_{i_n,1}} \exp \left( \sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right) \quad \text{if } 1 \leq p \leq k,$$

while the first order right factor of (3.0.2) admits the solution  $u_0$  given by:

$$u_0(x) = e^{-\left(\sum_{j=0}^{h+1} \frac{b_j}{j+1} x^{j+1}\right)} \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp \left( \sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i} \right),$$

which is a particular solution of the equation (3.2.47).

In all these expressions,  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  $m_{i_l}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

**Proposition 3.2.2 (Sufficient condition for the factorization of (3.2.47))** Consider the equation (3.2.47) and assume that the polynomial

$$R_{k+h}(x) = \sum_{l=0}^{k+h} \rho_l x^l \tag{3.2.57}$$

satisfies the relation

$$R_{k+h}(x) = \mathcal{L}_{10}(x)\mathcal{L}_{20}(x) + \mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \tag{3.2.58}$$

with

$$\begin{cases} \mathcal{L}_{11}(x) = 1 & \text{if } p = 0, \\ \mathcal{L}_{11}(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}} & \text{if } 1 \leq p \leq k, \end{cases}$$

and  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  explicitly given by one of the two following situations:

(i)

$$\mathcal{L}_{10}(x) = \sum_{j=0}^{h+1} b_j x^j \quad \text{if } p = 0,$$

$$\begin{aligned} \mathcal{L}_{10}(x) &= \sum_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}-1} \left[ \mu_{i_n,1} + \sum_{j=1}^{m_{i_n}-1} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right] \prod_{\substack{l=1 \\ l \neq n}}^q (x - \lambda_{i_l})^{m_{i_l}} \\ &+ \left( \sum_{j=0}^{h+1} b_j x^j \right) \prod_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}} \quad \text{if } 1 \leq p \leq k, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-1} [(\mu_{j_n,1} - m_{j_n}) \\ &+ \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q} (x - \lambda_{j_l})^{m_{j_l}}; \end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{L}_{10}(x) &= 0 \quad \text{if } p=0, \\
\mathcal{L}_{10}(x) &= \sum_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}-1} \left[ \mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1} \frac{\mu_{i_n,i+1}}{(x - \lambda_{i_n})^i} \right] \prod_{\substack{l=1 \\ l \neq n}}^q (x - \lambda_{i_l})^{m_{i_l}} \\
&\quad \text{if } 1 \leq p \leq k; \\
\mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-1} [(\mu_{j_n,1} - m_{j_n}) \\
&\quad + \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q} (x - \lambda_{j_l})^{m_{j_l}} \\
&\quad + \left( \sum_{j=0}^{h+1} b_j x^j \right) \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}},
\end{aligned}$$

where  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  
 $m_{i_n}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

Then, the equation (3.2.47) can be written in the form (3.0.2) where

$$\mathcal{L}_{21}(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}} \quad (3.2.59)$$

is such that

$$\mathcal{L}_{11}(x) \mathcal{L}_{21}(x) = \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}. \quad (3.2.60)$$

**Proof.** It is similar to that of the Proposition 3.1.2. ■

**Example 3.2.1** Consider the biconfluent Heun equation [44, 91]

$$u''(x) + \left( b_0 + b_1 x + \frac{\mu_{1,1}}{x - \lambda_1} \right) u'(x) + \frac{\rho_0 + \rho_1 x}{x - \lambda_1} u(x) = 0, \quad (3.2.61)$$

where  $b_0, b_1, \lambda_1, \mu_{1,1}, \rho_0, \rho_1$  are constants such that  $b_1 \neq 0$ . Then, the equation (3.2.61) gives two formal factorizable classes:

(i) First class,  $\rho_0 = 2b_1$ ,  $\rho_1 = b_0 - b_1 \lambda_1$  :

$$\begin{aligned}
\mathcal{L}_{11}(x) &= 1, & \mathcal{L}_{21}(x) &= (x - \lambda_1), \\
\mathcal{L}_{10}(x) &= 0, & \mathcal{L}_{20}(x) &= k_0 + k_1 x + k_2 x^2,
\end{aligned}$$

$$k_0 = \mu_{1,1} - b_0 \lambda_1 - 1, \quad k_1 = b_0 - b_1 \lambda_1, \quad k_2 = b_1.$$

Two particular solutions of the equation (3.2.61) are given by

$$\begin{aligned}
u_0(x) &= (x - \lambda_1)^{1-\mu_{1,1}} e^{-b_0 x - \frac{1}{2} b_1 x^2}, \\
u_1(x) &= u_0(x) \int (x - \lambda_1)^{\mu_{1,1}-2} e^{b_0 x + \frac{1}{2} b_1 x^2} dx.
\end{aligned}$$

(ii) Second class,  $\rho_0 = b_0(\mu_{1,1} - 1)$ ,  $\rho_1 = b_1(\mu_{1,1} - 1)$  :

$$\begin{aligned} \mathcal{L}_{11}(x) &= 1, & \mathcal{L}_{21}(x) &= (x - \lambda_1), \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, & \mathcal{L}_{20}(x) &= k_0, \end{aligned}$$

$$h_0 = b_0, \quad h_1 = b_1, \quad k_0 = \mu_{1,1} - 1.$$

Two particular solutions of the equation (3.2.61) are provided by

$$\begin{aligned} u_0(x) &= (x - \lambda_1)^{1-\mu_{1,1}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_1)^{\mu_{1,1}-2} e^{-b_0 x - \frac{1}{2} b_1 x^2} dx. \end{aligned}$$

### 3.3 Third class of factorizable 2<sup>nd</sup> order linear ODEs

In this section, we deal with the classes of factorizable second order linear ODEs of the type

$$P_{k+2}(x) u''(x) + Q_{k+1}(x) u'(x) + R_k(x) u(x) = 0, \quad k \in \mathbb{N} \quad (3.3.62)$$

explicitly written as

$$\left( \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i} \right) u''(x) + \left( \sum_{j=0}^{k+1} \gamma_j x^j \right) u'(x) + \left( \sum_{l=0}^k \rho_l x^l \right) u(x) = 0, \quad (3.3.63)$$

where  $\sum_{i=1}^{p_0} m_i = k + 2$ , or, equivalently, in the canonical form:

$$u''(x) + \left( \sum_{i=1}^{p_0} \sum_{j=1}^{m_i} \frac{\mu_{i,j}}{(x - \lambda_i)^j} \right) u'(x) + \frac{\sum_{l=0}^k \rho_l x^l}{\prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}} u(x) = 0. \quad (3.3.64)$$

**Proposition 3.3.1 (Factorizability necessary condition of (3.3.62))** *Let equation (3.3.62) be decomposable into the form (3.0.2). Then, the degrees of polynomials  $\mathcal{L}_{ij}$  satisfy the following relations:*

$$\deg \mathcal{L}_{11} + \deg \mathcal{L}_{21} = k + 2 \quad \text{and} \quad (3.3.65)$$

$$\begin{cases} \deg \mathcal{L}_{20} = k - p + 1, & 1 \leq p \leq k + 1 \\ \deg \mathcal{L}_{10} = j, & 0 \leq j \leq p - 1, \end{cases} \quad (3.3.66)$$

where  $p = \deg \mathcal{L}_{11}$ .

**Proof.** The system (3.0.3)-(3.0.5) becomes:

$$\mathcal{L}_{11} \mathcal{L}_{21} = P_{k+2}, \quad (3.3.67)$$

$$\mathcal{L}_{10} \mathcal{L}_{21} + \mathcal{L}_{11} (\mathcal{L}_{21})_x + \mathcal{L}_{11} \mathcal{L}_{20} = Q_{k+1}, \quad (3.3.68)$$

$$\mathcal{L}_{10} \mathcal{L}_{20} + \mathcal{L}_{11} (\mathcal{L}_{20})_x = R_k. \quad (3.3.69)$$

The identification of both sides of the equation (3.3.67) yields:

$$\deg (\mathcal{L}_{11} \mathcal{L}_{21}) = \deg (P_{k+2})$$

which implies

$$\deg(\mathcal{L}_{11}) + \deg(\mathcal{L}_{21}) = k + 2. \quad (3.3.70)$$

Since  $p = \deg \mathcal{L}_{11}$ , we get from the relation (3.3.70):

$$\deg(\mathcal{L}_{21}) = k + 2 - p. \quad (3.3.71)$$

The equation (3.3.68) also allows to write:

$$\deg(\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20}) = \deg(Q_{k+1})$$

which implies

$$\begin{aligned} k + 1 &= \max \{ \deg(\mathcal{L}_{10}\mathcal{L}_{21}), \deg(\mathcal{L}_{11}(\mathcal{L}_{21})_x), \deg(\mathcal{L}_{11}\mathcal{L}_{20}) \} \\ &= \max \{ \deg(\mathcal{L}_{10}) + \deg(\mathcal{L}_{21}), \deg(\mathcal{L}_{11}) + \deg((\mathcal{L}_{21})_x), \\ &\quad \deg(\mathcal{L}_{11}) + \deg(\mathcal{L}_{20}) \} \\ &= \max \{ \deg(\mathcal{L}_{10}) + \deg(\mathcal{L}_{21}), \deg(\mathcal{L}_{11}) + [\deg(\mathcal{L}_{21}) - 1], \\ &\quad \deg(\mathcal{L}_{11}) + \deg(\mathcal{L}_{20}) \}. \end{aligned} \quad (3.3.72)$$

The substitution of (3.3.71) into (3.3.72) gives:

$$\begin{aligned} k + 1 &= \max \{ \deg(\mathcal{L}_{10}) + k + 2 - p, p + [(k + 2 - p) - 1], p + \deg(\mathcal{L}_{20}) \} \\ &= \max \{ \deg(\mathcal{L}_{10}) + k + 2 - p, k + 1, p + \deg(\mathcal{L}_{20}) \}. \end{aligned}$$

Therefore,

$$\begin{cases} \deg(\mathcal{L}_{10}) + k + 2 - p \leq k + 1 \\ p + \deg(\mathcal{L}_{20}) \leq k + 1, \end{cases}$$

that is

$$\begin{cases} \deg(\mathcal{L}_{10}) = j, & 0 \leq j \leq p - 1 \\ \deg(\mathcal{L}_{20}) = i, & 0 \leq i \leq k + 1 - p. \end{cases} \quad (3.3.73)$$

Besides, the identification of both sides of the equation (3.3.69) leads to:

$$\deg(\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x) = \deg(R_k)$$

which implies

$$\begin{aligned} k &= \max \{ \deg(\mathcal{L}_{10}\mathcal{L}_{20}), \deg(\mathcal{L}_{11}(\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg(\mathcal{L}_{10}) + \deg(\mathcal{L}_{20}), \deg(\mathcal{L}_{11}) + \deg((\mathcal{L}_{20})_x) \} \\ &= \max \{ \deg(\mathcal{L}_{10}) + \deg(\mathcal{L}_{20}), p + [\deg(\mathcal{L}_{20}) - 1] \} \equiv m. \end{aligned}$$

- If  $m = \deg(\mathcal{L}_{10}) + \deg(\mathcal{L}_{20})$  then  $\deg(\mathcal{L}_{20}) = k - j$  which yields by the first equality of (3.3.73)  $k + 1 - p \leq k - j \leq k$ . Therefore, by the second equality of (3.3.73) we must have  $\deg(\mathcal{L}_{20}) = k + 1 - p$ .
- If  $m = p + [\deg(\mathcal{L}_{20}) - 1]$  then  $\deg(\mathcal{L}_{20}) = k + 1 - p$ .

■  
 The polynomials  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  are characterized by  $(j + 1) + (k + 1 - p + 1) = k + j - p + 3$  constants,  $0 \leq j \leq p - 1$ . The results of Proposition 3.3.2 are determined in the case where  $j = p - 1$  because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (3.3.68) only. After substitution of polynomials  $\mathcal{L}_{11}$ ,  $\mathcal{L}_{10}$ ,  $\mathcal{L}_{21}$ ,  $\mathcal{L}_{20}$  determined by equations (3.3.67) and (3.3.68) into the equation (3.3.69), a simple identification of coefficients gives a set of relations expressing the  $\rho_l$  as functions of the constants  $\lambda_i$  and  $\mu_{i,j}$ . For each of such relations, the corresponding first order left factor of (3.0.2) admits the solution  $v_1$  given by:

$$v_1(x) = 1 \quad \text{if } p = 0,$$

$$v_1(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{-\mu_{i_n,1}} \exp \left( \sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j} \right) \quad \text{if } 1 \leq p \leq k + 1,$$

while the first order right factor of (3.0.2) possesses the solution  $u_0$  given by:

$$u_0(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n} - \mu_{j_n,1}} \exp \left( \sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i} \right)$$

which is a particular solution of equation (3.3.64).  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  $m_{i_l}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

**Proposition 3.3.2 (Sufficient condition for the factorization of (3.3.64))** Consider the equation (3.3.64) and assume that the polynomial

$$R_k(x) = \sum_{l=0}^k \rho_l x^l \tag{3.3.74}$$

satisfies the relation

$$R_k(x) = \mathcal{L}_{10}(x)\mathcal{L}_{20}(x) + \mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \tag{3.3.75}$$

with

$$\begin{cases} \mathcal{L}_{11}(x) = 1 & \text{if } p = 0, \\ \mathcal{L}_{11}(x) = \prod_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}} & \text{if } 1 \leq p \leq k + 1, \end{cases} \tag{3.3.76}$$

and  $\mathcal{L}_{10}$  and  $\mathcal{L}_{20}$  explicitly given by

$$\mathcal{L}_{10}(x) = 0 \quad \text{if } p = 0,$$

$$\mathcal{L}_{10}(x) = \sum_{n=1}^q (x - \lambda_{i_n})^{m_{i_n}-1} \left[ \mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1} \frac{\mu_{i_n,i+1}}{(x - \lambda_{i_n})^i} \right] \prod_{\substack{l=1 \\ l \neq n}}^q (x - \lambda_{i_l})^{m_{i_l}}$$

if  $1 \leq p \leq k + 1$ ;

$$\mathcal{L}_{20}(x) = \sum_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}-1} [(\mu_{j_n,1} - m_{j_n}) + \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q} (x - \lambda_{j_l})^{m_{j_l}}.$$

where  $\mu_{i_n,l}, \mu_{j_n,l} \in \{\mu_{1,1}, \dots, \mu_{p_0,m_{p_0}}\}$ ,  $p = \sum_{l=1}^q m_{i_l}$ ,  $1 \leq q \leq p_0$ ;  $m_{i_l}, m_{j_l} \in \{m_1, \dots, m_{p_0}\}$ ;  $\lambda_{i_n} \neq \lambda_{j_n}$ ,  $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \dots, \lambda_{p_0}\}$ .

Then, the equation (3.1.19) can be written in the form (3.0.2) where

$$\mathcal{L}_{21}(x) = \prod_{n=1}^{p_0-q} (x - \lambda_{j_n})^{m_{j_n}} \quad (3.3.77)$$

is such that

$$\mathcal{L}_{11}(x) \mathcal{L}_{21}(x) = \prod_{i=1}^{p_0} (x - \lambda_i)^{m_i}. \quad (3.3.78)$$

**Proof.** It is similar to that of the Proposition 3.1.2. ■

**Example 3.3.1** Consider the following second order linear ODE

$$u''(x) + \left( \frac{\mu_{1,1}}{x-\lambda_1} + \frac{\mu_{1,2}}{(x-\lambda_1)^2} + \frac{\mu_{2,1}}{x-\lambda_2} + \frac{\mu_{3,1}}{x-\lambda_3} \right) u'(x) + \frac{\rho_0 + \rho_1 x + \rho_2 x^2}{(x-\lambda_1)^2(x-\lambda_2)(x-\lambda_3)} u(x) = 0, \quad (3.3.79)$$

where  $\mu_{1,1}, \mu_{1,2}, \mu_{2,1}, \mu_{3,1}, \lambda_1, \lambda_2, \lambda_3, \rho_0, \rho_1, \rho_2$  are constants such that  $\lambda_i \neq \lambda_j$  for  $i \neq j$ . When  $\mu_{1,2} = 0$ ,  $\mu_{1,1} = 1$  and  $\mu_{2,1} = \mu_{3,1} = \frac{1}{2}$ , (3.3.79) is an extension of the Wangerin's equation [76]. Then,

(i) One of the factorizable classes is characterized by

$$\rho_2 = \mu_{2,1} - 2 + \mu_{3,1} + \mu_{1,1}\mu_{2,1} - 2\mu_{1,1} + \mu_{3,1}\mu_{1,1},$$

$$\begin{aligned} \rho_0 &= \mu_{2,1}\lambda_1^2 - 2\lambda_1^2 + \mu_{3,1}\lambda_1^2 - \mu_{1,1}\lambda_1\lambda_3 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_3 - \mu_{1,1}\lambda_1\lambda_2 \\ &+ \mu_{3,1}\mu_{1,1}\lambda_1\lambda_2 + \mu_{1,2}\lambda_3 - \mu_{1,2}\mu_{2,1}\lambda_3 + \mu_{1,2}\lambda_2 - \mu_{3,1}\mu_{1,2}\lambda_2, \end{aligned}$$

$$\begin{aligned} \rho_1 &= -2\mu_{2,1}\lambda_1 + 4\lambda_1 - 2\mu_{3,1}\lambda_1 - \mu_{1,1}\lambda_1\mu_{2,1} + 2\mu_{1,1}\lambda_1 \\ &- \mu_{3,1}\mu_{1,1}\lambda_1 + \mu_{1,2}\mu_{2,1} - 2\mu_{1,2} + \mu_{3,1}\mu_{1,2} + \mu_{1,1}\lambda_3 \\ &- \mu_{1,1}\mu_{2,1}\lambda_3 + \mu_{1,1}\lambda_2 - \mu_{3,1}\mu_{1,1}\lambda_2 : \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x - \lambda_1)^2, & \mathcal{L}_{21}(x) &= (x - \lambda_2)(x - \lambda_3), \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, & \mathcal{L}_{20}(x) &= k_0 + k_1 x, \end{aligned}$$

$$\begin{aligned} h_1 &= \mu_{1,1}, & h_0 &= -\mu_{1,1}\lambda_1 + \mu_{1,2}, \\ k_1 &= \mu_{2,1} - 2 + \mu_{3,1}, & k_0 &= \lambda_3 - \mu_{2,1}\lambda_3 + \lambda_2 - \mu_{3,1}\lambda_2. \end{aligned}$$

Two particular solutions of equation (3.3.79) are given by

$$\begin{aligned} u_0(x) &= (x - \lambda_3)^{1-\mu_{3,1}} (x - \lambda_2)^{1-\mu_{2,1}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_3)^{\mu_{3,1}-2} (x - \lambda_2)^{\mu_{2,1}-2} (x - \lambda_1)^{-\mu_{1,1}} e^{\frac{\mu_{1,2}}{x-\lambda_1}} dx. \end{aligned}$$

(ii) Another factorizable class is given by

$$\begin{aligned}\rho_2 &= -2 + \mu_{1,1} - 2\mu_{2,1} + \mu_{1,1}\mu_{2,1} - 2\mu_{3,1} + \mu_{3,1}\mu_{1,1}, \\ \rho_0 &= -2\lambda_2\lambda_3 + \mu_{1,1}\lambda_2\lambda_3 - 2\mu_{3,1}\lambda_1\lambda_2 - \mu_{3,1}\mu_{1,2}\lambda_2 + \mu_{3,1}\mu_{1,1}\lambda_1\lambda_2 \\ &\quad - 2\mu_{2,1}\lambda_1\lambda_3 - \mu_{1,2}\mu_{2,1}\lambda_3 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_3, \\ \rho_1 &= 2\lambda_2 - \mu_{1,1}\lambda_2 + 2\lambda_3 - \mu_{1,1}\lambda_3 + 2\mu_{3,1}\lambda_2 - \mu_{3,1}\mu_{1,1}\lambda_2 \\ &\quad + 2\mu_{2,1}\lambda_3 - \mu_{1,1}\mu_{2,1}\lambda_3 + 2\mu_{2,1}\lambda_1 + \mu_{1,2}\mu_{2,1} - \mu_{1,1}\lambda_1\mu_{2,1} \\ &\quad + 2\mu_{3,1}\lambda_1 + \mu_{3,1}\mu_{1,2}] - \mu_{3,1}\mu_{1,1}\lambda_1 : \end{aligned}$$

$$\begin{aligned}\mathcal{L}_{11}(x) &= (x - \lambda_2)(x - \lambda_3), & \mathcal{L}_{21}(x) &= (x - \lambda_1)^2, \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, & \mathcal{L}_{20}(x) &= k_0 + k_1 x,\end{aligned}$$

$$\begin{aligned}h_1 &= \mu_{2,1} + \mu_{3,1}, & h_0 &= -\mu_{3,1}\lambda_2 - \mu_{2,1}\lambda_3, \\ k_1 &= -2 + \mu_{1,1}, & k_0 &= 2\lambda_1 + \mu_{1,2} - \mu_{1,1}\lambda_1.\end{aligned}$$

Two particular solutions of the equation (3.3.79) can be written as:

$$\begin{aligned}u_0(x) &= (x - \lambda_1)^{2-\mu_{1,1}} e^{\frac{\mu_{1,2}}{x-\lambda_1}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_1)^{\mu_{1,1}-4} (x - \lambda_2)^{-\mu_{2,1}} (x - \lambda_3)^{-\mu_{3,1}} e^{-\frac{\mu_{1,2}}{x-\lambda_1}} dx.\end{aligned}$$

**Example 3.3.2** Consider the following second order linear ODE

$$\begin{aligned}u''(x) + \left( \frac{\mu_{1,1}}{x-\lambda_1} + \frac{\mu_{1,2}}{(x-\lambda_1)^2} + \frac{\mu_{2,1}}{x-\lambda_2} + \frac{\mu_{2,2}}{(x-\lambda_2)^2} + \frac{\mu_{3,1}}{x-\lambda_3} \right) u'(x) \\ + \frac{\rho_0 + \rho_1 x + \rho_2 x^2 + \rho_3 x^3}{(x-\lambda_1)^2(x-\lambda_2)^2(x-\lambda_3)} u(x) = 0,\end{aligned}\tag{3.3.80}$$

where  $\mu_{1,1}, \mu_{1,2}, \mu_{2,1}, \mu_{2,2}, \mu_{3,1}, \lambda_1, \lambda_2, \lambda_3, \rho_0, \rho_1, \rho_2, \rho_3$  are constants such that  $\lambda_i \neq \lambda_j$  for  $i \neq j$ . When  $\mu_{1,2} = \mu_{2,2} = \mu_{2,1} = 0, \mu_{1,1} = 1$  and  $\mu_{3,1} = \frac{1}{2}$ , (3.3.80) is an extension

of the Heine's equation [76]. Then, one of the factorizable classes is characterized by

$$\rho_3 = -6 + 2\mu_{3,1} + 2\mu_{2,1} - 3\mu_{1,1} + \mu_{3,1}\mu_{1,1} + \mu_{1,1}\mu_{2,1},$$

$$\begin{aligned} \rho_0 &= -2\mu_{3,1}\lambda_1^2\lambda_2 + 4\lambda_1^2\lambda_2 + 2\lambda_1^2\lambda_3 - \mu_{2,1}\lambda_1^2\lambda_2 + \mu_{2,2}\lambda_1^2 - \mu_{2,1}\lambda_1^2\lambda_3 \\ &+ 2\mu_{1,1}\lambda_1\lambda_2\lambda_3 + \mu_{1,1}\lambda_1\lambda_2^2 - \mu_{3,1}\mu_{1,1}\lambda_1\lambda_2^2 + \mu_{1,1}\lambda_1\mu_{2,2}\lambda_3 - \mu_{1,1}\lambda_1\mu_{2,1}\lambda_2\lambda_3 \\ &- 2\mu_{1,2}\lambda_2\lambda_3 - \mu_{1,2}\lambda_2^2 + \mu_{3,1}\mu_{1,2}\lambda_2^2 - \mu_{1,2}\mu_{2,2}\lambda_3 + \mu_{1,2}\mu_{2,1}\lambda_2\lambda_3, \end{aligned}$$

$$\begin{aligned} \rho_1 &= -2\mu_{1,1}\lambda_1\lambda_3 - 4\mu_{1,1}\lambda_1\lambda_2 - 2\mu_{1,1}\lambda_2\lambda_3 + 2\mu_{2,1}\lambda_1\lambda_3 + 2\mu_{2,1}\lambda_1\lambda_2 - 4\lambda_1\lambda_3 \\ &- 8\lambda_1\lambda_2 + 4\mu_{3,1}\lambda_1\lambda_2 + 4\mu_{1,2}\lambda_2 + 2\mu_{1,2}\lambda_3 + 2\mu_{3,1}\lambda_1^2 + 2\mu_{2,1}\lambda_1^2 - \mu_{1,1}\lambda_2^2 \\ &- 2\mu_{2,2}\lambda_1 - 2\mu_{3,1}\mu_{1,2}\lambda_2 + \mu_{3,1}\mu_{1,1}\lambda_2^2 - 6\lambda_1^2 + \mu_{1,2}\mu_{2,2} + 2\mu_{3,1}\mu_{1,1}\lambda_1\lambda_2 \\ &- \mu_{1,1}\mu_{2,2}\lambda_3 - \mu_{1,2}\mu_{2,1}\lambda_3 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_2 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_3 + \mu_{1,1}\mu_{2,1}\lambda_2\lambda_3 \\ &- \mu_{1,1}\lambda_1\mu_{2,2} - \mu_{1,2}\mu_{2,1}\lambda_2, \end{aligned}$$

$$\begin{aligned} \rho_2 &= 12\lambda_1 - 4\mu_{3,1}\lambda_1 - 4\mu_{2,1}\lambda_1 - 2\mu_{3,1}\lambda_2 + 4\lambda_2 + 2\lambda_3 - \mu_{2,1}\lambda_2 + \mu_{2,2} \\ &- \mu_{2,1}\lambda_3 + 3\mu_{1,1}\lambda_1 - \mu_{3,1}\mu_{1,1}\lambda_1 - \mu_{1,1}\lambda_1\mu_{2,1} - 3\mu_{1,2} + \mu_{3,1}\mu_{1,2} + \mu_{1,2}\mu_{2,1} \\ &- 2\mu_{3,1}\mu_{1,1}\lambda_2 + 4\mu_{1,1}\lambda_2 + 2\mu_{1,1}\lambda_3 - \mu_{1,1}\mu_{2,1}\lambda_2 + \mu_{1,1}\mu_{2,2} - \mu_{1,1}\mu_{2,1}\lambda_3 : \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x - \lambda_1)^2, & \mathcal{L}_{21}(x) &= (x - \lambda_2)^2(x - \lambda_3), \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, & \mathcal{L}_{20}(x) &= k_0 + k_1 x + k_2 x^2, \end{aligned}$$

$$k_2 = -3 + \mu_{3,1} + \mu_{2,1}, \quad h_1 = \mu_{1,1}, \quad h_0 = -\mu_{1,1}\lambda_1 + \mu_{1,2},$$

$$\begin{aligned} k_1 &= -2\mu_{3,1}\lambda_2 + 4\lambda_2 + 2\lambda_3 - \mu_{2,1}\lambda_2 + \mu_{2,2} - \mu_{2,1}\lambda_3, \\ k_0 &= -2\lambda_2\lambda_3 - \lambda_2^2 + \mu_{3,1}\lambda_2^2 - \mu_{2,2}\lambda_3 + \mu_{2,1}\lambda_2\lambda_3. \end{aligned}$$

Two particular solutions of equation (3.3.80) are given by

$$\begin{aligned} u_0(x) &= (x - \lambda_3)^{1-\mu_{3,1}}(x - \lambda_2)^{2-\mu_{2,1}} e^{\frac{\mu_{2,2}}{x-\lambda_2}}, \\ u_1(x) &= u_0(x) \int (x - \lambda_3)^{\mu_{3,1}-2} (x - \lambda_2)^{\mu_{2,1}-4} (x - \lambda_1)^{-\mu_{1,1}} e^{-\frac{\mu_{2,2}}{x-\lambda_2}} e^{\frac{\mu_{1,2}}{x-\lambda_1}} dx. \end{aligned}$$

# Function Expansion Methods to Solve Autonomous NLPDEs

---

In this paper, we propose some algorithms for analytical solution construction to nonlinear polynomial partial differential equations with constant function coefficients. These schemes are based on one-(single), two- (double) or three- (triple) function expansion methods. Most of the existing expansion function methods are well recovered from the mentioned schemes. The effectiveness of these methods has been tested on some nonlinear partial differential equations (NLPDEs) describing important phenomena in physics.

Consider the constant coefficient partial differential equations

$$F\left(u^{(s)}(x)\right) = 0, \quad (4.0.1)$$

where the nonzero positive integer  $s$  is the order of the equation and  $x = (x^1, \dots, x^n)$  are independent variables. The dependent variable  $u = u(x)$  is a scalar valued function. If

$$F\left(u^{(s)}(x)\right) = P\left(u_{(k_1)}[h_1](x), u_{(k_2)}[h_2](x), \dots, u_{(k_r)}[h_r](x)\right), \quad (4.0.2)$$

where  $r, k_i, h_i \in \mathbb{N}$  with  $\max\{k_i, i = 1, \dots, r\} = s$ ,  $h_i \in \{1, 2, \dots, p_{k_i}\}$  and  $P$  is a polynomial whose the indeterminates are the  $r$  functions  $u_{(k_i)}[h_i](x)$ , the equation (4.0.1) becomes

$$P\left(u_{(k_1)}[h_1](x), u_{(k_2)}[h_2](x), \dots, u_{(k_r)}[h_r](x)\right) = 0 \quad (4.0.3)$$

which is called a polynomial autonomous partial differential equation.

The change of variables  $u(x) = v(\xi)$  with  $\xi = \alpha_1 x^1 + \alpha_2 x^2 + \dots + \alpha_n x^n$ , where  $\alpha_i \in \mathbb{R}$  transforms (4.0.3) into an ordinary differential equation

$$Q\left(v_{(k_1)}[1](\xi), v_{(k_2)}[1](\xi), \dots, v_{(k_r)}[1](\xi)\right) = 0, \quad (4.0.4)$$

where  $Q$  is also a polynomial function whose the indeterminates are  $v_{(k_i)}[1](\xi)$ ,  $i = 1, \dots, r$ . If the polynomial  $Q$  has only one monomial, then the equation (4.0.4) is simply solved by successive integrations. Thus, without loss of generality, one can assume that the polynomial  $Q$  is a sum of at least two monomials. We propose in this paper to search for a solution of the equation (4.0.4) by using single, double or triple function expansion methods.

## 4.1 Single function expansion method

It consists to seek the function  $v = v(\xi)$ , solution of the equation (4.0.4), into the form

$$v = A(F) + F_{(1)}[1]B(F), \quad (4.1.5)$$


---

where

$$A(F) = \sum_{i=-m}^m a_i F^i, \quad B(F) = \sum_{i=-\hat{m}}^{\hat{m}} b_i F^i$$

and the function  $F = F(\xi)$  is a solution to an auxiliary differential equation which is in one of the forms

$$F_{(1)}[1] = R \quad \text{or} \quad (F_{(1)}[1])^2 = R, \tag{4.1.6}$$

$R$  being a rational function in  $F$  and  $m, \hat{m} \in \mathbb{N}$ ,  $a_i, b_i \in \mathbb{R}$ . The higher order derivatives of  $F$  can be written into the form

$$F_{(k)}[1] = R^{0,k} + F_{(1)}[1] R^{1,k},$$

where the positive integer  $k \geq 2$  and  $R^{i,k}$ ,  $i = 0, 1$ , are also rational functions in  $F$ .

One can adopt the following iterative process in practice for the determination of different parameters of  $v$ .

**Estimation of the integers  $m, \hat{m}$ .**

Let  $M_1, M_2, \dots, M_\nu$  be the monomials of  $Q$  such that  $M_1$  contains the highest order derivative of the function  $v(\xi)$  and let  $M_2$  be a nonlinear or linear monomial.  $M_2$  is linear only in the case where all remaining monomials are linear. Substitute in (4.0.4) the expression (4.1.5) of  $v(\xi)$  along with (4.1.6) and write the result in the form

$$Q = \frac{1}{T} \sum_{i=1}^{\nu} (K_i + F_{(1)}[1] L_i), \tag{4.1.7}$$

where  $K_i, L_i, T$  are polynomials in  $F$  whose degrees depend linearly on  $m, \hat{m}$  such that

$$M_i = \frac{1}{T} (K_i + F_{(1)}[1] L_i). \tag{4.1.8}$$

Solve the system of linear algebraic equations obtained by balancing the degree of  $K_1$  with that of  $K_2$  and the degree of  $L_1$  with that of  $L_2$  in  $F$  to determine the values of integers  $m, \hat{m}$ .

**Estimation of the constants  $a_i, b_i, \alpha_i$ .**

Introduce into (4.1.7) the obtained values of  $m, \hat{m}$ . Write the result in the form

$$Q = \frac{1}{T} (K + F_{(1)}[1] L), \tag{4.1.9}$$

where  $K = \sum_{i=1}^{\nu} K_i$  and  $L = \sum_{i=1}^{\nu} L_i$ . Set to zero all coefficients of distinct monomials in  $K$  and  $L$ . This gives a system of algebraic equations whose the unknowns are the constants  $a_i, b_i$  and  $\alpha_i$ .

**Remark 4.1.1** *We have the tanh-expansion method if we take  $F(\xi) = \tanh(\xi)$ , the tan-expansion method if  $F(\xi) = \tan(\xi)$ , the exp-expansion method if  $F(\xi) = \exp(\xi)$ , the  $\frac{G'}{G}$ -expansion method if  $F(\xi) = \frac{G'(\xi)}{G(\xi)}$ , where  $G''(\xi) = \alpha G'(\xi) + \beta G(\xi)$  with  $\alpha, \beta \in \mathbb{R}$ . It is noticeable that in all these cases  $F'(\xi)$  is a polynomial function in  $F(\xi)$  and hence a rational function as required.*

## 4.2 Double function expansion method

It aims at finding the function  $v = v(\xi)$ , solution of the equation (4.0.4), into the form

$$v = A(F, G) + F_{(1)}[1]B(F, G) + G_{(1)}[1]C(F, G) + F_{(1)}[1]G_{(1)}[1]D(F, G), \quad (4.2.10)$$

where

$$A(F, G) = \sum_{i=-m_1}^{m_1} \sum_{j=-m_2}^{m_2} a_{i,j} F^i G^j, \quad B(F, G) = \sum_{i=-\hat{m}_1}^{\hat{m}_1} \sum_{j=-\hat{m}_2}^{\hat{m}_2} b_{i,j} F^i G^j,$$

$$C(F, G) = \sum_{i=-\check{m}_1}^{\check{m}_1} \sum_{j=-\check{m}_2}^{\check{m}_2} c_{i,j} F^i G^j, \quad D(F, G) = \sum_{i=-\bar{m}_1}^{\bar{m}_1} \sum_{j=-\bar{m}_2}^{\bar{m}_2} d_{i,j} F^i G^j$$

and the functions  $F = F(\xi)$  and  $G = G(\xi)$  are solutions of an auxiliary system of differential equations of the form

$$F_{(1)}[1] = R_1 \quad \text{or} \quad (F_{(1)}[1])^2 = R_1, \quad (4.2.11)$$

$$G_{(1)}[1] = R_2 \quad \text{or} \quad (G_{(1)}[1])^2 = R_2, \quad (4.2.12)$$

$R_k$ ,  $k = 1, 2$  being rational functions in  $F$ ,  $G$  and  $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2 \in \mathbb{N}$ ,  $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in \mathbb{R}$ . The higher order derivatives of  $F$  and  $G$  can be written into the form

$$F_{(k)}[1] = R_1^{0,k} + F_{(1)}[1] R_1^{1,k} + G_{(1)}[1] R_1^{2,k} + F_{(1)}[1] G_{(1)}[1] R_1^{3,k},$$

$$G_{(k)}[1] = R_2^{0,k} + F_{(1)}[1] R_2^{1,k} + G_{(1)}[1] R_2^{2,k} + F_{(1)}[1] G_{(1)}[1] R_2^{3,k},$$

where the positive integer  $k \geq 2$  and  $R_j^{i,k}$ ,  $i = 0, 1, 2, 3$ ,  $j = 1, 2$ , are also rational functions in  $F$ ,  $G$ .

One can adopt the following iterative process in practice for the determination of different parameters of  $v$ .

### Estimation of the integers $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$ .

Let  $M_1, M_2, \dots, M_\nu$  be the monomials of  $Q$  such that  $M_1$  contains the highest order derivative of the function  $v(\xi)$  and let  $M_2$  be a nonlinear or linear monomial.  $M_2$  is linear only in the case where all remaining monomials are linear. Substitute in (4.0.4) the expression (4.2.10) of  $v(\xi)$  along with (4.2.11)-(4.2.12) and write the result in the form

$$Q = \frac{1}{T} \sum_{i=1}^{\nu} (K_i + F_{(1)}[1] L_i + G_{(1)}[1] S_i + F_{(1)}[1] G_{(1)}[1] J_i), \quad (4.2.13)$$

where  $K_i, L_i, S_i, J_i, T$  are polynomials in  $F, G$  whose the degrees linearly depend on  $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$  such that

$$M_i = \frac{1}{T} (K_i + F_{(1)}[1] L_i + G_{(1)}[1] S_i + F_{(1)}[1] G_{(1)}[1] J_i). \quad (4.2.14)$$

Solve the system of linear algebraic equations obtained by balancing the degree of  $K_1$  with that of  $K_2$ , the degree of  $L_1$  with that of  $L_2$ , the degree of  $S_1$  with that of  $S_2$  and the degree of  $J_1$  with that of  $J_2$  in  $F$  and  $G$  to determine the values of integers  $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$ .

**Estimation of the constants**  $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, \alpha_i$ .

Introduce into (4.2.13) the obtained values of  $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$ . Write the result in the form

$$Q = \frac{1}{T} (K + F_{(1)}[1]L + G_{(1)}[1]S + F_{(1)}[1]G_{(1)}[1]J), \quad (4.2.15)$$

where  $K = \sum_{i=1}^{\nu} K_i$ ,  $L = \sum_{i=1}^{\nu} L_i$ ,  $S = \sum_{i=1}^{\nu} S_i$  and  $J = \sum_{i=1}^{\nu} J_i$ . Set to zero all coefficients of distinct monomials in  $K, L, S$  and  $J$ . This gives a system of algebraic equations whose the unknowns are the constants  $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$  and  $\alpha_i$ .

**Remark 4.2.1** We have the (sinh, cosh)-expansion method if we take  $F(\xi) = \sinh(\xi)$  and  $G(\xi) = \cosh(\xi)$ , the (sin, cos)-expansion method if  $F(\xi) = \sin(\xi)$  and  $G(\xi) = \cos(\xi)$ . We obtain a more general formulation of the  $\left(\frac{H'}{H}, \frac{1}{H}\right)$ -expansion method by setting  $F(\xi) = \frac{H'(\xi)}{H(\xi)}$  and  $G(\xi) = \frac{1}{H(\xi)}$ , where  $H''(\xi) + \lambda H(\xi) = \mu$  with  $\lambda, \mu \in \mathbb{R}$ , and a more general formulation of the projective Riccati equation expansion method if we choose  $F(\xi), G(\xi)$  such that  $F'(\xi) = \alpha F(\xi)G(\xi)$  and  $G'(\xi) = \mu + \alpha^2 G^2(\xi) - \beta F(\xi)$  with  $\alpha, \beta, \mu \in \mathbb{R}^*$ . In all these cases,  $F'(\xi)$  and  $G'(\xi)$  are also polynomial functions in  $F(\xi), G(\xi)$  and hence rational functions as required.

### 4.3 Triple function expansion method

Here, we need to express the function  $v = v(\xi)$ , solution of the equation (4.0.4), into the form

$$\begin{aligned} v = & A + B^{1,1}F_{(1)}[1] + B^{2,1}G_{(1)}[1] + B^{3,1}H_{(1)}[1] + C^{1,1}F_{(1)}[1]G_{(1)}[1] \\ & + C^{2,1}F_{(1)}[1]H_{(1)}[1] + C^{3,1}G_{(1)}[1]H_{(1)}[1] + D F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] \end{aligned} \quad (4.3.16)$$

where  $A, B^{1,1}, B^{2,1}, B^{3,1}, C^{1,1}, C^{2,1}, C^{3,1}, D$  are functions of  $F, G, H$  given by

$$\begin{aligned} A &= \sum_{i=-m_1}^{m_1} \sum_{j=-m_2}^{m_2} \sum_{k=-m_3}^{m_3} a_{i,j,k} F^i G^j H^k, & B^{1,1} &= \sum_{i=-\hat{m}_{1,1}}^{\hat{m}_{1,1}} \sum_{j=-\hat{m}_{1,2}}^{\hat{m}_{1,2}} \sum_{k=-\hat{m}_{1,3}}^{\hat{m}_{1,3}} b_{i,j,k}^{1,1} F^i G^j H^k, \\ B^{2,1} &= \sum_{i=-\hat{m}_{2,1}}^{\hat{m}_{2,1}} \sum_{j=-\hat{m}_{2,2}}^{\hat{m}_{2,2}} \sum_{k=-\hat{m}_{2,3}}^{\hat{m}_{2,3}} b_{i,j,k}^{2,1} F^i G^j H^k, & B^{3,1} &= \sum_{i=-\hat{m}_{3,1}}^{\hat{m}_{3,1}} \sum_{j=-\hat{m}_{3,2}}^{\hat{m}_{3,2}} \sum_{k=-\hat{m}_{3,3}}^{\hat{m}_{3,3}} b_{i,j,k}^{3,1} F^i G^j H^k, \\ C^{1,1} &= \sum_{i=-\check{m}_{1,1}}^{\check{m}_{1,1}} \sum_{j=-\check{m}_{1,2}}^{\check{m}_{1,2}} \sum_{k=-\check{m}_{1,3}}^{\check{m}_{1,3}} c_{i,j,k}^{1,1} F^i G^j H^k, & C^{2,1} &= \sum_{i=-\check{m}_{2,1}}^{\check{m}_{2,1}} \sum_{j=-\check{m}_{2,2}}^{\check{m}_{2,2}} \sum_{k=-\check{m}_{2,3}}^{\check{m}_{2,3}} c_{i,j,k}^{2,1} F^i G^j H^k, \\ C^{3,1} &= \sum_{i=-\bar{m}_{3,1}}^{\bar{m}_{3,1}} \sum_{j=-\bar{m}_{3,2}}^{\bar{m}_{3,2}} \sum_{k=-\bar{m}_{3,3}}^{\bar{m}_{3,3}} c_{i,j,k}^{3,1} F^i G^j H^k, & D &= \sum_{i=-\bar{m}_1}^{\bar{m}_1} \sum_{j=-\bar{m}_2}^{\bar{m}_2} \sum_{k=-\bar{m}_3}^{\bar{m}_3} d_{i,j,k} F^i G^j H^k \end{aligned}$$

and  $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$  are positive integers,  $a_{i,j,k}, b_{i,j,k}^{l,1}, c_{i,j,k}^{l,1}, d_{i,j,k} \in \mathbb{R}$ ; the functions  $F = F(\xi)$ ,  $G = G(\xi)$  and  $H = H(\xi)$  are solutions of the following auxiliary system of differential equations:

$$F_{(1)}[1] = R_1 \quad \text{or} \quad (F_{(1)}[1])^2 = R_1, \quad (4.3.17)$$

$$G_{(1)}[1] = R_2 \quad \text{or} \quad (G_{(1)}[1])^2 = R_2, \quad (4.3.18)$$

$$H_{(1)}[1] = R_3 \quad \text{or} \quad (H_{(1)}[1])^2 = R_3, \quad (4.3.19)$$

$R_k$ ,  $k = 1, 2, 3$  being rational functions in  $F, G, H$ . The higher order derivatives of  $F, G$  and  $H$  can be written into the form

$$\begin{aligned} F_{(k)}[1] &= R_1^{0,k} + F_{(1)}[1] R_1^{1,k} + G_{(1)}[1] R_1^{2,k} + H_{(1)}[1] R_1^{3,k} + F_{(1)}[1]G_{(1)}[1] R_1^{4,k} \\ &\quad + F_{(1)}[1]H_{(1)}[1] R_1^{5,k} + G_{(1)}[1]H_{(1)}[1] R_1^{6,k} + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] R_1^{7,k}, \\ G_{(k)}[1] &= R_2^{0,k} + F_{(1)}[1] R_2^{1,k} + G_{(1)}[1] R_2^{2,k} + H_{(1)}[1] R_2^{3,k} + F_{(1)}[1]G_{(1)}[1] R_2^{4,k} \\ &\quad + F_{(1)}[1]H_{(1)}[1] R_2^{5,k} + G_{(1)}[1]H_{(1)}[1] R_2^{6,k} + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] R_2^{7,k}, \\ H_{(k)}[1] &= R_3^{0,k} + F_{(1)}[1] R_3^{1,k} + G_{(1)}[1] R_3^{2,k} + H_{(1)}[1] R_3^{3,k} + F_{(1)}[1]G_{(1)}[1] R_3^{4,k} \\ &\quad + F_{(1)}[1]H_{(1)}[1] R_3^{5,k} + G_{(1)}[1]H_{(1)}[1] R_3^{6,k} + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] R_3^{7,k}, \end{aligned}$$

where the positive integer  $k \geq 2$  and  $R_j^{i,k}$ ,  $i = 0, 1, 2, 3, 4, 5, 6, 7$ ,  $j = 1, 2, 3$ , are also rational functions in  $F, G, H$ . In practice, the following iterative process leads to the determination of different parameters of  $v$ .

**Estimation of the integer  $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$ .**

Let  $M_1, M_2, \dots, M_\nu$  be the monomials of  $Q$  such that  $M_1$  contains the highest order derivative of the function  $v(\xi)$  and let  $M_2$  be a nonlinear or linear monomial.  $M_2$  is linear only in the case where all remaining monomials are linear. Substitute in (4.0.4) the expression (4.3.16) of  $v(\xi)$  along with (4.3.17)-(4.3.19) and write the result in the form

$$\begin{aligned} Q &= \frac{1}{T} \sum_{i=1}^{\nu} (K_i + F_{(1)}[1] L_{1,i} + G_{(1)}[1] L_{2,i} + H_{(1)}[1] L_{3,i} + F_{(1)}[1]G_{(1)}[1] S_{1,i}) \\ &\quad + \frac{1}{T} \sum_{i=1}^{\nu} (F_{(1)}[1]H_{(1)}[1] S_{2,i} + G_{(1)}[1]H_{(1)}[1] S_{3,i} + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] J_i), \end{aligned} \quad (4.3.20)$$

where  $K_i, L_{j,i}, S_{j,i}, J_i, T$  are polynomials in  $F, G, H$  whose the degrees linearly depend on  $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$  and

$$\begin{aligned} M_i &= \frac{1}{T} (K_i + F_{(1)}[1] L_{1,i} + G_{(1)}[1] L_{2,i} + H_{(1)}[1] L_{3,i} + F_{(1)}[1]G_{(1)}[1] S_{1,i}) \\ &\quad + \frac{1}{T} (F_{(1)}[1]H_{(1)}[1] S_{2,i} + G_{(1)}[1]H_{(1)}[1] S_{3,i} + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] J_i). \end{aligned} \quad (4.3.21)$$

Solve the system of linear algebraic equations obtained by balancing the degree of  $K_1$  with that of  $K_2$ , the degree of  $L_{j,1}$  with that of  $L_{j,2}$ , the degree of  $S_{j,1}$  with that of  $S_{j,1}$  and the degree of  $J_1$  with that of  $J_2$  in  $F, G$  and  $H$  to determine the values of the integers  $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$ .

**Estimation of the constants  $a_{i,j,k}, b_{i,j,k}^{l,1}, c_{i,j,k}^{l,1}, d_{i,j,k}, \alpha_i$ .**

Introduce into (4.3.20) the obtained values of  $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$ . Write the result in the form

$$\begin{aligned} Q &= \frac{1}{T} (K + F_{(1)}[1] L_1 + G_{(1)}[1] L_2 + H_{(1)}[1] L_3 + F_{(1)}[1]G_{(1)}[1] S_1) \\ &\quad + \frac{1}{T} (F_{(1)}[1]H_{(1)}[1] S_2 + G_{(1)}[1]H_{(1)}[1] S_3 + F_{(1)}[1]G_{(1)}[1]H_{(1)}[1] J). \end{aligned} \quad (4.3.22)$$

where  $K = \sum_{i=1}^{\nu} K_i$ ,  $L_j = \sum_{i=1}^{\nu} L_{j,i}$ ,  $S_j = \sum_{i=1}^{\nu} S_{j,i}$  and  $J = \sum_{i=1}^{\nu} J_i$ . Set to zero all coefficients of distinct monomials in  $K, L_j, S_j$  and  $J$ . This gives a system of algebraic equations whose the unknowns are the constants  $a_{i,j,k}, b_{i,j,k}^{l,1}, c_{i,j,k}^{l,1}, d_{i,j,k}$  and  $\alpha_i$ .

**Remark 4.3.1** We have the (sn, cn, dn)-expansion method if we take  $F(\xi) = \text{sn}(\xi)$ ,  $G(\xi) = \text{cn}(\xi)$  and  $H(\xi) = \text{dn}(\xi)$ , where  $\text{sn}(\xi) = \text{sn}(\xi, k)$ ,  $\text{cn}(\xi) = \text{cn}(\xi, k)$  and  $\text{dn}(\xi) = \text{dn}(\xi, k)$  with  $0 < k < 1$  are the basis Jacobi elliptic functions [24]. The function sn, cn and dn are solutions of the first order ordinary differential equations

$$(w'(\xi))^2 = (1 - w^2(\xi))(1 - k^2w^2(\xi)), \quad (4.3.23)$$

$$(w'(\xi))^2 = (1 - w^2(\xi))(k^2w^2(\xi) + 1 - k^2), \quad (4.3.24)$$

$$(w'(\xi))^2 = (1 + w^2(\xi))(w^2(\xi) + k^2 - 1), \quad (4.3.25)$$

respectively. When  $k \rightarrow 0$  the Jacobi functions degenerate to the trigonometric functions, i.e.,

$$\lim_{k \rightarrow 0} \text{sn}(\xi, k) = \sin(\xi), \quad \lim_{k \rightarrow 0} \text{cn}(\xi, k) = \cos(\xi), \quad \lim_{k \rightarrow 0} \text{dn}(\xi, k) = 1.$$

When  $k \rightarrow 1$ , the Jacobi functions degenerate to the hyperbolic functions, i.e.,

$$\lim_{k \rightarrow 1} \text{sn}(\xi, k) = \tanh(\xi), \quad \lim_{k \rightarrow 1} \text{cn}(\xi, k) = \frac{1}{\cosh(\xi)}, \quad \lim_{k \rightarrow 1} \text{dn}(\xi, k) = \frac{1}{\cosh(\xi)}.$$

The function sn, cn and dn have the algebraic properties

$$\text{sn}^2(\xi, k) + \text{cn}^2(\xi, k) = 1, \quad k^2\text{sn}^2(\xi, k) + \text{dn}^2(\xi, k) = 1,$$

$$k^2\text{cn}^2(\xi, k) + 1 - k^2 = \text{dn}^2(\xi, k), \quad \text{cn}^2(\xi, k) + (1 - k^2)\text{sn}^2(\xi, k) = \text{dn}^2(\xi, k)$$

and the differential properties

$$\frac{d}{d\xi} \text{sn}(\xi, k) = \text{cn}(\xi, k) \text{dn}(\xi, k), \quad \frac{d}{d\xi} \text{cn}(\xi, k) = -\text{sn}(\xi, k) \text{dn}(\xi, k),$$

$$\frac{d}{d\xi} \text{dn}(\xi, k) = -k^2 \text{sn}(\xi, k) \text{cn}(\xi, k).$$

## 4.4 Application to some relevant NLPDEs in physics

### (A) Case 1: Burger-Fisher equation

Let us use the single function expansion method to analyze the Burger-Fisher equation [102]

$$u_{xx} + uu_x - u_t + u - u^2 = 0, \quad (4.4.26)$$

where  $u = u(t, x)$ . We first substitute into (4.4.26) the variables  $u(t, x) = v(\xi)$ ,  $\xi = \alpha t + \beta x$ , with  $\alpha, \beta \in \mathbb{R}$  to obtain

$$\beta^2 v_{\xi\xi} + \beta vv_{\xi} - \alpha v_{\xi} + v - v^2 = 0. \quad (4.4.27)$$

We take the expansion (4.1.5) to look for the solution of the equation (4.4.27) with the assumption that the function  $F = F(\xi)$  satisfies the auxiliary equation  $F'(\xi) = 1 - F^2(\xi)$ . Balancing the degree of the nonlinear term  $vv_{\xi}$  with that of the linear term  $v_{\xi\xi}$  yields  $m = \hat{m} = 1$ . This allows to take the ansatz

$$v(\xi) = \sum_{i=-1}^1 a_i F^i(\xi) + F'(\xi) \sum_{j=-1}^1 b_j F^j(\xi), \quad (4.4.28)$$

where  $a_i, b_j$  are constants to determine. Here  $F(\xi) = \tanh(\xi)$  and (4.4.28) becomes

$$\begin{aligned} v(\xi) &= -b_1 \tanh^3(\xi) - b_0 \tanh^2(\xi) + (a_1 + b_1 - b_{-1}) \tanh(\xi) \\ &+ a_0 + b_0 + \frac{a_{-1} + b_{-1}}{\tanh(\xi)}. \end{aligned} \quad (4.4.29)$$

Substitute (4.4.29) into (4.4.27). Then, collect and set to zero the coefficients of each power of  $\tanh^i(\xi)$ . Solving the resulting system with respect to the parameters  $a_i, b_j, \alpha, \beta$ , several solution sets are obtained leading to the following solutions for the equation (4.4.26):  $\epsilon \in \{-1, +1\}$

$$\begin{aligned} u_1(t, x) &= 0, \\ u_2(t, x) &= 1, \\ u_3(t, x) &= \frac{1}{2} + \frac{\frac{\epsilon}{2}}{\tanh\left(\frac{\epsilon}{2}t\right)}, \\ u_4(t, x) &= \frac{1}{2} + \frac{\epsilon}{2} \tanh\left(\frac{\epsilon}{2}t\right), \\ u_5(t, x) &= \frac{1}{2} + \frac{\frac{\epsilon}{2}}{\tanh\left[\frac{\epsilon}{4}\left(\frac{5}{2}t + x\right)\right]}, \\ u_6(t, x) &= \frac{1}{2} + \frac{\epsilon}{2} \tanh\left[\frac{\epsilon}{4}\left(\frac{5}{2}t + x\right)\right], \\ u_7(t, x) &= \frac{1}{2} + \frac{\epsilon}{4} \tanh\left(\frac{\epsilon}{4}t\right) + \frac{\frac{\epsilon}{4}}{\tanh\left(\frac{\epsilon}{4}t\right)}, \\ u_8(t, x) &= \frac{1}{2} + \frac{\epsilon}{4} \tanh\left[\frac{\epsilon}{8}\left(\frac{5}{2}t + x\right)\right] + \frac{\frac{\epsilon}{4}}{\tanh\left[\frac{\epsilon}{8}\left(\frac{5}{2}t + x\right)\right]}. \end{aligned}$$

(B) Case 2: Burger-Fisher equation

Consider now the double function expansion method for the analysis of the Burger-Fisher equation [102]

$$u_{xx} + uu_x - u_t + u - u^2 = 0, \quad (4.4.30)$$

where  $u = u(t, x)$ . We first substitute into (4.4.30) the variables  $u(t, x) = v(\xi)$ ,  $\xi = \alpha t + \beta x$ , with  $\alpha, \beta \in \mathbb{R}$  to obtain

$$\beta^2 v_{\xi\xi} + \beta v v_{\xi} - \alpha v_{\xi} + v - v^2 = 0. \quad (4.4.31)$$

We take the expansion (4.2.10) to look for the solution of the equation (4.4.31) assuming that the functions  $F = F(\xi)$  and  $G = G(\xi)$  satisfy the auxiliary equation  $F'(\xi) = G(\xi)$  and  $G'(\xi) = F(\xi)$ . Balancing the degree of the nonlinear term  $vv_{\xi}$  with that of the linear term  $v_{\xi\xi}$  yields  $m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \check{m}_1 = \check{m}_2 = \bar{m}_1 = \bar{m}_2 = 2$ . This allows to take the ansatz

$$v = \sum_{i=-2}^2 \sum_{j=-2}^2 \{a_{i,j} + b_{i,j}F_{(1)}[1] + c_{i,j}G_{(1)}[1] + d_{i,j}F_{(1)}[1]G_{(1)}[1]\} F^i G^j, \quad (4.4.32)$$

where  $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$  are constants to determine. Note that  $F(\xi) = \cosh(\xi)$  and  $G(\xi) = \sinh(\xi)$ . Substitute (4.4.32) into (4.4.31), collect and set to zero the coefficients of each

power of  $\cosh^i(\xi) \sinh^j(\xi)$ . Solving the resulting system with respect to the parameters  $a_{i,j}$ ,  $b_{i,j}$ ,  $c_{i,j}$ ,  $d_{i,j}$ ,  $\alpha$ ,  $\beta$ , several solution sets are obtained leading, in addition to the solutions found in the Case 1, to the following solutions of equation (4.4.30), ( $a$  is an arbitrary constant):

$$\begin{aligned} u_9(t, x) &= a [\sinh(2t + x) + \cosh(2t + x)], \\ u_{10}(t, x) &= 1 + a [\sinh(t + x) + \cosh(t + x)], \\ u_{11}(t, x) &= a \left[ \sinh \left( t + \frac{1}{2}x \right) + \cosh \left( t + \frac{1}{2}x \right) \right]^2, \\ u_{12}(t, x) &= 1 + a \left[ \sinh \left( \frac{1}{2}t + \frac{1}{2}x \right) + \cosh \left( \frac{1}{2}t + \frac{1}{2}x \right) \right]^2. \end{aligned}$$

(C) Case 3: Fifth-order KdV equations

Consider the well known fifth-order KdV (fKdV) equations [102] in its standard form:

$$u_t + \sigma u^2 u_x + \delta u_x u_{2x} + \rho u u_{3x} + u_{5x} = 0, \quad (4.4.33)$$

where  $\sigma, \delta, \rho$  are arbitrary nonzero real parameters and  $u = u(t, x)$  is a sufficiently smooth function. A variety of the fKdV equations can be retrieved from this equation by changing the real values of the parameters  $\sigma, \delta$  and  $\rho$ . However, five well known forms of the fKdV equations are of particular interest in the literature. There are:

(C1) The Sawada-Kotera (SK) equation [93] given by

$$u_t + 5u^2 u_x + 5u_x u_{2x} + 5u u_{3x} + u_{5x} = 0; \quad (4.4.34)$$

(C2) The Caudrey-Dodd-Gibbon equation [26] given by

$$u_t + 180u^2 u_x + 30u_x u_{2x} + 30u u_{3x} + u_{5x} = 0; \quad (4.4.35)$$

(C3) The Lax equation [71] provided by

$$u_t + 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x} = 0; \quad (4.4.36)$$

(C4) The Kaup-Kupersmidt (KK) equation [69] expressed as

$$u_t + 20u^2 u_x + 25u_x u_{2x} + 10u u_{3x} + u_{5x} = 0. \quad (4.4.37)$$

(C5) The Ito equation [60] written as

$$u_t + 2u^2 u_x + 6u_x u_{2x} + 3u u_{3x} + u_{5x} = 0. \quad (4.4.38)$$

It is important to note that there is another significant fifth-order equation that appears in the literature in the form

$$u_t + \sigma u u_x + \delta u_{3x} - \rho u_{5x} = 0, \quad (4.4.39)$$

where  $\sigma, \delta, \rho$  are constants. This equation is called the Kawahara equation. The standard form of the Kawahara equation [63] is a fifth order KdV equation of the form

$$u_t + 6u u_x + u_{3x} - u_{5x} = 0, \quad (4.4.40)$$

that describes a model for plasma waves, capillary-gravity water waves. The Kawahara equation appears in the theory of shallow water with surface tension and in the theory of magneto acoustic waves in a cold collision free plasma. In [45] we have studied the Kawahara equation (4.4.39) using the symmetry group analysis method.

In the new variables  $y = \alpha t + \beta x$  and  $v(y) = u(t, x)$ , (4.4.33) is reduced to the ordinary differential equation

$$\alpha v_y + \beta^5 v_{5y} + \beta^3 \rho v v_{3y} + \beta \sigma v^2 v_y + \beta^3 \delta v_y v_{2y} = 0 \quad (4.4.41)$$

while equation (4.4.39) is reduced to

$$\alpha v_y + \beta \sigma v v_y + \beta^3 \delta v_{3y} - \beta^5 \rho v_{5y} = 0. \quad (4.4.42)$$

The single function expansion method suggests to take the ansatze

$$v(y) = \sum_{i=-2}^2 a_i F^i(y) + F'(y) \sum_{j=-2}^2 b_j F^j(y) \quad (4.4.43)$$

for the equation (4.4.41) and the ansatze

$$v(y) = \sum_{i=-4}^4 a_i F^i(y) + F'(y) \sum_{j=-4}^4 b_j F^j(y) \quad (4.4.44)$$

for the equation (4.4.42), where  $a_i, b_j$  are constants to determine and  $F$  is a sufficiently smooth function. Here, we assume that the function  $F = F(y)$  satisfies the auxiliary equation  $F'(y) = 1 - F^2(y)$ , i.e.  $F(y) = \tanh(y)$ . Thus, (4.4.43) becomes

$$\begin{aligned} v(y) &= -b_2 \tanh^4(y) - b_1 \tanh^3(y) + (a_2 - b_0 + b_2) \tanh^2(y) + (a_1 - b_{-1} + b_1) \tanh(y) \\ &+ \frac{a_{-1} + b_{-1}}{\tanh(y)} + \frac{a_{-2} + b_{-2}}{\tanh^2(y)} \end{aligned} \quad (4.4.45)$$

and (4.4.44) becomes

$$\begin{aligned} v(y) &= -b_4 \tanh^6(y) - b_3 \tanh^5(y) + (a_4 - b_2 + b_4) \tanh^4(y) + (a_3 - b_1 + b_3) \tanh^3(y) \\ &+ (a_2 - b_0 + b_2) \tanh^2(y) + a_0 - b_{-2} + b_0 + \frac{a_{-1} - b_{-3} + b_{-1}}{\tanh(y)} \\ &+ \frac{a_{-2} - b_{-4} + b_{-2}}{\tanh^2(y)} + \frac{a_{-3} + b_{-3}}{\tanh^3(y)} + \frac{a_{-4} + b_{-4}}{\tanh^4(y)}. \end{aligned} \quad (4.4.46)$$

Substitute (4.4.45) into (4.4.41) and (4.4.46) into (4.4.42). Then, collect and set to zero the coefficients of each power of  $\tanh(y)$ . Solving the resulting system of algebraic equations, several sets of parameters  $\{a_i, b_j, \alpha, \beta\}$  are obtained leading to the following results:

(i) Analytical solutions to the Kawahara equation (4.4.40):

$$\begin{aligned}
u_1(t, x) &= a_0; \\
u_2(t, x) &= a_0 - \frac{35}{169} \left[ \tanh^2(\varphi(t, x)) - \frac{1}{2} \tanh^4(\varphi(t, x)) \right]; \\
u_3(t, x) &= a_0 - \frac{35}{169} \left[ \frac{1}{\tanh^2(\varphi(t, x))} - \frac{1}{2} \frac{1}{\tanh^4(\varphi(t, x))} \right]; \\
u_4(t, x) &= a_0 + \frac{35}{5408} \left[ \tanh^4(\psi(t, x)) + \frac{1}{\tanh^4(\psi(t, x))} \right] \\
&\quad - \frac{35}{1352} \left[ \tanh^2(\psi(t, x)) + \frac{1}{\tanh^2(\psi(t, x))} \right],
\end{aligned}$$

where  $\varphi(t, x) = \frac{3\sqrt{13}(338a_0-23)}{4394}t - \frac{\sqrt{13}}{26}x$  and  $\psi(t, x) = \frac{3\sqrt{13}(2704a_0-9)}{70304}t - \frac{\sqrt{13}}{52}x$ .

(ii) Analytical solutions to the Sawada-Kotera equation (4.4.34):

$$\begin{aligned}
u_1(t, x) &= a_0; \\
u_2(t, x) &= 8\beta^2 - 12\beta^2 \tanh^2(\varphi(t, x)); \\
u_3(t, x) &= 8\beta^2 - 12\beta^2 \frac{1}{\tanh^2(\varphi(t, x))}; \\
u_4(t, x) &= a_0 - 6\beta^2 \tanh^2(\bar{\varphi}(t, x)); \\
u_5(t, x) &= a_0 - 6\beta^2 \frac{1}{\tanh^2(\bar{\varphi}(t, x))}; \\
u_6(t, x) &= 8\beta^2 - 12\beta^2 \left[ \tanh^2(\psi(t, x)) + \frac{1}{\tanh^2(\psi(t, x))} \right]; \\
u_7(t, x) &= a_0 - 6\beta^2 \left[ \tanh^2(\bar{\psi}(t, x)) + \frac{1}{\tanh^2(\bar{\psi}(t, x))} \right],
\end{aligned}$$

where

$$\begin{aligned}
\varphi(t, x) &= 16\beta^5 t - \beta x, \quad \psi(t, x) = 256\beta^5 t - \beta x, \\
\bar{\varphi}(t, x) &= \beta (76\beta^4 + a_0^2 - 40\beta^2 a_0) t - \beta x, \quad \bar{\psi}(t, x) = \beta (16\beta^4 + 5a_0^2 - 40\beta^2 a_0) t - \beta x.
\end{aligned}$$

(iii) Analytical solutions to the Caudrey-Dodd-Gibbon equation (4.4.35):

$$\begin{aligned}
u_1(t, x) &= a_0; \\
u_2(t, x) &= \frac{4}{3}\beta^2 - 2\beta^2 \tanh^2(\varphi(t, x)); \\
u_3(t, x) &= \frac{4}{3}\beta^2 - 2\beta^2 \frac{1}{\tanh^2(\varphi(t, x))}; \\
u_4(t, x) &= a_0 - \beta^2 \tanh^2(\psi(t, x)); \\
u_5(t, x) &= a_0 - \beta^2 \frac{1}{\tanh^2(\psi(t, x))}; \\
u_6(t, x) &= \frac{4}{3}\beta^2 - 2\beta^2 \left[ \tanh^2(\bar{\varphi}(t, x)) + \frac{1}{\tanh^2(\bar{\varphi}(t, x))} \right]; \\
u_7(t, x) &= a_0 - \beta^2 \left[ \tanh^2(\bar{\psi}(t, x)) + \frac{1}{\tanh^2(\bar{\psi}(t, x))} \right],
\end{aligned}$$

where

$$\begin{aligned}\varphi(t, x) &= 16\beta^5 t - \beta x, & \psi(t, x) &= 4\beta (19\beta^4 + 45a_0^2 - 60\beta^2 a_0) t - \beta x, \\ \bar{\varphi}(t, x) &= 256\beta^5 t - \beta x, & \bar{\psi}(t, x) &= 4\beta (4\beta^4 + 45a_0^2 - 60\beta^2 a_0) t - \beta x.\end{aligned}$$

(iv) Analytical solutions to the Lax equation (4.4.36):

$$\begin{aligned}u_1(t, x) &= a_0; \\ u_2(t, x) &= 4\beta^2 - 6\beta^2 \tanh^2(\varphi(t, x)); \\ u_3(t, x) &= 4\beta^2 - 6\beta^2 \frac{1}{\tanh^2(\varphi(t, x))}; \\ u_4(t, x) &= a_0 - 2\beta^2 \tanh^2(\psi(t, x)); \\ u_5(t, x) &= a_0 - 2\beta^2 \frac{1}{\tanh^2(\psi(t, x))}; \\ u_6(t, x) &= 4\beta^2 - 2\beta^2 \left[ \tanh^2(\phi(t, x)) + 3 \frac{1}{\tanh^2(\phi(t, x))} \right]; \\ u_7(t, x) &= 4\beta^2 - 2\beta^2 \left[ 3 \tanh^2(\phi(t, x)) + \frac{1}{\tanh^2(\phi(t, x))} \right]; \\ u_8(t, x) &= 4\beta^2 - 6\beta^2 \left[ \tanh^2(\bar{\varphi}(t, x)) + \frac{1}{\tanh^2(\bar{\varphi}(t, x))} \right]; \\ u_9(t, x) &= a_0 - 2\beta^2 \left[ \tanh^2(\bar{\psi}(t, x)) + \frac{1}{\tanh^2(\bar{\psi}(t, x))} \right],\end{aligned}$$

where

$$\begin{aligned}\varphi(t, x) &= 56\beta^5 t - \beta x, & \psi(t, x) &= 2\beta (28\beta^4 + 15a_0^2 - 40\beta^2 a_0) t - \beta x, \\ \bar{\varphi}(t, x) &= 896\beta^5 t - \beta x, & \phi(t, x) &= 336\beta^5 t - \beta x, \\ \bar{\psi}(t, x) &= 2\beta (48\beta^4 + 15a_0^2 - 40\beta^2 a_0) t - \beta x.\end{aligned}$$

(v) Analytical solutions to the Kaup-Kupersmidt equation (4.4.37):

$$\begin{aligned}u_1(t, x) &= a_0; \\ u_2(t, x) &= \beta^2 - \frac{3}{2}\beta^2 \tanh^2(\varphi(t, x)); \\ u_3(t, x) &= \beta^2 - \frac{3}{2}\beta^2 \frac{1}{\tanh^2(\varphi(t, x))}; \\ u_4(t, x) &= 8\beta^2 - 12\beta^2 \tanh^2(\psi(t, x)); \\ u_5(t, x) &= 8\beta^2 - 12\beta^2 \frac{1}{\tanh^2(\psi(t, x))}; \\ u_6(t, x) &= \beta^2 - \frac{3}{2}\beta^2 \left[ \tanh^2(\bar{\varphi}(t, x)) + \frac{1}{\tanh^2(\bar{\varphi}(t, x))} \right]; \\ u_7(t, x) &= 8\beta^2 - 12\beta^2 \left[ \tanh^2(\bar{\psi}(t, x)) + \frac{1}{\tanh^2(\bar{\psi}(t, x))} \right],\end{aligned}$$

where

$$\begin{aligned}\varphi(t, x) &= \beta^5 t - \beta x, & \bar{\varphi}(t, x) &= 16\beta^5 t - \beta x, \\ \psi(t, x) &= 176\beta^5 t - \beta x, & \bar{\psi}(t, x) &= 2816\beta^5 t - \beta x.\end{aligned}$$

(vi) Analytical solutions to the Ito equation (4.4.38):

$$\begin{aligned}
u_1(t, x) &= a_0; \\
u_2(t, x) &= 4\beta^2 - 6\beta^2 \tanh^2(\beta x); \\
u_3(t, x) &= 4\beta^2 - 6\beta^2 \frac{1}{\tanh^2(\beta x)}; \\
u_4(t, x) &= 20\beta^2 - 30\beta^2 \tanh^2(96\beta^5 t - \beta x); \\
u_5(t, x) &= 20\beta^2 - 30\beta^2 \frac{1}{\tanh^2(96\beta^5 t - \beta x)}; \\
u_6(t, x) &= 4\beta^2 - 6\beta^2 \left[ \tanh^2(\beta x) + \frac{1}{\tanh^2(\beta x)} \right]; \\
u_7(t, x) &= 20\beta^2 - 30\beta^2 \left[ \tanh^2(1536\beta^5 t - \beta x) + \frac{1}{\tanh^2(1536\beta^5 t - \beta x)} \right].
\end{aligned}$$

Finally, let us emphasize that all these methods involve cumbersome computations and hence do require the use of powerful computers. Except for this disadvantage, our analysis can be straightforwardly extended to multiple function expansion methods and to the investigation of systems of partial differential equations.

# Group Invariant Solutions of Differential Equations

---

In this chapter, we describe the process to find one-parameter Lie symmetry groups of transformations for differential equations [37, 81]. Then we show how to use these symmetries for the integration of differential equations.

## 5.1 One-parameter groups of transformations

### Lie group transformations

**Definition 5.1.1** Let  $z = (z^1, \dots, z^p) \in \mathbb{R}^p$ . A set  $G$  of transformations

$$T_a : \quad \tilde{z}^i = \varphi^i(z; a), \quad i = 1, 2, \dots, p,$$

where  $a$  is a real parameter which continuously takes values in a neighbourhood  $\mathcal{D} \subset \mathbb{R}$  of  $a = 0$  and  $\varphi^i$  are differentiable functions such that  $\varphi^i(z; 0) = z^i$ , is a continuous one-parameter (local) Lie group of transformations in  $\mathbb{R}^p$  provided the group properties of closure, identity and inverse are satisfied, namely:

(i) **Closure:** If  $T_a, T_b \in G$  and  $a, b \in \mathcal{D}' \subset \mathcal{D}$ , then

$$T_a T_b = T_c \in G, \quad c = \sigma(a, b) \in \mathcal{D}.$$

(ii) **Identity:** There exists  $T_0 \in G$  such that

$$T_0 T_a = T_a T_0 = T_a$$

for any  $a \in \mathcal{D}' \subset \mathcal{D}$ .  $T_0$  is known as the identity of the group.

(iii) **Inverse:** For any  $T_a \in G$ ,  $a \in \mathcal{D}' \subset \mathcal{D}$ , there exists  $T_a^{-1} = T_{a^{-1}} \in G$ , with  $a^{-1} \in \mathcal{D}$  such that

$$T_a^{-1} T_a = T_a T_a^{-1} = T_0.$$

In the sequel, we use the term "one-parameter group" to mean a continuous one-parameter (local) group.

**Example 5.1.1** The set of transformations

$$T_a : \quad \tilde{x} = x + ax_0, \quad a \in \mathbb{R}$$

where  $x_0$  is a fixed real number, forms a one-parameter group.

---

- The composition of the above transformations is

$$T_a T_b : \quad \tilde{x} = x + (a + b)x_0.$$

therefore  $\tilde{x} = x + cx_0$ , where  $c = \sigma(a, b) = a + b$ .

- Since  $\tilde{x} = x$  when  $a = 0$ ,  $T_0$  is the identity.
- It can be shown that

$$T_{-a} T_a = T_a T_{-a} = T_0.$$

Thus, the group property of inverses holds.

Hence,  $T_a$  is a one-parameter group known as the group of translations.

**Example 5.1.2** The set of transformations

$$T_a : \quad (\tilde{x}, \tilde{y}) = \left( \frac{x}{1 - ax}, \frac{y}{1 - ax} \right), \quad -1 < a < 1,$$

where  $(x, y) \in \{(x, y) \in \mathbb{R}^2 : |x| < 1, y \in \mathbb{R}\}$ , forms a one-parameter group.

- The composition of the above transformations is

$$T_a T_b : \quad (\tilde{x}, \tilde{y}) = \left( \frac{x}{1 - (a + b)x}, \frac{y}{1 - (a + b)x} \right).$$

Therefore  $(\tilde{x}, \tilde{y}) = \left( \frac{x}{1 - cx}, \frac{y}{1 - cx} \right)$ , where  $c = \sigma(a, b) = a + b$ .

- Since  $(\tilde{x}, \tilde{y}) = (x, y)$  when  $a = 0$ ,  $T_0$  is the identity.
- It can be shown that

$$T_{-a} T_a = T_a T_{-a} = T_0.$$

Thus, the group property of inverse holds.

Hence,  $T_a$  is a one-parameter group known as the projective group.

**Example 5.1.3** The set of transformations

$$T_a : \quad (\tilde{x}, \tilde{y}) = (x \cos a - y \sin a, x \sin a + y \cos a), \quad a \in \mathbb{R},$$

forms a one-parameter group.

- The composition of the above transformations is

$$T_a T_b : \quad (\tilde{x}, \tilde{y}) = (x \cos(a + b) - y \sin(a + b), x \sin(a + b) + y \cos(a + b)).$$

Therefore  $(\tilde{x}, \tilde{y}) = (x \cos c - y \sin c, x \sin c + y \cos c)$ , where  $c = \sigma(a, b) = a + b$ .

- Since  $(\tilde{x}, \tilde{y}) = (x, y)$  when  $a = 0$ ,  $T_0$  is the identity.
- It can be shown that

$$T_{-a} T_a = T_a T_{-a} = T_0.$$

Thus, the group property of inverse holds.

Hence,  $T_a$  is a one-parameter group known as the group of rotations in plane.

**Example 5.1.4** The set of transformations

$$T_a : \quad \tilde{x} = (1 + a)x, \quad a > -1,$$

forms a one-parameter group.

- The composition of the transformations is

$$T_a T_b : \quad \tilde{\tilde{x}} = (1 + a)(1 + b)x = (1 + a + b + ab)x.$$

Therefore  $\tilde{\tilde{x}} = (1 + c)x$ , where  $c = \sigma(a, b) = a + b + ab$ .

- Since  $\tilde{x} = x$  when  $a = 0$ ,  $T_0$  is the identity.
- It can be shown that

$$T_{\frac{-a}{1+a}} T_a = T_a T_{\frac{-a}{1+a}} = T_0.$$

Thus, the group property of the inverse holds.

Hence,  $T_a$  is a one-parameter group known as the scaling or dilatation group.

The choice of the parameter of a one-parameter group is not unique. Different choices of this parameter might produce different composition laws. It is desirable to have an additive composition law.

**Definition 5.1.2** A group parameter  $a$  is canonical if the group composition law is additive, that is  $\sigma(a, b) = a + b$ .

The following theorem shows that a canonical parameter can always be constructed.

**Theorem 5.1.1** For any composition law  $\sigma(a, b)$ , there exists a canonical parameter  $\bar{a}$  defined by

$$\bar{a} = \int_0^a \frac{da'}{\chi(a')},$$

where

$$\chi(a') = \left. \frac{\partial \sigma(a', b)}{\partial b} \right|_{b=0}.$$

**Example 5.1.5** Consider the following one-parameter group of transformations from Example 5.1.4,

$$T_a : \quad \tilde{x} = (1 + a)x, \quad a > -1.$$

It has been shown that the composition law is  $\sigma(a, b) = a + b + ab$ . Hence the parameter  $a$  is not canonical. Using Theorem 5.1.1 above we have

$$\chi(a') = \left. \frac{\partial \sigma(a', b)}{\partial b} \right|_{b=0} = \left. \frac{\partial (a' + b + a'b)}{\partial b} \right|_{b=0} = 1 + a'$$

and

$$\bar{a} = \int_0^a \frac{da'}{1 + a'} = \ln(1 + a).$$

Therefore written in the new parameter, the group consists of transformations

$$T_{\bar{a}} : \quad \tilde{x} = e^{\bar{a}} x, \quad \bar{a} \geq 0.$$

It can be checked that the composition law is now  $\bar{\sigma}(\bar{a}, \bar{b}) = \bar{a} + \bar{b}$ . Thus the new parameter is canonical.

From now, we assume that the group parameters are canonical.

In the context of differential equations, we denote by  $x = (x^1, \dots, x^n)$  the vector of independent variables and by  $u = (u^1, \dots, u^m)$  the vector of dependent variables. Then, the tuple  $(x, u) = (x^1, \dots, x^n, u^1, \dots, u^m)$  is viewed as a local coordinate chart in an  $(n + m)$ -dimensional manifold  $M$ . The subsequent notations for a one-parameter Lie transformation group is

$$\tilde{x}^i = \varphi^i(x, u; a), \quad \tilde{u}^\alpha = \psi^\alpha(x, u; a), \quad (5.1.1)$$

where  $\varphi^i(x, u; 0) = x^i$ ,  $i = 1, 2, \dots, n$   $\psi^\alpha(x, u; 0) = u^\alpha$ ,  $\alpha = 1, 2, \dots, m$  and  $a \in \mathbb{R}$  is the parameter.

### Infinitesimal generator and Lie equations

The Lie theory allows the construction of one-parameter group from their first order approximations

$$\tilde{x}^i = x^i + a\xi^i(x, u) + \dots, \quad \tilde{u}^\alpha = u^\alpha + a\eta^\alpha(x, u) + \dots. \quad (5.1.2)$$

Equation (5.1.2) is the first order Taylor expansion of (5.1.1) about  $a = 0$  with the initial conditions

$$\varphi^i(x, u; 0) = x^i, \quad \psi^\alpha(x, u; 0) = u^\alpha.$$

Hence we have

$$\xi^i(x, u) = \left. \frac{\partial \varphi^i(x, u; a)}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, u) = \left. \frac{\partial \psi^\alpha(x, u; a)}{\partial a} \right|_{a=0}.$$

The vector  $(\xi^i, \eta^\alpha)$  is the tangent vector to the curve  $(\tilde{x}, \tilde{u})$  parametrized by  $a$ . The first order approximations (5.1.2) can be written as

$$\tilde{x}^i = (1 + aX)x^i + \dots, \quad \tilde{u}^\alpha = (1 + aX)u^\alpha + \dots,$$

where

$$X = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (5.1.3)$$

The differential operator (5.1.3) is called the infinitesimal generator of the group or vector fields in the space  $M$ . It is also known as the symmetry generator.

**Theorem 5.1.2** *For any given infinitesimal generator (5.1.3) the associated one-parameter group is obtained by the solution of the Lie equations*

$$\frac{d\tilde{x}^i}{da} = \xi^i(\tilde{x}, \tilde{u}), \quad \frac{d\tilde{u}^\alpha}{da} = \eta^\alpha(\tilde{x}, \tilde{u}) \quad (5.1.4)$$

subject to the initial conditions  $\tilde{x}^i|_{a=0} = x^i$ ,  $\tilde{u}^\alpha|_{a=0} = u^\alpha$ .

The solution of Lie equations (5.1.4) involves exponentiating the generator  $X$ , i.e.,

$$\tilde{x}^i = \exp(aX)x^i, \quad \tilde{u}^\alpha = \exp(aX)u^\alpha, \quad (5.1.5)$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2!}X^2 + \dots = \sum_{j=0}^{+\infty} \frac{a^j}{j!} X^j \quad (5.1.6)$$

is known as the Lie series operator.

**Example 5.1.6** *If we have*

$$X = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u},$$

*which group corresponds to it? Of course we have to integrate (5.1.4), that is,*

$$\frac{d\tilde{x}}{da} = \tilde{x}, \quad \frac{d\tilde{u}}{da} = \tilde{u}.$$

*The solution with initial values  $\tilde{x}(0) = x$  and  $\tilde{u}(0) = u$  is obviously the scaling transformations*

$$\tilde{x} = e^a x, \quad \tilde{u} = e^a u.$$

**Example 5.1.7** *Consider the following infinitesimal generator*

$$X = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}.$$

*The corresponding Lie equations are*

$$\frac{d\tilde{x}}{da} = -\tilde{u}, \quad \frac{d\tilde{u}}{da} = \tilde{x}$$

*subject to  $\tilde{x}(0) = x$  and  $\tilde{u}(0) = u$ . The solution of these equations leads to the one-parameter group of rotations*

$$\tilde{x} = x \cos a - u \sin a, \quad \tilde{u} = x \sin a + u \cos a.$$

*Alternatively, we can use the exponential map (5.1.5) in the form*

$$\tilde{x} = e^{aX} x, \quad \tilde{u} = e^{aX} u,$$

*where the generator  $X$  is as given above and  $e^{aX}$  is defined by (5.1.6).*

## 5.2 Invariance criteria

### Prolongation formulas

If we want to apply a point transformation (5.1.1) to a system of differential equations

$$F_\nu \left( x, u^{(s)}(x) \right) = 0, \quad \nu = 1, 2, \dots, l, \quad (5.2.7)$$

we have to know how to transform the derivatives  $u_{(k)}^\alpha[h]$ , that is how to extend (or prolong) the point transformation to the derivatives.

**Proposition 5.2.1** *The extension (prolongation) up to the  $s$ -th order derivatives of the infinitesimal generator (5.1.3) is*

$$Pr^{(s)}X = X + \sum_{\alpha=1}^m \sum_{k=1}^s \sum_{h=1}^{p_k} \eta_{k,h}^\alpha \left( x, u^{(k)} \right) \frac{\partial}{\partial u_{(k)}^\alpha[h]}, \quad (5.2.8)$$

*where the functions  $\eta_{k,h}^\alpha$  are given by the following formula*

$$\eta_{k,h}^\alpha \left( x, u^{(s)} \right) = \left( \eta^\alpha - \sum_{i=1}^n \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i} \right)_{(k)} [h] + \sum_{i=1}^n \xi^i(x, u) \frac{\partial u_{(k)}^\alpha[h]}{\partial x^i}. \quad (5.2.9)$$

*The quantity  $\eta^\alpha - \sum_{i=1}^n \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i}$  is the so-called Lie characteristic function.*

Furthermore,  $\text{Pr}^{(s)}X$  is also an infinitesimal generator to the  $s$ -th order extension of the point transformation (5.1.1), namely,

$$\tilde{x}^i = \varphi^i(x, u; a), \quad \tilde{u}^\alpha = \psi^\alpha(x, u; a), \quad \tilde{u}_{(k)}^\alpha[h] = \psi_{k,h}^\alpha(x, u^{(s)}; a) \quad (5.2.10)$$

where

$$\varphi^i(x, u; 0) = x^i, \quad \psi^\alpha(x, u; 0) = u^\alpha, \quad \psi_{k,h}^\alpha(x, u^{(s)}; 0) = u_{(k)}^\alpha[h],$$

$$\xi^i(x, u) = \frac{\partial \varphi^i(x, u; a)}{\partial a} \Big|_{a=0}, \quad \eta^\alpha(x, u) = \frac{\partial \psi^\alpha(x, u; a)}{\partial a} \Big|_{a=0},$$

$$\eta_{k,h}^\alpha(x, u^{(s)}) = \frac{\partial \psi_{k,h}^\alpha(x, u^{(s)}; a)}{\partial a} \Big|_{a=0}$$

with  $i = 1, 2, \dots, n$ ,  $\alpha = 1, 2, \dots, m$ ,  $k = 1, 2, \dots, s$ ,  $h = 1, 2, \dots, p_k$ .

**Example 5.2.1** Consider the operator

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \quad (5.2.11)$$

in which the function  $u$  depends on the variable  $x$ . The first prolongation of the operator (5.2.11) is

$$\text{Pr}^{(1)}X = X + \eta_{1,1} \frac{\partial}{\partial u_x},$$

where  $\eta_{1,1}(x, u, u_x)$  is

$$\eta_{1,1} = \eta_x + u_x \eta_u - u_x \xi_x - u_x^2 \xi_u. \quad (5.2.12)$$

The second prolongation of the operator (5.2.11) is

$$\text{Pr}^{(2)}X = X + \eta_{1,1} \frac{\partial}{\partial u_x} + \eta_{2,1} \frac{\partial}{\partial u_{2x}},$$

where  $\eta_{1,1}(x, u, u_x)$  is given by (5.2.12) and  $\eta_{2,1}(x, u, u_x, u_{2x})$  is

$$\begin{aligned} \eta_{2,1} &= \eta_{2x} + 2u_x \eta_{xu} - u_x \xi_{2x} - 2u_x^2 \xi_{xu} + u_x^2 \eta_{uu} - u_x^3 \xi_{uu} \\ &+ u_{2x} \eta_u - 2u_{2x} \xi_x - 3u_{2x} u_x \xi_u. \end{aligned} \quad (5.2.13)$$

**Example 5.2.2** Consider the operator

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (5.2.14)$$

in which the function  $u$  depends on the two variables  $t$  and  $x$ . The second prolongation of the operator (5.2.14) is

$$\text{Pr}^{(2)}X = X + \eta_{1,1} \frac{\partial}{\partial u_t} + \eta_{1,2} \frac{\partial}{\partial u_x} + \eta_{2,1} \frac{\partial}{\partial u_{2t}} + \eta_{2,2} \frac{\partial}{\partial u_{tx}} + \eta_{2,3} \frac{\partial}{\partial u_{2x}},$$

where  $\eta_{1,1}(t, x, u, u_t, u_x)$ ,  $\eta_{1,2}(t, x, u, u_t, u_x)$ ,  $\eta_{2,1}(t, x, u, u_t, u_x, u_{2t}, u_{tx}, u_{2x})$ ,  $\eta_{2,2}(t, x, u, u_t, u_x, u_{2t}, u_{tx}, u_{2x})$  and  $\eta_{2,3}(t, x, u, u_t, u_x, u_{2t}, u_{tx}, u_{2x})$  are given by

$$\eta_{1,1} = \eta_t + u_t \eta_u - u_t \xi_t^1 - u_t^2 \xi_u^1 - u_x \xi_t^2 - u_x u_t \xi_u^2, \quad (5.2.15)$$

$$\eta_{1,2} = \eta_x + u_x \eta_u - u_t \xi_x^1 - u_t u_x \xi_u^1 - u_x \xi_x^2 - u_x^2 \xi_u^2, \quad (5.2.16)$$

$$\begin{aligned} \eta_{2,1} &= \eta_{2t} + 2u_t \eta_{tu} - u_t \xi_{2t}^1 - 2u_t^2 \xi_{tu}^1 - u_x \xi_{2t}^2 - 2u_x u_t \xi_{tu}^2 + u_t^2 \eta_{2u} - u_t^3 \xi_{2u}^1 \\ &- u_x u_t^2 \xi_{2u}^2 + u_{2t} \eta_u - 2u_{2t} \xi_t^1 - 3u_{2t} u_t \xi_u^1 - u_{2t} u_x \xi_u^2 - 2u_{tx} \xi_t^2 - 2u_{tx} u_t \xi_u^2, \end{aligned} \quad (5.2.17)$$

$$\begin{aligned} \eta_{2,2} &= \eta_{tx} + u_x \eta_{tu} - u_t \xi_{tx}^1 - u_t u_x \xi_{tu}^1 - u_x \xi_{tx}^2 - u_x^2 \xi_{tu}^2 + u_t \eta_{xu} + u_t u_x \eta_{2u} \\ &- u_t^2 u_x \xi_{2u}^1 - u_t u_x \xi_{xu}^2 - u_t u_x^2 \xi_{2u}^2 - u_{2t} \xi_x^1 - u_{2t} u_x \xi_u^1 + u_{tx} \eta_u - 2u_{tx} u_t \xi_u^1 \\ &- u_{tx} \xi_x^2 - 2u_{tx} u_x \xi_u^2 - u_{tx} \xi_t^1 - u_{2x} \xi_t^2 - u_{2x} u_t \xi_u^2 - u_t^2 \xi_{xu}^1, \end{aligned} \quad (5.2.18)$$

$$\begin{aligned} \eta_{2,3} &= \eta_{2x} + 2u_x \eta_{xu} - u_t \xi^{12x} - 2u_t u_x \xi_{xu}^1 - u_x \xi_{2x}^2 - 2u_x^2 \xi_{xu}^2 + u_x^2 \eta_{2u} - u_t u_x^2 \xi_{2u}^1 \\ &- u_x^3 \xi_{2u}^2 - 2u_{tx} \xi_x^1 - 2u_{tx} u_x \xi_u^1 + u_{2x} \eta_u - u_{2x} u_t \xi_u^1 - 2u_{2x} \xi_x^2 - 3u_{2x} u_x \xi_u^2. \end{aligned} \quad (5.2.19)$$

**Example 5.2.3** Consider the operator

$$X = \xi^1(x, y, u, v) \frac{\partial}{\partial x} + \xi^2(x, y, u, v) \frac{\partial}{\partial y} + \eta^1(x, y, u, v) \frac{\partial}{\partial u} + \eta^2(x, y, u, v) \frac{\partial}{\partial v} \quad (5.2.20)$$

in which the functions  $u$  and  $v$  depend on the two variables  $x$  and  $y$ . The first prolongation of the operator (5.2.20) is

$$Pr^{(1)}X = X + \eta_{1,1}^1 \frac{\partial}{\partial u_x} + \eta_{1,2}^1 \frac{\partial}{\partial u_y} + \eta_{1,1}^2 \frac{\partial}{\partial v_x} + \eta_{1,2}^2 \frac{\partial}{\partial v_y},$$

where  $\eta_{1,1}^1(x, y, u, u_x, u_y, v, v_x, v_y)$ ,  $\eta_{1,2}^1(x, y, u, u_x, u_y, v, v_x, v_y)$ ,  $\eta_{1,1}^2(x, y, u, u_x, u_y, v, v_x, v_y)$ , and  $\eta_{1,2}^2(x, y, u, u_x, u_y, v, v_x, v_y)$  are given by

$$\eta_{1,1}^1 = \eta_x^1 + u_x \eta_u^1 + v_x \eta_v^1 - u_x \xi_x^1 - u_x^2 \xi_u^1 - u_x v_x \xi_v^1 - u_y \xi_x^2 - u_y u_x \xi_u^2 - u_y v_x \xi_v^2, \quad (5.2.21)$$

$$\eta_{1,2}^1 = \eta_y^1 + u_y \eta_u^1 + v_y \eta_v^1 - u_x \xi_y^1 - u_x u_y \xi_u^1 - u_x v_y \xi_v^1 - u_y \xi_y^2 - u_y^2 \xi_u^2 - u_y v_y \xi_v^2, \quad (5.2.22)$$

$$\eta_{1,1}^2 = \eta_x^2 + u_x \eta_u^2 + v_x \eta_v^2 - v_x \xi_x^1 - v_x u_x \xi_u^1 - v_x^2 \xi_v^1 - v_y \xi_x^2 - v_y u_x \xi_u^2 - v_y v_x \xi_v^2, \quad (5.2.23)$$

$$\eta_{1,2}^2 = \eta_y^2 + u_y \eta_u^2 + v_y \eta_v^2 - v_x \xi_y^1 - v_x u_y \xi_u^1 - v_x v_y \xi_v^1 - v_y \xi_y^2 - v_y u_y \xi_u^2 - v_y^2 \xi_v^2. \quad (5.2.24)$$

## Determining equations

**Definition 5.2.1** A group action (5.1.1) is a symmetry transformation, or for short a symmetry of a differential equation (5.2.7) if it maps solutions into solutions; the image  $\tilde{u} = \tilde{u}(\tilde{x})$  of any solution  $u = u(x)$  is again a solution. In other words, if the system (5.2.7) does not change (is invariant) under the transformation (5.2.10),  $s$ -th order extension of (5.1.1), then

$$F_\nu(\tilde{x}, \tilde{u}^{(s)}(\tilde{x})) = 0, \quad \nu = 1, 2, \dots, l, \quad (5.2.25)$$

and (5.1.1) is a symmetry group of the system (5.2.7).

By differentiating (5.2.25) with respect to the parameter  $a$ , we obtain the following symmetry condition:

**Proposition 5.2.2** *If  $X$  is an infinitesimal generator for a symmetry group of a differential equation (5.2.7), then*

$$Pr^{(s)}X \left[ F_\nu \left( x, u^{(s)}(x) \right) \right] = 0, \quad \nu = 1, 2, \dots, l. \quad (5.2.26)$$

For the converse of this proposition, the system (5.2.7) must satisfy the maximal rank condition.

**Definition 5.2.2** *A system of differential equations of the form (5.2.7) is said to be of maximal rank if the matrix*

$$J_F \left( x, u^{(s)} \right) = \left( \frac{\partial F_\nu \left( x, u^{(s)} \right)}{\partial x^i}, \frac{\partial F_\nu \left( x, u^{(s)} \right)}{\partial u_{(k)}^\alpha [h]} \right) \quad (5.2.27)$$

*is of rank  $l$  whenever  $F_\nu \left( x, u^{(s)}(x) \right) = 0$ ,  $\nu = 1, \dots, l$ . The matrix  $J_F$  has  $l$  lines ( $\nu = 1, \dots, l$ ),  $n + q_s$  columns ( $i = 1, \dots, n$ ,  $\alpha = 1, \dots, m$ ,  $k = 0, 1, \dots, s$  and  $h = 1, \dots, p_k$ ) and is also called the Jacobian matrix of the system (5.2.7).*

The maximal rank condition gives an assurance that a system of differential equations can be written in solved form, meaning that the derivatives can be isolated on the left hand side. In order to illustrate the maximal rank condition, we calculate the Jacobian matrix for the heat and Laplace equations.

**Example 5.2.4** *The Laplace equation*

$$F \left( x, y, u^{(2)}(x, y) \right) \equiv u_{xx} + u_{yy} = 0$$

*is of maximal rank, since the Jacobian matrix with respect to all the variables*

$$(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) \quad \text{is} \quad J_F = (0, 0; 0, 0; 0, 0; 1, 0, 1),$$

*which is clearly of rank 1 everywhere. However, the rather silly equivalent equation*

$$\tilde{F} \left( x, y, u^{(2)}(x, y) \right) \equiv (u_{xx} + u_{yy})^2 = 0$$

*is not of maximal rank, since*

$$J_{\tilde{F}} = (0, 0; 0, 0; 0, 0; 2(u_{xx} + u_{yy}), 0, 2(u_{xx} + u_{yy}))$$

*vanishes whenever  $(u_{xx} + u_{yy})^2 = 0$ .*

**Example 5.2.5** *For the heat equation*

$$F \left( t, x, u^{(2)}(t, x) \right) \equiv u_t - u_{xx} = 0,$$

*the Jacobian matrix with respect to all variables is*

$$J_F = \left( \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial u}; \frac{\partial F}{\partial u_t}, \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_{tt}}, \frac{\partial F}{\partial u_{tx}}, \frac{\partial F}{\partial u_{xx}} \right) = (0, 0; 0, 1, 0; 0, 0, -1)$$

*which is of rank 1 on  $F = 0$ . Hence the maximal rank condition is satisfied.*

**Proposition 5.2.3** *Let  $X$  be a generator of a point transformation. Suppose the system of differential equations (5.2.7) is of maximal rank. If*

$$\text{Pr}^{(s)}X \left[ F_\nu \left( x, u^{(s)}(x) \right) \right] = 0, \quad \nu = 1, 2, \dots, l, \quad (5.2.28)$$

*whenever  $F_\nu(x, u^{(s)}(x)) = 0$ ,  $\nu = 1, \dots, l$ , then the transformation generated by  $X$  is the symmetry group of the system (5.2.7).*

In practice, the symmetry condition (5.2.28) will often turn to be the condition that  $\text{Pr}^{(s)}X(F_\nu)$  is a linear combination of  $F_\nu$  :

$$\text{Pr}^{(s)}X(F_\nu) = \sum_{\beta=1}^l \lambda_{\nu\beta} F_\beta, \quad \nu = 1, 2, \dots, l \quad (5.2.29)$$

with some (nonconstant) coefficients  $\lambda_{\nu\beta}$ .

Equations (5.2.28) are the so-called determining equations. In general the determining equations comprise an over-determined system of linear homogeneous partial differential equations for the unknown coordinates  $\xi^i$  and  $\eta^\alpha$  of the symmetry generator  $X$ . The solutions of the determining system form a vector space, that is, any finite linear combination of symmetries is again a symmetry. This stems from the fact that the determining equations are linear.

## 5.3 Determination of Lie group of differential equations

### Lie algebra of operators

The symmetries have an additional beautiful structure: they form a Lie algebra.

**Definition 5.3.1** *A Lie algebra is formed by a vector space  $L$  over a field,  $\mathbb{F}$ , say, together with a binary operation  $[\cdot, \cdot]$ , called Lie bracket defined on  $L$  such that the following properties hold:*

(i) *Bilinearity.* If  $X_1, X_2, X_3 \in L$ , then

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3]$$

*for scalars  $a$  and  $b$  in  $\mathbb{F}$ .*

(ii) *Skew-Symmetry.* If  $X_1, X_2 \in L$ , then

$$[X_1, X_2] = -[X_2, X_1]$$

(iii) *Jacobi Identity.* If  $X_1, X_2, X_3 \in L$ , then

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0.$$

Consider a Lie algebra  $L$ . If the vector space  $L$  is finite-dimensional, its dimension is the dimension of the Lie algebra.

We take here  $\mathbb{F}$  to be the field of real numbers  $\mathbb{R}$ . We define the Lie bracket  $[\cdot, \cdot]$  on the set of vector fields  $\mathcal{V}$  as

$$[X_1, X_2] = X_1X_2 - X_2X_1, \quad \forall X_1, X_2 \in \mathcal{V}, \quad (5.3.30)$$

where for  $j = 1, 2$ ,

$$X_j = \sum_{i=1}^n \xi^{i,j}(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^{\alpha,j}(x, u) \frac{\partial}{\partial u^\alpha}.$$

The binary operation (5.3.30) makes the space of vector fields  $\mathcal{V}$  a Lie algebra.

**Theorem 5.3.1** *The set of all symmetries forms a Lie algebra called a symmetry Lie algebra.*

### Lie symmetry algorithm

Determining the Lie point symmetry of the system (5.2.7) means finding functions  $\xi^i(x, u)$  and  $\eta^\beta(x, u)$  of an infinitesimal generator  $X$  such that the symmetry conditions  $\text{Pr}^{(s)}X(F_\nu) = 0$  are satisfied. Proposition 5.2.3, when coupled with the prolongation formulae (5.2.8) along with (5.2.9), provides an effective computational procedure for finding the most general symmetry group of almost any system of partial differential equations of interest. In this procedure, one lets the coefficients  $\xi^i(x, u)$ ,  $\eta^\beta(x, u)$  of the infinitesimal generator  $X$  of a hypothetical one-parameter symmetry group of the system be unknown functions of  $x$  and  $u$ . The coefficients  $\eta_{k,h}^\alpha$  of the prolonged infinitesimal generator  $\text{Pr}^{(s)}X$  will be certain explicit expressions involving the partial derivatives of the coefficients  $\xi^i$  and  $\eta^\alpha$  with respect to both  $x$  and  $u$ . The infinitesimal criterion invariance (5.2.28) will thus involve  $x$ ,  $u$  and the derivatives of  $u$  with respect to  $x$ , as well as  $\xi^i$ ,  $\eta^\alpha$  and their partial derivatives with respect to  $x$  and  $u$ . After eliminating any dependencies among the derivatives of the function  $u$  caused by the system itself (since (5.2.28) need only to hold on solutions of the system), we can then equate the coefficients of the remaining unconstrained partial derivatives of  $u$  to zero. This will result in a large number of elementary partial differential equations for the coefficient functions  $\xi^i$ ,  $\eta^\alpha$  of the infinitesimal generator, called the determining equations for the symmetry group of the given system.

### Symmetry calculations

The following example illustrates the Lie algorithm for calculating symmetries of differential equations.

**Example 5.3.1** *Consider the equation*

$$u_{xx} = \frac{\alpha}{u^3}, \quad \alpha \neq 0 \quad (5.3.31)$$

*which is a special case of the Ermakov-Pinney equation.*

*According to Lie algorithm the vector field*

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \quad (5.3.32)$$

*is a symmetry generator of (5.3.31) if and only if we have the symmetry condition*

$$\text{Pr}^{(2)}X \left[ u_{xx} - \frac{\alpha}{u^3} \right] \Big|_{(5.3.31)} = 0,$$

*where*

$$\text{Pr}^{(2)}X = X + \eta_{1,1} \frac{\partial}{\partial u_x} + \eta_{2,1} \frac{\partial}{\partial u_{2x}}.$$

The variables  $\eta_{1,1}$  and  $\eta_{2,1}$  are given by (5.2.12)-(5.2.13).

However, the term involving  $\eta_{1,1}$  does not contribute in the calculations because there is no  $u_x$  appearing in the equation under consideration. Thus we have the determining equation

$$\left(\frac{3\alpha}{u^4}\eta + \eta_{2,1}\right) \Big|_{(5.3.31)} = 0. \quad (5.3.33)$$

Upon expansion of (5.3.33) we have

$$\eta_{2,1} = \frac{3\alpha}{u^4}\eta + \eta_{2x} + u_x(2\eta_{xu} - \xi_{2x}) + u_x^2(\eta_{2u} - 2\xi_{xu}) - u_x^3\xi_{2u} + \frac{\alpha}{u^3}(\eta_u - 2\xi_x) - \frac{3\alpha}{u^3}u_x\xi_u = 0.$$

The separation of the above expression with respect to powers of  $u_x$  yields the equations

$$\begin{aligned} (u_x)^3 : \quad & \xi_{2u} = 0, \\ (u_x)^2 : \quad & \eta_{2u} - 2\xi_{xu} = 0, \\ (u_x)^1 : \quad & 2\eta_{xu} - \xi_{2x} - \frac{3\alpha}{u^3}\xi_u = 0, \\ (u_x)^0 : \quad & \frac{3\alpha}{u^4}\eta + \eta_{2x} + \frac{\alpha}{u^3}(\eta_u - 2\xi_x) = 0. \end{aligned}$$

The general solution of the above system is

$$\xi = c_0x^2 + 2xc_1 + c_2, \quad \eta = (c_0x + c_1)u, \quad c_0, c_1, c_2 \in \mathbb{R}.$$

Thus the symmetry Lie algebra of (5.3.31) is generated by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_3 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}.$$

Hence the symmetry Lie algebra is three-dimensional and the Lie bracket of the symmetry generators is

$$[X_1, X_2] = 2X_1, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = 2X_3.$$

## 5.4 Change of variables

### Invariants

Consider a one-parameter group of transformations (5.1.1) with generator  $X$  given in (5.1.3). Suppose that  $(x, u)$  belongs to an  $(n + m)$ -dimensional manifold  $M$ .

**Definition 5.4.1** A function  $g : M \rightarrow \mathbb{R}$  is called an invariant of the group action (5.1.1) if

$$g(\tilde{x}, \tilde{u}) = g(x, u), \quad \forall (x, u) \in M.$$

**Proposition 5.4.1** A smooth real valued function  $g : M \rightarrow \mathbb{R}$  is an invariant for the one-parameter group action (5.1.1) with generator  $X$  if and only if

$$X(g(x, u)) = 0, \quad \forall (x, u) \in M.$$

**Definition 5.4.2** Let  $g^1, \dots, g^k$  be smooth real-valued functions defined on the manifold  $M$ . Then

(a)  $g^1, \dots, g^k$  are called functionally dependent if for each  $(x, u) \in M$  there is a neighbourhood  $\Lambda \times U$  of  $(x, u)$  and a smooth real-valued function  $H(z^1, \dots, z^k)$ , not identically zero on any open subset of  $\mathbb{R}^k$ , such that

$$H(g^1(x, u), \dots, g^k(x, u)) = 0, \quad \forall (x, u) \in \Lambda \times U.$$

(b)  $g^1, \dots, g^k$  are called functionally independent if there are not functionally dependent when restricted to any open subset  $\Lambda \times U \subset M$ .

**Proposition 5.4.2** Let  $g^1, \dots, g^k$  be smooth real-valued functions defined on the  $(n + m)$ -dimensional manifold  $M$ . Then  $g^1, \dots, g^k$  are functionally dependent if and only if the matrix

$$J(x, u) = \left( \frac{\partial g^\beta(x, u)}{\partial x^i}, \frac{\partial g^\beta(x, u)}{\partial u^\alpha} \right) \quad (5.4.34)$$

has rank strictly less than  $k$  for all  $(x, u) \in M$ .

The matrix  $J(x, u)$  defined by (5.4.34) has  $k$  lines ( $\beta = 1, \dots, k$ ),  $n + m$  columns ( $i = 1, \dots, n, \alpha = 1, \dots, m$ ) and is called the Jacobian matrix associated to the functions  $g^1, \dots, g^k$ .

**Proposition 5.4.3** Let  $(x_0, u_0) \in M$  such that  $X|_{(x_0, u_0)} \neq 0$ . Then there exist  $n + m - 1$  functionally independent invariants, hence  $n + m - 1$  functionally independent solutions to the linear homogeneous first order partial differential equations

$$X(g) \equiv \sum_{i=1}^n \xi^i(x, u) \frac{\partial g}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x, u) \frac{\partial g}{\partial u^\alpha} = 0 \quad (5.4.35)$$

in a neighbourhood of  $(x_0, u_0)$ .

The general solution of (5.4.35) can be found by integrating the corresponding characteristic system of ordinary differential equations, which is

$$\frac{dx^1}{\xi^1(x, u)} = \dots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \dots = \frac{du^m}{\eta^m(x, u)}. \quad (5.4.36)$$

The general solution of (5.4.36) can be written in the form

$$\zeta^1(x, u) = c_1, \dots, \zeta^{m+n-1}(x, u) = c_{m+n-1}$$

in which  $c_1, \dots, c_{m+n-1}$  are constants of integration and the  $\zeta^i(x, u)$  are functions independent of the  $c_j$ 's. It is then easily seen that the functions  $\zeta^1, \dots, \zeta^{m+n-1}$  are the required functionally independent solutions of (5.4.35). These functions form a basis of invariants of the one-parameter group generated by  $X$ . Note that the basis is not unique. Any other invariant, i.e. any other solution of (5.4.35), will necessarily be a function of  $\zeta^1, \dots, \zeta^{m+n-1}$ .

We illustrate this technique with some examples.

**Example 5.4.1** Consider a group  $G$  of dilatations in  $\mathbb{R}^3$  which has infinitesimal generator

$$X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + cu \frac{\partial}{\partial u}, \quad a, b, c \in \mathbb{R}^*.$$

The characteristic system is

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{du}{cu}.$$

or equivalently

$$\begin{cases} \frac{dx}{ax} = \frac{dy}{by} \\ \frac{dx}{ax} = \frac{du}{cu} \end{cases}$$

The solution to the first equation is

$$yx^{-\frac{b}{a}} = c_1$$

for  $c_1$  an arbitrary constant. The solution to the second equation is

$$ux^{-\frac{c}{a}} = c_2$$

for  $c_2$  an arbitrary constant. Thus, the functions

$$\zeta^1(x, y, u) = yx^{-\frac{b}{a}}, \quad \zeta^2(x, y, u) = ux^{-\frac{c}{a}}$$

provide a complete set of functionally independent invariants of the group  $G$ .

**Example 5.4.2** Consider the group  $G$  of rotations in  $\mathbb{R}^3$ . The operator of rotation in  $\mathbb{R}^3$  around the  $(u)$  axis is

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

The corresponding characteristic system is

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{du}{0}.$$

The first integral is clear:  $u = c_1$ . The second integral, found by the integration of the equation  $x dy + y dx = 0$ , is  $x^2 + y^2 = c_2$ . Therefore, the basis of invariants is

$$\{\zeta^1(x, y, u) = u, \zeta^2(x, y, u) = x^2 + y^2\}.$$

**Example 5.4.3** Consider the projective group  $G$  in  $\mathbb{R}^2$ . Its infinitesimal generator is

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.$$

The characteristic system consists of one equation

$$\frac{dx}{x^2} = \frac{dy}{xy},$$

with the obvious first integral  $\frac{y}{x} = c$  for  $c$  an arbitrary constant. Thus,  $\zeta^1(x, y, u) = \frac{y}{x}$ , or any function thereof, is the single independent invariant of the group  $G$ .

### Canonical variables

**Definition 5.4.3** *Two groups are similar if one can be transformed into the other by means of a suitable point change of variables.*

Using the chain rule we find that under the change of variables

$$y^i = h^i(x, u), \quad i = 1, 2, \dots, n, \quad v^\alpha = \theta^\alpha(x, u), \quad \alpha = 1, 2, \dots, m,$$

the generator  $X$  transforms as

$$\begin{aligned} X &= \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \\ &= \sum_{i=1}^n \xi^i \left( \sum_{k=1}^n \frac{\partial h^k}{\partial x^i} \frac{\partial}{\partial y^k} + \sum_{\beta=1}^m \frac{\partial \theta^\beta}{\partial x^i} \frac{\partial}{\partial v^\beta} \right) + \sum_{\alpha=1}^m \eta^\alpha \left( \sum_{k=1}^n \frac{\partial h^k}{\partial u^\alpha} \frac{\partial}{\partial y^k} + \sum_{\beta=1}^m \frac{\partial \theta^\beta}{\partial u^\alpha} \frac{\partial}{\partial v^\beta} \right) \\ &= \sum_{k=1}^n \left( \sum_{i=1}^n \xi^i \frac{\partial h^k}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha \frac{\partial h^k}{\partial u^\alpha} \right) \frac{\partial}{\partial y^k} + \sum_{\beta=1}^m \left( \sum_{i=1}^n \xi^i \frac{\partial \theta^\beta}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha \frac{\partial \theta^\beta}{\partial u^\alpha} \right) \frac{\partial}{\partial v^\beta}. \end{aligned}$$

In more suggestive form, we write

$$X = \sum_{k=1}^n X(y^k) \frac{\partial}{\partial y^k} + \sum_{\beta=1}^m X(v^\beta) \frac{\partial}{\partial v^\beta}.$$

Then the coefficient operators  $X(y^k)$  and  $X(v^\beta)$  are written in terms of the new variables  $(y, v)$  which can be reduced to translation along one of the directions and are known as canonical variables.

**Proposition 5.4.4** *Assume that  $X$  is not vanishing at a point  $(x_0, u_0) \in M : X|_{(x_0, u_0)} \neq 0$ . Then there is a local coordinate chart  $y = (y^1, \dots, y^{n+m})$  at  $(x_0, u_0)$  such that in terms of these coordinates, the generators (5.1.3) take the simple form*

$$X = \frac{\partial}{\partial y^1}. \quad (5.4.37)$$

(5.4.37) is called the normal or canonical form of the generator  $X$ .

**Example 5.4.4** *The canonical variables  $(y, v)$  for translations along the  $y^1$  axis are determined by the solution of the system*

$$X(y^1) = 1, \quad X(y^2) = 0, \dots, X(y^n) = 0, \quad X(v^1) = 0, \dots, X(v^m) = 0. \quad (5.4.38)$$

In order to solve the system (5.4.38), we require a basis of invariants and a solution of  $X(y^1) = 1$ .

## 5.5 The use of symmetries to ordinary differential equations

### First order ordinary differential equations

Consider the first order ordinary differential equation

$$\frac{du}{dx} = F(x, u), \quad (5.5.39)$$

where  $u = u(x)$  is a real-valued function of one independent variable  $x$ . It will be shown that if this equation is invariant under a one-parameter group of transformations, then it can be integrated by quadrature (indefinite integral). Consider the transformation

$$\tilde{x} = \varphi(x, u; a), \quad \tilde{u} = \psi(x, u; a), \quad (5.5.40)$$

where  $\varphi(x, u; 0) = x$ ,  $\psi(x, u; 0) = u$  which is a one-parameter symmetry group of (5.5.39) on an open subset  $\Lambda \times U \subset \mathbb{R}^2$ , with the infinitesimal generator

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}. \quad (5.5.41)$$

Let  $(x_0, u_0) \in \Lambda \times U$  such that  $X|_{(x_0, u_0)} \neq 0$ . According to the Proposition 5.4.4 (normal form), we can introduce new coordinates

$$y = \phi(x, u), \quad w = \zeta(x, u) \quad (5.5.42)$$

near  $(x_0, u_0)$  such that in the  $(y, w)$ -coordinates the generator  $X$  has the simple translational form  $X = \frac{\partial}{\partial w}$  provided  $\phi$  and  $\zeta$  satisfy the linear partial differential equations

$$X(\phi) \equiv \xi(x, u) \frac{\partial \phi}{\partial x} + \eta(x, u) \frac{\partial \phi}{\partial u} = 0, \quad X(\zeta) \equiv \xi(x, u) \frac{\partial \zeta}{\partial x} + \eta(x, u) \frac{\partial \zeta}{\partial u} = 1. \quad (5.5.43)$$

The first prolongation  $\text{Pr}^{(1)}X = X = \frac{\partial}{\partial w}$  also. Thus in the new coordinate system, in order to be invariant, the differential equation must be independent of  $w$ , so (5.5.39) is equivalent to the elementary equation

$$\frac{dw}{dy} = H(y) \quad (5.5.44)$$

for some function  $H$ . This equation is trivially integrated by quadrature, with

$$w = \int H(y) dy + c$$

for some constant  $c$ . Re-substituting the expressions (5.5.42) for  $w$  and  $y$ , we obtain a solution  $u = f(x)$  for our original equation in implicit form.

**Example 5.5.1** *A homogeneous equation is one of the form*

$$\frac{du}{dx} = F\left(\frac{u}{x}\right),$$

where  $F$  only depends on the ratio of  $u$  to  $x$ . Such an equation has the group of scaling transformations with infinitesimal generator

$$X = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

The new coordinates  $y, w$  satisfying (5.5.43) are given by

$$y = \frac{u}{x}, \quad w = \ln x.$$

Employing the chain rule, we find

$$\frac{du}{dx} = \frac{\frac{du}{dy}}{\frac{dx}{dy}} = \frac{x(1 + yw_y)}{x w_y} = \frac{1 + yw_y}{w_y},$$

so the equation takes the form

$$\frac{dw}{dy} = \frac{1}{F(y) - y}.$$

This has the solution

$$w = \int \frac{dy}{F(y) - y} + c,$$

which in turn defines  $u$  implicitly as a function of  $x$  once we set  $w = \ln x, y = \frac{u}{x}$ . For example, if the equation is

$$\frac{du}{dx} = \frac{u^2 + 2xu}{x^2} = \left(\frac{u}{x}\right)^2 + 2\frac{u}{x},$$

so  $F(y) = y^2 + 2y$ , then, in the coordinates  $y = \frac{u}{x}, w = \ln x$ , we have

$$\frac{dw}{dy} = \frac{1}{y^2 + y}.$$

The solution is

$$w = -\ln(1 + y^{-1}) + c,$$

This can be solved explicitly for  $u$  as a function of  $x$ :

$$u = \frac{x^2}{\tilde{c} - x},$$

where  $\tilde{c} = e^c$ .

## Higher order ordinary differential equations

Symmetry group can also be used to aid in solving higher order ordinary differential equations. Let

$$F\left(x, u^{(s)}(x)\right) = F\left(x, u, \frac{du}{dx}, \dots, \frac{d^s u}{dx^s}\right) = 0 \quad (5.5.45)$$

be a single  $s$ -th order differential equation involving the real-valued dependent variable  $u$  with one independent variable  $x$ . The basic result in this case is that if we know a one-parameter symmetry group of this equation, then we can reduce the order of the equation by one. To see this, we first choose coordinates  $y = \phi(x, u), w = \zeta(x, u)$  as in (5.5.42) such that the group transforms into a group of translation with infinitesimal generator  $X = \frac{\partial}{\partial w}$ . Employing the chain rule, we can express the derivatives of  $u$  with respect to  $y$ ,

$$\frac{d^k u}{dx^k} = \delta_k \left( y, w, \frac{dw}{dy}, \dots, \frac{d^k w}{dy^k} \right)$$

for certain function  $\delta_k$ . Substituting these expressions into our equation (5.5.45), we find an equivalent  $s$ -th order equation

$$\tilde{F}\left(y, w^{(s)}(y)\right) = \tilde{F}\left(y, w, \frac{dw}{dy}, \dots, \frac{d^s w}{dy^s}\right) = 0 \quad (5.5.46)$$

in terms of the new coordinates  $y$  and  $w$ . Moreover, since the original equation (5.5.45) has a symmetry group, so does the transformed equation. In terms of the  $(y, w)$ -coordinates, the infinitesimal generator has the trivial prolongation  $\text{Pr}^{(1)}X = X = \frac{\partial}{\partial w}$ . The infinitesimal criterion of invariance implies

$$\text{Pr}^{(1)}X\left(\tilde{F}\right) = \frac{\partial \tilde{F}}{\partial w} = 0$$

whenever  $\tilde{F} = 0$ . This means that  $\tilde{F}$  must be independent of  $w$ , i.e.

$$\tilde{F} = \tilde{F}\left(y, \frac{dw}{dy}, \dots, \frac{d^s w}{dy^s}\right).$$

Now, setting  $v = \frac{dw}{dy}$  we have an  $(s-1)$ -th order equation for  $v$

$$\tilde{F}\left(y, v, \frac{dv}{dy}, \dots, \frac{d^{s-1}v}{dy^{s-1}}\right) = 0, \quad (5.5.47)$$

whose solution provides the general solution to our original equation. Namely, if  $v = h(y)$  is a solution of (5.5.47), then  $w = \int h(y) dy + c$  is a solution of (5.5.46), and hence, replacing  $w$  and  $y$  by their expressions in terms of  $x$  and  $u$ , implicitly defines a solution of the original equation.

**Example 5.5.2** Consider a homogeneous second order linear equation

$$u_{xx} + p(x)u_x + q(x)u = 0. \quad (5.5.48)$$

This is invariant under the group of scale transformations with infinitesimal generator

$$X = u \frac{\partial}{\partial u}.$$

The coordinates  $(y, w)$  which normalize  $X$  are given by  $y = x$ ,  $w = \ln u$  (provided  $u \neq 0$ ), with

$$X = \frac{\partial}{\partial w}$$

in these coordinates. We have

$$u = e^w, \quad u_x = w_x e^w, \quad u_{xx} = (w_{xx} + w_x^2) e^w,$$

so the equation becomes

$$w_{xx} + w_x^2 + p(x)w_x + q(x) = 0,$$

which is independent of  $w$ . We have thus reconstructed the well-known transformation between a linear second order equation and a first order Riccati equation; namely  $z = w_x = \frac{u_x}{u}$  changes (5.5.48) into the Riccati equation

$$z_x = -z^2 - p(x)z - q(x).$$

### Systems of first order ordinary differential equations

If we know a one-parameter symmetry group of a system of first order ordinary differential equations, then we can find a solution by quadrature from the solution to a first order system with one fewer equation in it. A higher order system can always be replaced by an equivalent first order system, so we are justified in restricting our attention to the latter case.

**Proposition 5.5.1** *Let*

$$\frac{du^\nu}{dx} = F_\nu(x, u), \quad \nu = 1, \dots, m \quad (5.5.49)$$

*be a first order system of  $m$  ordinary differential equations. Suppose that (5.5.49) has a one-parameter symmetry group generated by*

$$X = \xi(x, u) \frac{\partial}{\partial x} + \sum_{\nu=1}^m \eta^\nu(x, u) \frac{\partial}{\partial u^\nu}. \quad (5.5.50)$$

*Then there is a change of variables  $(y, w) = \psi(x, u)$  under which the system takes the form*

$$\frac{dw^\nu}{dy} = H_\nu(y, w^1(y), \dots, w^{m-1}(y)), \quad \nu = 1, \dots, m. \quad (5.5.51)$$

*Thus the system is reduced to a system of  $m-1$  ordinary differential equations for  $w^1, \dots, w^{m-1}$  together with the quadrature*

$$w^m(y) = \int H_m(y, w^1(y), \dots, w^{m-1}(y)) dy + c.$$

**Proof.** We can locally find new coordinates  $y = \phi(x, u)$ ,  $w^\nu = \zeta^\nu(x, u)$ ,  $\nu = 1, \dots, m$ , such that  $X = \frac{\partial}{\partial w^m}$  in these coordinates. In fact,

$$\phi(x, u), \zeta^1(x, u), \dots, \zeta^{m-1}(x, u)$$

will be a complete set of functionally independent invariants of the symmetry group, so

$$X(\zeta^m) = 1, \quad X(\phi) = X(\zeta^\nu) = 0, \quad \nu = 1, \dots, m-1.$$

It is then a simple matter to check that the equivalent first order system for  $w^1, \dots, w^m$  is invariant under the translation group generated by  $X = \frac{\partial}{\partial w^m}$  if and only if the right hand sides are all independent of  $w^m$ , i.e. it is of the form (5.5.51). ■

**Example 5.5.3** *Consider an autonomous system of two equations*

$$\frac{du}{dx} = F(u, v), \quad \frac{dv}{dx} = H(u, v).$$

*Clearly*

$$X = \frac{\partial}{\partial x}$$

*generates a one-parameter symmetry group, so we can reduce this to a single first order equation plus a quadrature. The new coordinates are  $y = u$ ,  $w = v$  and  $z = x$ , in which we are viewing  $w$  and  $z$  as function of  $y$ . Then*

$$\frac{du}{dx} = \frac{1}{\frac{dz}{dy}}, \quad \frac{dv}{dx} = \frac{\frac{dw}{dy}}{\frac{dz}{dy}},$$

so we have the equivalent system

$$\frac{dw}{dy} = \frac{H(y, w)}{F(y, w)}, \quad \frac{dz}{dy} = \frac{1}{F(y, w)}.$$

We thus are left with a single first order equation for  $w = w(y)$ ; the corresponding value of  $z = z(y)$  is determined by a quadrature:

$$z = \int \frac{dy}{F(y, w)} + c.$$

If we revert to our original variables  $x, u, v$  we see that we just have the equation

$$\frac{dv}{du} = \frac{H(u, v)}{F(u, v)}$$

together with the quadrature

$$x = \int \frac{du}{F(u, v(u))} + c.$$

## 5.6 The use of symmetries to partial differential equations

### Generation of solutions by symmetry transformations

By definition, a symmetry transformation maps a solution into a solution. So if we have already known a (special) solution of partial differential equation, we can apply the symmetry transformation to obtain (possibly) new solution. To carry out this, we need the transformation

$$\tilde{x}^i = \varphi^i(x, u; a), \quad \tilde{u}^\alpha = \psi^\alpha(x, u; a) \quad (5.6.52)$$

corresponding to the infinitesimal generator

$$X = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (5.6.53)$$

That is, these transformations must satisfy the first order differential equations

$$\frac{d\tilde{x}^i}{da} = \xi^i(\tilde{x}(a), \tilde{u}(a)), \quad \frac{d\tilde{u}^\alpha}{da} = \eta^\alpha(\tilde{x}(a), \tilde{u}(a)) \quad (5.6.54)$$

with initial values  $\tilde{x}^i(0) = x^i$  and  $\tilde{u}^\alpha = u^\alpha$ . We also have to hand the inverse transformations

$$x^i = \bar{\varphi}^i(\tilde{x}, \tilde{u}; a), \quad \tilde{u}^\alpha = \bar{\psi}^\alpha(\tilde{x}, \tilde{u}; a). \quad (5.6.55)$$

Assume that (5.6.52) is a symmetry group of the system

$$F(x, u^{(s)}(x)) = 0. \quad (5.6.56)$$

To apply these transformations to a special solution

$$u = f(x) \iff G(x, u) = 0,$$

where  $G(x, u) = u - f(x)$ , one simply has to replace  $x$  and  $u$  by means of (5.6.55) to obtain

$$G[\bar{\varphi}(\tilde{x}, \tilde{u}; a), \bar{\psi}(\tilde{x}, \tilde{u}; a)] = 0$$

which can be resolved with respect to the  $\tilde{u}$  components to yield the transformed and hopefully new solution

$$\tilde{u} = g(\tilde{x}; a).$$

**Example 5.6.1** Consider the heat equation

$$u_{xx} - u_t = 0$$

and its symmetry generator

$$X = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - \left( \frac{x^2}{4} + \frac{t}{2} \right) u \frac{\partial}{\partial u}. \quad (5.6.57)$$

For this special symmetry, the differential equation (5.6.54) for the finite transformations  $\tilde{x}$ ,  $\tilde{t}$  and  $\tilde{u}$  can be read as

$$\frac{d\tilde{x}}{da} = \tilde{x}\tilde{t}, \quad \frac{d\tilde{t}}{da} = \tilde{t}^2, \quad \frac{d\tilde{u}}{da} = - \left( \frac{\tilde{x}^2}{4} + \frac{\tilde{t}}{2} \right) \tilde{u}. \quad (5.6.58)$$

The first two of these equations yield

$$\frac{d\tilde{x}}{d\tilde{x}} = \frac{d\tilde{t}}{d\tilde{t}},$$

which is integrated by

$$\frac{\tilde{x}}{x} = \frac{\tilde{t}}{t}$$

(remember that  $x$ ,  $t$  and  $u$  are initial values of  $\tilde{x}$ ,  $\tilde{t}$  and  $\tilde{u}$ , respectively). Substituting this back into (5.6.58), we obtain

$$\frac{d\tilde{x}}{da} = \frac{\tilde{x}^2 t}{x}, \quad \frac{d\tilde{t}}{da} = \tilde{t}^2$$

and thus

$$\tilde{x} = \frac{x}{1 - at}, \quad \tilde{t} = \frac{t}{1 - at}. \quad (5.6.59)$$

The last of the three equations (5.6.58) can now be written as

$$\frac{d\tilde{u}}{\tilde{u}} = - \frac{x^2 da}{4(1 - at)^2} - \frac{t da}{2(1 - at)} \quad (5.6.60)$$

which yields

$$\tilde{u} = u \sqrt{1 - at} e^{-\frac{ax^2}{4(1-at)}}. \quad (5.6.61)$$

Equations (5.6.59) and (5.6.61) are the finite symmetry transformations. Their inverse is

$$x = \frac{\tilde{x}}{1 + a\tilde{t}}, \quad t = \frac{\tilde{t}}{1 + a\tilde{t}}, \quad u = \tilde{u} \sqrt{1 + a\tilde{t}} e^{\frac{a\tilde{x}^2}{4(1+a\tilde{t})}}. \quad (5.6.62)$$

Now we can apply this symmetry transformation to any solution of the heat equation, for example, to the trivial solution  $u = A = \text{const}$ . We immediately get, from this and (5.6.62),

$$u = \frac{A}{\sqrt{1+at}} e^{-\frac{ax^2}{4(1+at)}}$$

(after having dropped the tilde); that is, we have generated a nontrivial two-parameter solution from the one-parameter solution  $u = A$ .

### Reduction of variables by symmetry transformations

Consider the system of differential equations

$$F_\beta(x, u^{(s)}(x)) = 0, \quad \beta = 1, 2, \dots, l, \quad (5.6.63)$$

where  $(x^1, \dots, x^n)$  and  $u = (u^1, \dots, u^m)$ . Assume that (5.6.63) has a one-parameter group whose infinitesimal generator is  $X$ . By a suitable change of variables,  $X$  can be transformed into its normal form. One first has to introduce  $n + m$  new variables

$$z(x, u), y(x, u) = (y^1(x, u), \dots, y^{n-1}(x, u)), v(x, u) = (v^1(x, u), \dots, v^m(x, u))$$

defined by

$$X(z) = 1, X(y^i) = X(v^\alpha) = 0, \quad i = 1, 2, \dots, n-1, \quad \alpha = 1, 2, \dots, m$$

instead of the original  $n$  independent and  $m$  dependent variables  $x$  and  $u$ . From that set of new variables, the  $n$  variables  $\{z, y^1, \dots, y^{n-1}\}$  are viewed as independent and the  $m$  variables  $\{v^1, \dots, v^m\}$  are viewed as dependent. It will then be possible to rewrite the original system (5.6.63) as

$$H_\beta(y, v^{(s)}(y)) = 0, \quad \beta = 1, 2, \dots, l, \quad (5.6.64)$$

where the  $H_\beta$  and  $v$  do not depend on  $z$ . Indeed, in the new coordinates,  $X$  takes the normal form  $X = \frac{\partial}{\partial z}$ . By construction,

$$X(v^\alpha) = \frac{\partial v^\alpha}{\partial z} = 0,$$

i.e.  $v^\alpha$  is independent of  $z$ . Furthermore, the infinitesimal invariance criterion

$$\text{Pr}^{(s)}(H_\beta) = \frac{\partial H_\beta}{\partial z} = 0$$

also implies that the functions  $H_\beta$  are independent of  $z$ . So we see that the existence of symmetry for a system of partial differential equations ensures the possibility of diminishing by one the number of variables on which the solution depends.

**Example 5.6.2** *One of the symmetry group of the heat equation*

$$u_t = u_{xx}$$

is generated by

$$X = 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}$$

and has global invariants

$$y = t, \quad v = ue^{\frac{x^2}{4t}}, \quad t > 0.$$

We find

$$u_t = \left( v_y + \frac{x^2}{4t^2}v \right) e^{-\frac{x^2}{4t}}, \quad u_{xx} = \left( \frac{x^2}{4t^2} - \frac{1}{2t} \right) v e^{-\frac{x^2}{4t}}.$$

Therefore, for the heat equation, the reduced equation for this group is a first order ordinary differential equation

$$2yv_y + v = 0,$$

despite the fact that the heat equation is a second order partial differential equation. The solution is

$$v(y) = \frac{k}{\sqrt{y}}.$$

Hence a group invariant solution is

$$u(x, t) = \frac{k}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

## Part II

# Approximate Analytical Solutions of Nonlinear Models

---



---

**Abstract:** In this part, we present the Adomian decomposition method which is a powerful tool to handle analytic approximate solutions to differential equations. Then we success to extend this method to systems of nonlinear partial differential equations with more unknowns than equations, i.e to under determined systems. These solutions are constructed in the form of series which converge if there exists a solution to the problem. The approach is illustrated on some examples. The main results of this part have been published [52].

---



# Adomian Method for PDEs

---

## 6.1 Brief description of the Adomian method

The Adomian decomposition method allows the resolution of various functional equations of algebraic, integral, differential, integro-differential and partial differential types. This method can be adapted as well to linear as to nonlinear problems. Let  $E$  be a Banach functional space. Consider the problem of finding a function  $u \in E$  which satisfies the functional equation (written into the canonical form [95])

$$u - N u = f, \quad (6.1.1)$$

where  $N : E \rightarrow E$  is a nonlinear operator and  $f \in E$  is a known function. The Adomian method consists to look for a solution  $u$  as a series

$$u = \sum_{n=0}^{+\infty} u_n \quad (6.1.2)$$

and to decompose the nonlinear term  $N u$  into a series

$$N u = \sum_{n=0}^{+\infty} A_n. \quad (6.1.3)$$

The  $A_n$  are called Adomian polynomials and are obtained by the following formula [3, 29, 30, 68]

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}. \quad (6.1.4)$$

The substitution of (6.1.2)-(6.1.3) into (6.1.1) gives

$$\sum_{n=0}^{+\infty} u_n = f + \sum_{n=0}^{+\infty} A_n. \quad (6.1.5)$$

This yields by identification

$$u_0 = f, \quad u_1 = A_0, \quad u_{n+1} = A_n, \quad n \geq 1. \quad (6.1.6)$$

In practice, all terms of series (6.1.2) cannot be determined, and the solution is approximated by the truncated series

$$\varphi_n = \sum_{i=0}^{n-1} u_i. \quad (6.1.7)$$


---

However, one can ask the following question: To which conditions the method converges? One deduces from the relation (6.1.5) that

$$\sum_{n=0}^{+\infty} A_n < +\infty \iff \sum_{n=0}^{+\infty} u_n < +\infty. \quad (6.1.8)$$

Therefore, the convergence conditions are in connection with the nonlinear operator  $N$  from which the Adomian polynomials  $A_n$  are defined. In addition, an analysis of several results concerning the convergence of the Adomian method [1, 2, 27, 28, 59, 88, 87] allows us to conclude that the properties satisfied by the operator  $N$  so that there exists a solution to the problem (6.1.1) also ensure the convergence of the series (6.1.3) and hence the convergence of the series (6.1.2). In other words, the Adomian decomposition method converges whenever a solution to the problem exists. Now, we describe the use of this method for the solution investigation for partial differential equations.

## 6.2 Theoretical considerations

Consider the partial differential equation

$$G(x, u^{(s)}(x)) = f(x) \quad (6.2.9)$$

where the nonzero positive integer  $s$  is the order of equation,  $x = (x^1, \dots, x^n)$  are independent variables,  $u = u(x)$  is the dependent variable,  $f$  is a continuous function and  $G$  is a differentiable function with respect to its arguments. Suppose that (6.2.9) can be written into the form

$$u_{(k_0)}[h_0](x) + F(x, u_{(k_1)}[h_1](x), u_{(k_2)}[h_2](x), \dots, u_{(k_r)}[h_r](x)) = f(x), \quad (6.2.10)$$

where  $r, k_i, h_i \in \mathbb{N}$  with  $\max\{k_i, i = 0, \dots, r\} = s$ ,  $h_i \in \{1, 2, \dots, p_{k_i}\}$  and  $F$  is a differential function with respect to its  $r$  arguments. Let  $L$  be the linear differential operator such that  $Lu = u_{(k_0)}[h_0]$  and

$$L = \frac{\partial^{k_0}}{(\partial x^{i_1})^{l_1} (\partial x^{i_2})^{l_2} \dots (\partial x^{i_{s_0}})^{l_{s_0}}}$$

with  $1 \leq i_1 < i_2 < \dots < i_{s_0} \leq n$ ,  $l_1 + l_2 + \dots + l_{s_0} = k_0$  and  $l_i \neq 0$ . In order to minimize the number of operations to perform in practice, it is suitable that  $L$  be the smallest order linear differential operator in  $G$ . Set  $I = \{i_1, i_2, \dots, i_{s_0}\}$ . Define  $L^{-1}$ , the inverse of the operator  $L$  by

$$\begin{aligned} L^{-1}v(x) &= \left( \int_{x_1^{i_1}}^{x_1^{i_1}} \int_{x_2^{i_1}}^{z_{l_1-1}^{i_1}} \dots \int_{x_{l_1}^{i_1}}^{z_1^{i_1}} \right) \left( \int_{x_1^{i_2}}^{x_1^{i_2}} \int_{x_2^{i_2}}^{z_{l_2-1}^{i_2}} \dots \int_{x_{l_2}^{i_2}}^{z_1^{i_2}} \right) \\ &\dots \left( \int_{x_1^{i_{s_0}}}^{x_1^{i_{s_0}}} \int_{x_2^{i_{s_0}}}^{z_{l_{s_0}-1}^{i_{s_0}}} \dots \int_{x_{l_{s_0}}^{i_{s_0}}}^{z_1^{i_{s_0}}} \right) v(y^1, y^2, \dots, y^n) \\ &\quad \left( dz_0^{i_1} dz_1^{i_1} \dots dz_{l_1-1}^{i_1} \right) \left( dz_0^{i_2} dz_1^{i_2} \dots dz_{l_2-1}^{i_2} \right) \dots \left( dz_0^{i_{s_0}} dz_1^{i_{s_0}} \dots dz_{l_{s_0}-1}^{i_{s_0}} \right), \end{aligned}$$

where  $y^j = z_0^j$  if  $j \in I$  and  $y^j = x^j$  otherwise. The constants  $x_k^i$ ,  $k = 1, 2, \dots, l_i$  as well as the order of integration are closely related to the choice of boundary conditions in such a way

that  $L^{-1}(Lu) = u(x) + g(x)$ . Here,  $g$  is a function completely determined by the boundary conditions. Thus, applying  $L^{-1}$  to both sides of (6.2.10) yields

$$u(x) = -g(x) + L^{-1} f(x) - L^{-1} F. \tag{6.2.11}$$

The Adomian decomposition method consists in searching for the function  $u$  as an infinite series

$$u(x) = \sum_{\tau=0}^{+\infty} u_{\tau}(x), \tag{6.2.12}$$

where the  $u_{\tau}$  are functions to determine. Express the differentiable function  $F$  as a series

$$F = \sum_{\tau=0}^{+\infty} A_{\tau}, \tag{6.2.13}$$

where the  $A_{\tau}$  are multivariate polynomials also called Adomian polynomials

$$A_{\tau} = \frac{1}{\tau!} \left[ \frac{d^{\tau}}{d\lambda^{\tau}} F \left( x, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_1)}[h_1], \sum_{i=0}^{\tau} \lambda^i u_{i,(k_2)}[h_2], \dots, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_r)}[h_r] \right) \right]_{\lambda=0}. \tag{6.2.14}$$

Notice that for a linear term, i.e.  $F = u_{(k)}[h]$ , its Adomian polynomials are  $A_{\tau} = u_{\tau,(k)}[h]$ . The substitution in (6.2.11) of the expressions (6.2.12) and (6.2.13) along with (6.2.14) yields

$$\sum_{\tau=0}^{+\infty} u_{\tau} = -g(x) + L^{-1} f(x) - L^{-1} \left( \sum_{\tau=0}^{+\infty} A_{\tau} \right). \tag{6.2.15}$$

The functions  $u_{\tau}$  can be obtained using the following recurrence relation

$$u_0 = -g(x) + L^{-1} f(x), \quad u_{\tau+1} = -L^{-1} A_{\tau}, \quad \tau \geq 0 \tag{6.2.16}$$

since the  $A_{\tau}$  depend only on  $u_0, u_1, \dots, u_{\tau}$ . If the functions  $u_{\tau}$  are determined for all  $\tau \in \mathbb{N}$ , then a solution  $u$  of equation (6.2.10) is immediately formed using the series (6.2.12). Otherwise, if only a finite number, says  $\nu$ , of the functions  $u_{\tau}$  is found, on can use the partial sum

$$\phi_{\nu}(x) = \sum_{\tau=0}^{\nu-1} u_{\tau}(x) \tag{6.2.17}$$

as an analytical expression of an approximative solution to equation (6.2.10).

### 6.3 Application to Kompaneets equation

The Kompaneets equation [58], also known as the photon diffusion equation, is written as

$$\frac{\partial u}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial u}{\partial x} + u + u^2 \right) \right], \tag{6.3.18}$$

where  $u = u(t, x)$ . The expanded form of equation (6.3.18) is

$$u_t = x^2 u_{xx} + (4x + x^2) u_x + 4xu + 2x^2 uu_x + 4xu^2. \tag{6.3.19}$$

Using Adomian decomposition method, we solve this Kompaneets equation subject to the initial condition

$$u(0, x) = \frac{1}{x}. \quad (6.3.20)$$

In an operator form, equation (6.3.19) becomes

$$L u = F(x, u, u_x, u_{xx}) = x^2 u_{xx} + (4x + x^2) u_x + 4xu + 2x^2 u u_x + 4xu^2, \quad (6.3.21)$$

where  $L$  is defined by

$$L = \frac{\partial}{\partial t}. \quad (6.3.22)$$

The inverse operator  $L^{-1}$  is identified by

$$L^{-1} v(t, x) = \int_0^t v(s, x) ds. \quad (6.3.23)$$

Applying  $L^{-1}$  to both sides of (6.3.21) and using the initial condition (6.3.20), we obtain

$$\begin{aligned} u(t, x) &= \frac{1}{x} + L^{-1} F(x, u, u_x, u_{xx}) \\ &= \frac{1}{x} + L^{-1} [x^2 u_{xx} + (4x + x^2) u_x + 4xu + 2x^2 u u_x + 4xu^2]. \end{aligned} \quad (6.3.24)$$

The Adomian decomposition method suggests that  $u(t, x)$  can be defined by

$$u(t, x) = \sum_{\tau=0}^{+\infty} u_{\tau}(t, x) \quad (6.3.25)$$

and the nonlinear term  $F(x, u, u_x, u_{xx})$  by

$$F = \sum_{\tau=0}^{+\infty} A_{\tau}, \quad (6.3.26)$$

where the  $A_{\tau}$  are Adomian polynomials obtained by the formula

$$A_{\tau} = \frac{1}{\tau!} \left[ \frac{d^{\tau}}{d\lambda^{\tau}} F \left( x, \sum_{i=0}^{\tau} \lambda^i u_{i,(0)}[1], \sum_{i=0}^{\tau} \lambda^i u_{i,(1)}[2], \sum_{i=0}^{\tau} \lambda^i u_{i,(2)}[3] \right) \right]_{\lambda=0} \quad (6.3.27)$$

explicitly giving:

$$\begin{aligned} A_0 &= 4xu_{0,x} + 4xu_0 + 4xu_0^2 + x^2 u_{0,2x} + x^2 u_{0,x} + 2x^2 u_0 u_{0,x}, \\ A_1 &= 4xu_{1,x} + 4xu_1 + 8xu_0 u_1 + x^2 u_{1,2x} + x^2 u_{1,x} + 2x^2 u_1 u_{0,x} + 2x^2 u_0 u_{1,x}, \\ A_2 &= 4xu_{2,x} + 4xu_2 + 4xu_1^2 + 2x^2 u_0 u_{2,x} + x^2 u_{2,2x} + 8xu_0 u_2 \\ &\quad + 2x^2 u_2 u_{0,x} + x^2 u_{2,x} + 2x^2 u_1 u_{1,x}, \\ A_3 &= 4xu_{3,x} + 4xu_3 + 8xu_1 u_2 + 2x^2 u_2 u_{1,x} + 2x^2 u_3 u_{0,x} + 8xu_0 u_3 \\ &\quad + 2x^2 u_1 u_{2,x} + 2x^2 u_0 u_{3,x} + x^2 u_{3,2x} + x^2 u_{3,x}, \\ A_4 &= 4xu_{4,x} + 4xu_4 + 4xu_2^2 + 2x^2 u_2 u_{2,x} + x^2 u_{4,2x} + x^2 u_{4,x} + 2x^2 u_4 u_{0,x} \\ &\quad + 8xu_1 u_3 + 2x^2 u_1 u_{3,x} + 2x^2 u_0 u_{4,x} + 2x^2 u_3 u_{1,x} + 8xu_0 u_4, \\ A_5 &= 4xu_{5,x} + 4xu_5 + 8xu_2 u_3 + x^2 u_{5,2x} + 8xu_1 u_4 + 2x^2 u_5 u_{0,x} + 2x^2 u_1 u_{4,x} \\ &\quad + 2x^2 u_4 u_{1,x} + 2x^2 u_0 u_{5,x} + x^2 u_{5,x} + 8xu_0 u_5 + 2x^2 u_2 u_{3,x} + 2x^2 u_3 u_{2,x}; \end{aligned}$$

and so on. Using the above assumptions gives

$$\sum_{\tau=0}^{+\infty} u_{\tau} = \frac{1}{x} + L^{-1} \left( \sum_{\tau=0}^{+\infty} A_{\tau} \right) \quad (6.3.28)$$

from which one derives the recursive relation

$$u_0(t, x) = \frac{1}{x}, \quad u_{\tau+1}(t, x) = L^{-1} A_{\tau}, \quad \tau \geq 0. \quad (6.3.29)$$

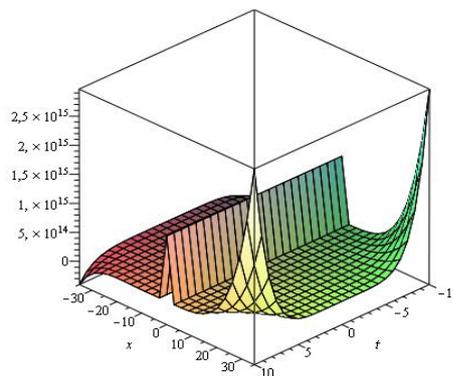
The first components are determined by:

$$\begin{aligned} u_1(t, x) &= 3t, & u_2(t, x) &= (9 + 6x)t^2, \\ u_3(t, x) &= (18 + 48x + 10x^2)t^3, \\ u_4(t, x) &= (27 + 216x + 155x^2 + 15x^3)t^4, \\ u_5(t, x) &= \left( \frac{162}{5} + 1232x^2 + \frac{3456}{5}x + \frac{1956}{5}x^3 + 21x^4 \right) t^5, \\ u_6(t, x) &= \left( \frac{162}{5} + 1728x + \frac{19988}{3}x^2 + 4874x^3 + \frac{4242}{5}x^4 + 28x^5 \right) t^6 \end{aligned}$$

permitting to write an approximate solution to Kompaneets Eq. (6.3.19) under (6.3.20) as

$$\begin{aligned} \phi(t, x) &= u_0(t, x) + u_1(t, x) + u_2(t, x) + u_3(t, x) + u_4(t, x) + u_5(t, x) + u_6(t, x) \\ &= \frac{1}{x} + 3t + (9 + 6x)t^2 + (18 + 48x + 10x^2)t^3 + (27 + 216x + 155x^2 + 15x^3)t^4 \\ &\quad + \left( \frac{162}{5} + 1232x^2 + \frac{3456}{5}x + \frac{1956}{5}x^3 + 21x^4 \right) t^5 \\ &\quad + \left( \frac{162}{5} + 1728x + \frac{19988}{3}x^2 + 4874x^3 + \frac{4242}{5}x^4 + 28x^5 \right) t^6 \end{aligned}$$

whose behavior is shown in the Figure 6.1.



Function  $\phi$  versus  $x$  and  $t$  on the range  $t = -10 \dots 10$ ,  $x = -35 \dots 35$

Figure 6.1: Function  $\phi$ , approximate solution of the Kompaneets equation, versus  $x$  and  $t$ .

## 6.4 Application to Novikov equation

The Novikov equation is expressed as follows [11]:

$$u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}, \quad (6.4.30)$$

where  $u = u(t, x)$ . Using Adomian decomposition method, we solve this equation subject to the initial condition

$$u(0, x) = x. \quad (6.4.31)$$

In an operator form, equation (6.4.30) becomes

$$Lu = F(x, u, u_x, u_{xx}, u_{xxt}, u_{xxx}) = u_{xxt} - 4u^2u_x + 3uu_xu_{xx} + u^2u_{xxx}, \quad (6.4.32)$$

where  $L$  is defined by

$$L = \frac{\partial}{\partial t} \quad (6.4.33)$$

which its inverse operator  $L^{-1}$  given by (6.3.23).

Applying  $L^{-1}$  to both sides of (6.4.32) and using the initial condition (6.4.31), we find

$$\begin{aligned} u(t, x) &= x + L^{-1} F(x, u, u_x, u_{xx}, u_{xxt}, u_{xxx}) \\ &= x + L^{-1} (u_{xxt} - 4u^2u_x + 3uu_xu_{xx} + u^2u_{xxx}). \end{aligned} \quad (6.4.34)$$

The Adomian polynomials  $A_\tau$  are given by the formula

$$\begin{aligned} A_\tau &= \frac{1}{\tau!} \left[ \frac{d^\tau}{d\lambda^\tau} F \left( x, \sum_{i=0}^{\tau} \lambda^i u_{i,(0)}[1], \sum_{i=0}^{\tau} \lambda^i u_{i,(1)}[2], \sum_{i=0}^{\tau} \lambda^i u_{i,(2)}[3], \right. \right. \\ &\quad \left. \left. \sum_{i=0}^{\tau} \lambda^i u_{i,(3)}[3], \sum_{i=0}^{\tau} \lambda^i u_{i,(3)}[4] \right) \right]_{\lambda=0}. \end{aligned} \quad (6.4.35)$$

from which are deduced the explicit expressions:

$$\begin{aligned} A_0 &= u_{0,t2x} - 4u_0^2u_{0,x} + 3u_0u_{0,x}u_{0,2x} + u_0^2u_{0,3x}, \\ A_1 &= u_{1,t2x} - 8u_0u_{0,x}u_1 + u_0^2u_{1,3x} + 2u_0u_{0,3x}u_1 - 4u_0^2u_{1,x} + 3u_0u_{0,x}u_{1,2x} \\ &\quad + 3u_0u_{1,x}u_{0,2x} + 3u_1u_{0,x}u_{0,2x}, \\ A_2 &= u_{2,t2x} - 4u_1^2u_{0,x} + 3u_0u_{1,x}u_{1,2x} - 4u_0^2u_{2,x} + 2u_0u_{1,3x}u_1 + 3u_1u_{0,x}u_{1,2x} \\ &\quad - 8u_0u_{1,x}u_1 + 3u_0u_{0,x}u_{2,2x} + 3u_2u_{0,x}u_{0,2x} + 2u_0u_{0,3x}u_2 + 3u_0u_{2,x}u_{0,2x} \\ &\quad - 8u_0u_{0,x}u_2 + u_1^2u_{0,3x} + 3u_1u_{1,x}u_{0,2x} + u_0^2u_{2,3x}, \\ A_3 &= u_{3,t2x} + 3u_0u_{0,x}u_{3,2x} + 2u_0u_{2,3x}u_1 + 3u_1u_{2,x}u_{0,2x} + 3u_1u_{0,x}u_{2,2x} - 8u_0u_{2,x}u_1 \\ &\quad + u_0^2u_{3,3x} - 4u_0^2u_{3,x} + 2u_1u_{0,3x}u_2 + 2u_0u_{1,3x}u_2 + 3u_1u_{1,x}u_{1,2x} + 3u_0u_{2,x}u_{1,2x} \\ &\quad - 8u_0u_{1,x}u_2 - 4u_1^2u_{1,x} + u_1^2u_{1,3x} - 8u_0u_{0,x}u_3 + 2u_0u_{0,3x}u_3 + 3u_0u_{1,x}u_{2,2x} \\ &\quad + 3u_0u_{3,x}u_{0,2x} + 3u_3u_{0,x}u_{0,2x} + 3u_2u_{0,x}u_{1,2x} + 3u_2u_{1,x}u_{0,2x} - 8u_1u_{0,x}u_2; \end{aligned}$$

and so on. Use the relation

$$\sum_{\tau=0}^{+\infty} u_\tau = x + L^{-1} \left( \sum_{\tau=0}^{+\infty} A_\tau \right) \quad (6.4.36)$$

from which results the following recursive relation

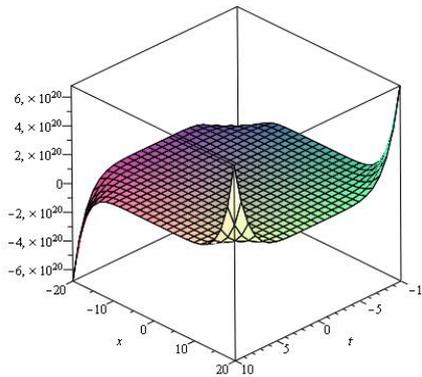
$$u_0(t, x) = x, \quad u_{\tau+1}(t, x) = L^{-1} A_{\tau}, \quad \tau \geq 0 \quad (6.4.37)$$

generating the functions:

$$\begin{aligned} u_1(t, x) &= -4x^2t, & u_2(t, x) &= (-12x + 32x^3)t^2 - 8t, \\ u_3(t, x) &= (400x^2 - 320x^4)t^3 + 224xt^2, \\ u_4(t, x) &= (744x - 9856x^3 + 3584x^5)t^4 + (864 - 4992x^2)t^3, \\ u_5(t, x) &= \left(-\frac{326688}{5}x^2 + 214272x^4 - 43008x^6\right)t^5 + (-73728x + 101376x^3)t^4 - \frac{30208}{3}t^3, \\ u_6(t, x) &= \left(-\frac{407328}{5}x + \frac{16407808}{5}x^3 - 4347904x^5 + 540672x^7\right)t^6 \\ &+ \left(-\frac{695616}{5} + 3563520x^2 - 1955840x^4\right)t^5 + \frac{1908224}{3}xt^4 \end{aligned}$$

which serve to express an approximate analytical solution to the Novikov equation (6.4.30) with the initial condition (6.4.31) as follows: see Figure 6.2 for its graphic representation.

$$\begin{aligned} \phi(t, x) &= u_0(t, x) + u_1(t, x) + u_2(t, x) + u_3(t, x) + u_4(t, x) + u_5(t, x) + u_6(t, x) \\ &= x - (4x^2 + 8)t + (32x^3 + 212x)t^2 - \left(320x^4 + 4592x^2 + \frac{27616}{3}\right)t^3 \\ &+ \left(\frac{1689272}{3}x + 91520x^3 + 3584x^5\right)t^4 \\ &+ \left(-43008x^6 - 1741568x^4 + \frac{17490912}{5}x^2 - \frac{695616}{5}\right)t^5 \\ &+ \left(-\frac{407328}{5}x + \frac{16407808}{5}x^3 - 4347904x^5 + 540672x^7\right)t^6. \end{aligned}$$



Function  $\phi$  versus  $x$  and  $t$  on the range  $t = -10 \dots 10$ ,  $x = -20 \dots 20$

Figure 6.2: Function  $\phi$ , approximate solution of the Novikov equation, versus  $x$  and  $t$ .



# Adomian Method for Determined Systems of PDEs

## 7.1 Theoretical considerations

Consider the system of partial differential equations

$$G_j \left( x, u^{(s)}(x) \right) = f_j(x), \quad j = 1, 2, \dots, m \quad (7.1.1)$$

where the nonzero positive integer  $s$  is the order of system,  $x = (x^1, \dots, x^n)$  are independent variables,  $u = u(x) = (u^1(x), u^2(x), \dots, u^m(x))$  are the dependent variables, the  $f_j$  are continuous functions and the  $G_j$  are differentiable functions with respect to their arguments. Suppose that (7.1.1) can be written into the form

$$u_{(k_j,0)}^j [h_{j,0}](x) + F_j \left( x, u_{(k_j,1)}^{\alpha_{j,1}} [h_{j,1}](x), u_{(k_j,2)}^{\alpha_{j,2}} [h_{j,2}](x), \dots, u_{(k_j,r_j)}^{\alpha_{j,r_j}} [h_{j,r_j}](x) \right) = f_j(x), \quad (7.1.2)$$

where  $r_j, k_{j,i}, h_{j,i} \in \mathbb{N}$  with  $\max \{k_{j,i} \mid j = 1, 2, \dots, m, i = 0, \dots, r_j\} = s$ ,  $h_{j,i} \in \{1, 2, \dots, p_{k_{j,i}}\}$ ,  $\alpha_{j,i} \in \{1, 2, \dots, m\}$  and  $F_j$  is a differentiable function with respect to its  $r_j$  arguments. Let  $L_j$  be the linear differential operator such that  $L_j v = v_{(k_j,0)} [h_{j,0}]$  and

$$L_j = \frac{\partial^{k_{j,0}}}{(\partial x^{\beta_{j,1}})^{l_{j,1}} (\partial x^{\beta_{j,2}})^{l_{j,2}} \dots (\partial x^{\beta_{j,s_j}})^{l_{j,s_j}}}$$

with  $1 \leq \beta_{j,1} < \beta_{j,2} < \dots < \beta_{j,s_j} \leq n$ ,  $l_{j,1} + l_{j,2} + \dots + l_{j,s_j} = k_{j,0}$  and  $l_{j,i} \neq 0$ . As in the previous case, it is suitable that  $L_j$  be the smallest order linear differential operator in  $G_j$ . Set  $I_j = \{\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,s_j}\}$ . Define  $L_j^{-1}$ , the inverse of the operator  $L_j$ , by

$$\begin{aligned} L_j^{-1} v(x) &= \left( \int_{x_1^{\beta_{j,1}}}^{x^{\beta_{j,1}}} \int_{x_2^{\beta_{j,1}}}^{z_{l_{j,1}-1}^{\beta_{j,1}}} \dots \int_{x_{l_{j,1}}^{\beta_{j,1}}}^{z_1^{\beta_{j,1}}} \right) \left( \int_{x_1^{\beta_{j,2}}}^{x^{\beta_{j,2}}} \int_{x_2^{\beta_{j,2}}}^{z_{l_{j,2}-1}^{\beta_{j,2}}} \dots \int_{x_{l_{j,2}}^{\beta_{j,2}}}^{z_1^{\beta_{j,2}}} \right) \\ &\dots \left( \int_{x_1^{\beta_{j,s_j}}}^{x^{\beta_{j,s_j}}} \int_{x_2^{\beta_{j,s_j}}}^{z_{l_{j,s_j}-1}^{\beta_{j,s_j}}} \dots \int_{x_{l_{j,s_j}}^{\beta_{j,s_j}}}^{z_1^{\beta_{j,s_j}}} \right) v(y^1, y^2, \dots, y^n) \left( dz_0^{\beta_{j,1}} dz_1^{\beta_{j,1}} \dots dz_{l_{j,1}-1}^{\beta_{j,1}} \right) \\ &\left( dz_0^{\beta_{j,2}} dz_1^{\beta_{j,2}} \dots dz_{l_{j,2}-1}^{\beta_{j,2}} \right) \dots \left( dz_0^{\beta_{j,s_j}} dz_1^{\beta_{j,s_j}} \dots dz_{l_{j,s_j}-1}^{\beta_{j,s_j}} \right), \end{aligned}$$

where  $y^i = z_0^i$  if  $i \in I_j$  and  $y^i = x^i$  otherwise. The constants  $x_k^i$  and the order of integration are closely related to the choice of boundary conditions in such a way that  $L_j^{-1}(L_j v) = v(x) + g_j(x)$ . Here,  $g_j$  is a function completely determined by the boundary conditions. Thus, applying  $L_j^{-1}$  to both sides of (7.1.2) yields

$$u^j(x) = -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j. \quad (7.1.3)$$

Now as required in this case, the functions  $u^j$  can be derived as infinite series:

$$u^j(x) = \sum_{\tau=0}^{+\infty} u_{\tau}^j(x), \quad (7.1.4)$$

where the  $u_{\tau}^j$  are functions to determine. Express the differentiable functions  $F_j$  as series

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \quad (7.1.5)$$

where the  $A_{j,\tau}$  are multivariate Adomian polynomials:

$$A_{j,\tau} = \frac{1}{\tau!} \left[ \frac{d^{\tau}}{d\lambda^{\tau}} F_j \left( x, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,1})}^{\alpha_{j,1}} [h_{j,1}], \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,2})}^{\alpha_{j,2}} [h_{j,2}], \dots, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,r_j})}^{\alpha_{j,r_j}} [h_{j,r_j}] \right) \right]_{\lambda=0}. \quad (7.1.6)$$

The substitution in (7.1.3) of the expressions (7.1.4) and (7.1.5) along with (7.1.6) yields

$$\sum_{\tau=0}^{+\infty} u_{\tau}^j = -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right). \quad (7.1.7)$$

The functions  $u_{\tau}^j$  can be then obtained using the following recurrence relation

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad u_{\tau+1}^j = -L_j^{-1} A_{j,\tau}, \quad \tau \geq 0, \quad j = 1, 2, \dots, m \quad (7.1.8)$$

since the  $A_{j,\tau}$  depend only on  $u_0^{\beta}, u_1^{\beta}, \dots, u_{\tau}^{\beta}, \beta = 1, 2, \dots, m$ . If the functions  $u_{\tau}^j$  are determined for all  $\tau \in \mathbb{N}$ , then a solution  $u$  of system (7.1.2) is immediately formed using the series (7.1.4). Otherwise, if only a finite number, says  $\nu$ , of the functions  $u_{\tau}^j$  is found, one can use the partial sums

$$\phi_{\nu}^j(x) = \sum_{\tau=0}^{\nu-1} u_{\tau}^j(x) \quad (7.1.9)$$

and consider the function  $\phi_{\nu} = (\phi_{\nu}^1, \phi_{\nu}^2, \dots, \phi_{\nu}^m)$  as an analytical approximate solution to the system of differential equations (7.1.2).

## 7.2 Application to Ginzburg-Landau equations

Consider the Ginzburg-Landau (GL) equation [85, 86]

$$\begin{cases} \eta_1 \left( \frac{\partial \psi}{\partial t} + i\kappa \phi \psi \right) + \left( \frac{i}{\kappa} \nabla + A \right)^2 \psi - \psi + |\psi|^2 \psi = 0 \\ \eta_2 \left( \frac{\partial A}{\partial t} + \nabla \phi \right) + \text{curl curl } A + \frac{i}{2\kappa} (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) + |\psi|^2 A = 0, \end{cases} \quad (7.2.10)$$

where

$$\begin{aligned} A(t, \cdot) : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (A^1(t, x, y, z), A^2(t, x, y, z), A^3(t, x, y, z)) \end{aligned}$$

is the real vector-valued magnetic potential;

$$\begin{aligned} \psi(t, \cdot) : \mathbb{R}^3 &\longrightarrow \mathbb{C} \\ (x, y, z) &\longmapsto \psi^1(t, x, y, z) + i\psi^2(t, x, y, z) \end{aligned}$$

is the complex scalar-valued order parameter;

$$\begin{aligned}\phi(t, \cdot) : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto \phi(t, x, y, z)\end{aligned}$$

is the real scalar-valued electric potential;  $\eta_1, \eta_2$  are some given relaxation parameters and  $\kappa$  is the Ginzburg-Landau parameter. Assume that  $\phi = -\operatorname{div}(A) = -(A_x^1 + A_y^2 + A_z^3)$ . Adopting the following notation:  $u^1 = A^1$ ,  $u^2 = A^2$ ,  $u^3 = A^3$ ,  $u^4 = \psi^1$ ,  $u^5 = \psi^2$ , and after some computation and identification, equations (7.2.10) become the determined system

$$u_t^j = F_j(u, u_{(1)}, u_{(2)}), \quad j = 1, 2, 3, 4, 5, \quad (7.2.11)$$

where

$$\begin{aligned}F_1 &= -\frac{1}{\eta_2} \left[ -\eta_2 u_{2x}^1 - u_{2y}^1 - u_{2z}^1 + (1 - \eta_2)(u_{xy}^2 + u_{xz}^3) - \frac{1}{\kappa}(u^4 u_x^5 - u^5 u_x^4) \right. \\ &\quad \left. + u^1 \left( (u^4)^2 + (u^5)^2 \right) \right],\end{aligned}$$

$$\begin{aligned}F_2 &= -\frac{1}{\eta_2} \left[ -u_{2x}^2 - \eta_2 u_{2y}^2 - u_{2z}^2 + (1 - \eta_2)(u_{xy}^1 + u_{yz}^3) - \frac{1}{\kappa}(u^4 u_y^5 - u^5 u_y^4) \right. \\ &\quad \left. + u^2 \left( (u^4)^2 + (u^5)^2 \right) \right],\end{aligned}$$

$$\begin{aligned}F_3 &= -\frac{1}{\eta_2} \left[ -u_{2x}^3 - u_{2y}^3 - \eta_2 u_{2z}^3 + (1 - \eta_2)(u_{xz}^1 + u_{yz}^2) - \frac{1}{\kappa}(u^4 u_z^5 - u^5 u_z^4) \right. \\ &\quad \left. + u^3 \left( (u^4)^2 + (u^5)^2 \right) \right],\end{aligned}$$

$$\begin{aligned}F_4 &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2}(u_{2x}^4 + u_{2y}^4 + u_{2z}^4) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u^5 (u_x^1 + u_y^2 + u_z^3) \right. \\ &\quad \left. - \frac{1}{\kappa}(u^1 u_x^5 + u^2 u_y^5 + u^3 u_z^5) + u^4 \left( (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (u^5)^2 - 1 \right) \right],\end{aligned}$$

$$\begin{aligned}F_5 &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2}(u_{2x}^5 + u_{2y}^5 + u_{2z}^5) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u^4 (u_x^1 + u_y^2 + u_z^3) \right. \\ &\quad \left. - \frac{1}{\kappa}(u^1 u_x^4 + u^2 u_y^4 + u^3 u_z^4) + u^5 \left( (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (u^5)^2 - 1 \right) \right].\end{aligned}$$

Using Adomian decomposition method, we solve this system subject to the initial conditions

$$u^j(0, x, y, z) = x + y + z, \quad j = 1, 2, 3, 4, 5. \quad (7.2.12)$$

Consider the differential operator  $L$

$$L = \frac{\partial}{\partial t} \quad (7.2.13)$$

and define its inverse operator  $L^{-1}$  by

$$L^{-1} v(t, x, y, z) = \int_0^t v(s, x, y, z) ds. \quad (7.2.14)$$

Applying  $L^{-1}$  to both sides of (7.2.11) and using the initial condition (7.2.12), we find

$$u^j(t, x, y, z) = x + y + z + L^{-1} F_j(u, u_{(1)}, u_{(2)}), \quad j = 1, 2, 3, 4, 5. \quad (7.2.15)$$

The Adomian decomposition method suggests that the functions  $u^j$  can be seek as

$$u^j(t, x, y, z) = \sum_{\tau=0}^{+\infty} u_{\tau}^j(t, x, y, z) \quad (7.2.16)$$

while the differentiable functions  $F_j$  as series

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \quad (7.2.17)$$

where the Adomian polynomials  $A_{j,\tau}$  are obtained by the formula

$$A_{j,\tau} = \frac{1}{\tau!} \left[ \frac{d^{\tau}}{d\lambda^{\tau}} F_j \left( \sum_{i=0}^{\tau} \lambda^i u_i, \sum_{i=0}^{\tau} \lambda^i u_{i,(1)}, \sum_{i=0}^{\tau} \lambda^i u_{i,(2)} \right) \right]_{\lambda=0} \quad (7.2.18)$$

from which we get:

$$\begin{aligned} A_{1,0} &= -\frac{1}{\eta_2} \left[ -\eta_2 u_{0,2x}^1 - u_{0,2y}^1 - u_{0,2z}^1 + (1 - \eta_2) (u_{0,xy}^2 + u_{0,xz}^3) - \frac{1}{\kappa} (u_0^4 u_{0,x}^5 - u_0^5 u_{0,x}^4) \right. \\ &\quad \left. + u_0^1 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right], \end{aligned}$$

$$\begin{aligned} A_{2,0} &= -\frac{1}{\eta_2} \left[ -u_{0,2x}^2 - \eta_2 u_{0,2y}^2 - u_{0,2z}^2 + (1 - \eta_2) (u_{0,xy}^1 + u_{0,yz}^3) - \frac{1}{\kappa} (u_0^4 u_{0,y}^5 - u_0^5 u_{0,y}^4) \right. \\ &\quad \left. + u_0^2 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right], \end{aligned}$$

$$\begin{aligned} A_{3,0} &= -\frac{1}{\eta_2} \left[ -u_{0,2x}^3 - u_{0,2y}^3 - \eta_2 u_{0,2z}^3 + (1 - \eta_2) (u_{0,xz}^1 + u_{0,yz}^2) - \frac{1}{\kappa} (u_0^4 u_{0,z}^5 - u_0^5 u_{0,z}^4) \right. \\ &\quad \left. + u_0^3 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right], \end{aligned}$$

$$\begin{aligned} A_{4,0} &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{0,2x}^4 + u_{0,2y}^4 + u_{0,2z}^4) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_0^5 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) \right. \\ &\quad - \frac{1}{\kappa} (u_0^1 u_{0,x}^5 + u_0^2 u_{0,y}^5 + u_0^3 u_{0,z}^5) \\ &\quad \left. + u_0^4 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right], \end{aligned}$$

$$\begin{aligned} A_{5,0} &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{0,2x}^5 + u_{0,2y}^5 + u_{0,2z}^5) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_0^4 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) \right. \\ &\quad - \frac{1}{\kappa} (u_0^1 u_{0,x}^4 + u_0^2 u_{0,y}^4 + u_0^3 u_{0,z}^4) \\ &\quad \left. + u_0^5 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right], \end{aligned}$$

$$\begin{aligned}
A_{1,1} &= -\frac{1}{\eta_2} \left[ -\eta_2 u_{1,2x}^1 - u_{1,2y}^1 - u_{1,2z}^1 + (1 - \eta_2) (u_{1,xy}^2 + u_{1,xz}^3) + u_1^1 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right. \\
&\quad \left. - \frac{1}{\kappa} (u_1^4 u_{0,x}^5 + u_0^4 u_{1,x}^5 - u_1^5 u_{0,x}^4 - u_0^5 u_{1,x}^4) + u_0^1 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) \right],
\end{aligned}$$

$$\begin{aligned}
A_{2,1} &= -\frac{1}{\eta_2} \left[ -u_{1,2x}^2 - \eta_2 u_{1,2y}^2 - u_{1,2z}^2 + (1 - \eta_2) (u_{1,xy}^1 + u_{1,yz}^3) + u_1^2 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right. \\
&\quad \left. - \frac{1}{\kappa} (u^4 u_{0,y}^5 + u_0^4 u_{1,y}^5 - u_1^5 u_{0,y}^4 - u_0^5 u_{1,y}^4) + u_0^2 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) \right],
\end{aligned}$$

$$\begin{aligned}
A_{3,1} &= -\frac{1}{\eta_2} \left[ -u_{1,2x}^3 - u_{1,2y}^3 - \eta_2 u_{1,2z}^3 + (1 - \eta_2) (u_{1,xz}^1 + u_{1,yz}^2) + u_1^3 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right. \\
&\quad \left. - \frac{1}{\kappa} (u_1^4 u_{0,z}^5 + u_0^4 u_{1,z}^5 - u_1^5 u_{0,z}^4 - u_0^5 u_{1,z}^4) + u_0^3 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) \right],
\end{aligned}$$

$$\begin{aligned}
A_{4,1} &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{1,2x}^4 + u_{1,2y}^4 + u_{1,2z}^4) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_1^5 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) \right. \\
&\quad + u_0^4 (2u_0^1 u_1^1 + 2u_0^3 u_1^3 + 2u_0^2 u_1^2 + 2u_0^5 u_1^5 + 2u_0^4 u_1^4) \\
&\quad + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_0^5 (u_{1,x}^1 + u_{1,y}^2 + u^3 1, z) \\
&\quad - \frac{1}{\kappa} (u_1^1 u_{0,x}^5 + u_0^1 u_{1,x}^5 + u_1^2 u_{0,y}^5 + u_0^2 u_{1,y}^5 + u_1^3 u_{0,z}^5 + u_0^3 u_{1,z}^5) \\
&\quad \left. + u_1^4 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right],
\end{aligned}$$

$$\begin{aligned}
A_{5,1} &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{1,2x}^5 + u_{1,2y}^5 + u_{1,2z}^5) + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_1^4 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) \right. \\
&\quad + u_0^5 (2u_0^1 u_1^1 + 2u_0^3 u_1^3 + 2u_0^2 u_1^2 + 2u_0^5 u_1^5 + 2u_0^4 u_1^4) \\
&\quad + \left( \kappa \eta_1 - \frac{1}{\kappa} \right) u_0^5 (u_{1,x}^1 + u_{1,y}^2 + u^3 1, z) \\
&\quad - \frac{1}{\kappa} (u_1^1 u_{0,x}^4 + u_0^1 u_{1,x}^4 + u_1^2 u_{0,y}^4 + u_0^2 u_{1,y}^4 + u_1^3 u_{0,z}^4 + u_0^3 u_{1,z}^4) \\
&\quad \left. + u_1^5 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right];
\end{aligned}$$

and so on. Then, from

$$\sum_{\tau=0}^{+\infty} u_{\tau}^j = x + y + z + L^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right) \quad (7.2.19)$$

we find the recursive relation

$$u_0^j = x + y + z, \quad u_{\tau+1}^j = L^{-1} A_{j,\tau}, \quad \tau \geq 0, \quad j = 1, 2, 3, 4, 5. \quad (7.2.20)$$

The first components of the latter are determined in the form, setting  $X = x + y + z$  :

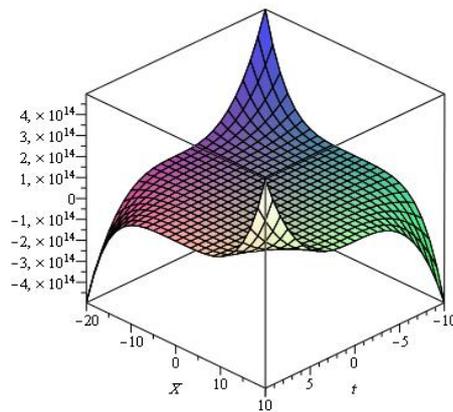
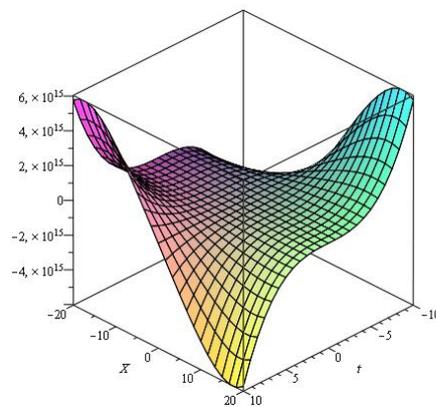
$$u_1^1 = u_1^2 = u_1^3 = -\frac{2X^3}{\eta_2} t,$$

$$\begin{aligned}
u_1^4 = u_1^5 &= \left[ -\frac{5X^3}{\eta_1} + \left( -3\kappa + \frac{6}{\kappa\eta_1} + \frac{1}{\eta_1} \right) X \right] t, \\
u_2^1 = u_2^2 = u_2^3 &= \left[ \left( \frac{10}{\eta_2\eta_1} + \frac{2}{\eta_2^2} \right) X^5 + \left( -\frac{2}{\eta_2\eta_1} - \frac{12}{\eta_2\eta_1\kappa} + \frac{6\kappa}{\eta_2} \right) X^3 - \frac{18X}{\eta_2} \right] t^2, \\
u_2^4 = u_2^5 &= \left[ \left( \frac{45}{2\eta_1^2} + \frac{6}{\eta_2\eta_1} \right) X^5 + \left( -\frac{12}{\eta_2\eta_1\kappa} - \frac{7}{\eta_1^2} + \frac{21\kappa}{\eta_1} - \frac{57}{\eta_1^2\kappa} + \frac{9\kappa}{\eta_2} \right) X^3 \right. \\
&\quad \left. + \left( -\frac{27}{\eta_1^2\kappa^2} + \frac{6}{\eta_1^2\kappa} - \frac{3\kappa}{\eta_1} + \frac{9\kappa^2}{2} + \frac{1}{2\eta_1^2} - \frac{18}{\eta_1} \right) X \right] t^2, \\
u_3^1 = u_3^2 = u_3^3 &= \left[ \left( -\frac{12}{\eta_2\eta_1} + \frac{36\kappa}{\eta_2} - \frac{72}{\eta_2\eta_1\kappa} \right) X + \left( -\frac{140}{3\eta_2\eta_1^2} - \frac{4}{3\eta_2^3} - \frac{28}{\eta_2^2\eta_1} \right) X^7 \right. \\
&\quad \left. + \left( \frac{8\kappa}{\eta_2\eta_1} - \frac{16}{\eta_2\kappa\eta_1^2} + \frac{248}{\eta_2\eta_1} - \frac{4}{3\eta_2\eta_1^2} - \frac{12\kappa^2}{\eta_2} + \frac{52}{\eta_2^2} + \frac{12}{\eta_2\kappa^2\eta_1^2} \right) X^3 \right. \\
&\quad \left. + \left( \frac{16}{\eta_2\eta_1^2} + \frac{116}{\eta_2\kappa\eta_1^2} - \frac{48\kappa}{\eta_2\eta_1} - \frac{24\kappa}{\eta_2^2} + \frac{4}{\eta_2^2\eta_1} + \frac{40}{\eta_2^2\kappa\eta_1} \right) X^5 \right] t^3, \\
u_3^4 = u_3^5 &= \left[ \left( -\frac{58}{\eta_2\eta_1^2} - \frac{8}{\eta_2^2\eta_1} - \frac{235}{2\eta_1^3} \right) X^7 \right. \\
&\quad \left. + \left( -\frac{10\kappa}{\eta_2^2} - \frac{291\kappa}{2\eta_1^2} + \frac{12}{\eta_2^2\kappa\eta_1} + \frac{10}{\eta_2\eta_1^2} + \frac{97}{2\eta_1^3} - \frac{137\kappa}{\eta_2\eta_1} + \frac{240}{\eta_2\kappa\eta_1^2} + \frac{426}{\eta_1^3\kappa} \right) X^5 \right. \\
&\quad \left. + \left( \frac{267}{\eta_1^2} - \frac{45\kappa^2}{\eta_2} - \frac{35}{6\eta_1^3} - \frac{20}{\eta_2\kappa\eta_1^2} - \frac{89}{\eta_1^3\kappa} + \frac{231}{\eta_1^3\kappa^2} + \frac{204}{\eta_2\eta_1} + \frac{35\kappa}{\eta_1^2} - \frac{105\kappa^2}{2\eta_1} \right. \right. \\
&\quad \left. \left. - \frac{24}{\eta_2\kappa^2\eta_1^2} + \frac{15\kappa}{\eta_2\eta_1} \right) X^3 + \left( \frac{27\kappa}{\eta_1} + \frac{117}{\eta_1^2\kappa} - \frac{72}{\eta_1^2\kappa^3\eta_2} + \frac{3}{\eta_1^3\kappa} + \frac{18}{\eta_2\eta_1\kappa} - \frac{9\kappa^3}{2} \right. \right. \\
&\quad \left. \left. + \frac{1}{6\eta_1^3} - \frac{18}{\eta_1^2} - \frac{3\kappa}{2\eta_1^2} - \frac{396}{\eta_1^3\kappa^3} + \frac{9\kappa^2}{2\eta_1} - \frac{39}{\eta_1^3\kappa^2} + \frac{18\kappa}{\eta_2} \right) X \right] t^3.
\end{aligned}$$

Therefore, an analytical approximate solution to the Ginzburg-Landau equations (7.2.10) under the assumption  $\phi = -\text{div}(A) = -(A_x^1 + A_y^2 + A_z^3)$  with the initial condition (7.2.12) is expressed as

$$A^1 = A^2 = A^3 \simeq u_0^1 + u_1^1 + u_2^1 + u_3^1, \quad \psi^1 = \psi^2 \simeq u_0^4 + u_1^4 + u_2^4 + u_3^4.$$

For fixed parameters  $\eta_1 = \eta_2 = 1$  and  $\kappa = 10^{-3}$ , the graphic representation is shown in the Figure 7.1.

Function  $A^1 = A^2 = A^3$  versus  $X$  and  $t$ .Function  $\psi^1 = \psi^2$  versus  $X$  and  $t$ .Figure 7.1: Approximate solution of the Ginzburg-Landau system (7.2.11) versus  $X = x + y + z$  and  $t$ .



# Adomian Method for Under Determined Systems of PDEs

## 8.1 Theoretical considerations

Consider the system of partial differential equations

$$G_j \left( x, u^{(s)}(x), \tilde{u}^{(s)}(x) \right) = f_j(x), \quad j = 1, 2, \dots, m \quad (8.1.1)$$

where the nonzero positive integer  $s$  is the order of system,  $x = (x^1, \dots, x^n)$  are independent variables,  $u = u(x) = (u^1(x), u^2(x), \dots, u^m(x))$ ,  $\tilde{u} = \tilde{u}(x) = (\tilde{u}^1(x), \tilde{u}^2(x), \dots, \tilde{u}^{\tilde{m}}(x))$  are the dependent variables, the  $f_j$  are continuous functions and the  $G_j$  are differentiable functions with respect to their arguments. Suppose that (8.1.1) can be written into the form (8.1.2) for  $\tilde{m} < m$

$$\begin{aligned} f_j &= u_{(k_{j,0})}^j [h_{j,0}] + \tilde{u}_{(k_{j,0})}^j [h_{j,0}] \\ &+ F_j \left( x, u_{(k_{j,1})}^{\alpha_{j,1}} [h_{j,1}], \dots, u_{(k_{j,r_j})}^{\alpha_{j,r_j}} [h_{j,r_j}], \tilde{u}_{(\tilde{k}_{j,1})}^{\tilde{\alpha}_{j,1}} [\tilde{h}_{j,1}], \dots, \tilde{u}_{(\tilde{k}_{j,\tilde{r}_j})}^{\tilde{\alpha}_{j,\tilde{r}_j}} [\tilde{h}_{j,\tilde{r}_j}] \right), \\ &j = 1, 2, \dots, \tilde{m}, \end{aligned} \quad (8.1.2)$$

$$\begin{aligned} f_j &= u_{(k_{j,0})}^j [h_{j,0}] + F_j \left( x, u_{(k_{j,1})}^{\alpha_{j,1}} [h_{j,1}], \dots, u_{(k_{j,r_j})}^{\alpha_{j,r_j}} [h_{j,r_j}], \tilde{u}_{(\tilde{k}_{j,1})}^{\tilde{\alpha}_{j,1}} [\tilde{h}_{j,1}], \dots, \tilde{u}_{(\tilde{k}_{j,\tilde{r}_j})}^{\tilde{\alpha}_{j,\tilde{r}_j}} [\tilde{h}_{j,\tilde{r}_j}] \right), \\ &j = \tilde{m} + 1, \tilde{m} + 2, \dots, m \end{aligned}$$

or into the form (8.1.3) for  $\tilde{m} \geq m$

$$\begin{aligned} f_j &= u_{(k_{j,0})}^j [h_{j,0}] + \tilde{u}_{(k_{j,0})}^j [h_{j,0}] \\ &+ F_j \left( x, u_{(k_{j,1})}^{\alpha_{j,1}} [h_{j,1}], \dots, u_{(k_{j,r_j})}^{\alpha_{j,r_j}} [h_{j,r_j}], \tilde{u}_{(\tilde{k}_{j,1})}^{\tilde{\alpha}_{j,1}} [\tilde{h}_{j,1}], \dots, \tilde{u}_{(\tilde{k}_{j,\tilde{r}_j})}^{\tilde{\alpha}_{j,\tilde{r}_j}} [\tilde{h}_{j,\tilde{r}_j}] \right), \\ &j = 1, 2, \dots, m - 1, \end{aligned} \quad (8.1.3)$$

$$\begin{aligned} f_m &= u_{(k_{m,0})}^m [h_{m,0}] + \sum_{\tilde{j}=m}^{\tilde{m}} \tilde{u}_{(k_{m,0})}^{\tilde{j}} [h_{m,0}] \\ &+ F_m \left( x, u_{(k_{j,1})}^{\alpha_{j,1}} [h_{j,1}], \dots, u_{(k_{j,r_j})}^{\alpha_{j,r_j}} [h_{j,r_j}], \tilde{u}_{(\tilde{k}_{j,1})}^{\tilde{\alpha}_{j,1}} [\tilde{h}_{j,1}], \dots, \tilde{u}_{(\tilde{k}_{j,\tilde{r}_j})}^{\tilde{\alpha}_{j,\tilde{r}_j}} [\tilde{h}_{j,\tilde{r}_j}] \right), \end{aligned}$$

where  $r_j, k_{j,i}, h_{j,i}, \tilde{r}_j, \tilde{k}_{j,i}, \tilde{h}_{j,i} \in \mathbb{N}$  with  $\max \{k_{j,i}, \tilde{k}_{j,i}\} = s$ ,  $h_{j,i} \in \{1, 2, \dots, p_{k_{j,i}}\}$ ,  $\tilde{h}_{j,i} \in \{1, 2, \dots, p_{\tilde{k}_{j,i}}\}$ ,  $\alpha_{j,i} \in \{1, 2, \dots, m\}$ ,  $\tilde{\alpha}_{j,i} \in \{1, 2, \dots, \tilde{m}\}$  and  $F_j$  is a differentiable function

with respect to its  $r_j$  arguments. Let  $L_j$  be the linear differential operator such that  $L_j v = v_{(k_{j,0})}[h_{j,0}]$  and

$$L_j = \frac{\partial^{k_{j,0}}}{(\partial x^{\beta_{j,1}})^{l_{j,1}} (\partial x^{\beta_{j,2}})^{l_{j,2}} \dots (\partial x^{\beta_{j,s_j}})^{l_{j,s_j}}}$$

with  $1 \leq \beta_{j,1} < \beta_{j,2} < \dots < \beta_{j,s_j} \leq n$ ,  $l_{j,1} + l_{j,2} + \dots + l_{j,s_j} = k_{j,0}$  and  $l_{j,i} \neq 0$ . As required,  $L_j$  has to be the smallest order linear differential operator in  $G_j$ . Set  $I_j = \{\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,s_j}\}$ . Define  $L_j^{-1}$ , the inverse of the operator  $L_j$  by

$$\begin{aligned} L_j^{-1}v(x) &= \left( \int_{x_1^{\beta_{j,1}}}^{x_1^{\beta_{j,1}}} \int_{x_2^{\beta_{j,1}}}^{z_{l_{j,1}-1}^{\beta_{j,1}}} \dots \int_{x_{l_{j,1}}^{\beta_{j,1}}}^{z_1^{\beta_{j,1}}} \right) \left( \int_{x_1^{\beta_{j,2}}}^{x_1^{\beta_{j,2}}} \int_{x_2^{\beta_{j,2}}}^{z_{l_{j,2}-1}^{\beta_{j,2}}} \dots \int_{x_{l_{j,2}}^{\beta_{j,2}}}^{z_1^{\beta_{j,2}}} \right) \\ &\dots \left( \int_{x_1^{\beta_{j,s_j}}}^{x_1^{\beta_{j,s_j}}} \int_{x_2^{\beta_{j,s_j}}}^{z_{l_{j,s_j}-1}^{\beta_{j,s_j}}} \dots \int_{x_{l_{j,s_j}}^{\beta_{j,s_j}}}^{z_1^{\beta_{j,s_j}}} \right) v(y^1, y^2, \dots, y^n) \left( dz_0^{\beta_{j,1}} dz_1^{\beta_{j,1}} \dots dz_{l_{j,1}-1}^{\beta_{j,1}} \right) \\ &\quad \left( dz_0^{\beta_{j,2}} dz_1^{\beta_{j,2}} \dots dz_{l_{j,2}-1}^{\beta_{j,2}} \right) \dots \left( dz_0^{\beta_{j,s_j}} dz_1^{\beta_{j,s_j}} \dots dz_{l_{j,s_j}-1}^{\beta_{j,s_j}} \right), \end{aligned}$$

where  $y^i = z_0^i$  if  $i \in I_j$  and  $y^i = x^i$  otherwise. The constants  $x_k^i$  as well as the order of integration are closely related to the choice of boundary conditions in such a way that  $L_j^{-1}(L_j v) = v(x) + g_j(x)$ . Here,  $g_j$  is a function completely determined by the boundary conditions. Thus, applying  $L_j^{-1}$  to both sides of (8.1.2) yields

$$\begin{aligned} u^j(x) + \tilde{u}^j(x) &= -g_j(x) - \tilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = 1, 2, \dots, \tilde{m}, \\ u^j(x) &= -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = \tilde{m} + 1, \tilde{m} + 2, \dots, m; \end{aligned} \quad (8.1.4)$$

and applying  $L_j^{-1}$  to both sides of (8.1.3) gives

$$\begin{aligned} u^j(x) + \tilde{u}^j(x) &= -g_j(x) - \tilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = 1, 2, \dots, m-1, \\ u^m(x) + \sum_{\tilde{j}=m}^{\tilde{m}} \tilde{u}^{\tilde{j}}(x) &= -g_m(x) - \sum_{\tilde{j}=m}^{\tilde{m}} \tilde{g}_{\tilde{j}}(x) + L_m^{-1} f_m(x) - L_m^{-1} F_m. \end{aligned} \quad (8.1.5)$$

The Adomian decomposition method allows to represent the functions  $u^j$  and  $\tilde{u}^j$  as infinite series

$$u^j(x) = \sum_{\tau=0}^{+\infty} u_\tau^j(x), \quad \tilde{u}^j(x) = \sum_{\tau=0}^{+\infty} \tilde{u}_\tau^j(x), \quad (8.1.6)$$

where the  $u_\tau^j$  and  $\tilde{u}_\tau^j$  are functions to determine. Express the differentiable functions  $F_j$  as series

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \quad (8.1.7)$$

where the  $A_{j,\tau}$  are multivariate polynomials also called Adomian polynomials

$$\begin{aligned} A_{j,\tau} &= \frac{1}{\tau!} \left[ \frac{d^\tau}{d\lambda^\tau} F_j \left( x, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,1})}^{\alpha_{j,1}}[h_{j,1}], \dots, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,r_j})}^{\alpha_{j,r_j}}[h_{j,r_j}], \right. \right. \\ &\quad \left. \left. \sum_{i=0}^{\tau} \lambda^i \tilde{u}_{i,(k_{j,1})}^{\tilde{\alpha}_{j,1}}[\tilde{h}_{j,1}], \dots, \sum_{i=0}^{\tau} \lambda^i \tilde{u}_{i,(k_{j,\tilde{r}_j})}^{\tilde{\alpha}_{j,\tilde{r}_j}}[\tilde{h}_{j,\tilde{r}_j}] \right) \right]_{\lambda=0}. \end{aligned} \quad (8.1.8)$$

The substitution in (8.1.4) of the expressions (8.1.6) and (8.1.7) along with (8.1.8) yields

$$\sum_{\tau=0}^{+\infty} u_{\tau}^j + \sum_{\tau=0}^{+\infty} \tilde{u}_{\tau}^j = -g_j(x) - \tilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right), \quad j = 1, 2, \dots, \tilde{m},$$

$$\sum_{\tau=0}^{+\infty} u_{\tau}^j = -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right), \quad j = \tilde{m} + 1, \tilde{m} + 2, \dots, m.$$

Now substituting in (8.1.5) the expressions (8.1.6) and (8.1.7) along with (8.1.8) yields

$$\sum_{\tau=0}^{+\infty} u_{\tau}^j + \sum_{\tau=0}^{+\infty} \tilde{u}_{\tau}^j = -g_j(x) - \tilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right),$$

$$j = 1, 2, \dots, m - 1,$$

$$\sum_{\tau=0}^{+\infty} u_{\tau}^m + \sum_{\tau=0}^{+\infty} \left( \sum_{\tilde{j}=m}^{\tilde{m}} \tilde{u}_{\tau}^{\tilde{j}} \right) = -g_m(x) - \sum_{\tilde{j}=m}^{\tilde{m}} \tilde{g}_{\tilde{j}}(x) + L_m^{-1} f_m(x) - L_m^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right).$$

The functions  $u_{\tau}^j$  and  $\tilde{u}_{\tau}^{\tilde{j}}$  can be obtained using the following recursive relations.

- For  $\tilde{m} < m$ , if  $j = 1, 2, \dots, \tilde{m}$

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad \tilde{u}_0^j = -\tilde{g}_j(x),$$

$$u_{\tau+1}^j = -L_j^{-1} (B_{j,\tau}^0), \quad \tilde{u}_{\tau+1}^j = -L_j^{-1} (B_{j,\tau}^1),$$

$$\tau \geq 0, \quad B_{j,\tau}^0 + B_{j,\tau}^1 = A_{j,\tau}$$

and if  $j = \tilde{m} + 1, \tilde{m} + 2, \dots, m$

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad u_{\tau+1}^j = -L_j^{-1} A_{j,\tau}, \quad \tau \geq 0.$$

- For  $\tilde{m} \geq m$ , if  $j = 1, 2, \dots, m - 1$

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad \tilde{u}_0^j = -\tilde{g}_j(x),$$

$$u_{\tau+1}^j = -L_j^{-1} (B_{j,\tau}^0), \quad \tilde{u}_{\tau+1}^j = -L_j^{-1} (B_{j,\tau}^1),$$

$$\tau \geq 0, \quad B_{j,\tau}^0 + B_{j,\tau}^1 = A_{j,\tau}$$

and if  $\tilde{j} = m, m + 1, \dots, \tilde{m}$

$$u_0^m = -g_m(x) + L_m^{-1} f_m(x), \quad \tilde{u}_0^{\tilde{j}} = -\tilde{g}_{\tilde{j}}(x),$$

$$u_{\tau+1}^m = -L_m^{-1} (B_{m,\tau}^0), \quad \tilde{u}_{\tau+1}^{\tilde{j}} = -L_m^{-1} (B_{m,\tau}^{\tilde{j}-m+1}),$$

$$\tau \geq 0, \quad B_{m,\tau}^0 + \sum_{i=1}^{\tilde{m}-m+1} B_{m,\tau}^i = A_{m,\tau}$$

since the  $A_{j,\tau}$  depend only on  $u_0^{\beta}, \tilde{u}_0^{\tilde{\beta}}, u_1^{\beta}, \tilde{u}_1^{\tilde{\beta}}, \dots, u_{\tau}^{\beta}, \tilde{u}_{\tau}^{\tilde{\beta}}, \beta = 1, 2, \dots, m, \tilde{\beta} = 1, 2, \dots, \tilde{m}$ .

If the functions  $u_\tau^j$  and  $\tilde{u}_\tau^j$  are determined for all  $\tau \in \mathbb{N}$ , then a solution  $(u, \tilde{u})$  of system (8.1.2) or (8.1.3) is immediately formed using the series (8.1.6). Otherwise, if only a finite number, says  $\nu$ , of the functions  $u_\tau^j$  and  $\tilde{u}_\tau^j$  is found, one can use the partial sums

$$\phi_\nu^j(x) = \sum_{\tau=0}^{\nu-1} u_\tau^j(x), \quad \tilde{\phi}_\nu^j(x) = \sum_{\tau=0}^{\nu-1} \tilde{u}_\tau^j(x) \quad (8.1.15)$$

and consider the function  $\phi_\nu = \left( \phi_\nu^1, \phi_\nu^2, \dots, \phi_\nu^m, \tilde{\phi}_\nu^1, \tilde{\phi}_\nu^2, \dots, \tilde{\phi}_\nu^m \right)$  as an approximate solution to the system of differential equations (8.1.2) or (8.1.3).

## 8.2 Application to Ginzburg-Landau equations

Consider again the Ginzburg-Landau equations (7.2.10) but without any assumption on the function  $\phi$ . Adopting the following notation:  $u^1 = A^1$ ,  $u^2 = A^2$ ,  $u^3 = A^3$ ,  $u^4 = \psi^1$ ,  $u^5 = \psi^2$ ,  $\tilde{u} = \phi$  and after some computation and identification, the equations (7.2.10) become the under determined system

$$\begin{aligned} u_t^4 + \tilde{u}_t &= F_4^0(u, u_{(1)}, \tilde{u}_{(1)}, u_{(2)}) + F_4^1(u, \tilde{u}, u_{(1)}, u_{(2)}), \\ u_t^5 &= F_5(u, \tilde{u}, u_{(1)}, u_{(2)}), \\ u_t^j &= F_j(u, u_{(1)}, \tilde{u}_{(1)}, u_{(2)}), \quad j = 1, 2, 3, \end{aligned} \quad (8.2.16)$$

where

$$\begin{aligned} F_4^0 &= -\frac{1}{\eta_1} \left[ -\eta_1 \tilde{u}_t - \frac{1}{\kappa^2} (u_{2x}^4 + u_{2y}^4 + u_{2z}^4) \right. \\ &\quad \left. + u^4 \left( (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (u^5)^2 - 1 \right) \right], \end{aligned}$$

$$F_4^1 = -\frac{1}{\eta_1} \left[ -\eta_1 \kappa \tilde{u} u^5 - \frac{1}{\kappa} u^5 (u_x^1 + u_y^2 + u_z^3) - \frac{1}{\kappa} (u^1 u_x^5 + u^2 u_y^5 + u^3 u_z^5) \right],$$

$$\begin{aligned} F_5 &= -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{2x}^5 + u_{2y}^5 + u_{2z}^5) - \frac{1}{\kappa} u^4 (u_x^1 + u_y^2 + u_z^3) - \eta_1 \kappa \tilde{u} u^4 \right. \\ &\quad \left. - \frac{1}{\kappa} (u^1 u_x^4 + u^2 u_y^4 + u^3 u_z^4) + u^5 \left( (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (u^5)^2 - 1 \right) \right], \end{aligned}$$

$$F_1 = -\frac{1}{\eta_2} \left[ u_{xy}^2 - u_{2y}^1 - u_{2z}^1 + u_{xz}^3 - \frac{1}{\kappa} (u^4 u_x^5 - u^5 u_x^4) + \eta_2 \tilde{u}_x + u^1 \left( (u^4)^2 + (u^5)^2 \right) \right],$$

$$F_2 = -\frac{1}{\eta_2} \left[ u_{yz}^3 - u_{2z}^2 - u_{2x}^2 + u_{xy}^1 - \frac{1}{\kappa} (u^4 u_y^5 - u^5 u_y^4) + \eta_2 \tilde{u}_y + u^2 \left( (u^4)^2 + (u^5)^2 \right) \right],$$

$$F_3 = -\frac{1}{\eta_2} \left[ u_{xz}^1 - u_{2x}^3 - u_{2y}^3 + u_{yz}^2 - \frac{1}{\kappa} (u^4 u_z^5 - u^5 u_z^4) + \eta_2 \tilde{u}_z + u^3 \left( (u^4)^2 + (u^5)^2 \right) \right].$$

Using Adomian decomposition method, we solve this system subject to the initial conditions

$$\tilde{u}(0, x, y, z) = u^j(0, x, y, z) = x + y + z, \quad j = 1, 2, 3, 4, 5. \quad (8.2.17)$$

For that, consider the differential operator  $L$

$$L = \frac{\partial}{\partial t} \quad (8.2.18)$$

and define its inverse operator  $L^{-1}$  by

$$L^{-1} v(t, x, y, z) = \int_0^t v(s, x, y, z) ds. \quad (8.2.19)$$

Applying  $L^{-1}$  to both sides of (8.2.16) and using the initial conditions (8.2.17), we find

$$\begin{aligned} u^4 + \tilde{u} &= 2(x + y + z) + L^{-1} F_4^0(u, u_{(1)}, \tilde{u}_{(1)}, u_{(2)}) + L^{-1} F_4^1(u, \tilde{u}, u_{(1)}, u_{(2)}), \\ u^5 &= x + y + z + L^{-1} F_5(u, \tilde{u}, u_{(1)}, u_{(2)}), \\ u^j &= x + y + z + L^{-1} F_j(u, u_{(1)}, \tilde{u}_{(1)}, u_{(2)}), \quad j = 1, 2, 3. \end{aligned} \quad (8.2.20)$$

The Adomian decomposition method suggests that the functions  $u^j, \tilde{u}$  can be sought as

$$u^j(t, x, y, z) = \sum_{\tau=0}^{+\infty} u_\tau^j(t, x, y, z), \quad \tilde{u}(t, x, y, z) = \sum_{\tau=0}^{+\infty} \tilde{u}_\tau(t, x, y, z) \quad (8.2.21)$$

while the differentiable functions  $F_j, F_4^0, F_4^1$  can be developed in a series:

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \quad F_4^0 = \sum_{\tau=0}^{+\infty} B_{4,\tau}^0, \quad F_4^1 = \sum_{\tau=0}^{+\infty} B_{4,\tau}^1, \quad (8.2.22)$$

where the  $A_{j,\tau}, B_{4,\tau}^0, B_{4,\tau}^1$  are Adomian polynomials obtained by the formula

$$A_{j,\tau} = \frac{1}{\tau!} \left[ \frac{d^\tau}{d\lambda^\tau} F_j \left( \sum_{i=0}^{\tau} \lambda^i u_i, \sum_{i=0}^{\tau} \lambda^i \tilde{u}_i, \sum_{i=0}^{\tau} \lambda^i u_{i,(1)}, \sum_{i=0}^{\tau} \lambda^i \tilde{u}_{i,(1)}, \sum_{i=0}^{\tau} \lambda^i u_{i,(2)} \right) \right]_{\lambda=0}, \quad (8.2.23)$$

$$B_{4,\tau}^k = \frac{1}{\tau!} \left[ \frac{d^\tau}{d\lambda^\tau} F_4^k \left( \sum_{i=0}^{\tau} \lambda^i u_i, \sum_{i=0}^{\tau} \lambda^i \tilde{u}_i, \sum_{i=0}^{\tau} \lambda^i u_{i,(1)}, \sum_{i=0}^{\tau} \lambda^i \tilde{u}_{i,(1)}, \sum_{i=0}^{\tau} \lambda^i u_{i,(2)} \right) \right]_{\lambda=0}. \quad (8.2.24)$$

The explicit expressions of the Adomian polynomials  $A_{j,\tau}, B_{4,\tau}^0, B_{4,\tau}^1$  for the nonlinear functions  $F_j, F_4^0, F_4^1$  can be now calculated using the formula (8.2.23)-(8.2.24). There results the following:

$$\begin{aligned} B_{4,0}^0 &= -\frac{1}{\eta_1} \left[ -\eta_1 \tilde{u}_{0,t} - \frac{1}{\kappa^2} (u_{0,2x}^4 + u_{0,2y}^4 + u_{0,2z}^4) \right. \\ &\quad \left. + u_0^4 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right], \end{aligned}$$

$$B_{4,0}^1 = -\frac{1}{\eta_1} \left[ -\eta_1 \kappa \tilde{u}_0 u_0^5 - \frac{1}{\kappa} u_0^5 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) - \frac{1}{\kappa} (u_0^1 u_{0,x}^5 + u_0^2 u_{0,y}^5 + u_0^3 u_{0,z}^5) \right],$$

$$A_{5,0} = -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa^2} (u_{0,2x}^5 + u_{0,2y}^5 + u_{0,2z}^5) - \frac{1}{\kappa} u_0^4 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) - \eta_1 \kappa \tilde{u}_0 u_0^4 \right. \\ \left. - \frac{1}{\kappa} (u_0^1 u_{0,x}^4 + u_0^2 u_{0,y}^4 + u_0^3 u_{0,z}^4) + u_0^5 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right],$$

$$A_{1,0} = -\frac{1}{\eta_2} \left[ u_{0,xy}^2 - u_{0,2y}^1 - u_{0,2z}^1 + u_{0,xz}^3 - \frac{1}{\kappa} (u_0^4 u_{0,x}^5 - u_0^5 u_{0,x}^4) + \eta_2 \tilde{u}_{0,x} \right. \\ \left. + u_0^1 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

$$A_{2,0} = -\frac{1}{\eta_2} \left[ u_{0,yz}^3 - u_{0,2z}^2 - u_{0,2x}^2 + u_{0,xy}^1 - \frac{1}{\kappa} (u_0^4 u_{0,y}^5 - u_0^5 u_{0,y}^4) + \eta_2 \tilde{u}_{0,y} \right. \\ \left. + u_0^2 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

$$A_{3,0} = -\frac{1}{\eta_2} \left[ u_{0,xz}^1 - u_{0,2x}^3 - u_{0,2y}^3 + u_{0,yz}^2 - \frac{1}{\kappa} (u_0^4 u_{0,z}^5 - u_0^5 u_{0,z}^4) + \eta_2 \tilde{u}_{0,z} \right. \\ \left. + u_0^3 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

$$B_{4,1}^0 = -\frac{1}{\eta_1} \left[ -\eta_1 \tilde{u}_{1,t} + u_1^4 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right. \\ \left. - \frac{1}{\kappa^2} (u_{1,2x}^4 + u_{1,2y}^4 + u_{1,2z}^4) + u_0^4 (2u_0^1 u_1^1 + 2u_0^3 u_1^3 + 2u_0^2 u_1^2 + 2u_0^5 u_1^5 + 2u_0^4 u_1^4) \right],$$

$$B_{4,1}^1 = -\frac{1}{\eta_1} \left[ -\frac{1}{\kappa} u_1^5 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) - \frac{1}{\kappa} u_0^5 (u_{1,x}^1 + u_{1,y}^2 + u_{1,z}^3) - \eta_1 \kappa \tilde{u}_1 u_0^5 - \eta_1 \kappa \tilde{u}_0 u_1^5 \right. \\ \left. - \frac{1}{\kappa} (u_1^1 u_{0,x}^5 + u_0^1 u_{1,x}^5 + u_1^2 u_{0,y}^5 + u_0^2 u_{1,y}^5 + u_1^3 u_{0,z}^5 + u_0^3 u_{1,z}^5) \right],$$

$$A_{5,1} = -\frac{1}{\eta_1} \left[ -\eta_1 \kappa \tilde{u}_1 u_0^4 - \frac{1}{\kappa^2} (u_{1,2x}^5 + u_{1,2y}^5 + u_{1,2z}^5) - \frac{1}{\kappa} u_1^4 (u_{0,x}^1 + u_{0,y}^2 + u_{0,z}^3) \right. \\ \left. - \eta_1 \kappa \tilde{u}_0 u_1^4 + u_0^5 (2u_0^1 u_1^1 + 2u_0^3 u_1^3 + 2u_0^2 u_1^2 + 2u_0^5 u_1^5 + 2u_0^4 u_1^4) - \frac{1}{\kappa} u_0^4 (u_{1,x}^1 + u_{1,y}^2 + u_{1,z}^3) \right. \\ \left. - \frac{1}{\kappa} (u_1^1 u_{0,x}^4 + u_0^1 u_{1,x}^4 + u_1^2 u_{0,y}^4 + u_0^2 u_{1,y}^4 + u_1^3 u_{0,z}^4 + u_0^3 u_{1,z}^4) \right. \\ \left. + u_1^5 \left( (u_0^1)^2 + (u_0^2)^2 + (u_0^3)^2 + (u_0^4)^2 + (u_0^5)^2 - 1 \right) \right],$$

$$A_{1,1} = -\frac{1}{\eta_2} \left[ u_{1,xy}^2 - u_{1,2y}^1 - u_{1,2z}^1 + u_{1,xz}^3 - \frac{1}{\kappa} (u_1^4 u_{0,x}^5 + u_0^4 u_{1,x}^5 - u_1^5 u_{0,x}^4 - u_0^5 u_{1,x}^4) \right. \\ \left. + u_0^1 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) + \eta_2 \tilde{u}_{1,x} + u_1^1 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

$$A_{2,1} = -\frac{1}{\eta_2} \left[ u_{1,yz}^3 - u_{1,2z}^2 - u_{1,2x}^2 + u_{1,xy}^1 - \frac{1}{\kappa} (u_1^4 u_{0,y}^5 + u_0^4 u_{1,y}^5 - u_1^5 u_{0,y}^4 - u_0^5 u_{1,y}^4) \right. \\ \left. + u_0^2 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) + \eta_2 \tilde{u}_{1,y} + u_1^2 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

$$A_{3,1} = -\frac{1}{\eta_2} \left[ u_{1,xz}^1 - u_{1,2x}^3 - u_{1,2y}^3 + u_{1,yz}^2 - \frac{1}{\kappa} (u_1^4 u_{0,z}^5 + u_0^4 u_{1,z}^5 - u_1^5 u_{0,z}^4 - u_0^5 u_{1,z}^4) \right. \\ \left. + u_0^3 (2u_0^4 u_1^4 + 2u_0^5 u_1^5) + \eta_2 \tilde{u}_{1,z} + u_1^3 \left( (u_0^4)^2 + (u_0^5)^2 \right) \right],$$

and so on. Using the above assumptions, we have

$$\sum_{\tau=0}^{+\infty} u_\tau^4 + \sum_{\tau=0}^{+\infty} \tilde{u}_\tau = 2(x+y+z) + L^{-1} \left( \sum_{\tau=0}^{+\infty} B_{j,\tau}^0 \right) + L^{-1} \left( \sum_{\tau=0}^{+\infty} B_{j,\tau}^1 \right), \\ \sum_{\tau=0}^{+\infty} u_\tau^j = x+y+z + L^{-1} \left( \sum_{\tau=0}^{+\infty} A_{j,\tau} \right), \quad j = 1, 2, 3, 5$$

which induces the recursive relation

$$u_0^4 = x+y+z, \quad u_{\tau+1}^4 = L^{-1} (B_{4,\tau}^0), \quad \tau \geq 0, \quad (8.2.25)$$

$$\tilde{u}_0 = x+y+z, \quad \tilde{u}_{\tau+1} = L^{-1} (B_{4,\tau}^1), \quad \tau \geq 0, \quad (8.2.26)$$

$$u_0^j = x+y+z, \quad u_{\tau+1}^j = L^{-1} (A_{j,\tau}), \quad \tau \geq 0, \quad j = 1, 2, 3, 5. \quad (8.2.27)$$

The first components are then determined (we have set  $x+y+z = X$ ) as:

$$u_1^4 = \left( -\frac{5X^3}{\eta_1} + \frac{X}{\eta_1} \right) t, \quad u_1^5 = \left[ \left( \frac{6}{\eta_1 \kappa} + \frac{1}{\eta_1} \right) X + \kappa X^2 - \frac{5X^3}{\eta_1} \right] t, \\ u_1^1 = u_1^2 = u_1^3 = \left( -1 - \frac{2X^3}{\eta_2} \right) t, \quad \tilde{u}_1 = \left( \frac{6X}{\eta_1 \kappa} + \kappa X^2 \right) t, \\ u_2^3 = \left[ \frac{6X^5}{\eta_2^2} + \frac{3X^2}{\eta_2} + \left( \frac{30}{\eta_2 \eta_1} - \kappa - \frac{12}{\eta_2^2} \right) X - \frac{3}{\eta_1 \kappa} - \frac{\kappa}{\eta_2} \right] t^2, \\ u_2^1 = \left[ \left( \frac{4}{\eta_2^2} + \frac{5}{\eta_2 \eta_1} \right) X^5 - \frac{X^3}{\eta_2 \eta_1} + \frac{2X^2}{\eta_2} + \left( \frac{6}{\eta_2^2} - \frac{15}{\eta_2 \eta_1} - \kappa \right) X - \frac{\kappa}{\eta_2} - \frac{3}{\eta_1 \kappa} \right] t^2, \\ \tilde{u}_2 = \left[ -\frac{\kappa X^4}{\eta_2} + \left( -\frac{16}{\kappa \eta_2 \eta_1} + \frac{\kappa^2}{2} - \frac{20}{\eta_1^2 \kappa} \right) X^3 + \frac{9X^2}{2\eta_1} + \left( \frac{6}{\eta_1^2 \kappa^2} - \frac{\kappa}{2} + \frac{2}{\eta_1^2 \kappa} \right) X - \frac{2}{\eta_1 \kappa} \right] t^2, \\ u_2^2 = \left[ \left( \frac{4}{\eta_2^2} + \frac{5}{\eta_2 \eta_1} \right) X^5 - \frac{\kappa X^4}{\eta_2} + \left( -\frac{6}{\kappa \eta_2 \eta_1} - \frac{1}{\eta_2 \eta_1} \right) X^3 \right. \\ \left. + \frac{2X^2}{\eta_2} + \left( \frac{6}{\eta_2^2} - \frac{15}{\eta_2 \eta_1} - \kappa \right) X + \frac{2\kappa}{\eta_2} - \frac{3}{\eta_1 \kappa} \right] t^2, \\ u_2^4 = \left[ \left( \frac{11}{\eta_2 \eta_1} + \frac{10}{\eta_1^2} \right) X^5 - \frac{\kappa X^4}{\eta_1} + \left( -\frac{1}{\eta_2 \eta_1} - \frac{2}{\eta_1^2} - \frac{6}{\eta_1^2 \kappa} \right) X^3 \right. \\ \left. + \frac{11X^2}{2\eta_1} - \frac{18X}{\kappa^2 \eta_1 \eta_2} - \frac{1}{2\eta_1} \right] t^2 + \left( \frac{6X}{\eta_1 \kappa} + \kappa X^2 \right) t, \\ u_2^5 = \left[ \left( \frac{11}{\eta_2 \eta_1} + \frac{10}{\eta_1^2} \right) X^5 + \left( -\frac{\kappa}{\eta_2} - \frac{\kappa}{\eta_1} \right) X^4 + \left( -\frac{16}{\kappa \eta_2 \eta_1} - \frac{2}{\eta_1^2} - \frac{1}{\eta_2 \eta_1} + \frac{\kappa^2}{2} - \frac{26}{\eta_1^2 \kappa} \right) X^3 \right. \\ \left. + \frac{10X^2}{\eta_1} + \left( -\frac{18}{\kappa^2 \eta_1 \eta_2} + \frac{6}{\eta_1^2 \kappa^2} + \frac{2}{\eta_1^2 \kappa} - \frac{\kappa}{2} \right) X - \frac{2}{\eta_1 \kappa} - \frac{1}{2\eta_1} \right] t^2,$$

$$\begin{aligned}
u_3^1 = & \left[ -\frac{\kappa X^4}{\eta_2} + \frac{2\kappa}{\eta_2} - \frac{6X^3}{\kappa\eta_2\eta_1} \right] t^2 + \left[ \frac{\kappa}{6} - \frac{2}{3\eta_1^2\kappa} - \frac{1}{3\eta_2^2} + \frac{2}{3\eta_2\eta_1} - \frac{2}{\eta_1^2\kappa^2} \right. \\
& + \left( -\frac{20}{3\eta_2\eta_1^2} - \frac{28}{3\eta_2^3} - \frac{24}{\eta_2^2\eta_1} \right) X^7 + \left( \frac{4}{\kappa\eta_2\eta_1^2} + \frac{8}{3\kappa\eta_2^3} + \frac{4}{3\eta_2\eta_1^2} + \frac{4}{\eta_2^2\eta_1} - \frac{8}{3\kappa\eta_2^2\eta_1} \right) X^5 \\
& + \left( -\frac{5}{\eta_2^2} - \frac{31}{3\eta_2\eta_1} \right) X^4 + \left( \frac{30}{\eta_2^2\eta_1} + \frac{2}{3\kappa\eta_2^2\eta_1} + \frac{16}{\kappa^2\eta_2^2\eta_1} + \frac{200}{3\eta_2\eta_1^2} - \frac{68}{3\eta_2^3} + \frac{8\kappa}{3\eta_2} \right) X^3 \\
& - \left( \frac{\kappa^2}{\eta_2} + \frac{2}{3\eta_2} - \frac{32}{\kappa\eta_2^2\eta_1} + \frac{4}{\eta_2\eta_1^2} + \frac{3}{\eta_1} - \frac{28}{\kappa\eta_2\eta_1^2} \right) X + \left( \frac{2\kappa}{3\eta_2\eta_1} + \frac{2\kappa}{3\eta_2^2} \right) X^6 \\
& + \left. \left( -\frac{\kappa^2}{2} + \frac{1}{3\kappa\eta_2^2} + \frac{20}{\eta_1^2\kappa} + \frac{5}{3\eta_2\eta_1} + \frac{10\kappa}{3\eta_2^2} - \frac{4\kappa}{\eta_2\eta_1} + \frac{20}{\kappa\eta_2\eta_1} \right) X^2 \right] t^3,
\end{aligned}$$

$$\begin{aligned}
u_3^2 = & -\frac{\kappa}{\eta_2} t^2 + \left[ \frac{\kappa}{6} - \frac{2}{\eta_1^2\kappa^2} - \frac{2}{3\eta_1^2\kappa} - \frac{1}{3\eta_2^2} + \frac{29}{3\eta_2\eta_1} + \left( -\frac{20}{3\eta_2\eta_1^2} - \frac{28}{3\eta_2^3} - \frac{24}{\eta_2^2\eta_1} \right) X^7 \right. \\
& + \left( \frac{4\kappa}{\eta_2^2} + \frac{2\kappa}{3\eta_2\eta_1} \right) X^6 + \left( \frac{52}{3\kappa\eta_2\eta_1^2} + \frac{8}{3\kappa\eta_2^3} + \frac{4}{3\eta_2\eta_1^2} + \frac{24}{\kappa\eta_2^2\eta_1} + \frac{4}{\eta_2^2\eta_1} - \frac{\kappa^2}{3\eta_2} \right) X^5 \\
& - \left( \frac{92}{\kappa\eta_2\eta_1^2} + \frac{64}{\kappa\eta_2^2\eta_1} + \frac{4}{\eta_2\eta_1^2} + \frac{2}{3\eta_2} - \frac{2\kappa^2}{\eta_2} + \frac{3}{\eta_1} \right) X + \left( -\frac{5}{\eta_2^2} - \frac{40}{3\eta_2\eta_1} \right) X^4 \\
& + \left( \frac{30}{\eta_2^2\eta_1} - \frac{68}{3\eta_2^3} + \frac{2}{3\kappa\eta_2^2\eta_1} + \frac{200}{3\eta_2\eta_1^2} - \frac{4}{\eta_2\eta_1^2\kappa^2} - \frac{4}{3\kappa\eta_2\eta_1^2} + \frac{13\kappa}{3\eta_2} + \frac{16}{\kappa^2\eta_2^2\eta_1} \right) X^3 \\
& + \left. \left( \frac{5}{3\eta_2\eta_1} + \frac{20}{\eta_1^2\kappa} + \frac{1}{3\kappa\eta_2^2} - \frac{26\kappa}{3\eta_2^2} - \frac{\kappa^2}{2} + \frac{88}{3\kappa\eta_2\eta_1} - \frac{4\kappa}{\eta_2\eta_1} \right) X^2 \right] t^3,
\end{aligned}$$

$$\begin{aligned}
u_3^3 = & -\frac{\kappa}{\eta_2} t^2 + \left[ \frac{\kappa}{6} + \frac{11}{3\eta_2^2} - \frac{31}{3\eta_2\eta_1} - \frac{2}{3\eta_1^2\kappa} - \frac{2}{\eta_1^2\kappa^2} + \left( -\frac{20}{3\eta_2^2\eta_1} - \frac{52}{3\eta_2^3} \right) X^7 \right. \\
& + \left( -\frac{8}{3\kappa\eta_2^2\eta_1} + \frac{8}{3\kappa\eta_2^3} + \frac{4}{3\eta_2^2\eta_1} \right) X^5 - \frac{35X^4}{3\eta_2^2} + \frac{2\kappa X^6}{3\eta_2^2} \\
& + \left( \frac{4}{\kappa^2\eta_2^2\eta_1} - \frac{400}{3\eta_2\eta_1^2} + \frac{2}{3\kappa\eta_2^2\eta_1} + \frac{160}{3\eta_2^3} + \frac{10\kappa}{3\eta_2} - \frac{80}{\eta_2^2\eta_1} \right) X^3 \\
& + \left( \frac{20}{\eta_1^2\kappa} - \frac{\kappa^2}{2} + \frac{4\kappa}{\eta_2^2} + \frac{1}{3\kappa\eta_2^2} + \frac{22}{\kappa\eta_2\eta_1} + \frac{8\kappa}{\eta_2\eta_1} \right) X^2 \\
& + \left. \left( -\frac{2}{\eta_2} + \frac{32}{\kappa\eta_2^2\eta_1} - \frac{\kappa^2}{\eta_2} + \frac{8}{\eta_2\eta_1^2} + \frac{64}{\kappa\eta_2\eta_1^2} - \frac{3}{\eta_1} \right) X \right] t^3,
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_3 = & \left[ \left( -\frac{\kappa^2}{3\eta_2} - \frac{24}{\kappa^3\eta_1^2\eta_2} + \frac{4}{3\kappa^2\eta_1^3} + \frac{50}{\kappa\eta_2\eta_1^2} - \frac{20}{\kappa\eta_2^2\eta_1} - \frac{22}{3\eta_1} + \frac{4}{\eta_1^3\kappa^3} \right) X \right. \\
& + \frac{2\kappa X^6}{\eta_2^2} + \left( \frac{94}{\kappa\eta_2\eta_1^2} + \frac{40}{\kappa\eta_1^3} + \frac{52}{\kappa\eta_2^2\eta_1} - \frac{\kappa^2}{\eta_2} \right) X^5 - \frac{4}{3\eta_2\eta_1} - \frac{20}{3\eta_1^2\kappa^2} - \frac{1}{\eta_1^2\kappa} \\
& - \left( \frac{28}{3\kappa\eta_2\eta_1^2} + \frac{112}{3\eta_2\eta_1^2\kappa^2} - \frac{13\kappa}{6\eta_1} - \frac{\kappa}{\eta_2} + \frac{128}{3\kappa^2\eta_1^3} + \frac{16}{3\kappa\eta_1^3} \right) X^3 + \left( \frac{1}{6\kappa^3} - \frac{43}{3\eta_2\eta_1} - \frac{10}{\eta_1^2} \right) X^4 \\
& + \left. \left( \frac{15}{\kappa\eta_2\eta_1} + \frac{55}{2\eta_1^2\kappa} + \frac{2}{3\eta_1^2} - \frac{5\kappa^2}{6} + \frac{10\kappa}{\eta_2\eta_1} - \frac{4\kappa}{\eta_2^2} \right) X^2 \right] t^3 + \left[ \frac{6X}{\eta_1^2\kappa^2} + \frac{3X^2}{2\eta_1} \right] t^2,
\end{aligned}$$

$$\begin{aligned}
u_3^5 &= \left[ \frac{6X}{\eta_1^2 \kappa^2} - \frac{6X^3}{\eta_1^2 \kappa} - \frac{\kappa X^4}{\eta_1} + \frac{3X^2}{2\eta_1} \right] t^2 + \left[ \frac{6}{\kappa^2 \eta_1 \eta_2} - \frac{2}{\eta_1^2 \kappa} + \frac{5}{3\eta_2 \eta_1} - \frac{\kappa}{3\eta_2 \eta_1} - \frac{20}{3\eta_1^2 \kappa^2} \right. \\
&+ \left. \left( \frac{16}{3\eta_2 \eta_1^2} + \frac{2}{\eta_2^2 \eta_1} - \frac{4\kappa^2}{3\eta_2} + \frac{28}{3\eta_1^3} + \frac{40}{\kappa \eta_2^2 \eta_1} + \frac{244}{3\kappa \eta_1^3} - \frac{2\kappa^2}{3\eta_1} + \frac{440}{3\kappa \eta_2 \eta_1^2} \right) X^5 \right. \\
&+ \left. \left( -\frac{4}{\eta_2^2 \eta_1} - \frac{\kappa}{3\eta_1} + \frac{4}{3\kappa^2 \eta_1^3} - \frac{24}{\kappa^3 \eta_1^2 \eta_2} - \frac{31}{3\eta_1} + \frac{10}{\eta_2 \eta_1^2} + \frac{16}{\kappa \eta_2^2 \eta_1} + \frac{2\kappa^2}{3\eta_2} \right. \right. \\
&- \left. \left. \frac{40}{\kappa \eta_2 \eta_1^2} + \frac{4}{\eta_1^3 \kappa^3} \right) X + \left( \frac{41\kappa}{6\eta_1} + \frac{120}{\kappa^2 \eta_2^2 \eta_1} - \frac{176}{3\kappa^2 \eta_1^3} + \frac{2\kappa}{3\eta_2} - \frac{40}{3\kappa \eta_2 \eta_1^2} - \frac{50}{\eta_2 \eta_1^2} \right. \right. \\
&+ \left. \left. \frac{20}{\eta_2^2 \eta_1} - \frac{112}{3\eta_2 \eta_1^2 \kappa^2} - \frac{2}{3\eta_1^3} - \frac{32}{3\kappa \eta_1^3} \right) X^3 - \left( \frac{30}{\eta_1^3} + \frac{94}{3\eta_2^2 \eta_1} + \frac{104}{3\eta_2 \eta_1^2} \right) X^7 \right. \\
&+ \left. \left( \frac{13\kappa}{3\eta_2 \eta_1} + \frac{4\kappa}{3\eta_2^2} + \frac{14\kappa}{3\eta_1^2} \right) X^6 - \left( \frac{\kappa}{3\eta_2 \eta_1} + \frac{31}{\eta_1^2} - \frac{\kappa^3}{6} + \frac{2\kappa}{3\eta_1^2} + \frac{43}{\eta_2 \eta_1} \right) X^4 \right. \\
&+ \left. \left( -\frac{10\kappa}{3\eta_2 \eta_1} + \frac{8}{3\eta_1^2} + \frac{1}{\eta_2 \eta_1} - \frac{5\kappa^2}{6} + \frac{12}{\kappa \eta_2 \eta_1} + \frac{2\kappa}{\eta_2^2} + \frac{263}{6\eta_1^2 \kappa} \right) X^2 \right] t^3, \\
u_3^4 &= \left[ \left( -\frac{\kappa}{\eta_2} - \frac{\kappa}{\eta_1} \right) X^4 + \left( -\frac{16}{\kappa \eta_2 \eta_1} - \frac{26}{\eta_1^2 \kappa} + \frac{\kappa^2}{2} \right) X^3 + \frac{9X^2}{2\eta_1} + \left( \frac{6}{\eta_1^2 \kappa^2} - \frac{\kappa}{2} + \frac{2}{\eta_1^2 \kappa} \right) X \right. \\
&- \left. \frac{2}{\eta_1 \kappa} \right] t^2 + \left[ \left( -\frac{43}{\eta_2 \eta_1^2} - \frac{30}{\eta_1^3} - \frac{28}{\eta_2^2 \eta_1} \right) X^7 + \left( \frac{14\kappa}{3\eta_1^2} + \frac{13\kappa}{3\eta_2 \eta_1} \right) X^6 \right. \\
&+ \left. \left( \frac{28}{3\eta_1^3} + \frac{4}{3\eta_2^2 \eta_1} + \frac{124}{3\kappa \eta_1^3} - \frac{2\kappa^2}{3\eta_1} + \frac{98}{3\kappa \eta_2 \eta_1^2} + \frac{26}{3\eta_2 \eta_1^2} \right) X^5 \right. \\
&- \left. \left( \frac{12}{\kappa \eta_2 \eta_1} + \frac{10\kappa}{3\eta_2 \eta_1} - \frac{2}{\eta_1^2} - \frac{2}{3\eta_2 \eta_1} - \frac{49}{3\eta_1^2 \kappa} \right) X^2 + \left( -\frac{2\kappa}{3\eta_1^2} - \frac{20}{\eta_2 \eta_1} - \frac{21}{\eta_1^2} - \frac{\kappa}{3\eta_2 \eta_1} \right) X^4 \right. \\
&- \left. \left( \frac{36}{\kappa^3 \eta_1^2 \eta_2} + \frac{6}{\eta_2 \eta_1^2 \kappa^2} + \frac{3}{\eta_1} + \frac{5}{\eta_2 \eta_1^2} + \frac{\kappa}{3\eta_1} - \frac{2}{\eta_2^2 \eta_1} \right) X + \frac{4}{\kappa^2 \eta_1 \eta_2} - \frac{1}{\eta_1^2 \kappa} + \frac{2\kappa}{3\eta_2 \eta_1} \right. \\
&+ \left. \left( -\frac{2}{\kappa \eta_2 \eta_1^2} + \frac{80}{\kappa^2 \eta_2^2 \eta_1} - \frac{16}{3\kappa \eta_1^3} - \frac{10}{\eta_2^2 \eta_1} + \frac{74}{3\eta_2 \eta_1^2} + \frac{124}{\eta_2 \eta_1^2 \kappa^2} + \frac{14\kappa}{3\eta_1} - \frac{2}{3\eta_1^3} - \frac{16}{\kappa^2 \eta_1^3} \right) X^3 \right] t^3.
\end{aligned}$$

Therefore, we deduce an approximate solution to the Ginzburg-Landau equations (7.2.10) subject to the initial condition (8.2.17) as follows:

$$A^1 = A^2 = A^3 \simeq u_0^1 + u_1^1 + u_2^1 + u_3^1, \quad \phi \simeq \tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3,$$

$$\psi^1 \simeq u_0^4 + u_1^4 + u_2^4 + u_3^4, \quad \psi^2 \simeq u_0^5 + u_1^5 + u_2^5 + u_3^5.$$

The corresponding graphic is shown in the Figure 8.1 for fixed parameters  $\eta_1 = \eta_2 = 1$  and  $\kappa = 10^{-3}$ .

In this chapter, we have shown that the Adomian decomposition method is also adaptable to provide approximate solution to under-determined systems of NLPDEs.

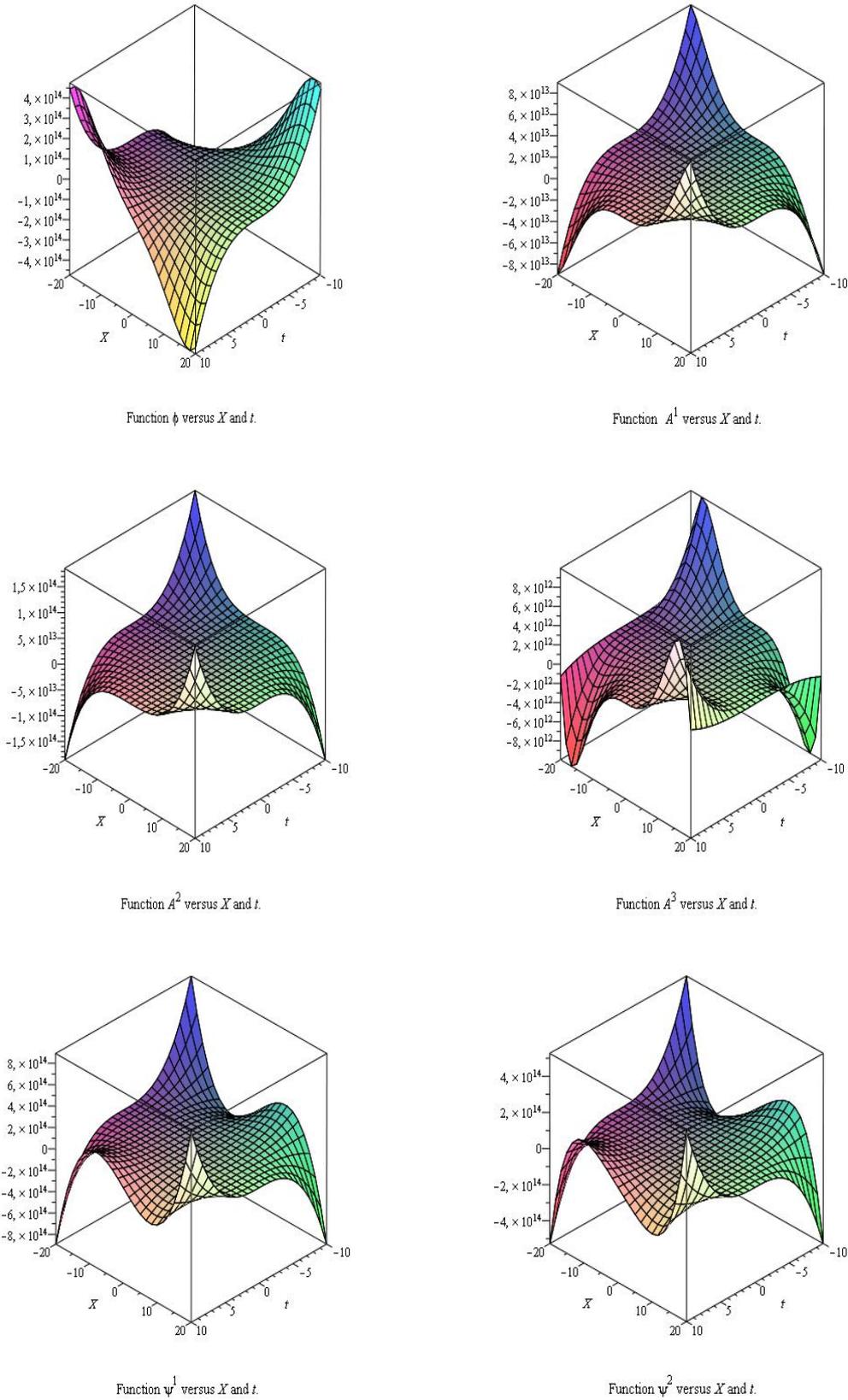


Figure 8.1: Approximate solution of the Ginzburg-Landau system (8.2.16) versus  $X = x + y + z$  and  $t$ .

## Part III

# Symmetry Groups and Conservation Laws of PDEs





# Symmetry Groups, Noether Theorem and Conservation Laws

---

In this chapter, we extend the Ibragimov's conservation theorem for PDEs [57] to under determined systems of differential equations. The concepts of adjoint equation and formal Lagrangian for a system of differential equations whose the number of equations is equal to or lower than the number of dependent variables are defined. It is proved that the system given by an equation and its adjoint is associated with a variational problem (with or without classical Lagrangian) and inherits all Lie-point symmetries from the original equation. Accordingly, a Noether theorem for conservation laws can be formulated. The main results of this chapter have been published [48].

## 9.1 Basic definitions and Noether theorem

Consider  $X$ , an  $n$ -dimensional independent variable space, and  $U$ , an  $m$ -dimensional dependent variable space. Let  $x = (x^1, \dots, x^n) \in X$  and  $u = (u^1, \dots, u^m) \in U$ . Let  $s \in \mathbb{N}$ , we denote by  $X \times U^{(s)}$ , the total space whose coordinates are denoted by  $(x, u^{(s)})$ , encompassing the independent variables  $x$  and the dependent variables with their derivatives up to order  $s$ , globally denoted by  $u^{(s)}$ .

**Definition 9.1.1 (Total derivative operator)** *The total derivative operator is a differential operator denoted  $D_{x^i}$  whose action on a smooth function  $P = P(x, u^{(s)}(x))$  is*

$$D_{x^i} P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^m \sum_{k=1}^s \sum_{h=1}^{p_k} \frac{\partial u_{(k)}^\alpha [h]}{\partial x^i} \frac{\partial P}{\partial u_{(k)}^\alpha [h]}. \quad (9.1.1)$$

**Definition 9.1.2 (Euler-Lagrange operator)** *The Euler-Lagrange operator is a differential operator denoted  $\frac{\delta}{\delta u}$ , whose action on a smooth function  $P = P(x, u^{(s)}(x))$  is*

$$\frac{\delta}{\delta u} P = \left( \frac{\delta}{\delta u^1} P, \dots, \frac{\delta}{\delta u^m} P \right),$$

where for  $\alpha = 1, \dots, m$

$$\frac{\delta}{\delta u^\alpha} P = \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial P(x, u^{(s)}(x))}{\partial u_{(k)}^\alpha [h]} \right)_{(k)} [h]. \quad (9.1.2)$$

Consider a functional  $\mathcal{F}$  defined by the variational integral

$$\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s)}(x)) dx, \quad (9.1.3)$$

where  $\Lambda$  is a connected open subset of  $X$  and  $L$  defined on  $\Lambda \times U^{(s)}$  is a smooth function called the Lagrangian of the functional  $\mathcal{F}$ .

---

**Theorem 9.1.1** *The Euler-Lagrange equations [81] associated to the functional  $\mathcal{F}$  are*

$$\frac{\delta}{\delta u^\alpha} L(x, u^{(s)}(x)) = 0, \quad \alpha = 1, \dots, m. \quad (9.1.4)$$

**Definition 9.1.3 (Variational symmetry group)** *A local group of transformations  $G$  acting on  $\Lambda \times U$  is a variational symmetry group for the functional  $\mathcal{F}$  if for all  $(x, u) \in \Lambda \times U$  and  $g \in G$  such that  $g \cdot (x, u) = (\tilde{x}, \tilde{u}) \in \tilde{\Lambda} \times U$  with  $\tilde{\Lambda} \subset \Lambda$ , then*

$$\int_{\tilde{\Lambda}} L(\tilde{x}, \tilde{u}^{(s)}(\tilde{x})) d\tilde{x} = \int_{\Lambda} L(x, u^{(s)}(x)) dx. \quad (9.1.5)$$

**Theorem 9.1.2 (Variational infinitesimal test)** *A local group of transformations  $G$  acting on  $\Lambda \times U$  is a variational symmetry group for the functional  $\mathcal{F}$  if and only if there exists a  $n$ -tuple  $B = (B^1, \dots, B^n)$  such that*

$$\text{Pr}^{(s)} X(L) + L \text{Div} \xi = \text{Div} B \quad (9.1.6)$$

for every infinitesimal generator  $X$  of  $G$

$$X = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta_\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

**Theorem 9.1.3 ([81])** *If  $G$  is a variational symmetry group for the functional  $\mathcal{F}$ , then  $G$  is a symmetry group for the Euler-Lagrange equations (9.1.4).*

**Theorem 9.1.4 (Noether theorem)** *Let  $G$  be a one-parameter variational symmetry group for the functional  $\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s)}(x)) dx$ , and*

$$X = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be an infinitesimal generator of  $G$ . Then the Euler-Lagrange equations (9.1.4) have a conservation law:

$$\text{Div} C \equiv D_{x^1} C^1 + \dots + D_{x^n} C^n = 0 \quad (9.1.7)$$

whenever  $u$  satisfies the Euler-Lagrange equations. The components of the conserved vector  $C = (C^1(x, u^{(s)}), \dots, C^n(x, u^{(s)}))$  are explicitly given by:

$$\begin{aligned} C^i &= -B^i + \xi^i L + \sum_{\alpha=1}^m \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} W_{(k)}^\alpha[h] \\ &\times \left[ \sum_{k'=k}^s (-1)^{k'} \sum_{h'=1}^{p_{k'}} \left( \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k'+1)}^\alpha [n(h'-1) + i]} \right)_{(k')} [h'] \right] \end{aligned} \quad (9.1.8)$$

with  $B = (B^1, \dots, B^n)$  fulfilling the condition (9.1.6), and

$$W^\alpha = \eta^\alpha - \sum_{j=1}^n \xi^j(x, u) \frac{\partial u^\alpha}{\partial x^j}, \quad \alpha = 1, \dots, m. \quad (9.1.9)$$

## 9.2 Conservation laws for determined systems of NLPDEs

The new conservation laws theorem established by Ibragimov [57] concerns a system of  $s$ -order PDEs of the form

$$F_\alpha(x, u^{(s)}) = 0, \quad \alpha = 1, \dots, m, \quad (9.2.10)$$

where  $F_\alpha$  are differentiable functions having  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m$  dependent variables  $u = (u^1, \dots, u^m)$ ,  $u = u(x)$ .

**Definition 9.2.1 (Adjoint equation)** *The adjoint system of (9.2.10) is defined by*

$$F_\alpha^*(x, u^{(s)}, v^{(s)}) \equiv \frac{\delta}{\delta u^\alpha} \left[ \sum_{\beta=1}^m v^\beta F_\beta \right] = 0, \quad \alpha = 1, \dots, m, \quad (9.2.11)$$

where  $v = (v^1, \dots, v^m)$ ,  $v = v(x)$  are new dependent variables also called nonlocal variables.

**Theorem 9.2.1** *The global system formed by the equations of the original system (9.2.10) and that of the adjoint system (9.2.11) constitutes the Euler-Lagrange equations associated to a functional defined by a variational integral whose the Lagrangian is*

$$L(x, u^{(s)}, v^{(s)}) = \sum_{\beta=1}^m v^\beta F_\beta. \quad (9.2.12)$$

**Definition 9.2.2** *The system (9.2.10) is called self-adjoint if the substitution  $v = u$  in (9.2.11) gives*

$$F_\alpha^* = \sum_{\beta=1}^m \Gamma_{\alpha\beta}(x, u^{(s)}) F_\beta, \quad \alpha = 1, \dots, m. \quad (9.2.13)$$

*The system (9.2.10) is called quasi-self-adjoint if there exists a differentiable function  $h(u)$  such that equations (9.2.13) hold upon the substitution  $v = h(u)$  in (9.2.11).*

**Theorem 9.2.2 ([57])** *Consider the system (9.2.10) together with its adjoint (9.2.11). Let*

$$X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_\beta \frac{\partial}{\partial u^\beta} \quad (9.2.14)$$

*be an infinitesimal generator of a symmetry group for the original system (9.2.10), i.e.*

$$\text{Pr}^{(s)} X(F_\alpha) = \sum_{\beta=1}^m \lambda_{\alpha\beta}(x, u^{(s)}) F_\beta. \quad (9.2.15)$$

*Then, the variational problem defined by the Lagrangian*

$$L(x, u^{(s)}, v^{(s)}) = \sum_{\beta=1}^m v^\beta F_\beta \quad (9.2.16)$$

whose associated Euler-Lagrange equations are given by the global system (9.2.10)-(9.2.11) has a variational symmetry group generated by the operator

$$Y = X + \sum_{\beta=1}^m \eta_{\beta}^* \frac{\partial}{\partial v^{\beta}}, \quad (9.2.17)$$

where

$$\eta_{\beta}^* = - \left[ \sum_{\alpha=1}^m v^{\alpha} \lambda_{\alpha\beta} + v^{\beta} \sum_{i=1}^n D_{x^i} (\xi^i) \right]. \quad (9.2.18)$$

We can say that every infinitesimal generator  $X$  of the original system (9.2.10), provides a conservation law for the global system (9.2.10)-(9.2.11). Indeed, from the previous development, the construction of conservation laws for a system of PDEs possessing a symmetry group with generator  $X$  can be performed in three steps:

Step 1: Find the adjoint system.

Step 2: Extend the operator  $X$  to a generator  $Y$  of symmetry group for the global system.

Step 3: Use the Noether theorem to compute explicitly the components of a conserved vector for the global system.

In the case where the original system is quasi-self-adjoint, its conserved vectors are deduced from those of the global system by a suitable substitution of the nonlocal variables.

### Application to the modified Korteweg de Vries (mKdV) equation

For the mKdV equation

$$F(t, x, u^{(3)}) \equiv u_t + \beta u_{3x} + \alpha u^p u_x = 0, \quad (9.2.19)$$

one of the infinitesimal generators is the scaling transformation operator

$$X = -3t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + \frac{2}{p} \frac{\partial}{\partial u}. \quad (9.2.20)$$

The equality  $\text{Pr}^{(3)} X(F) = \frac{3p+2}{p} F$  implies that  $\lambda = \frac{3p+2}{p}$ . Thus, we deduce that a generator of symmetry group  $Y$  of the variational problem defined by the Lagrangian  $L(t, x, u^{(3)}, v) = v F(t, x, u^{(3)})$  is afforded by

$$Y = -3t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + \frac{2}{p} \frac{\partial}{\partial u} + \eta^* \frac{\partial}{\partial v}, \quad (9.2.21)$$

with  $\eta^* = \frac{p-2}{p} v$ . The adjoint equation  $F^*$  of (9.2.19) is given by

$$F^*(t, x, u^{(3)}, v^{(3)}) \equiv \frac{\delta}{\delta u} (v F) = -\alpha u^p v_x - v_t - \beta v_{3x} = 0. \quad (9.2.22)$$

The equality  $F^*|_{v=u} = -F$  implies that  $\Gamma = -1$ . Thus, we infer that the mKdV equation (9.2.19) is self-adjoint.

By the Noether theorem, the components of a conserved vector  $C = (C^1, C^2)$  for the system (9.2.19)-(9.2.22) are given by:

$$\begin{aligned} C^1 &= v \left( -3\beta t u_{3x} - 3\alpha t u^p u_x + \frac{2}{p} u + x u_x \right), \\ C^2 &= -x v u_t + \frac{2\alpha}{p} v u^{p+1} + \frac{2\beta}{p} u v_{2x} + \beta x u_x v_{2x} + 3\alpha t v u_t u^p + 3\beta t u_t v_{2x} \\ &\quad - \beta \left( \frac{2}{p} + 1 \right) v_x u_x - \beta x v_x u_{2x} - 3\beta t v_x u_{tx} + \beta \left( \frac{2}{p} + 2 \right) v u_{2x} + 3\beta t v u_{t2x}. \end{aligned}$$

The self-adjointness of the mKdV equation (9.2.19) guarantees that its conserved vector is  $C|_{v=u} = (C^1|_{v=u}, C^2|_{v=u})$  where

$$\begin{aligned} C^1|_{v=u} &= -3\beta t u u_{3x} - 3\alpha t u^{p+1} u_x + \frac{2}{p} u^2 + x u u_x, \\ C^2|_{v=u} &= -x u u_t + \frac{2\alpha}{p} u^{p+2} + 3\alpha t u_t u^{p+1} + 3\beta t u_t u_{2x} - \beta \left( \frac{2}{p} + 1 \right) u_x^2 \\ &\quad - 3\beta t u_x u_{tx} + \beta \left( \frac{4}{p} + 1 \right) u u_{2x} + 3\beta t u u_{t2x}. \end{aligned}$$

### 9.3 Conservation laws for under determined systems of NLPDEs

**Definition 9.3.1 (Adjoint equation)** Consider the system

$$F_\alpha \left( x, u^{(s)}, \tilde{u}^{(s)} \right) = 0, \quad \alpha = 1, \dots, m, \tag{9.3.23}$$

where  $F_\alpha$  are differentiable functions having  $n$  independent variables  $x = (x^1, \dots, x^n)$  and  $m + \tilde{m}$  dependent variables  $u = (u^1, \dots, u^m)$ ,  $\tilde{u} = (\tilde{u}^1, \dots, \tilde{u}^{\tilde{m}})$ ,  $u = u(x)$ ,  $\tilde{u} = \tilde{u}(x)$ ;  $u^{(s)}$  (resp.  $\tilde{u}^{(s)}$ ) is a vector encompassing dependent variable  $u$  (resp.  $\tilde{u}$ ) and their derivatives up to order  $s$ . We introduce the differentiable functions

$$\begin{aligned} F_\alpha^* &= \frac{\delta}{\delta u^\alpha} \left[ \sum_{\beta=1}^m v^\beta F_\beta + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m F_\nu \right) \right] \\ \tilde{F}_\alpha^* &= \frac{\delta}{\delta \tilde{u}^\alpha} \left[ \sum_{\beta=1}^m v^\beta F_\beta + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m F_\nu \right) \right], \end{aligned}$$

where  $v = (v^1, \dots, v^m)$  and  $\tilde{v} = (\tilde{v}^1, \dots, \tilde{v}^{\tilde{m}})$  are new dependent variables also called non local variables. Then, we define the corresponding system of adjoint equations by

$$F_\alpha^* \left( x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)} \right) = 0, \quad \alpha = 1, \dots, m \tag{9.3.24}$$

$$\tilde{F}_\alpha^* \left( x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)} \right) = 0, \quad \tilde{\alpha} = 1, \dots, \tilde{m}. \tag{9.3.25}$$

**Theorem 9.3.1** Any system of PDEs (9.3.23):  $F_\alpha(x, u^{(s)}, \tilde{u}^{(s)}) = 0$ ,  $\alpha = 1, \dots, m$ , considered together with their adjoint equations (9.3.24)-(9.3.25), has a Lagrangian. Namely, the Eqs (9.3.23)-(9.3.25) with  $2(m + \tilde{m})$  unknowns are the Euler-Lagrange equations with Lagrangian

$$L(x, u^{(s)}, \tilde{u}^{(s)}, v^{(s)}, \tilde{v}^{(s)}) = \sum_{\beta=1}^m v^\beta F_\beta + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m F_\nu \right).$$

**Proof.** It is immediate from the definitions of Euler-Lagrange equations and adjoint equations.

$$\frac{\delta L}{\delta v^\alpha} = F_\alpha, \quad \alpha = 1, \dots, m; \quad \frac{\delta L}{\delta \tilde{v}^{\tilde{\alpha}}} = \sum_{\nu=1}^m F_\nu, \quad \tilde{\alpha} = 1, \dots, \tilde{m}$$

and by definition of adjoint equations

$$\frac{\delta L}{\delta u^\alpha} = F_\alpha^*, \quad \alpha = 1, \dots, m; \quad \frac{\delta L}{\delta \tilde{u}^{\tilde{\alpha}}} = \tilde{F}_{\tilde{\alpha}}^*, \quad \tilde{\alpha} = 1, \dots, \tilde{m}.$$

■

**Definition 9.3.2** The system (9.3.23) is called self-adjoint if the substitution  $(v, \tilde{v}) = (u, \tilde{u})$  in its adjoint Eqs. (9.3.24)-(9.3.25) gives, for some differentiable functions  $\Gamma_{\alpha\nu}$  and  $\tilde{\Gamma}_{\tilde{\alpha}\nu}$ ,

$$F_\alpha^* = \sum_{\nu=1}^m \Gamma_{\alpha\nu}(x, u^{(s)}, \tilde{u}^{(s)}) F_\nu, \quad \alpha = 1, \dots, m, \quad (9.3.26)$$

$$\tilde{F}_{\tilde{\alpha}}^* = \sum_{\nu=1}^m \tilde{\Gamma}_{\tilde{\alpha}\nu}(x, u^{(s)}, \tilde{u}^{(s)}) F_\nu, \quad \tilde{\alpha} = 1, \dots, \tilde{m}. \quad (9.3.27)$$

The system (9.3.23) is called quasi-self-adjoint if there exist two differentiable functions  $h(u, \tilde{u})$  and  $\tilde{h}(u, \tilde{u})$  such that Eqs. (9.3.26)-(9.3.27) hold upon the substitution  $(v, \tilde{v}) = (h(u, \tilde{u}), \tilde{h}(u, \tilde{u}))$  in its adjoint Eqs. (9.3.24)-(9.3.25).

Provided these statements, we can now provide the main results of this chapter.

**Theorem 9.3.2** Consider the system (9.3.23). Then its adjoint Eqs. (9.3.24)-(9.3.25) inherits symmetries of equations (9.3.23). Namely, if the system (9.3.23) admits symmetry group generated by an operator

$$X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_\beta \frac{\partial}{\partial u^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}} \frac{\partial}{\partial \tilde{u}^{\tilde{\beta}}}, \quad (9.3.28)$$

where  $\xi^i = \xi^i(x, u, \tilde{u})$ ,  $\eta_\beta = \eta_\beta(x, u, \tilde{u})$ ,  $\tilde{\eta}_{\tilde{\beta}} = \tilde{\eta}_{\tilde{\beta}}(x, u, \tilde{u})$  such that

$$Pr^{(s)} X(F_\alpha) = \sum_{\beta=1}^m \lambda_{\alpha\beta}(x, u^{(s)}, \tilde{u}^{(s)}) F_\beta,$$

then the equations (9.3.24)-(9.3.25) have the generator of symmetries

$$Y = X + \sum_{\beta=1}^m \eta_\beta^* \frac{\partial}{\partial v^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}}^* \frac{\partial}{\partial \tilde{v}^{\tilde{\beta}}}, \quad (9.3.29)$$

where

$$\begin{aligned} \eta_\beta^* + \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\alpha}}^* &= - \left[ \sum_{\alpha=1}^m v^\alpha \lambda_{\alpha\beta} + v^\beta \sum_{i=1}^n D_{x^i}(\xi^i) \right. \\ &\quad \left. + \left( \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \sum_{\alpha=1}^m \lambda_{\alpha\beta} + \left( \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \left( \sum_{i=1}^n D_{x^i}(\xi^i) \right) \right] \end{aligned} \quad (9.3.30)$$

which is satisfied in particular for

$$\eta_\beta^* = - \left[ \sum_{\alpha=1}^m \left( v^\alpha + \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \lambda_{\alpha\beta} + v^\beta \sum_{i=1}^n D_{x^i}(\xi^i) \right], \quad \tilde{\eta}_{\tilde{\alpha}}^* = -\tilde{v}^{\tilde{\alpha}} \sum_{i=1}^n D_{x^i}(\xi^i).$$

**Proof.** By the variational infinitesimal test, we have:

$$\begin{aligned} 0 &= \text{Pr}^{(s)}Y(L) + L \sum_{i=1}^n D_{x^i}(\xi^i) \\ &= \sum_{\beta=1}^m \text{Pr}^{(s)}Y(v^\beta)F_\beta + \sum_{\beta=1}^m v^\beta \text{Pr}^{(s)}X(F_\beta) \\ &\quad + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \text{Pr}^{(s)}Y(\tilde{v}^{\tilde{\beta}}) \right) \left( \sum_{\nu=1}^m F_\nu \right) + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m \text{Pr}^{(s)}X(F_\nu) \right) \\ &\quad + \left[ \sum_{\beta=1}^m v^\beta F_\beta + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m F_\nu \right) \right] \sum_{i=1}^n D_{x^i}(\xi^i) \\ &= \sum_{\beta=1}^m \eta_\beta^* F_\beta + \sum_{\beta=1}^m \sum_{\alpha=1}^m v^\beta \lambda_{\beta\alpha} F_\alpha \\ &\quad + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}}^* \right) \left( \sum_{\nu=1}^m F_\nu \right) + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m \sum_{\alpha=1}^m \lambda_{\nu\alpha} F_\alpha \right) \\ &\quad + \left[ \sum_{\beta=1}^m v^\beta F_\beta + \left( \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{v}^{\tilde{\beta}} \right) \left( \sum_{\nu=1}^m F_\nu \right) \right] \sum_{i=1}^n D_{x^i}(\xi^i) \\ &= \sum_{\beta=1}^m \left[ \eta_\beta^* + \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\alpha}}^* + \sum_{\alpha=1}^m v^\alpha \lambda_{\alpha\beta} + \left( \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \sum_{\alpha=1}^m \lambda_{\alpha\beta} \right. \\ &\quad \left. + v^\beta \sum_{i=1}^n D_{x^i}(\xi^i) + \left( \sum_{\tilde{\alpha}=1}^{\tilde{m}} \tilde{v}^{\tilde{\alpha}} \right) \sum_{i=1}^n D_{x^i}(\xi^i) \right] F_\beta. \end{aligned}$$

Setting the coefficients of  $F_\beta$  to 0 ends the proof. ■

Finally, as in the previous section, we can say that every infinitesimal generator

$$X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{\beta=1}^m \eta_\beta \frac{\partial}{\partial u^\beta} + \sum_{\tilde{\beta}=1}^{\tilde{m}} \tilde{\eta}_{\tilde{\beta}} \frac{\partial}{\partial \tilde{u}^{\tilde{\beta}}} \quad (9.3.31)$$

of differential equations (9.3.23) provides a conservation law for the global system (9.3.23)-(9.3.24)-(9.3.25).

# Scaling Symmetries, Homotopy Operators and Conservation Laws

---

In this chapter, we propose an alternative direct algebraic method of constructing, for nonlinear evolution partial differential equations, conservation laws that depend not only on dependent variables and its derivatives but also explicitly on independent variables. As illustration, the fifth order Korteweg de Vries (fKdV) and modified  $(n + 1)$ -dimensional Zakharov-Kuznetsov (ZK) equations are probed. The main results of this chapter have been published [50].

## 10.1 Basic definitions and theorems

This section, mainly based on [81, 83], addresses relevant definitions, theorems and notations playing a central role in studying conservation laws for nonlinear PDEs. Consider a system of  $s$ -order PDEs

$$F_\nu \left( x, u^{(s)}(x) \right) = 0, \quad \nu = 1, \dots, m, \quad (10.1.1)$$

where  $x = (x^1, \dots, x^n)$  and  $u = (u^1, \dots, u^m)$  and  $X \times U^{(s)}$ , the space whose coordinates are denoted by  $(x, u^{(s)})$ , (encompassing the independent variables  $x$ , the dependent variables  $u$  and their derivatives up to order  $s$ ,  $u^{(s)}$ ).

**Definition 10.1.1 (Differential function)** *A function  $f$  defined on  $X \times U^{(s)}$  is called differential function if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments.*

**Definition 10.1.2 (Invariant scaling or dilatation group for a PDE)** *The system of  $s$ -order PDEs*

$$F_\nu \left( x, u^{(s)}(x) \right) = 0, \quad \nu = 1, \dots, m$$

*is said to be invariant under a dilatation group if there exist a nonzero parameter  $\lambda$  and vector constants  $(a^1, \dots, a^n) \in \mathbb{R}_*^n$ , and  $(b^1, \dots, b^m) \in \mathbb{R}_*^m$  such that*

$$F_\nu \left( \tilde{x}, \tilde{u}^{(s)}(\tilde{x}) \right) = 0, \quad \nu = 1, \dots, m$$

*with  $\tilde{x} = (\lambda^{a_1} x^1, \dots, \lambda^{a_n} x^n)$  and  $\tilde{u} = (\lambda^{b_1} u^1, \dots, \lambda^{b_m} u^m)$ .*

*Moreover, one can attribute weights to the variables and total derivatives as follows:*

$$w(x^i) = a_i, \quad w(u^j) = b_j, \quad w(D_{x^i}) = -a_i.$$

**Definition 10.1.3 (Rank of a monomial)**

---

1. A monomial is a term  $M$  of a differential polynomial function expressed as:

$$M = c \left( \prod_{i=1}^n (x^i)^{n_i} \right) \prod_{\alpha=1}^m \prod_{l=1}^{r_\alpha} \left( D_{x^1}^{k_{\alpha,l,1}} \cdots D_{x^n}^{k_{\alpha,l,n}} u^\alpha \right)^{m_{\alpha,l}}$$

with  $c \in \mathbb{R}_*$ ,  $n_i, r_\alpha, m_{\alpha,l}, k_{\alpha,l,j} \in \mathbb{N}$  such that  $(k_{\alpha,l,1}, \dots, k_{\alpha,l,n}) \neq (k_{\alpha,\hat{l},1}, \dots, k_{\alpha,\hat{l},n})$  for  $l \neq \hat{l}$ .

2. The rank of the monomial  $M$  is the real number

$$\text{rank}(M) = \sum_{i=1}^n n_i W(x^i) + \sum_{\alpha=1}^m \sum_{l=1}^{r_\alpha} m_{\alpha,l} \left[ W(u^\alpha) + \sum_{j=1}^n k_{\alpha,l,j} W(D_{x^j}) \right].$$

3. A differential polynomial function is said to be uniform in rank if all its monomials have the same rank.

**Proposition 10.1.1** *The differential functions defining a system of polynomial partial differential equations, invariant under a dilation group, are uniform in rank.*

**Definition 10.1.4 (Zeroth-Euler operator)** *Let  $f$  defined on  $X \times U^{(s)}$  be a differential function. The zeroth-Euler-Lagrange operator (also called the variational derivative) of  $f$  is given by*

$$\frac{\delta}{\delta u} f = \left( \frac{\delta}{\delta u^1} f, \dots, \frac{\delta}{\delta u^m} f \right),$$

where for  $\alpha = 1, \dots, m$

$$\frac{\delta}{\delta u^\alpha} f = \sum_{k_1=0}^s \cdots \sum_{k_n=0}^s (-D_{x^1})^{k_1} \cdots (-D_{x^n})^{k_n} \frac{\partial f}{\partial u_{k_1 x^1 \cdots k_n x^n}^\alpha}.$$

**Definition 10.1.5 (Total divergence)** *A total divergence of an  $n$ -dimensional differential function  $F = (F^1, \dots, F^n)$  is defined by:*

$$\text{Div } F = D_{x^1} F^1 + \cdots + D_{x^n} F^n.$$

**Theorem 10.1.1 (Exactness theorem)** *A differential function  $f$  defined on  $X \times U^{(s)}$  is exact if there exists a differential vector function  $C = (C^1(x, u^{(s-1)}), \dots, C^n(x, u^{(s-1)}))$  such that*

$$f = \text{Div } C.$$

$f$  is said to be exact (or a divergence) if and only if

$$\frac{\delta}{\delta u^\alpha} f = 0, \quad \alpha = 1, \dots, m.$$

**Theorem 10.1.2 (Characterization of divergence-equivalent terms)** *Two or more terms are divergence-equivalent if there exists a linear combination of these terms which is a divergence. The images of divergence-equivalent terms under the zeroth-Euler operator are linearly dependent.*

## 10.2 Homotopy operator formula

**Definition 10.2.1 (Higher-Euler operator)** Let  $f$  defined on  $X \times U^{(s)}$  be a differential function. The  $(i_1, \dots, i_n)$ -higher-Euler-Lagrange operator (also called the higher variational derivative) of  $f$  is given by

$$\frac{\delta^{(i_1, \dots, i_n)}}{\delta u} f = \left( \frac{\delta^{(i_1, \dots, i_n)}}{\delta u^1} f, \dots, \frac{\delta^{(i_1, \dots, i_n)}}{\delta u^m} f \right),$$

where for  $\alpha = 1, \dots, m$

$$\frac{\delta^{(i_1, \dots, i_n)}}{\delta u^\alpha} f = \sum_{k_1=i_1}^s \dots \sum_{k_n=i_n}^s \binom{k_1}{i_1} \dots \binom{k_n}{i_n} (-D_{x^1})^{k_1-i_1} \dots (-D_{x^n})^{k_n-i_n} \frac{\partial f}{\partial u_{k_1 x^1 \dots k_n x^n}^\alpha}.$$

**Definition 10.2.2 (Homotopy operator)** Let  $f$  defined on  $X \times U^{(s)}$  be an exact differential function with  $n$  independent variables  $x = (x^1, \dots, x^n)$ . The  $n$ -dimensional homotopy operator is an  $n$ -component vector,  $(H_u^{(x^1)} f, \dots, H_u^{(x^n)} f)$ , where for  $i = 1, \dots, n$

$$H_u^{(x^i)} f = \int_0^1 \sum_{\alpha=1}^m (I_{u^\alpha}^{(x^i)} f) [\lambda u] \frac{d\lambda}{\lambda}$$

with the integrands  $I_{u^\alpha}^{(x^i)} f$  defined as

$$I_{u^\alpha}^{(x^i)} f = \sum_{k_1=0}^s \dots \sum_{k_i=0}^{s-1} \dots \sum_{k_n=0}^s \frac{1+k_i}{1+k_1+\dots+k_n} D_{x^1}^{k_1} \dots D_{x^n}^{k_n} \left( u^\alpha \frac{\delta^{(k_1, \dots, k_i+1, \dots, k_n)}}{\delta u^\alpha} f \right).$$

The notation  $(I_{u^\alpha}^{(x^i)} f) [\lambda u]$  means that in  $I_{u^\alpha}^{(x^i)} f$ , all components of the function  $u$  as well as their derivatives are multiplied by  $\lambda$ .

**Theorem 10.2.1** Let  $f$  defined on  $X \times U^{(s)}$  be an exact differential function with  $n$  independent variables  $x = (x^1, \dots, x^n)$ . Then

$$\text{Div}^{-1} f = (H_u^{(x^1)} f, \dots, H_u^{(x^n)} f).$$

In one, two and three independent variables, the homotopy operator takes the following forms.

- **One-dimensional homotopy operator**,  $x = x^1$ ,  $H_u f$  :

$$H_u f = \int_0^1 \sum_{\alpha=1}^m (I_{u^\alpha} f) [\lambda u] \frac{d\lambda}{\lambda},$$

where

$$I_{u^\alpha} f = \sum_{k=1}^s \left( \sum_{i=0}^{k-1} u_{ix}^\alpha (-D_x)^{k-i-1} \right) \frac{\partial f}{\partial u_{kx}^\alpha} \quad \text{and} \quad D_x^{-1} f = H_u f.$$

- *Two-dimensional homotopy operator*,  $x = (x^1, x^2)$ ,  $(H_u^{(x^1)} f, H_u^{(x^2)} f)$ :

$$H_u^{(x^i)} f = \int_0^1 \sum_{\alpha=1}^m (I_{u^\alpha}^{(x^i)} f) [\lambda u] \frac{d\lambda}{\lambda}, \quad i = 1, 2,$$

with the integrands  $I_{u^\alpha}^{(x^i)} f$  defined as

$$\begin{aligned} I_{u^\alpha}^{(x^i)} f &= \sum_{k_i=1}^s \sum_{k_p=0}^s \sum_{l_i=0}^{k_i-1} \sum_{l_p=0}^{k_p} u_{l_i x^i l_p x^p}^j \frac{\binom{l_i + l_p}{l_i} \binom{k_i + k_p - l_i - l_p - 1}{k_i - l_i - 1}}{\binom{k_i + k_p}{k_i}} \\ &\times (-D_{x^i})^{k_i - l_i - 1} (-D_{x^p})^{k_p - l_p} \frac{\partial f}{\partial u_{k_i x^i k_p x^p}^\alpha}, \end{aligned}$$

where

$$p \in \{1, 2\} \setminus \{i\} \quad \text{and} \quad \text{Div}^{-1} f = (H_u^{(x^1)} f, H_u^{(x^2)} f).$$

- *Three-dimensional homotopy operator*,  $x = (x^1, x^2, x^3)$ ,  $(H_u^{(x^1)} f, H_u^{(x^2)} f, H_u^{(x^3)} f)$ :

$$H_u^{(x^i)} f = \int_0^1 \sum_{\alpha=1}^m (I_{u^\alpha}^{(x^i)} f) [\lambda u] \frac{d\lambda}{\lambda}, \quad i = 1, 2, 3$$

with the integrands  $I_{u^\alpha}^{(x^i)} f$  defined as

$$\begin{aligned} I_{u^\alpha}^{(x^i)} f &= \sum_{k_i=1}^s \sum_{k_p=0}^s \sum_{k_q=0}^s \sum_{l_i=0}^{k_i-1} \sum_{l_p=0}^{k_p} \sum_{l_q=0}^{k_q} u_{l_i x^i l_p x^p l_q x^q}^\alpha \frac{\binom{l_i + l_p + l_q}{l_i} \binom{l_p + l_q}{l_p}}{\binom{k_i + k_p + k_q}{k_i}} \\ &\times \frac{\binom{k_i + k_p + k_q - l_i - l_p - l_q - 1}{k_i - l_i - 1} \binom{k_p + k_q - l_p - l_q}{k_p - l_p}}{\binom{k_p + k_q}{k_p}} \\ &\times (-D_{x^i})^{k_i - l_i - 1} (-D_{x^p})^{k_p - l_p} (-D_{x^q})^{k_q - l_q} \frac{\partial f}{\partial u_{k_i x^i k_p x^p k_q x^q}^\alpha}, \end{aligned}$$

where

$$p, q \in \{1, 2, 3\} \setminus \{i\}, \quad q + 1 \equiv i [3], \quad p \neq q \quad \text{and} \quad \text{Div}^{-1} f = (H_u^{(x^1)} f, H_u^{(x^2)} f, H_u^{(x^3)} f).$$

### 10.3 Conservation laws of evolution NLPDEs

Let

$$(u^\nu)_t = F_\nu \left( t, x, u^{(s)}(t, x) \right), \quad \nu = 1, \dots, m \quad (10.3.2)$$

be a system of  $(n + 1)$ -dimensional  $s$ -order differential equations, where  $x = (x^1, \dots, x^n)$  and  $u = (u^1, \dots, u^m)$ .

**Definition 10.3.1 (Conservation law for a PDE)** *A conservation law of (10.3.2) is the first order linear partial differential equation*

$$D_t \rho + \text{Div } J = 0 \tag{10.3.3}$$

which is also satisfied on every solution  $u = u(t, x)$  of the system (10.3.2), where  $\rho$  is called conserved density whose the associated conserved flux is the vector differential function  $J = (J^1, \dots, J^n)$ .

Assuming that (10.3.2) is of maximal rank, we propose to find its conservation laws. The approach consists of three successive steps: *i*) determination of a scaling symmetry group; *ii*) construction of a density and *iii*) calculation of a flux.

### 10.3.1 Determination of a scaling symmetry group

A scaling symmetry group can be obtained using linear algebra, which avoids having to solve a system of over determined PDEs. This is done by solving an algebraic system of weights for independent and dependent variables and total derivatives appearing in the PDEs, assuming that each equation of the system of PDEs is uniform in rank and taking into account the fact that the values of these ranks are not necessary the same for distinct equations. In other words, one makes the hypothesis that the  $R_\nu$  monomials of the function  $F_\nu$  have the same rank and one forms a set of  $R_\nu - 1$  equations whose unknowns are the weights of variables and total derivatives present in the function  $F_\nu$ . The final algebraic system of weight equations results from  $\sum_{\nu=1}^m (R_\nu - 1)$  equations of  $m$  different determined sets.

- In the case when the system (10.3.2) has a scaling symmetry group, since the scaling parameter  $\lambda$  is arbitrary, in general the solutions of the weight system can be of the form:

$$\begin{aligned} w(u^j) &= h_j(w(D_{x^{i_1}}), \dots, w(D_{x^{i_k}})), \quad j = 1, \dots, m \\ w(D_t) &= g_0(w(D_{x^{i_1}}), \dots, w(D_{x^{i_k}})), \\ w(D_{x^l}) &= g_l(w(D_{x^{i_1}}), \dots, w(D_{x^{i_k}})), \quad l \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}. \end{aligned} \tag{10.3.4}$$

Setting  $w(D_{x^{i_1}}) = r_{i_1}, \dots, w(D_{x^{i_k}}) = r_{i_k}$ , the relations (10.3.4) give:

$$\begin{aligned} w(u^j) &= h_j(r_{i_1}, \dots, r_{i_k}) \equiv s_j, \quad j = 1, \dots, m \\ w(D_t) &= g_0(r_{i_1}, \dots, r_{i_k}) \equiv r_0, \\ w(D_{x^l}) &= g_l(r_{i_1}, \dots, r_{i_k}) \equiv r_l, \quad l \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}. \end{aligned}$$

Therefore, the weights of independent variables are  $w(t) = -r_0$ ,  $w(x_i) = -r_i$  and a scaling symmetry group of the system (10.3.2) is:

$$(t, x, u) \mapsto (\lambda^{-r_0}t, \lambda^{-r_1}x^1, \dots, \lambda^{-r_n}x^n, \lambda^{s_1}u^1, \dots, \lambda^{s_m}u^m).$$

- When the system (10.3.2) does not possess a scaling symmetry group, sometimes identified by the solution of the weight system under the form  $w(u^j) = w(D_t) = w(D_{x^i}) = 0$ , one can transform it into another PDEs system of the form

$$(u^\nu)_t = F_\nu(t, x, u^{(s)}, \tilde{u}), \quad \nu = 1, \dots, m, \tag{10.3.5}$$

where  $\tilde{u} = (\tilde{u}^1, \dots, \tilde{u}^q)$  is a vector of new dependent variables, which now has a scaling symmetry group. Such a transformation can be performed by one or both of the following actions:

- (i) Conversion of some arbitrary monomial coefficients of considered system into new dependent variables;
- (ii) Suitable identification of monomials in the system and their multiplication by new dependent variables.

By the same arguments as before, the solution of the corresponding weight system yields a scaling symmetry group of the new system (10.3.5) expressed as:

$$(t, x, u, \tilde{u}) \mapsto (\lambda^{-r_0}t, \lambda^{-r_1}x^1, \dots, \lambda^{-r_n}x^n, \lambda^{s_1}u^1, \dots, \lambda^{s_m}u^m, \lambda^{e_1}\tilde{u}^1, \dots, \lambda^{e_q}\tilde{u}^q).$$

### 10.3.2 Construction of a density and its associated flux

In the sequel, we consider a PDEs system (10.3.5) and treat the new dependent variables  $\tilde{u}^1, \dots, \tilde{u}^q$  as weighted constants. Once a scaling symmetry group of equation (10.3.5) is defined, one fixes the value  $r$  of the rank for the density  $\rho$  which can be constructed step by step as follows:

- (i) Use the independent variables  $x$ , the (eventual) supplementary variables  $\tilde{u}$ , the dependent variables  $u$  and their partial derivatives to form a set  $Q$  of rank  $r$  monomials which are not divergences or divergence-equivalent terms:  $Q = \{M_\mu : \text{rank}(M_\mu) = r\}$  in which the monomials  $M_\mu$  are of the form

$$M_\mu = t^{n_{\mu,0}} \left( \prod_{i=1}^n (x^i)^{n_{\mu,i}} \right) \left( \prod_{\tilde{\alpha}=1}^q (\tilde{u}^{\tilde{\alpha}})^{n_{\mu,\tilde{\alpha}}} \right) \prod_{\alpha=1}^m \prod_{l=1}^{r_{\mu,\alpha}} \left( D_{x^1}^{k_{\mu,\alpha,l,1}} \dots D_{x^n}^{k_{\mu,\alpha,l,n}} u^\alpha \right)^{m_{\mu,\alpha,l}},$$

where  $n_{\mu,0}, n_{\mu,i}, r_{\mu,\alpha}, k_{\mu,\alpha,l,j}, m_{\mu,\alpha,l} \in \mathbb{N}$ .

- (ii) Linearly combine the monomials of  $Q$  to form a rank  $r$  candidate density  $\rho$ . By the conservation law (10.3.3):  $D_t \rho = -\text{Div} J$ ,  $D_t \rho$  is a divergence with respect to the space variables. Therefore, the undetermined coefficients of  $\rho$  are found by solving the linear algebraic system formed by setting to zero the coefficients of monomials in

$$\frac{\delta}{\delta u^j} \left( D_t \rho |_{u_t=F(t,x,u(s),\tilde{u})} \right) = 0, \quad j = 1, \dots, m. \quad (10.3.6)$$

Provided a density  $\rho$ , the corresponding flux  $J$  can be computed by using the homotopy operator as follows:

$$J = (J^1, \dots, J^n) = \text{Div}^{-1} (-D_t \rho |_{u_t=F}). \quad (10.3.7)$$

### 10.3.3 Application to a fifth order KdV equation

Consider the (1 + 1)-dimensional fifth order KdV equation given by [45]

$$u_t = -\gamma u_{5x} - \beta u_{3x} - \alpha u^p u_x \equiv F(t, x, u^{(5)}). \quad (10.3.8)$$

**Scaling symmetry group for the equation (10.3.10)**

The uniformity condition for the rank applied to the function  $F$  leads to the weight system

$$\begin{aligned} w(D_t) + w(u) &= 5w(D_x) + w(u) \\ &= 3w(D_x) + w(u) \\ &= w(D_x) + (p+1)w(u) \end{aligned} \quad (10.3.9)$$

whose solution is  $w(u) = w(D_t) = w(D_x) = 0$  indicating that (10.3.8) does not possess a scaling symmetry group. In order to get round this situation, we substitute the parameter  $\beta$  by a new dependent variable  $\tilde{u}(t, x)$ , i.e.  $\beta \leftrightarrow \tilde{u}(t, x)$ . Equation (10.3.8) then becomes

$$u_t = -\gamma u_{5x} - \tilde{u} u_{3x} - \alpha u^p u_x \equiv F(t, x, u^{(5)}, \tilde{u}), \quad (10.3.10)$$

with the corresponding weight system

$$\begin{aligned} w(D_t) + w(u) &= 5w(D_x) + w(u) \\ &= 3w(D_x) + w(u) + w(\tilde{u}) \\ &= w(D_x) + (p+1)w(u) \end{aligned} \quad (10.3.11)$$

yielding the solution

$$w(D_t) = 5w(D_x), \quad w(u) = \frac{4}{p}w(D_x), \quad w(\tilde{u}) = 2w(D_x). \quad (10.3.12)$$

Setting  $w(D_x) = 1$  leads to

$$w(x) = -1, \quad w(D_t) = 5 = -w(t), \quad w(u) = \frac{4}{p}, \quad w(\tilde{u}) = 2 \quad (10.3.13)$$

from which we deduce the one-parameter dilatation group of equation (10.3.10) as

$$(t, x, u, \tilde{u}) \mapsto (\lambda^{-5}t, \lambda^{-1}x, \lambda^{\frac{4}{p}}u, \lambda^2\tilde{u}). \quad (10.3.14)$$

Let us now compute the relevant quantities.

**Conserved density  $\rho_1$  of rank  $r = 8$  and its associated flux  $J$  for (10.3.10) with  $p = 1$** 

We have

$$Q = \{u^2, \tilde{u}^2 u\}.$$

From  $Q$ , we form a candidate density  $\rho_1 = c_1 u^2 + c_2 \tilde{u}^2 u$ . Condition (10.3.6) is satisfied for any constants  $c_1$  and  $c_2$ . Seeking  $c_1 = c_2 = 1$  gives

$$\rho_1 = u^2 + \tilde{u}^2 u.$$

Let  $E_1 = -D_t \rho_1|_{u_t=F(t,x,u^{(5)},\tilde{u})}$ . Applying the one dimensional homotopy operator to the differential function  $E_1$ , we obtain the flux

$$J = 2\gamma \left( uu_{4x} - u_x u_{3x} + \frac{1}{2} u_{2x}^2 \right) + 2\tilde{u} \left( uu_{2x} - \frac{1}{2} u_x^2 \right) + \frac{2}{3} \alpha u^3 + \gamma \tilde{u}^2 u_{4x} \tilde{u}^3 u_{2x} + \frac{1}{2} \tilde{u}^2 \alpha u^2.$$

Conserved density  $\rho_2$  of rank  $r = 8$  and its associated flux  $J$  for (10.3.10) with  $p = 2$

We get

$$Q = \{u^4, u^2 u_{2x}, \tilde{u} u^3, \tilde{u}^2 u^2, u_{2x}^2, \tilde{u}^3 u\}.$$

From  $Q$ , a candidate density can be written as  $\rho_2 = c_1 u^4 + c_2 u^2 u_{2x} + c_3 \tilde{u} u^3 + c_4 \tilde{u}^2 u^2 + c_5 u_{2x}^2 + c_6 \tilde{u}^3 u$ . Condition (10.3.6) is satisfied if and only if  $c_1 = c_2 = c_3 = c_5 = 0$  and for any constants  $c_4$  and  $c_6$ . Setting  $c_4 = c_6 = 1$  gives

$$\rho_2 = \tilde{u}^2 u^2 + \tilde{u}^3 u.$$

Let  $E_2 = -D_t \rho_2|_{u_t=F(t,x,u^{(s)},\tilde{u})}$ . Applying the one dimensional homotopy operator to the differential function  $E_2$ , we obtain the flux

$$J = 2\tilde{u}^3 \left( u u_{2x} - \frac{1}{2} u_x^2 \right) + \tilde{u}^3 \gamma u_{4x} + \frac{1}{2} \tilde{u}^2 \alpha u^4 + 2\tilde{u}^2 \gamma \left( u u_{4x} - u_x u_{3x} + \frac{1}{2} u_{2x}^2 \right) \frac{1}{3} \tilde{u}^3 \alpha u^3 + \tilde{u}^4 u_{2x}.$$

## 10.4 Time-space dependent conservation laws of some evolution NLPDEs

We start this section with the following statement.

**Proposition 10.4.1** *If a differential function  $\rho$  such that*

$$\sum_{i=1}^n D_{x^i} \rho = \sum_{\nu=1}^m F_\nu \quad (10.4.15)$$

*is a conserved density for the system (10.3.5) whose associated flux is  $J = (J^1, \dots, J^n)$ , then*

$$\tilde{\rho} = t\rho + \frac{1}{n} \left( \sum_{j=1}^n x^j \right) \sum_{\nu=1}^m u^\nu \quad (10.4.16)$$

*is also a conserved density of the system (10.3.5) with the associated flux  $\tilde{J} = (\tilde{J}^1, \dots, \tilde{J}^n)$ , where*

$$\tilde{J}^i = tJ^i - \frac{1}{n} \left( \sum_{j=1}^n x^j \right) \rho. \quad (10.4.17)$$

**Proof.**

$$D_t \tilde{\rho} = t D_t \rho + \rho + \frac{1}{n} \left( \sum_{j=1}^n x^j \right) \sum_{\nu=1}^m (u^\nu)_t, \quad D_{x^i} \tilde{J}^i = t D_{x^i} J^i - \frac{1}{n} \rho - \frac{1}{n} \left( \sum_{j=1}^n x^j \right) D_{x^i} \rho.$$

Therefore,

$$D_t \tilde{\rho} + \text{Div} \tilde{J} = t [D_t \rho + \text{Div} J] + \frac{1}{n} \left( \sum_{j=1}^n x^j \right) \left[ \sum_{\nu=1}^m (u^\nu)_t - \sum_{i=1}^n D_{x^i} \rho \right] = 0$$

whenever  $(u^\nu)_t = F_\nu(t, x, u^{(s)}, \tilde{u})$ ,  $\nu = 1, \dots, m$ . ■

10.4.1 Case of a fifth order KdV equation

Consider the (1 + 1)-dimensional fifth order KdV equation (10.3.8) with the parameter  $p = 1$ , namely [45]

$$u_t = -\gamma u_{5x} - \beta u_{3x} - \alpha uu_x \equiv F(t, x, u^{(5)}). \tag{10.4.18}$$

The differential function  $\rho = -\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2$  satisfies

$$D_x \rho = -\gamma u_{5x} - \beta u_{3x} - \alpha uu_x = F(t, x, u^{(5)}).$$

Applying the one dimensional homotopy operator to the exact differential function

$$E = -D_t \rho|_{u_t=F(t,x,u^{(5)},\tilde{u})}$$

yields the flux

$$J = -\frac{1}{3}\alpha^2 u^3 - 2u_{6x}\beta\gamma - 2u\beta\alpha u_{2x} - u_{8x}\gamma^2 - 2u\gamma\alpha u_{4x} - \frac{1}{2}\beta\alpha u_x^2 - 3u_x\gamma\alpha u_{3x} - \frac{7}{2}u_{2x}^2\gamma\alpha - u_{4x}\beta^2.$$

That is  $(\rho, J)$  is a conserved vector of the equation (10.4.18) which satisfies the condition (10.4.15). By using relations (10.4.16) and (10.4.17), we obtain for this equation the following time-space dependent conserved density

$$\tilde{\rho} = t \left( -\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2 \right) + xu$$

with the associated flux

$$\begin{aligned} \tilde{J} = & t \left( -\frac{1}{3}\alpha^2 u^3 - 2u_{6x}\beta\gamma - 2u\beta\alpha u_{2x} - u_{8x}\gamma^2 - 2u\gamma\alpha u_{4x} - \frac{1}{2}\beta\alpha u_x^2 \right. \\ & \left. - 3u_x\gamma\alpha u_{3x} - \frac{7}{2}u_{2x}^2\gamma\alpha - u_{4x}\beta^2 \right) - x \left( -\gamma u_{4x} - \beta u_{2x} - \frac{\alpha}{2}u^2 \right). \end{aligned}$$

10.4.2 Case of the (2 + 1)-dimensional Zakharov-Kuznetsov equation

Consider the modified (2 + 1)-dimensional Zakharov-Kuznetsov (ZK) equation [83]

$$u_t = -\alpha(uu_x + uu_y) - \beta \left[ (u_{2x} + u_{2y})_x + (u_{2x} + u_{2y})_y \right] \equiv F(t, x, y, u^{(2)}). \tag{10.4.19}$$

The uniformity condition applied to the function  $F$  leads to the weight system

$$\begin{aligned} w(D_t) + w(u) &= 2w(u) + w(D_x) \\ &= 2w(u) + w(D_y) \\ &= w(u) + 3w(D_x) \\ &= w(u) + 3w(D_y) \\ &= w(u) + w(D_x) + 2w(D_y) \\ &= w(u) + 2w(D_x) + w(D_y) \end{aligned} \tag{10.4.20}$$

whose solution can be written as

$$w(D_t) = 3w(D_x), \quad w(D_y) = w(D_x), \quad w(u) = 2w(D_x).$$

Setting  $w(D_x) = 1$  provides

$$w(x) = -1, \quad w(D_t) = 3, \quad w(t) = -3, \quad w(D_y) = 1, \quad w(y) = -1, \quad w(u) = 2$$

from which we deduce the one-parameter scaling symmetry group of (10.4.19) as:

$$(t, x, y, u) \mapsto (\lambda^{-3}t, \lambda^{-1}x, \lambda^{-1}y, \lambda^2u).$$

The differential function  $\rho = -\alpha u^2 - \beta u_{2x} - \beta u_{2y}$  satisfies

$$D_x \rho + D_y \rho = -\alpha (uu_x + uu_y) - \beta \left[ (u_{2x} + u_{2y})_x + (u_{2x} + u_{2y})_y \right] = F(t, x, y, u^{(2)}).$$

Applying the two dimensional homotopy operator to the exact differential function

$$E = -D_t \rho|_{u_t = F(t, x, y, u^{(2)})}$$

we obtain the flux  $J = (J^1, J^2)$ , where

$$\begin{aligned} J^1 &= -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{x3y}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{6}{5}u_{2x2y}\beta^2 \\ &\quad - 2u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{1}{6}u_y^2\beta\alpha - \frac{1}{5}u_{4y}\beta^2, \\ J^2 &= -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_y^2\beta\alpha - \frac{2}{3}u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{6}{5}u_{2x2y}\beta^2 \\ &\quad - 2u\beta\alpha u_{2y} - \frac{4}{5}u_{x3y}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - u_{4y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2. \end{aligned}$$

That is  $(\rho, J^1, J^2)$  is a conserved vector of the equation (10.4.19) which satisfies the condition (10.4.15). By using relations (10.4.16) and (10.4.17), we compute for this equation the time-space dependent conserved density

$$\tilde{\rho} = t \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} \right) + \frac{1}{2}(x + y)u$$

with the associated flux  $\tilde{J} = (\tilde{J}^1, \tilde{J}^2)$ , where

$$\begin{aligned} \tilde{J}^1 &= t \left( -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{x3y}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{6}{5}u_{2x2y}\beta^2 \right. \\ &\quad \left. - 2u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{1}{6}u_y^2\beta\alpha - \frac{1}{5}u_{4y}\beta^2 \right) \\ &\quad - \frac{1}{2}(x + y) \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} \right), \\ \tilde{J}^2 &= t \left( -\frac{1}{3}u^3\alpha^2 - \frac{1}{2}u_y^2\beta\alpha - \frac{2}{3}u\beta\alpha u_{2x} - \frac{4}{5}u_{3xy}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{6}{5}u_{2x2y}\beta^2 \right. \\ &\quad \left. - 2u\beta\alpha u_{2y} - \frac{4}{5}u_{x3y}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - u_{4y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2 \right) \\ &\quad - \frac{1}{2}(x + y) \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} \right). \end{aligned}$$

10.4.3 Case of the (3 + 1)-dimensional Zakharov-Kuznetsov equation

Consider the modified (3 + 1)-dimensional Zakharov-Kuznetsov (ZK) equation [83]

$$\begin{aligned}
 u_t &= -\alpha (uu_x + uu_y + uu_z) - \beta \left[ (u_{2x} + u_{2y} + u_{2z})_x + (u_{2x} + u_{2y} + u_{2z})_y \right. \\
 &\quad \left. + (u_{2x} + u_{2y} + u_{2z})_z \right] \equiv F(t, x, y, z, u^{(2)}).
 \end{aligned}
 \tag{10.4.21}$$

The uniformity condition applied to the function  $F$  leads to the weight system

$$\begin{aligned}
 w(D_t) + w(u) &= 2w(u) + w(D_x), & w(D_t) + w(u) &= w(u) + w(D_x) + 2w(D_y) \\
 w(D_t) + w(u) &= 2w(u) + w(D_z), & w(D_t) + w(u) &= w(u) + w(D_y) + 2w(D_x) \\
 w(D_t) + w(u) &= w(u) + 3w(D_y), & w(D_t) + w(u) &= w(u) + w(D_z) + 2w(D_x) \\
 w(D_t) + w(u) &= 2w(u) + w(D_y), & w(D_t) + w(u) &= w(u) + w(D_x) + 2w(D_z) \\
 w(D_t) + w(u) &= w(u) + 3w(D_x), & w(D_t) + w(u) &= w(u) + w(D_y) + 2w(D_z) \\
 w(D_t) + w(u) &= w(u) + 3w(D_z), & w(D_t) + w(u) &= w(u) + w(D_z) + 2w(D_y)
 \end{aligned}$$

whose solution is

$$w(D_t) = 3w(D_x), \quad w(D_y) = w(D_x), \quad w(D_z) = w(D_x), \quad w(u) = 2w(D_x).$$

Setting  $w(D_x) = 1$  gives

$$w(D_t) = 3, \quad w(D_y) = w(D_z) = 1, \quad w(u) = 2, \quad w(x) = w(y) = w(z) = -1, \quad w(t) = -3$$

from which we deduce the associated one-parameter scaling symmetry group as:

$$(t, x, y, z, u) \mapsto (\lambda^{-3}t, \lambda^{-1}x, \lambda^{-1}y, \lambda^{-1}z, \lambda^2u).$$

The differential function  $\rho = -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z}$  satisfies

$$\begin{aligned}
 D_x \rho + D_y \rho + D_z \rho &= -\alpha (uu_x + uu_y + uu_z) - \beta \left[ (u_{2x} + u_{2y} + u_{2z})_x \right. \\
 &\quad \left. + (u_{2x} + u_{2y} + u_{2z})_y + (u_{2x} + u_{2y} + u_{2z})_z \right] = F(t, x, y, z, u^{(2)}).
 \end{aligned}$$

Applying the three dimensional homotopy operator to the exact differential function

$$E = -D_t \rho|_{u_t=F(t,x,y,z,u^{(2)})},$$

with the help of the computer algebra system Maple, yields the flux  $J = (J^1, J^2, J^3)$ , where

$$\begin{aligned}
 J^1 &= -\frac{4}{5}u_{x2yz}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{2}{5}u_{2y2z}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{4}{3}u\beta\alpha u_{xz} \\
 &\quad - \frac{6}{5}u_{2x2z}\beta^2 - 2u\beta\alpha u_{2x} - \frac{1}{5}u_{4z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{4}{5}u_{x3z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2z} \\
 &\quad - \frac{1}{3}\alpha^2 u^3 - \frac{6}{5}u_{2x2y}\beta^2 - \frac{4}{5}u_{3xz}\beta^2 - \frac{4}{5}u_{xy2z}\beta^2 - \frac{1}{5}u_{4y}\beta^2 - \frac{4}{5}u_{x3y}\beta^2 \\
 &\quad - \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_z\beta\alpha u_x - \frac{1}{6}u_z^2\beta\alpha - \frac{1}{2}u_x^2\beta\alpha - \frac{1}{6}u_y^2\beta\alpha,
 \end{aligned}$$

$$\begin{aligned}
 J^2 &= -\frac{1}{2}u_y^2\beta\alpha - \frac{4}{5}u_{3yz}\beta^2 - \frac{1}{6}u_z^2\beta\alpha - \frac{4}{5}u_{y3z}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{5}u_{2xyz}\beta^2 \\
 &- \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{xy2z}\beta^2 - \frac{1}{5}u_{4z}\beta^2 - \frac{4}{5}u_{x3y}\beta^2 - u_{4y}\beta^2 - \frac{2}{5}u_{2x2z}\beta^2 \\
 &- \frac{6}{5}u_{2x2y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2 - 2u\beta\alpha u_{2y} - \frac{6}{5}u_{2y2z}\beta^2 - \frac{1}{3}u_z\beta\alpha u_y \\
 &- \frac{1}{3}\alpha^2 u^3 - \frac{4}{3}u\beta\alpha u_{yz} - \frac{2}{3}u\beta\alpha u_{2x} - \frac{2}{3}u\beta\alpha u_{2z} - \frac{4}{5}u_{3xy}\beta^2, \\
 \\
 J^3 &= -\frac{1}{5}u_{4y}\beta^2 - \frac{4}{5}u_{y3z}\beta^2 - \frac{2}{5}u_{2x2y}\beta^2 - \frac{4}{5}u_{3yz}\beta^2 - \frac{6}{5}u_{2x2z}\beta^2 - \frac{6}{5}u_{2y2z}\beta^2 \\
 &- \frac{4}{5}u_{3xz}\beta^2 - \frac{1}{5}u_{4x}\beta^2 - \frac{4}{5}u_{x3z}\beta^2 - \frac{1}{2}u_z^2\beta\alpha - \frac{4}{5}u_{x2yz}\beta^2 - \frac{1}{6}u_y^2\beta\alpha \\
 &- \frac{4}{5}u_{2xyz}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{yz} - \frac{1}{3}\alpha^2 u^3 - \frac{1}{3}u_z\beta\alpha u_x - \frac{1}{3}u_z\beta\alpha u_y \\
 &- u_{4z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - 2u\beta\alpha u_{2z} - \frac{4}{3}u\beta\alpha u_{xz} - \frac{2}{3}u\beta\alpha u_{2x}.
 \end{aligned}$$

That is  $(\rho, J^1, J^2, J^3)$  is a conserved vector of the equation (10.4.21) which satisfies the condition (10.4.15). By using relations (10.4.16) and (10.4.17), we compute for this equation the time-space dependent conserved density

$$\tilde{\rho} = t \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z} \right) + \frac{1}{3}(x + y + z)u$$

with the associated flux  $\tilde{J} = (\tilde{J}^1, \tilde{J}^2, \tilde{J}^3)$ , where

$$\begin{aligned}
 \tilde{J}^1 &= t \left( -\frac{4}{5}u_{x2yz}\beta^2 - \frac{4}{3}u\beta\alpha u_{xy} - \frac{2}{5}u_{2y2z}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - u_{4x}\beta^2 - \frac{4}{3}u\beta\alpha u_{xz} \right. \\
 &- \frac{6}{5}u_{2x2z}\beta^2 - 2u\beta\alpha u_{2x} - \frac{1}{5}u_{4z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - \frac{4}{5}u_{x3z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2z} \\
 &- \frac{1}{3}\alpha^2 u^3 - \frac{6}{5}u_{2x2y}\beta^2 - \frac{4}{5}u_{3xz}\beta^2 - \frac{4}{5}u_{xy2z}\beta^2 - \frac{1}{5}u_{4y}\beta^2 - \frac{4}{5}u_{x3y}\beta^2 \\
 &- \left. \frac{4}{5}u_{3xy}\beta^2 - \frac{1}{3}u_z\beta\alpha u_x - \frac{1}{6}u_z^2\beta\alpha - \frac{1}{2}u_x^2\beta\alpha - \frac{1}{6}u_y^2\beta\alpha \right) \\
 &- \frac{1}{3}(x + y + z) \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z} \right),
 \end{aligned}$$

$$\begin{aligned}
 \tilde{J}^2 &= t \left( -\frac{1}{2}u_y^2\beta\alpha - \frac{4}{5}u_{3yz}\beta^2 - \frac{1}{6}u_z^2\beta\alpha - \frac{4}{5}u_{y3z}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{5}u_{2xyz}\beta^2 \right. \\
 &- \frac{4}{3}u\beta\alpha u_{xy} - \frac{4}{5}u_{xy2z}\beta^2 - \frac{1}{5}u_{4z}\beta^2 - \frac{4}{5}u_{x3y}\beta^2 - u_{4y}\beta^2 - \frac{2}{5}u_{2x2z}\beta^2 \\
 &- \frac{6}{5}u_{2x2y}\beta^2 - \frac{1}{3}u_y\beta\alpha u_x - \frac{1}{5}u_{4x}\beta^2 - 2u\beta\alpha u_{2y} - \frac{6}{5}u_{2y2z}\beta^2 - \frac{1}{3}u_z\beta\alpha u_y \\
 &- \left. \frac{1}{3}\alpha^2 u^3 - \frac{4}{3}u\beta\alpha u_{yz} - \frac{2}{3}u\beta\alpha u_{2x} - \frac{2}{3}u\beta\alpha u_{2z} - \frac{4}{5}u_{3xy}\beta^2 \right) \\
 &- \frac{1}{3}(x + y + z) \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z} \right),
 \end{aligned}$$

$$\begin{aligned}
 \tilde{J}^3 &= t \left( -\frac{1}{5}u_{4y}\beta^2 - \frac{4}{5}u_{y3z}\beta^2 - \frac{2}{5}u_{2x2y}\beta^2 - \frac{4}{5}u_{3yz}\beta^2 - \frac{6}{5}u_{2x2z}\beta^2 - \frac{6}{5}u_{2y2z}\beta^2 \right. \\
 &- \frac{4}{5}u_{3xz}\beta^2 - \frac{1}{5}u_{4x}\beta^2 - \frac{4}{5}u_{x3z}\beta^2 - \frac{1}{2}u_z^2\beta\alpha - \frac{4}{5}u_{x2yz}\beta^2 - \frac{1}{6}u_y^2\beta\alpha \\
 &- \frac{4}{5}u_{2xyz}\beta^2 - \frac{1}{6}u_x^2\beta\alpha - \frac{4}{3}u\beta\alpha u_{yz} - \frac{1}{3}\alpha^2 u^3 - \frac{1}{3}u_z\beta\alpha u_x - \frac{1}{3}u_z\beta\alpha u_y \\
 &- \left. u_{4z}\beta^2 - \frac{2}{3}u\beta\alpha u_{2y} - 2u\beta\alpha u_{2z} - \frac{4}{3}u\beta\alpha u_{xz} - \frac{2}{3}u\beta\alpha u_{2x} \right) \\
 &- \frac{1}{3}(x+y+z) \left( -\frac{1}{2}\alpha u^2 - \beta u_{2x} - \beta u_{2y} - \beta u_{2z} \right).
 \end{aligned}$$



## Part IV

# Extremum Conditions for Variational Problems





---

**Abstract:** This part addresses both necessary and relevant sufficient extremum conditions for a variational problem defined by a smooth Lagrangian, involving higher derivatives of several variable vector valued functions. A general formulation of first order necessary extremum conditions for variational problems with (or without) constraints is given. Global Legendre second order necessary extremum conditions are provided as well as a new general explicit formula for second order sufficient extremum condition which does not require the notion of conjugate points as in the Jacobi sufficient condition. The main results of this part have been submitted for publication [53].

---



# First Order Extremum Conditions

---

This chapter contains two parts. First, we briefly recall useful definitions and properties used in the sequel. Then, we analyze the variational problem with constraints, and give a general formulation of the first order necessary extremum condition which is rigorously proved.

## 11.1 Some known first order extremum conditions

### Holonomic constraints

We will consider functional of the form

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx, \quad (11.1.1)$$

where  $u \in \mathcal{C}^2(\bar{I}, \mathbb{R}^N)$ , and  $I = ]a, b[$ . We will demand that  $u$  satisfies a holonomic constraint

$$g(x, u(x)) = 0, \quad a \leq x \leq b. \quad (11.1.2)$$

**Theorem 11.1.1 ([84])** *Suppose that  $F \in \mathcal{C}^2(\bar{I} \times \Omega)$ , where  $\Omega$  is an open set in  $\mathbb{R}^{2N}$ . Suppose that  $g \in \mathcal{C}^2(\bar{I} \times W)$ , where  $W \subset \mathbb{R}^N$  and that  $\nabla_u g(x, u) \neq 0$  on the set where  $g(x, u(x)) = 0$ . Suppose that  $u \in \mathcal{C}^2(\bar{I}, W)$  is a local extremum for  $\mathcal{F}$ , subject to the holonomic constraint in (11.1.2). Then there is a function  $\lambda \in \mathcal{C}(\bar{I})$  such that  $u$  is an extremum of the functional*

$$\mathcal{G}(u) = \int_a^b [F(x, u(x), u'(x)) + \lambda(x)g(x, u(x))] dx. \quad (11.1.3)$$

**Remark 11.1.1** *The Lagrangian of the functional  $\mathcal{G}$  in (11.1.3) is*

$$G(x, u, u') = F(x, u, u') + \lambda(x)g(x, u)$$

and the Euler-Lagrange equations are

$$F_{u^j} + \lambda g_{u^j} - \frac{d}{dx} F_{u'^j} = 0, \quad j = 1, 2, \dots, N.$$

### Nonholonomic constraints

**Theorem 11.1.2 ([84])** *Suppose that  $F$ , and  $g^j$  for  $j = 1, 2, \dots, m$  belong to  $\mathcal{C}^3(\bar{I} \times \Omega, \mathbb{R})$ , where  $\Omega \in \mathbb{R}^{2N}$  and that  $u \in \mathcal{C}^2([a, b], \mathbb{R}^N)$  is a local extremum of the functional*

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx, \quad (11.1.4)$$


---

subject to the nonholonomic constraints

$$g^j(x, u(x), u'(x)) = 0, \quad j = 1, 2, \dots, m.$$

Suppose that the constraints together with  $u$  satisfy the following

(1) The matrix

$$D_u g(x, u, u') = \left( \frac{\partial g^j(x, u, u')}{\partial u'^k} \right)$$

has rank  $m$  for  $a \leq x \leq b$ ;

(2) The only solutions to the system of differential equations

$$\sum_{j=1}^m \left[ \left( g_{u'^k}^j - \frac{d}{dx} g_{u'^k}^j \right) \mu_j - g_{u'^k}^j \frac{d\mu_j}{dx} \right] = 0, \quad k = 1, 2, \dots, N$$

is  $\mu_1(x) = \mu_2(x) = \dots = \mu_m(x) = 0$ .

Then there exist functions  $\lambda_1, \lambda_2, \dots, \lambda_m$  defined on  $[a, b]$  such that  $u$  is an extremum for the functional with Lagrangian

$$G(x, u, u') = F(x, u, u') + \sum_{j=1}^m \lambda_j(x) g^j(x, u, u').$$

In this work, we give an answer to the following question: What do the results of the above theorems become when the vector-valued function  $u = (u^1, \dots, u^m)$  depends on several variables  $x = (x^1, \dots, x^n)$  and/or the Lagrangian of the used functional includes higher order derivatives of  $u$ ?

## 11.2 First variation and extremum conditions for unconstrained problems

Consider a functional of the form

$$\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s)}(x)) dx \quad (11.2.5)$$

where  $\Lambda$  is a connected subset of  $X$ . Let  $\Omega$  be an open subset of  $U^{(s)}$ . We assume that the function  $L$ , usually called the Lagrangian of the functional  $\mathcal{F}$ , is defined on the open subset  $\Lambda \times \Omega$  of  $X \times U^{(s)}$  and is continuous in all its  $n + q_s$  variables so that the variational integral (11.2.5) exists. The problem consists in finding conditions that the function  $u$  must satisfy in order to be a minimum or maximum of the functional  $\mathcal{F}$ , requiring that  $L \in \mathcal{C}^{s+1}(\Lambda \times \Omega, \mathbb{R})$ . For the integral in (11.2.5) to be defined, it is necessary that the function  $u \in \mathcal{C}_b^s(\Lambda, U)$ , where

$$\mathcal{C}_b^s(\Lambda, U) = \left\{ \psi \in \mathcal{C}^s(\Lambda, U) : \sum_{j=1}^m \sum_{k=0}^s \sum_{l=1}^{p_k} \sup_{x \in \Lambda} |\psi_{(k)}^j[l](x)| < +\infty \right\}.$$

In addition,  $L(x, u^{(s)}(x))$  must be defined for all  $x \in \Lambda$ . This means that  $u^{(s)}(x) \in \Omega$  for all  $x \in \Lambda$ . Such a function  $u$  will be said to be admissible for the functional  $\mathcal{F}$ .

**Definition 11.2.1** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a global minimum for  $\mathcal{F}$ , if  $\mathcal{F}(u) \leq \mathcal{F}(v)$  for every admissible function  $v$ .

**Definition 11.2.2** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a global maximum for  $\mathcal{F}$ , if  $\mathcal{F}(v) \leq \mathcal{F}(u)$  for every admissible function  $v$ .

A function which is either a global minimum or a global maximum is called a global extremum. To come up with the definition of local extremums for a functional, we need to have a measure of distance between two functions.

**Definition 11.2.3** Let  $\phi \in C_b^s(\Lambda, U)$ . We define the 0-norm of  $\phi$  by

$$\|\phi\|_0 = \sum_{j=1}^m \sup_{x \in \Lambda} |\phi^j(x)|$$

and the  $s$ -norm of  $\phi$  by

$$\|\phi\|_s = \sum_{j=1}^m \sum_{k=0}^s \sum_{l=1}^{p_k} \sup_{x \in \Lambda} |\phi_{(k)}^j[l](x)|.$$

Clearly, for  $s > 0$  the numbers  $\|\phi - \psi\|_0$  and  $\|\phi - \psi\|_s$  provide quite different measures of the distance between  $\phi$  and  $\psi$ . These measures lead to two different definitions of local minima.

**Definition 11.2.4** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a weak local minimum for  $\mathcal{F}$  if there is an  $\epsilon > 0$  such that  $\mathcal{F}(u) \leq \mathcal{F}(v)$  for all admissible functions  $v$  satisfying  $\|v - u\|_s < \epsilon$ .  $u$  is a strict weak local minimum if  $\mathcal{F}(u) < \mathcal{F}(v)$  for all such  $v$  with  $v \neq u$ .

**Definition 11.2.5** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a strong local minimum for  $\mathcal{F}$  if there is an  $\epsilon > 0$  such that  $\mathcal{F}(u) \leq \mathcal{F}(v)$  for all admissible functions  $v$  satisfying  $\|v - u\|_0 < \epsilon$ .  $u$  is a strict strong local minimum if  $\mathcal{F}(u) < \mathcal{F}(v)$  for all such  $v$  with  $v \neq u$ .

**Definition 11.2.6** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a weak local maximum for  $\mathcal{F}$  if there is an  $\epsilon > 0$  such that  $\mathcal{F}(u) \geq \mathcal{F}(v)$  for all admissible functions  $v$  satisfying  $\|v - u\|_s < \epsilon$ .  $u$  is a strict weak local maximum if  $\mathcal{F}(u) > \mathcal{F}(v)$  for all such  $v$  with  $v \neq u$ .

**Definition 11.2.7** A function  $u$  which is admissible for the functional  $\mathcal{F}$  is a strong local maximum for  $\mathcal{F}$  if there is an  $\epsilon > 0$  such that  $\mathcal{F}(u) \geq \mathcal{F}(v)$  for all admissible functions  $v$  satisfying  $\|v - u\|_0 < \epsilon$ .  $u$  is a strict strong local maximum if  $\mathcal{F}(u) > \mathcal{F}(v)$  for all such  $v$  with  $v \neq u$ .

A function which is either a weak local minimum or a weak local maximum is called a weak local extremum. A function which is either a strong local minimum or a strong local maximum is called a strong local extremum.

Without loss of generality, we can assume that  $\Lambda = \prod_{i=1}^n ]a^i, b^i[$  with  $a^i \leq b^i$ .

**Definition 11.2.8** A function  $\psi \in C(\Lambda, U)$  is said to have compact support in  $\Lambda$  if there is  $\epsilon > 0$  such that  $\psi(x) = 0$  for all  $x = (x^1, \dots, x^n)$  with  $x^i \in ]a^i, a^i + \epsilon[$  or  $x^i \in ]b^i - \epsilon, b^i[$  for some  $i \in \{1, 2, \dots, n\}$ . The set of all functions which are infinitely differentiable and have compact support in  $\Lambda$  is denoted by  $C_0^\infty(\Lambda, U)$ .

**Lemma 11.2.1** *Let  $f \in \mathcal{C}(\Lambda, \mathbb{R})$ . If  $\int_{\Lambda} f(x)\psi(x)dx = 0$  for all  $\psi \in \mathcal{C}_0^{\infty}(\Lambda, \mathbb{R})$ , then  $f(x) = 0$  for all  $x \in \Lambda$ .*

Given an admissible function  $u \in \mathcal{C}^s(\Lambda, U)$  and any  $\phi \in \mathcal{C}_0^{\infty}(\Lambda, U)$ , there is an  $\epsilon_0 > 0$  such that the function  $v = u + t\phi$  is admissible for all  $|t| < \epsilon_0$ . Therefore, the function

$$\Phi(t) = \mathcal{F}(u + t\phi) = \int_{\Lambda} L(x, u^{(s)}(x) + t\phi^{(s)}(x)) dx \quad (11.2.6)$$

is a well defined function of  $t$  for  $|t| < \epsilon_0$ . Throughout this paper,  $\epsilon_0$  stands for such a number.

Assume now that  $u \in \mathcal{C}^s(\Lambda, U)$  is a local extremum of  $\mathcal{F}$ . We may as well assume that  $u$  is a local minimum. We have  $\Phi(t) = \mathcal{F}(u + t\phi) \geq \mathcal{F}(u) = \Phi(0)$  for  $|t| < \epsilon_0$ , i.e. 0 is a local minimum for  $\Phi$ . Suppose that  $L \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$  implying that  $\Phi$  is also continuously differentiable and we must have

$$\Phi'(0) = 0. \quad (11.2.7)$$

We can calculate  $\Phi'$  by differentiating (11.2.6) with respect to  $t$  under the integral sign. Doing so and using the chain rule we get

$$\begin{aligned} \Phi'(t) &= \frac{d}{dt} \mathcal{F}(u + t\phi) \\ &= \frac{d}{dt} \int_{\Lambda} L(x, u^{(s)}(x) + t\phi^{(s)}(x)) dx \\ &= \int_{\Lambda} \frac{d}{dt} L(x, u^{(s)}(x) + t\phi^{(s)}(x)) dx \\ &= \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \phi_{(k)}^j[h](x) \frac{\partial L(x, u^{(s)}(x) + t\phi^{(s)}(x))}{\partial u_{(k)}^j[h]} dx. \end{aligned} \quad (11.2.8)$$

In particular at  $t = 0$  we get

$$\Phi'(0) = \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \phi_{(k)}^j[h](x) \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^j[h]} dx. \quad (11.2.9)$$

**Definition 11.2.9** *The first variation of  $\mathcal{F}$  in a neighborhood of  $u$  in the direction  $\phi$  is defined by*

$$\delta\mathcal{F}(u + t\phi, \phi) = \Phi'(t). \quad (11.2.10)$$

*In particular, the first variation of  $\mathcal{F}$  at  $u$  in the direction  $\phi$  is expressed by*

$$\delta\mathcal{F}(u, \phi) = \Phi'(0). \quad (11.2.11)$$

Notice that the first variation at  $u$  is defined in Definition 11.2.9 whether  $u$  is a local extremum or not. However, if  $u$  is a local extremum of  $\mathcal{F}$ , then by (11.2.7) and (11.2.11),  $\delta\mathcal{F}(u, \phi) = 0$ . We have proved the following first order necessary condition on a local extremum of  $\mathcal{F}$ .

**Proposition 11.2.1** *Suppose that  $L \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$ , and that  $u \in \mathcal{C}_b^s(\Lambda, U)$  is a local extremum for the functional  $\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s)}(x)) dx$ . Then*

$$\delta\mathcal{F}(u, \phi) = 0 \quad (11.2.12)$$

for all  $\phi \in \mathcal{C}_0^{\infty}(\Lambda, U)$ .

The condition in (11.2.12) is called the weak form of the Euler-Lagrange equations. A function  $u$  which satisfies (11.2.12) is called the weak extremum of  $\mathcal{F}$ .

Now assume that the Lagrangian  $L \in \mathcal{C}^{s+1}(\Lambda \times \Omega, \mathbb{R})$ , and  $u \in \mathcal{C}_b^{2s}(\Lambda, U)$ . Using the divergence theorem to successively integrate by parts (11.2.9) until all derivative actions on  $\phi^j$  are now moved into  $\frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^j[h]}$ , and taking into account that  $\phi^j \in \mathcal{C}_0^\infty(\Lambda, \mathbb{R})$ , we get

$$\delta\mathcal{F}(u, \phi) = \int_{\Lambda} \sum_{j=1}^m \left( \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] \right) \phi^j(x) dx. \tag{11.2.13}$$

If  $u$  is a weak local extremum, then (11.2.13) is equal to 0 for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$ . In particular if we take  $\phi = \psi e^l$ , where  $\psi \in \mathcal{C}_0^\infty(\Lambda, \mathbb{R})$  and  $e^l$  is the  $l$ -th vector of the canonical basis of  $\mathbb{R}^m$ , then we get

$$0 = \delta\mathcal{F}(u, \psi e^l) = \int_{\Lambda} \left( \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^l[h]} \right)_{(k)} [h] \right) \psi(x) dx$$

for all  $\psi \in \mathcal{C}_0^\infty(\Lambda, \mathbb{R})$ . By Lemma 11.2.1, we see that

$$\sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^l[h]} \right)_{(k)} [h] = 0$$

for all  $x \in \Lambda$  and  $l = 1, 2, \dots, m$ . Thus, we have proved the following theorem.

**Theorem 11.2.1** *Suppose that  $L \in \mathcal{C}^{s+1}(\Lambda \times \Omega, \mathbb{R})$ , and  $u \in \mathcal{C}_b^{2s}(\Lambda, U)$  is a local extremum for the functional  $\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s)}(x)) dx$ . Then*

$$\sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s)}(x))}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] = 0 \tag{11.2.14}$$

for all  $x \in \Lambda$  and  $j = 1, 2, \dots, m$ .

The equations (11.2.14) are called the Euler-Lagrange equations. A solution to the Euler-Lagrange equations is called an extremum for the functional  $\mathcal{F}$ .

### 11.3 First variation and extremum conditions for constrained problems

We want to find extremums for the functional

$$\mathcal{F}(u) = \int_{\Lambda} L(x, u^{(s_1)}(x)) dx \tag{11.3.15}$$

subject to constraints of the form

$$F_j(x, u^{(s_2)}(x)) = 0 \quad j = 1, 2, \dots, m' \tag{11.3.16}$$

for all  $x \in \Lambda$ . Let  $\Omega_i$  be open subsets of  $U^{(s_i)}$ ,  $i = 1, 2$  such that  $L$  is defined on  $\Lambda \times \Omega_1$  and  $F_j$  is defined on  $\Lambda \times \Omega_2$ . Constraints of type (11.3.16) are called holonomic constraints if

$s_2 = 0$ , and nonholonomic constraints if  $s_2 \geq 1$ . In this subsection, we examine these types of constrained variational problems.

For  $m = m'$ , i.e. the number of equations in the system formed by the constraints is equal to the number of unknowns, we exploit the fact that such a system appears for the Euler-Lagrange equations of some variational problems [56, 57] to prove our next result.

**Theorem 11.3.1** *Suppose that  $L \in \mathcal{C}^{s_1+1}(\Lambda \times \Omega_1, \mathbb{R})$ ,  $F_j \in \mathcal{C}^{s_2+1}(\Lambda \times \Omega_2, \mathbb{R})$  and that the function  $u \in \mathcal{C}_b^{2s}(\Lambda, U)$ ,  $s = \max(s_1, s_2)$ , verifies the constraints (11.3.16) and is a local extremum for the functional  $\mathcal{F}$  defined by (11.3.15). If a function  $\lambda(x) = (\lambda^1(x), \dots, \lambda^m(x))$  defined on  $\Lambda$  is solution of the system*

$$\sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial [\sum_{l=1}^m \lambda^l(x) F_l(x, u^{(s_2)}(x))]}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] = 0 \quad j = 1, 2, \dots, m \quad (11.3.17)$$

then  $u$  is a local extremum for the functional whose Lagrangian is

$$G(x, u^{(s)}(x)) = L(x, u^{(s_1)}(x)) + \sum_{l=1}^m \lambda^l(x) F_l(x, u^{(s_2)}(x)). \quad (11.3.18)$$

**Proof.** Consider the variational problem whose Lagrangian is defined by

$$G'(x, u^{(s)}(x), v(x)) = L(x, u^{(s_1)}(x)) + \sum_{l=1}^m v^l(x) F_l(x, u^{(s_2)}(x)), \quad (11.3.19)$$

where  $v(x) = (v^1(x), \dots, v^m(x))$  is viewed as dependent variable. The Euler-Lagrange equations of this variational problem are

$$P_j \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'(x, u^{(s)}(x), v(x))}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] = 0; \quad (11.3.20)$$

$$Q_j \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'(x, u^{(s)}(x), v(x))}{\partial v_{(k)}^j[h]} \right)_{(k)} [h] = 0, \quad (11.3.21)$$

$j = 1, 2, \dots, m$ . Taking into account (11.3.19), the expressions of  $P_j$  and  $Q_j$  give

$$P_j = P_{j,1} + P_{j,2}; \quad Q_j = Q_{j,1} + Q_{j,2}$$

where

$$\begin{aligned} P_{j,1} &= \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s_1)}(x))}{\partial u_{(k)}^j[h]} \right)_{(k)} [h]; \\ P_{j,2} &= \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial [\sum_{l=1}^m v^l(x) F_l(x, u^{(s_2)}(x))]}{\partial u_{(k)}^j[h]} \right)_{(k)} [h]; \\ Q_{j,1} &= \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s_1)}(x))}{\partial v_{(k)}^j[h]} \right)_{(k)} [h]; \end{aligned}$$

$$Q_{j,2} = \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial [\sum_{l=1}^m v^l(x) F_l(x, u^{(s_2)}(x))]}{\partial v_{(k)}^j[h]} \right)_{(k)} [h].$$

The  $P_{j,1}$  are expressions defining the Euler-Lagrange equations of the variational problem (11.3.15). Thus,  $P_{j,1} = 0$  since  $u$  is a local extremum for the functional  $\mathcal{F}$ . According to the relations (11.3.17), the expressions  $P_{j,2}$  vanish when  $v(x) = \lambda(x)$ . The expressions  $Q_{j,1}$  vanish since the Lagrangian  $L$  does not depend on  $v$  nor on its derivatives.

For  $j = 1, 2, \dots, m$ ,  $Q_{j,2} = F_j(x, u^{(s_2)}(x))$  and therefore vanish since the function  $u$  satisfies the constraints (11.3.16).

Finally, the Euler-Lagrange equations (11.3.20)-(11.3.21) are automatically verified if and only if  $v(x) = \lambda(x)$ . This proves that  $u$  is also a local extremum for the functional whose Lagrangian is  $G'(x, u^{(s)}(x), \lambda(x)) = G(x, u^{(s)}(x))$ . ■

For  $m' < m$ , we redefine the problem in the following manner: Find the extremums for the functional

$$\mathcal{F}(u, \tilde{u}) = \int_{\Lambda} L(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x)) dx \tag{11.3.22}$$

subject to the constraints

$$F_j(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)) = 0 \quad j = 1, 2, \dots, m \tag{11.3.23}$$

for all  $x \in \Lambda$ , where  $\tilde{u}(x) = (\tilde{u}^1(x), \dots, \tilde{u}^{\tilde{m}}(x)) \in \tilde{U}$ ,  $\tilde{U}$  being an  $\tilde{m}$ -dimensional space. Let  $\Omega_i$  be open subsets of  $U^{(s_i)}$  and  $\tilde{\Omega}_i$  be open subsets of  $\tilde{U}^{(s_i)}$ ,  $i = 1, 2$  such that  $L$  is defined on  $\Lambda \times \Omega_1 \times \tilde{\Omega}_1$  and  $F_j$  is defined on  $\Lambda \times \Omega_2 \times \tilde{\Omega}_2$ . Here, the number of equations in the system formed by the constraints is lower than the number of unknowns, i.e. the constraints form an under determined system. Such a system appears for the Euler-Lagrange equations of some variational problems [48]. We then prove the following result.

**Theorem 11.3.2** *Suppose that  $L \in \mathcal{C}^{s_1+1}(\Lambda \times \Omega_1 \times \tilde{\Omega}_1, \mathbb{R})$ ,  $F_j \in \mathcal{C}^{s_2+1}(\Lambda \times \Omega_2 \times \tilde{\Omega}_2, \mathbb{R})$  and that the function  $(u, \tilde{u}) \in \mathcal{C}_b^{2s}(\Lambda, U) \times \mathcal{C}_b^{2s}(\Lambda, \tilde{U})$ ,  $s = \max(s_1, s_2)$ , verifies the constraints (11.3.23) and is a local extremum for the functional  $\mathcal{F}$  defined by (11.3.22). If a function  $(\lambda(x), \tilde{\lambda}(x))$  defined on  $\Lambda$  with  $\lambda(x) = (\lambda^1(x), \dots, \lambda^m(x))$  and  $\tilde{\lambda}(x) = (\tilde{\lambda}^1(x), \dots, \tilde{\lambda}^{\tilde{m}}(x))$ , is solution to the system*

$$\sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial [\sum_{l=1}^m (\lambda^l(x) + \sum_{i=1}^{\tilde{m}} \tilde{\lambda}^i(x)) F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x))]}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] = 0; \tag{11.3.24}$$

$$\sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial [\sum_{l=1}^m (\lambda^l(x) + \sum_{i=1}^{\tilde{m}} \tilde{\lambda}^i(x)) F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x))]}{\partial \tilde{u}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h] = 0, \tag{11.3.25}$$

$j = 1, 2, \dots, m, \tilde{j} = 1, 2, \dots, \tilde{m}$ , then  $(u, \tilde{u})$  is a local extremum for the functional whose

Lagrangian is

$$\begin{aligned} G\left(x, u^{(s)}(x), \tilde{u}^{(s)}(x)\right) &= \sum_{l=1}^m \left( \lambda^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{\lambda}^{\tilde{l}}(x) \right) F_l\left(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)\right) \\ &+ L\left(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x)\right). \end{aligned} \quad (11.3.26)$$

**Proof.** Consider the variational problem whose Lagrangian is defined by

$$\begin{aligned} G'\left(x, u^{(s)}(x), \tilde{u}^{(s)}(x), v(x), \tilde{v}(x)\right) &= \sum_{l=1}^m \left( v^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{v}^{\tilde{l}}(x) \right) F_l\left(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)\right) \\ &+ L\left(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x)\right), \end{aligned} \quad (11.3.27)$$

where  $v(x) = (v^1(x), \dots, v^m(x))$ , and  $\tilde{v}(x) = (\tilde{v}^1(x), \dots, \tilde{v}^{\tilde{m}}(x))$  are viewed as dependent variables. The Euler-Lagrange equations of this variational problem are

$$P_j \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'}{\partial u_{(k)}^j[h]} \right)_{(k)} [h] = 0; \quad (11.3.28)$$

$$Q_{\tilde{j}} \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'}{\partial \tilde{u}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h] = 0, \quad (11.3.29)$$

$$R_j \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'}{\partial v_{(k)}^j[h]} \right)_{(k)} [h] = 0; \quad (11.3.30)$$

$$S_{\tilde{j}} \equiv \sum_{k=0}^s (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial G'}{\partial \tilde{v}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h] = 0, \quad (11.3.31)$$

$j = 1, 2, \dots, m$ ,  $\tilde{j} = 1, 2, \dots, \tilde{m}$ . Taking into account (11.3.27), the expressions of  $P_j$ ,  $Q_{\tilde{j}}$ ,  $R_j$  and  $S_{\tilde{j}}$  are given by

$$P_j = P_{j,1} + P_{j,2}; \quad Q_{\tilde{j}} = Q_{\tilde{j},1} + Q_{\tilde{j},2};$$

$$R_j = R_{j,1} + R_{j,2}; \quad S_{\tilde{j}} = S_{\tilde{j},1} + S_{\tilde{j},2}$$

where

$$P_{j,1} = \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L\left(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x)\right)}{\partial u_{(k)}^j[h]} \right)_{(k)} [h];$$

$$P_{j,2} = \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial \left[ \sum_{l=1}^m \left( v^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{v}^{\tilde{l}}(x) \right) F_l\left(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)\right) \right]}{\partial u_{(k)}^j[h]} \right)_{(k)} [h];$$

$$Q_{\tilde{j},1} = \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L\left(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x)\right)}{\partial \tilde{u}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h];$$

$$\begin{aligned}
Q_{\tilde{j},2} &= \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial \left[ \sum_{l=1}^m \left( v^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{v}^{\tilde{l}}(x) \right) F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)) \right]}{\partial \tilde{u}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h]; \\
R_{j,1} &= \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x))}{\partial v_{(k)}^j[h]} \right)_{(k)} [h]; \\
R_{j,2} &= \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial \left[ \sum_{l=1}^m \left( v^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{v}^{\tilde{l}}(x) \right) F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)) \right]}{\partial v_{(k)}^j[h]} \right)_{(k)} [h]; \\
S_{\tilde{j},1} &= \sum_{k=0}^{s_1} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial L(x, u^{(s_1)}(x), \tilde{u}^{(s_1)}(x))}{\partial \tilde{v}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h]; \\
S_{\tilde{j},2} &= \sum_{k=0}^{s_2} (-1)^k \sum_{h=1}^{p_k} \left( \frac{\partial \left[ \sum_{l=1}^m \left( v^l(x) + \sum_{\tilde{l}=1}^{\tilde{m}} \tilde{v}^{\tilde{l}}(x) \right) F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x)) \right]}{\partial \tilde{v}_{(k)}^{\tilde{j}}[h]} \right)_{(k)} [h].
\end{aligned}$$

The  $P_{j,1}$  and  $Q_{\tilde{j},1}$  are nothing but the Euler-Lagrange equations of the variational problem (11.3.22). Hence,  $P_{j,1} = 0$  and  $Q_{\tilde{j},1} = 0$  since  $(u, \tilde{u})$  is a local extremum for the functional  $\mathcal{F}$ .

According to the relations (11.3.24) and (11.3.25), the expressions  $P_{j,2}$  and  $Q_{\tilde{j},2}$  vanish when  $(v(x), \tilde{v}(x)) = (\lambda(x), \tilde{\lambda}(x))$ .

The expressions  $R_{j,1}$  and  $S_{\tilde{j},1}$  vanish since the Lagrangian  $L$  does not depend on  $v$  and  $\tilde{v}$  nor on their derivatives.

For  $j = 1, 2, \dots, m$ , and  $\tilde{j} = 1, 2, \dots, \tilde{m}$ ,  $R_{j,2} = F_j(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x))$  and  $S_{\tilde{j},2} = \sum_{l=1}^m F_l(x, u^{(s_2)}(x), \tilde{u}^{(s_2)}(x))$  therefore vanish since the function  $u$  satisfies the constraints (11.3.23).

Finally, the Euler-Lagrange equations (11.3.28)-(11.3.31) are automatically verified if and only if  $(v(x), \tilde{v}(x)) = (\lambda(x), \tilde{\lambda}(x))$ . This proves that  $u$  is also a local extremum for the functional whose Lagrangian is  $G'(x, u^{(s)}(x), \tilde{u}^{(s)}(x), \lambda(x), \tilde{\lambda}(x)) = G(x, u^{(s)}(x), \tilde{u}^{(s)}(x))$ . ■



# Second Order Extremum Conditions

---

The solutions to the Euler-Lagrange equations may also include (local) maxima, as well as other non-extremum critical functions. To distinguish between the possibilities, we need to formulate a second derivative test for the objective functional. In the calculus of variations, the second derivative of a functional is known as its second variation. This section contains relevant results which are new to our best knowledge of the literature. We investigate the second variation of a functional as well as the second order necessary and sufficient conditions that a function should satisfy to be either a minimum or a maximum.

## 12.1 Some known second order extremum conditions

### The Legendre condition

**Theorem 12.1.1 ([84])** *Suppose that  $u$  is a local, weak minimum for the functional*

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx.$$

*Then*

$$\sum_{j,k=1}^N F_{u^j u^k}(x, u(x), u'(x)) \xi^j \xi^k \geq 0, \quad \forall a \leq x \leq b, \quad \forall \xi \in \mathbb{R}^N. \quad (12.1.1)$$

The inequality in (12.1.1) is called the Legendre condition. As the theorem says, it is a necessary condition for  $u$  to be a weak minimum. The Legendre condition says that the matrix

$$F_{u'u'} = (F_{u^j u^k})$$

must be positive semi-definite at every point along a minimum.

### The Jacobi conditions

Consider the functional

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x)) dx, \quad (12.1.2)$$

where,  $u = (u^1, u^2, \dots, u^n)$ . Introduce the matrices

$$F_{uu} = (F_{u^i u^k}), \quad F_{uu'} = (F_{u^i u'^k}), \quad F_{u'u'} = (F_{u'^i u'^k}),$$

$$P = \frac{1}{2} F_{u'u'}, \quad Q = \frac{1}{2} \left( F_{uu} - \frac{d}{dx} F_{uu'} \right).$$


---

**Definition 12.1.1** *Let*

$$\begin{aligned} h^1 &= (h_{11}, h_{12}, \dots, h_{1n}) \\ h^2 &= (h_{21}, h_{22}, \dots, h_{2n}) \\ &\vdots \quad \dots \\ h^n &= (h_{n1}, h_{n2}, \dots, h_{nn}) \end{aligned} \quad (12.1.3)$$

be set of  $n$  solutions of the linear equations called the Jacobi system

$$-\frac{d}{dx}(Ph') + Qh = 0 \quad (12.1.4)$$

associated to the functional (12.1.2), where the  $i$ -th solution satisfies the initial conditions

$$h_{ik}(a) = 0, \quad h'_{ii}(a) = 1, \quad h'_{ik}(a) = 0, \quad k \neq i, \quad i, k = 1, 2, \dots, n.$$

Then the point  $\tilde{a}$ , ( $\tilde{a} \neq a$ ), is said to be conjugate to the point  $a$  if the determinant

$$\begin{vmatrix} h_{11}(x) & h_{12}(x) & \dots & h_{1n}(x) \\ h_{21}(x) & h_{22}(x) & \dots & h_{2n}(x) \\ \vdots & & \dots & \\ h_{n1}(x) & h_{n2}(x) & \dots & h_{nn}(x) \end{vmatrix}$$

vanishes for  $x = \tilde{a}$ .

**Theorem 12.1.2 ([35]) (Jacobi necessary condition).** *If the extremum  $u$  correspond to a minimum of the functional (12.1.2), and if the matrix  $P(x, u(x), u'(x))$  is positive definite along this extremum, then the open interval  $]a, b[$  contains no points conjugate to  $a$ .*

**Theorem 12.1.3 ([35]) (Jacobi sufficient condition).** *Suppose that for some curve  $\gamma$  with equation  $u = u(x)$ , the functional (12.1.2) satisfies the following conditions:*

(1) *The curve  $\gamma$  is an extremum, i.e., satisfies the system of Euler equations*

$$F_{u^i} - \frac{d}{dx}F_{u'^i} = 0, \quad i = 1, 2, \dots, n;$$

(2) *Along  $\gamma$  the matrix*

$$P(x) = \frac{1}{2}F_{u'u'}(x, u(x), u'(x))$$

*is positive definite;*

(3) *The interval  $[a, b]$  contains no points conjugate to the point  $a$ .*

*Then the functional (12.1.2) has a weak minimum for the curve  $\gamma$ .*

In this work, we give an answer to the following question: What do the results of the four above theorems become when the vector-valued function  $u = (u^1, \dots, u^m)$  depends on several variables  $x = (x^1, \dots, x^n)$  and/or the Lagrangian of the used functional includes higher order derivatives of  $u$ ? To our best knowledge of the literature, in this general situation, there is not

explicit method available to determine if a known extremum is a minimum, a maximum, or a saddle point. To fill this gap and provide a suitable answer to our main question, we establish a regular connection between the second variation of a functional and an operational square matrix. Therefore, by the well known result of the matrix theory, explicit formula for the necessary and sufficient extremum conditions can be derived without making use of the notion of conjugate points as in the Jacobi theorems. Furthermore, the matrices  $F_{uu}$ ,  $F_{uu'}$  and  $F_{u'u'}$  used in the above Legendre and Jacobi conditions are deduced as sub matrices of a general matrix associated with the second variation.

## 12.2 Second variation and conditions for local extremums

Consider a variational problem of the form (11.2.5) with the Lagrangian  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ . Define an  $m \times m$  block matrix  $A$  of second order partial derivatives of  $L$  by:

$$A = \left[ A^{jj'} \right]_{1 \leq j, j' \leq m} \tag{12.2.5}$$

with  $A^{jj'}$  being again an  $s \times s$  block matrix defined by

$$A^{jj'} = \left[ A_{kk'}^{jj'} \right]_{0 \leq k, k' \leq s},$$

where  $A_{kk'}^{jj'}$  is a  $p_k \times p_{k'}$  matrix defined by

$$A_{kk'}^{jj'} = \left[ \frac{\partial^2 L}{\partial u_{(k)}^j [h] \partial u_{(k')}^{j'} [h']} \right]_{\substack{1 \leq h \leq p_k \\ 1 \leq h' \leq p_{k'}}}. \tag{12.2.6}$$

Note that the matrix  $A$  is obviously symmetric by construction.

**Example 12.2.1** *Let us construct the matrix  $A^{jj'}$  for particular values of the integers  $n$  and  $s$ . If  $s = 1$ , then*

$$A^{jj'} = \begin{bmatrix} A_{00}^{jj'} & A_{01}^{jj'} \\ A_{10}^{jj'} & A_{11}^{jj'} \end{bmatrix}.$$

*In this case, we have for  $n = 1$ ,  $x = x^1$  :*

$$\begin{aligned} A_{00}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, & A_{01}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u_x^{j'}}, \\ A_{10}^{jj'} &= \frac{\partial^2 L}{\partial u_x^j \partial u^{j'}}, & A_{11}^{jj'} &= \frac{\partial^2 L}{\partial u_x^j \partial u_x^{j'}}, \end{aligned}$$

*thus*

$$A^{jj'} = \begin{bmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_x^{j'}} \\ \frac{\partial^2 L}{\partial u_x^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_x^j \partial u_x^{j'}} \end{bmatrix};$$

*for  $n = 2$ ,  $x = (x^1, x^2)$  :*

$$A_{00}^{jj'} = \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, \quad A_{01}^{jj'} = \left( \begin{array}{cc} \frac{\partial^2 L}{\partial u^j \partial u_{x^1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x^2}^{j'}} \end{array} \right),$$

$$A_{10}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} \end{pmatrix}, \quad A_{11}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \end{pmatrix},$$

thus

$$A^{jj'} = \begin{bmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \end{bmatrix}.$$

If  $s = 2$ , then

$$\text{with } A^{jj'} = \begin{bmatrix} A_{00}^{jj'} & A_{01}^{jj'} & A_{02}^{jj'} \\ A_{10}^{jj'} & A_{11}^{jj'} & A_{12}^{jj'} \\ A_{20}^{jj'} & A_{21}^{jj'} & A_{22}^{jj'} \end{bmatrix}.$$

In this case, we have for  $n = 1$ ,  $x = x^1$ :

$$\begin{aligned} A_{00}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, & A_{01}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u_x^{j'}}, & A_{02}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u_{2x}^{j'}}, \\ A_{10}^{jj'} &= \frac{\partial^2 L}{\partial u_x^j \partial u^{j'}}, & A_{11}^{jj'} &= \frac{\partial^2 L}{\partial u_x^j \partial u_x^{j'}}, & A_{12}^{jj'} &= \frac{\partial^2 L}{\partial u_x^j \partial u_{2x}^{j'}}, \\ A_{20}^{jj'} &= \frac{\partial^2 L}{\partial u_{2x}^j \partial u^{j'}}, & A_{21}^{jj'} &= \frac{\partial^2 L}{\partial u_{2x}^j \partial u_x^{j'}}, & A_{22}^{jj'} &= \frac{\partial^2 L}{\partial u_{2x}^j \partial u_{2x}^{j'}} \end{aligned}$$

thus

$$A^{jj'} = \begin{bmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_x^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{2x}^{j'}} \\ \frac{\partial^2 L}{\partial u_x^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_x^j \partial u_x^{j'}} & \frac{\partial^2 L}{\partial u_x^j \partial u_{2x}^{j'}} \\ \frac{\partial^2 L}{\partial u_{2x}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{2x}^j \partial u_x^{j'}} & \frac{\partial^2 L}{\partial u_{2x}^j \partial u_{2x}^{j'}} \end{bmatrix};$$

for  $n = 2$ ,  $x = (x^1, x^2)$ :

$$\begin{aligned} A_{00}^{jj'} &= \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, & A_{01}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_2}^{j'}} \end{pmatrix}, \\ A_{02}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_2}^{j'}} \end{pmatrix}, \\ A_{10}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} \end{pmatrix}, & A_{11}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \end{pmatrix}, \\ A_{12}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \end{pmatrix}, \\ A_{20}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_2}^j \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u^{j'}} \end{pmatrix}, & A_{21}^{jj'} &= \begin{pmatrix} \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} \end{pmatrix}, \end{aligned}$$

$$A_{22}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{2x_2}^{j'}} \end{pmatrix},$$

thus

$$A^{jj'} = \begin{bmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_2}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_2}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_1}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{x_1 x_2}^j \partial u_{2x_2}^{j'}} \\ \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{2x_1}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{x_1 x_2}^{j'}} & \frac{\partial^2 L}{\partial u_{2x_2}^j \partial u_{2x_2}^{j'}} \end{bmatrix}.$$

Let us recall the following formulation of the Taylor's theorem with the remainder, useful in the sequel.

**Theorem 12.2.1 ([84])** *Suppose that  $f \in C^2(I, \mathbb{R})$ ,  $a \in I$ , where  $I$  is an open interval. Then*

$$f(a+t) = f(a) + f'(a)t + f''(a)\frac{t^2}{2} + e(a,t)t^2,$$

where

$$e(a,t) = \int_0^1 [f''(a+\sigma t) - f''(a)] (1-\sigma) d\sigma.$$

We can apply this theorem to rewrite (11.2.6) as

$$\mathcal{F}(u+t\phi) = \Phi(t) = \Phi(0) + \Phi'(0)t + \frac{1}{2}t^2\Phi''(0) + e(0,t)t^2, \tag{12.2.7}$$

where

$$e(0,t) = \int_0^1 [\Phi''(\sigma t) - \Phi''(0)] (1-\sigma) d\sigma. \tag{12.2.8}$$

As already shown  $\Phi(0) = \mathcal{F}(u)$  and by definition  $\Phi'(0) = \delta\mathcal{F}(u, \phi)$ . The quantity  $\Phi''(t)$  can be found by differentiating (11.2.8) under the integral sign and using the chain rule:

$$\begin{aligned} \Phi''(t) &= \frac{d}{dt} \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \phi_{(k)}^j[h](x) \frac{\partial L(x, u^{(s)}(x) + t\phi^{(s)}(x))}{\partial u_{(k)}^j[h]} dx \\ &= \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \phi_{(k)}^j[h](x) \frac{d}{dt} \left( \frac{\partial L(x, u^{(s)}(x) + t\phi^{(s)}(x))}{\partial u_{(k)}^j[h]} \right) dx \\ &= \int_{\Lambda} \sum_{j,j'=1}^m \sum_{k,k'=0}^s \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \phi_{(k)}^j[h](x) \phi_{(k')}^{j'}[h'](x) \frac{\partial^2 L(x, u^{(s)}(x) + t\phi^{(s)}(x))}{\partial u_{(k)}^j[h] \partial u_{(k')}^{j'}[h']} dx \\ &= \int_{\Lambda} \phi^{(s)}(x) A(x, u^{(s)}(x) + t\phi^{(s)}(x))^T \phi^{(s)}(x) dx, \end{aligned} \tag{12.2.9}$$

where the notation  $^T(\cdot)$  denotes the transpose of  $(\cdot)$ . In particular at  $t = 0$ , we get

$$\begin{aligned}\Phi''(0) &= \int_{\Lambda} \sum_{j,j'=1}^m \sum_{k,k'=0}^s \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \phi_{(k)}^j[h](x) \phi_{(k')}^{j'}[h'](x) \frac{\partial^2 L(x, u^{(s)}(x))}{\partial u_{(k)}^j[h] \partial u_{(k')}^{j'}[h']} dx \\ &= \int_{\Lambda} \phi^{(s)}(x) A(x, u^{(s)}(x))^T \phi^{(s)}(x) dx.\end{aligned}\quad (12.2.10)$$

We then arrive at the following formulation.

**Definition 12.2.1** *The second variation of the functional  $\mathcal{F}$  in the neighborhood of  $u$  in the direction  $\phi$  is defined by*

$$\delta^2 \mathcal{F}(u + t\phi, \phi) = \Phi''(t). \quad (12.2.11)$$

*In particular, the second variation of  $\mathcal{F}$  at  $u$  in the direction  $\phi$  is given by*

$$\delta^2 \mathcal{F}(u, \phi) = \Phi''(0). \quad (12.2.12)$$

The Taylor expansion (12.2.7) can be now re-expressed as

$$\begin{aligned}\Phi(t) &= \mathcal{F}(u + t\phi) = \mathcal{F}(u) + t\delta\mathcal{F}(u, \phi) + \frac{1}{2}t^2\delta^2\mathcal{F}(u, \phi) \\ &\quad + t^2 \int_0^1 [\delta^2\mathcal{F}(u + \sigma t\phi, \phi) - \delta^2\mathcal{F}(u, \phi)] (1 - \sigma) d\sigma.\end{aligned}\quad (12.2.13)$$

Let us also recall the following two results which are important to prove the main results of this work.

**Lemma 12.2.1 ([84])** *Suppose that  $A = (a_{ij})$  is an  $N \times N$  matrix, and set  $\|A\| = \sqrt{\sum_{i,j=1}^N a_{ij}^2}$ . Then  $(\|\cdot\|)$  denotes the Euclidean norm  $\mathbb{R}^N$ )*

1.  $v \cdot w \leq \|v\| \|w\|$  for all  $v$  and  $w$  in  $\mathbb{R}^N$ .
2.  $|Av| \leq \|A\| \|v\|$  for all  $v$  in  $\mathbb{R}^N$ .
3.  $|v \cdot Aw| \leq \|A\| \|v\| \|w\|$  for all  $v$  and  $w$  in  $\mathbb{R}^N$ .

**Definition 12.2.2 (Positive semi-definite)** *A symmetric matrix  $A \in \mathbb{R}^{N^2}$  is called positive semi-definite if  $v \cdot Av \geq 0$  for all  $v \in \mathbb{R}^N$ .*

**Definition 12.2.3 (Positive definite)** *A symmetric matrix  $A \in \mathbb{R}^{N^2}$  is called positive definite if  $v \cdot Av > 0$  for all  $v \in \mathbb{R}^N \setminus \{0\}$ .*

**Lemma 12.2.2 ([84])** *Suppose that  $A$  is a positive definite  $N \times N$  matrix. Then there is a constant  $k > 0$  such that  $v \cdot Av \geq k\|v\|^2$  for all  $v$  in  $\mathbb{R}^N$ .*

There results the following.

**Lemma 12.2.3** *Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $u \in \mathcal{C}_b^s(\Lambda, U)$  is admissible for  $\mathcal{F}$ . Then for any  $\epsilon > 0$ , there is  $\delta > 0$  such that*

$$|e(0, t)| \leq \frac{\epsilon}{2} \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \left| \phi_{(k)}^j[h](x) \right|^2 dx = \frac{\epsilon}{2} \int_{\Lambda} \left\| \phi^{(s)}(x) \right\|^2 dx$$

for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$  and  $|t| \leq \epsilon_0$  such that

(i)  $u + t\phi$  is admissible for  $\mathcal{F}$ ,

(ii)  $|t|\sigma \|\phi^{(s)}(x)\| < \delta$  for all  $x \in \Omega$  and  $0 \leq \sigma \leq 1$ .

Here,  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^{q_s}$ .

**Proof.** Using (12.2.9) and (12.2.10), we see that

$$\begin{aligned} e(0, t) &= \int_0^1 (1 - \sigma) [\delta^2 \mathcal{F}(u + \sigma t \phi, \phi) - \delta^2 \mathcal{F}(u, \phi)] d\sigma \\ &= \int_0^1 (1 - \sigma) \int_{\Omega} \phi^{(s)}(x) \left[ A \left( x, u^{(s)}(x) + \sigma t \phi^{(s)}(x) \right) \right. \\ &\quad \left. - A \left( x, u^{(s)}(x) \right) \right]^T \phi^{(s)}(x) dx d\sigma. \end{aligned}$$

Each entry of the matrix  $A$  is a second derivative of the function  $L$  with respect to the coordinates in  $\Omega$ .

Let  $\epsilon > 0$ . Since all the second derivatives of  $L$  with respect to the variables in  $\Omega$  are continuous, then the matrix  $A$  is continuous and there exists  $\delta > 0$  such that, for all  $\phi \in \mathcal{C}_0^\infty(\Omega, U)$ ,

$$\left\| \left( u^{(s)}(x) + t\sigma \phi^{(s)}(x) \right) - u^{(s)}(x) \right\| = |t|\sigma \|\phi^{(s)}(x)\| < \delta \implies \|B(x, \sigma, t)\| < \epsilon$$

for all  $x \in \Omega$ ,  $|t| \leq \epsilon_0$  and  $0 \leq \sigma \leq 1$ , where

$$B(x, \sigma, t) = A \left( x, u^{(s)}(x) + \sigma t \phi^{(s)}(x) \right) - A \left( x, u^{(s)}(x) \right).$$

Therefore, using the continuity of the bilinear form induced by the matrix  $A$  (see the third property of Lemma 12.2.1), we obtain the required result:

$$\begin{aligned} |e(0, t)| &\leq \epsilon \int_0^1 (1 - \sigma) d\sigma \int_{\Omega} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \left| \phi_{(k)}^j[h](x) \right|^2 dx \\ &\leq \frac{\epsilon}{2} \int_{\Omega} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \left| \phi_{(k)}^j[h](x) \right|^2 dx = \frac{\epsilon}{2} \int_{\Omega} \left\| \phi^{(s)}(x) \right\|^2 dx. \end{aligned}$$

Here,  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^{q_s}$  since  $\phi^{(s)}(x) \in \mathbb{R}^{q_s}$  for all  $x \in \Omega$ . ■

**Theorem 12.2.2** Suppose that  $L \in \mathcal{C}^2(\Omega \times \Omega, \mathbb{R})$ , and  $u \in \mathcal{C}_b^s(\Omega, U)$  is admissible for  $\mathcal{F}$ .

1. If  $u$  is a weak local minimum for  $\mathcal{F}$ , then for all  $\phi \in \mathcal{C}_0^\infty(\Omega, U)$ , we have

$$\delta \mathcal{F}(u, \phi) = 0 \quad \text{and} \quad \delta^2 \mathcal{F}(u, \phi) \geq 0. \tag{12.2.14}$$

2. If  $u$  is a weak local extremum for  $\mathcal{F}$  and there is a constant  $k > 0$  such that

$$\delta^2 \mathcal{F}(u, \phi) \geq k \int_{\Omega} \left\| \phi^{(s)}(x) \right\|^2 dx \tag{12.2.15}$$

for all  $\phi \in \mathcal{C}_0^\infty(\Omega, U)$ , then  $u$  is a strict weak local minimum.

**Proof.** For the first part of Theorem 12.2.2, the assumption that  $u$  is a weak local minimum for  $\mathcal{F}$  implies that  $t = 0$  is a local minimum for the function  $\Phi(t) = \mathcal{F}(u + t\phi)$ . Consequently,  $0 = \Phi'(0) = \delta\mathcal{F}(u, \phi)$ . The Taylor expansion (12.2.7) of  $\Phi$  gives

$$\Phi''(0) = 2 \frac{\Phi(t) - \Phi(0)}{t^2} + 2e(0, t)$$

which leads to

$$0 \leq \lim_{t \rightarrow 0} 2 \frac{\Phi(t) - \Phi(0)}{t^2} = \Phi''(0) = \delta^2\mathcal{F}(u, \phi)$$

since  $\lim_{t \rightarrow 0} e(0, t) = 0$  and  $\Phi(t) \geq \Phi(0)$  for all  $t \neq 0$ .

For the second part, we suppose that  $v \in \mathcal{C}_b^s(\Lambda, U)$  is admissible for  $\mathcal{F}$ . Let  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$  so that  $v = u + t\phi$  for some  $t \in \mathbb{R}$  such that  $|t| \leq \epsilon_0$ . Then

$$\mathcal{F}(v) = \mathcal{F}(u + t\phi) = \mathcal{F}(u) + t\delta\mathcal{F}(u, \phi) + \frac{1}{2}t^2\delta^2\mathcal{F}(u, \phi) + t^2e(0, t),$$

where

$$e(0, t) = \int_0^1 [\delta^2\mathcal{F}(u + \sigma t\phi, \phi) - \delta^2\mathcal{F}(u, \phi)] (1 - \sigma) d\sigma.$$

By assumption,  $u$  is a weak extremum, so  $\delta\mathcal{F}(u, \phi) = 0$ . By Lemma 12.2.3, there is  $\epsilon > 0$  such that

$$|e(0, t)| \leq \frac{k}{4} \int_\Lambda \|\phi^{(s)}(x)\|^2 dx$$

provided  $|t|\sigma\|\phi^{(s)}(x)\| < \epsilon$  for all  $x \in \Lambda$  and  $0 \leq \sigma \leq 1$ . Therefore, using (12.2.15), if  $\|v - u\|_s = |t|\|\phi\|_s < \epsilon$  we have

$$\begin{aligned} \mathcal{F}(v) &\geq \mathcal{F}(u) + \frac{t^2}{2}\delta^2\mathcal{F}(u, \phi) - t^2|e(0, t)| \\ &\geq \mathcal{F}(u) + \frac{kt^2}{2} \int_\Lambda \|\phi^{(s)}(x)\|^2 dx - \frac{kt^2}{4} \int_\Lambda \|\phi^{(s)}(x)\|^2 dx \\ &= \mathcal{F}(u) + \frac{kt^2}{4} \int_\Lambda \|\phi^{(s)}(x)\|^2 dx. \end{aligned}$$

If  $v \neq u$ , then the integral on the right hand side is positive, and we have  $\mathcal{F}(v) > \mathcal{F}(u)$ . Therefore  $u$  is a strict weak local minimum. ■

In part (1) of Theorem 12.2.2, the fact that the second variations must be nonnegative is a necessary condition for  $u$  to be a local minimum.

**Theorem 12.2.3** *Let  $\Lambda$  be a bounded connected subset of  $X$ . Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $u \in \mathcal{C}_b^s(\Lambda, U)$  is admissible for  $\mathcal{F}$ . If  $u$  is a weak local extremum for  $\mathcal{F}$  and*

$$\delta^2\mathcal{F}(u, \phi) \geq 0 \tag{12.2.16}$$

for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$ , then  $u$  is a weak local minimum.

**Proof.** Let  $\hat{\epsilon} > 0$ . Suppose that  $v \in \mathcal{C}_b^s(\Lambda, U)$  is admissible for  $\mathcal{F}$ . Let  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$  so that  $v = u + t\phi$  for some  $t \in \mathbb{R}$  such that  $|t| \leq \epsilon_0$ . Then

$$\mathcal{F}(v) = \mathcal{F}(u + t\phi) = \mathcal{F}(u) + t\delta\mathcal{F}(u, \phi) + \frac{1}{2}t^2\delta^2\mathcal{F}(u, \phi) + t^2e(0, t),$$

where

$$e(0, t) = \int_0^1 [\delta^2 \mathcal{F}(u + \sigma t \phi, \phi) - \delta^2 \mathcal{F}(u, \phi)] (1 - \sigma) d\sigma.$$

By assumption,  $u$  is a weak extremum, so  $\delta \mathcal{F}(u, \phi) = 0$ . By Lemma 12.2.3, there is  $\epsilon > 0$  such that

$$|e(0, t)| \leq \frac{\hat{\epsilon}}{2} \int_{\Lambda} \|\phi^{(s)}(x)\|^2 dx \leq \frac{\hat{\epsilon}}{2} \text{mes}(\Lambda) \|\phi\|_s^2$$

provided  $|t| \|\sigma\| \|\phi^{(s)}(x)\| < \epsilon$  for all  $x \in \Lambda$  and  $0 \leq \sigma \leq 1$ . Therefore, using (12.2.16), if  $\|v - u\|_s = |t| \|\phi\|_s < \epsilon$  with  $\phi \neq 0$ , we have

$$\begin{aligned} \mathcal{F}(v) &\geq \mathcal{F}(u) + \frac{t^2}{2} \delta^2 \mathcal{F}(u, \phi) - t^2 |e(0, t)| \\ &\geq \mathcal{F}(u) - \frac{\hat{\epsilon}}{2} t^2 \text{mes}(\Lambda) \|\phi\|_s^2 \\ &\geq \mathcal{F}(u) - \frac{\hat{\epsilon}}{2} \epsilon^2 \text{mes}(\Lambda). \end{aligned}$$

Thus,

$$\mathcal{F}(v) \geq \mathcal{F}(u) - \lim_{\hat{\epsilon} \rightarrow 0} \left[ \frac{\hat{\epsilon}}{2} \epsilon^2 \text{mes}(\Lambda) \right] = \mathcal{F}(u).$$

We have  $\mathcal{F}(v) \geq \mathcal{F}(u)$ . Therefore  $u$  is a weak local minimum. ■

## Second variation and Legendre necessary conditions

According to Theorem 12.2.2, if  $u$  is a weak local minimum for the functional  $\mathcal{F}$ , then  $\delta^2 \mathcal{F}(u, \phi) \geq 0$  for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$ . Here we find some natural and nontrivial consequences of that condition.

Construct nonzero functions  $\psi_l$  by  $\psi_0 = 1$  and for  $l = 1, 2, \dots, s$

$$\psi_l(y) = \begin{cases} 0 & -\infty < y \leq -1 \\ 1 - y^l \text{sign}(y) & -1 \leq y \leq +1 \\ 0 & +1 \leq y < +\infty \end{cases}$$

if  $l$  is odd, and

$$\psi_l(y) = \begin{cases} 0 & -\infty < y \leq -1 \\ 1 - y^l & -1 \leq y \leq +1 \\ 0 & +1 \leq y < +\infty \end{cases}$$

if  $l$  is even.

It is clear that  $\psi_l \in \mathcal{C}^\infty(\mathbb{R} \setminus \{-1, 1\}, \mathbb{R})$  and satisfy  $\psi_l(y) = 0$  for all  $y$  with  $|y| > 1$ , i.e.  $\psi_l \in \mathcal{C}_0^\infty(\mathbb{R} \setminus \{-1, 1\}, \mathbb{R})$ . Furthermore,  $\psi_l \in \mathcal{C}(\mathbb{R}, \mathbb{R})$  with  $\psi_l(-1) = \psi_l(1) = 0$ . We also have

$$(\psi_l)_{(l)}[1](y) \in \{-l!, 0, +l!\} \quad \forall y \in \mathbb{R} \tag{12.2.17}$$

that is  $(\psi_l)_{(l)}[1]$  is constant on  $\mathbb{R}$ . Thus,

$$(\psi_l)_{(l+\nu)}[1](y) = 0 \quad \forall y \in \mathbb{R}, \nu \geq 1. \tag{12.2.18}$$

Let  $x_0 = (x_0^1, \dots, x_0^n) \in \Lambda$ . Since  $\Lambda$  is an open subset of  $\mathbb{R}^n$ , there is  $r_0 > 0$  such that  $B(x_0, r_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r_0\} \subset \Lambda$ . Let  $\xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m$  and  $0 < \epsilon < \frac{r_0}{\sqrt{n}}$ .

Set  $\phi_l(x) = (\phi_l^1(x), \dots, \phi_l^m(x)) \cdot \mathcal{X}_{B(x_0, r_0)}(x)$  where  $\mathcal{X}_{B(x_0, r_0)}$  is the characteristic function of the set  $B(x_0, r_0)$  and

$$\phi_l^j(x) = \xi^j \epsilon^l \sum_{i=1}^n \psi_l \left( \frac{x^i - x_0^i}{\epsilon} \right). \quad (12.2.19)$$

Clearly, the support of  $\phi_l$  is a compact contained in  $\Lambda$ .

We have for all  $k \in \mathbb{N}$  and  $h = 1, 2, \dots, p_k$

$$\left( \phi_l^j \right)_{(k)} [h](x) = \xi^j \epsilon^{l-k} (\psi_l)_{(k)} [1] \left( \frac{x^{i(h)} - x_0^{i(h)}}{\epsilon} \right) \quad (12.2.20)$$

for some  $i(h) \in \{1, 2, \dots, n\}$ . Using (12.2.17) and (12.2.18), we see that

$$\left( \phi_l^j \right)_{(l)} [h] \in \{-\xi^j l, 0, +\xi^j l\} \quad \forall h = 1, 2, \dots, p_l \quad (12.2.21)$$

and for  $\nu \geq 1$

$$\left( \phi_l^j \right)_{(l+\nu)} [h] = 0 \quad \forall h = 1, 2, \dots, p_{l+\nu}. \quad (12.2.22)$$

Therefore, if  $u$  is a weak local minimum for the functional  $\mathcal{F}$ , we have for all  $l = 0, 1, 2, \dots, s$

$$0 \leq \delta^2 \mathcal{F}(u, \phi_l) = I_1 + I_2 + I_3, \quad (12.2.23)$$

where

$$I_1 = \int_{\|x-x_0\| \leq r_0} \sum_{j,j'=1}^m \sum_{k,k'=l+1}^s \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \left( \phi_l^j \right)_{(k)} [h](x) \left( \phi_l^{j'} \right)_{(k')} [h'](x) \frac{\partial^2 L(x, u^{(s)}(x))}{\partial u_{(k)}^j [h] \partial u_{(k')}^{j'} [h']} dx; \quad (12.2.24)$$

$$I_2 = \int_{\|x-x_0\| \leq r_0} \sum_{j,j'=1}^m \sum_{h,h'=1}^{p_l} \left( \phi_l^j \right)_{(l)} [h](x) \left( \phi_l^{j'} \right)_{(l)} [h'](x) \frac{\partial^2 L(x, u^{(s)}(x))}{\partial u_{(l)}^j [h] \partial u_{(l)}^{j'} [h']} dx; \quad (12.2.25)$$

$$I_3 = 2 \int_{\|x-x_0\| \leq r_0} \sum_{j,j'=1}^m \sum_{k=0}^{l-1} \sum_{k'=k+1}^l \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \left( \phi_l^j \right)_{(k)} [h](x) \left( \phi_l^{j'} \right)_{(k')} [h'](x) \frac{\partial^2 L(x, u^{(s)}(x))}{\partial u_{(k)}^j [h] \partial u_{(k')}^{j'} [h']} dx. \quad (12.2.26)$$

Of course, if  $l = 0$  there is not the integral  $I_3$ . If  $s = 0$   $I_1$  and  $I_3$  do not exist. By (12.2.22), we see that  $I_1 = 0$ . Using (12.2.19) and (12.2.20) in (12.2.25), we have

$$I_2 = \int_{\|x-x_0\| \leq r_0} \sum_{j,j'=1}^m \xi^j \xi^{j'} \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x, u^{(s)}(x))}{\partial u_{(l)}^j [h] \partial u_{(l)}^{j'} [h']} \times (\psi_l)_{(l)} [1] \left( \frac{x^{i(h)} - x_0^{i(h)}}{\epsilon} \right) (\psi_l)_{(l)} [1] \left( \frac{x^{i(h')} - x_0^{i(h')}}{\epsilon} \right) dx. \quad (12.2.27)$$

If we set  $x = x_0 + \epsilon y$  then  $dx = \epsilon^n dy$  and using (12.2.17),  $I_2$  satisfies

$$I_2 \leq \epsilon^n (l!)^2 \int_{\|y\| \leq 1} \sum_{j,j'=1}^m \xi^j \xi^{j'} \left( \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x_0 + \epsilon y, u^{(s)}(x_0 + \epsilon y))}{\partial u_{(l)}^j[h] \partial u_{(l)}^{j'}[h']} \right) dy. \tag{12.2.28}$$

In a similar way,  $I_3$  becomes

$$I_3 = 2\epsilon^n \epsilon^{2l-(k+k')} \int_{\|y\| \leq 1} \sum_{j,j'=1}^m \xi^j \xi^{j'} \left( \sum_{k=0}^{l-1} \sum_{k'=k+1}^l \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} (\psi_l)_{(k)} [1] \left( y^{i(h)} \right) \right. \\ \left. \times (\psi_l)_{(k')} [1] \left( y^{i(h')} \right) \frac{\partial^2 L(x_0 + \epsilon y, u^{(s)}(x_0 + \epsilon y))}{\partial u_{(k)}^j[h] \partial u_{(k')}^{j'}[h']} \right) dy. \tag{12.2.29}$$

Substituting (12.2.28) and (12.2.29) into (12.2.23), we have

$$0 \leq \sum_{j,j'=1}^m \xi^j \xi^{j'} \left[ (l!)^2 \int_{\|y\| \leq 1} \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x_0 + \epsilon y, u^{(s)}(x_0 + \epsilon y))}{\partial u_{(l)}^j[h] \partial u_{(l)}^{j'}[h']} dy \right. \\ \left. + \epsilon^{2l-(k+k')} \int_{\|y\| \leq 1} \sum_{k=0}^{l-1} \sum_{k'=k+1}^l \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} (\psi_l)_{(k)} [1] \left( y^{i(h)} \right) \right. \\ \left. \times (\psi_l)_{(k')} [1] \left( y^{i(h')} \right) \frac{\partial^2 L(x_0 + \epsilon y, u^{(s)}(x_0 + \epsilon y))}{\partial u_{(k)}^j[h] \partial u_{(k')}^{j'}[h']} dy \right]. \tag{12.2.30}$$

We have  $\epsilon^{2l-(k+k')} \rightarrow 0$  as  $\epsilon \rightarrow 0$  since in  $I_3$ ,  $2l - (k + k') \geq 1$ .

Therefore, as  $\epsilon \rightarrow 0$ , the second term in (12.2.30) vanishes and it remains

$$0 \leq (l!)^2 \sum_{j,j'=1}^m \xi^j \xi^{j'} \left( \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x_0, u^{(s)}(x_0))}{\partial u_{(l)}^j[h] \partial u_{(l)}^{j'}[h']} \right) \int_{\|y\| \leq 1} dy \tag{12.2.31}$$

from which we deduce

$$\sum_{j,j'=1}^m \left( \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x_0, u^{(s)}(x_0))}{\partial u_{(l)}^j[h] \partial u_{(l)}^{j'}[h']} \right) \xi^j \xi^{j'} \geq 0. \tag{12.2.32}$$

Since  $x_0 \in \Lambda$  and  $\xi \in \mathbb{R}^m$  are arbitrary, we have proved the following theorem

**Theorem 12.2.4** *Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $\bar{u}$  is a weak local minimum for  $\mathcal{F}$ . Then for all  $x \in \Lambda$  and  $\xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m$ ,*

$$\sum_{j,j'=1}^m \left( \sum_{h,h'=1}^{p_l} \frac{\partial^2 L(x, \bar{u}^{(s)}(x))}{\partial u_{(l)}^j[h] \partial u_{(l)}^{j'}[h']} \right) \xi^j \xi^{j'} \geq 0 \quad l = 0, 1, 2, \dots, s, \tag{12.2.33}$$

*i.e. for all  $x \in \Lambda$ , the square matrices  $A_{ll}^{jj'}(x, \bar{u}^{(s)}(x))$ ,  $l = 0, 1, 2, \dots, s$ , are positive semi-definite.*

The inequalities in (12.2.33) are called the general forms of Legendre conditions. They define by Theorem 12.2.4 new necessary conditions for  $\bar{u}$  to be a weak local minimum of  $\mathcal{F}$ . We say that the function  $\bar{u}$  satisfies the strict Legendre conditions if the matrices  $A_{ll}^{jj'}(x, \bar{u}^{(s)}(x))$ ,  $l = 0, 1, 2, \dots, s$ , are positive definite, uniformly for all  $x \in \Lambda$ .

## Second variation and relevant sufficient conditions

Part (2) of Theorem 12.2.2 gives us a sufficient condition for a function to be a minimum. However, the conditions involving the second variations are not easy to satisfy. So, the results of this subsection are useful as they imply the condition (12.2.15).

**Theorem 12.2.5** *Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $u$  is a weak local extremum for  $\mathcal{F}$ . If the matrix  $A(x, u^{(s)}(x))$  defined by (12.2.5) is positive definite for all  $x \in \Lambda$ , then  $u$  is a strict weak local minimum.*

**Proof.** By (12.2.10), (12.2.12) and Lemma 12.2.2, for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$  we have

$$\delta^2 \mathcal{F}(u, \phi) = \int_{\Lambda} \phi^{(s)}(x) A(x, u^{(s)}(x))^T \phi^{(s)}(x) dx \geq k \int_{\Lambda} \|\phi^{(s)}(x)\|^2 dx$$

for some  $k > 0$ . By part (2) of Theorem 12.2.2,  $u$  is a strict weak minimum for  $\mathcal{F}$ . ■

**Theorem 12.2.6** *Let  $\Lambda$  be a bounded connected subset of  $X$ . Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $u$  is a weak local extremum for  $\mathcal{F}$ . If the matrix  $A(x, u^{(s)}(x))$  defined by (12.2.5) is semi-positive definite for all  $x \in \Lambda$ , then  $u$  is a weak local minimum.*

**Proof.** By hypothesis, the function  $V(x; \phi) = \phi^{(s)}(x) A(x, u^{(s)}(x))^T \phi^{(s)}(x)$  is continuous and positive on  $\Lambda$  for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$ . Therefore, by (12.2.10) and (12.2.12), for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$ , we have

$$\delta^2 \mathcal{F}(u, \phi) = \int_{\Lambda} \phi^{(s)}(x) A(x, u^{(s)}(x))^T \phi^{(s)}(x) dx \geq 0.$$

Thus, by Theorem 12.2.3,  $u$  is a weak minimum for  $\mathcal{F}$ . ■ The second variation of  $\mathcal{F}$  is given by

$$\begin{aligned} \delta^2 \mathcal{F}(u, \phi) &= \int_{\Lambda} \sum_{j, j'=1}^m \sum_{k, k'=0}^s \phi_{(k)}^j(x) A_{kk'}^{jj'}(x, u^{(s)}(x))^T \phi_{(k')}^{j'}(x) dx \\ &= \int_{\Lambda} \sum_{j, j'=1}^m \sum_{k=0}^s \left[ \phi_{(k)}^j A_{kk}^{jj'}{}^T \phi_{(k)}^{j'} + 2 \sum_{\substack{k'=0 \\ k' \neq k}}^s \phi_{(k)}^j A_{kk'}^{jj'}{}^T \phi_{(k')}^{j'} \right] dx \\ &= I_1 + 2I_2, \end{aligned} \tag{12.2.34}$$

where the matrices  $A_{kk'}^{jj'}(x, u^{(s)}(x))$  are defined by (12.2.6) and

$$I_1 = \int_{\Lambda} \sum_{j, j'=1}^m \sum_{k=0}^s \phi_{(k)}^j A_{kk}^{jj'}{}^T \phi_{(k)}^{j'} dx; \tag{12.2.35}$$

$$I_2 = \int_{\Lambda} \sum_{j, j'=1}^m \sum_{k=0}^s \sum_{\substack{k'=0 \\ k' \neq k}}^s \phi_{(k)}^j A_{kk'}^{jj'}{}^T \phi_{(k')}^{j'} dx. \tag{12.2.36}$$

Integral  $I_1$  can be rewritten as

$$\begin{aligned} I_1 &= \int_{\Lambda} \sum_{j=1}^m \left[ \sum_{k=0}^s \phi_{(k)}^j A_{kk}^{jj}{}^T \phi_{(k)}^j + 2 \sum_{\substack{j'=1 \\ j' \neq j}}^m \sum_{k=0}^s \phi_{(k)}^j A_{kk}^{jj'}{}^T \phi_{(k)}^{j'} \right] dx \\ &= J_1 + 2J_2, \end{aligned} \tag{12.2.37}$$

$$J_1 = \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \phi_{(k)}^j A_{kk}^{jj} T \phi_{(k)}^j dx; \tag{12.2.38}$$

$$J_2 = \int_{\Lambda} \sum_{j=1}^m \sum_{\substack{j'=1 \\ j' \neq j}}^m \sum_{k=0}^s \phi_{(k)}^j A_{kk}^{jj'} T \phi_{(k)}^{j'} dx. \tag{12.2.39}$$

Thus, the second variation can be written as

$$\delta^2 \mathcal{F}(u, \phi) = I_1 + 2 I_2 = J_1 + 2 J_2 + 2 I_2 = J_1 + 2 (J_2 + I_2). \tag{12.2.40}$$

We can now prove the following new sufficient condition.

**Theorem 12.2.7** *Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ , and  $u$  is a weak local extremums for  $\mathcal{F}$ . If*

- (i)  $J_2 + I_2 \geq 0$ , and
- (ii) *the square matrices  $A_{kk}^{jj'}(x, u^{(s)}(x))$  are positive definite for all  $x \in \Lambda$ , i.e., satisfy the strict Legendre conditions,*

*then  $u$  is a strict weak local minimum for  $\mathcal{F}$ .*

**Proof.** We have shown that

$$\delta^2 \mathcal{F}(u, \phi) = J_1 + 2 (J_2 + I_2), \tag{12.2.41}$$

where  $J_1$ ,  $J_2$  and  $I_2$  are defined by (12.2.38), (12.2.39) and (12.2.36), respectively. By condition (ii), using the Lemma 12.2.2, there exist constants  $\alpha_k^j > 0$  such that

$$\begin{aligned} J_1 &\geq \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \alpha_k^j \phi_{(k)}^j T \phi_{(k)}^j dx \\ &\geq \alpha \int_{\Lambda} \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \left( \phi_{(k)}^j [h](x) \right)^2 dx = \alpha \int_{\Lambda} \left\| \phi^{(s)}(x) \right\|^2 dx, \end{aligned} \tag{12.2.42}$$

where  $0 < \alpha = \min \left\{ \alpha_k^j, 1 \leq j \leq m, 0 \leq k \leq s \right\}$ . By condition (i) and the inequality (12.2.42), the second variation (12.2.41) satisfies for all  $\phi \in \mathcal{C}_0^\infty(\Lambda, U)$  the inequality

$$\delta^2 \mathcal{F}(u, \phi) \geq \alpha \int_{\Lambda} \left\| \phi^{(s)}(x) \right\|^2 dx. \tag{12.2.43}$$

Consequently, by the second part of Theorem 12.2.2,  $u$  is a weak minimum for  $\mathcal{F}$ . ■

**Corollary 12.2.1** *Suppose that  $L \in \mathcal{C}^2(\Lambda \times \Omega, \mathbb{R})$ ,  $u$  is a weak extremum for  $\mathcal{F}$ . If*

- (a) *for all  $k \neq k'$ , the bilinear forms defined on  $\mathbb{R}^{p_k} \times \mathbb{R}^{p_{k'}}$  by the matrices  $A_{kk'}^{jj'}(x, u^{(s)}(x))$  are positive for all  $x \in \Lambda$ , and*
- (b) *the square matrices  $A_{kk}^{jj'}(x, u^{(s)}(x))$  are positive definite for all  $x \in \Lambda$ ,*

*then  $u$  is a strict weak local minimum for  $\mathcal{F}$ .*

**Proof.** It suffices to show that condition (i) in Theorem 12.2.7 is satisfied. We have

$$J_2 = \int_{\Lambda} \sum_{j=1}^m \sum_{\substack{j'=1 \\ j' \neq j}}^m \sum_{k=0}^s \sum_{h, h'=1}^{p_k} \phi_{(k)}^j[h] A_{kk}^{jj'}[h, h']^T \phi_{(k)}^{j'}[h'] dx \geq 0 \quad (12.2.44)$$

since by condition (b) the integrand is always positive;

$$I_2 = \int_{\Lambda} \sum_{j, j'=1}^m \sum_{k=0}^s \sum_{\substack{k'=0 \\ k' \neq k}}^s \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \phi_{(k)}^j[h] A_{kk'}^{jj'}[h, h']^T \phi_{(k')}^{j'}[h'] dx, \quad (12.2.45)$$

since by condition (a) the integrand is always positive. Therefore  $J_2 + I_2 \geq 0$ . ■

### 12.3 Applications to some concrete examples

To conclude this work, let us analyze some applications.

**Example 12.3.1** Consider the problem of finding extremum point  $u = u(x)$  with  $x \in [a, b]$ , of the functional  $\mathcal{F}$  defined by

$$\mathcal{F}(u) = \int_a^b \sqrt{1 + u_x(x)^2} dx.$$

The Lagrangian of this functional is

$$L(x, u^{(1)}) = \sqrt{1 + u_x^2}.$$

The extremum must satisfy the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) = 0$$

which gives

$$\frac{u_{x,x}}{(1 + u_x^2)^{\frac{3}{2}}} = 0.$$

The general solution of this equation is  $u(x) = c_1x + c_2$ , where  $c_1$  and  $c_2$  are constants determined by the given end point constraints.

Determine the matrix  $A$  associated to the second variation of this problem.

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix},$$

where

$$A_{00} = \frac{\partial^2 L}{\partial u \partial u} = 0, \quad A_{01} = \frac{\partial^2 L}{\partial u \partial u_x} = 0, \quad A_{10} = \frac{\partial^2 L}{\partial u_x \partial u} = 0,$$

$$A_{11} = \frac{\partial^2 L}{\partial u_x \partial u_x} = \frac{1}{(1 + u_x^2)^{\frac{3}{2}}}.$$

Thus,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{(1 + u_x^2)^{\frac{3}{2}}} \end{bmatrix}.$$

It is clear that the matrix  $A$  is positive semi-definite. Therefore, the found function  $u$ , solution to the Euler-Lagrange equation, is a minimum point to the functional  $\mathcal{F}$ .

Note here that the Legendre necessary conditions are well satisfied. Indeed,  $A_{00} \geq 0$  and  $A_{11} \geq 0$ .

**Example 12.3.2** Consider the problem of finding extremum point  $u = u(x)$  with  $x \in [a, b]$ , of the functional  $\mathcal{F}$  defined by

$$\mathcal{F}(u) = \int_a^b u(x) \sqrt{1 + u_x(x)^2} dx.$$

The Lagrangian of this functional is

$$L(x, u^{(1)}) = u \sqrt{1 + u_x^2}.$$

The extremum must satisfy the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) = 0$$

which gives

$$\frac{1 + u_x^2 - u u_{x,x}}{(1 + u_x^2)^{\frac{3}{2}}} = 0.$$

The general solution of this equation is  $u(x) = c_1 \cosh\left(\frac{x+c_2}{c_1}\right)$ , where  $c_1$  and  $c_2$  are constants determined by the given end conditions.

Determine the matrix  $A$  associated with the second variation of this problem.

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix},$$

where

$$A_{00} = \frac{\partial^2 L}{\partial u \partial u} = 0, \quad A_{01} = \frac{\partial^2 L}{\partial u \partial u_x} = \frac{u_x}{\sqrt{1 + u_x^2}}, \quad A_{10} = \frac{\partial^2 L}{\partial u_x \partial u} = \frac{u_x}{\sqrt{1 + u_x^2}},$$

$$A_{11} = \frac{\partial^2 L}{\partial u_x \partial u_x} = \frac{u}{(1 + u_x^2)^{\frac{3}{2}}}.$$

Thus,

$$A = \begin{bmatrix} 0 & \frac{u_x}{\sqrt{1 + u_x^2}} \\ \frac{u_x}{\sqrt{1 + u_x^2}} & \frac{u}{(1 + u_x^2)^{\frac{3}{2}}} \end{bmatrix}.$$

It is clear that the matrix  $A$  is neither positive semi-definite nor negative semi-definite (i.e.,  $-A$  is not positive semi-definite). Hence, the found function  $u$ , solution to the Euler-Lagrange equation, is neither a minimum point nor a maximum point to the functional  $\mathcal{F}$ . Therefore, we can conclude that this function  $u$  is an instable equilibrium point.

**Example 12.3.3** Let  $\Lambda$  be a connected subset of  $\mathbb{R}^2$ . Consider the problem of finding the function  $(u^1, u^2)$ , where  $u^1 = u^1(x^1, x^2)$ ,  $u^2 = u^2(x^1, x^2)$  with  $(x^1, x^2) \in \Lambda$ , which is an extremum of the functional  $\mathcal{F}$  defined by

$$\mathcal{F}(u^1, u^2) = \int_{\Lambda} L(x^1, x^2, u^{1(1)}(x^1, x^2), u^{2(1)}(x^1, x^2)) dx^1 dx^2,$$

where the Lagrangian  $L$  is

$$L = (u^1)^2 + (u^2)^2 + (u_{x^1}^1)^2 + (u_{x^2}^1)^2 + (u_{x^1}^2)^2 + (u_{x^2}^2)^2 + \frac{1}{2}(u^1 u^2 - u_{x^1}^1 u_{x^2}^1 - u_{x^1}^2 u_{x^2}^2).$$

Extremum of the functional  $\mathcal{F}$  must satisfy the Euler-Lagrange equations

$$\begin{aligned} \frac{\partial L}{\partial u^1} - \frac{\partial}{\partial x^1} \left( \frac{\partial L}{\partial u_{x^1}^1} \right) - \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial u_{x^2}^1} \right) &= 0 \\ \frac{\partial L}{\partial u^2} - \frac{\partial}{\partial x^1} \left( \frac{\partial L}{\partial u_{x^1}^2} \right) - \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial u_{x^2}^2} \right) &= 0 \end{aligned}$$

which give the system

$$\begin{aligned} 2u^1 + \frac{1}{2}u^2 - 2u_{2x^1}^1 + u_{x^1 x^2}^1 - 2u_{2x^2}^1 &= 0 \\ 2u^2 + \frac{1}{2}u^1 - 2u_{2x^1}^2 + u_{x^1 x^2}^2 - 2u_{2x^2}^2 &= 0. \end{aligned}$$

The general solution to this system is

$$\begin{aligned} u^1(x^1, x^2) &= c_5 e^{-\frac{\sqrt{5}}{2}x^1} + c_6 e^{-\frac{\sqrt{5}}{2}x^2} + c_7 e^{\frac{\sqrt{5}}{2}x^1} + c_8 e^{\frac{\sqrt{5}}{2}x^2} \\ &\quad - \left( c_1 e^{-\frac{\sqrt{3}}{2}x^1} + c_2 e^{-\frac{\sqrt{3}}{2}x^2} + c_3 e^{\frac{\sqrt{3}}{2}x^1} + c_4 e^{\frac{\sqrt{3}}{2}x^2} \right) \\ u^2(x^1, x^2) &= c_1 e^{-\frac{\sqrt{3}}{2}x^1} + c_2 e^{-\frac{\sqrt{3}}{2}x^2} + c_3 e^{\frac{\sqrt{3}}{2}x^1} + c_4 e^{\frac{\sqrt{3}}{2}x^2} \\ &\quad + c_5 e^{-\frac{\sqrt{5}}{2}x^1} + c_6 e^{-\frac{\sqrt{5}}{2}x^2} + c_7 e^{\frac{\sqrt{5}}{2}x^1} + c_8 e^{\frac{\sqrt{5}}{2}x^2}, \end{aligned}$$

where the constants  $c_i$  are determined by the given boundary conditions.

Determine the matrix  $A$  associated with the second variation of the functional  $\mathcal{F}$ .

$$A = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}$$

with  $A^{jj'}$  defined by

$$A^{jj'} = \begin{bmatrix} A_{00}^{jj'} & A_{01}^{jj'} \\ A_{10}^{jj'} & A_{11}^{jj'} \end{bmatrix}.$$

We have:

$$\begin{aligned} A_{00}^{11} &= \frac{\partial^2 L}{\partial u^1 \partial u^1} = 2, & A_{00}^{12} &= \frac{\partial^2 L}{\partial u^1 \partial u^2} = \frac{1}{2}, \\ A_{00}^{21} &= \frac{\partial^2 L}{\partial u^2 \partial u^1} = \frac{1}{2}, & A_{00}^{22} &= \frac{\partial^2 L}{\partial u^2 \partial u^2} = 2, \\ A_{10}^{11} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x^1}^1 \partial u^1} \\ \frac{\partial^2 L}{\partial u_{x^2}^1 \partial u^1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & A_{10}^{12} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x^1}^1 \partial u^2} \\ \frac{\partial^2 L}{\partial u_{x^2}^1 \partial u^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
A_{10}^{21} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u^1} \\ \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u^1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & A_{10}^{22} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u^2} \\ \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
A_{01}^{11} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u^1 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u^1 \partial u_{x_2}^1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, & A_{01}^{12} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u^1 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u^1 \partial u_{x_2}^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\
A_{01}^{21} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u^2 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u^2 \partial u_{x_2}^1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, & A_{01}^{22} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u^2 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u^2 \partial u_{x_2}^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\
A_{11}^{12} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^1 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u_{x_1}^1 \partial u_{x_2}^2} \\ \frac{\partial^2 L}{\partial u_{x_2}^1 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u_{x_2}^1 \partial u_{x_2}^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_{11}^{21} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u_{x_2}^1} \\ \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u_{x_2}^1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_{11}^{11} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^1 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u_{x_1}^1 \partial u_{x_2}^1} \\ \frac{\partial^2 L}{\partial u_{x_2}^1 \partial u_{x_1}^1} & \frac{\partial^2 L}{\partial u_{x_2}^1 \partial u_{x_2}^1} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}, \\
A_{11}^{22} &= \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u_{x_1}^2 \partial u_{x_2}^2} \\ \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u_{x_1}^2} & \frac{\partial^2 L}{\partial u_{x_2}^2 \partial u_{x_2}^2} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix},
\end{aligned}$$

which give

$$A^{12} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^{21}, \quad A^{11} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix} = A^{22}$$

and hence

$$A = \begin{bmatrix} 2 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 2 \end{bmatrix}.$$

It is easy to see that the matrices  $A_{kk}^{jj'}$ ,  $k = 0, 1$ , are all semi-positive definite. This implies that the Legendre necessary conditions for minimum point are satisfied. Furthermore, the matrix  $A$  is positive definite. Thus, we can well conclude that the found solution  $(u^1, u^2)$  to the Euler-Lagrange equations is effectively a minimum point for the functional  $\mathcal{F}$ .

**Example 12.3.4** Let  $\Lambda$  be a connected subset of  $\mathbb{R}^2$ . Consider the problem of finding a function  $u = u(x^1, x^2)$  with  $(x^1, x^2) \in \Lambda$ , which is an extremum of the functional  $\mathcal{F}$  defined by

$$\mathcal{F}(u) = \int_{\Lambda} L(x^1, x^2, u^{(2)}(x^1, x^2)) dx^1 dx^2$$

whose Lagrangian  $L$  is

$$L = u^2 + u_{x^1}^2 + u_{x^2}^2 + u_{2x^1}^2 + u_{x^1 x^2}^2 + u_{2x^2}^2 - \frac{1}{2}(u_{x^1} u_{x^2} + u_{2x^1} u_{x^1 x^2} + u_{2x^1} u_{2x^2} + u_{x^1 x^2} u_{2x^2}).$$

The extremum must satisfy the Euler-Lagrange equation

$$\begin{aligned}
0 &= \frac{\partial L}{\partial u} - \frac{\partial}{\partial x^1} \left( \frac{\partial L}{\partial u_{x^1}} \right) - \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial u_{x^2}} \right) + \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^1} \left( \frac{\partial L}{\partial u_{2x^1}} \right) \\
&+ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial u_{x^1 x^2}} \right) + \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial u_{2x^2}} \right)
\end{aligned} \tag{12.3.46}$$

which gives the equation

$$2u - 2u_{2x^1} + u_{x^1 x^2} - 2u_{2x^2} + 2u_{4x^1} - u_{3x^1 x^2} + u_{2x^1 2x^2} - u_{x^1 3x^2} + 2u_{4x^2} = 0.$$

The general solution to this equation is

$$\begin{aligned}
u(x^1, x^2) &= e^{-\frac{\sqrt{3}}{2}x^1} \left[ c_1 \cos\left(\frac{1}{2}x^1\right) + c_2 \sin\left(\frac{1}{2}x^1\right) \right] \\
&+ e^{\frac{\sqrt{3}}{2}x^1} \left[ c_3 \cos\left(\frac{1}{2}x^1\right) + c_4 \sin\left(\frac{1}{2}x^1\right) \right] \\
&+ e^{-\frac{\sqrt{3}}{2}x^2} \left[ c_5 \cos\left(\frac{1}{2}x^2\right) + c_6 \sin\left(\frac{1}{2}x^2\right) \right] \\
&+ e^{\frac{\sqrt{3}}{2}x^2} \left[ c_7 \cos\left(\frac{1}{2}x^2\right) + c_8 \sin\left(\frac{1}{2}x^2\right) \right],
\end{aligned}$$

where the constants  $c_i$  are determined by the boundary conditions.

Determine the matrix  $A$  associated with the second variation of  $\mathcal{F}$ :

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{00} = \frac{\partial^2 L}{\partial u \partial u} = 2, \quad A_{02} = \begin{bmatrix} \frac{\partial^2 L}{\partial u \partial u_{2x^1}} & \frac{\partial^2 L}{\partial u \partial u_{x^1 x^2}} & \frac{\partial^2 L}{\partial u \partial u_{2x^2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x^1} \partial u} \\ \frac{\partial^2 L}{\partial u_{x^2} \partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} \frac{\partial^2 L}{\partial u \partial u_{x^1}} & \frac{\partial^2 L}{\partial u \partial u_{x^2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x^1} \partial u_{x^1}} & \frac{\partial^2 L}{\partial u_{x^1} \partial u_{x^2}} \\ \frac{\partial^2 L}{\partial u_{x^2} \partial u_{x^1}} & \frac{\partial^2 L}{\partial u_{x^2} \partial u_{x^2}} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{2x^1} \partial u} \\ \frac{\partial^2 L}{\partial u_{x^1 x^2} \partial u} \\ \frac{\partial^2 L}{\partial u_{2x^2} \partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{x^1} \partial u_{2x^1}} & \frac{\partial^2 L}{\partial u_{x^1} \partial u_{x^1 x^2}} & \frac{\partial^2 L}{\partial u_{x^1} \partial u_{2x^2}} \\ \frac{\partial^2 L}{\partial u_{x^2} \partial u_{2x^1}} & \frac{\partial^2 L}{\partial u_{x^2} \partial u_{x^1 x^2}} & \frac{\partial^2 L}{\partial u_{x^2} \partial u_{2x^2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{2x^1} \partial u_{x^1}} & \frac{\partial^2 L}{\partial u_{2x^1} \partial u_{x^2}} \\ \frac{\partial^2 L}{\partial u_{x^1 x^2} \partial u_{x^1}} & \frac{\partial^2 L}{\partial u_{x^1 x^2} \partial u_{x^2}} \\ \frac{\partial^2 L}{\partial u_{2x^2} \partial u_{x^1}} & \frac{\partial^2 L}{\partial u_{2x^2} \partial u_{x^2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} \frac{\partial^2 L}{\partial u_{2x1} \partial u_{2x1}} & \frac{\partial^2 L}{\partial u_{2x1} \partial u_{x1x2}} & \frac{\partial^2 L}{\partial u_{2x1} \partial u_{2x2}} \\ \frac{\partial u_{x1x2} \partial u_{2x1}}{\partial^2 L} & \frac{\partial u_{x1x2} \partial u_{x1x2}}{\partial^2 L} & \frac{\partial u_{x1x2} \partial u_{2x2}}{\partial^2 L} \\ \frac{\partial u_{2x2} \partial u_{2x1}}{\partial^2 L} & \frac{\partial u_{2x2} \partial u_{x1x2}}{\partial^2 L} & \frac{\partial u_{2x2} \partial u_{2x2}}{\partial^2 L} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix}.$$

Thus,

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix}.$$

It is easy to see that the matrices  $A_{kk}$ ,  $k = 0, 1, 2$  are all semi-positive definite. This implies that the Legendre necessary conditions for minimum point are satisfied. Furthermore, the matrix  $A$  is positive definite. Thus, we can well conclude that the found solution  $u$  to the Euler-Lagrange equations is effectively a minimum point for the functional  $\mathcal{F}$ .



# Conclusion

There were three main purposes to this dissertation. Firstly, investigate and improve some well-known methods to determine exact or analytical approximate solutions to nonlinear partial differential equations and systems. Secondly, study and extend some techniques of construction of conservation laws for nonlinear partial differential equations and systems. Thirdly, develop and generalize methods to find critical functions of functionals defined by variational integrals. Then, elaborate very efficient tests to check whether a critical function is a genuine extremum or not.

For our first objective, we have described the factorization process of NLPDEs and provided sufficient conditions for the factorization in the particular case of second order equations. Three classes of factorizable second order linear ODEs characterized by the degrees of their corresponding polynomial function coefficients are probed. We have also proposed some algorithmic processes, based on the expansion of either one function (single function expansion method) or two functions (double function expansion method) or three functions (triple function expansion method), for the construction of exact solutions to nonlinear polynomial partial differential equations with constant function coefficients. Most of the existing expansion function methods have been well recovered from the mentioned principles. Then, we have described the process to find one-parameter Lie symmetry groups of transformations for differential equations and shown its use in the integration of differential equations. In order to handle analytical approximate solutions, we have focussed on the Adomian decomposition method and extended it to under determined systems of NLPDEs.

Concerning our second purpose, we have extended the Ibragimov non local conservation theorem [57] for partial differential equations to under determined systems of differential equations. Afterward, we have proposed an alternative direct algebraic method of constructing, for nonlinear evolution partial differential equations, conservation laws that depend not only on dependent variables and its derivatives but also explicitly on independent variables.

About the third and last aim of our thesis, we have addressed both necessary and relevant sufficient extremum conditions for a variational problem defined by a smooth Lagrangian, involving higher derivatives of several variable vector valued functions. A general formulation of first order necessary extremum conditions for variational problems with (or without) constraints has been given. Global Legendre second order necessary extremum conditions have been provided as well as new general explicit formula for second order sufficient extremum condition which do not requires the notion of conjugate points as in the Jacobi sufficient condition. In the fourth part of this thesis, we have elaborated more general and powerful tests to determine the nature of extremum for unconstrained or constrained optimization problems where a system of partial differential equations appears as an essential part of the constraints. One of the purposes of our further investigations will be to create a Maple package that will systematically specify the nature of a critical function to such optimization problems.

---



# Bibliography

- [1] K. Abbaoui, and Y. Cherruault, *Comput. Math. Appl.* 5, (1994) 103-109. (Cited on page 80.)
  - [2] K. Abbaoui, and Y. Cherruault, *Math. Compact. Model.* 20, (1994) 69-73. (Cited on page 80.)
  - [3] K. Abbaoui, *Les fondements mathématiques de la méthode décompositionnelle d'Adomian et application la résolution de problèmes issus de la biologie et de la médecine*, Thèse, Université Pierre et Marie Curie-ParisVI, (1995). (Cited on page 79.)
  - [4] M. A. Abdou and A. Elhanbaly, *Commun. Nonlinear Sci. Numer. Simul.* 12, (2007) 1229-1241. (Cited on page xiv.)
  - [5] M. A. Abdou, *Electronic Journal of Theoretical Physics* 4 No. 14, (2007) 17-30. (Cited on page xiv.)
  - [6] M. A. Abdou, *Journal of Nonlinear Dynamics* 52 (1-2), (2008) 95-102. (Cited on page xiv.)
  - [7] G. Adomian, Kluwer Academic Publisher, Boston, 1994. (Cited on page xvi.)
  - [8] G. Adomian, *J. Math. Anal. Appl.* 135, (1988) 501-544. (Cited on page xvi.)
  - [9] G. Adomian, *Appl. Math. Lett.* 11 (3), (1998) 31-133. (Cited on page xvi.)
  - [10] F. Allan, *J. Appl. Math. Comput.* 190, (2007) 6-14. (Cited on page xvi.)
  - [11] N. W. Andrew Hone and Jacek Szmigielski, *Dynamics of PDE*, Vol.6 No.3, (2009) 253-289. (Cited on pages xiii and 84.)
  - [12] R. W. Anthernton and G. M. Homsy, *Stud. Appl. Math.* 54, (1975) pp 31-60. (Cited on page xvii.)
  - [13] S. C. Anco and G.W. Bluman, *Europ. J. Appl. Math.* 13 (2002), pp. 545-566. (Cited on page xvii.)
  - [14] S. C. Anco and G.W. Bluman, *Europ. J. Appl. Math.* 13 (2002), pp. 567-585. (Cited on page xvii.)
  - [15] A. Baquero and L. Solanilla, *Balkan J. Geom. Appl.* 8 (2003). (Cited on page xviii.)
  - [16] E. Beke, *Mathematische Annalen* 45, (1894), 185-195. (Cited on page xiv.)
  - [17] A. Bekir, *Phys. Scr.* 77, (2008). (Cited on page xiv.)
  - [18] A. Bekir, *Phys. Lett. A* 372, (2008) 3400-3406. (Cited on page xiv.)
  - [19] L. M. Berkovich, Factorization of some classes of nonlinear ordinary differential equations: methods and algorithms, Poster Session Abstracts of the 1998 International Symposium on Symbolic and Algebraic Computation ISSAC'1998, Rostock, Germany, Aug. 13-15, 1998, George Labahn, Mohamed Omar rayes, Editors, 1-15. (Cited on page xiv.)
-

- [20] L. M. Berkovich, *Appl. Anal. Discrete Math.* **1**, (2007) pp. 122-149. (Cited on page [xiv](#).)
- [21] L. M. Berkovich, *Factorization and transformations of ordinary differential equations*, *Saratov University Publ.*, 1989. (Cited on page [xiv](#).)
- [22] Bessel-Hagen *Mathem. Ann.*, **84**, pp. 258-276, 1921. English trans. in *Archives of ALGA*, **3**, ALGA publ., BTH, Karlskrona, Sweden, (2006), pp. 33-51. (Cited on page [xvi](#).)
- [23] L. Bin and Z. Hong-Qing, *Commun. Theor. Phys. (Beijing, China)* **50**, (2008) 814-820. (Cited on page [xiv](#).)
- [24] C. A. Briot and J. C. Bouquet, *Théorie des fonctions elliptiques*, Deuxième édition, Paris, Gauthier-Villars, 1875. (Cited on page [46](#).)
- [25] F. Cap, Scientific report NR. 9, Research Grant Nr. NGR-52-046-001, Institute of Theoretical Physics, University of Innsbruck, Innsbruck, Austria, August (1966). (Cited on page [xiii](#).)
- [26] P. J. Caudrey, R. K. Dodd, and J. D. Gibbon, *Proc. Roy. Soc. Lond. A* **351**, (1976) 407-422. (Cited on page [48](#).)
- [27] Y. Cherruault, *Convergence of Adomian method*, *Kybernotes* **18** (20), (1990) 31-38. (Cited on pages [xvi](#) and [80](#).)
- [28] Y. Cherruault and G. Adomian, *Math. Comput. Modelling*, **18** (12), (1993) 103-106. (Cited on pages [xvi](#) and [80](#).)
- [29] Y. Cherruault, *Modèles et méthodes mathématiques pour les sciences du vivant*, Presses Universitaires de France (P.U.F), (1998a). (Cited on page [79](#).)
- [30] Y. Cherruault, *Optimisation : Méthodes locales et globales*, Presses Universitaires de France (P.U.F), (1999). (Cited on page [79](#).)
- [31] Fred Cooper, Avinash Khare, Uday Sukhatme, Los Alamos, *Supersymmetry in quantum mechanics*, World scientific publishing Co. pte. Ltd. (2001). (Cited on page [xiv](#).)
- [32] W. Deng-Shan, R. Yu-Jie and Z. Hong-Qing, *Appl. Math. E-Notes* **5**, (2005) 157-163. (Cited on page [xiv](#).)
- [33] A. Elgarayhi and A. A. Karawia, *Int. J. Nonlinear Sci.* Vol.7 No.4, (2009) 414-419. (Cited on page [xiv](#).)
- [34] S. A. El-Wakil, E. M. Abulwafa, A. Elhanbaly and M. A. Abdou, *Chaos Solitons and Fractals* **33**, (2007) 1512-1522. (Cited on page [xiv](#).)
- [35] S. Fomin and I. Gelfand, *Calculus of Variations*, (Prentice-Hall, 1963). (Cited on pages [xviii](#) and [144](#).)
- [36] C. Fox, *An introduction to the calculus of variations*, (Oxford university press, 1950). (Cited on page [xviii](#).)
- [37] S. Hans, *Differential equations: Their solution using symmetries*, Cambridge University press 1989. (Cited on pages [xv](#) and [53](#).)

- [38] J. H. He and X. H. Wu, Chaos Solitons and Fractals v30, (2006) pp. 700-708. (Cited on page [xiv](#).)
- [39] J. H. He, Int. NonLinear Mech. 344, (1999) pp. 699-708. (Cited on page [xvi](#).)
- [40] J. H. He, Appl. Math. Comput. 114 (2), (2000) pp. 115-123. (Cited on page [xvi](#).)
- [41] J. H. He, Appl. Math. Comput. 135 (2003) pp. 73-79. (Cited on page [xvi](#).)
- [42] Hill, Rev. Mod. Phys., 23, (1951) pp. 253-260. (Cited on page [xvii](#).)
- [43] M. N. Hounkonnou, K. Sodoga, and E. C. Azatassou, J. Phys. A: Math. Gen., vol. **38** (2005) pp. 371-390. (Cited on page [xiv](#).)
- [44] M. N. Hounkonnou, A. Ronveaux and K. Sodoga, Appl. Math. Comput. **189**, (2007) pp. 816-820. (Cited on pages [xiv](#), [29](#), [30](#) and [34](#).)
- [45] M. N. Hounkonnou and P. A. Dkengne Sielenou, Int. J. Contemp. Math. Sci. **4** (35), (2009) pp 1719-1738. (Cited on pages [xv](#), [49](#), [120](#) and [123](#).)
- [46] M. N. Hounkonnou and P. A. Dkengne Sielenou, Afr. Diaspora J. Math. Volume 8, Number 2 (2009) pp. 137-147. (Cited on page [xiv](#).)
- [47] M. N. Hounkonnou and P. A. Dkengne Sielenou, XXVIII Workshop on Geometrical Methods in Physics. AIP Conference Proceedings, Volume 1191, (2009) pp. 104-109. (Cited on page [xiv](#).)
- [48] M. N. Hounkonnou and P. A. Dkengne Sielenou, XXIX Workshop on Geometric Methods in Physics. AIP Conference Proceedings, Volume 1307, (2010) pp. 83-88. (Cited on pages [107](#) and [139](#).)
- [49] M. N. Hounkonnou and P. A. Dkengne Sielenou, SUT J. Math. Volume 46 Number 2, (2010) pp. 205-229. (Cited on page [23](#).)
- [50] M. N. Hounkonnou and P. A. Dkengne Sielenou, Commun. Math. Anal., *Special Volume in Honor of Prof. Peter Lax* Volume 8 Number 3, (2010) pp. 102-119. (Cited on page [115](#).)
- [51] M. N. Hounkonnou and P. A. Dkengne Sielenou, Commun. Math. Anal., *Special Volume in Honor of Prof. Stephen Smale* Volume 10, Number 1, (2011) pp. 53-74. (Cited on pages [3](#) and [13](#).)
- [52] M. N. Hounkonnou and P. A. Dkengne Sielenou, Afr. Diaspora J. Math., *Special Volume in Honor of Profs. C. Corduneanu, A. Fink, and S. Zaidman* Volume 12, Number 2, (2011) pp. 73-103. (Cited on page [77](#).)
- [53] M. N. Hounkonnou and P. A. Dkengne Sielenou, Extremum conditions for functionals involving higher derivatives of several variable vector valued functions, Preprint submitted to SUT Journal of Mathematical, (2011). (Cited on page [131](#).)
- [54] M. N. Hounkonnou and A. Ronveaux, Commun. Math. Anal. Volume **11**, Number 1, (2011) pp. 121-136. (Cited on page [xiv](#).)

- [55] N. H. Ibragimov, 1 (3) (1969) pp. 350-359. English trans. in Theor. Math. Phys. 1 (3), (1969), pp. 267-276. (Cited on page xvii.)
- [56] N. H. Ibragimov, J. Math. Anal. Appl. 318, (2) (2006). (Cited on pages xvii and 138.)
- [57] N. H. Ibragimov, J. Math. Anal. Appl. **333**, (1) (2007). (Cited on pages xvii, 107, 109, 138 and 163.)
- [58] N. H. Ibragimov, J. Phys. A: Math. Theor. 43, (2010) 502001. (Cited on pages xiii and 81.)
- [59] L. E. Ibrahim, Appl. Math. E-Notes 7, (2007) 214-221. (Cited on page 80.)
- [60] M. Ito, J. Phys. Soc. Japan 49 (2), (1980) 771-778. (Cited on page 48.)
- [61] C. G. J. Jacobi, Vorlesungen ber Dynamik, 2nd ed., Reimer, Berlin, 1884. (Cited on page xvi.)
- [62] G. Junker, and P. Roy Annals of Physics **270** 1, (1998) pp. 155-177. (Cited on page xiv.)
- [63] T. Kawahara, J. Phys. Soc. Japan 33, (1972) 260-264. (Cited on page 48.)
- [64] R. Khamitova, Symmetries and conservation laws, PhD Thesis, University of the Witwatersrand, Johannesburg, (2009). (Cited on page xvii.)
- [65] H. Kheiri, A. Jabbari, Acta universitatis apulensis 22, (2010) 185-194. (Cited on page xiv.)
- [66] F. Klein, Math. Phys. Kl. Heft 2, (1918) pp. 171-189. (Cited on page xvi.)
- [67] W. Koepf, *Hypergeometric Summation*, Vieweg, Braunschweig-Weisbaden (1998). (Cited on page 23.)
- [68] B. O. Konfe, *Nouvelles méthodes mathématiques Alienor et Adomian, pour la Biomédecine*, Thèse, Université de Ouagadougou (Burkina Faso), (2005). (Cited on page 79.)
- [69] B. A. Kuperschmidt, Phys. Lett. A 102, (1984) 213-215. (Cited on page 48.)
- [70] P. S. Laplace, Celestial Mechanics: Paris **1** (1798). (Cited on page xvii.)
- [71] P. D. Lax, Commun. Pure Appl. Math., 21, (1968) 467-490. (Cited on page 48.)
- [72] B. Li and C. Jinlan, Journal of Henan University of Science and Technology (Natural Science) 26 (5), (2005) 80-83. (Cited on page xiv.)
- [73] L. Ling-xiao, L. Er-qiang and W. Ming-liang, Appl. Math. J. Chinese Univ. 25 (4), (2010) 454-462. (Cited on page xiv.)
- [74] S. Liu, Z. Fu and Q. Zhao, Phys. Lett. A. 289 (1), (2001) 69-74. (Cited on page xiv.)
- [75] J. B. Liu and K. Q. Yang, Chaos Solitons Fractals 22 (1), (2004) 111-121. (Cited on page xiv.)
- [76] P. Moon and D. E. Spencer, *Field Theory for Engineers*, New York: Van Nostrand, (1961). (Cited on pages xiii, 38 and 40.)

- [77] W. Malfliet and W. Hereman, Phys. Scr. 54 (1996). (Cited on page xiv.)
- [78] R. Naz, F. M. Mahomed and D. P. Mason, Appl. Math. Comput. **205**, (2008) pp. 212-230. (Cited on page xvi.)
- [79] R. Naz, Symmetry solutions and conservation laws for some partial differential equations in field mechanics, PhD Thesis, Växjö University, Sweden (2008). (Cited on page xvii.)
- [80] E. Noether, Invariant Variation problems. *Nachr. v. d. Ges. d. Wiss. zu Göttingen* (1918), pp 235-257. (Cited on pages xvi and xvii.)
- [81] P. J. Olver, *Applications of Lie groups to differential equations*, 2nd ed., Springer, 1993. (Cited on pages xv, xvii, 53, 108 and 115.)
- [82] P. J. Olver, The calculus of variations. <http://www.math.umn.edu/~olver/appl.html> (2010). (Cited on pages xvii and xviii.)
- [83] L. D. Poole, Symbolic computation of conservation laws of nonlinear partial differential equations using homotopy operators, *PhD Thesis, Colodaro School of Mines* (2007). (Cited on pages xvii, 115, 123 and 125.)
- [84] J. C. Polking, Calculus of Variations. <http://math.rice.edu/~polking/Math410/> (2010). (Cited on pages xviii, 133, 143, 147 and 148.)
- [85] D. Qiang, Global existence and uniqueness of solution of the time-dependent Ginzburg-Landau equations in superconductivity, Appl. Anal. 52, (1994) 1-17. (Cited on pages xiii and 88.)
- [86] D. Qiang, Numerical approximation of Ginzburg-Landau for superconductivity, J. Math. Phys. 46 (2005). (Cited on pages xiii and 88.)
- [87] R. Rajaram and M. Najafi, Appl. Math. Comput. 212, (2009) pp. 72-81. (Cited on page 80.)
- [88] M. M. Rodrigues and E. M. Rocha, Acta Appl. Math. 112, (2010) pp. 299-308. (Cited on page 80.)
- [89] A. Ronveaux, Rev. Questions Sci. **172**, (2001) pp. 409-416. (Cited on page xiv.)
- [90] A. Ronveaux, Appl. Math. Comput. **141**, (2003) pp. 177-184. (Cited on page xiv.)
- [91] A. Ronveaux, Editor, *Heun's Differential Equations*, (Oxford University Press), Oxford and New York, (1995). (Cited on pages 29, 30 and 34.)
- [92] V. Rosenhaus, J. Phys. A: Math. Gen. **39** (2006), pp. 7693-7703. (Cited on page xvi.)
- [93] K. Sawada and T. Kotera, Progr. Theoret. Phys. 51, (1974) pp. 1355-1367. (Cited on page 48.)
- [94] J. M. Sanz-Serna, J. Comput. Phys. **47**, (1982) pp 199-210. (Cited on page xvi.)
- [95] V. Seng, *Recherche de formes canoniques d'Adomian pour la résolution d'équations fonctionnelles non linéaires par la méthode décompositionnelle*, Thèse, Université Pierre et Marie Curie (ParisVI), (1997). (Cited on page 79.)

- [96] S. Shi, *Acta Mathematica Sinica*, Vol. 1 No. 4, (1985) pp. 348-355. (Not cited.)
- [97] H. Steudel, *Zeit Naturforsch A* **17**, (1962) pp 129-132. (Cited on page xvii.)
- [98] V. M. Tkachuk, *J. Phys. A: Math. Gen.* **32**, (1999) 1291. (Cited on page xiv.)
- [99] M. Van Hoeij, *J. Symbolic Computation* **24**, (1997) pp. 537-561. (Cited on page xiv.)
- [100] A. M. Wazwaz, *Appl. Math. Comput.* 187, (2007) pp. 1131-1142. (Cited on page xiv.)
- [101] A. M. Wazwaz, *Appl. Math. Comput.* 188, (2007) pp. 1467-1475. (Cited on page xiv.)
- [102] A. M. Wazwaz, *Partial differential equations and solitary wave theory*, Springer, 2009. (Cited on pages xiii, xvi, 46, 47 and 48.)
- [103] M. Wen-Xiu, H. Tingwen and Y. Zhang, *Phys. Scr.* 82 (2010). (Cited on page xiv.)
- [104] T. Wolf, Application of CRACK in the classification of integrable systems, *CRM Proceedings and Lectures Notes* **37**, (2004) pp. 283-300. (Cited on page xvi.)
- [105] T. Wolf, *Trends in Mathematics: Differential Equations with Symbolic Computation* (2005) pp. 291-306. (Cited on page xvi.)
- [106] T. Wolf, *Euro. J. Appl. Math.* **13**, (2002) pp. 129-152. (Cited on page xvii.)
- [107] R. X. Yao and Z. B. Li. *Commun. Thero. Phys.* **41**, (2004) pp 487. (Cited on page xvii.)
- [108] R. X. Yao and Z. B. Li. *Chinese Journal of Physics* **42**, (2004) pp 315-322. (Cited on page xvii.)
- [109] S. Zhang, *Chaos Solitons Fractals* 32 (4), (2007) pp. 1375-1383. (Cited on page xiv.)
- [110] S. Zhang, *Appl. Math. Comput.* 189 (1), (2007) pp. 836-843. (Cited on page xiv.)
- [111] J. Zhang, X. L. Wei and Y. J., *Phys. Lett. A* 372, (2008) pp. 3653-3658. (Cited on page xiv.)
- [112] D. Zwillinger, *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, (1997). (Cited on page xiii.)

