Microscopic derivation of degenerated diffusion phenomena

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HABILITATION À DIRIGER LES RECHERCHES

Discipline : Mathématiques

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présentée et soutenue publiquement le 4 décembre 2019 par

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Description microscopique de phénomènes diffusifs dégénérés

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à Michele et Lorenzo

Remerciements

Je voudrais tout d'abord remercier chaleureusement Massimiliano Gubinelli, Herbert Spohn et Cristina Toninelli d'avoir accepté de rapporter ce manuscrit, malgré la lourde tâche que ce travail représente, et de l'avoir fait avec autant d'attention. Je remercie sincèrement Stephan De Bièvre, Giambattista Giacomin, Mylène Maida et Ellen Saada d'avoir accepté de faire partie du jury, bien que nos préoccupations mathématiques soient parfois distantes (et, pour certains, d'avoir bravé les dangers en prenant le train la veille d'une grève redoutable !).

Stefano Olla et Cédric Bernardin méritent un remerciement tout particulier pour leur accompagnement fidèle et bienveillant depuis mon entrée dans le monde de la recherche. Leur soutien constant m'a permise d'arriver jusqu'ici.

Je dois beaucoup à l'ensemble de mes collaborateurs et collaboratrices, avec lesquels j'ai énormément de plaisir à travailler, avec lesquels j'ai tant appris (et encore tant à apprendre !). Dans l'ordre alphabétique : Oriane Blondel, Clément Cancès, Clément Erignoux, Max Fathi, Tertuliano Franco, Patricia Gonçalves, Milton Jara, Tomasz Perkowski, Christian Olivera, Nicolas Perkowski, Makiko Sasada. Sans vous, ce manuscrit ne serait pas ce qu'il est. Les projets ne manquent pas, et je m'en réjouis ! Beaucoup d'autres mathématiciens et mathématiciennes ont contribué d'une manière ou d'une autre à mes travaux, aux détours d'un café ou entre deux exposés lors d'une conférence. Grâce à nos discussions, ma recherche avance, bien que leurs contributions passent (pour le moment) inaperçues. Au risque d'en oublier, en voici quelques-uns, qui aujourd'hui me semblent incontournables : Giada Basile, Guillaume Barraquand, Grégoire Ferré, Tadahisa Funaki, Maxime Herda, François Huveneers, Claudio Landim, Jani Lukkarinen, Vivien Lecomte, Jan Maas, Gunther Schütz, Chi Tran, et enfin Gabriel Stoltz, digne d'un remerciement spécial pour tout le temps passé à discuter de nos projets, dans des lieux aussi variés qu'incongrus.

Je remercie tous ceux qui me suivent, de près ou de loin, dans mes pérégrinations mathématiques. Il serait vain de vouloir tous les citer, mais j'adresse un salut particulier à Antoine Gloria, qui m'a fait confiance et m'a ouvert les portes d'Inria. À tous les membres des équipes Rapsodi et Mephysto, qui m'ont réservé un accueil des plus chaleureux (fidèle à la réputation lilloise !), et rendent mon travail plus enjoué (merci à Flore sans qui les pauses ☕ ne seraient peut-être pas ce qu'elles sont aujourd'hui), sans oublier Karine, Charlotte et Coralie qui s'occupent si bien de nous. Aux membres du laboratoire Paul Painlevé et en particulier ceux des équipes Probas-Stats et ANEDP, à tous ceux de l'UMPA qui m'ont soutenue à mes tous débuts. Aux ami(e)s mathématicien(ne)s de longue date que je croise régulièrement, et toujours avec plaisir.

Enfin, cette nouvelle étape n'aurait pas été franchie sans le soutien inconditionnel de mes ami(e)s et de ma famille : merci à celles et ceux qui m'ont offert leur canapé ou chambre d'amis (beaucoup se reconnaîtront), qui m'ont ragaillardie dans les moments difficiles, et qui m'encouragent quoiqu'il arrive. À maman, et Germain, toujours présents, à ma soeur Perrine, toujours proche. À Michele et Lorenzo, sans qui ma vie ne serait pas aussi ensoleillée.

Contents

CHAPTER 1 Introduction

Describing the physical world at the microscopic level is an exciting scientific challenge. For about one century, we have been able to represent a gas that disperses at our macroscopic scale as a collection of a very large number N of molecules which interact with each other. Mathematically, the connection between the two scales (micro/macro) is formulated as a scaling limit problem, which is called *local equilibrium* in the literature. Its mathematical formulation can be seen as a law of large numbers which permits almost sure predictions when the number of molecules becomes very large. The main goal is to prove that the empirical measure of the microscopic system, properly re-scaled in time and space converges, as N goes to infinity, to a deterministic measure whose density is solution to a partial differential equation generally called *hydrodynamic equation*. When the microscopic dynamics of the atoms is described by the purely deterministic laws of classical mechanics, the exact mathematical implementation of these scaling limits is extremely demanding, mainly because of the lack of mathematical framework out of equilibrium, where dynamics plays a major role. For that reason, all the microscopic models which are presented here are either purely stochastic, or supplemented by a stochastic part which keeps the conservation laws intact and improve space-time mixing properties.

This problematic has inspired the research I have carried out so far. In this manuscript, Chapters 2, 3 and 4 are intended to review the works [12, 13, 21, 22, 23, 43, 58, 60, 74], which have been concluded after my PhD thesis (*i.e.* since 2014), and Chapter 5 is devoted to the exposition of some works in preparation and other perspectives for the future. In the rest of the introduction I will precise the general background and describe the main content of the manuscript: Section 1.1 presents three distinct physical phenomena for which I and coauthors have obtained rigorous microscopic derivations. The microscopic models under investigation will then be described in Section 1.2. In Section 1.3 I will shortly expose other works which will not be discussed in detail, namely [41, 73, 87].

1.1 Microscopic derivation of physical phenomena

1.1.1 Anomalous diffusion phenomena (Chapter 2)

The *diffusion* of a macroscopic quantity ρ, if *standard*, is generally governed by an equation resulting in an operator of the form $\mathrm{div}(\mathrm{D}(\rho)\nabla\rho)$ where $\mathrm{D}(\cdot)$ is the diffusion coefficient matrix.

In particular, the *porous medium equation* and the *Stefan problem* (which will be described below, see (1.2) and (1.3)) belong to this family. Equivalently, in thermodynamics, diffusive heat transport is characterized by the *Fourier law*, which states that the heat flux and temperature gradients are linearly related by the conductivity coefficient. In porous media propagation, its equivalent is the *Darcy law*.

However, several models have been recently put forward, for which the diffusive aspect is of nonlocal nature, and is called *anomalous*. Generally, the Laplacian operator is replaced by fractional Laplacians $-(-\Delta)^s$ with $s \in (0,1)$, which in particular feature long-distance interactions. The understanding of this anomalous behaviour has attracted a lot of interest in recent years, not only because it is expected to clarify the fundamental ingredients behind the Fourier law, but also because of its technological applications, especially in low-dimensional materials. In particular the one-dimensional setting continues to puzzle physicists and mathematicians, from both theoretical and numerical points of view (see *e.g.* the reviews [80, 35, 82] and the lecture notes [79]).

In a general framework, the transport properties are commonly investigated through:

1. the *non-equilibrium stationary state* (NESS). In that case, the system is connected to heat baths at different temperatures, and one measures the temperature profile and the energy current in the steady state. The dependence of the current w.r.t. the microscopic system size indicates whether the diffusion is normal or not. This approach is usually implemented via numerical simulations. Any advance towards a mathematical confirmation of these simulation results is challenging.
2. the *correlations of conserved quantities*. In non-equilibrium stationary states of diffusive systems, they are generically long-range. Their dependence on the microscopic system size can be related to the nonlocal nature of the large deviation function of the conserved quantity [34]. Many predictions have been made, but very few have been verified.

1.1.2 Randomly growing interfaces (Chapter 3)

Besides hydrodynamic equations (which are PDEs), the macroscopic laws that can arise from microscopic systems can also be stochastic partial differential equations (SPDEs) if one is looking at the *fluctuations* around the mean (which may be the deterministic hydrodynamic equation, but also an equilibrium value if the system starts at equilibrium). Among the referenced SPDEs is the *Kardar-Parisi-Zhang* (KPZ) *equation* which has been first introduced more than thirty years ago in [70] as the *universal* law describing the fluctuations of randomly growing

interfaces of one-dimensional stochastic dynamics close to a stationary state. Since then, it has generated an intense research activity among the physics and mathematics community. Recently, new results [29, 31, 37, 55, 57] have revealed the strong connection between particle systems and the macroscopic description of growing interfaces. In particular, the *weak KPZ universality conjecture* [28, 92, 94] states that the fluctuations of a large class of one-dimensional microscopic interface growth models are ruled at the macroscopic scale by solutions of the KPZ equation, which reads as follows:

$$\mathrm{d}h = \sigma\Delta h\,\mathrm{d}t + \overline{\lambda}(\nabla h)^2\,\mathrm{d}t + \sqrt{\mathrm{D}}\,\mathrm{d}\mathcal{W}, \tag{1.1}$$

where $h : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is the height of the interface arising from a stable bulk phase in contact with a metastable one, $\overline{\lambda} \in \mathbb{R}$, $\sigma, \mathrm{D} > 0$ are material parameters and $\partial_t \mathcal{W}$ is a standard space time white noise, modeling the randomness in transitions from metastable to stable.

1.1.3 Multiphase medium (Chapter 4)

Finally, the evolution of a *multiphase medium*, for instance the joint evolution of liquid and solid phases, is a complex physical phenomenon, which often features absorbing states and phase transitions.

Let us consider an ideal gas flowing in a homogeneous medium. The evolution of its scaled density $\rho : \mathbb{R}^d \times \mathbb{R}_+ \to [0,1]$ can be described by the *porous medium equation* (PME) written as

$$\partial_t \rho = \mathrm{div}(\rho^{m-1}\,\nabla\rho), \tag{1.2}$$

where div and ∇ are respectively the divergence and gradient operators in $\mathbb{R}^d$. The constant $m > 1$ regulates the behavior of the diffusion coefficient close to vanishing densities: indeed, the diffusion coefficient in (1.2) is equal to $\mathrm{D}(\rho) = \rho^{m-1}$. It is known that, starting from an initial density ρ_0 with compact support, the solution $\rho(x,t)$ is nonnegative and has compact support in the space variables for each positive t. Therefore there are interfaces separating the regions where ρ is positive from those where it is zero.

Let us now assume that the homogeneous medium is subject to a phase change. In order to describe the temperature distribution in the medium, J. Stefan in [96] introduced what is now called the *Stefan problem*: an equation which rules the evolution of both phases, with the separation interface allowed to evolve in time. To be more precise, consider first that heat linearly diffuses and that the setting is one-dimensional. Let us define the liquid region as the domain where $\rho > 0$ and the solid region where $\rho = 0$. Then, the mathematical formulation of the Stefan problem is the following: find a curve $x = \Gamma(t)$ and a function $\rho(x,t) \geqslant 0$ such that

$$\begin{cases} \partial_t \rho = \mathrm{D}\partial^2_{xx}\rho, & \text{if } 0 < x < \Gamma(t), \\ \rho(x,t) = 0, & \text{if } x \geqslant \Gamma(t), \end{cases} \tag{1.3}$$

and

$$\frac{d\Gamma}{dt} = -\partial_x \rho(\Gamma(t), t), \tag{1.4}$$

where $D > 0$ is a diffusion constant, and with initial and boundary conditions. Physically speaking, the second condition on the free boundary $\Gamma(t)$ corresponds to the presence of latent heat at the phase transition. The mathematical solution to (1.3) is usually obtained *via* the weak formulation and the regularity of $\Gamma(t)$ can be analyzed (see *e.g.* [85] for a review).

1.2 Choice of microscopic models

In Chapters 2, 3 and 4, several mathematical results about the microscopic description of the three phenomena described in the previous section will be exposed. Before that, let me introduce in this section the microscopic models which are involved.

1.2.1 Exclusion processes

The simplest interacting particle systems that have been investigated in the last decades are *exclusion processes* with a *conservative* stochastic dynamics [81]. In these models, single particles occupy the sites of a lattice, and each particle can jump at a given rate to a neighbouring vacant site. More precisely, to each lattice site is attached an independent random Poisson clock. When the clock rings, if there is a particle at this site, then it jumps or not to a neighbouring site, provided that the target site is empty and depending on the local configuration (see Figure 1.1).

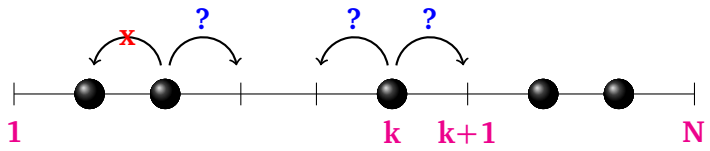

Figure 1.1: One-dimensional exclusion process with N sites ($N \in \mathbb{N}$ being the scaling parameter).

The total number of particles is conserved along the evolution. This class of models is rich enough to describe many physical systems and capture very complex features. In particular, the dynamics becomes intricate (and interesting) if one assumes that:

- the particle dynamics is *locally* perturbed, by adding *microscopic defects*. This will be the case in Section 3.4.2 of this manuscript. Depending on the type of perturbation, the macroscopic laws can hold different boundary conditions, as can be seen *e.g.* in [44, 45];
- there is a *weak asymmetry* in the system: more precisely, the difference between the rates for a particle to jump to its right and left neighbours is not zero, but proportional to $N^{-\gamma}$ where $\gamma > 0$. The literature about the *weakly asymmetric simple exclusion process* is huge. We invite the reader to read Section 2.3 where a recent result will be detailed, and several references will be given;
- the jumps are submitted to *kinetic constraints*, making the particle system belonging to the family of *kinetically constrained lattice gases* (KCLG). These models are particularly

suitable to describe moving boundary problems. In Section 3.4.1 we will assume that the particle mobility increases as the local density increases. At the microscopic level, this can be translated by the following local constraint: a particle can hop to its vacant right neighbour only if its left neighbour is also occupied, and *vice versa* [49, 8].

1.2.2 Chains of coupled oscillators

Since Fermi-Pasta-Ulam-Tsingou [42], the one-dimensional chain of anharmonic oscillators has been identified as a natural model for studying macroscopic thermodynamic laws. This model is originally purely deterministic: the configurations of positions/momenta follow an Hamiltonian evolution, so that the total energy of the system is preserved.

Figure 1.2: One-dimensional coupled oscillators.

A well chosen anharmonicity in the interaction (which, in particular, makes the dynamical system non-integrable) is expected to cancel all integrals of motion in the hydrodynamic limit, except for the momentum, volume and energy. If the chain is furthermore unpinned, a superdiffusive evolution of the thermal energy is expected. Despite the local character of the microscopic dynamics, some macroscopic nonlocality appears from the conservation of several quantities, which have different characteristic speeds of propagation and are nonlinearly related. Its rigorous derivation is a very hard problem, which is still out of reach [24] despite many years of intense research activity. One of the most demanding question is the mathematical understanding of the microscopic systems which have more than one conserved quantity, and notably out of equilibrium.

In the last decades, mathematical results have been obtained by adding *energy conserving random perturbations* to the Hamiltonian dynamics [7, 73, 72, 14]. In particular, the analytical tractability of the harmonic chain provides many insights for the understanding of theoretical aspects of heat conduction. Indeed, the stochastic noise models the effects of nonlinearities of the anharmonic chains, and the resulting dynamics turns out to have similar macroscopic behaviour [6]. In this context, the articles [72, 14] provide the first rigorous derivation of the hydrodynamic limit of anomalous heat conduction.

1.3 Other topics

Three works [41, 73, 87] will not be exposed in this *Habilitation*. Let me give here a few words about their content.

In [41] in collaboration with M. Fathi, we propose a new approach to the well-known convergence towards the hydrodynamic limit for the *symmetric simple exclusion process* (SSEP). More

precisely, we characterize any possible limit of its empirical density measures as solutions to the heat equation by passing to the limit in the *gradient flow* structure of the particle system: indeed, equivalent formulations of the gradient flow property for discrete Markov chains have recently been proposed in [84, 86, 38]. These gradient flow structures are a powerful tool to study convergence of sequences of dynamics to some limit. Two main strategies are existing: one of them consists in using the discrete (in time) approximation schemes suggested by the gradient flow structure (see for example [2]). The second one, which we use in [41], consists in characterizing gradient flows in terms of a relation between the energy function and its variations, and passing to the limit in this characterization.

In [73] in collaboration with T. Komorowski and S. Olla, we study the macroscopic behaviour of a chain of N coupled harmonic oscillators, perturbed with random flips of momenta (see also Section 2.1 below for a description of the model). We prove that in a diffusive space-time scaling limit the profiles corresponding to the two conserved quantities – energy and volume stretch – converge to the solution of a diffusive nonlinear system of differential equations. We follow an approach based on *Wigner distributions*, which permit to control the energy distribution over various frequency modes and provide a natural separation between mechanical and thermal energies. In the macroscopic limit we prove that locally the thermal energy spectrum has a constant density equal to the local thermal energy (or temperature), *i.e.* that the system is, at macroscopic positive times, in local equilibrium, even though it is not at initial time.

In [87] in collaboration with C. Olivera, we derive a limit process which belongs to the family of nonlocal PDEs, and is related to anomalous diffusions. More precisely, we study the asymptotic behaviour of a system of particles which interact *moderately*, *i.e.* an intermediate situation between weak and strong interaction, and which are submitted to random scattering. We prove a law of large numbers for the empirical density process, which in the macroscopic limit follows a fractional conservation law. The latter is a generalization of convection-diffusion equations, and appears in physical models, in areas like hydrodynamics and molecular biology.

1.4 Notations

Here follows a list of notations which will be used throughout the manuscript.

FUNCTIONAL ANALYSIS:

- We denote by
  - $\mathcal{C}(\mathrm{X},\mathrm{Y})$ the space of continuous functions from X to Y, and $\mathcal{C}(\mathrm{X})$ the space of *real–valued* continuous functions from X to $\mathbb{R}$.
  - $\mathcal{C}^\infty(\mathbb{R}^d)$ the space of real–valued infinitely differentiable functions, and $\mathcal{C}_c^\infty(\mathbb{R}^d)$ the space of real–valued infinitely differentiable functions with compact support.
  - $\mathcal{C}^\infty([0,1])$ the space of real–valued functions $\varphi : [0,1] \to \mathbb{R}$ such that φ is continuous in $[0,1]$ as well as all its derivatives.
- We abbreviate by $\mathrm{d}^k\varphi$ the k–th derivative of φ, and for $k = 1$ (resp. $k = 2$) we simply denote it by $\nabla\varphi$ (resp. $\Delta\varphi$) (even in one-dimensional cases).

- For any measurable space (U, ν) we denote by $\mathbf{L}^2(\nu)$ or $\mathbf{L}^2(\mathrm{U})$ the usual $\mathbf{L}^2$–space with norm $\|\cdot\|_{\mathbf{L}^2(\nu)}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathbf{L}^2(\nu)}$. Whenever the integration variable u may be unclear to the reader, we enlighten it by denoting $\mathbf{L}^2_u(\nu)$ or $\mathbf{L}^2_u(\mathrm{U})$.
- We denote by $\mathbf{L}^p(\mathrm{U};\mathrm{V})$ the usual $\mathbf{L}^p$–space of functions from U to V, with $p \in (0,\infty]$. The norm $\|\cdot\|_p$ is the associated $\mathbf{L}^p$-norm, whenever the integration spaces are clear to the reader. Otherwise, the $\mathbf{L}^p(\mathrm{U})$-norm will be denoted by $\|\cdot\|_{\mathbf{L}^p(\mathrm{U})}$.
- Given $\varphi : \mathbb{R} \to \mathbb{R}$, we denote (whenever these limits exist):

$$\varphi(0^+) = \lim_{\substack{u\to 0\\ u>0}} \varphi(u) \quad \text{and} \quad \varphi(0^-) = \lim_{\substack{u\to 0\\ u<0}} \varphi(u),$$

COMPARISONS: Given two real–valued functions f and g depending on $u \in \mathbb{R}^d$ we write

- $f \lesssim g$ or $g \gtrsim f$ if there exists a constant $\mathrm{C} > 0$, which does not depend on the parameters involved in f and g, such that $f \leqslant \mathrm{C}g$. If $f \lesssim g$ and $f \gtrsim g$, we write $f \simeq g$.
- $f = \mathcal{O}(g)$ in the neighborhood of u_0 if $|f| \lesssim |g|$ in the neighborhood of u_0
- $f = o(g)$ in the neighborhood of u_0 if $\lim_{u\to u_0} f(u)/g(u) = 0$.

FOURIER ANALYSIS:

- Let $\widehat{\psi} : \mathbb{R} \to \mathbb{C}$ be the Fourier transform of a function $\psi \in \mathbf{L}^1(\mathbb{R})$, which is defined by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}} e^{-2i\pi u\xi}\, \psi(u)\, du, \qquad \xi \in \mathbb{R}. \tag{1.5}$$

OTHERS:

- We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of positive integers and by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the set of non-negative integers. The discrete torus of size $\mathrm{N} \in \mathbb{N}$ is denoted by $\mathbb{T}_\mathrm{N} = \mathbb{Z}/\mathrm{N}\mathbb{Z}$ and the continuous torus is denoted by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- Sometimes it will be convenient to precise the dependence of the constant C on some extra parameters and this will be done by the standard notation $\mathrm{C}(\lambda)$ if λ is the extra parameter.
- We denote by $\mathbf{1}_\mathrm{A}(\eta)$ the indicator function that equals 1 if $\eta \in \mathrm{A}$ and 0 otherwise. The function $\delta_{k,\ell}$ is the indicator function that equals 1 if $k = \ell$ and 0 otherwise.
- For any complex number $z \in \mathbb{C}$, we denote by $\sqrt{z}$ its principal square root, which has positive real part: if $z = re^{i\theta}$ with $r \geqslant 0$ and $\theta \in (-\pi, \pi]$, then its principal square root is $\sqrt{z} = \sqrt{r}e^{i\theta/2}$.
- Assume that $\Omega = \mathrm{X}^{\mathbb{Z}}$ is some configuration space (in this text, mainly $\mathrm{X} = \{0,1\}$ or $\mathrm{X} = \mathbb{R}$). For any $k \in \mathbb{Z}$, we denote by τ_k the translated operator that acts on a function $f : \Omega \to \mathbb{R}$ as $(\tau_k f)(\eta) = f(\tau_k \eta)$, and $\tau_k\eta$ is the configuration obtained from η by shifting: for $\ell \in \mathbb{Z}$, $(\tau_k\eta)_\ell = \eta_{k+\ell}$. We say that a function defined on Ω is *local* if it depends on the variable $\omega \in \Omega$ only through a finite number of $\{\omega_k\}_{k\in\mathbb{Z}}$.
- For any function $\varphi : \mathbb{R} \to \mathbb{R}$ and $\mathrm{N} \in \mathbb{N}$ a scaling parameter, we define its *discrete gradient* and *discrete Laplacian* as follows: for all $k \in \mathbb{Z}$,

$$\nabla_\mathrm{N}\varphi\big(\tfrac{k}{\mathrm{N}}\big) = \mathrm{N}\Big(\varphi\big(\tfrac{k+1}{\mathrm{N}}\big) - \varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big),$$
$$\Delta_\mathrm{N}\varphi\big(\tfrac{k}{\mathrm{N}}\big) = \mathrm{N}\Big(\nabla_\mathrm{N}\varphi\big(\tfrac{k}{\mathrm{N}}\big) - \nabla_\mathrm{N}\varphi\big(\tfrac{k-1}{\mathrm{N}}\big)\Big).$$

CHAPTER 2 From normal diffusion to superdiffusion

This chapter is based on [12, 13, 74]*, written in collaboration with C. Bernardin, P. Gonçalves, M. Jara, T. Komorowski and S. Olla. Some parts are largely borrowed from these papers.*

Contents

2.1 Microscopic models with several local conservation laws

It is well known that the class of low-dimensional lattices of coupled classical oscillators contains systems which exhibit anomalous diffusion. In that case, the *low dimension* and the presence of *several conservation laws* give rise to unusual transport properties, and in particular to superdiffusivity (as explained for instance in the lecture notes [79]).

2.1.1 Deterministic oscillator chain

The microscopic model at the core of this chapter is the one-dimensional oscillators chain of size $N+1$ (with $N \in \mathbb{N}$, being the scaling parameter). The oscillators are described by their real-valued *positions* denoted by $q_0, \dots, q_N$ and their *momenta* $p_0, \dots, p_N$. We assume that each particle has unit mass. Since the setting is one-dimensional, the definition of the dynamics will be more conveniently rewritten in terms of the *stretches* defined by $r_j = q_j - q_{j-1}$, for any $j = 1, \dots, N$. The total *Hamiltonian*, or *energy*, of the system is given by

$$\mathcal{H}_N = \frac{1}{2}\sum_{j=0}^{N} p_j^2 + \sum_{j=1}^{N} V(r_j),$$

where V is the *interaction* potential, to be chosen further. We are interested in the time evolution of the configurations

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_N, p_0, \dots, p_N) \in \mathbb{R}^N \times \mathbb{R}^{N+1}.$$

The *deterministic* chain dynamics follows the Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}r_j}{\mathrm{d}t} = p_j - p_{j-1} & j \in \{1,\dots,\mathrm{N}\}, \\ \dfrac{\mathrm{d}p_j}{\mathrm{d}t} = \mathrm{V}'(r_{j+1}) - \mathrm{V}'(r_j), & j \in \{1,\dots,\mathrm{N}-1\}, \end{cases} \tag{2.1}$$

and, at the boundaries,

$$\frac{\mathrm{d}p_{\mathrm{N}}}{\mathrm{d}t} = -\mathrm{V}'(r_{\mathrm{N}}), \qquad \frac{\mathrm{d}p_0}{\mathrm{d}t} = \mathrm{V}'(r_1). \tag{2.2}$$

Boundary conditions will be imposed along the way (see the following sections).

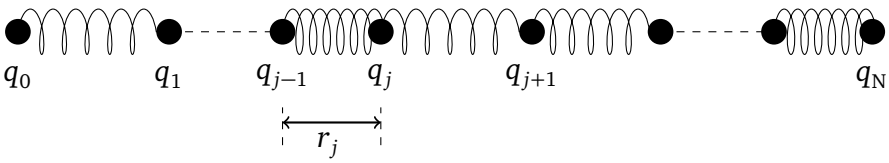

Figure 2.1: One-dimensional coupled oscillators.

Three quantities are *locally conserved* by the dynamics: the stretch, the momentum, and the energy. Let us define the *local energy*

$$\mathcal{E}_j = \begin{cases} \frac{1}{2}p_j^2 + \mathrm{V}(r_j) & \text{if } j \in \{1,\dots,\mathrm{N}\} \\ \frac{1}{2}p_0^2 & \text{if } j = 0. \end{cases}$$

One can easily write the three local conservation laws:

$$\frac{\mathrm{d}r_j}{\mathrm{d}t} = \mathfrak{J}^r_{j-1,j} - \mathfrak{J}^r_{j,j+1} \qquad \text{with} \quad \mathfrak{J}^r_{j,j+1} = -p_j \quad \text{for any } j \in \{0,\dots,\mathrm{N}\}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = \mathfrak{J}^p_{j-1,j} - \mathfrak{J}^p_{j,j+1} \qquad \text{with} \quad \mathfrak{J}^p_{j,j+1} = \begin{cases} -\mathrm{V}'(r_{j+1}) & \text{if } j \in \{0,\dots,\mathrm{N}-1\} \\ 0 & \text{if } j = -1 \text{ or } j = \mathrm{N} \end{cases}$$

$$\frac{\mathrm{d}\mathcal{E}_j}{\mathrm{d}t} = \mathfrak{J}^e_{j-1,j} - \mathfrak{J}^e_{j,j+1} \qquad \text{with} \quad \mathfrak{J}^e_{j,j+1} = \begin{cases} -p_j\mathrm{V}'(r_{j+1}) & \text{if } j \in \{0,\dots,\mathrm{N}-1\}, \\ 0 & \text{if } j = -1 \text{ or } j = \mathrm{N}. \end{cases}$$

If the momenta are assumed to satisfy the periodicity condition $p_0 = p_{\mathrm{N}}$, then these three quantities are also globally conserve: namely, the total stretch $\mathcal{R}_{\mathrm{N}}$, the total momentum $\mathcal{P}_{\mathrm{N}}$, and the total energy $\mathcal{H}_{\mathrm{N}}$, respectively given by

$$\mathcal{R}_{\mathrm{N}} = \sum_{j=1}^{\mathrm{N}} r_j, \qquad \mathcal{P}_{\mathrm{N}} = \sum_{j=0}^{\mathrm{N}} p_j, \quad \text{and} \quad \mathcal{H}_{\mathrm{N}} = \sum_{j=0}^{\mathrm{N}} \mathcal{E}_j = \frac{1}{2}\sum_{j=0}^{\mathrm{N}} p_j^2 + \sum_{j=1}^{\mathrm{N}} \mathrm{V}(r_j),$$

do not evolve in time under the system (2.1)–(2.2).

Recently, Bernardin and Stoltz [16] introduced a simplified version of this system, which is more tractable and has significant qualitative similarities. Their model is defined as follows: the Hamiltonian dynamics is simplified in order to keep only two conserved quantities. More precisely, positions and momenta are replaced by one real-valued vector field denoted by $\omega_1, \dots, \omega_N$. The deterministic dynamics is then governed by

$$\dot{\omega}_j = V'(\omega_{j+1}) - V'(\omega_{j-1}), \qquad j \in \{1, \dots, N\}, \tag{2.3}$$

with periodic boundary conditions: $\omega_{N+k} = \omega_k$ for any $k \in \mathbb{Z}$. Therefore, recalling that $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ is the discrete torus of size N, the configurations are in this case

$$\omega = (\omega_1, \dots, \omega_N) \in \mathbb{R}^{\mathbb{T}_N}.$$

The two conservation laws are called *volume* and *energy* and are respectively given by

$$\mathcal{V}_N = \sum_{j \in \mathbb{T}_N} \omega_j, \qquad \mathcal{E}_N = \sum_{j \in \mathbb{T}_N} V(\omega_j).$$

2.1.2 Stochastic noises

From about twenty years, it has been proposed to perturb the Hamiltonian dynamics with stochastic interactions that keep some conservation laws intact, like for instance random exchanges of momentum between nearest neighbours. These models have the advantage to be solvable, keeping at the same time the features of deterministic models. Different stochastic perturbations will be considered: either conservative *Poissonian* noises, which are discrete in nature, or conservative *Brownian* noises. We define them in the rest of this section.

⋄ To the Hamiltonian system (2.1)–(2.2) we will often add a random mechanism that conserves the energy and stretch (but not the volume): any two nearest neighbour particles exchange their momenta randomly in such a way that the total energy $\mathcal{H}_N$ is conserved, but not the total momentum $\mathcal{P}_N$. The new equations of the microscopic dynamics in the bulk of the chain are then given as follows: let $\{w_{j,j+1}(t)\ ;\ j = 0, \dots, N-1\}$ be independent, standard, one-dimensional Wiener processes, attached to the (unoriented) bonds $\{j, j+1\}$ of the chain, and let $\sigma > 0$ regulate the intensity of the random perturbation.

The new dynamics is given by

$$\begin{aligned} dr_j(t) &= \big(p_j - p_{j-1}\big)(t)\, dt, \qquad j \in \{1, \dots, N\} \\ dp_j(t) &= \big(V'(r_{j+1}) - V'(r_j)\big)(t)\, dt - \sigma p_j(t)\, dt \\ &\quad + \sqrt{\sigma}\big(p_{j-1}(t) dw_{j-1,j}(t) - p_{j+1}(t) dw_{j,j+1}(t)\big), \qquad j \in \{1, \dots, N-1\}. \end{aligned} \tag{2.4}$$

At the boundaries we have in this case

$$\begin{aligned} dp_0(t) &= V'(r_1)(t)\, dt - \tfrac{1}{2}\sigma p_0(t)\, dt - \sqrt{\sigma} p_1 dw_{0,1}(t) \\ dp_N(t) &= -V'(r_N)(t)\, dt - \tfrac{1}{2}\sigma p_N(t)\, dt + \sqrt{\sigma} p_{N-1}(t) dw_{N-1,N}(t). \end{aligned} \tag{2.5}$$

It will be always more convenient to define the dynamics *via* its generator, which is given here by

$$\mathcal{L} = \mathcal{A} + \sigma\mathcal{S}$$

where $\mathcal{A}$ is the generator of the deterministic Hamiltonian dynamics and acts on functions $f : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ as

$$(\mathcal{A}f)(\mathbf{r},\mathbf{p}) = \sum_{j=1}^{N}(p_j - p_{j-1})\frac{\partial f}{\partial r_j} + \sum_{j=1}^{N-1}\big(V'(r_{j+1}) - V'(r_j)\big)\frac{\partial f}{\partial p_j} + V'(r_1)\frac{\partial f}{\partial p_0} - V'(r_N)\frac{\partial f}{\partial p_N}, \quad (2.6)$$

and $\mathcal{S}$ is the generator of the stochastic noise defined as

$$\mathcal{S} = \sum_{j=0}^{N-1}(\mathcal{X}_j \circ \mathcal{X}_j), \quad (2.7)$$

where $\mathcal{X}_j$ is the momentum exchange operator:

$$\mathcal{X}_j = p_{j+1}\,\frac{\partial}{\partial p_j} - p_j\,\frac{\partial}{\partial p_{j+1}}.$$

⋄ Equivalently, to the simplified system (2.3) we will add the same mechanism, which randomly exchanches the nearest neighbour values $\omega_j, \omega_{j\pm1}$ in such a way that the total energy $\mathcal{E}_N$ is conserved, but not the volume $\mathcal{V}_N$. With a little abuse of notation, we define the generators of the dynamics on the configurations $\omega \in \mathbb{R}^{\mathbb{T}_N}$ by keeping the same letters (since we will never use them simultaneously): in that case, the generator of the Hamiltonian dynamics reads as

$$\mathcal{A} = \sum_{j\in\mathbb{T}_N}\big(V'(\omega_{j+1}) - V'(\omega_{j-1})\big)\frac{\partial}{\partial \omega_j}$$

and the generator of the stochastic noise is

$$\mathcal{S} = \sum_{j\in\mathbb{T}_N}(\mathcal{Y}_j \circ \mathcal{Y}_j) \quad (2.8)$$

where

$$\mathcal{Y}_j = \omega_{j+1}\,\frac{\partial}{\partial \omega_j} - \omega_j\,\frac{\partial}{\partial \omega_{j+1}}.$$

⋄ In order to observe distinct diffusion behaviours (standard or anomalous), we will also consider a stochastic perturbation which preserves *both* energy and volume. For that reason, we introduce a different noise whose generator is now defined as

$$\widetilde{\mathcal{S}} = \sum_{j\in\mathbb{T}_N}(\mathcal{Z}_j \circ \mathcal{Z}_j) \quad (2.9)$$

where

$$\mathcal{Z}_j = (\omega_{j+1} - \omega_j)\,\frac{\partial}{\partial \omega_{j-1}} + (\omega_j - \omega_{j-1})\,\frac{\partial}{\partial \omega_{j+1}} + (\omega_{j-1} - \omega_{j+1})\,\frac{\partial}{\partial \omega_j}.$$

Note that the diffusion operator $\mathcal{Z}_j$ is the generator of a Brownian motion on the circle

$$\big\{(\omega_{j-1},\omega_j,\omega_{j+1})\in\mathbb{R}^3\,;\ \omega_{j-1}^2+\omega_j^2+\omega_{j+1}^2=1,\ \omega_{j-1}+\omega_j+\omega_{j+1}=0\big\}.$$

$\diamond$ Finally, the discrete (Poissonian) version of the stochastic noise generated by $\widetilde{\mathcal{S}}$ can be described as follows: nearest neighbour variables ω_x,ω_{x+1} are exchanged at independently distributed random Poissonian times. In that way, both $\mathcal{E}_N$ and $\mathcal{V}_N$ are conserved by the dynamics, and the generator of the stochastic noise is now given for any $f:\mathbb{R}^{\mathbb{T}_N}\to\mathbb{R}$ by

$$\widetilde{\mathcal{S}}f(\omega)=\sum_{j\in\mathbb{T}_N}\big(f(\omega^{j,j+1})-f(\omega)\big), \tag{2.10}$$

where we denote by $\omega^{j,j+1}$ the configuration which is obtained from ω by exchanging ω_j and ω_{j+1}, keeping the other values identical, namely:

$$(\omega^{j,j+1})_k=\begin{cases}\omega_{j+1} & \text{if } k=j;\\ \omega_j & \text{if } k=j+1;\\ \omega_k & \text{otherwise.}\end{cases}$$

2.1.3 Predictions and recent results

For each one of these models, the main objectives are: to define the transport coefficients (like the *thermal conductivity*), to prove the propagation of *local equilibrium* and to obtain the diffusive equations governing the macroscopic evolution of the conserved quantities.

a) When energy normally diffuses

In [68] a *rotor* chain has been investigated as an example of Hamiltonian system with two conserved quantities (angular momentum and energy), for which the thermal conductivity is finite (and therefore energy diffuses). In that paper, numerics shows an unexpected behaviour of the chain when the latter is connected at the boundaries to two thermostats, and a mechanical force imposes an average angular momentum at one boundary: the stationary temperature profile coincides with the values of the thermostats, but in the middle of the chain it raises to a much higher value. This behaviour is related to the presence of two conserved quantities. Since the rotor model is too difficult to be treated analytically, we started to work in [74] with the *harmonic* chain, *i.e.* the dynamical system (2.1) with $V(r)=V_{\rm har}(r)=\frac{1}{2}r^2$, which is perturbed by the Brownian momentum exchange noise (2.7) (which destroys the conservation law of the momentum but keeps the other two conservation laws – energy and stretch – intact). The boundary conditions are similar to [68]: the boundaries of the chain are connected to two Langevin thermostats and an external force acting on one boundary puts the system in a *non-equilibrium stationary state* (NESS). The result that we obtain in [74] will be described below in Section 2.2.

b) When energy superdiffusion is expected

The very recent *nonlinear fluctuating hydrodynamics theory* developed by Spohn [97, 95] has brought a new insight to approach the problem of anomalous diffusion in the case of Hamiltonian lattice systems with several conservation laws. In the case of the simplified model (2.3), the equilibrium correlation functions generally present a *two-peak* structure, for which different universality classes can be achieved from a good choice of the model parameters (and in particular, the potential V). In order to describe the predictions, we abbreviate:

- **(D,D)** - when both peaks have Gaussian shape and broaden in time as $\sqrt{t}$,
- **(D,$\frac{3}{2}$L)** - when one peak has Gaussian shape, the other one shows characteristics of a Lévy walk and broadens in time as $t^{2/3}$,
- **(KPZ, $\frac{5}{3}$L)** - when one peak exhibits correlations as those in the KPZ universality class and broadens in time as $t^{2/3}$, the other one shows characteristics of a Lévy walk and broadens as $t^{3/5}$,
- **(D, mKPZ)** - when one peak has Gaussian shape, the other one broadens in time as $t^{2/3}$ and is close to the KPZ universality class, in the sense that its scaling function is slightly *modified* due to the presence of the diffusive peak.

The invariant measures of the Markov process $\{\omega(t)\ ;\ t \geqslant 0\}$ generated by (2.3) plus a volume-preserving stochastic noise $\widetilde{\mathcal{S}}$ (as in (2.9) or (2.10)) are the *Gibbs* product measures

$$\mu^{\mathrm{N}}_{\tau,\mathrm{T}}(\mathrm{d}\omega) = \prod_{j\in\mathbb{T}_{\mathrm{N}}} \frac{1}{\mathrm{Z}_{\tau,\mathrm{T}}} \exp\Big(-\mathrm{T}^{-1}\left(\mathrm{V}(\omega_j)+\tau\omega_j\right)\Big)\mathrm{d}\omega_j,$$

which are parametrized by the tension $\tau = -\int \mathrm{V}'(\omega_j)\mu^{\mathrm{N}}_{\tau,\mathrm{T}}(\mathrm{d}\omega)$ and the temperature $\mathrm{T} > 0$. Assume that the interaction potential V is the Fermi-Pasta-Ulam-Tsingou (FPUT) potential

$$\mathrm{V}_{\mathrm{FPUT}}(r) = \frac{1}{2}r^2 + \frac{1}{3}\alpha r^3 + \frac{1}{4}\beta r^4.$$

We give in the following table the predictions which have been done in [95] in function of the parameters α, β for a specific value of the thermodynamic quantities τ, T:

	$\mathrm{V}_{\mathrm{FPUT}}$ with $\tau = 1$ and $\mathrm{T} = \frac{1}{2}$
(D,D)	*none*
(D,$\frac{3}{2}$L)	$\alpha = 0,\ \beta = 1$
(KPZ, $\frac{5}{3}$L)	$\alpha = 2,\ \beta = 1$
(D, mKPZ)	*none*

Another choice for the potential gives rise to other behaviours: for instance, the exponential potential $\mathrm{V}_{\exp}(r) = b^{-2}(e^{-br} + br - 1)$ (which derives from the Toda chain) always leads to the universality class **(KPZ, $\frac{5}{3}$L)**. This behaviour is quite different from the original Toda chain, since in the latter case, all the correlation functions are expected to be ballistic [76]. All these predictions have been verified numerically [76, 95], but few of them have been rigorously proved.

In fact, only the harmonic case (*i.e.* $\alpha = \beta = 0$ in V_{FPUT}, which is also V_{har} introduced above), additionnally perturbed by a volume-preserving stochastic noise, has been successfully treated in [72, 14]: if the stochastic noise is generated by $\widetilde{\mathcal{S}}$, either Brownian (2.9) or Poissonian (2.10), then the behaviour is proved to be the same as in the case $\alpha = 0$ and $\beta = 1$, namely $\mathbf{(D,\frac{3}{2}L)}$. The approaches to solve the harmonic case deeply rely on the linearity of the dynamics, which authorizes the use of powerful tools coming from the Fourier analysis. In [13] we manage to prove rigorously that this behaviour persists for small nonlinear interactions, more precisely $\alpha = 0$ and $\beta = \beta_N \to 0$ (at some vanishing speed), still perturbed with the same noise generated by $\widetilde{\mathcal{S}}$. We will expose this result in Section 2.3.

Finally, in [12] we consider the harmonic simplified model (as before, *i.e.* take V_{har} in (2.3)), but now perturbed with two stochastic Brownian noises: $\mathcal{S}$ given by (2.8) and $\widetilde{\mathcal{S}}$ given by (2.9): both conserve the energy, but only the second one preserves the volume. If the first one vanishes identically, the volume is conserved, the energy transport is superdiffusive and described by a Lévy process governed by a fractional Laplacian: this is exactly the case $\mathbf{(D,\frac{3}{2}L)}$ already derived in [72, 14]. Otherwise, the volume conservation is destroyed, and the energy normally diffuses. When the intensity σ of the first noise vanishes with the chain size, *i.e.* $\sigma = \sigma_N \to 0$, we show in [12] that the macroscopic behaviour depends on the evanescence speed of σ_N, and we prove the existence of an interpolation process which lies between the normal diffusion regime and the fractional superdiffusion regime for the energy. This result will be detailed in Section 2.4.

2.2 Non-equilibrium stationary state

As explained in the previous section, the chain of coupled rotors has been studied numerically in [68], when Langevin thermostats are applied at both ends, while a constant force is applied to one end and the position of the rotor on the opposite side is kept fixed. While heat flows from the thermostats, work is performed by the torque, increasing the mechanical energy, which is then transformed into thermal energy by the dynamics of the rotors. The stationary temperature profiles observed numerically in [68] present a maximum inside the chain higher than the temperature of both thermostats. Furthermore, a negative linear response for the energy flux has been observed for certain values of the external parameters. This phenomenon is referred to in the literature as an *uphill diffusion*, see [75] or [27] and references therein. These numerical results have been confirmed in [69], as well as an instability of the system when thermostats are at zero temperature.

The present section, based on [74] (written in collaboration with T. Komorowski and S. Olla), aims at describing a similar phenomenon for the NESS of a different model, but with similar characteristics: the conserved quantities evolve macroscopically in the same diffusive time scale, and their macroscopic evolution is governed by a system of coupled diffusive equations. More precisely, we consider the chain of unpinned harmonic oscillators governed by the generator (already accelerated in a diffusive way)

$$\mathcal{L} = N^2(\mathcal{A} + \sigma\mathcal{S} + \mathcal{L}^{\text{bnd}}) \tag{2.11}$$

where

- $\mathcal{A}$ has been defined in the previous section in (2.6), with $V(r) = V_{ham}(r) = \frac{1}{2}r^2$,
- $\mathcal{S}$ it the Brownian noise given in (2.7) which destroys the momentum preservation,
- $\mathcal{L}^{bnd}$ encodes our assumptions at the boundaries: at the two opposite ends of the chain, two Langevin thermostats are attached, and moreover a constant force $\overline{\tau}_+$ acts on the last particle of the chain. Therefore, $\mathcal{L}^{bnd}$ acts on functions $f : \mathbb{R}^N \times \mathbb{R}^{N+1}$ as

$$(\mathcal{L}^{bnd} f)(\mathbf{r}, \mathbf{p}) = \overline{\tau}_+ \frac{\partial f}{\partial p_N} + \frac{\gamma}{2}\Big(T_- \frac{\partial^2 f}{\partial p_0^2} - p_0 \frac{\partial f}{\partial p_0}\Big) + \frac{\gamma}{2}\Big(T_+ \frac{\partial^2 f}{\partial p_N^2} - p_N \frac{\partial f}{\partial p_N}\Big).$$

  The parameter $\gamma > 0$ regulates the intensity of the Langevin thermostats, and $T_-, T_+ > 0$ are their temperatures. See Figure 2.2 for an illustration.

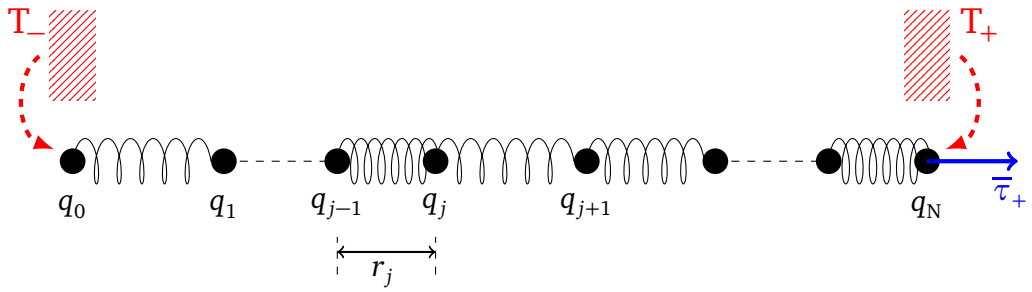

Figure 2.2: Oscillator chains with heat baths and one boundary force.

This system has only two conserved quantities: total energy and volume. Since the random mechanism does not conserve the total momentum, the macroscopic behaviour of these two quantities is diffusive, and the non stationary hydrodynamic limit with periodic boundary conditions (no thermostats or exterior force present) has been proven previously by us in [73]. The action of this constant force puts the system out of equilibrium, even when the temperatures of the thermostats are equal. As in the rotor chain, the exterior force performs positive work on the system, that increases the mechanical energy (concentrated on low frequency modes). The random mechanism transforms the mechanical energy into the thermal one (uniformly distributed in all frequencies, when the system is in a local equilibrium), which is eventually dissipated by the thermostats. This transfer of mechanical into thermal energy happens in the bulk of the system and is already completely predicted by the solution of the macroscopic diffusive system of equations obtained in the hydrodynamic limit [73], see also [9] for a similar model without boundary conditions.

From the microscopic energy conservation law there exist microscopic energy currents $\mathfrak{J}_{j,j+1}$ which satisfy

$$N^{-2}\mathcal{L}\Big(\tfrac{1}{2}\big(p_j^2 + r_j^2\big)\Big) = \mathfrak{J}_{j-1,j} - \mathfrak{J}_{j,j+1}, \qquad \text{for any } j \in \{0, \dots, N\} \tag{2.12}$$

and are given by

$$\mathfrak{J}_{j,j+1} = -p_j r_{j+1} + \frac{\sigma}{2}(p_j^2 - p_{j+1}^2), \qquad \text{if } j \in \{0, ..., N-1\}, \tag{2.13}$$

while at the boundaries

$$\mathfrak{J}_{-1,0} = \frac{\gamma}{2}\big(\mathrm{T}_- - p_0^2\big), \qquad \mathfrak{J}_{\mathrm{N},\mathrm{N}+1} = -\frac{\gamma}{2}\big(\mathrm{T}_+ - p_{\mathrm{N}}^2\big) - \overline{\tau}_+ p_{\mathrm{N}}. \tag{2.14}$$

Suppose now that $\{r(t,u), e_{\mathrm{th}}(t,u)\}$, with $(t,u) \in \mathbb{R}_+ \times [0,1]$, are the macroscopic profiles of *elongation* and *thermal energy* of the macroscopic system, obtained in the diffusive scaling limit. Precisely, define the thermal energy per particle as

$$\mathcal{E}_j^{\mathrm{th}} = \frac{1}{2}p_j^2 + \frac{1}{2}\big(r_j^2 - (\mathbb{E}[r_j])^2\big),$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to the law of the process. The profiles $r(t,\cdot), e_{\mathrm{th}}(t,\cdot)$ are the expected large N limits (in the weak formulation sense) of

$$\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} r_j(t)\delta_{j/\mathrm{N}}(\cdot), \quad \text{and} \quad \frac{1}{\mathrm{N}}\sum_{j=0}^{\mathrm{N}} \mathcal{E}_j^{\mathrm{th}}(t)\delta_{j/\mathrm{N}}(\cdot).$$

If both convergences do hold at time $t = 0$ to some given profiles $r_0(u)$ and $\mathcal{T}_0(u)$, then we expect that they satisfy the following system of equations[1],

$$\partial_t r(t,u) = \sigma^{-1}\,\partial_{uu}^2 r(t,u) \tag{2.15}$$

$$\partial_t e_{\mathrm{th}}(t,u) = \frac{1}{2}\big(\sigma^{-1}+\sigma\big)\,\partial_{uu}^2 e_{\mathrm{th}}(t,u) + \sigma^{-1}\left(\partial_u r(t,u)\right)^2, \qquad (t,u)\in\mathbb{R}_+\times[0,1], \tag{2.16}$$

with boundary conditions

$$\begin{aligned} r(t,0) &= 0, & r(t,1) &= \overline{\tau}_+, \\ e_{\mathrm{th}}(t,0) &= \mathrm{T}_-, & e_{\mathrm{th}}(t,1) &= \mathrm{T}_+, \end{aligned}$$

and with the initial condition

$$r(0,u) = r_0(u), \qquad e_{\mathrm{th}}(0,u) = \mathcal{T}_0(u).$$

When $\overline{\tau}_+ = 0$ and $\mathrm{T}_- = \mathrm{T}_+ = \mathrm{T}$, the system is said *in equilibrium* and the stationary probability distribution is given explicitly by the homogeneous Gibbs measure

$$\nu_{\mathrm{T}}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) = g_{\mathrm{T}}(\mathbf{r},\mathbf{p})\mathrm{d}p_0\prod_{j=1}^{\mathrm{N}}\mathrm{d}p_j\mathrm{d}r_j$$

where

$$g_{\mathrm{T}}(\mathbf{r},\mathbf{p}) = \frac{e^{-p_0^2/(2\mathrm{T})}}{\sqrt{2\pi\mathrm{T}}}\prod_{j=1}^{\mathrm{N}}\frac{e^{-(p_j^2+r_j^2)/(2\mathrm{T})}}{2\pi\mathrm{T}}. \tag{2.17}$$

If $\overline{\tau}_+ \neq 0$ or $\mathrm{T}_- \neq \mathrm{T}_+$, the stationary measure exists and is unique, but it is not explicitly computable. More precisely, we know that there exists a unique stationary probability distribution μ_{ss} on $\mathbb{R}^{\mathrm{N}}\times\mathbb{R}^{\mathrm{N}+1}$ for the microscopic dynamics generated by (2.11). As a consequence $\langle\mathcal{L}\mathrm{F}\rangle_{\mathrm{ss}} = 0$ for any function F in the domain of the operator $\mathcal{L}$, where hereafter, we denote

$$\langle\mathrm{F}\rangle_{\mathrm{ss}} = \int_{\mathbb{R}^{\mathrm{N}}\times\mathbb{R}^{\mathrm{N}+1}} \mathrm{F}\,\mathrm{d}\mu_{\mathrm{ss}}.$$

[1] See also Theorems 3.7 and 3.8 of [73] for a similar model which gives a similar coupled diffusive system for every value of σ.

The proof of the existence and uniqueness of a stationary state follows from the same argument as the one used in [15, Appendix A] for $\overline{\tau}_+ = 0$. The fact that in our case $\overline{\tau}_+$ does not vanish requires only minor modifications.

The corresponding *stationary* profiles, denoted respectively by $r_{\rm ss}(u)$ and $e_{\rm th,ss}(u)$, will solve the stationary version of equations (2.15) and (2.16), *i.e.*:

$$r_{\rm ss}(u) = \overline{\tau}_+ u \tag{2.18}$$

$$\left(\sigma^{-1} + \sigma\right)\partial^2_{uu} e_{\rm th,ss}(u) + 2\sigma^{-1}\,\overline{\tau}_+^2 = 0, \tag{2.19}$$

with the boundary conditions

$$e_{\rm th,ss}(0) = \mathrm{T}_-, \qquad e_{\rm th,ss}(1) = \mathrm{T}_+.$$

In other words

$$e_{\rm th,ss}(u) = \frac{\overline{\tau}_+^2}{1+\sigma^2}\, u(1-u) + (\mathrm{T}_+ - \mathrm{T}_-)u + \mathrm{T}_-. \tag{2.20}$$

Taking the average with respect to the stationary state in (2.12), we get the *stationary microscopic energy current*

$$\langle \mathfrak{J}_{j,j+1} \rangle_{\rm ss} =: \overline{j}_s, \qquad \text{for any } j \in \{-1, \dots, \mathrm{N}\}. \tag{2.21}$$

The *macroscopic stationary energy current* is defined as the limit of $\mathrm{N}\overline{j}_s$, as $\mathrm{N} \to +\infty$. It equals, see Theorem 2.2.2 below,

$$\mathrm{J}_{\rm ss} = -\frac{1}{2}(\sigma^{-1} + \sigma)(\mathrm{T}_+ - \mathrm{T}_-) - \frac{\overline{\tau}_+^2}{2\sigma}.$$

Observe that the energy current can flow against the temperature gradient if $\mathrm{T}_- > \mathrm{T}_+$ and $|\overline{\tau}_+|$ is large enough (*uphill diffusion*). Assuming $\mathrm{T}_+ \geqslant \mathrm{T}_-$ the maximum stationary temperature $e^{\max}_{\rm th,ss}$ is reached at

$$u_{\max} = \left(\frac{1}{2} + \frac{1+\sigma^2}{\overline{\tau}_+^2}(\mathrm{T}_+ - \mathrm{T}_-)\right) \wedge 1$$

which implies that, if the condition $2(1+\sigma^2)(\mathrm{T}_+ - \mathrm{T}_-) \leqslant \overline{\tau}_+^2$ is satisfied, then the chain reaches the maximum temperature inside ($u_{\max} < 1$), and the maximal temperature is equal to

$$e^{\max}_{\rm th,ss} = \frac{(\mathrm{T}_+ - \mathrm{T}_-)}{2} + \mathrm{T}_- + \frac{\overline{\tau}_+^2}{4(1+\sigma^2)} \geqslant \mathrm{T}_+.$$

Note that this does not depend on the sign of $\overline{\tau}_+$.

The work [74] is devoted to the proof of such a phenomenon, when $\sigma = 1$. This restriction is technical and will be further enlightened. We prove, see Theorem 2.2.3 below, that the energy and the volume stretch profiles converge to the stationary solution $r_{\rm ss}(u)$ and $e_{\rm th,ss}(u)$.

2.2.1 Statement of the main results

Let us start with the following:

THEOREM 2.2.1 (Stationary elongation profile). *The following uniform convergence holds:*

$$\sup_{u\in[0,1]} \left|\langle r_{[\mathrm{N}u]}\rangle_{\mathrm{ss}} - r_{\mathrm{ss}}(u)\right| \xrightarrow[\mathrm{N}\to\infty]{} 0,$$

where $r_{\mathrm{ss}}(u) = \overline{\tau}_+ u$. *In particular, for any continuous test function* $\varphi : [0,1] \to \mathbb{R}$,

$$\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\langle r_j\rangle_{\mathrm{ss}} \xrightarrow[\mathrm{N}\to\infty]{} \int_{[0,1]} \varphi(u) r_{\mathrm{ss}}(u)\,\mathrm{d}u.$$

Proof. The averages under the stationary state $\langle r_j\rangle_{\mathrm{ss}}$ and $\langle p_j\rangle_{\mathrm{ss}}$ are computable explicitly, see [74, Proposition 4.1]. It turns out that $\langle p_j\rangle_{\mathrm{ss}}$ is constant in j and equals $\overline{p}_s = \overline{\tau}_+/(\sigma\mathrm{N}+\gamma)$. We also have

$$\mathrm{N}\big(\langle r_{j+1}\rangle_{\mathrm{ss}} - \langle r_j\rangle_{\mathrm{ss}}\big) = \mathrm{N}\sigma\overline{p}_s \xrightarrow[\mathrm{N}\to\infty]{} \overline{\tau}_+, \qquad \text{for } j\in\{1,\dots,\mathrm{N}-1\}$$

and $\langle r_1\rangle_{\mathrm{ss}} \xrightarrow[\mathrm{N}\to\infty]{} 0$, $\langle r_{\mathrm{N}}\rangle_{\mathrm{ss}} \xrightarrow[\mathrm{N}\to\infty]{} \overline{\tau}_+$, which is enough to conclude.

Concerning the *stationary energy flow* and the validity of the Fourier law we have the following result on the macroscopic stationary energy current.

THEOREM 2.2.2 (Stationary energy current and Fourier law).

$$\mathrm{N}\overline{j}_s \xrightarrow[\mathrm{N}\to\infty]{} -\frac{1}{2}(\sigma^{-1}+\sigma)(\mathrm{T}_+ - \mathrm{T}_-) - \frac{\overline{\tau}_+^2}{2\sigma}. \tag{2.22}$$

Note that Theorems 2.2.1 and 2.2.2 are valid for any $\sigma > 0$. We now state our last main result about the stationary energy profile, which we are able to prove only for $\sigma = 1$.

THEOREM 2.2.3 (Stationary energy profile). *Assume that* $\sigma = 1$. *For any continuous test function* $\varphi : [0,1] \to \mathbb{R}$,

$$\mathcal{H}_{\mathrm{N}}(\varphi) = \frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\big\langle \tfrac{1}{2}(p_j^2 + r_j^2)\big\rangle_{\mathrm{ss}} \xrightarrow[\mathrm{N}\to\infty]{} \int_{[0,1]} \varphi(u)\Big(e_{\mathrm{th,ss}}(u) + \tfrac{1}{2}r_{\mathrm{ss}}^2(u)\Big)\,\mathrm{d}u, \tag{2.23}$$

where

$$\begin{aligned} r_{\mathrm{ss}}(u) &= \overline{\tau}_+ u,\\ e_{\mathrm{th,ss}}(u) &= \tfrac{1}{2}\overline{\tau}_+^2\, u(1-u) + (\mathrm{T}_+ - \mathrm{T}_-)u + \mathrm{T}_+. \end{aligned}$$

In [74, Appendix A] we give a proof for the non-stationary macroscopic evolution of the energy and the volume stretch profiles in the diffusive space-time scaling, *i.e.* (2.15) and (2.16). As for the NESS, the proof is rigorous only for $\sigma = 1$, for similar reasons. The corresponding result with periodic boundary conditions was contained in [9].

In the next section we briefly comment the proofs of Theorems 2.2.2 and 2.2.3.

2.2.2 Sketch of the proof

One of the main characteristics of this model is the existence of an explicit *fluctuation-dissipation relation*, which permits to write the stationary current $\overline{j}_s$ as a discrete gradient of some local function, as given in the following proposition (which follows by a direct calculation, see [74, Proposition 4.2]):

PROPOSITION 2.2.4 (Decomposition of the stationary current). *We can write $\overline{j}_s$ as a discrete gradient, namely*

$$\overline{j}_s = h(j+1) - h(j), \qquad j \in \{1, \dots, \mathrm{N}-1\}, \tag{2.24}$$

with

$$h(j) = -\frac{1}{2\sigma}\Big(\langle r_j^2\rangle_{\mathrm{ss}} + \langle p_{j-1}p_j\rangle_{\mathrm{ss}}\Big) - \frac{\sigma}{4}\Big(\langle p_j^2\rangle_{\mathrm{ss}} + \langle p_{j-1}^2\rangle_{\mathrm{ss}}\Big), \qquad j \in \{1, \dots, \mathrm{N}\}. \tag{2.25}$$

REMARK *2.2.1*. From (2.24), the function $h(j)$ is a *harmonic* function, namely:

$$h(j+1) + h(j-1) - 2h(j) = 0, \qquad \text{for any } j \in \{2, \dots, \mathrm{N}-1\}. \tag{2.26}$$

We can now sketch the proof of Theorem 2.2.2. Sum the identity (2.24) from $j = 1$ to $\mathrm{N}-1$ and apply (2.25) to express $h(\mathrm{N})$ and $h(1)$. We obtain:

$$(\mathrm{N}-1)\overline{j}_s = h(\mathrm{N}) - h(1) = \Big\langle -\tfrac{1}{2\sigma}(p_{\mathrm{N}-1}p_{\mathrm{N}} + r_{\mathrm{N}}^2) - \tfrac{\sigma}{4}(p_{\mathrm{N}-1}^2 + p_{\mathrm{N}}^2)\Big\rangle_{\mathrm{ss}} + \Big\langle \tfrac{1}{2\sigma}(p_1p_0 + r_1^2) + \tfrac{\sigma}{4}(p_1^2 + p_0^2)\Big\rangle_{\mathrm{ss}}.$$

The presence of the Langevin thermostats at the boundaries permit to get the following second order moment bounds: there exists $\mathrm{C} > 0$ (which does not depend on N) such that $\langle p_j^2\rangle_{\mathrm{ss}} \leqslant \mathrm{C}$ for $j = 0, 1, \mathrm{N}-1, \mathrm{N}$ and $\langle r_j^2\rangle_{\mathrm{ss}} \leqslant \mathrm{C}$ for $j = 1, \mathrm{N}$. This implies that

$$\big|\overline{j}_s\big| \leqslant \frac{\mathrm{C}}{\mathrm{N}},$$

and gives the right order for the Fourier law. More refined estimates permit to compute exactly the limits of $h(1)$ and $h(\mathrm{N})$, and to prove Theorem 3.4.1 (see [74, Proposition 5.9]).

We finally sketch the proof of Theorem 2.2.3: straightforward computations, using the definition (2.25) of h, yield

$$\begin{aligned}\Big\langle \tfrac{1}{2}(p_j^2 + r_j^2)\Big\rangle_{\mathrm{ss}} = &-\frac{2\sigma}{1+\sigma^2}h(j) + \frac{\sigma^2}{2(1+\sigma^2)}\Big(\langle p_j^2\rangle_{\mathrm{ss}} - \langle p_{j-1}^2\rangle_{\mathrm{ss}}\Big) \\ &- \frac{1}{1+\sigma^2}\langle p_jp_{j-1}\rangle_{\mathrm{ss}} + \frac{1-\sigma^2}{2(1+\sigma^2)}\Big(\langle p_j^2\rangle_{\mathrm{ss}} - \langle r_j^2\rangle_{\mathrm{ss}}\Big).\end{aligned} \tag{2.27}$$

Therefore, the microscopic energy profile can be decomposed as the sum of four terms:

$$\mathcal{H}_{\mathrm{N}}(\varphi) = \frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\Big\langle \tfrac{1}{2}(p_j^2 + r_j^2)\Big\rangle_{\mathrm{ss}} = \mathcal{H}_{\mathrm{N}}^{h}(\varphi) + \mathcal{H}_{\mathrm{N}}^{\nabla}(\varphi) + \mathcal{H}_{\mathrm{N}}^{\mathrm{corr}}(\varphi) + \mathcal{H}_{\mathrm{N}}^{\mathrm{m}}(\varphi), \tag{2.28}$$

where

$$\mathcal{H}_{\mathrm{N}}^{h}(\varphi) = -\frac{2\sigma}{1+\sigma^2}\,\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)h(j),$$

$$\mathcal{H}_{\mathrm{N}}^{\nabla}(\varphi) = \frac{\sigma^2}{2(1+\sigma^2)}\,\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\big(\langle p_j^2\rangle_{\mathrm{ss}} - \langle p_{j-1}^2\rangle_{\mathrm{ss}}\big),$$

$$\mathcal{H}_{\mathrm{N}}^{\mathrm{corr}}(\varphi) = -\frac{1}{1+\sigma^2}\,\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\langle p_{j-1}p_j\rangle_{\mathrm{ss}},$$

$$\mathcal{H}_{\mathrm{N}}^{\mathrm{m}}(\varphi) = \frac{1-\sigma^2}{2(1+\sigma^2)}\,\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\big(\langle p_j^2\rangle_{\mathrm{ss}} - \langle r_j^2\rangle_{\mathrm{ss}}\big).$$

Note that, if $\sigma = 1$, then $\mathcal{H}_{\mathrm{N}}^{\mathrm{m}} \equiv 0$. If $\sigma \neq 1$, we conjecture that this last term vanishes as $\mathrm{N} \to \infty$, but we are not able to prove it at the moment. The limits of the other three terms are summarized in the following proposition:

PROPOSITION 2.2.5. *For any continuous test function $\varphi : [0,1] \to \mathbb{R}$,*

$$\mathcal{H}_{\mathrm{N}}^{h}(\varphi) \xrightarrow[\mathrm{N}\to\infty]{} \int_{[0,1]} \varphi(u)\Big(\frac{\overline{\tau}_+^2}{1+\sigma^2}u + (\mathrm{T}_+ - \mathrm{T}_-)u + \mathrm{T}_-\Big)\,\mathrm{d}u, \tag{2.29}$$

$$\mathcal{H}_{\mathrm{N}}^{\nabla}(\varphi) \xrightarrow[\mathrm{N}\to\infty]{} 0, \tag{2.30}$$

$$\mathcal{H}_{\mathrm{N}}^{\mathrm{corr}}(\varphi) \xrightarrow[\mathrm{N}\to\infty]{} 0. \tag{2.31}$$

The complete proof of the proposition is given in [74], let me only comment on the ideas used in the argument:

- the limit (2.29) is concluded using the fact that h is harmonic,
- the limit (2.30) is a consequence of the presence of a discrete gradient $\langle p_j^2\rangle_{\mathrm{ss}} - \langle p_{j-1}^2\rangle_{\mathrm{ss}}$ inside the sum (an integration by parts allows one to conclude),
- the limit (2.31) is shown thanks to several *second order moment bounds*, which are detailed in [74, Section 5].

With the help of Proposition 2.2.5 the proof of Theorem 2.2.3 becomes straightforward. Assume that $\sigma = 1$. From the decomposition (2.27) and Proposition 2.2.5, we get

$$\begin{aligned}
\frac{1}{\mathrm{N}}\sum_{j=1}^{\mathrm{N}} \varphi\big(\tfrac{j}{\mathrm{N}}\big)\big\langle \tfrac{1}{2}(p_j^2 + r_j^2)\big\rangle_{\mathrm{ss}} &\xrightarrow[\mathrm{N}\to\infty]{} \int_{[0,1]} \varphi(u)\Big(\tfrac{\overline{\tau}_+^2}{2}u + (\mathrm{T}_+ - \mathrm{T}_-)u + \mathrm{T}_-\Big)\,\mathrm{d}u \\
&= \int_{[0,1]} \varphi(u)\Big(\tfrac{\overline{\tau}_+^2}{2}u(1-u) + (\mathrm{T}_+ - \mathrm{T}_-)u + \mathrm{T}_- + \tfrac{\overline{\tau}_+^2}{2}u^2\Big)\,\mathrm{d}u \\
&= \int_{[0,1]} \varphi(u)\Big(e_{\mathrm{th,ss}}(u) + \tfrac{1}{2}r_{\mathrm{ss}}^2(u)\Big)\,\mathrm{d}u,
\end{aligned}$$

from (2.18) and (2.20). Thus (2.23) follows.

2.3 Weak anharmonicity

As mentioned at the beginning of this chapter, in the work [13] – written in collaboration with C. Bernardin, P. Gonçalves and M. Jara – we consider a small quartic nonlinear perturbation of the linear Hamiltonian simplified model with conservative noise. In the absence of nonlinearities, it is proved in [14] that the model belongs to the universality class described by a skew 3/4-fractional diffusion equation for the energy and a normal diffusion for the volume, *i.e.* $\mathbf{(D,\frac{3}{2}L)}$ with the notation of Section 2.1.3. According to Spohn's theory [95], if the nonlinear perturbation is driven by an even potential and if the tension is null, the model still belongs to the same universality class. In this section we explain how to rigorously prove that this is indeed the case for *small* nonlinear perturbations.

The strategy of our proof follows the general scheme introduced in [14]. The success of this strategy, in the linear case, is due to the fact that the n-point correlation functions form a complicated but in any case, a closed system. Dealing with nonlinear potentials the problem is much harder since this last property is lost and we have to manage the control of a hierarchy. In [13] we are able to quantify the intensity with which one can perturb the linear system so as to cut the hierarchy and to recover the linear system, see Theorem 2.3.2 and Theorem 2.3.3 below. The control of the error terms produced by this cut-off requires several standard techniques of interacting particle systems as well as some *ad-hoc* estimates. These estimates depend heavily on the form of the potential but the scheme of the proof should work out for other types of potentials. Despite the fact that the techniques we use only allow us to obtain the results for a small intensity of the anharmonicity, we expect that the results remain valid for any intensity, see Remark 2.3.1 and Remark 2.3.2 below. Observe also that we are only considering the case of a perturbation given by an even potential with a zero tension. If one of these conditions is not respected we expect to reach a different universality class, following the heuristic predictions given in [95].

The dynamics of the interacting oscillators is defined on the whole line $\mathbb{Z}$ and is described by a Markov process on the state space $\Omega = \mathbb{R}^{\mathbb{Z}}$. A typical configuration is $\omega = \{\omega_k\,;\, k \in \mathbb{Z}\} \in \Omega$. The small anharmonicity is regulated by the parameter $\beta > 0$, and the system is studied at equilibrium. More precisely, the infinitesimal generator of the model is $\mathcal{L}_\beta = \mathcal{A}_\beta + \widetilde{\mathcal{S}}$ where $\mathcal{A}_\beta$ is the generator of the Hamiltonian dynamics, and $\widetilde{\mathcal{S}}$ is the generator of the Poissonian energy conserving noise, as already defined in Section 2.1: here they read

$$\mathcal{A}_\beta = \sum_{k\in\mathbb{Z}} \Big((\omega_{k+1} - \omega_{k-1}) + \beta(\omega_{k+1}^3 - \omega_{k-1}^3) \Big) \frac{\partial}{\partial\, \omega_k}$$

and for any local function $f : \Omega \to \mathbb{R}$ we let

$$(\widetilde{\mathcal{S}} f)(\omega) = \sum_{k\in\mathbb{Z}} \Big(f(\omega^{k,k+1}) - f(\omega) \Big),$$

as in (2.10) but on the infinite line. The one-site energy is the sum of the kinetic energy and

the potential energy, namely: for each $k \in \mathbb{Z}$, the energy of the atom k is

$$e_\beta(\omega_k) \qquad \text{where} \quad e_\beta(u) = \frac{u^2}{2} + \beta\,\frac{u^4}{4}.$$

By usual techniques (see [16] and references therein) we can show the existence of a Markov process $\{\omega^\beta(t)\ ;\ t \geqslant 0\}$ with state space $\Omega' \subset \Omega$ and generator $\mathcal{L}_\beta$ which has a family of invariant measures, namely the (infinite volume) Gibbs equilibrium measures given by

$$\nu^\beta_{\mathrm{T},\tau}(\mathrm{d}\omega) = \prod_{k\in\mathbb{Z}} \frac{\exp\big(-\mathrm{T}^{-1}\,(e_\beta(\omega_k) - \tau\omega_k)\big)}{\mathrm{Z}_\beta(\mathrm{T},\tau)}\mathrm{d}\omega_k,$$

which satisfy $\nu^\beta_{\mathrm{T},\tau}(\Omega') = 1$, and $\mathrm{Z}_\beta(\mathrm{T},\tau)$ is the normalization constant. These measures are associated to the two conserved quantities – volume and energy – which are formally given by

$$\mathcal{V} = \sum_{k\in\mathbb{Z}} \omega_k, \qquad \mathcal{E}_\beta = \sum_{k\in\mathbb{Z}} e_\beta(\omega_k).$$

For any $(\mathrm{T},\tau) \in (0,+\infty)\times\mathbb{R}$ and $\beta > 0$, we denote by $\langle\varphi\rangle^\beta_{\mathrm{T},\tau}$ the average of $\varphi : \Omega \to \mathbb{R}$ with respect to $\nu^\beta_{\mathrm{T},\tau}$ and by $\langle\cdot,\cdot\rangle^\beta_{\mathrm{T},\tau}$ the corresponding $\mathbf{L}^2$–inner product. Let us define

$$\mathfrak{e}_\beta(\mathrm{T},\tau) = \big\langle e_\beta(\omega_0)\big\rangle^\beta_{\mathrm{T},\tau} \qquad \text{and} \qquad \mathfrak{v}_\beta(\mathrm{T},\tau) = \big\langle \omega_0\big\rangle^\beta_{\mathrm{T},\tau}. \tag{2.32}$$

From here on, we consider the dynamics described by the accelerated generator $\mathrm{N}^a\mathcal{L}_{\beta_\mathrm{N}}$ (therefore the system evolves on the time scale $t\mathrm{N}^a$ for some $a > 0$), where β_N now depends also on the scaling parameter in such a way that $\lim_{\mathrm{N}\to\infty}\beta_\mathrm{N} = 0$. The dependence of β_N with respect to the scaling parameter N will be precised later on. We assume that the dynamics starts from the Gibbs equilibrium measure $\nu^{\beta_\mathrm{N}}_{\mathrm{T},0}$ at temperature $\mathrm{T} > 0$ and tension $\tau = 0$, and we look at its evolution during a time interval $[0,\overline{t}]$, where $\overline{t} > 0$ is fixed. The law of the resulted process

$$\Big\{\omega^{\beta_\mathrm{N}}_k(t\mathrm{N}^a)\ ;\ k\in\mathbb{Z}\Big\}_{t\in[0,\overline{t}]}$$

is simply denoted by $\mathbb{P}$, and the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. For the sake of readability, from now on we denote $\omega^{\beta_\mathrm{N}}_k(t\mathrm{N}^a)$ simply by $\omega_k(t\mathrm{N}^a)$.

2.3.1 Energy and volume fluctuation fields

We define, for any test function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$ and $\omega \in \Omega$,

$$\mathcal{E}^\mathrm{N}(\varphi,\omega) = \frac{1}{\sqrt{\mathrm{N}}}\sum_{k\in\mathbb{Z}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(e_{\beta_\mathrm{N}}(\omega_k) - \mathfrak{e}_{\beta_\mathrm{N}}(\mathrm{T},0)\Big),$$

and the dynamical *energy fluctuation field* by

$$\mathcal{E}^\mathrm{N}_t(\varphi) = \mathcal{E}^\mathrm{N}\big(\varphi,\omega(t\mathrm{N}^a)\big),$$

for any $t \in [0, \overline{t}]$. Similarly we define, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\omega \in \Omega$

$$\mathcal{V}^{\mathrm{N}}(\varphi, \omega) = \frac{1}{\sqrt{\mathrm{N}}} \sum_{k \in \mathbb{Z}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(\omega_k - \mathfrak{v}_{\beta_{\mathrm{N}}}(\mathrm{T}, 0)\Big),$$

and the dynamical *volume fluctuation field* by

$$\mathcal{V}_t^{\mathrm{N}}(\varphi) = \mathcal{V}^{\mathrm{N}}\big(\varphi, \omega(t\mathrm{N}^a)\big).$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a fixed function. We want to study the behaviour as $\mathrm{N} \to \infty$ of the *correlation energy and volume fields* given for any test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$\mathbf{E}_t^{\mathrm{N}}(\varphi) = \mathbb{E}\big[\mathcal{E}_0^{\mathrm{N}}(\psi)\, \mathcal{E}_t^{\mathrm{N}}(\varphi)\big] = \mathbb{E}\Big[\mathcal{E}_0^{\mathrm{N}}(\psi) \times \frac{1}{4\sqrt{\mathrm{N}}} \sum_{k \in \mathbb{Z}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(\big[2\omega_k^2 + \beta_{\mathrm{N}}\omega_k^4\big](t\mathrm{N}^a) - 4\mathfrak{e}_{\beta_{\mathrm{N}}}(\mathrm{T}, 0)\Big)\Big],$$

$$\mathbf{V}_t^{\mathrm{N}}(\varphi) = \mathbb{E}\big[\mathcal{V}_0^{\mathrm{N}}(\psi)\, \mathcal{V}_t^{\mathrm{N}}(\varphi)\big] = \mathbb{E}\Big[\mathcal{V}_0^{\mathrm{N}}(\psi) \times \frac{1}{\sqrt{\mathrm{N}}} \sum_{k \in \mathbb{Z}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(\omega_k(t\mathrm{N}^a) - \mathfrak{v}_{\beta_{\mathrm{N}}}(\mathrm{T}, 0)\Big)\Big].$$

2.3.2 Statement of the main results

Let us assume now that

$$\beta_{\mathrm{N}} = \frac{c}{\mathrm{N}^b} \qquad \text{for some } c, b > 0,$$

and recall that the time scale is $t\mathrm{N}^a$, with $a > 0$. Our main convergence theorems depend on the range of the parameters (a, b). Recall that $\tau = 0$. In the nonlinear fluctuating theory framework developed in [97], this choice implies in particular the identification of the *sound mode* with the volume, and the *heat mode* with the energy.

THEOREM 2.3.1 (Volume fluctuations in the time scale $t\mathrm{N}^a$ with $a \leqslant 1$).

Let us fix $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$, and $t > 0$. The macroscopic volume behaviour follows the dichotomy:

1. *If $a < 1$, then*

$$\lim_{\mathrm{N}\to\infty} \mathbf{V}_t^{\mathrm{N}}(\varphi) = \lim_{\mathrm{N}\to\infty} \mathbf{V}_0^{\mathrm{N}}(\varphi) = \mathrm{T} \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\, \mathrm{d}u\mathrm{d}v.$$

2. *If $a = 1$, then*

$$\lim_{\mathrm{N}\to\infty} \mathbf{V}_t^{\mathrm{N}}(\varphi) = \mathrm{T} \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\widehat{\mathrm{P}}_t(u - v)\, \mathrm{d}u\mathrm{d}v,$$

where $\{\widehat{\mathrm{P}}_t\}_{t \geqslant 0}$ is the semi-group generated by the transport operator -2∇.

In order to study the fluctuations in a longer time scale $a > 1$, we first need to recenter the fluctuation field in a frame moving with some specific velocity. To that aim, let us denote $\chi_{\mathrm{N}} = \langle \omega_0^2 \rangle_{\mathrm{T},0}^{\beta_{\mathrm{N}}}$ which satisfies $\chi_{\mathrm{N}} \to \mathrm{T}$ as $\mathrm{N} \to \infty$ (as shown in [13, Appendix A]). We then define

$$\mathbf{c}_{\mathrm{N}} = -2 - 6\chi_{\mathrm{N}}\beta_{\mathrm{N}}$$

which is essentially the *sound mode velocity* at first order in β_{N}. We now introduce the new volume fluctuation field $\widetilde{\mathbf{V}}_t^{\mathrm{N}}(\varphi)$, which is defined on a moving reference frame as follows:

$$\widetilde{\mathbf{V}}_t^{\mathrm{N}}(\varphi) = \mathbb{E}\Big[\mathcal{V}_0^{\mathrm{N}}(\psi) \times \frac{1}{\sqrt{\mathrm{N}}} \sum_{k \in \mathbb{Z}} \varphi\big(\tfrac{k - \mathbf{c}_{\mathrm{N}} t\mathrm{N}^a}{\mathrm{N}}\big)\Big(\omega_k(t\mathrm{N}^a) - \mathfrak{v}_{\beta_{\mathrm{N}}}(\mathrm{T}, 0)\Big)\Big]. \tag{2.33}$$

THEOREM 2.3.2 (Volume fluctuations in the time scale $t\mathrm{N}^a$ with $a > 1$).

Let us fix $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$, and $t > 0$. Let

$$a^*(b) = \begin{cases} 2b+1\,, & \text{if } b \in [0, \frac{1}{2}], \\ 2\,, & \text{if } b \geqslant \frac{1}{2}. \end{cases}$$

For any $b > 0$ two cases hold:

1. *If $a < a^*(b)$, then*

$$\lim_{\mathrm{N}\to\infty} \widetilde{\mathbf{V}}_t^{\mathrm{N}}(\varphi) = \lim_{n\to\infty} \widetilde{\mathbf{V}}_0^{\mathrm{N}}(\varphi) = \mathrm{T} \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\,\mathrm{d}u\mathrm{d}v.$$

2. *If $b > \frac{1}{2}$ and $a = a^*(b) = 2$ then*

$$\lim_{\mathrm{N}\to\infty} \widetilde{\mathbf{V}}_t^{\mathrm{N}}(\varphi) = \mathrm{T} \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\widetilde{\mathrm{P}}_t(u-v)\,\mathrm{d}u\mathrm{d}v,$$

where $\{\widetilde{\mathrm{P}}_t\}_{t\geqslant 0}$ is the semi-group generated by the Laplacian operator Δ.

These two theorems establish the following picture which is summarized in Figure 2.3:

- In the time scale $t\mathrm{N}^a$, $a < 1$, the volume field does not evolve.
- In the hyperbolic time scale $t\mathrm{N}$ ($a = 1$), the initial fluctuations of the volume field are transported with velocity -2.
- We then define a new volume field in a frame moving at velocity $\mathbf{c}_{\mathrm{N}} = -2 - 6\chi_{\mathrm{N}}\beta_{\mathrm{N}}$ which takes into account the first order term in β_{N} of the sound velocity. The new field does not evolve up to time scale $t\mathrm{N}^2$ for $b > \frac{1}{2}$ and up to the time scale $t\mathrm{N}^{2b+1}$ for $0 \leqslant b \leqslant \frac{1}{2}$.
- For $b > \frac{1}{2}$, in the diffusive time scale, the evolution is driven by a heat equation.

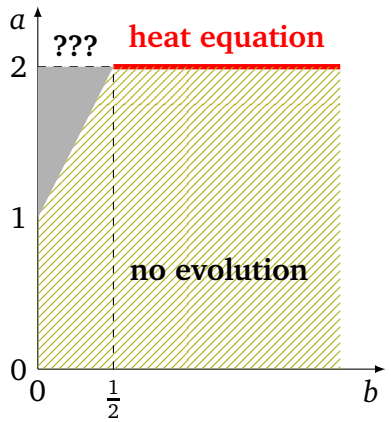

Figure 2.3: VOLUME FLUCTUATIONS: value of the time scale exponent a as a function of the anharmonicity exponent b.

REMARK *2.3.1.* For $0 \leqslant b \leqslant \frac{1}{2}$ we conjecture in fact that the evolution is trivial up to the time scale $t\mathrm{N}^2$ (a proof that there is no evolution in the light gray zone of Figure 2.3 is thus missing). Our conjecture is supported by the following consideration: for $b = 0$, *i.e.* β_{N} of order one, according to Spohn's nonlinear fluctuating hydrodynamics theory [97, 95] and the computations of [13, Appendix A], the fluctuations of the volume field should still belong to the diffusive universality class. Therefore, at $b = 0$, the time scale for which the sound evolution takes place should be $a = 2$. Assuming that the exponent $a := a^*(b)$ of the time scale on which there is evolution of the sound mode is continuous and linear in $b \in [0, \frac{1}{2}]$, we would get that $a^*(b) = 2$ for $b \in [0, \frac{1}{2}]$.

Let us now turn to the macroscopic evolution the energy fluctuation field.

THEOREM 2.3.3 (Energy fluctuations).

Let us fix $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$, and $t > 0$. We have the following two cases:

1. *If $a < \frac{3}{2}$ and $b > \frac{1}{4}$, then the macroscopic energy fluctuation field does not evolve:*

$$\lim_{\mathrm{N}\to\infty} \mathbf{E}_t^{\mathrm{N}}(\varphi) = \lim_{\mathrm{N}\to\infty} \mathbf{E}_0^{\mathrm{N}}(\varphi) = 2\mathrm{T}^2 \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\,\mathrm{d}u\mathrm{d}v.$$

2. *If $a = \frac{3}{2}$ and $b > \frac{1}{4}$, then*

$$\lim_{\mathrm{N}\to\infty} \mathbf{E}_t^{\mathrm{N}}(\varphi) = 2\mathrm{T}^2 \iint_{\mathbb{R}^2} \varphi(u)\psi(v)\mathrm{P}_t(u-v)\,\mathrm{d}u\mathrm{d}v,$$

where $\{\mathrm{P}_t\,;\, t \geqslant 0\}$ is the semi-group generated by the infinitesimal generator of an asymmetric $\frac{3}{2}$–stable Lévy process

$$\mathbb{L} = -\tfrac{1}{\sqrt{2}}\big((-\Delta)^{\frac{3}{4}} - \nabla(-\Delta)^{\frac{1}{4}}\big). \tag{2.34}$$

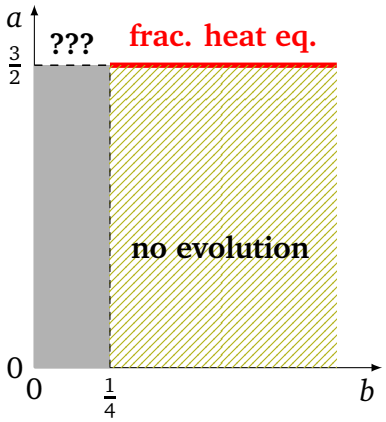

Figure 2.4: ENERGY FLUCTUATIONS: value of the time scale exponent a as a function of the anharmonicity exponent b.

REMARK *2.3.2.* Note that the operator $\mathbb{L}$ in (2.34) is the same as in [14], which treats the case $\beta_N = 0$. This theorem shows that if the nonlinearity is sufficiently weak, *i.e.* $\beta_N = o(N^{-1/4})$, then the energy fluctuation field starts to evolve only in the time scale $tN^{3/2}$ and that in this time scale its evolution is the same as in the linear case ($\beta_N = 0$). Similarly to what we explained in Remark 2.3.1 we expect that the result remains valid for $b \in [0, \frac{1}{4}]$, see Figure 2.4.

2.3.3 Sketch of the proof

a) Volume field

The proof of Theorem 2.3.1 and Theorem 2.3.2 is mainly based on the following differential equation which governs the evolution of the volume fluctuation field:

PROPOSITION 2.3.4. *For any function* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_t^N(\varphi) = \mathbf{V}_t^N\Big(2N^{a-2}\Delta_N\varphi - 2N^{a-1}\,\nabla_N\varphi\Big) + \beta_N\mathbf{V}_t^{3,N}\Big(N^{a-2}\Delta_N\varphi\, -\, 2N^{a-1}\nabla_N\varphi\Big),$$

where

$$\mathbf{V}_t^{3,N}(\varphi) = \mathbb{E}\Big[\mathcal{V}_0^N(\psi) \times \frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}}\varphi\big(\tfrac{k}{N}\big)\,\omega_k^3(tN^a)\Big]. \tag{2.35}$$

Using this last proposition, we obtain that in the case $a < 1$, $\mathbf{V}_t^N(\varphi) = \mathbf{V}_0^N(\varphi) + o(1)$ and no evolution holds.

Taking the hyperbolic time scale ($a = 1$), we get that the evolution of the volume field is such that for any $t \in [0, \overline{t}]$

$$\mathbf{V}_t^N(\varphi) - \mathbf{V}_0^N(\varphi) = -2\int_0^t \mathbf{V}_s^N(\nabla_N\varphi)\mathrm{d}s + o(\beta_N).$$

Thus, in the hyperbolic time scale, the initial fluctuations are transported with a velocity -2, and Theorem 2.3.1 is proved. This result seems to indicate that the sound velocity is -2. In fact this is not totally correct since a more accurate value of the sound velocity is given in [13, Appendix A] and it is equal to -2 only at 0–th order. Taking into account the first order correction in β_N in the sound mode velocity is fundamental in order to establish Theorem 2.3.2: this explains the definition of $\widetilde{\mathbf{V}}_t^N(\varphi)$ as given in (2.33).

The time evolution equation given by Proposition 2.3.4 can be rewritten in the new reference frame as:

$$\widetilde{\mathbf{V}}_t^N(\varphi) - \widetilde{\mathbf{V}}_0^N(\varphi) = \int_0^t \Big(\widetilde{\mathbf{V}}_s^N\Big(N^{a-2}\varphi'' + 6\chi_N\beta_N N^{a-1}\varphi'\Big) - 2N^{a-1}\beta_N\widetilde{\mathbf{V}}_s^{3,N}(\varphi')\Big)\mathrm{d}s + \varepsilon_N(t) \tag{2.36}$$

where $\chi_N = \langle\omega_0^2\rangle_{T,0}^{\beta_N}$, and where we have $\sup_{t\leqslant\overline{t}}\varepsilon_N(t) \to 0$ as $N \to \infty$. An argument based on Cauchy-Schwarz inequality shows that the term $\int_0^t \widetilde{\mathbf{V}}_s^N(N^{a-2}\varphi'')\mathrm{d}s$ vanishes as soon as $a < 2$ but remains if $a = 2$.

Finally, a clever decomposition of the local function $f(\omega) = \omega_0^3$ (see [13, Section 5.2]) shows:

- If $a < a^*(b)$, the term $\mathrm{N}^{a-1}\beta_{\mathrm{N}} \int_0^t \widetilde{\mathbf{V}}_s^{3,\mathrm{N}}(\varphi')\mathrm{d}s$ can be rewritten as

$$3\mathrm{N}^{a-1}\beta_{\mathrm{N}}\chi_{\mathrm{N}} \int_0^t \widetilde{\mathbf{V}}_s^{\mathrm{N}}(\varphi')\,\mathrm{d}s + \widetilde{\varepsilon}_{\mathrm{N}}(t),$$

  with $\mathbb{E}\big[\big(\mathrm{N}^{a-1}\beta_{\mathrm{N}}\,\widetilde{\varepsilon}_{\mathrm{N}}(t)\big)^2\big] \to 0$ as $\mathrm{N} \to \infty$.
  With the constant prefactor 2 this transport term cancels the term $6\chi_{\mathrm{N}}\beta_{\mathrm{N}}\mathrm{N}^{a-1}\widetilde{\mathbf{V}}_s^{\mathrm{N}}(\varphi')$ appearing in (2.36).
- If $a = 2$ and $b > \frac{1}{2}$, only the term $\int_0^t \widetilde{\mathbf{V}}_s^{\mathrm{N}}(\varphi'')\mathrm{d}s$ survives. All the other terms give a zero contribution.

More details are available in [13, Section 5].

b) Energy field

The general strategy introduced in [14] and further used in several works (in particular, in [12] presented in the next section), is based on the introduction of a *bidimensional (quadratic) correlation field*, defined for any $\mathrm{H} \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, as follows:

$$\mathbf{Q}_t^{2,\mathrm{N}}(\mathrm{H}) = \mathbb{E}\Big[\mathcal{E}_0^{\mathrm{N}}(\psi) \times \frac{1}{\mathrm{N}}\sum_{k\neq\ell} \mathrm{H}\big(\tfrac{k}{\mathrm{N}}, \tfrac{\ell}{\mathrm{N}}\big)\omega_k\omega_\ell(t\mathrm{N}^a)\Big], \tag{2.37}$$

see also (2.40) and Section 2.4.3 below. However, since the underlying model here is *nonlinear* the time evolution of the pair (energy field; quadratic field) is not closed (contrary to the previous works) and we have to deal with some hierarchy. This is the main difference with previous studies ([14, 11, 12, 72]) whose success was very dependent of this closeness due to the linear interactions.

Therefore, in [13, Section 3.2] we introduce three additional auxiliary fields which will appear in the time evolution equation satisfied by $\mathbf{E}_t^{\mathrm{N}}(\varphi)$, and we estimate the contributions of each of them separately, by means of Cauchy-Schwartz arguments, Kipnis-Varadhan inequalities and resolution of a Poisson equation. This procedure is quite technical: more details are available in [13, Section 6.1].

2.4 Interpolation between standard & anomalous diffusion

With respect to the previous model, we add to the simplified dynamics a second stochastic interaction which conserves only the energy and we scale down the strength of this second noise by $\gamma_{\mathrm{N}} \to 0$. The general idea of this assumption is the following: if the total generator $\mathcal{L}$ of the microscopic Markov process can be decomposed as $\mathcal{L} = \mathcal{L}_{\mathrm{Diff}} + \mathcal{L}_{\mathrm{SupDiff}}$, where $\mathcal{L}_{\mathrm{Diff}}$ generates a diffusion (*e.g.* the exchange noise $\mathcal{S}$) and $\mathcal{L}_{\mathrm{SupDiff}}$ evolves in a superdiffusive time scale (for instance $\widetilde{\mathcal{S}}$), then the crossover is obtained by accelerating the time differently

for both parts, *i.e.*: instead of considering the accelerated process generated by $N^a\mathcal{L}$, look at $N^2\mathcal{L}_{\text{Diff}} + N^b\mathcal{L}_{\text{SupDiff}}$, with b wisely chosen, so that both generators interact at the same level and contribute equally to the macroscopic limit. In particular, we prove in [12] – written in collaboration with C. Bernardin, P. Gonçalves and M. Jara – and explain in this section, that for a particular scaling parameter b, the macroscopic energy fluctuations follow an evolution equation of the form $\partial_t u = \mathbb{L}_c u$, where $\mathbb{L}_c$ has a Fourier representation[2] of the form

$$\widehat{\mathbb{L}_c}(\xi) = -\frac{4\pi^2\xi^2}{\sqrt{c + 2i\pi\xi}}.$$

In particular, note that $\mathbb{L}_c \to -c\left\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\right\}$ as $c \to 0$ and $\sqrt{c}\mathbb{L}_c \to \Delta$ as $c \to \infty$, providing in this way a crossover between anomalous and normal diffusion of energy. Note as well that the interpolation between the fractional and normal Laplacians can be understood as an ultraviolet cut-off at modes of order $\mathcal{O}(c)$: low modes behave diffusively, while high modes behave superdiffusively.

In [11] (written by us plus M. Sasada during my PhD thesis), another version of this model was considered. An almost complete phase diagram was obtained, although the interpolating part of the diagram described here was missing. However, the methods presented in [12] allow to complete the phase diagram in [11]. Moreover, with respect to [14] and [11], the model considered here has a Brownian noise instead of a Poissonian noise. At the level of the correlation function, the choice of a Poissonian or a Brownian noise does not make a sensitive difference. However, at the level of the Gaussian fluctuations, key tightness estimates do not hold for Poissonian noises due to rare events that may introduce huge discontinuities on the observables we are interested in. We believe that at the level of finite-dimensional distributions the same process still describes the scaling limit of energy fluctuations in the model, even with the Poissonian noise considered in [11]. However, it is not clear whether the obstructions in order to prove tightness are technical or intrinsic to those kind of noises.

2.4.1 The model

Let me now define more precisely the model under investigation. For that purpose let $\lambda > 0$ and $\gamma_N > 0$ (depending on the scale parameter $N \in \mathbb{N}$). We consider the system of diffusions evolving in infinite volume, on the state space $\Omega = \mathbb{R}^{\mathbb{Z}}$. For the sake of clarity we place ourselves in the time scale where the crossover occurs, which turns out to be the time scale $N^{3/2}$. Other time scales can be considered, see [12, 11]. Therefore, the process is generated by the operator $N^{3/2}\mathcal{L}_N$, where $\mathcal{L}_N$ is decomposed as the sum $\mathcal{L}_N = \mathcal{A} + \lambda\mathcal{S}_1 + \gamma_N\mathcal{S}_2$, where

$$\mathcal{A} = \sum_{k\in\mathbb{Z}} \big(\omega_{k+1} - \omega_{k-1}\big)\frac{\partial}{\partial\omega_k}$$
$$\mathcal{S}_1 = \sum_{k\in\mathbb{Z}} (\mathcal{Z}_k \circ \mathcal{Z}_k), \qquad \mathcal{S}_2 = \sum_{k\in\mathbb{Z}} (\mathcal{Y}_k \circ \mathcal{Y}_k),$$

[2] Remember that the *Fourier transform* has been defined in (1.5).

with the family of operators $\{\mathcal{Z}_k, \mathcal{Y}_k\}_{k\in\mathbb{Z}}$ as defined in Section 2.1: recall that they are given by

$$\mathcal{Z}_k = (\omega_{k+1}-\omega_k)\frac{\partial}{\partial\,\omega_{k-1}} + (\omega_k-\omega_{k-1})\frac{\partial}{\partial\,\omega_{k+1}} + (\omega_{k-1}-\omega_{k+1})\frac{\partial}{\partial\,\omega_k},$$
$$\mathcal{Y}_k = \omega_{k+1}\frac{\partial}{\partial\,\omega_k} - \omega_k\frac{\partial}{\partial\,\omega_{k+1}}.$$

The Markov process generated by the accelerated operator $\mathrm{N}^{3/2}\mathcal{L}_\mathrm{N}$ is denoted by $\omega^\mathrm{N}(t) = \{\omega_k^\mathrm{N}(t)\,;\,k\in\mathbb{Z}\}$. This diffusion has a family $\{\mu_\mathrm{T}\,;\,\mathrm{T}>0\}$ of invariant measures given by the infinite volume Gibbs product measures

$$\mu_\mathrm{T}(\mathrm{d}\omega) = \prod_{k\in\mathbb{Z}}\sqrt{\frac{1}{2\pi\mathrm{T}}}\exp\Big(-\frac{\omega_k^2}{2\mathrm{T}}\Big)\,\mathrm{d}\omega_k,$$

where as before T represents the temperature, and we denote by $\langle f\rangle_\mathrm{T}$ the average of $f:\Omega\to\mathbb{R}$ with respect to μ_T.

Once again, the law of the process $\{\omega_k^\mathrm{N}(t)\,;\,k\in\mathbb{Z}\}_{t\geqslant 0}$ starting from the invariant measure μ_T is denoted by $\mathbb{P}$, and the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. Under μ_T, the averaged energy per site equals $\langle\omega_k^2\rangle_\mathrm{T} = \mathrm{T}$, and the averaged volume per site equals $\langle\omega_k\rangle_\mathrm{T} = 0$.

2.4.2 Statement of the main results

We assume that the strength of the second noise scales as

$$\gamma_\mathrm{N} = \frac{c}{\mathrm{N}} \tag{2.38}$$

for some $c>0$. The general case $\gamma_\mathrm{N} = c/\mathrm{N}^b$, $b\geqslant 0$ is not written in [12] but can be easily adapted from [11], using the same methods.

The *energy fluctuation field* $\mathcal{E}_t^\mathrm{N}$ is defined as the distribution–valued process given for any $\varphi:\mathbb{R}\to\mathbb{R}$ in the usual Schwartz space $\mathcal{S}(\mathbb{R})$ of test functions by

$$\mathcal{E}_t^\mathrm{N}(\varphi) = \frac{1}{\sqrt{\mathrm{N}}}\sum_{k\in\mathbb{Z}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(\big(\omega_k^\mathrm{N}(t)\big)^2-\mathrm{T}\Big). \tag{2.39}$$

For fixed t and φ, the random variables $\mathcal{E}_t^\mathrm{N}(\varphi)$ satisfy a central limit theorem: they converge to a centered normal random variable of variance $2\mathrm{T}^2\|\varphi\|_2^2$, where $\|\cdot\|_2$ denotes the usual norm of the Hilbert space $\mathbf{L}^2(\mathbb{R})$.

Our main goal is to obtain a convergence result for the $\mathcal{S}'(\mathbb{R})$-valued process $\{\mathcal{E}_t^\mathrm{N}\}_{t\geqslant 0}$. As noted in [14], it turns out that the analysis of the *correlation field*

$$\mathcal{C}_t^\mathrm{N}(\mathrm{H}) = \frac{1}{\mathrm{N}^{3/4}}\sum_{k,\ell\in\mathbb{Z}}\mathrm{H}\big(\tfrac{k+\ell}{2\mathrm{N}},\tfrac{|\ell-k|}{\sqrt{\mathrm{N}}}\big)\Big(\omega_k^\mathrm{N}(t)\omega_\ell^\mathrm{N}(t)-\delta_{k,\ell}\mathrm{T}\Big) \tag{2.40}$$

will play a fundamental role on the derivation of the scaling limit of $\mathcal{E}_t^\mathrm{N}$. Above, $\mathrm{H}:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ is taken as a smooth function. The non-isotropic scaling is crucial in order to see the scaling

limit of $\mathcal{E}_t^{\mathrm{N}}$. Note that, contrary to the previous section, we are able to prove a convergence result for the process $\{\mathcal{E}_t^{\mathrm{N}}\}_{t\geqslant 0}$ and not only for its space-time correlations.

In order to state the main result, we now define the expected macroscopic operator $\mathbb{L}_c$ as follows. For any $\varphi \in \mathcal{S}(\mathbb{R})$, we define $\mathbb{L}_c\varphi$ via the action of the operator $\mathbb{L}_c$ on Schwartz spaces: precisely, the operator $\mathbb{L}_c$ acts on the Fourier transform of φ as:

$$\widehat{\mathbb{L}_c\varphi}(\xi) = \frac{1}{2\sqrt{3\lambda}} \frac{(2i\pi\xi)^2}{\sqrt{c+i\pi\xi}} \widehat{\varphi}(\xi), \qquad \xi \in \mathbb{R}. \tag{2.41}$$

This operator has nice properties, stated in the next proposition:

PROPOSITION 2.4.1. *The operator $\mathbb{L}_c$ is the generator of a Lévy process. It leaves the space $\mathcal{S}(\mathbb{R})$ invariant, and its Lévy-Khintchine representation is given by*

$$(\mathbb{L}_c\varphi)(u) = \int_{\mathbb{R}} \Big(\varphi(u-y) - \varphi(u) + y\varphi'(u)\Big)\Pi_c(\mathrm{d}y), \tag{2.42}$$

where Π_c is the measure on $\mathbb{R}$ defined by

$$\Pi_c(\mathrm{d}y) = -\frac{4c^{5/2}}{\sqrt{6\lambda\pi}}\, e^{-2cy}\left(\frac{3}{16(cy)^{5/2}} + \frac{1}{2(cy)^{3/2}} + \frac{1}{(cy)^{1/2}}\right)\mathbf{1}_{(0,+\infty)}(y)\mathrm{d}y. \tag{2.43}$$

We show in [12] that $\mathbb{L}_c\varphi$ can equivalently be defined as follows: for any $u \in \mathbb{R}$,

$$(\mathbb{L}_c\varphi)(u) = -2\partial_u \mathrm{F}(u,0), \tag{2.44}$$

where $\mathrm{F} : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is the function such that its Fourier transform with respect to its first variable, namely

$$\mathcal{F}_\xi(v) = \int_{\mathbb{R}} e^{-2i\pi u\xi}\mathrm{F}(u,v)\mathrm{d}u, \qquad \xi \in \mathbb{R}, v \geqslant 0,$$

is given by

$$\mathcal{F}_\xi(v) = -\frac{1}{4\sqrt{3\lambda}} \frac{(2i\pi\xi)\widehat{\varphi}(\xi)}{\sqrt{c+i\pi\xi}} \exp\left(-\sqrt{\frac{c+i\pi\xi}{3\lambda}}v\right), \qquad v \geqslant 0. \tag{2.45}$$

The function F defined in this way satisfies the integrability conditions

$$\int_{\mathbb{R}\times\mathbb{R}^+} \mathrm{F}^2(u,v)\,\mathrm{d}u\mathrm{d}v < \infty \quad \text{and} \quad \int_{\mathbb{R}\times\mathbb{R}^+} (\partial_v\mathrm{F})^2(u,v)\,\mathrm{d}u\mathrm{d}v < \infty. \tag{2.46}$$

Moreover the function F is solution of the Laplace equation

$$\begin{cases} \big(6\lambda\partial^2_{vv}\mathrm{F} - \partial_u\mathrm{F} - 2c\mathrm{F}\big)(u,v) = 0, & \text{for } u \in \mathbb{R}, v > 0, \\ 12\lambda\partial_v\mathrm{F}(u,0) = \varphi'(u), & \text{for } u \in \mathbb{R}. \end{cases} \tag{2.47}$$

Let $\mathbb{L}_c^\star$ be the adjoint of $\mathbb{L}_c$ in $\mathbf{L}^2(\mathbb{R})$ and $\mathfrak{S} = \frac{1}{2}(\mathbb{L}_c + \mathbb{L}_c^\star)$ be its symmetric part. Let us fix a time horizon $\overline{t} > 0$. We now explain the meaning of a *stationary solution of the infinite dimensional generalized Ornstein-Uhlenbeck equation driven by* $\mathbb{L}_c$, written formally as follows:

$$\partial_t\mathcal{E}_t = \mathbb{L}_c^\star(\mathcal{E}_t) + \sqrt{2\mathrm{T}^2(-\mathfrak{S})}\,\dot{\mathcal{W}}_t, \tag{2.48}$$

where $\{\dot{\mathcal{W}}_t\}_{t\in[0,\overline{t}]}$ is a $\mathcal{S}'(\mathbb{R})$–valued space time white noise.

DEFINITION 2.4.1. *We say that an $\mathcal{S}'(\mathbb{R})$-valued process $\{\mathcal{E}_t\}_{t\in[0,\overline{t}]}$ is* T-stationary *if, for any time $t \in [0,\overline{t}]$, the $\mathcal{S}'(\mathbb{R})$-valued random variable $\mathcal{E}_t$ is a white noise (in space) of variance $2\mathrm{T}^2$, namely: for any $\varphi \in \mathcal{S}(\mathbb{R})$, the real-valued random variable $\mathcal{E}_t(\varphi)$ has a normal distribution of mean zero and variance $2\mathrm{T}^2\|\varphi\|_2^2$.*

PROPOSITION 2.4.2. *There exists a unique (in law) stochastic process $\{\mathcal{E}_t\}_{t\in[0,\overline{t}]}$ with trajectories in $\mathcal{C}([0,\overline{t}],\mathcal{S}'(\mathbb{R}))$ called* stationary solution of (2.48) *if:*

1. *$\{\mathcal{E}_t\}_{t\in[0,\overline{t}]}$ is* T*-stationary;*
2. *for any differentiable function $\varphi : [0,\overline{t}] \to \mathcal{S}(\mathbb{R})$, the process*

$$\mathcal{E}_t(\varphi_t) - \mathcal{E}_0(\varphi_0) - \int_0^t \mathcal{E}_s\big((\partial_s + \mathbb{L}_c)\varphi_s\big)\,\mathrm{d}s$$

is a continuous martingale with respect to the natural filtration associated to $\mathcal{E}_\cdot$, namely

$$\sigma\big(\mathcal{E}_s(\varphi)\,;\, s \leqslant t, \varphi \in \mathcal{S}^{\text{test}}\big),$$

of quadratic variation

$$2\mathrm{T}^2 \int_0^t \int_{\mathbb{R}} \varphi_s(u)(-\mathfrak{S}\varphi_s)(u)\,\mathrm{d}u\mathrm{d}s.$$

Proof. Thanks to the fact that $\mathbb{L}$ is the generator of a Lévy process, the same argument used in [53, Appendix B] can be worked out here to prove the uniqueness of such solutions.

Roughly speaking, the main result of [12] states that the energy fluctuations described by $\mathcal{E}_t^{\mathrm{N}}$ (defined in (2.39)) satisfy an approximate martingale problem, which, in the limit $\mathrm{N} \to \infty$ becomes the martingale characterization of the limiting process described in Proposition 2.4.2. It can be precisely formulated as follows:

THEOREM 2.4.3. *The sequence of processes $\{\mathcal{E}_t^{\mathrm{N}}\,;\, t \in [0,\overline{t}]\}_{\mathrm{N}\in\mathbb{N}}$ converges in law as $\mathrm{N} \to \infty$ to the stationary solution of the infinite dimensional Ornstein-Uhlenbeck process given by (2.48), with respect to the weak topology of $\mathcal{C}([0,\overline{t}],\mathcal{S}'(\mathbb{R}))$.*

2.4.3 Sketch of the proof

Our proof of the convergence of the energy fluctuation field (Theorem 2.4.3) follows the usual scheme of convergence in law of stochastic processes: we show tightness of the processes $\mathcal{E}_t^{\mathrm{N}}$ in a suitable topology, then we prove that any limit point of the sequence $\{\mathcal{E}_t^{\mathrm{N}}\}_{\mathrm{N}\in\mathbb{N}}$ satisfies a weak formulation of the equation (2.48) and then we rely on the uniqueness result given by Proposition 2.4.2.

One technical difficulty comes from what is known in the literature as the *Replacement Lemma*: it is not very difficult to write down a martingale decomposition for $\mathcal{E}_t^{\mathrm{N}}$ that should heuristically converge to the martingale problem associated to $\mathcal{E}_t$. However the drift term of this martingale decomposition involves the energy current $\omega_k^{\mathrm{N}}(t)\omega_{k+1}^{\mathrm{N}}(t)$. This current is *not* a

function of the energy and therefore we say that the martingale problem for $\mathcal{E}_t^{\mathrm{N}}$ is not closed. To overcome that difficulty, we need to replace the current $\omega_k^{\mathrm{N}}(t)\omega_{k+1}^{\mathrm{N}}(t)$ by a function of the energy. This is accomplished by studying the relation between the energy fluctuations and the fluctuations of the *correlation field* given by (2.40), which is one of the main conceptual innovations in [14].

More precisely, fix $\varphi \in \mathcal{S}(\mathbb{R})$ and let F be defined as in (2.44), which in particular satisfies the Laplace equation (2.47). With this definition, a straightforward computation (see [12, Section 3.1]) shows the following martingale decomposition:

$$\mathcal{E}_t^{\mathrm{N}}(\varphi) - \mathcal{E}_0^{\mathrm{N}}(\varphi) = -\int_0^t \sum_{k\in\mathbb{Z}} \omega_k^{\mathrm{N}}(s)\omega_{k+1}^{\mathrm{N}}(s)(24\lambda)\partial_v \mathrm{F}\big(\tfrac{k}{\mathrm{N}}, 0\big)\mathrm{d}s + \mathcal{M}_{t,\mathrm{N}}^{\mathcal{E}}(\varphi) + \int_0^t \varepsilon_{\mathrm{N}}(s)\mathrm{d}s, \quad (2.49)$$

where $\mathcal{M}_{t,\mathrm{N}}^{\mathcal{E}}(\varphi)$ is a martingale whose quadratic variation vanishes as $\mathrm{N} \to \infty$, and ε_{N} satisfies

$$\lim_{\mathrm{N}\to\infty} \mathbb{E}\Big[\Big(\int_0^t \varepsilon_{\mathrm{N}}(s)\mathrm{d}s\Big)^2\Big] = 0, \qquad \text{and} \qquad \lim_{\mathrm{N}\to\infty} \sup_{t\in[0,\bar{t}]} \mathbb{E}\Big[\big|\varepsilon_{\mathrm{N}}(t)\big|^2\Big] < +\infty. \quad (2.50)$$

The presence of the first term in the right hand side of (2.49) explains why the correlation field (2.40) plays an important role, in particular when $\mathrm{H} = \mathrm{F}$. Writing in the same way the martingale decomposition for $\mathcal{C}_t^{\mathrm{N}}(\mathrm{F})$, and using the Laplace equation (2.47), one obtains (after several replacements, see [12, Section 3.3]) the following:

$$\mathcal{E}_t^{\mathrm{N}}(\varphi) - \mathcal{E}_0^{\mathrm{N}}(\varphi) = -2\int_0^t \mathcal{E}_s^{\mathrm{N}}\big(\partial_u \mathrm{F}(\cdot, 0)\big)\mathrm{d}s + \widetilde{\mathcal{M}}_{t,\mathrm{N}} + \mathcal{N}_{t,\mathrm{N}} + \int_0^t \widetilde{\varepsilon}_{\mathrm{N}}(s)\mathrm{d}s, \quad (2.51)$$

where

(i) $\widetilde{\mathcal{M}}_{t,\mathrm{N}}$ is a martingale whose quadratic variation converges, in mean, towards

$$2t\mathrm{T}^2 \int_{\mathbb{R}\times\mathbb{R}^+} \big(8c\mathrm{F}^2 + 24\lambda(\partial_v \mathrm{F})^2\big)(u, v)\mathrm{d}u\mathrm{d}v = 2t\mathrm{T}^2 \int_{\mathbb{R}} \varphi(u)(-\mathfrak{S}\varphi)(u)\mathrm{d}u,$$

with $\mathfrak{S}$ the symmetric part of $\mathbb{L}_c$;

(ii) $\mathcal{N}_{t,\mathrm{N}}$ is a martingale whose quadratic variation vanishes ;

(iii) $\widetilde{\varepsilon}_{\mathrm{N}}$ satisfies (2.50).

Recalling that, by definition (2.44) we have $\partial_u \mathrm{F}(\cdot, 0) = \mathbb{L}_c\varphi$, and using the characterization of the limit process given in Proposition 2.4.2, the conclusion of Theorem 2.4.3 follows after proving tightness of each process which appears in the above decomposition (2.51). This is done in [12, Section 4].

CHAPTER 3 The weak KPZ universality conjecture

This chapter is based on [23, 43, 58, 60]*, written in collaboration with O. Blondel, T. Franco, P. Gonçalves, M. Jara and N. Perkowski. Some parts are largely borrowed from these papers.*

Contents

3.1 Introduction

The first breakthrough towards the weak KPZ universality conjecture is due to Bertini and Giacomin: in their seminal paper [19], they show that the so-called *Cole-Hopf solution*[1] can be obtained as a scaling limit of the *weakly asymmetric exclusion process* (briefly described in Section 1.2). Their approach consists in performing the Cole-Hopf transformation at the microscopic level, following [50], and then showing that this microscopic transformation solves a linear equation (similarly to what happens at the macroscopic level). Since then, this strategy has been used in more and more sophisticated models (see [29, 30, 31, 33] for a non-exhaustive list of related works), however its applicability is limited to a specific class of particle systems.

3.1.1 Stochastic Burgers equation and energy solutions

More precisely, another way to look at the KPZ equation is via the *stochastic Burgers equation* (SBE), which is obtained from (1.1) by taking its derivative: if $\mathcal{Y}_t = \nabla h_t$, then $\mathcal{Y}_t$ satisfies

$$\mathrm{d}\mathcal{Y}(t,u) = \sigma\Delta\mathcal{Y}(t,u)\,\mathrm{d}t + \overline{\lambda}\nabla\big(\mathcal{Y}^2(t,u)\big)\,\mathrm{d}t + \sqrt{\mathrm{D}}\,\nabla\mathrm{d}\mathcal{W}_t, \tag{3.1}$$

[1] The so-called *Cole-Hopf transform* is originally known to be very useful in the study of the Burgers (deterministic KPZ) equation: this transformation turns the *nonlinear* Burgers equation into the *linear* heat conduction equation.

which has of course the same regularity issues as the KPZ equation. Nevertheless, this formulation is well adapted to derive KPZ behaviour from microscopic models. Indeed, the work initiated by Gonçalves and Jara in [55] has introduced a new tool, called *second order Boltzmann-Gibbs principle*, which makes the nonlinear term $\nabla(\mathcal{Y}^2(t,u))$ of the SBE naturally emerge from the underlying microscopic dynamics: the suitably rescaled conservative field subsequentially converges, as the size of the microscopic system goes to infinity, to random fields which are solutions $\mathcal{Y}$ of a generalized martingale problem for (3.1), where the singular nonlinear drift $\nabla(\mathcal{Y}^2(t,u))$ is a well-defined space time distributional random field. Gonçalves and Jara in [55] (see also [63]) called them *energy solutions*.

Recently, Gubinelli and Perkowski [65] proved uniqueness of these energy solutions. As a significant consequence, the proof of the weak KPZ universality conjecture could be concluded for many new models. The notion of energy solutions, compared to the methods of [19] or [66, 62], is strongly based on *stationarity*. In particular, this approach towards the weak KPZ universality has so far been worked out for stationary initial conditions or bounded entropy perturbations thereof. On the other hand, number of models do not admit a microscopic Cole-Hopf transformation, which prevents the use of the methods of [19], and in many cases they do not have the structure of a semilinear SPDE, which means that the pathwise approach of [66, 62] does not apply either.

3.1.2 Microscopic tools

At the microscopic level, the classical ("first order") Boltzmann-Gibbs principle, introduced by Brox and Rost in [25], states that the space time fluctuations of any local field associated to a conservative model can be written as a linear functional of the conservative field. Going one step further, the second order Boltzmann-Gibbs principle introduced in [55] investigates the first order correction to this limit: it is given by a quadratic functional of the conservative field. The error estimates are sharp enough in order to make sense of the singular term $\mathcal{Y}^2(t,u)$. The initial proof of [55] was based on a *multiscale analysis*, assuming that the underlying particle system is of exclusion type and for which a spectral gap inequality holds. Then, it has been extended to other dynamics, for example zero-range models [57], non-simple exclusion processes [56], semilinear SPDEs [64], or interacting diffusions [37].

3.1.3 Outline of the chapter

Let me now outline the works which will be exposed in this chapter: in [58] we give a more general proof of the second order Boltzmann-Gibbs principle which relies on a proper decomposition of the antisymmetric part of the current of the system in terms of polynomial functions. This new proof works for instance for integrable Hamiltonian one-dimensional chains [58]. Then, we modified and adapted it to kinetically constrained models [23], and exclusion processes with defects [43]. Finally, we have been looking for a rigorous microscopic derivation of the KPZ equation with boundary conditions, from an interacting particle system in contact

with stochastic reservoirs. This is more subtle than expected, because the boundary conditions do not behave canonically, as already noted in [51]. For that purpose, we first need to extend the notion of energy solutions to the SBE (3.1) by adding Dirichlet boundary conditions, and prove existence and uniqueness of such solutions. Then, we need to modify the proof of the second order Boltzmann-Gibbs principle to take into account the boundaries and the finite microscopic lattice.

In Section 3.2 I will give a rigorous definition of the stationary energy solutions to (3.1) in a quite general setting, so that it will fit all the mentioned works, and I will state the corresponding existence/uniqueness result (Theorem 3.2.2). Then, in Section 3.3 I will present the new proof (in its first version) of the second order Boltzmann-Gibbs principle given in [58], and in Section 3.4 I will explain how these different results have been adapted to three degenerate contexts in [23, 43, 60].

3.2 Notion of solutions for two SPDEs

All the microscopic models considered in this chapter present a *weak asymmetry* of strength $\propto N^{-\gamma}$ regulated by a parameter $\gamma > 0$. Depending on the range of γ, at the macroscopic level, we are able to see two important SPDEs arising: the Ornstein-Uhlenbeck process, and the stochastic Burgers equation. For both of them, we need to define what we intend by a solution, and to prove that this solution is unique.

Since at the end of this chapter I will present three different models with degenerate features, let me define here these solutions in a somewhat general context. Precisely, throughout this section we consider a *space* $\mathcal{S}^{\text{test}} \subset \mathcal{C}^\infty(X) \cap \mathbf{L}^2(X)$ *of real-valued test functions* $\varphi : X \to \mathbb{R}$ provided with a topology and $(\mathcal{S}^{\text{test}})'$ its topological dual, *i.e.* the set of linear functionals on $\mathcal{S}^{\text{test}}$ which are continuous in the topology of $\mathcal{S}^{\text{test}}$. In our examples we will take $X = [0,1]$ or $\mathbb{R}$, and the definition of $\mathcal{S}^{\text{test}}$ may involve boundary conditions (see Section 3.4).

Now, let $-\Delta$ be the closure of the Laplacian operator $-\Delta : \mathcal{S}^{\text{test}} \to \mathbf{L}^2(X)$ as an unbounded operator in the Hilbert space $\mathbf{L}^2(X)$ whose eigenvalues are $\lambda_m = (m\pi)^2$ and the associated eigenfunctions form an orthonormal basis of $\mathbf{L}^2(X)$. Assume that $\Delta(\mathcal{S}^{\text{test}}) \subset \mathcal{S}^{\text{test}}$ (which will be always the case in our examples), so that we can define Δ^{test} as the Laplacian acting on $\mathcal{Y} \in (\mathcal{S}^{\text{test}})'$ as follows:

$$\Delta^{\text{test}}\mathcal{Y}(\varphi) = \mathcal{Y}(\Delta\varphi), \qquad \text{for any } \varphi \in \mathcal{S}^{\text{test}}. \tag{3.2}$$

We denote by $T_t^{\text{test}} : \mathcal{S}^{\text{test}} \to \mathcal{S}^{\text{test}}$ the semi-group associated to Δ. Finally, we will also need to consider the *gradient* acting on some linear functional $\mathcal{Z}$ as follows:

$$\nabla^{\text{test}}\mathcal{Z}(\varphi) = -\mathcal{Z}(\nabla\varphi), \qquad \text{for any } \varphi \in \mathcal{S}^{\text{test}}. \tag{3.3}$$

In our examples (of Section 3.4) it may happen that $\nabla\varphi$ does not belong to $\mathcal{S}^{\text{test}}$ but in another topological space $\widetilde{\mathcal{S}}^{\text{test}}$, therefore in (3.3) $\mathcal{Z}$ needs to be taken as an element of $(\widetilde{\mathcal{S}}^{\text{test}})'$. In any case, whenever $\mathcal{S}^{\text{test}}$ and $\widetilde{\mathcal{S}}^{\text{test}}$ will be made explicit, it will be clear that (3.2) and (3.3) perfectly make sense.

3.2.1 Ornstein-Uhlenbeck process

We are now ready to give a well defined notion of solution to the *generalized Ornstein-Uhlenbeck process* described by the formal SPDE

$$\mathrm{d}\mathcal{Y}_t = \sigma\Delta^{\mathrm{test}}\mathcal{Y}_t\,\mathrm{d}t + \sqrt{\mathrm{D}}\,\nabla^{\mathrm{test}}\mathrm{d}\mathcal{W}_t, \tag{3.4}$$

with $\sigma, \mathrm{D} > 0$ and where $\{\mathcal{W}_t\}_{t\in[0,\overline{t}]}$ is a standard Brownian motion, with covariance given for any $\varphi, \psi \in \mathcal{S}^{\mathrm{test}}$ by

$$\mathbb{E}\big[\mathcal{W}_s(\varphi)\mathcal{W}_t(\psi)\big] = (s\wedge t)\langle\varphi,\psi\rangle_{\mathrm{L}^2(\mathrm{X})}, \tag{3.5}$$

and such that $\nabla^{\mathrm{test}}\mathcal{W}_t$ is a well defined object in $(\mathcal{S}^{\mathrm{test}})'$.

The following proposition aims at defining in a unique way the stochastic process solution to (2.48) (recall also Proposition 2.4.2 of Chapter 2 for a similar definition):

Proposition 3.2.1. *There exists a unique (in law) stochastic process $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$, with trajectories in $\mathcal{C}([0,\overline{t}],(\mathcal{S}^{\mathrm{test}})')$, called* Ornstein-Uhlenbeck process solution of (3.4)*, such that:*

1. *the process $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$ is said* stationary*, in the following sense:*
   for any $t\in[0,\overline{t}]$, the random variable $\mathcal{Y}_t$ is a $(\mathcal{S}^{\mathrm{test}})'$–valued space white noise with covariance given on $\varphi,\psi\in\mathcal{S}^{\mathrm{test}}$ by

$$\mathbb{E}\big[\mathcal{Y}_t(\varphi)\mathcal{Y}_t(\psi)\big] = \frac{\mathrm{D}}{2\sigma}\langle\varphi,\psi\rangle_{\mathrm{L}^2(\mathrm{X})}\,;$$

2. *for any $\varphi\in\mathcal{S}^{\mathrm{test}}$,*

$$\mathcal{M}_t(\varphi) = \mathcal{Y}_t(\varphi) - \mathcal{Y}_0(\varphi) - \sigma\int_0^t \mathcal{Y}_s(\Delta\varphi)\,\mathrm{d}s$$

   is a continuous martingale with respect to the natural filtration associated to $\mathcal{Y}_\cdot$, namely

$$\mathcal{F}_t = \mathfrak{S}\big(\mathcal{Y}_s(\varphi)\,;\, s\leqslant t,\ \varphi\in\mathcal{S}^{\mathrm{test}}\big), \tag{3.6}$$

   of quadratic variation

$$\big\langle\mathcal{M}(\varphi)\big\rangle_t = t\mathrm{D}\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathrm{X})}.$$

Moreover, for every function $\varphi\in\mathcal{S}^{\mathrm{test}}$, the stochastic process $\{\mathcal{Y}_t(\varphi)\}_{t\in[0,\overline{t}]}$ is Gaussian, and the distribution of $\mathcal{Y}_t(\varphi)$ conditionally to $\{\mathcal{F}_u\,;\,u<s\}$, is normal of mean $\mathcal{Y}_s(\mathrm{T}^{\mathrm{test}}_{\sigma(t-s)}\,\varphi)$ and variance

$$\mathrm{D}\int_0^{t-s}\big\|\nabla(\mathrm{T}^{\mathrm{test}}_{\sigma r}\,\varphi)\big\|^2_{\mathrm{L}^2(\mathrm{X})}\,\mathrm{d}r.$$

Proof of Proposition 3.2.1. In all the specific examples of [58, 23, 43, 60], we do not need to prove this statement, since it is quite standard, and can be found in [71, 77].

3.2.2 Stochastic Burgers equation

Now we define the notion of *stationary energy solutions* for the *stochastic Burgers equation*, formally written as

$$\mathrm{d}\mathcal{Y}_t = \sigma\Delta^{\mathrm{test}}\,\mathcal{Y}_t\,\mathrm{d}t + \overline{\lambda}\nabla^{\mathrm{test}}\left(\mathcal{Y}_t^2\right)\mathrm{d}t + \sqrt{\mathrm{D}}\,\nabla^{\mathrm{test}}\,\mathrm{d}\mathcal{W}_t, \tag{3.7}$$

with $\sigma, \mathrm{D} > 0$ and $\overline{\lambda} \in \mathbb{R}$, where $\{\mathcal{W}_t\}_{t\in[0,\overline{t}]}$ is a Brownian motion with covariance (3.5), and such that $\nabla^{\mathrm{test}}\mathcal{W}_t$ is a well defined object in $(\mathcal{S}^{\mathrm{test}})'$.

Note that $\nabla^{\mathrm{test}}\left(\mathcal{Y}_t^2\right)$ so far does not have a precise meaning but this will be rigorously given below in Theorem 3.2.2. For any $u \in \mathrm{X}$, let $\iota_\varepsilon(u)$ ($\varepsilon > 0$) be an approximation as $\varepsilon \to 0$ of the Delta function at point u[2].

The following theorem aims at defining uniquely the stochastic process solution to (3.7), as given for the first time in [55, 63]. This result is not true in general: we need to specify $\mathcal{S}^{\mathrm{test}}$ and to be very careful about the definitions of Δ^{test} and ∇^{test}. Nevertheless let me state it as a theorem, and give right away the references where it has been proved exactly in this form:

THEOREM 3.2.2. *There exists a unique (in law) stochastic process $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$ with trajectories in $\mathcal{C}([0,\overline{t}],(\mathcal{S}^{\mathrm{test}})')$, called* stationary energy solution *of (3.7), such that:*

1. *the process $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$ is said* stationary*, in the following sense:*
   for any $t \in [0,\overline{t}]$, the random variable $\mathcal{Y}_t$ is a $(\mathcal{S}^{\mathrm{test}})'$–valued space white noise with covariance given on $\varphi, \psi \in \mathcal{S}^{\mathrm{test}}$ by
   $$\mathbb{E}\big[\mathcal{Y}_t(\varphi)\mathcal{Y}_t(\psi)\big] = \frac{\mathrm{D}}{2\sigma}\langle\varphi,\psi\rangle_{\mathrm{L}^2(\mathrm{X})}\,;$$
2. *the process $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$ satisfies the following* energy estimate*:*
   there exists $\kappa > 0$ such that for any $\varphi \in \mathcal{S}^{\mathrm{test}}$, any $\delta, \varepsilon \in (0,1)$ with $\delta < \varepsilon$, and any $s, t \in [0,\overline{t}]$ with $s < t$,
   $$\mathbb{E}\Big[\big(\mathcal{A}^\varepsilon_{s,t}(\varphi) - \mathcal{A}^\delta_{s,t}(\varphi)\big)^2\Big] \leqslant \kappa(t-s)\varepsilon\,\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathrm{X})}, \tag{3.8}$$
   where
   $$\mathcal{A}^\varepsilon_{s,t}(\varphi) = -\int_s^t\int_{\mathrm{X}}\Big(\mathcal{Y}_r\big(\iota_\varepsilon(u)\big)\Big)^2\,\nabla\varphi(u)\,\mathrm{d}u\mathrm{d}r\,; \tag{3.9}$$
3. *for any $\varphi \in \mathcal{S}^{\mathrm{test}}$ and any $t \in [0,\overline{t}]$, the process*
   $$\mathcal{M}_t(\varphi) = \mathcal{Y}_t(\varphi) - \mathcal{Y}_0(\varphi) - \sigma\int_0^t \mathcal{Y}_s(\Delta\varphi)\,\mathrm{d}s - \overline{\lambda}\mathcal{A}_t(\varphi)$$
   is a continuous martingale with respect to the natural filtration (3.6) associated to $\mathcal{Y}_\cdot$, and of quadratic variation
   $$\big\langle\mathcal{M}(\varphi)\big\rangle_t = t\mathrm{D}\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathrm{X})}\,,$$

[2]For instance, if $\mathrm{X} = \mathbb{R}$ it may be simply given by $\iota_\varepsilon(u)(v) = \varepsilon^{-1}\mathbf{1}_{(u,u+\varepsilon]}(v)$, but we will see other choices in Section 3.4.

where the process $\{\mathcal{A}_t\}_{t\in[0,\overline{t}]}$ *is obtained through the following limit, which holds in the* $\mathbf{L}^2$*-sense:*

$$\mathcal{A}_t(\varphi) = \lim_{\varepsilon\to 0} \mathcal{A}^{\varepsilon}_{0,t}(\varphi)\,; \tag{3.10}$$

4. *the reversed process* $\{\mathcal{Y}_{\overline{t}-t}\}_{t\in[0,\overline{t}]}$ *satisfies item* (3) *with* $\overline{\lambda}$ *replaced by* $-\overline{\lambda}$.

Proof. The existence and uniqueness of such energy solutions has been proved in several contexts, originally in [65] and then in [37, 60], see also Section 3.4.3.

REMARK *3.2.1.* There is a way to make sense of $\mathcal{Y}_r(\iota_\varepsilon(u))$ (which is *a priori* not obvious since $\iota_\varepsilon(u)$ may not be a test function), as explained in [55, Remark 4].

REMARK *3.2.2.* The definition of the Ornstein-Uhlenbeck process of Proposition 3.2.1 and of the energy solution to the stochastic Burgers equation (3.7) when $\overline{\lambda} = 0$, are equivalent. The only part which is not obvious is that the Ornstein-Uhlenbeck process $\mathcal{Y}$ satisfies conditions *2.* and *4.* in Theorem 3.2.2, but this usually follows from the convergence results of Theorem 3.3.5 below.

3.3 Microscopic dynamics and density fluctuation field

In this section I adopt the general framework of [58], written in collaboration with P. Gonçalves and M. Jara: let $\mathrm{N} \in \mathbb{N}$ be the scaling parameter and fix $a \in (0,2]$ which will regulate the time scale. At the microscopic level, we are interested in the evolution of a Markov process $\{\eta^{\mathrm{N}}_{t\mathrm{N}^a}(k)\,;\ k\in\mathbb{Z}\}_{t\geqslant 0}$ in the accelerated time scale $t\mathrm{N}^a$, defined through its infinitesimal generator $\mathrm{N}^a\mathcal{L}_{\mathrm{N}}$. This process belongs to the class of conservative one-dimensional interacting particle systems, with state space $\Omega = \mathrm{Y}^{\mathbb{Z}}$. For instance, if the model is of exclusion type, then $\mathrm{Y} = \{0,1\}$ (so that there is at most one particle per site), whereas for Hamiltonian oscillators, $\mathrm{Y} = \mathbb{R}$ or $\mathbb{R}^2$ (the dynamics being on positions and velocities). For now we are considering evolution on the *infinite* lattice $\mathbb{Z}$, but we will also investigate the finite case in Section 3.4.3.

3.3.1 General assumptions

We start by stating three general assumptions which, if they are satisfied, directly imply the convergence results which will be given in Theorem 3.3.5. The first one involves the invariant measures, more precisely:

ASSUMPTION *3.3.1 (Invariant measures).* We assume that the Markov process has a family of invariant measures denoted by $\{\nu_\rho\,;\ \rho\in\mathrm{I}\}$, where I represents the range of values for the parameter. These measures are associated to the conserved quantity: $\sum_{k\in\mathbb{Z}}\eta(k)$, which we call *density*. For any $\rho\in\mathrm{I}$ we assume:

(i) ν_ρ is a product measure on Ω;

(ii) ν_ρ is invariant by translation, so that $\int_\Omega \eta(k)\nu_\rho(\mathrm{d}\eta) = \rho$ for any $k\in\mathbb{Z}$;

(iii) ν_ρ has finite first moments, *i.e.*

$$\int_\Omega |\eta(0)|^m \nu_\rho(\mathrm{d}\eta) < +\infty, \qquad \text{for } m = 2,3,4.$$

Assumption 3.3.1 mainly relies on translation invariance properties of the microscopic dynamics, which is not always true. For instance, when microscopic defects are added (like in [43]), *i.e.* the microscopic dynamics is locally perturbed, this assumption fails, see Section 3.4.2.

NOTATION *3.3.1.* We now fix $\rho \in \mathrm{I}$. Let $\mathcal{D}(\mathbb{R}_+, \Omega)$ be the Skorokhod path space, *i.e.* the space of càdlàg trajectories endowed with the weak topology. We denote by $\mathbb{P}^{\mathrm{N}}_\rho$ the probability measure on $\mathcal{D}(\mathbb{R}_+, \Omega)$ induced by the initial measure ν_ρ and the Markov process $\{\eta^{\mathrm{N}}_{t\mathrm{N}^a}(k)\ ;\ k \in \mathbb{Z}\}_{t\geqslant 0}$. Its expectation is denoted by $\mathbb{E}^{\mathrm{N}}_\rho$. From now on, we write $\eta_{t\mathrm{N}^a}$ for $\eta^{\mathrm{N}}_{t\mathrm{N}^a}$, for the sake of clarity. Let also $\overline{\eta}(k) = \eta(k) - \rho$ and more generally, $\overline{f}(\eta) = f(\eta) - \int f(\eta)\nu_\rho(\mathrm{d}\eta)$ be the centered variables, defined for any function $f : \Omega \to \mathbb{R}$. Finally we denote by $\chi(\rho)$ the variance:

$$\chi(\rho) = \int_\Omega \big(\eta(0) - \rho\big)^2 \ \nu_\rho(\mathrm{d}\eta). \tag{3.11}$$

The generator $\mathcal{L}_{\mathrm{N}}$ can be decomposed in $\mathbf{L}^2(\nu_\rho)$ into the sum of its symmetric and antisymmetric parts, more precisely we write $\mathcal{L}_{\mathrm{N}} = \mathcal{A}_{\mathrm{N}} + \mathcal{S}_{\mathrm{N}}$, where

$$\mathcal{S}_{\mathrm{N}} = \tfrac{1}{2}(\mathcal{L}_{\mathrm{N}} + \mathcal{L}^*_{\mathrm{N}}) \qquad \text{and} \qquad \mathcal{A}_{\mathrm{N}} = \tfrac{1}{2}(\mathcal{L}_{\mathrm{N}} - \mathcal{L}^*_{\mathrm{N}}),$$

with $\mathcal{L}^*_{\mathrm{N}}$ being the adjoint of $\mathcal{L}_{\mathrm{N}}$ in $\mathbf{L}^2(\nu_\rho)$. By the local conservation law, for any $k \in \mathbb{Z}$, there exists a function $j^{\mathrm{N}}_{k,k+1}$ defined on Ω such that

$$\mathcal{L}_{\mathrm{N}}\eta(k) = j^{\mathrm{N}}_{k-1,k}(\eta) - j^{\mathrm{N}}_{k,k+1}(\eta) \tag{3.12}$$

and $j^{\mathrm{N}}_{k,k+1}$ is the instantaneous current at the bond $\{k, k+1\}$. To fix notation we denote

$$\mathcal{S}_{\mathrm{N}}\eta(k) = j^{\mathrm{N,S}}_{k-1,k}(\eta) - j^{\mathrm{N,S}}_{k,k+1}(\eta) \qquad \text{and} \qquad \mathcal{A}_{\mathrm{N}}\eta(k) = j^{\mathrm{N,A}}_{k-1,k}(\eta) - j^{\mathrm{N,A}}_{k,k+1}(\eta),$$

so that $j^{\mathrm{N}}_{k,k+1} = j^{\mathrm{N,S}}_{k,k+1} + j^{\mathrm{N,A}}_{k,k+1}$. We denote by $\mathcal{D}_{\mathrm{N}}(f)$ the Dirichlet form associated to the Markov process, which is defined on local functions $f \in \mathbf{L}^2(\nu_\rho)$ as

$$\mathcal{D}_{\mathrm{N}}(f) = -\int_\Omega f(\eta)\mathcal{L}_{\mathrm{N}}f(\eta)\nu_\rho(\mathrm{d}\eta) = -\int_\Omega f(\eta)\mathcal{S}_{\mathrm{N}}f(\eta)\nu_\rho(\mathrm{d}\eta).$$

The second assumption is the following:

ASSUMPTION *3.3.2 (Dirichlet form).* For any $\mathrm{N} \in \mathbb{N}$, there exists a bounded function $\zeta^{\mathrm{N}}_{0,1} : \Omega \to [\delta, \delta^{-1}]$ with $\delta > 0$ such that the Dirichlet form takes the form

$$\mathcal{D}_{\mathrm{N}}(f) = \sum_{k\in\mathbb{Z}} \mathbf{I}^{\mathrm{N}}_{k,k+1}(f),$$

where

$$\mathbf{I}^{\mathrm{N}}_{k,k+1}(f) = \int_{\Omega} \zeta^{\mathrm{N}}_{k,k+1}(\eta)\left(f(\eta^{k,k+1}) - f(\eta)\right)^2 \nu_\rho(\mathrm{d}\eta), \tag{3.13}$$

with $\zeta^{\mathrm{N}}_{k,k+1}(\eta) = \tau_k \zeta^{\mathrm{N}}_{0,1}(\eta)$ and $\eta^{k,\ell}$ is the configuration obtained from η after exchanging the occupation values $\eta(k)$ and $\eta(\ell)$ as follows:

$$(\eta^{k,\ell})(j) = \begin{cases} \eta(\ell) & \text{if } j = k, \\ \eta(k) & \text{if } j = \ell, \\ \eta(j) & \text{otherwise.} \end{cases} \tag{3.14}$$

Assumption 3.3.2 may look restrictive but is actually valid for many models of interest. For example, a significant amount of lattice gas dynamics, either symmetric or asymmetric, with positive jumps rates, fall into this category. However, when kinetic constraints are added, as in [23], this assumption fails and one has to find *ad-hoc* arguments to adapt the proof (see Section 3.4.1).

Finally, our last assumption is related to the decomposition of the recentered current.

ASSUMPTION *3.3.3 (Instantaneous current).* There exists a local function $h : \Omega \to \mathbb{R}$ and a constant λ_{N}, such that for every $k \in \mathbb{Z}$,

$$\overline{j^{\mathrm{N,S}}_{k,k+1}}(\eta) = \tau_k h(\eta) - \tau_{k+1} h(\eta) \tag{3.15}$$

$$\overline{j^{\mathrm{N,A}}_{k,k+1}}(\eta) = \lambda_{\mathrm{N}}\, \overline{\eta}(k)\overline{\eta}(k+1) + \tau_k g_{\mathrm{N}}(\eta), \tag{3.16}$$

where $g_{\mathrm{N}} : \Omega \to \mathbb{R}$ is a local function such that for all $\varphi \in \mathcal{S}^{\mathrm{test}}$,

$$\lim_{\mathrm{N}\to\infty} \mathbb{E}^{\mathrm{N}}_\rho\left[\left(\int_0^t \sqrt{\mathrm{N}} \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}}\varphi\left(\tfrac{k}{\mathrm{N}}\right)\tau_k g_{\mathrm{N}}(\eta_{s\mathrm{N}^a})\mathrm{d}s\right)^2\right] = 0, \tag{3.17}$$

and moreover, denoting $\mathrm{H}(\rho) = \int h(\eta)\nu_\rho(\mathrm{d}\eta)$,

$$\lim_{\mathrm{N}\to\infty} \mathbb{E}^{\mathrm{N}}_\rho\left[\left(\int_0^t \frac{1}{\sqrt{\mathrm{N}}} \sum_{k\in\mathbb{Z}} \Delta_{\mathrm{N}}\varphi\left(\tfrac{k}{\mathrm{N}}\right)\left(\tau_k h(\eta_{s\mathrm{N}^a}) - \mathrm{H}(\rho) - \mathrm{H}'(\rho)(\eta_{s\mathrm{N}^a}(k) - \rho)\right)\mathrm{d}s\right)^2\right] = 0. \tag{3.18}$$

REMARK *3.3.1.* The condition (3.18) is not very hard to get, since one can usually apply the classical Boltzmann-Gibbs principle introduced in [25].

However, the condition (3.17) may look strong: what me mean here is that the error produced by g_{N} is small compared to the one induced by the degree two polynomials $\overline{\eta}(k)\overline{\eta}(k+1)$. This is the case for example for polynomials of degree greater or equal than 3, under Assumptions 3.3.1 and 3.3.2, as we show in [58, Theorem 5].

3.3.2 Density fluctuation field and martingale decomposition

Our process of interest is the *density fluctuation field*, defined on functions φ in some space of test functions $\mathcal{S}^{\text{test}}$, as

$$\mathcal{Y}_t^{\mathrm{N}}(\varphi) = \frac{1}{\sqrt{\mathrm{N}}} \sum_{k\in\mathbb{Z}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\big(\eta_{t\mathrm{N}^a}(k) - \rho\big).$$

By Dynkin's formula, for any $\varphi \in \mathcal{S}^{\text{test}}$

$$\mathrm{M}_t^{\mathrm{N}}(\varphi) = \mathcal{Y}_t^{\mathrm{N}}(\varphi) - \mathcal{Y}_0^{\mathrm{N}}(\varphi) - \int_0^t \mathrm{N}^a \mathcal{L}_{\mathrm{N}}(\mathcal{Y}_s^{\mathrm{N}}(\varphi))\mathrm{d}s$$

is a martingale. A simple computation based on the conservation law (3.12) and an integration by part shows that the integral part of $\mathrm{M}_t^{\mathrm{N}}(\varphi)$ can be written as

$$\int_0^t \mathrm{N}^a \mathcal{L}_{\mathrm{N}}(\mathcal{Y}_s^{\mathrm{N}}(\varphi))\mathrm{d}s = \int_0^t \frac{\mathrm{N}^{a-1}}{\sqrt{\mathrm{N}}} \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\Big(j_{k,k+1}^{\mathrm{N}}(\eta_{s\mathrm{N}^a}) - \mathbb{E}_\rho^{\mathrm{N}}\big[j_{k,k+1}^{\mathrm{N}}(\eta)\big]\Big)\mathrm{d}s. \tag{3.19}$$

According to Assumption 3.3.3 and by a summation by parts, we can rewrite the integral part (3.19) in the following way:

$$\tfrac{1}{2}\mathrm{H}'(\rho) \int_0^t \frac{\mathrm{N}^{a-2}}{\sqrt{\mathrm{N}}} \sum_{k\in\mathbb{Z}} \Delta_{\mathrm{N}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\overline{\eta}_{s\mathrm{N}^a}(k)\mathrm{d}s \tag{3.20}$$

$$+ \lambda_{\mathrm{N}} \int_0^t \frac{\mathrm{N}^{a-1}}{\sqrt{\mathrm{N}}} \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\overline{\eta}_{s\mathrm{N}^a}(k)\overline{\eta}_{s\mathrm{N}^a}(k+1)\mathrm{d}s, \tag{3.21}$$

plus a term which is negligible in $\mathbf{L}^2(\mathbb{P}_\rho^{\mathrm{N}})$ and given in (3.17) and (3.18) (recall that $a \leqslant 2$). The treatment of the term (3.21) is the main technical novelty, which we expose now.

3.3.3 The second order Boltzmann-Gibbs principle

For any $\mathbf{v} \in \ell^2(\mathbb{Z})$, we denote: $\|\mathbf{v}\|_{2,\mathrm{N}}^2 = \mathrm{N}^{-1}\sum_{k\in\mathbb{Z}} \mathbf{v}^2(k) < \infty$.

Under Assumptions 3.3.1 and 3.3.2, the following result holds true:

Theorem 3.3.1 (Second order Boltzmann-Gibbs principle in infinite volume). *There exists a constant $\mathrm{C} = \mathrm{C}(\rho) > 0$ such that, for any $\mathrm{L} \in \mathbb{N}$ and $t > 0$, and for any function $\mathbf{v} \in \ell^2(\mathbb{Z})$:*

$$\mathbb{E}_\rho^{\mathrm{N}}\Bigg[\bigg(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\Big(\overline{\eta}_{s\mathrm{N}^a}(k)\overline{\eta}_{s\mathrm{N}^a}(k+1) - \big(\overrightarrow{\eta}_{s\mathrm{N}^a}^{(\mathrm{L})}(k)\big)^2 + \frac{\chi(\rho)}{\mathrm{L}}\Big)\mathrm{d}s\bigg)^2\Bigg]$$
$$\leqslant \mathrm{C}t\bigg(\frac{\mathrm{L}}{\mathrm{N}^{a-1}} + \frac{t\mathrm{N}}{\mathrm{L}^2}\bigg)\|\mathbf{v}\|_{2,\mathrm{N}}^2 \tag{3.22}$$

where $\overrightarrow{\eta}^{(\mathrm{L})}(k)$ is the average centered configuration on a box of size L situated to the right of the site k, namely:

$$\overrightarrow{\eta}^{(\mathrm{L})}(k) = \frac{1}{\mathrm{L}} \sum_{\ell=k+1}^{k+\mathrm{L}} \overline{\eta}(\ell). \tag{3.23}$$

REMARK *3.3.2.* Last result can be extended to higher degree polynomials, provided that higher moments are finite: more precisely, if one wants to replace in (3.22) the function $\overline{\eta}(k)\overline{\eta}(k+1)$ with a polynomial of degree d, then condition (iii) in Assumption 3.3.1 has to be replaced by

$$\int_\Omega |\eta(0)|^m \nu_\rho(\mathrm{d}\eta) < +\infty, \qquad \text{for } m = 2, \dots, 2d.$$

This generalization has been done rigorously in [58].

REMARK *3.3.3.* The initial proof of the second order Boltzmann-Gibbs principle of [55], which uses a spectral gap inequality estimate, permits to get the same kind of result for a larger class of functions. More precisely, the function

$$\overline{\eta}(k)\overline{\eta}(k+1) - \big(\overrightarrow{\eta}^{(\mathrm{L})}(k)\big)^2 + \frac{\chi(\rho)}{\mathrm{L}}$$

which appears in (3.22) can be replaced with

$$\tau_k f(\eta) - \Psi_f(\rho) - \Psi_f'(\rho)\big(\eta(k) - \rho\big), \quad \text{where} \quad \Psi_f(\rho) = \int_\Omega f(\eta)\nu_\rho(\mathrm{d}\eta),$$

for any local function $f : \Omega \to \mathbb{R}$. However, in all our examples (Section 3.4), the spectral gap assumption will not be satisfied. Fortunately, we only need to look at specific local fields whose additive functionals can be written as polynomials. Therefore, Theorem 3.3.1, even dealing with specific local functions, will be enough to conclude.

3.3.4 Sketch of the proof of Theorem 3.3.1

In this section we give elements for the proof of the second-order Boltzmann-Gibbs principle. This part may be skipped by the reader at first reading.

In the initial proof of the second order Boltzmann-Gibbs principle in [55], the *multiscale analysis* was exposed in two main steps: the first one is reminiscent from the well-known *one-block estimate* ([71, Chapter 5, Lemma 3.1]), which consists in replacing a local function by its average on a microscopic block; the second one is reminiscent from the *two-blocks estimate* ([71, Chapter 5, Lemma 3.2]) and consists in a key iterative bound to replace the aforementioned average on a microscopically big block by an average on a macroscopically small block. Our new proof [58] consists in following step by step this argument, after decomposing suitably the polynomials, in such a way that there is no need to apply a spectral gap inequality.

Let me illustrate how this new proof works: first, let us choose a site k. Roughly speaking, we want to replace the local function $\overline{\eta}(k)\overline{\eta}(k+1)$ by the square of the average to its right $\big(\overrightarrow{\eta}^{\mathrm{L}}(k)\big)^2$ (see Figure 3.1). The proof will make use of the sites situated between $k+\mathrm{L}+1$ and $k+2\mathrm{L}$, as explained below.

Before going into the proof details, let us simplify some notations: for a function $g : \Omega \to \mathbb{R}$ we denote by $\|g\|_2$ its $\mathbf{L}^2(\nu_\rho)$–norm: $\|g\|_2^2 = \int_\Omega g^2(\eta)\nu_\rho(\mathrm{d}\eta)$. In the following, $\mathrm{C} := \mathrm{C}(\rho)$

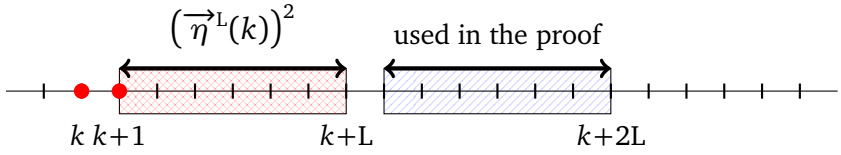

Figure 3.1: Replacement for the local function $\overline{\eta}(k)\overline{\eta}(k+1)$.

denotes a constant that does not depend on N nor on t nor on the sizes of all boxes involved, and which may change from line to line. We fix a measurable function $\mathbf{v} \in \ell^2(\mathbb{Z})$. Finally we define, for any local function $g : \Omega \to \mathbb{R}$

$$\mathfrak{I}_{t,\mathrm{N}}(g) = \mathbb{E}^{\mathrm{N}}_{\rho}\Big[\Big(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\tau_k g(\eta_{s\mathrm{N}^a})\,\mathrm{d}s\Big)^2\Big].$$

With this definition, (3.22) follows from showing that for any $\mathrm{N}, \ell \in \mathbb{N}$ and any $t > 0$,

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{L}}) \leqslant \mathrm{C}t\Big(\frac{\mathrm{L}}{\mathrm{N}^{a-1}} + \frac{t\mathrm{N}}{\mathrm{L}^2}\Big)\|\mathbf{v}\|^2_{2,\mathrm{N}} \tag{3.24}$$

where the local function g_{L} is given by $g_{\mathrm{L}}(\eta) = \overline{\eta}(0)\overline{\eta}(1) - \big(\overrightarrow{\eta}^{\mathrm{L}}(0)\big)^2 + \chi(\rho)/\mathrm{L}$. We are now going to decompose g_{L} as the sum of several local functions, for which the estimates are simpler. With a small abuse of language, we say that, at each step, we *replace* a local function with another one. More precisely, let $\mathrm{L}_0 \leqslant \mathrm{L}$ and assume first (to simplify) that $\mathrm{L} = 2^{\mathrm{M}}\mathrm{L}_0$ for some integer $\mathrm{M} \in \mathbb{N}$. Denote $\mathrm{L}_m = 2^m\mathrm{L}_0$ for $m \in \{0, \dots, \mathrm{M}\}$.

One can easily check the decomposition

$$\begin{aligned}
g_{\mathrm{L}}(\eta) = \quad & \overline{\eta}(0)\Big(\overline{\eta}(1) - \overrightarrow{\eta}^{\mathrm{L}_0}(\mathrm{L}_0)\Big) && (3.25)\\
& + \overrightarrow{\eta}^{\mathrm{L}_0}(\mathrm{L}_0)\Big(\overline{\eta}(0) - \overrightarrow{\eta}^{\mathrm{L}_0}(0)\Big) && (3.26)\\
& + \sum_{m=0}^{\mathrm{M}-1} \overrightarrow{\eta}^{\mathrm{L}_m}(0)\Big(\overrightarrow{\eta}^{\mathrm{L}_m}(\mathrm{L}_m) - \overrightarrow{\eta}^{2\mathrm{L}_m}(2\mathrm{L}_m)\Big) && (3.27)\\
& + \sum_{m=0}^{\mathrm{M}-1} \overrightarrow{\eta}^{2\mathrm{L}_m}(2\mathrm{L}_m)\Big(\overrightarrow{\eta}^{\mathrm{L}_m}(0) - \overrightarrow{\eta}^{2\mathrm{L}_m}(0)\Big) && (3.28)\\
& + \overrightarrow{\eta}^{\mathrm{L}}(0)\Big(\overrightarrow{\eta}^{\mathrm{L}}(\mathrm{L}) - \overline{\eta}(\mathrm{L}+1)\Big) && (3.29)\\
& + \overrightarrow{\eta}^{\mathrm{L}}(0)\Big(\overline{\eta}(\mathrm{L}+1) - \overline{\eta}(0)\Big) && (3.30)\\
& + \overrightarrow{\eta}^{\mathrm{L}}(0)\Big(\overline{\eta}(0) - \overrightarrow{\eta}^{\mathrm{L}}(0)\Big) + \frac{\chi(\rho)}{\mathrm{L}}. && (3.31)
\end{aligned}$$

For example, in (3.25) we say that we *replace* $\overline{\eta}(1)$ by $\overrightarrow{\eta}^{\mathrm{L}_0}(\mathrm{L}_0)$, while $\overline{\eta}(0)$ is considered as *fixed*. Seven terms appear from (3.25) to (3.31). Let us denote them by order of appearance

as follows:

$$g_{\mathrm{I}}(\eta), \quad g_{\mathrm{II}}(\eta), \quad g_{\mathrm{III}}(\eta), \quad g_{\mathrm{IV}}(\eta), \quad g_{\mathrm{V}}(\eta), \quad g_{\mathrm{VI}}(\eta), \quad g_{\mathrm{VII}}(\eta).$$

The decomposition above can naturally be written for $\tau_k g_{\mathrm{L}}$ by translating each term. Let us now illustrate the first steps of the decomposition: in Figure 3.2 below, we use the arrows as symbols for the replacements we perform, and we illustrate the consecutive replacements from (3.25) to (3.27), the latter corresponding to $\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m) \mapsto \overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)$.

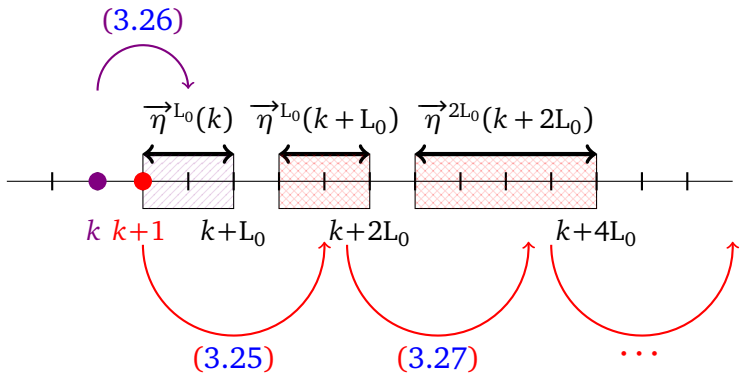

Figure 3.2: Illustration of steps (3.25)–(3.27).

Similarly, one can also see in (3.28) that $\overrightarrow{\eta}^{\mathrm{L}_m}(k)$ is replaced with $\overrightarrow{\eta}^{2\mathrm{L}_m}(k)$ as we illustrate in Figure 3.3.

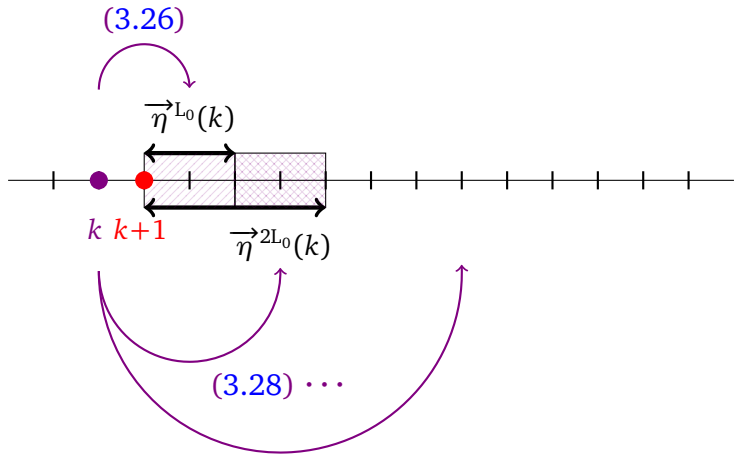

Figure 3.3: Illustration of step (3.28): successive replacements.

The role of the pre-factors $\overrightarrow{\eta}^{\mathrm{L}_m}(k)$ in (3.27) and $\overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)$ in (3.28) can be roughly understood as follows: these local functions have a variance of order $(\mathrm{L}_m)^{-1}$ under ν_ρ, which compensates the price to pay when one tries to replace $\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m)$ by $\overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)$ and $\overrightarrow{\eta}^{\mathrm{L}_m}(k)$ by $\overrightarrow{\eta}^{2\mathrm{L}_m}(k)$. More precisely, this compensation is optimal if the support of the pre-factor does not intersect the set of sites which are used in the replacement: for example, the support of $\overrightarrow{\eta}^{\mathrm{L}_m}(k)$ is $\{k+1,\dots,k+\mathrm{L}_m\}$ and it does not intersect $\{k+\mathrm{L}_m+1,\dots,k+4\mathrm{L}_m\}$, which corresponds to the sites used in the replacement $\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m) \mapsto \overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)$, see (3.27).

Let us go back to our goal estimate (3.24). From the standard convexity inequality $(a_1 + \cdots + a_p)^2 \leqslant p(a_1^2 + \cdots a_p^2)$, one can see that (3.24) follows from seven independent estimates. More precisely, it is enough to prove that

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{w}}) \leqslant \mathrm{C}t\left(\frac{\mathrm{L}}{\mathrm{N}^{a-1}} + \frac{t\mathrm{N}}{\mathrm{L}^2}\right)\|\mathbf{v}\|_{2,\mathrm{N}}^2, \quad \text{for any } \mathrm{w} \in \{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V},\mathrm{VI},\mathrm{VII}\}. \tag{3.32}$$

There is one further step in two particular cases: for $\mathrm{w} = \mathrm{III}$ and $\mathrm{w} = \mathrm{IV}$, we also use Minkowski's inequality, and we bound as follows:

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{III}}) \leqslant \mathrm{C}\left(\sum_{m=0}^{\mathrm{M}-1}\left\{\mathbb{E}_\rho^{\mathrm{N}}\left[\left(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\overrightarrow{\eta}_{s\mathrm{N}^a}^{\mathrm{L}_m}(k)\Big(\overrightarrow{\eta}_{s\mathrm{N}^a}^{\mathrm{L}_m}(k+\mathrm{L}_m) - \overrightarrow{\eta}_{s\mathrm{N}^a}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)\Big)\mathrm{d}s\right)^2\right]\right\}^{1/2}\right)^2 \tag{3.33}$$

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{IV}}) \leqslant \mathrm{C}\left(\sum_{m=0}^{\mathrm{M}-1}\left\{\mathbb{E}_\rho^{\mathrm{N}}\left[\left(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\overrightarrow{\eta}_{s\mathrm{N}^a}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)\Big(\overrightarrow{\eta}_{s\mathrm{N}^a}^{\mathrm{L}_m}(k) - \overrightarrow{\eta}_{s\mathrm{N}^a}^{2\mathrm{L}_m}(k)\Big)\mathrm{d}s\right)^2\right]\right\}^{1/2}\right)^2.$$

Finally, the proof of (3.32) can almost be resumed in one general statement, which we are going to apply several times. Let us state here our main estimate:

PROPOSITION 3.3.2. *Let* $\mathbf{A}, \mathbf{B}$ *be two subsets of* $\mathbb{Z}$*, and let us denote by* $\#\mathbf{B}$ *the cardinality of* $\mathbf{B}$*. Consider* $g : \Omega \to \mathbb{R}$ *a local function whose support does not intersect* $\mathbf{B}$*, namely:* $\mathrm{Supp}(g) \cap \mathbf{B} = \emptyset$*, and which has mean zero with respect to* ν_ρ*. Then, there exists* $\mathrm{C} > 0$ *such that, for any* $\mathrm{N} \in \mathbb{N}$ *and* $t > 0$*,*

$$\mathbb{E}_\rho^{\mathrm{N}}\left[\left(\int_0^t \sum_{k\in\mathbf{A}}\Big\{\mathbf{v}(k)\tau_k g(\eta_{s\mathrm{N}^a})\sum_{\ell\in\tau_k\mathbf{B}}\big(\overline{\eta}_{s\mathrm{N}^a}(\ell) - \overline{\eta}_{s\mathrm{N}^a}(\ell+1)\big)\Big\}\mathrm{d}s\right)^2\right] \leqslant \frac{\mathrm{C}t(\#\mathbf{B})^2}{\mathrm{N}^{a-1}}\,\|g\|_2^2\,\|v\|_{2,\mathrm{N}}^2.$$

This proposition is proved in [58] in a slightly different form, see also [60] for a closer version.

Now, let me explain how we can apply Proposition 3.3.2 in order to estimate $\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{w}})$, for $\mathrm{w} \in \{\mathrm{I},\ldots,\mathrm{VI}\}$. The only estimate that has to be considered separately is the one involving g_{VII}. We prove that the assumptions of Proposition 3.3.2 are satisfied for g_{III} (using (3.33)) and we let the reader check the other ones. First, recall (3.33) and note that

$$\overrightarrow{\eta}^{\mathrm{L}_m}(k)\Big(\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m) - \overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)\Big) = \frac{\overrightarrow{\eta}^{\mathrm{L}_m}(k)}{2\mathrm{L}_m}\sum_{\ell=k+1}^{k+\mathrm{L}_m}\sum_{j=\ell+\mathrm{L}_m}^{\ell+2\mathrm{L}_m-1}(\overline{\eta}(j) - \overline{\eta}(j+1)) \tag{3.34}$$

$$+ \frac{\overrightarrow{\eta}^{\mathrm{L}_m}(k)}{2\mathrm{L}_m}\sum_{\ell=k+1}^{k+\mathrm{L}_m}\sum_{j=\ell+\mathrm{L}_m}^{\ell+3\mathrm{L}_m-1}(\overline{\eta}(j) - \overline{\eta}(j+1)). \tag{3.35}$$

The above identity can be easily obtained by splitting each average $\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m)$ and $\overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m)$ in two parts, as follows:

$$\overrightarrow{\eta}^{\mathrm{L}_m}(k+\mathrm{L}_m) = \frac{1}{2\mathrm{L}_m}\sum_{\ell=1}^{\mathrm{L}_m}\overline{\eta}(\ell+k+\mathrm{L}_m) + \frac{1}{2\mathrm{L}_m}\sum_{\ell=1}^{\mathrm{L}_m}\overline{\eta}(\ell+k+\mathrm{L}_m)$$

$$\overrightarrow{\eta}^{2\mathrm{L}_m}(k+2\mathrm{L}_m) = \frac{1}{2\mathrm{L}_m}\sum_{\ell=1}^{\mathrm{L}_m}\overline{\eta}(\ell+k+2\mathrm{L}_m) + \frac{1}{2\mathrm{L}_m}\sum_{\ell=1}^{\mathrm{L}_m}\overline{\eta}(\ell+k+3\mathrm{L}_m),$$

and subtracting the sums one by one. Let us first deal with (3.34): we can use Proposition 3.3.2 with

$$g(\eta) = \frac{\overrightarrow{\eta}^{\mathrm{L}_m}(0)}{2\mathrm{L}_m}, \quad \text{and} \quad \mathbf{B} = \bigcup_{\ell=1}^{\mathrm{L}_m} \big\{\ell + \mathrm{L}_m, \dots, \ell + 2\mathrm{L}_m - 1\big\}.$$

Note that $\|g\|_2^2 = \mathrm{C}/\mathrm{L}_m^3$ and $\#\mathbf{B} = \mathrm{L}_m^2$, and remember that $\mathrm{L}_m = 2^m \mathrm{L}_0$. Next, we deal with (3.35): we only need to change $\mathbf{B}$ which now reads

$$\mathbf{B} = \bigcup_{\ell=1}^{\mathrm{L}_m} \big\{\ell + \mathrm{L}_m, \dots, \ell + 3\mathrm{L}_m - 1\big\}, \quad \text{hence} \quad \#\mathbf{B} = 2\mathrm{L}_m^2.$$

Therefore, one can see from (3.33) and Proposition 3.3.2 that

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{III}}) \leqslant \mathrm{C}\bigg(\sum_{m=0}^{\mathrm{M}-1}\bigg(\frac{t\mathrm{L}_m^4}{\mathrm{N}^{a-1}}\frac{1}{\mathrm{L}_m^2}\|\mathbf{v}\|_{2,\mathrm{N}}^2\bigg)^{1/2}\bigg)^2 \leqslant \mathrm{C}t\frac{\mathrm{L}_{\mathrm{M}}}{\mathrm{N}^{a-1}}\|\mathbf{v}\|_{2,\mathrm{N}}^2.$$

Performing similar arguments and using Proposition 3.3.2 together with Minkowski's inequality, we get that, for any $\mathrm{L}, \mathrm{N} \in \mathbb{N}$ and any $t > 0$,

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{I}}) \leqslant \mathrm{C}t\frac{\mathrm{L}_0^2}{\mathrm{N}^{a-1}}\|\mathbf{v}\|_{2,\mathrm{N}}^2, \qquad \mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{II}}) \leqslant \mathrm{C}t\frac{\mathrm{L}_0}{\mathrm{N}^{a-1}}\|\mathbf{v}\|_{2,\mathrm{N}}^2,$$
$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{w}}) \leqslant \mathrm{C}t\frac{\mathrm{L}}{\mathrm{N}^{a-1}}\|\mathbf{v}\|_{2,\mathrm{N}}^2, \qquad \text{for any } \mathrm{w} \in \{\mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}\}.$$

Finally, we have to estimate the last remaining term involving g_{VII}, which is treated separately. More precisely, in [58, 60] we prove the following:

Proposition 3.3.3. *For any $\mathrm{L}, \mathrm{N} \in \mathbb{N}$ and any $t > 0$,*

$$\mathfrak{I}_{t,\mathrm{N}}(g_{\mathrm{VII}}) = \mathbb{E}_{\rho}^{\mathrm{N}}\bigg[\bigg(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\Big(\overrightarrow{\eta}_{s\mathrm{N}^a}^{\mathrm{L}}(k)\Big[\overline{\eta}_{s\mathrm{N}^a}(k) - \overrightarrow{\eta}_{s\mathrm{N}^a}^{\mathrm{L}}(k)\Big] + \frac{\chi(\rho)}{\mathrm{L}}\Big)\mathrm{d}s\bigg)^2\bigg]$$
$$\leqslant \mathrm{C}t\bigg(\frac{\mathrm{L}}{\mathrm{N}^{a-1}} + \frac{t\mathrm{N}}{\mathrm{L}^2}\bigg)\|\mathbf{v}\|_{2,\mathrm{N}}^2. \quad (3.36)$$

Putting together the previous estimates into our decomposition (3.25)–(3.31) of g_{L}, we obtain the final bound (3.24) and Theorem 3.3.1 follows.

Let me now present various of its applications.

3.3.5 Consequences of the Boltzmann-Gibbs principle

a) Superdiffusive systems

We consider systems which fulfill Assumptions 3.3.1, 3.3.2 and 3.3.3, and which evolve superdiffusively, so that $a < 2$. Recall from above that

$$\mathrm{M}_t^{\mathrm{N}}(\varphi) = \mathcal{Y}_t^{\mathrm{N}}(\varphi) - \mathcal{Y}_0^{\mathrm{N}}(\varphi) + \tfrac{1}{2}\int_0^t \frac{\mathrm{N}^{a-2}}{\sqrt{\mathrm{N}}}\sum_{k\in\mathbb{Z}} \Delta_{\mathrm{N}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\tau_k h(\eta_{s\mathrm{N}^a})\mathrm{d}s \quad (3.37)$$
$$+ \lambda_{\mathrm{N}}\int_0^t \frac{\mathrm{N}^{a-1}}{\sqrt{\mathrm{N}}}\sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}}\varphi\big(\tfrac{k}{\mathrm{N}}\big)\overline{\eta}_{s\mathrm{N}^a}(k)\overline{\eta}_{s\mathrm{N}^a}(k+1)\mathrm{d}s, \quad (3.38)$$

plus a term which is negligible in $\mathbf{L}^2(\mathbb{P}^n_\rho)$ and given in (3.17). Since $a < 2$ and h is a local function, a simple computation shows that the first time integral (3.37) vanishes in $\mathbf{L}^2(\mathbb{P}^N_\rho)$, as N goes to infinity.

We now treat the second one (3.38): in all the examples that we know (in particular [54, 10]), the second order Boltzmann-Gibbs principle stated in Theorem 3.3.1 and the Cauchy-Schwarz inequality, permit to show the following: for any $a < \frac{4}{3}$ the term (3.38) vanishes in $\mathbf{L}^2(\mathbb{P}^N_\rho)$, as N goes to infinity. More details are given in [58, Section 3.1.2] when we treat some concrete example. It is still an open problem to prove the same result for any $a < 2$. The precise statement is the following:

THEOREM 3.3.4 (Trivial limit). *Fix $a < \frac{4}{3}$ and a time horizon $\overline{t} > 0$. The sequence of processes $\{\mathcal{Y}^N_t \; ; \; t \in [0,\overline{t}]\}_{N\in\mathbb{N}}$ converges in distribution with respect to the Skorokhod topology of the path space $\mathcal{D}([0,\overline{t}] \; ; \; (\mathcal{S}^{\text{test}})')$, as $N \to \infty$, to the (constant) process $\mathcal{Y}_t$ given on $\varphi \in \mathcal{S}^{\text{test}}$ by $\mathcal{Y}_t(\varphi) = \mathcal{Y}_0(\varphi)$.*

b) Diffusive systems

We now consider systems which fulfill Assumptions 3.3.1, 3.3.2 and 3.3.3 and which evolve diffusively so that $a = 2$. If we add a weak asymmetry to the system given by $\lambda N^{-\gamma}$, for some $\gamma > \frac{1}{2}$, then this can be translated in the decomposition (3.16) of the antisymmetric current as taking $\lambda_N = \lambda N^{-\gamma}$. The contribution in the last integral (3.38) will therefore read as

$$\lambda \int_0^t \frac{\sqrt{N}}{N^\gamma} \sum_{k\in\mathbb{Z}} \nabla_N \varphi\big(\tfrac{k}{N}\big) \overline{\eta}_{sN^a}(k) \overline{\eta}_{sN^a}(k+1) ds.$$

In this case, as a consequence of the second order Boltzmann-Gibbs principle, one can show a crossover on the fluctuations which depends on the strength γ of the asymmetry.

THEOREM 3.3.5 (Crossover fluctuations). *Fix a time horizon $\overline{t} > 0$. The sequence of processes $\{\mathcal{Y}^N_t \; ; \; t \in [0,\overline{t}]\}_{N\in\mathbb{N}}$ converges in distribution with respect to the Skorokhod topology of the path space $\mathcal{D}([0,\overline{t}] \; ; \; (\mathcal{S}^{\text{test}})')$, as $N \to \infty$, to*

- *if $\gamma > \frac{1}{2}$: the unique Ornstein-Uhlenbeck process as defined in Proposition 3.2.1, with*

$$\sigma = \tfrac{1}{2} H'(\rho) \qquad \text{and} \qquad D = H'(\rho)\chi(\rho);$$

- *if $\gamma = \frac{1}{2}$: the unique energy solution of the stochastic Burgers equation as defined in Theorem 3.2.2, with*

$$\sigma = \tfrac{1}{2} H'(\rho), \qquad \overline{\lambda} = \lambda \qquad \text{and} \qquad D = H'(\rho)\chi(\rho).$$

Proof. Again, using Assumption 3.3.3 the first integral term (3.37) can we rewritten as in (3.20):

$$\tfrac{1}{2} H'(\rho) \int_0^t \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}} \Delta_N \varphi\big(\tfrac{k}{N}\big) \overline{\eta}_{sN^a}(k) ds = \tfrac{1}{2} H'(\rho) \int_0^t \mathcal{Y}^N_s(\Delta_N \varphi) ds$$

plus a term which is negligible in $\mathbf{L}^2(\mathbb{P}^{\mathrm{N}}_\rho)$. Let me now briefly explain how the nonlinearity emerges in the case $\gamma = \frac{1}{2}$. We use the second order Boltzmann-Gibbs principle with $\mathrm{L} = \varepsilon \mathrm{N}$, $\varepsilon > 0$ (and $a = 2$). This choice makes the right hand side of (3.22) vanish. In that case, note that (3.38) read exactly as in (3.22) since $\lambda_{\mathrm{N}} \mathrm{N}^{a-1}/\sqrt{\mathrm{N}} = \lambda \mathrm{N}^{a-\gamma-3/2} = \lambda$, and therefore it can be replaced with the sum of two terms: first,

$$\lambda \int_0^t \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}} \varphi\Big(\tfrac{k}{\mathrm{N}}\Big) \Big(\overrightarrow{\eta}^{\varepsilon \mathrm{N}}_{s\mathrm{N}^2}(k)\Big)^2 \, \mathrm{d}s = \lambda \int_0^t \frac{1}{\mathrm{N}} \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}} \varphi\Big(\tfrac{k}{\mathrm{N}}\Big) \Big(\mathcal{Y}^{\mathrm{N}}_s\Big(\iota_\varepsilon\Big(\tfrac{k}{\mathrm{N}}\Big)\Big)\Big)^2 \, \mathrm{d}s, \tag{3.39}$$

where for any $u \in \mathbb{R}$, we chose $\iota_\varepsilon(u) : \mathbb{R} \to \mathbb{R}$ defined as $\iota_\varepsilon(u)(v) = \varepsilon^{-1} \mathbf{1}_{(u,u+\varepsilon]}(v)$. This last term (3.39) gives rise to the nonlinear term appearing in the SBE (see *2.* in Theorem 3.2.2). Second, the additional term

$$\int_0^t \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}} \varphi\Big(\tfrac{k}{\mathrm{N}}\Big) \frac{\chi(\rho)}{\varepsilon \mathrm{N}} \, \mathrm{d}s$$

equals 0 since the sum of the discrete gradients vanishes. To sum up, the decomposition (3.37)–(3.38) rewrites

$$\mathcal{Y}^{\mathrm{N}}_t(\varphi) - \mathcal{Y}^{\mathrm{N}}_0(\varphi) + \tfrac{1}{2}\mathrm{H}'(\rho) \int_0^t \mathcal{Y}^{\mathrm{N}}_s(\Delta_{\mathrm{N}}\varphi) \mathrm{d}s + \lambda \int_0^t \frac{1}{\mathrm{N}} \sum_{k\in\mathbb{Z}} \nabla_{\mathrm{N}} \varphi\Big(\tfrac{k}{\mathrm{N}}\Big) \Big(\mathcal{Y}^{\mathrm{N}}_s\Big(\iota_\varepsilon\Big(\tfrac{k}{\mathrm{N}}\Big)\Big)\Big)^2 \, \mathrm{d}s$$

plus vanishing sequences in $\mathbf{L}^2(\mathbb{P}^{\mathrm{N}}_\rho)$. One then needs to pass to the limit $\mathrm{N} \to \infty$ in each of of these terms, so as to recover the characterization given in Theorem 3.2.2.

3.4 Improvements to handle three degenerate models

The three models considered in [23, 43, 60] can not be treated using the general results of the previous sections, since at least one of the Assumptions 3.3.1–3.3.3 will fail. They are models of exclusion type, meaning that $\mathrm{X} = \{0,1\}$ and the configuration space is $\Omega = \{0,1\}^\Lambda$ where $\Lambda \subset \mathbb{Z}$. Take $\eta \in \Omega$. We say there is a particle at k (resp. k is empty) if $\eta(k) = 1$ (resp. $\eta(k) = 0$). Moreover, we are looking at the microscopic dynamics in the diffusive time scale, meaning $a = 2$.

3.4.1 Kinetic constraints

In [23], written in collaboration with O. Blondel and P. Gonçalves, we consider microscopic dynamics with kinetic constraints, which are not ergodic and provide blocked states. With a more refined proof of the second order Boltzmann-Gibbs principle, we are able to prove Theorem 3.3.5 for microscopic dynamics which allow degenerate exchange rates.

More precisely, the dynamics is defined on the full lattice ($\Lambda = \mathbb{Z}$), and can be described as follows: if there is a particle at the site $k \in \mathbb{Z}$ and if $k+1$, resp. $k-1$, is unoccupied, then it jumps to $k+1$, resp. $k-1$, at rate

$$p(1) c_{k,k+1}(\eta), \qquad \text{resp.} \quad p(-1) c_{k,k-1}(\eta).$$

Here, $c_{k,k+1}(\eta) = c_{k+1,k}(\eta)$ represents a *constraint* which is given as follows: fix an integer $m \geqslant 2$, then the jump takes place only if there are at least $m-1$ particles around k, namely

$$c_{k,k+1}(\eta) = c_{k+1,k}(\eta) = \sum_{\ell=1}^{m} \prod_{\substack{j=-(m-\ell) \\ j\neq 0,1}}^{\ell} \eta(k+j). \tag{3.40}$$

For the sake of simplicity we focus in this manuscript on the case $m = 2$ (but the general case is treated in [23]), in which case the last formula reduces to

$$c_{k,k+1}(\eta) = c_{k+1,k}(\eta) = \eta(k-1)\eta(k+2).$$

If $p(1) = p(-1) = \frac{1}{2}$, this model is symmetric and is the one of [59] investigated by Gonalves, Landim and Toninelli.

We consider the *weakly asymmetric* version of this model, namely $p(\pm 1) = \frac{1}{2} \pm \frac{\lambda}{N^\gamma}$, and the generator $\mathcal{L}_N$ of the microscopic process acts on local functions $f : \Omega \to \mathbb{R}$ as

$$(\mathcal{L}_N f)(\eta) = \sum_{\substack{k,\ell\in\mathbb{Z} \\ |k-\ell|=1}} c_{k,\ell}(\eta) p(\ell-k)\eta(k)(1-\eta(\ell))\big(f(\eta^{k,\ell}) - f(\eta)\big), \tag{3.41}$$

where $p(k) = 0$ for any $|k| \neq 1$ and $\eta^{k,\ell}$ has been already defined in (3.14).

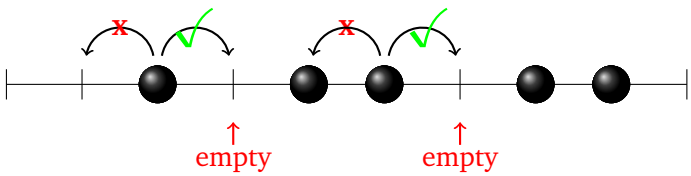

Figure 3.4: Allowed jumps are denoted by ✓. Forbidden jumps are denoted by x.

a) Blocked configurations

Note that the jump rates are zero if the constraint is not satisfied. In particular, there are configurations which never evolve under the dynamics (*blocked configurations*): those in which the distance between any two occupied sites is bigger than 2. As a consequence, the process admits an infinite number of invariant measures: in particular, they contain the Dirac measures supported on configurations which cannot evolve under the dynamics.

Let ν_ρ be the Bernoulli product measure with a constant parameter $\rho \in [0,1]$. It is not difficult to see that ν_ρ is invariant. The most important ingredient for us is the existence of a *mobile cluster* inside finite boxes, which occurs with very high probability, see (3.42) below. This mobile cluster, which has been introduced in [59], is a collection of m neighbouring particles, and it has the property that there exists a sequence of nearest neighbour jumps with positive rates (which we call *allowed path*) which permits to shift the mobile cluster to any other position

on $\mathbb{Z}$. As noted in [59], another crucial property of this mobile cluster is that, for any k, ℓ that do not belong to the cluster, there exists an allowed path that transports the cluster to the vicinity of k, ℓ and it permits to exchange $\eta(k)$ with $\eta(\ell)$ while restoring the other occupation variables to their initial value (in other words, a sequence of transitions with positive rates that turn η into $\eta^{k,\ell}$). Moreover, we can choose this allowed path in such a way that no bond is used more than six times. We refer to Figure 3.5 for an illustration of this dynamical path.

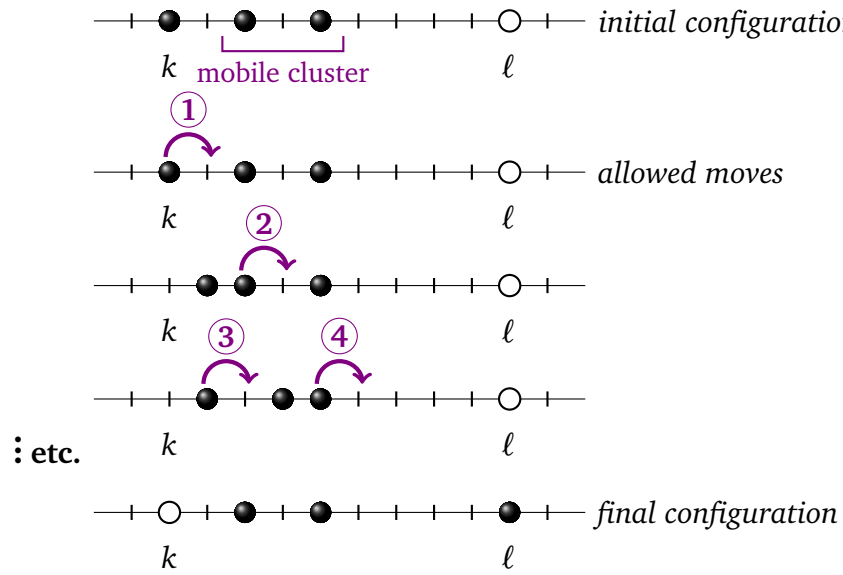

Figure 3.5: Illustration of allowed moves using the mobile cluster, in order to exchange $\eta(k)$ with $\eta(\ell)$.

Almost surely under ν_ρ, there are no blocked configurations. However, given a spatial region of interest, the closest mobile cluster might be very far, and thus virtually useless. In order to control this, we introduce the notion of *good* and *bad* configurations inside a finite box. Fix $k \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and let $\Lambda_k^\ell = \{k+1, ..., k+\ell\}$ be the box of size ℓ to the right of k. We define the (*good*) set

$$\begin{aligned}\mathcal{G}_\ell(k) &= \Big\{\eta \in \Omega\,;\ \eta \text{ contains a mobile cluster in } \Lambda_k^\ell\Big\} \\ &= \Big\{\eta \in \Omega\,;\ \sum_{j=k+1}^{k+\ell-2} \big(\eta(j)\eta(j+1) + \eta(j)\eta(j+2)\big) + \eta(k+\ell-1)\eta(k+\ell) > 0\Big\}.\end{aligned}$$

Under ν_ρ, the probability of the *bad* set $\mathcal{B}_\ell(k) = \Omega \backslash \mathcal{G}_\ell(k)$ rapidly decreases to 0 as $\ell \to \infty$ (see also [59]). Indeed, when $\rho \in (0,1)$:

$$\nu_\rho\big(\Omega \backslash \mathcal{G}_\ell(k)\big) \leqslant (1-\rho^2)^{\lfloor \ell/2 \rfloor}. \tag{3.42}$$

c) Microscopic current

As already seen, the generator $\mathcal{L}_N$ can be decomposed in the Hilbert space $\mathbf{L}^2(\nu_\rho)$ into its symmetric and antisymmetric parts: $\mathcal{L}_N = \mathcal{S}_N + \mathcal{A}_N$. The microscopic instantaneous current,

which is also decomposed has $j^{\mathrm{N}}_{k,k+1}(\eta) = j^{\mathrm{N,S}}_{k,k+1}(\eta) + j^{\mathrm{N,A}}_{k,k+1}(\eta)$, is here given by the following proposition (proved in [23, Proposition 2.2]):

PROPOSITION 3.4.1 (Instantaneous current). *There exist*

- *a local function* $h : \Omega \to \mathbb{R}$,
- *and three local functions* $\mathbf{P}_j : \Omega \to \mathbb{R}$ *(which do not depend on* N*), where, for any* $j = 1, 2, 3$*, the function* $\mathbf{P}_j$ *is a homogeneous multivariate polynomial of degree* j *in the recentered variables* $\{\overline{\eta}(-1), \dots, \overline{\eta}(2)\}$*, such that, for every* $k \in \mathbb{Z}$*,*

$$\overline{j}^{\mathrm{N,S}}_{k,k+1}(\eta) = \tau_k h(\eta) - \tau_{k+1} h(\eta), \tag{3.43}$$

$$\overline{j}^{\mathrm{N,A}}_{k,k+1}(\eta) = \mathrm{N}^{-\gamma} \sum_{j=1}^{3} \tau_k \mathbf{P}_j(\eta). \tag{3.44}$$

It turns out that the expectation of the instantaneous current, in the diffusive time scale, with respect to the equilibrium measure ν_ρ, equals

$$\nu_\rho\big(\mathrm{N}^2 j^{\mathrm{N}}_{k,k+1}\big) = \lambda \mathrm{N}^{2-\gamma} m \rho^{m-1} \chi(\rho) =: \mathbf{c}_{\mathrm{N}}.$$

In other words, the fluctuations travel at velocity $\mathbf{c}_{\mathrm{N}}$. Therefore, in order to see a non-trivial evolution of the density fluctuations in the diffusive time scale N^2, we need to recenter the density fluctuation field by removing $\mathbf{c}_{\mathrm{N}}$ as follows:

$$\widetilde{\mathcal{Y}}^{\mathrm{N}}_t(\varphi) = \frac{1}{\sqrt{\mathrm{N}}} \sum_{k \in \mathbb{Z}} \varphi\big(\tfrac{k - \mathbf{c}_{\mathrm{N}} t}{\mathrm{N}}\big)\big(\eta_{t\mathrm{N}^2}(k) - \rho\big). \tag{3.45}$$

In the following we still denote the field by $\mathcal{Y}^{\mathrm{N}}_t$ for simplicity of notation. Actually, a simple computation shows that the choice $\rho = \frac{2}{3}$ implies that $\mathbf{c}_{\mathrm{N}}$ vanishes. To lighten further computations we will often assume $\rho = \frac{2}{3}$, but we stress that the results hold for any ρ.

d) Assumptions 3.3.1–3.3.3

Recall Notation 3.3.1, and assume $a = 2$. We now check the three assumptions:

- Assumption 3.3.1 is satisfied since ν_ρ is the Bernoulli product measure ;
- Assumption 3.3.2 about the Dirichlet form is not satisfied, because $\mathbf{I}^{\mathrm{N}}_{k,k+1}(f)$ writes in the form (3.13) with $\zeta^{\mathrm{N}}_{k,k+1}(\eta) = c_{k,k+1}(\eta)$, which is not uniformly bounded away from 0 ;
- Assumption 3.3.3 is not trivial:
  - the symmetric current can be decomposed as (3.15) from Proposition 3.4.1 above, but it is not clear that (3.18) holds, because of the presence of bad configurations ;
  - in the same way, the explicit expressions for the polynomials P_j in Proposition 3.4.1 permit to isolate the quadratic function $\overline{\eta}(k)\overline{\eta}(k+1)$ as in (3.16), and to get an explicit formula for g_{N}, but the convergence (3.17) is not easy to prove, for the same reason (presence of bad configurations).

Therefore, in what follows we explain how to adapt the proof of the second order Boltzmann-Gibbs principle to microscopic models which do not satisfy Assumption 3.3.2 because of kinetic constraints, and how to prove (3.17) and (3.18). In fact, the same idea applies to all the estimates, since all the terms write on the form

$$\mathbb{E}_{\rho}^{N}\Big[\Big(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\tau_k g(\eta_{sN^2})ds\Big)^2\Big].$$

In each one of these terms, we first need to restrict the set of configurations on which the integrals are performed. More precisely, we separate the set of configurations into two sets: the *irreducible* component that contains all configurations with at least one mobile cluster in a suitable position, and the remaining configurations, which have small weight under the equilibrium measure ν_ρ, as follows:

Lemma 3.4.2 (Restriction of the set of configurations). *Let $g : \Omega \to \mathbb{R}$ be a mean zero local function in $\mathbf{L}^2(\nu_\rho)$ whose support is contained in $\{-L+1, ..., L\}$. Then, there exists $C > 0$ such that, for any $t > 0$, any integer $L_0 \leqslant L$ and any $N \in \mathbb{N}$*

$$\mathbb{E}_{\rho}^{N}\Big[\Big(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\,\tau_k g(\eta_{sN^2})\,ds\Big)^2\Big] \leqslant Ct^2\,NL\,(1-\rho^2)^{L_0/2}\,\mathrm{Var}(g)\,\|\mathbf{v}\|_{2,N}^2 + 2\mathbb{E}_{\rho}^{N}\Big[\Big(\int_0^t \sum_{k\in\mathbb{Z}} \mathbf{v}(k)\,\tau_k g(\eta_{sN^2})\,\mathbf{1}_{\mathcal{G}_{L_0}(k+L)}(\eta_{sN^2})\,ds\Big)^2\Big], \quad (3.46)$$

where $\mathrm{Var}(g)$ is the variance of g with respect to the measure ν_ρ.

Remark *3.4.1.* Let us give some highlights for our choice to condition on the set $\mathcal{G}_{L_0}(k+L)$ in (3.46). Fix $k \in \mathbb{Z}$. In the last integral (3.46) we impose the configuration $\tau_k\eta_{sN^2}$ to belong to $\mathcal{G}_{L_0}(k+L)$. In other words, we want to find at time sN^2 a mobile cluster in the box $\{k+L+1, ..., k+L+L_0\}$. Thus, the condition involves a box of size L_0 which does not intersect the support of $\tau_k g(\eta)$, which is by assumption $\{k-L+1, ..., k+L\}$. We note that we could have chosen to use the condition $\mathbf{1}_{\mathcal{G}_{L_0}(k-L-L_0)}(\eta)$, which asks for a mobile cluster in the box $\{k-L-L_0+1, ..., k-L\}$. The result would be exactly the same.

Remark *3.4.2.* Note that whenever $L_0 = N^\theta$, with $\theta > 0$, and L remains polynomial in N, the first term on the right hand side of (3.46) vanishes as $N \to \infty$.

Provided with Lemma 3.4.2 (which is proved in [23, Lemma 5.4], the same strategy as in [59] can be worked out: a *path argument* (see [23, Lemma 5.7]) explains how the presence of a good box (guaranteed by Lemma 3.4.2), and of a mobile cluster, helps us to deal with the possible degeneracy of rates. Then, in order to prove Theorem 3.3.1, we use the same decomposition idea (3.25)–(3.31), but, when computing $\mathfrak{I}_{t,N}(g_w)$, we first apply Lemma 3.4.2. After that, we estimate the contributions of all the terms by using the same ideas as in Section 3.3.4.

e) Final result

Finally, the limit crossover result given by Theorem 3.3.5 holds with parameters:

$$\sigma = \rho, \qquad \mathrm{D} = \mathrm{D}(\rho) = 2\rho^2(1-\rho), \qquad \overline{\lambda} = \frac{1}{2}\lambda \mathrm{D}''(\rho).$$

The setting is the infinite one without any boundary condition, *i.e.* $\mathcal{S}^{\text{test}} = \mathcal{S}(\mathbb{R})$ is the usual Schwartz space, and Δ the usual Laplacian in the infinite space. The uniqueness result of Theorem 3.2.2 is therefore the same as in [65].

3.4.2 Microscopic defects

In [43] written in collaboration with T. Franco and P. Gonçalves, we investigate the standard weakly asymmetric exclusion process that we perturb at one particular bond, in order to consider microscopic dynamics with local defects. The dynamics is, again on, the full lattice ($\Lambda = \mathbb{Z}$) and can be described as follows: a particle at the site k jumps to $k+1$ (resp. $k-1$) at rate

$$\frac{1}{2} + \frac{\lambda}{2\mathrm{N}^\gamma}, \qquad \text{resp.} \quad \frac{1}{2} - \frac{\lambda}{2\mathrm{N}^\gamma},$$

but the jump rate from -1 to 0 (resp. 0 to -1) is equal to

$$\frac{\alpha}{2\mathrm{N}^\beta} + \frac{\lambda}{2\mathrm{N}^\gamma}, \qquad \text{resp.} \quad \frac{\alpha}{2\mathrm{N}^\beta} - \frac{\lambda}{2\mathrm{N}^\gamma},$$

with $\alpha > 0$ and $\beta \geqslant 0$; see Figure 3.6. In order to have positive rates we have to impose some conditions on the parameters $\lambda, \beta, \alpha, \gamma$: either $\gamma > \beta$ or ($\beta = \gamma$ and $\alpha \geqslant \lambda$).

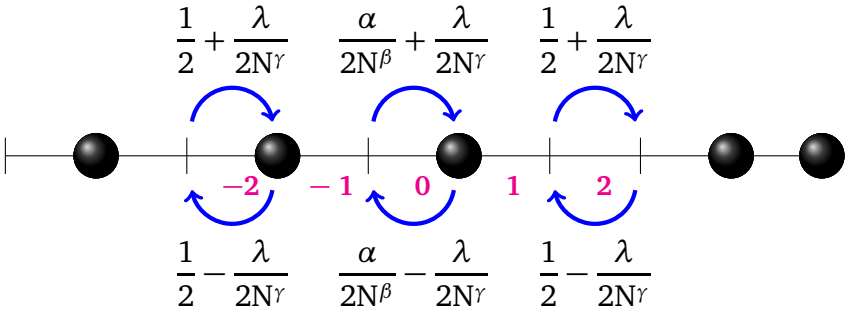

Figure 3.6: Illustration of the jump rates. The bond of vertices $\{-1, 0\}$ has particular rates.

For the choice:

- $\lambda = \beta = 0$ and $\alpha = 1$, we recover the symmetric simple exclusion process (SSEP) ;
- $\lambda = 0$, $\alpha > 0$ and $\beta \geqslant 0$ we recover the SSEP with a slow bond (without the weak asymmetry) which has been investigated in [46]. The authors proved a phase transition depending on the regime of β: for $\beta < 1$ the limit is the same Ornstein-Uhlenbeck process as in the SSEP limit, for $\beta = 1$ the limit is an Ornstein-Uhlenbeck process defined on a Fréchet space $\mathcal{S}^{\text{test}}$ in which the functions have a boundary condition of Robin's type; and for $\beta > 1$ the limit is an Ornstein-Uhlenbeck process defined on a Fréchet space $\mathcal{S}^{\text{test}}$ in which the functions have a boundary condition of Neumann's type.

In [43] we prove the crossover result as given in Theorem 3.3.5 for the regime $\beta \leqslant \frac{1}{2}$; in the other cases we obtain the limiting processes as in [46]:

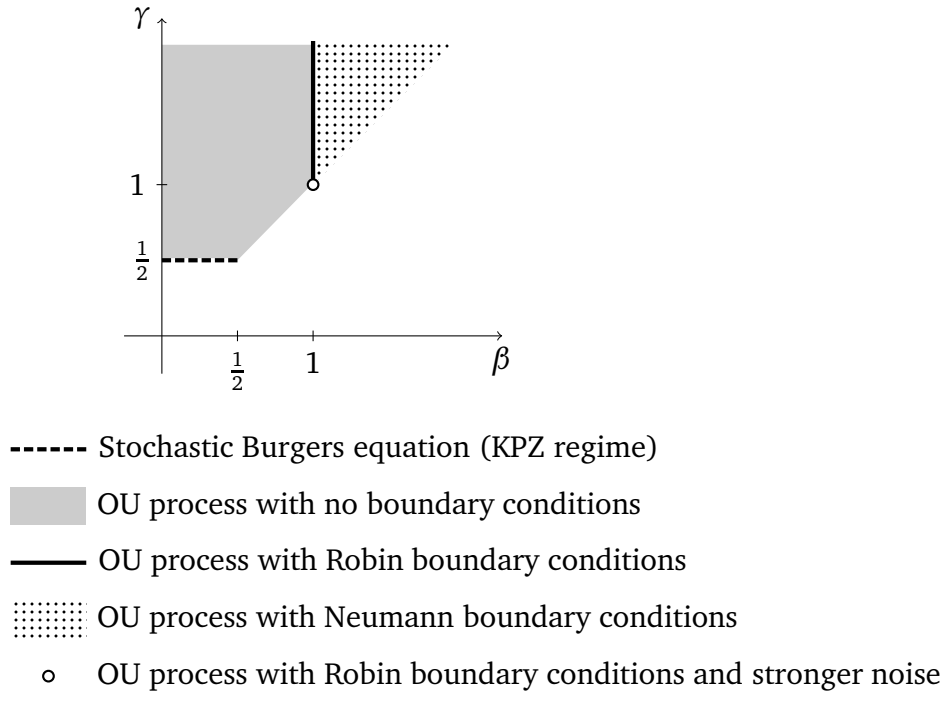

Figure 3.7: Macroscopic density fluctuations. In the region $\beta > \gamma$ the process is not defined (negative rates). OU stands for Ornstein-Uhlenbeck.

a) Assumptions 3.3.1–3.3.3

We now check the three assumptions:

- Assumption 3.3.1 is satisfied, the invariant measure is again the Bernoulli measure ν_ρ ;
- Assumption 3.3.2 about the Dirichlet form is not satisfied, because in this case $\mathbf{I}^{N}_{k,k+1}(f)$ writes in the form (3.13) with

$$\zeta^{N}_{k,k+1}(\eta) = \begin{cases} \dfrac{1}{2} + \dfrac{\lambda}{2N^{\gamma}}\,; & k \neq -1 \\ \dfrac{\alpha}{2N^{\beta}} + \dfrac{\lambda}{2N^{\gamma}}\,; & k = -1. \end{cases}$$

  which, for $k = -1$, is not uniformly bounded away from 0.
  This implies that the second order Boltzmann-Gibbs principle cannot be straightforwardly obtained. In fact, the estimate which is given in the right hand side of (3.22) in Theorem 3.3.1 is modified and needs to be replaced by

$$Ct\left(\frac{L}{N^{a-1}} + \frac{N^{\beta}}{\alpha N^{a-1}} + \frac{tN}{L^2}\right)\|\mathbf{v}\|^2_{2,N} + Ct\left(\frac{N^{\beta}(\log_2(L))^2}{\alpha N^{a-1}}\right)\frac{1}{N}\sum_{k\neq -1}\mathbf{v}^2(k). \tag{3.47}$$

This $\log_2(\mathrm{L})$ factor comes from the estimation of (3.27) and (3.28) in the multiscale decomposition (note that $\mathrm{M} = \log_2(\mathrm{L})$), which has to take into account the slow rate. More details are available in [43, Section 6] ;

- Assumption 3.3.3 is not trivial:
  - the antisymmetric current satisfies (3.16) with $g_{\mathrm{N}} \equiv 0$;
  - but the symmetric part writes

$$j^{\mathrm{N,S}}_{k,k+1}(\eta) = \begin{cases} \dfrac{\mathrm{N}^2}{2}(\eta(k) - \eta(k+1)) \, ; & k \neq -1 \\ \dfrac{\alpha \mathrm{N}^2}{2\mathrm{N}^\beta}(\eta(-1) - \eta(0)) \, ; & k = -1, \end{cases}$$

and it is not clear how to get (3.18). However, this feature had already been treated in [46]: the decomposition above is exactly the reason why boundary conditions emerge at the limit, in the Ornstein-Uhlenbeck processes. In [46] the authors make its definition rigorous by introducing new spaces $\mathcal{S}^{\mathrm{test}}$ of test functions, which we repeat below for the sake of completeness.

b) Spaces of test functions with boundary conditions

First, fix $\alpha > 0$. For any function $\varphi : \mathbb{R} \to \mathbb{R}$ we define the norm:

$$\|\varphi\|^2_{2,\beta} = \int_{\mathbb{R}} \varphi^2(u) \mathrm{d}u + \mathbf{1}_{\beta=1}\left(\frac{\varphi^2(0)}{\alpha} + \mathbf{1}_{\gamma=1} \frac{\lambda \varphi^2(0)}{\alpha^2} \right).$$

Let $\mathbf{L}^2_\beta(\mathbb{R})$ be the space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\|\varphi\|_{2,\beta} < +\infty$. When $\beta \neq 1$, the norm $\|\cdot\|_{2,\beta}$ is the usual $\mathbf{L}^2(\mathbb{R})$–norm with respect to the Lebesgue measure, and for the sake of simplicity we will rewrite it as $\|\cdot\|_2$. Despite the norm above depends on λ, α and γ, for simplicity of notation we do not index on them.

Definition 3.4.1. *We define $\mathcal{S}(\mathbb{R}\backslash\{0\})$ as the space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that:*

(i) φ is smooth on $\mathbb{R}\backslash\{0\}$, i.e. $\varphi \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\})$,

(ii) φ is continuous from the right at 0,

(iii) for all non-negative integers k, ℓ, the function φ satisfies

$$\|\varphi\|_{k,\ell} = \sup_{u \neq 0} \left| (1 + |u|^\ell) \frac{\mathrm{d}^k \varphi}{\mathrm{d}u^k}(u) \right| < \infty. \tag{3.48}$$

We are now ready to introduce the spaces $\mathcal{S}^{\mathrm{test}}$:

Definition 3.4.2. *We define three spaces of test functions depending on the range of β: let $\mathcal{S}^{\mathrm{test}} = \mathcal{S}_\beta(\mathbb{R})$ be*

$$\begin{cases} \mathcal{S}(\mathbb{R}) \quad \textit{(the usual Schwartz space)} & \textit{if } \beta < 1 \\ \left\{ \varphi \in \mathcal{S}(\mathbb{R}\backslash\{0\}) \, ; \, \mathrm{d}^{2k+1}\varphi(0^+) = \mathrm{d}^{2k+1}\varphi(0^-) = \alpha\left(\mathrm{d}^{2k}\varphi(0^+) - \mathrm{d}^{2k}\varphi(0^-) \right), \forall \, k \in \mathbb{N}_0 \right\} & \textit{if } \beta = 1 \\ \left\{ \varphi \in \mathcal{S}(\mathbb{R}\backslash\{0\}) \, ; \, \mathrm{d}^{2k+1}\varphi(0^+) = \mathrm{d}^{2k+1}\varphi(0^-) = 0, \forall \, k \in \mathbb{N}_0 \right\} & \textit{if } \beta > 1 \end{cases}$$

Finally, we need to define the operators Δ^{test} and ∇^{test}:

DEFINITION 3.4.3. *We define* $\nabla^{\text{test}} = \nabla_\beta : \mathcal{S}_\beta(\mathbb{R}) \to \mathcal{S}(\mathbb{R}\backslash\{0\})$ *and* $\Delta^{\text{test}} = \Delta_\beta : \mathcal{S}_\beta(\mathbb{R}) \to \mathcal{S}_\beta(\mathbb{R})$ *as the operators acting on functions* $\varphi \in \mathcal{S}_\beta(\mathbb{R})$ *as*

$$\nabla_\beta \varphi(u) = \begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}u}(u), & \text{if } u \neq 0, \\ \dfrac{\mathrm{d}\varphi}{\mathrm{d}u}(0^+), & \text{if } u = 0, \end{cases} \quad \text{and} \quad \Delta_\beta \varphi(u) = \begin{cases} \dfrac{\mathrm{d}^2\varphi}{\mathrm{d}u^2}(u), & \text{if } u \neq 0, \\ \dfrac{\mathrm{d}^2\varphi}{\mathrm{d}u^2}(0^+), & \text{if } u = 0. \end{cases}$$

When $\beta < 1$, the operator ∇_β (resp. Δ_β) coincides with the usual gradient operator ∇ (resp. Laplacian operator Δ). Under Definitions 3.4.2 and 3.4.3, if $\varphi \in \mathcal{S}_\beta(\mathbb{R})$, then $\Delta_\beta \mathrm{T}_t^\beta \varphi \in \mathcal{S}_\beta(\mathbb{R})$, where T_t^β is the semi-group of the PDE associated with the macroscopic evolution.

c) Final result

With these definitions, and the new second order Boltzmann-Gibbs principle with the estimate given in (3.47), we obtain Theorem 3.3.5 as follows:

- if $(\beta \leqslant \frac{1}{2}$ and $\gamma > \frac{1}{2})$ OR $(\gamma \geqslant \beta > \frac{1}{2})$, the macroscopic fluctuation field is the unique Ornstein-Uhlenbeck process as defined in Proposition 3.2.1 with $\mathcal{S}^{\text{test}} = \mathcal{S}_\beta(\mathbb{R})$ and the parameters
$$\sigma = \tfrac{1}{2}, \qquad \mathrm{D} = \chi(\rho).$$
- if $(\gamma = \frac{1}{2}$ and $\beta \leqslant \frac{1}{2})$, the macroscopic fluctuation field is the unique energy solution of the SBE as defined in Theorem 3.2.2 with $\mathcal{S}^{\text{test}} = \mathcal{S}(\mathbb{R})$ and
$$\sigma = \tfrac{1}{2}, \qquad \overline{\lambda} = \lambda, \qquad \mathrm{D} = \chi(\rho).$$

3.4.3 Stochastic reservoirs

Finally, in [60] written in collaboration with P. Gonçalves and N. Perkowski, our goal is to go beyond the weak KPZ universality conjecture and to derive the KPZ equation with boundary conditions, from an interacting particle system in contact with stochastic reservoirs. We want to legitimate the choice done at the macroscopic level for the KPZ/SBE equation from the microscopic description of the system. For that purpose, we extend the notion of energy solutions to the stochastic Burgers equation (3.1) by adding Dirichlet boundary conditions: in other words we prove Theorem 3.2.2 with a space $\mathcal{S}^{\text{test}}$ which involves boundary conditions.

We also introduce the notion of energy solutions to two related SPDEs: the KPZ equation with Neumann boundary conditions and the stochastic heat equation with Robin boundary conditions. We prove their existence and uniqueness, and we rigorously establish the formal links between the three equations. This is more subtle than expected: a formal computation suggests that the Cole-Hopf transform of the KPZ equation with Neumann boundary conditions should also satisfy Neumann boundary conditions, but we show that instead it satisfies Robin boundary conditions.

Finally, we associate an interface growth model to our microscopic model, roughly speaking by integrating it in the space variable, and show that it converges to the energy solution of the KPZ equation, thereby giving a physical justification of the Neumann boundary conditions.

a) WASEP with reservoirs

At the microscopic level, the model that we consider is the following: contrarily to the previous models, the system of particles evolves on the *finite* set $\Lambda_N = \{1, \dots, N-1\}$ of size $N \in \mathbb{N}$. The Markov process $\eta_t = \{\eta_t(k) \; ; \; k \in \Lambda_N\}$ uses the following possible moves: for any $k \notin \{1, N-1\}$, a particle at site k can attempt to jump to its neighbouring sites $k-1$ or $k+1$, provided that they are empty. Similarly, a particle at site 1 can jump to its right neighbour 2 or it can leave the system, and the particle at site $N-1$ can jump to its left neighbour $N-2$ or it can leave the system. Moreover, attached to the extremities of Λ_N there are two reservoirs of particles: one at site 0, which can send a particle to site 1 (if this site is empty), and the other one at site N, which can send a particle to site $N-1$ (if this site is empty). Another interpretation of the boundary dynamics could be given as follows: particles can either be created (resp. annihilated) at the sites $x = 1$ or $x = N-1$ if the site is empty (resp. occupied).

As before, all possible moves are encoded into the generator, and we are interested in the accelerated Markov process $\{\eta_{tN^2}\}$ in the diffusive time scale. The state space is $\Omega = \Omega_N = \{0,1\}^{\Lambda_N}$, and the generator $\mathcal{L}_N$ is given here by $\mathcal{L}_N = \mathcal{L}_N^{\text{bulk}} + \mathcal{L}_N^{\text{bnd}}$, where $\mathcal{L}_N^{\text{bulk}}$ and $\mathcal{L}_N^{\text{bnd}}$ given below encode respectively the dynamics on the bulk, and on the left/right boundaries.

For any $\lambda > 0$ and $\gamma \geqslant \frac{1}{2}$ we define $\mathcal{L}_N^{\text{bulk}}$ and $\mathcal{L}_N^{\text{bnd}}$ acting on functions $f : \Omega_N = \{0,1\}^{\Lambda_N} \to \mathbb{R}$ as follows: first,

$$\big(\mathcal{L}_N^{\text{bulk}} f\big)(\eta) = \sum_{k=1}^{N-2} \Big(\Big(1 + \frac{\lambda}{N^\gamma}\Big)\eta(k)\big(1-\eta(k+1)\big) + \eta(k+1)\big(1-\eta(k)\big) \Big) \big(f(\eta^{k,k+1}) - f(\eta)\big),$$

Second, the generator of the dynamics at the boundaries is given by

$$\begin{aligned}\big(\mathcal{L}_N^{\text{bnd}} f\big)(\eta) = &\Big(\Big(1 + \frac{\lambda}{N^\gamma}\Big)\overline{\rho}\big(1-\eta(1)\big) + \eta(1)\big(1-\overline{\rho}\big) \Big) \big(f(\sigma^1\eta) - f(\eta)\big) \\ &+ \Big(\Big(1 + \frac{\lambda}{N^\gamma}\Big)\eta(N-1)\big(1-\overline{\rho}\big) + \overline{\rho}\big(1-\eta(N-1)\big) \Big) \big(f(\sigma^{N-1}\eta) - f(\eta)\big)\end{aligned}$$

where $\overline{\rho} \in (0,1)$ and

$$(\sigma^k\eta)(\ell) = \begin{cases} 1-\eta(\ell) & \text{if } \ell = k, \\ \eta(\ell) & \text{if } \ell \neq k. \end{cases}$$

We refer to Figure 3.8 for an illustration of the dynamics.

The invariant measure for these dynamics is the Bernoulli product measure on $\{0,1\}^{\Lambda_N}$ of parameter $\rho = \overline{\rho}$, which we denote by ν_ρ^N. We prove in [60] the equivalent of Theorem 3.3.5 where the SPDEs are defined with Dirichlet boundary conditions, and assuming that the initial density is $\rho = \frac{1}{2}$ (this assumption, which aims at removing a transport phenomenon inside the system, will be explained below).

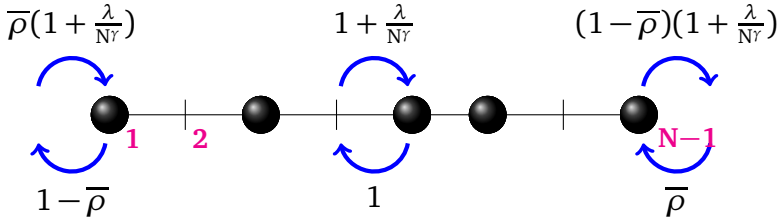

Figure 3.8: Illustration of the jump rates. The leftmost and rightmost rates are the entrance/exiting rates.

b) Assumptions 3.3.1–3.3.3

Here, Assumptions 3.3.1 and Assumption 3.3.2 do not cause any trouble, since the invariant measures are the Bernoulli product measures and the rates are uniformly bounded away from zero. The main difficulty is to adapt the proof of the second order Boltzmann-Gibbs principle in finite volume, and, even more complex, to prove the uniqueness result about the energy solutions given by Theorem 3.2.2. First, Theorem 3.3.1 rewrites in the case of finite volume (and diffusive scale $a=2$) as follows:

THEOREM 3.4.3 (Second order Boltzmann-Gibbs principle in finite volume, with $a=2$). *There exists a constant $C>0$ such that, for any $N,\ell\in\mathbb{N}$ such that $\ell<\frac{N}{4}$ and any $t>0$,*

$$\mathbb{E}^N_\rho\Big[\Big(\int_0^t\sum_{k=1}^{N-2}\mathbf{v}(k)\Big(\overline{\eta}_{sN^2}(k)\overline{\eta}_{sN^2}(k+1)-\mathbb{Q}(x,\ell,\eta_{sN^2})\Big]\,ds\Big)^2\Big]\leqslant Ct\Big(\frac{\ell}{N}+\frac{tN}{\ell^2}\Big)\|\mathbf{v}\|^2_{2,N},\qquad(3.49)$$

where

$$\mathbb{Q}(k,\ell,\eta)=\begin{cases}\big(\overrightarrow{\eta}^\ell(k)\big)^2-\dfrac{\chi(\rho)}{\ell}, & \text{if } k\in\{1,\dots,N-2\ell-1\}\,;\\[2ex] \big(\overleftarrow{\eta}^\ell(k)\big)^2-\dfrac{\chi(\rho)}{\ell}, & \text{if } k\in\{N-2\ell,\dots,N-2\},\end{cases}\qquad(3.50)$$

and $\overrightarrow{\eta}^\ell(k)$ has been defined in (3.23), and $\overleftarrow{\eta}^\ell(k)$ is the average centered configuration on a box of size ℓ situated to the left of the site $k\in\Lambda_N$, namely

$$\overleftarrow{\eta}^\ell(k)=\frac{1}{\ell}\sum_{m=k-\ell}^{k-1}\overline{\eta}(m).$$

The assumption $\ell<\frac{N}{4}$ ensures that one of the two conditions in (3.50) is always satisfied and $\overrightarrow{\eta}^\ell(k)$ and $\overleftarrow{\eta}^\ell(k)$ are always well defined. The proof goes exactly along the lines of Section 3.3.4, but here we need to split all the estimates in two cases, depending whether we are dealing with a site k which is closer to the right or to the left boundary. More precisely, let us choose a site k which is not too close to the right boundary, in the sense that there are at least 2ℓ sites between k and $N-1$, then we can replace the local function $\overline{\eta}(k)\overline{\eta}(k+1)$ by the square of the average to its right $\big(\overrightarrow{\eta}^\ell(k)\big)^2$ (see Figure 3.9). The main reason to keep at least 2ℓ sites between k and $N-1$ is because the proof makes use of the sites situated between $k+\ell+1$ and

$k+2\ell$. Otherwise, when $k+2\ell > \mathrm{N}-1$, we replace the same local function by the square of the average to its left $(\overleftarrow{\eta}^{\ell}(k))^2$ (see Figure 3.10).

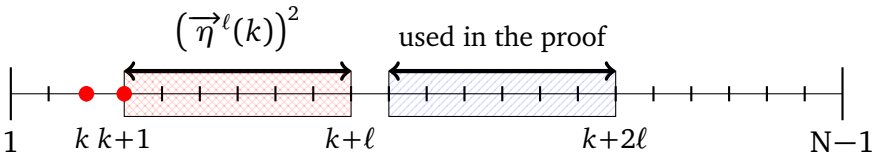

Figure 3.9: Replacement for the local function $\overline{\eta}(k)\overline{\eta}(k+1)$ when $k+2\ell \leqslant \mathrm{N}-1$.

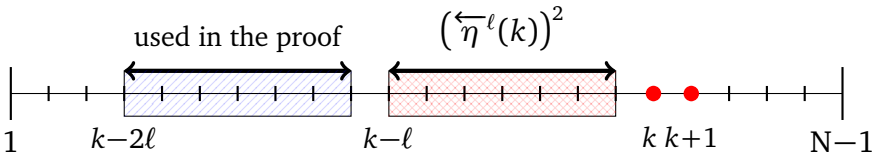

Figure 3.10: Replacement for the local function $\overline{\eta}(k)\overline{\eta}(k+1)$ when $k+2\ell > \mathrm{N}-1$.

The rest of the proof is really similar to what we already did in Section 3.3.4, more details can be found in [60, Section 5].

Concerning the uniqueness of energy solutions, the proof is much more involved, and we now expose some of its ideas in the next section.

c) Energy solutions with boundary conditions

We first need to introduce the suitable space for test functions: let

$$\mathcal{S}_{\mathrm{Dir}} = \left\{ \varphi \in \mathcal{C}^{\infty}([0,1]) \, ; \, \mathrm{d}^{2k}\varphi(0) = \mathrm{d}^{2k}\varphi(1) = 0, \text{ for any } k \in \mathbb{N}_0 \right\} \tag{3.51}$$

$$\mathcal{S}_{\mathrm{Neu}} = \left\{ \varphi \in \mathcal{C}^{\infty}([0,1]) \, ; \, \mathrm{d}^{2k+1}\varphi(0) = \mathrm{d}^{2k+1}\varphi(1) = 0, \text{ for any } k \in \mathbb{N}_0 \right\}, \tag{3.52}$$

both equipped with the family of seminorms $\left\{ \sup_{u\in[0,1]} |\mathrm{d}^k\varphi(u)| \right\}_{k\in\mathbb{N}_0}$. Then $\mathcal{S}_{\mathrm{Dir}}$ and $\mathcal{S}_{\mathrm{Neu}}$ are Fréchet spaces, and we write $\mathcal{S}'_{\mathrm{Dir}}$ and $\mathcal{S}'_{\mathrm{Neu}}$ for their topological duals.

Now, let $-\Delta$ be the closure of the Laplacian operator $-\Delta : \mathcal{S}_{\mathrm{Dir}} \to \mathbf{L}^2([0,1])$ as an unbounded operator in the Hilbert space $\mathbf{L}^2([0,1])$. It is a positive and self-adjoint operator whose eigenvalues and eigenfunctions are given for any integer $m \geqslant 1$, respectively by $\lambda_m = (m\pi)^2$ and $e_m(u) = \sqrt{2}\sin(m\pi u)$. The set $\{e_m \, ; \, m \geqslant 1\}$ forms an orthonormal basis of $\mathbf{L}^2([0,1])$.

Then, similarly to what we did in Section 3.2, we define Δ_{Neu} (resp. Δ_{Dir}) as the Neumann (resp. Dirichlet) Laplacian acting on $\varphi \in \mathcal{S}_{\mathrm{Neu}}$ and $\mathcal{Y} \in \mathcal{S}'_{\mathrm{Neu}}$ (resp. $\varphi \in \mathcal{S}_{\mathrm{Dir}}$ and $\mathcal{Y} \in \mathcal{S}'_{\mathrm{Dir}}$) as follows:

$$\Delta_{\mathrm{Neu}}\mathcal{Y}(\varphi) = \mathcal{Y}(\Delta\varphi), \qquad \text{resp.} \qquad \Delta_{\mathrm{Dir}}\mathcal{Y}(\varphi) = \mathcal{Y}(\Delta\varphi).$$

Finally, the last operator denoted by ∇_{Dir}, is the gradient operator defined as follows: given $\mathcal{Y} \in \mathcal{S}'_{\text{Neu}}$ and $\varphi \in \mathcal{S}_{\text{Dir}}$ we set

$$\nabla_{\text{Dir}}\mathcal{Y}(\varphi) = -\mathcal{Y}(\nabla\varphi). \tag{3.53}$$

The general stochastic Burgers equation (3.7) then reads as

$$\mathrm{d}\mathcal{Y}_t = \sigma\Delta_{\text{Dir}}\,\mathcal{Y}_t\,\mathrm{d}t + \overline{\lambda}\nabla_{\text{Dir}}\left(\mathcal{Y}_t^2\right)\mathrm{d}t + \sqrt{\mathrm{D}}\,\nabla_{\text{Dir}}\,\mathrm{d}\mathcal{W}_t, \tag{3.54}$$

where $\{\mathcal{W}_t\}_{t\in[0,\overline{t}]}$ is a $\mathcal{S}'_{\text{Neu}}$–valued Brownian motion with covariance (3.5). Since $\mathcal{W}_t$ is $\mathcal{S}'_{\text{Neu}}$–valued, then $\nabla_{\text{Dir}}\mathcal{W}_t$ is a well defined object in $\mathcal{S}'_{\text{Dir}}$.

Finally, the right choice for the approximation of the identity ι_ε given in Theorem 3.2.2 is

$$\iota_\varepsilon(u)(v) = \begin{cases} \varepsilon^{-1}\,\mathbf{1}_{]u,u+\varepsilon]}(v) & \text{if } u\in[0,1-2\varepsilon), \\ \varepsilon^{-1}\,\mathbf{1}_{[u-\varepsilon,u[}(v) & \text{if } u\in[1-2\varepsilon,1]. \end{cases}$$

Before going any further into the scheme of the proof, we need to introduce the *stochastic heat equation with Robin boundary conditions*, and we take this opportunity to define also the *KPZ equation with Neumann boundary conditions*, since they all are closely related.

d) KPZ equation and stochastic heat equation

We define the notion of solution for *the KPZ equation with Neumann boundary condition*, which is formally given by

$$\mathrm{d}\mathcal{Z}_t = \sigma\Delta_{\text{Neu}}\,\mathcal{Z}_t\mathrm{d}t + \overline{\lambda}\,(\nabla\mathcal{Z}_t)^{\diamond 2}\,\mathrm{d}t + \sqrt{\mathrm{D}}\,\mathrm{d}\mathcal{W}_t, \tag{3.55}$$

with $\sigma, \mathrm{D} > 0$ and $\overline{\lambda}\in\mathbb{R}$, and where $\{\mathcal{W}_t\}_{t\in[0,\overline{t}]}$ is a standard $\mathcal{S}'_{\text{Neu}}$–valued Brownian motion with covariance (3.5) and $(\nabla\mathcal{Z}_t)^{\diamond 2}$ denotes a *renormalized square* which will have a precise meaning in Theorem 3.4.4 below.

Definition 3.4.4. *For $\varepsilon > 0$ and $\mathcal{Z}\in\mathcal{C}([0,1])$, let us define*

$$\nabla_\varepsilon\mathcal{Z}(u) = \begin{cases} \varepsilon^{-1}(\mathcal{Z}(u+\varepsilon)-\mathcal{Z}(u)), & \text{if } u\in[0,1-2\varepsilon), \\ \varepsilon^{-1}(\mathcal{Z}(u)-\mathcal{Z}(u-\varepsilon)), & \text{if } u\in[1-2\varepsilon,1]. \end{cases} \tag{3.56}$$

The following theorem aims at defining uniquely the stochastic process solution to (3.55):

Theorem 3.4.4. *Let Z be a random variable with values in $\mathcal{C}([0,1])$, such that $\nabla_{\text{Dir}}\mathrm{Z}$ is a white noise with variance $\mathrm{D}/(2\sigma)$ and such that $\sup_{u\in[0,1]}\mathbb{E}[e^{2\mathrm{Z}(u)}] < \infty$.*

There exists a unique (in law) stochastic process $\{\mathcal{Z}_t\}_{t\in[0,\overline{t}]}$ with trajectories in $\mathcal{C}([0,\mathrm{T}],\mathcal{S}'_{\text{Neu}})$, called almost stationary energy solution *of (3.55), such that:*

1. $\mathrm{law}(\mathcal{Z}_0) = \mathrm{law}(\mathrm{Z})$;
2. *there exists a stationary energy solution $\{\mathcal{Y}_t\}_{t\in[0,\overline{t}]}$ to the stochastic Burgers equation (3.7) – with trajectories in $\mathcal{C}([0,\mathrm{T}],\mathcal{S}_{\text{Dir}})$ – such that $\nabla_{\text{Dir}}\mathcal{Z} = \mathcal{Y}$;*

3. *for any* $\varphi \in \mathcal{S}_{\text{Neu}}$ *and any* $t \in [0, \overline{t}]$*, the process*

$$\mathcal{N}_t(\varphi) = \mathcal{Z}_t(\varphi) - \mathcal{Z}_0(\varphi) - \sigma \int_0^t \mathcal{Z}_s(\Delta\varphi)\,\mathrm{d}s - \overline{\lambda}\mathcal{B}_t(\varphi)$$

is a continuous martingale with respect to the natural filtration associated to $\mathcal{Z}$*., defined as in* (3.6) *but with* $\varphi \in \mathcal{S}_{\text{Neu}}$ *and with* $\mathcal{Z}_s$ *in the place of* $\mathcal{Y}_s$*, of quadratic variation*

$$\left\langle \mathcal{N}(\varphi) \right\rangle_t = t\mathrm{D}\|\varphi\|^2_{\mathbf{L}^2([0,1])},$$

where the process $\{\mathcal{B}_t\}_{t\in[0,\overline{t}]}$ *is obtained through the following limit, which holds in the* $\mathbf{L}^2$*-sense:*

$$\mathcal{B}_t(\varphi) = \lim_{\varepsilon\to 0} \int_0^t \int_0^1 \left((\nabla_\varepsilon \mathcal{Z}_s(u))^2 - \frac{\mathrm{D}}{2\sigma\varepsilon} \right) \varphi(u)\mathrm{d}u\mathrm{d}s \tag{3.57}$$

where $\nabla_\varepsilon \mathcal{Z}_s(u)$ *has been defined in* (3.56).

REMARK *3.4.3.* Note that we did not require $\mathcal{Z}$ to be a continuous function in u, so it is not obvious that $\nabla_\varepsilon \mathcal{Z}(u)$ as defined in (3.56) is well defined. But $\nabla_\varepsilon \mathcal{Z}(u) = \mathcal{Y}(\iota_\varepsilon(u))$, and as discussed before the right hand side can be made sense of with [55, Remark 4].

REMARK *3.4.4.* By "almost stationary" we mean that the derivative of $\mathcal{Z}$ is stationary: It is a (multiple of the) white noise for all times, so $\mathcal{Z}$ is a Brownian motion for all times. But the starting point $\mathcal{Z}_t(0)$ of this Brownian motion will change with time.

The proof of Theorem 3.4.4 is of the same difficulty as the proof of Theorem 3.2.2 for the SBE with Dirichlet boundary conditions. They both rely on a third SPDE, the *stochastic heat equation (SHE) with Robin boundary conditions*, which is written as

$$\mathrm{d}\Phi_t = \sigma\Delta_{\text{Rob}}\Phi_t\,\mathrm{d}t + \sqrt{\mathrm{D}}\,\Phi_t\,\mathrm{d}\mathcal{W}_t, \tag{3.58}$$

with $\sigma, \mathrm{D} > 0$ and $\{\mathcal{W}_t\}_{t\in[0,\overline{t}]}$ is a standard $\mathcal{S}'_{\text{Neu}}$–valued Brownian motion with covariance (3.5), and where we want to impose (formally):

$$\nabla\Phi_t(0) = \alpha\Phi_t(0), \qquad \nabla\Phi_t(1) = \beta\Phi_t(1), \tag{3.59}$$

with $\alpha, \beta \in \mathbb{R}$. To see how we should implement these boundary conditions, consider $f \in \mathcal{C}^2([0,1])$ and $\varphi \in \mathcal{S}_{\text{Neu}}$, and assume that f satisfies the Robin boundary conditions $\nabla f(0) = \alpha f(0)$, and $\nabla f(1) = \beta f(1)$. Integrating by parts twice and using the Neumann boundary conditions for φ and the Robin boundary conditions for f, we obtain

$$\int_0^1 \Delta f(u)\varphi(u)\mathrm{d}u = \int_0^1 f(u)\Delta\varphi(u)\mathrm{d}u - \alpha f(0)\varphi(0) + \beta f(1)\varphi(1).$$

This suggests to define the operator

$$\Delta_{\text{Rob}} : \mathcal{S}'_{\text{Neu}} \cap \mathcal{C}([0,1]) \to \mathcal{S}'_{\text{Neu}}, \qquad \Delta_{\text{Rob}} f(\varphi) = f(\Delta\varphi) - \alpha f(0)\varphi(0) + \beta f(1)\varphi(1),$$

and in principle this leads to a definition of solutions to (3.58). But for technical reasons we do not want to assume *a priori* our solution to be continuous in (t,u). This means we could change the values of $\Phi_t(0)$ and $\Phi_t(1)$ without changing $\Phi_t(\varphi)$, so we cannot hope to get uniqueness without further assumptions. Let us introduce a suitable class of processes for which the boundary term is well defined: we write $\Phi \in \mathcal{L}^2_C([0,\overline{t}])$ if Φ takes values in the Borel measurable functions on $[0,\overline{t}]\times[0,1]$, if

$$\sup_{(u,t)\in[0,1]\times[0,\overline{t}]} \mathbb{E}\big[\Phi_t(u)^2\big] < \infty,$$

and if for all $t>0$ the process $u \mapsto \Phi_t(u)$ is continuous in $\mathbf{L}^2(\mathbb{P})$, where $\mathbb{P}$ is the law of the process $\Phi_\cdot$ and above $\mathbb{E}$ is the expectation w.r.t. to $\mathbb{P}$. For $\Phi \in \mathcal{L}^2_C([0,T])$ we cannot change the value of $\Phi_t(0)$ without changing Φ_t in an environment of 0, and this would also change $\Phi_t(\varphi)$ for some test functions φ.

Proposition 3.4.5. *Let* F *be a random variable with values in the Borel measurable functions on* $[0,1]$*, such that* $\sup_{u\in[0,1]}\mathbb{E}[F(u)^2]<\infty$.

Then there exists at most one (law of a) process $\{\Phi_t\}_{t\in[0,\overline{t}]} \in \mathcal{L}^2_C([0,T])$*, called* weak solution *to* (3.58) *with boundary condition* (3.59)*, such that:*

1. $\mathrm{law}(\Phi_0) = \mathrm{law}(F)$*;*
2. *for any* $\varphi \in \mathcal{S}_{\mathrm{Neu}}$ *and any* $t\in[0,T]$*, the process*

$$\mathfrak{M}_t(\varphi) = \Phi_t(\varphi) - \Phi_0(\varphi) - \sigma\int_0^t \Phi_s(\Delta\varphi)\,\mathrm{d}s - \sigma\int_0^t (-\alpha\Phi_s(0)\varphi(0)+\beta\Phi_s(1)\varphi(1))\mathrm{d}s \quad (3.60)$$

is a continuous martingale of quadratic variation

$$\big\langle \mathfrak{M}(\varphi)\big\rangle_t = D\int_0^t\int_0^1 |\Phi_s(u)\varphi(u)|^2 \mathrm{d}u\mathrm{d}s\,.$$

Proof of Proposition 3.4.5. To obtain the uniqueness of such solutions is much easier that for the SBE or the KPZ equation as defined in Theorem 3.2.2 and Theorem 3.4.4 respectively. It is given in [60, Appendix D].

We conclude this section by making the link between the KPZ equation (3.55) and the stochastic heat equation (3.58), with their respective boundary conditions. This link is, as expected, done through the Cole-Hopf transformation, although the boundary conditions are linked in a more complicated way than one might guess naively.

Proposition 3.4.6 ([60, Proposition 3.13]). *Let* $\mathcal{Z}$ *be the almost stationary energy solution of the KPZ equation* (3.55) *with Neumann boundary condition as defined in Theorem 3.4.4. Then, for any* $t\in[0,T]$ *we have*

$$\mathcal{Z}_t = \frac{\sigma}{\overline{\lambda}}\log\Phi_t + \frac{D^2\overline{\lambda}^3}{48\sigma^4}t,$$

where Φ *solves the stochastic heat equation*

$$\mathrm{d}\Phi_t = \sigma\Delta_{\mathrm{Rob}}\Phi_t\,\mathrm{d}t + \frac{\sqrt{\mathrm{D}}\,\overline{\lambda}}{\mathrm{A}}\Phi_t\,\mathrm{d}\mathcal{N}_t$$

with the Robin boundary condition

$$\nabla\Phi_t(0) = -\frac{\mathrm{D}\overline{\lambda}^2}{4\sigma^3}\Phi_t(0), \qquad \nabla\Phi_t(1) = \frac{\mathrm{D}\overline{\lambda}^2}{4\sigma^3}\Phi_t(1),$$

and where above $\mathcal{N}_t$ *has been defined in item 3. of Theorem 3.4.4.*

In that sense $\mathcal{Z}$ can be interpreted as a Cole-Hopf solution to the KPZ equation with inhomogeneous Neumann boundary conditions, and then the question arises which of the two formulations (inhomogeneous or homogeneous Neumann conditions) accurately describes the behaviour of our stochastic process at the boundary. While of course the main difficulty with the KPZ equation is that its solutions are not differentiable and in particular we cannot evaluate $\nabla\mathcal{Z}_t(0)$ pointwise, we show in [60, Proposition 3.14] that after averaging a bit in time there are canonical interpretations for $\nabla\mathcal{Z}_t(0)$ and $\nabla\mathcal{Z}_t(1)$, and both indeed vanish. We also show in [60, Proposition 3.15] that the formal change of the boundary conditions from Neumann to Robin in the exponential transformation $\Phi = e^{\mathcal{Z}}$ is reflected in the "pointwise" boundary behaviour of Φ (again after averaging in time).

This should be compared with the recent work of Gerencsér and Hairer [51] who show that the classical solution $\mathcal{Z}^\varepsilon$ to the KPZ equation,

$$\mathrm{d}\mathcal{Z}^\varepsilon_t = \Delta\mathcal{Z}^\varepsilon_t\,\mathrm{d}t + \big((\nabla\mathcal{Z}^\varepsilon_t)^2 - c^\varepsilon\big)\mathrm{d}t + \mathrm{d}\mathcal{W}^\varepsilon_t\,, \tag{3.61}$$

with Neumann boundary condition $\nabla\mathcal{Z}^\varepsilon_t(0) = \nabla\mathcal{Z}^\varepsilon_t(1) = 0$ and where $\mathcal{W}^\varepsilon$ is a mollification of $\mathcal{W}$ and $\{c^\varepsilon\}_{\varepsilon>0}$ is a sequence of diverging constants, may converge to different limits satisfying different boundary conditions as $\varepsilon \to 0$, depending on which mollifier was used for $\mathcal{W}^\varepsilon$. But if the noise is only mollified in space and white in time, then the limit is always the same and it agrees with the Cole-Hopf solution with formal Neumann boundary condition $\nabla\mathcal{Z}_t(0) = \nabla\mathcal{Z}_t(1) = 0$. Since our results show that the Cole-Hopf solution with formal boundary condition $\nabla\mathcal{Z}_t(0) = -\nabla\mathcal{Z}_t(1) = -(\overline{\lambda})^2/4$ satisfies $\nabla\mathcal{Z}_t(0) = \nabla\mathcal{Z}_t(1) = 0$ in the "physical" sense, this suggests that in order to obtain the correct limit it is not only necessary to subtract a large diverging constant c^ε in (3.61), but additionally one should perform a boundary renormalization. Indeed, under boundary conditions the solution $\mathcal{Z}^\varepsilon$ to (3.61) is not spatially homogeneous, and therefore there is no reason to renormalize it by subtracting a constant c^ε and instead the renormalization might also be spatially inhomogeneous.

e) Uniqueness of energy solutions

In [60, Section 6] we prove uniqueness of energy solutions, namely Theorem 3.2.2 and Theorem 3.4.4, following the same strategy used in [65] for proving the uniqueness of energy

solutions on the real line: we mollify an energy solution $\mathcal{Y}$ to the SBE, which we denote by $\mathcal{Y}^n$, then we find a suitable anti-derivative $\mathcal{Z}^n$ of $\mathcal{Y}^n$, and let $\Phi^n = e^{\mathcal{Z}^n}$. Then we take the mollification away and show that $\Phi = \lim_n \Phi^n$ solves (a version of) the SHE. Since uniqueness holds for solutions to the SHE, $\mathcal{Y} = \nabla \log \Phi$ must be unique. But while the strategy is the same as in [65], the technical details are considerably more involved because our space is no longer translation invariant and many of the tools of [65] break down.

Moreover, the dynamics of Φ^n contain a singular term that converges to Dirac deltas at the boundaries, a new effect which can be interpreted as a change of boundary conditions: formally we would expect that $\nabla\Phi_t(0) = e^{\mathcal{Z}_t(0)}\nabla\mathcal{Z}_t(0) = 0$ because $\nabla\mathcal{Z}_t(0) = 0$ by the Neumann boundary conditions for $\mathcal{Z}$. However, the singular term in the dynamics changes the boundary conditions to Robin's type, $\nabla\Phi_t(0) = -c\Phi_t(0)$, $\nabla\Phi_t(1) = c\Phi_t(1)$ for a constant $c \in \mathbb{R}$ which depends on the parameters $\sigma, \overline{\lambda}, \mathrm{D}$ (see Proposition 3.4.6).

f) Final results

Most of our arguments work for any $\rho \in (0,1)$. However, we restrict the statement of the convergence results to the case $\rho = \frac{1}{2}$. This restriction is not only convenience of notation: otherwise we would pick up an additional diverging transport term in the martingale decomposition. In the periodic case or if the underlying lattice is $\mathbb{Z}$ we can kill that term by observing our system in a moving frame, see [55] for instance, and also the definition (3.45) in the previous model, but of course this does not work in finite volume with boundary conditions. Therefore, we need to assume either $\gamma \geqslant \frac{3}{2}$ or $\rho = \frac{1}{2}$. Since we are mostly interested in the case $\gamma = \frac{1}{2}$, we take $\rho = \frac{1}{2}$.

Finally, the convergence result towards both SPDEs with Dirichlet boundary condition, as stated in Theorem 3.3.5 is obtained in the case $\overline{\rho} = \frac{1}{2}$, with the following parameters:

$$\sigma = 1, \qquad \mathrm{D} = \tfrac{1}{2}, \qquad \overline{\lambda} = \lambda.$$

Moreover, as we already mentioned, the KPZ equation and the SBE are closely related, and this relation can also be seen at the microscopic level. There is a natural *height function* $h(k)$ which integrates the density fluctuation field in the space variable, *i.e.* it describes a curve h from $\{1,\dots,\mathrm{N}\}$ to $\mathbb{R}$ which satisfies $h(k+1) = h(k) + (\eta(k) - \rho)$. This suggests the definition $h(k) = \sum_{\ell=1}^{k-1}(\eta(\ell) - \rho)$, which however has the disadvantage that if a new particle enters at site 1, then $h(k)$ increases by 1 for all k. Therefore, we set

$$h_t^{\mathrm{N}}(k) = h_t^{\mathrm{N}}(1) + \sum_{\ell=1}^{k-1}(\eta_t^{\mathrm{N}}(\ell) - \rho), \qquad \text{for any } k \in \{1,\dots,\mathrm{N}\}, \tag{3.62}$$

where, by definition, $h_0^{\mathrm{N}}(1) = 0$ and

- $h_t^{\mathrm{N}}(1)$ increases by 1 whenever a particle at site 1 leaves the system to the left,
- $h_t^{\mathrm{N}}(1)$ decreases by 1 whenever a new particle enters the system at the site 1.

In other words $h_t^{\mathrm{N}}(1)$ is exactly the net number of particles removed from the system at the left boundary until time $t\mathrm{N}^2$. The weak asymmetry in the particle system causes the height

function to slowly decrease because $\lambda > 0$ (it would grow if $\lambda < 0$), and at the first order the decrease is of order $\mathbf{c}_{\mathrm{N}} t$ for

$$\mathbf{c}_{\mathrm{N}} = -\mathrm{N}^{2-\gamma}\, \rho(1-\rho)\lambda. \tag{3.63}$$

For instance, with our stationary dynamics, one can easily see that

$$\mathbb{E}^{\mathrm{N}}_{\rho}\big[h^{\mathrm{N}}_t(k)\big] = \mathbb{E}^{\mathrm{N}}_{\rho}\big[h^{\mathrm{N}}_0(k)\big] - \frac{\lambda}{\mathrm{N}^{\gamma}}\rho(1-\rho)t = \mathbf{c}_{\mathrm{N}} t, \qquad \text{for any } k \in \{1,\dots,\mathrm{N}\}.$$

In the case $\gamma = \frac{1}{2}$ (where we will see KPZ behaviour) note that $\mathbf{c}_{\mathrm{N}} = -\mathrm{N}^{3/2}\rho(1-\rho)\lambda$. So while on the microscopic scale the decrease is negligible, it becomes important on time scales of order N^2. We investigate the fluctuations around the flat interface $\mathbf{c}_{\mathrm{N}} t$, namely:

Definition 3.4.5. *For any $t > 0$, let $\mathcal{Z}^n_t$ be the* height fluctuation field *which acts on functions $\varphi \in \mathcal{S}_{\mathrm{Neu}}$ as*

$$\mathcal{Z}^{\mathrm{N}}_t(\varphi) = \frac{1}{\mathrm{N}^{3/2}} \sum_{k=1}^{\mathrm{N}} \varphi\big(\tfrac{k}{\mathrm{N}}\big)\big(h^{\mathrm{N}}_{t\mathrm{N}^2}(k) - \mathbf{c}_{\mathrm{N}} t\big).$$

Remark *3.4.5.* As expected, $\mathcal{Z}^{\mathrm{N}}_t$ and $\mathcal{Y}^{\mathrm{N}}_t$ are related: if $\varphi \in \mathcal{S}_{\mathrm{Dir}}$, then $\nabla\varphi \in \mathcal{S}_{\mathrm{Neu}}$ and a simple computation, based on a summation by parts, shows that $\mathcal{Z}^{\mathrm{N}}_t(\widetilde{\nabla}_{\mathrm{N}}\varphi)$ can be written as $-\mathcal{Y}^{\mathrm{N}}_t(\varphi)$ plus the boundary terms:

$$-\tfrac{1}{\sqrt{\mathrm{N}}}\varphi\big(\tfrac{1}{\mathrm{N}}\big)\big(h^{\mathrm{N}}_{t\mathrm{N}^2}(1) - \mathbf{c}_{\mathrm{N}} t\big) + \tfrac{1}{\sqrt{\mathrm{N}}}\varphi\big(\tfrac{\mathrm{N}}{\mathrm{N}}\big)\big(h^{\mathrm{N}}_{t\mathrm{N}^2}(\mathrm{N}) - \mathbf{c}_{\mathrm{N}} t\big),$$

where, above, $\widetilde{\nabla}_{\mathrm{N}}\varphi$ is essentially a discrete gradient: $\widetilde{\nabla}_{\mathrm{N}}\varphi(\frac{k}{\mathrm{N}}) = \mathbf{1}_{\{1,\dots,\mathrm{N}-1\}}(k)\ \nabla_{\mathrm{N}}\varphi(\frac{k}{\mathrm{N}})$, in particular $\widetilde{\nabla}_{\mathrm{N}}\varphi(\frac{\mathrm{N}}{\mathrm{N}}) = 0$. Since $\varphi \in \mathcal{S}_{\mathrm{Dir}}$, it is not difficult to show that the last expression vanishes in $\mathbf{L}^2(\mathbb{P}^{\mathrm{N}}_{\rho})$ as $\mathrm{N} \to \infty$.

The counterpart of Theorem 3.3.5 for the height process of our model is:

Theorem 3.4.7. *The sequence of processes $\{\mathcal{Z}^{\mathrm{N}}_t \ ;\ t \in [0,\overline{t}]\}_{\mathrm{N}\in\mathbb{N}}$ converges in distribution as $\mathrm{N} \to \infty$ towards:*

1. *if $\gamma > \frac{1}{2}$, the unique almost stationary energy solution of the KPZ equation (3.55) with Neumann boundary conditions (defined in Theorem 3.4.4) with $\sigma = 1$, $\overline{\lambda} = 0$ and $\mathrm{D} = \frac{1}{2}$;*
2. *if $\gamma = \frac{1}{2}$, the unique almost stationary energy solution of the KPZ equation (3.55) with Neumann boundary conditions (defined in Theorem 3.4.4) with $\sigma = 1$, $\overline{\lambda} = \lambda$ and $\mathrm{D} = \frac{1}{2}$.*

CHAPTER 4 | Microscopic dynamics with degenerate jump rates

This chapter is based on [21, 22]*, written in collaboration with O. Blondel, C. Cancès, C. Erignoux and M. Sasada. Some parts are largely borrowed from these papers.*

Contents

All the models presented in this chapter fall into the category of *exclusion processes with periodic boundary conditions*. Therefore, if $N \in \mathbb{N}$ is the scaling parameter, the configuration space is always $\Omega_N = \{0,1\}^{\mathbb{T}_N}$, and the particle dynamics is as follows: on the periodic domain $\mathbb{T}_N$, we associate with each site a random Poissonian clock. When the clock at site $k \in \mathbb{T}_N$ rings, if there is a particle sitting at this site k, that particle jumps to $k-1$ or $k+1$ if some local constraint is satisfied. More precisely, the infinitesimal generator ruling the evolution in time of this Markov process is given by $\mathcal{L}_N$, which acts on functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$\mathcal{L}_N f(\eta) = \sum_{k \in \mathbb{T}_N} r_{k,k+1}(\eta)\big(f(\eta^{k,k+1}) - f(\eta)\big), \tag{4.1}$$

where the constraint and the exclusion rule are encoded in the rates $r_{k,k+1}$ and $\eta^{k,k+1}$ denotes the configuration obtained from η by exchanging the states of sites $k, k+1$, defined in (3.14). Note that the dynamics conserves the total number of particles $\sum_{k \in \mathbb{T}_N} \eta(k)$.

4.1 Fluid propagation into porous medium

In [21] written in collaboration with O. Blondel, C. Cancès and M. Sasada, we derive for the first time a moving boundary problem from a conservative and degenerate microscopic dynamics (belonging to the family of kinetically constrained lattice gases). We obtain as macroscopic limit the porous medium equation (1.2), which reads in dimension one as

$$\partial_t \rho = \partial^2_{uu}(\rho^m), \tag{4.2}$$

with few restrictions on the initial profile. The PME (4.2) has already been derived from a degenerate and conservative dynamics in [59], for any integer $m \geqslant 2$. In [21] we improve their

results with the derivation of solutions with moving interfaces, more precisely we include the possibility for the initial condition to vanish. Let me note here that the model is the *symmetric version* of the kinetically constrained exclusion process already described in Section 3.4.1 of this manuscript (but instead of looking at the macroscopic density fluctuations around equilibrium, we are now interested in the non-equilibrium behavior, starting from an arbitrary density profile, and in the derivation of the hydrodynamic limits).

Here follows the definition of the model: the generator $\mathcal{L}_N$ of the particle dynamics reads as in (4.1) with the rate

$$r_{k,k+1}(\eta) = \Big(\sum_{\ell=1}^{m} \prod_{\substack{j=-(m-\ell) \\ j\neq 0,1}}^{\ell} \eta(k+j)\Big)\Big(\eta(k)\big(1-\eta(k+1)\big) + \eta(k+1)\big(1-\eta(k)\big)\Big).$$

For the sake of simplicity we will often consider in this manuscript the case $m = 2$ (but the general case is treated in [21]), in which case the last formula reduces to

$$r_{k,k+1}(\eta) = \eta(k-1)\eta(k+2)\Big(\eta(k)\big(1-\eta(k+1)\big) + \eta(k+1)\big(1-\eta(k)\big)\Big).$$

An illustration of possible jumps is provided in Figure 4.1.

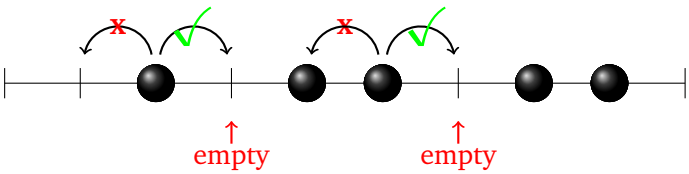

Figure 4.1: Allowed jumps are denoted by ✓. Forbidden jumps are denoted by **x**.

Our choice of initial condition and the lack of ergodicity of the particle dynamics make the usual techniques (like the entropy method, or the relative entropy method originally due to Yau [100], which are explained in detail in [71]) fail. We propose in [21] a nontrivial extension of the *relative entropy method* to degenerate intervals and singular solutions.

The main idea of the relative entropy method is the following: since the particle system has a family of product invariant measures indexed by the density (here, the Bernoulli product measure ν_ρ^N), one can use the non-homogeneous product measures $\nu_{\rho(t,u)}^N$ with slowly varying parameter associated with the solution $\rho(t,u)$ to the PME (4.2), and compare it to the state at macroscopic time t of the diffusively accelerated Markov process. The latter is denoted below by μ_t^N, it is a probability law on $\{0,1\}^{\mathbb{T}_N}$. If one expects the PME to be the correct hydrodynamic equation, these two measures should be close, and this can be seen from the investigation of the time evolution of the relative entropy $H(\mu_t^N | \nu_{\rho(t,u)}^N)$ (see (4.24) for the definition).

In our case, two obstacles appear straight away. The first one is that $\rho(t,u)$ can take values 0 and 1, and therefore the above relative entropy will generally be infinite. Indeed, Yau designed this method for the linear heat equation $\partial_t \rho = \Delta\rho$ which can be wisely rewritten as

$\partial_t \rho = \partial_u\big(\rho(1-\rho)\partial_u(f(\rho))\big)$, with $f(\rho) = \log(\rho/(1-\rho))$ being the macroscopic entropy [52]. Therefore, the application of Yau's method to the derivation of the PME is also based on the reformulation of $\partial_t \rho = \partial_{uu}(\rho^m)$ into $\partial_t \rho = \partial_u(m\rho^m(1-\rho)\partial_u(f(\rho)))$, with the same function f which degenerates at $\rho = 0$ and $\rho = 1$. We note however that f is *not* the natural physical entropy for the PME, as explained in [90]. The second obstacle is that the solution $\rho(t,u)$ has poor analytic properties as soon as the initial condition $\rho(0,\cdot) = \rho^{\mathrm{ini}}(\cdot)$ vanishes, which complicates the control of the time evolution of the entropy. To remove these obstacles, we modify the original investigation by considering an approximation of $\rho(t,u)$, denoting ahead by $\rho_{\mathrm{N}}(t,u)$, which satisfies two important properties:

1. it is bounded away from 0 and 1 and regular;
2. the sequence (ρ_{N}) uniformly converges to ρ on compactly supported time intervals.

As we will briefly see in the next sections, these two properties are not enough to apply straightforwardly Yau's method: we also need sharp controls on several derivatives of ρ. Moreover, the usual *one-block estimate* (which is at the core of the relative entropy method) requires understanding the interface between the positivity set of ρ and its complement, and needs very refined additional arguments. These are the main ingredients of our proof.

4.1.1 Statement of the main results

The initial configuration is random, distributed according to some initial probability measure μ_0^{N} on Ω_{N}. We denote by $\{\eta_t^{\mathrm{N}}\}_{t\geqslant 0}$ the Markov process generated by $\mathrm{N}^2\mathcal{L}_{\mathrm{N}}$ and starting from the initial state μ_0^{N}. For any fixed $t \geqslant 0$, the probability law of $\{\eta_t^{\mathrm{N}}(k)\ ;\ k \in \mathbb{T}_{\mathrm{N}}\}$ on the state space Ω_{N} is denoted by μ_t^{N}. In the following we also denote by $\mathbb{P}_{\mu_0^{\mathrm{N}}}$ the probability measure on the space of trajectories $\mathcal{D}(\mathbb{R}_+, \Omega_{\mathrm{N}})$ induced by the initial state μ_0^{N} and the accelerated Markov process $\{\eta_t^{\mathrm{N}}\}_{t\geqslant 0}$. Its corresponding expectation is denoted by $\mathbb{E}_{\mu_0^{\mathrm{N}}}$.

For any $\alpha \in [0,1]$, let ν_α^{N} be the Bernoulli product measure on $\{0,1\}^{\mathbb{T}_{\mathrm{N}}}$ and ν_α be the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$. We denote by E_α the expectation with respect to ν_α, and note that $\mathrm{E}_\alpha[\eta(0)] = \alpha$. One can easily check that the product measures $\{\nu_\alpha^{\mathrm{N}}\ ;\ \alpha \in [0,1]\}$ are *reversible* for the Markov process $\{\eta_t^{\mathrm{N}}\}$.

As the size N of the system goes to infinity, the discrete torus $\mathbb{T}_{\mathrm{N}}$ tends to the full lattice $\mathbb{Z}$. Therefore, we will need to consider functions on the space $\{0,1\}^{\mathbb{Z}}$. Let $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ be a *local* function (therefore f is necessarily bounded). We then denote by $\overline{f}(\alpha)$ its average with respect to the measure ν_α:

$$\overline{f}(\alpha) = \mathrm{E}_\alpha[\varphi(\eta)].$$

Note that $\alpha \mapsto \overline{f}(\alpha)$ is continuous for every local function f.

Let us now define the *non-homogeneous product measure* $\nu_{\rho(\cdot)}^{\mathrm{N}}$ on $\{0,1\}^{\mathbb{T}_{\mathrm{N}}}$ associated with a density profile $\rho : \mathbb{T} \to [0,1]$, whose marginal at site $k \in \mathbb{T}_{\mathrm{N}}$ is given by

$$\nu_{\rho(\cdot)}^{\mathrm{N}}\big\{\eta\ :\ \eta(k) = 1\big\} = 1 - \nu_{\rho(\cdot)}^{\mathrm{N}}\big\{\eta\ :\ \eta(k) = 0\big\} = \rho\big(\tfrac{k}{\mathrm{N}}\big). \tag{4.3}$$

We denote by $\mathbb{E}_{\rho(\cdot)}^{\mathrm{N}}$ the expectation with respect to $\nu_{\rho(\cdot)}^{\mathrm{N}}$.

Let $\rho^{\mathrm{ini}} \in \mathrm{L}^\infty(\mathbb{T};[0,1])$ be an initial density profile. As already pointed out by Gonçalves *et al.* [59], the underlying macroscopic equation is expected to be the PME

$$\begin{cases} \partial_t \rho = \partial^2_{uu}(\rho^m) & \text{in } \mathbb{R}_+ \times \mathbb{T}, \\ \rho_{|_{t=0}} = \rho^{\mathrm{ini}} & \text{in } \mathbb{T}. \end{cases} \tag{4.4}$$

This equation is of degenerate parabolic type. It is well known that the notion of strong solution is not suitable to get the well-posedness of the problem (4.4) unless ρ^{ini} remains bounded away from 0. Indeed, the space derivative of ρ may be discontinuous at the boundary of the set $\{\rho > 0\}$ (see for instance [99]). This motivates the introduction of the following notion of weak solutions.

Definition 4.1.1. *A function $\rho \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{T};[0,1])$ is said to be a weak solution to (4.4) corresponding to the initial profile ρ^{ini} if $\partial_u(\rho^2) \in \mathbf{L}^2(\mathbb{R}_+ \times \mathbb{T})$ and*

$$\iint_{\mathbb{R}_+\times\mathbb{T}} \rho\ \partial_t \xi \,\mathrm{d}u\mathrm{d}t + \int_{\mathbb{T}} \rho^{\mathrm{ini}}\ \xi(0,\cdot)\mathrm{d}u - \iint_{\mathbb{R}_+\times\mathbb{T}} \partial_u(\rho^m)\partial_u \xi \,\mathrm{d}u\mathrm{d}t = 0, \tag{4.5}$$

for all $\xi \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{T})$ with compact support.

What we call a weak solution corresponds to what is called an *energy solution* in Vàzquez' monograph (see [99, Section 5.3.2]). The classical existence theory based on compactness arguments (see [99, Theorem 5.5]) can be extended to our periodic setting without any trouble[1].

As it will appear in the sequel, the so-called *pressure*, denoted by ϖ in what follows, plays an important role. It is related to the density ρ by the monotone relation

$$\varpi = \varpi(\rho) = \frac{m}{m-1}\rho^{m-1}.$$

If $m = 2$, then ϖ is proportional to the density, but it is a very particular case. The equation (4.4) then rewrites

$$\partial_t \rho - \partial_u(\rho\, \partial_u \varpi) = 0.$$

We denote $\varpi^{\mathrm{ini}} = \frac{m}{m-1}\left(\rho^{\mathrm{ini}}\right)^{m-1}$, and we now state our assumption on the initial condition.

Assumption *4.1.1 (The initial profile).* We assume that:

- The initial pressure profile ϖ^{ini} is Lipschitz continuous: there exists $\mathrm{C}_{\mathrm{Lip}} > 0$ such that

$$\left\|\partial_u \varpi^{\mathrm{ini}}\right\|_\infty \leqslant \mathrm{C}_{\mathrm{Lip}}. \tag{4.7}$$

[1] The uniqueness of the weak solution and the fact that they remain bounded between 0 and 1 are consequences of the following $\mathbf{L}^1$-contraction/comparison principle (see [99, Proposition 6.1]): let ρ^{ini} and $\check{\rho}^{\mathrm{ini}}$ be two initial profiles in $\mathbf{L}^\infty(\mathbb{T};[0,1])$, and let ρ and $\check{\rho}$ be corresponding weak solutions, then

$$\int_{\mathbb{T}} \left(\rho(t,u) - \check{\rho}(t,u)\right)^+ \mathrm{d}u \leqslant \int_{\mathbb{T}} \left(\rho^{\mathrm{ini}}(u) - \check{\rho}^{\mathrm{ini}}(u)\right)^+ \mathrm{d}u, \qquad \text{for any } t \geqslant 0, \tag{4.6}$$

where $a^+ = \max(a,0)$ denotes the positive part of a. In the above relation, we have used the fact that any weak solution to (4.4) belongs to $\mathcal{C}(\mathbb{R}_+;\mathbf{L}^1(\mathbb{T}))$ (see for instance [26]).

- The initial positivity set
$$\mathcal{P}_0 = \{u \in \mathbb{T} \,:\, \rho^{\mathrm{ini}}(u) > 0\} \tag{4.8}$$
has a finite number of connected components.

Note that this assumption is less restrictive than the one given in [59], where ρ^{ini} was supposed to be uniformly bounded away from 0 and 1. In particular, we authorize vanishing initial profiles. Our main result reads as follows:

THEOREM 4.1.1. *We assume that the initial microscopic configuration* $\{\eta_0(k) \,:\, k \in \mathbb{T}_N\}$ *is distributed according to* $\mu_0^N = \nu^N_{\rho^{\mathrm{ini}}(\cdot)}$, *with* ρ^{ini} *satisfying Assumption 4.1.1.*

Then, the following local equilibrium convergence *holds at any macroscopic time* $t > 0$*: for any continuous function* $\varphi : \mathbb{T} \to \mathbb{R}$, *any local function* $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{E}_{\mu_0^N}\left[\left|\frac{1}{N}\sum_{k\in\mathbb{T}_N} \varphi\left(\tfrac{k}{N}\right)\tau_k f\left(\eta_t^N\right) - \int_{\mathbb{T}} \varphi(u)\overline{f}\left(\rho(t,u)\right)\mathrm{d}u\right|\right] = 0, \tag{4.9}$$

where ρ *is the unique weak solution of* (4.4) *in the sense of Definition 4.1.1.*

REMARK *4.1.1.* The porous medium equation (4.4) admits *fundamental solutions*, which are usually called *Barenblatt* (or ZKB) solutions. An explicit form for the Barenblatt solution is

$$\rho^{\mathrm{B}}(t,x) = t^{-\frac{1}{m+1}}\left(\left(\mathrm{C} - \frac{(m-1)x^2}{2m(m+1)t^{\frac{2}{m+1}}}\right)^+\right)^{\frac{1}{m-1}}$$

for each $\mathrm{C} > 0$. In particular, for sufficiently small $t_0 > 0$, its support is contained in $\mathbb{T}$ and $\rho^{\mathrm{ini}}(\cdot) = \rho^{\mathrm{B}}(t_0,\cdot)$ satisfies Assumption 4.1.1.

4.1.2 Sketch of the proof

If the porous medium equation starts with an initial profile which vanishes, then the solution at any later time can have discontinuous gradients across the *interfaces* at which the function becomes positive. This is a problem when one tries to prove hydrodynamic limits. The best way to tackle discontinuity problems is to slightly perturb the initial condition, by making it positive, and bounded away from 1.

a) The porous medium equation (PME)

We start with some properties of the unique weak solution $\rho(t,u)$ to (4.4). Our first statement is related to the continuity of the weak solutions to the porous medium equation. Such a regularity result can be deduced from [99, Section 7.7 and Section 15.1]. It is also a straightforward consequence of the forthcoming Proposition 4.1.6.

PROPOSITION 4.1.2 (Regularity of the solution). *The unique weak solution to* (4.4) *is continuous on* $\mathbb{R}_+ \times \mathbb{T}$, *and the corresponding pressure* $\varpi = \frac{m}{m-1}\rho^{m-1}$ *is Lipschitz continuous.*

For any $t \geqslant 0$, we denote by $\mathcal{P}_t = \{u \in \mathbb{T} \,:\, \rho(t,u) > 0\}$ the positivity set of $\rho(t,\cdot)$, which is an open subset of $\mathbb{T}$ since $\rho(t,\cdot)$ is continuous. Finally we denote by

$$\Gamma_t = \partial \mathcal{P}_t = \overline{\mathcal{P}_t} \setminus \mathcal{P}_t \tag{4.10}$$

the *interface* between the positivity set $\mathcal{P}_t$ of $\rho(t,\cdot)$ and the complementary

$$\mathcal{Z}_t = \overbrace{\{u \in \mathbb{T} \,:\, \rho(t,u) = 0\}}^{\circ} = \mathbb{T} \setminus \overline{\mathcal{P}_t} \tag{4.11}$$

of its support. Note that Γ_t is closed, and is a nowhere dense set, but it can *a priori* have positive Lebesgue measure. Actually, Lemma 4.1.4 below shows that from our assumption (4.8) on $\mathcal{P}_0$, this does not happen and that the Lebesgue measure of Γ_t vanishes for any $t > 0$.

REMARK *4.1.2.* The derivatives of the pressure ϖ can have jump discontinuities on the so-called *free boundary*

$$\bigcup_{t \in (0,T]} \{t\} \times \Gamma_t,$$

but both ϖ and ρ are smooth outside of this set at positive times. We refer the reader to [99, Chapter 14] for the general theory and also [99, Chapter 4] for several examples.

In what follows, the notation Leb stands for the usual Lebesgue measure restricted on $\mathbb{T}$, and $|B|$ denotes the cardinality of the discrete subset $B \subset \mathbb{T}_N$.

PROPOSITION 4.1.3 (Positivity intervals). *For any $\delta > 0$ and $t \in [0,T]$ we set*

$$\Gamma_t(\delta) = \overline{\{u \in \mathbb{T} \,:\, 0 < \rho(t,u) < \delta\}}. \tag{4.12}$$

We have

$$\int_0^T \mathrm{Leb}\,(\Gamma_t(\delta))\,\mathrm{d}t \underset{\delta \to 0}{\longrightarrow} 0. \tag{4.13}$$

Proof of Proposition 4.1.3. The proof follows from the following technical lemma, which is proved in [21, Appendix A].

LEMMA 4.1.4 (Connected components of the positivity set). *For any $t > 0$, $\mathcal{P}_t$ has a finite number of connected components.*

From last lemma, since $\mathcal{P}_t$ has a finite number of connected components for any $t > 0$, we know that Γ_t is a finite union of points, and therefore $\mathrm{Leb}(\Gamma_t) = 0$. Since

$$\Gamma_t = \bigcap_{\delta > 0} \overline{\{u \in \mathbb{T} \,:\, 0 < \rho(t,u) < \delta\}} = \bigcap_{\delta > 0} \Gamma_t(\delta),$$

it follows from the monotonicity of the Lebesgue measure that

$$0 = \mathrm{Leb}(\Gamma_t) = \lim_{\delta \to 0} \mathrm{Leb}(\Gamma_t(\delta)), \qquad \text{for any } t \in [0,T].$$

Moreover, since $\Gamma_t(\delta) \subset \mathbb{T}$, we get that $\mathrm{Leb}(\Gamma_t(\delta)) \leqslant 1$ for all $t \in [0,T]$. Hence (4.13) follows from Lebesgue's dominated convergence Theorem.

b) The regularized initial condition

In order to prove Theorem 4.1.1, we need to introduce a *regularized approximate solution* to the PME. This is the goal of this section.

Let $(\varepsilon_N)_{N\in\mathbb{N}}$ be a vanishing sequence such that $\varepsilon_N \in (0, \frac{1}{2})$. The rate at which $\varepsilon_N \to 0$ will be precised later on. Let $h \in \mathcal{C}^\infty(\mathbb{R})$ be a positive symmetric function of unit mass, with support contained in $(-1,1)$ and such that $\partial_y h(y) \leqslant 0$ if $y \geqslant 0$. Denote $C_h = \|h\|_\infty$. From our assumptions, $\|\partial_y h\|_1 = 2C_h$.

Let us now define the *regularizing approximation of the unit*:

$$h_N(y) = \varepsilon_N^{-1}\, h(\varepsilon_N^{-1} y), \qquad y \in \mathbb{R},$$

which satisfies $\mathrm{Supp}(h_N) \subset (-\varepsilon_N, \varepsilon_N)$, and also

$$\|h_N\|_1 = 1, \qquad \|h_N\|_\infty = \frac{C_h}{\varepsilon_N}, \qquad \|\partial_y h_N\|_1 = \frac{2C_h}{\varepsilon_N}. \tag{4.14}$$

From here several steps are necessary to define the approximate initial data ρ_N^{ini}. First, we introduce the truncated initial density and pressure defined by

$$\begin{aligned}
\widetilde{\rho}_N^{\mathrm{ini}} &= \max\left\{\varepsilon_N,\ \min\left(1-\varepsilon_N,\ \rho^{\mathrm{ini}}\right)\right\},\\
\widetilde{\varpi}_N^{\mathrm{ini}} &= \tfrac{m}{m-1}\left(\widetilde{\rho}_N^{\mathrm{ini}}\right)^{m-1} = \max\left\{\tfrac{m}{m-1}\varepsilon_N^{m-1},\ \min\left(\tfrac{m}{m-1}(1-\varepsilon_N)^{m-1},\ \varpi^{\mathrm{ini}}\right)\right\}.
\end{aligned}$$

The *truncated and regularized initial data* are then defined by

$$\varpi_N^{\mathrm{ini}} = \widetilde{\varpi}_N^{\mathrm{ini}} \star h_N, \qquad \rho_N^{\mathrm{ini}} = \left(\tfrac{m-1}{m}\varpi_N^{\mathrm{ini}}\right)^{\frac{1}{m-1}}, \tag{4.15}$$

where $\star$ is the usual convolution product on $\mathbb{T}$. This approximation procedure is designed so that the following properties hold:

1. *Regularity*: ρ_N^{ini} and ϖ_N^{ini} are smooth on $\mathbb{T}$;
2. *Boundedness away from* 0 *and* 1: $\varepsilon_N \leqslant \rho_N^{\mathrm{ini}} \leqslant 1-\varepsilon_N$;
3. *Lipschitz regularity of the regularized pressure*: $\left\|\partial_u \varpi_N^{\mathrm{ini}}\right\|_\infty \leqslant C_{\mathrm{Lip}}$, where C_{Lip} has been introduced in (4.7).
4. *Uniform convergence towards the initial profiles*:

$$\left\|\varpi_N^{\mathrm{ini}} - \varpi^{\mathrm{ini}}\right\|_\infty \leqslant (m + C_{\mathrm{Lip}})\varepsilon_N \xrightarrow[N\to\infty]{} 0 \tag{4.16}$$

$$\left\|\rho_N^{\mathrm{ini}} - \rho^{\mathrm{ini}}\right\|_\infty \leqslant C_{\mathrm{ini}}\,(\varepsilon_N)^{\frac{1}{m-1}} \xrightarrow[N\to\infty]{} 0. \tag{4.17}$$

with $C_{\mathrm{ini}} = \left(\frac{(m-1)}{m}(m + C_{\mathrm{Lip}})\right)^{\frac{1}{m-1}}$.

Our assumptions imply:

$$\left\|\partial_u \rho_N^{\mathrm{ini}}\right\|_\infty \leqslant \frac{C_{\mathrm{Lip}}}{m}(\varepsilon_N)^{2-m}, \tag{4.18}$$

therefore ρ_N^{ini} is uniformly Lipschitz only in the case $m = 2$. If $m \geqslant 3$ the right hand side above goes to infinity as $N \to \infty$.

Let us now define the *regularized solution* ρ_N on $\mathbb{R}_+ \times \mathbb{T}$ as the solution to

$$\begin{cases} \partial_t \rho_N = \partial_{uu}^2 \big((\rho_N)^m \big) & \text{in } \mathbb{R}_+ \times \mathbb{T}, \\ (\rho_N)_{|_{t=0}} = \rho_N^{ini} & \text{in } \mathbb{T}. \end{cases} \tag{4.19}$$

This solution will play a central role in the proof of Theorem 4.1.1, as well as the corresponding regularized pressure:

$$\varpi_N = \frac{m}{m-1} (\rho_N)^{m-1}. \tag{4.20}$$

Let start here with two major properties of ρ_N.

PROPOSITION 4.1.5. *Fix a time horizon line $T > 0$. Problem (4.19) admits a unique strong solution $\rho_N \in \mathcal{C}^\infty([0,T] \times \mathbb{T})$ which satisfies*

$$\varepsilon_N \leqslant \rho_N \leqslant 1 - \varepsilon_N. \tag{4.21}$$

PROPOSITION 4.1.6 (Uniform convergence). *The sequence $(\rho_N)_{N \in \mathbb{N}}$ converges uniformly in the space $[0,T] \times \mathbb{T}$ towards the unique weak solution to (4.4).*

c) Relative entropy method

In the following, for any two probability measures μ, ν on $\{0,1\}^{\mathbb{T}_N}$ we denote by $H(\mu|\nu)$ the relative entropy of μ with respect to ν, defined as usual by

$$H(\mu|\nu) = \sup_f \left\{ \int f \, d\mu - \log \int e^f \, d\nu \right\},$$

where the supremum is carried over all real–valued functions. The following *entropy inequality* is going to be useful: for any $\gamma > 0$, we have

$$\int f \, d\mu \leqslant \frac{1}{\gamma} \left(\log \int e^{\gamma f} \, d\nu + H(\mu|\nu) \right). \tag{4.22}$$

Fix $\alpha \in (0,1)$ and an invariant measure ν_α. We introduce the density

$$\psi_t^N(\eta) = \frac{d\nu^N_{\rho_N(t,\cdot)}}{d\nu_\alpha}(\eta) = \frac{1}{Z_t^N} \exp\left(\sum_{k \in \mathbb{T}_N} \eta(k) \, \lambda_N\big(t, \tfrac{k}{N}\big) \right),$$

where

$$\lambda_N(t,u) = \log\left(\frac{\rho_N(t,u)(1-\alpha)}{\alpha(1-\rho_N(t,u))} \right), \tag{4.23}$$

and Z_t^N is the normalization constant. Note that λ_N is well defined thanks to Proposition 4.1.5. Recall moreover that μ_t^N is the distribution of the accelerated process at time tN^2 and denote its density with respect to ν_α as

$$f_t^N = \frac{d\mu_t^N}{d\nu_\alpha}.$$

Finally, we are interested in the relative entropy

$$\mathcal{H}_N(t) = H\big(\mu_t^N | \nu^N_{\rho_N(t,\cdot)}\big) = \int f_t^N(\eta) \log\left(\frac{f_t^N(\eta)}{\psi_t^N(\eta)}\right) d\nu_\alpha(\eta). \tag{4.24}$$

The proof of Theorem 4.1.1 is based on the investigation of the time evolution of the entropy $\mathcal{H}_N(t)$. This strategy is inspired by the *relative entropy method* which is exposed in details for instance in [71, Chapter 6]. However, in our case the standard method cannot work: the usual scheme works with the relative entropy of μ_t^N with respect to the product measure $\nu^N_{\rho(t,\cdot)}$, associated with the *true* weak solution of the PME (4.4). As we have seen in Section 4.1.2, this solution has poor regularity properties, and more importantly, it can vanish on non-trivial intervals. This would make the relative entropy take infinite values for presumably long times.

This is why we work with a different relative entropy: here, $\mathcal{H}_N(t)$ defined in (4.24) involves the non-homogeneous product measure $\nu^N_{\rho_N(t,\cdot)}$, which is associated with the *regularized* solution ρ_N, defined in (4.19). Since ρ_N is smooth and bounded away from 0 and 1, the relative entropy is always finite. Since (ρ_N) uniformly converges to ρ on $[0,T]\times\mathbb{T}$ (from Proposition 4.1.6), one might believe that the arguments of [71] can be easily adapted. However, one needs much more than uniform convergence. In particular, sharp controls on the derivatives of ρ_N are also needed, as explained in details in [21].

Let us conclude with two important results concerning $\mathcal{H}_N(t)$, which imply Theorem 4.1.1. First of all, at $t = 0$, the initial relative entropy is negligible with respect to N as $N \to \infty$, namely:

Lemma 4.1.7 (Initial entropy).

$$\mathcal{H}_N(0) = H\big(\mu_0^N | \nu^N_{\rho_N^{ini}(\cdot)}\big) = H\big(\nu^N_{\rho^{ini}} | \nu^N_{\rho_N^{ini}(\cdot)}\big) = \mathcal{O}\big(N(\varepsilon_N)^{\frac{1}{m-1}} |\log \varepsilon_N|\big) = o(N), \text{ as } N \to \infty.$$

This lemma is proved in [21, Lemma 3.7]. Next, we are able to control the entropy production on a finite time interval. This is where we need to make an assumption on the convergence speed of (ε_N):

Assumption *4.1.2 (Convergence speed of ε_N).*

$$\lim_{N\to\infty} N(\varepsilon_N)^{6m-6} = +\infty. \tag{4.25}$$

Proposition 4.1.8 (Entropy production). *Under Assumption 4.1.2, there exists a constant $\kappa > 0$ such that*

$$\mathcal{H}_N(T) \leqslant \kappa \int_0^T \mathcal{H}_N(s) ds + o_T(N),$$

where $o_T(N)$ stands for a sequence of real numbers $C(T,N)$ such that $C(T,N)/N \to 0$ as $N \to \infty$.

This is the main technical result which is proved in [21], and requires two main ingredients:

1. **Sharp estimates** on several $\mathbf{L}^p$–norms of
   (a) the density ρ_N and its space derivatives $\partial_u \rho_N$, $\partial^2_{uu}\rho_N$, $\partial^3_{uuu}\rho_N$;

(b) also of the pressure ϖ_N and its space derivatives of same order.

In particular, we need to know how they behave with respect to ε_N. Here is one example:

$$\iint_{[0,\bar{t}]\times\mathbb{T}} \left|\partial^2_{uu}\rho_N\right|^2 dt\,du \leqslant C_0(\varepsilon_N)^{5-3m},$$

where C_0 is an explicit constant which depends on m and C_{Lip}. All the estimates are provided in [21, Proposition 4.1, Proposition 4.2]. They also explain the origin of Assumption 4.1.2.

2. A new **one-block estimate** which takes into account the degeneracy of the limit profile (which can vanish), and for which new arguments are needed. It is stated as follows:

LEMMA 4.1.9 (One-block estimate). *Let $\varepsilon > 0$. For every local function $g : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ there exists $\gamma_0 > 0$ and $L_0 < \infty$ such that: for any $\ell \geqslant L_0$ there exists $N_0 = N_0(\ell)$ such that for any $N \geqslant N_0$ we have*

$$\int_0^{\bar{t}} \int \frac{1}{N} \sum_{k\in\mathbb{T}_N} \tau_k V_{\ell,g}(\eta) f_t^N(\eta)\, \nu_\alpha(d\eta)dt \leqslant \frac{1}{\gamma_0 N} \int_0^{\bar{t}} \mathcal{H}_N(t)dt + \varepsilon, \tag{4.26}$$

where

$$V_{\ell,g}(\eta) = \left| \frac{1}{2\ell+1} \sum_{|j|\leqslant\ell} \tau_j g(\eta) - \overline{g}\big(\eta^{(\ell)}(0)\big) \right|.$$

Let me give the main steps of the proof: we first need to split the left hand side in (4.26) in two terms, which consider two distinct sets of configurations. More precisely, for $k \in \mathbb{T}_N$, $\ell \in \mathbb{N}$, let

$$\mathcal{Q}_{k,\ell} = \Big\{\eta : \sum_{j=k-\ell}^{k+\ell-1} \eta(j)\eta(j+1) \geqslant 1\Big\}$$

the set of configurations in which there are two neighbouring particles within distance ℓ of k (in particular the box of radius ℓ around k contains a *mobile cluster*). We split the left hand side in (4.26) as follows:

$$\int_0^{\bar{t}} \int \frac{1}{N} \sum_{k\in\mathbb{T}_N} \tau_k V_{\ell,g}(\eta) \mathbf{1}_{\mathcal{Q}_{k,\ell}}(\eta) f_t^N(\eta)\, \nu_\alpha(d\eta)dt \tag{4.27}$$

$$+ \int_0^{\bar{t}} \int \frac{1}{N} \sum_{k\in\mathbb{T}_N} \tau_k V_{\ell,g}(\eta) \mathbf{1}_{\mathcal{Q}^c_{k,\ell}}(\eta) f_t^N(\eta)\, \nu_\alpha(d\eta)dt. \tag{4.28}$$

As indicated in [59, Section 3.1], the restriction to the irreducible set $\mathcal{Q}_{k,\ell}$ in (4.27) allows us to repeat standard arguments, and to conclude that

$$\limsup_{\ell\to\infty} \limsup_{N\to\infty} \int_0^{\bar{t}} \int \frac{1}{N} \sum_{k\in\mathbb{T}_N} \tau_k V_{\ell,g}(\eta) \mathbf{1}_{\mathcal{Q}_{k,\ell}}(\eta) f_t^N(\eta)\, \nu_\alpha(d\eta)dt = 0. \tag{4.29}$$

The treatment of the other term (4.28) is less standard: first, by the entropy inequality (4.22), the term inside the time integral $\int_0^{\overline{t}}$ can be bounded above by

$$\frac{\mathrm{H}\big(\mu_t^{\mathrm{N}} | \nu^{\mathrm{N}}_{\rho_{\mathrm{N}}(t,\cdot)}\big)}{\gamma \mathrm{N}} + \frac{1}{\gamma \mathrm{N}} \log \int \exp\Big(\gamma \sum_{k \in \mathbb{T}_{\mathrm{N}}} \tau_k \mathrm{V}_{\ell,g}(\eta) \mathbf{1}_{\mathcal{Q}^c_{k,\ell}}(\eta) \Big) \nu^{\mathrm{N}}_{\rho_{\mathrm{N}}(t,\cdot)}(\mathrm{d}\eta) \tag{4.30}$$

for any $\gamma > 0$. Recall that $\varepsilon > 0$ is fixed. We need to show that we can choose $\gamma > 0$ such that

$$\limsup_{\ell \to \infty} \limsup_{\mathrm{N} \to \infty} \int_0^{\overline{t}} \frac{1}{\gamma \mathrm{N}} \log \int \exp\Big(\gamma \sum_{k \in \mathbb{T}_{\mathrm{N}}} \tau_k \mathrm{V}_{\ell,g}(\eta) \mathbf{1}_{\mathcal{Q}^c_{k,\ell}}(\eta) \Big) \nu^{\mathrm{N}}_{\rho_{\mathrm{N}}(t,\cdot)}(\mathrm{d}\eta) \mathrm{d}t \leqslant \varepsilon. \tag{4.31}$$

Now, contrary to [59], we made no assumption to ensure that $\nu^{\mathrm{N}}_{\rho(t,\cdot)}(\mathcal{Q}^c_{k,\ell})$ decays exponentially in ℓ for all k. In fact, this is plain wrong when $\rho(t,\cdot)$ vanishes on an interval. Therefore, the strategy worked out in [21] is to split $\mathbb{T}_{\mathrm{N}}$ into three sets of points: the *good* ones, the *almost zeroes* and the *bad* ones. Namely, for any $\delta > 0$, and any vanishing sequence (α_{N}) such that $\alpha_{\mathrm{N}} \leqslant \delta$, let

$$\begin{aligned}
\mathrm{G}_t^{\mathrm{N},\ell}(\delta) &= \Big\{ k \in \mathbb{T}_{\mathrm{N}} \,:\, \rho_{\mathrm{N}}(t,\cdot) \geqslant \delta \quad \text{on } \Big[\tfrac{k-\ell-\ell_0}{\mathrm{N}}, \tfrac{k+\ell+\ell_0}{\mathrm{N}} \Big] \Big\}, \\
\mathrm{Z}_t^{\mathrm{N},\ell}(\alpha_{\mathrm{N}}) &= \Big\{ k \in \mathbb{T}_{\mathrm{N}} \,:\, \rho_{\mathrm{N}}(t,\cdot) \leqslant \alpha_{\mathrm{N}} \quad \text{on } \Big[\tfrac{k-\ell-\ell_0}{\mathrm{N}}, \tfrac{k+\ell+\ell_0}{\mathrm{N}} \Big] \Big\}, \\
\mathrm{B}_t^{\mathrm{N},\ell}(\delta, \alpha_{\mathrm{N}}) &= \mathbb{T}_{\mathrm{N}} \setminus \big(\mathrm{G}_t^{\mathrm{N},\ell}(\delta) \cup \mathrm{Z}_t^{\mathrm{N},\ell}(\alpha_{\mathrm{N}}) \big).
\end{aligned}$$

The parameters $\delta > 0$ and $\alpha_{\mathrm{N}} \to 0$ will be chosen ahead. We want to study the limit as $\mathrm{N} \to \infty$ of the cardinality of these sets of points (renormalized by N). For that purpose, let us introduce the following sets: for any $\delta > 0$, $t \in [0, \mathrm{T}]$, let

$$\begin{aligned}
\mathcal{G}_t(\delta) &= \Big\{ u \in \mathbb{T} \,:\, \rho(t,u) \geqslant \delta \Big\}, \\
\mathcal{Z}_t(\delta) &= \overbrace{\Big\{ u \in \mathbb{T} \,:\, \rho(t,u) = 0 \Big\}}^{\circ} \cup \Gamma_t(\delta), \\
\mathcal{B}_t(\delta) &= \Big\{ u \in \mathbb{T} \,:\, 0 < \rho(t,u) < \delta \Big\} \subset \Gamma_t(\delta),
\end{aligned}$$

where $\Gamma_t(\delta)$ has been defined in (4.12). Note first that

$$\mathbb{T} \setminus \big(\mathcal{G}_t(\delta) \cup \mathcal{Z}_t \big) = \mathcal{B}_t(\delta) \cup \Gamma_t,$$

where $\mathcal{Z}_t$ and Γ_t have been defined respectively in (4.11) and (4.10). Therefore, since $\mathrm{Leb}(\Gamma_t) = 0$ (which comes from the proof of Proposition 4.1.3) the two remaining sets above have the same Lebesgue measure:

$$\mathrm{Leb}\big(\mathcal{B}_t(\delta) \big) = \mathrm{Leb}\Big(\mathbb{T} \setminus \big(\mathcal{G}_t(\delta) \cup \mathcal{Z}_t \big) \Big). \tag{4.32}$$

In [21, Lemma 5.3] we prove the following: there exists a vanishing sequence (δ_{N}) (which depends on ρ_{N}) such that, for any $\ell, \ell_0 \in \mathbb{N}$ fixed, and $\delta > 0$, the following

convergences hold:

$$\lim_{N\to\infty}\frac{1}{N}\left|G_t^{N,\ell}(\delta-\delta_N)\right| = \mathrm{Leb}\big(\mathcal{G}_t(\delta)\big) \tag{4.33}$$

$$\lim_{N\to\infty}\frac{1}{N}\left|Z_t^{N,\ell}(\delta_N)\right| = \mathrm{Leb}\big(\mathcal{Z}_t\big) \tag{4.34}$$

and therefore from (4.32):

$$\lim_{N\to\infty}\frac{1}{N}\left|B_t^{N,\ell}(\delta-\delta_N,\delta_N)\right| = \mathrm{Leb}\big(\mathcal{B}_t(\delta)\big).$$

Then the proof of (4.26) is concluded as follows: we fix $\delta>0$ as a parameter that will vanish *after* letting $N\to\infty$ and $\ell\to\infty$. Take the expression under the limit in the left hand side of (4.31), and take N sufficiently large such that $\delta-\delta_N>\delta_N$. We divide the sum that appears there into three sums:

- one over $B_t^{N,\ell}(\delta-\delta_N,\delta_N)$,
- one over $Z_t^{N,\ell}(\delta_N)$,
- and the last one over $G_t^{N,\ell}(\delta-\delta_N)$,

since by definition their union gives $\mathbb{T}_N$. Finally, careful estimates, together with (once again) Proposition 4.1.3, imply the desired result. More details are in the proof of [21, Lemma 5.2].

d) Conclusion

From Gronwall's inequality and Lemma 4.1.7, we conclude:

COROLLARY 4.1.10. *For any $t>0$,*

$$H\big(\mu_t^N \mid \nu^N_{\rho_N(t,\cdot)}\big) = \mathcal{H}_N(t) = o_t(N), \qquad \text{as } N\to\infty.$$

Then, one has to prove that Corollary 4.1.10 is sufficient to get the local equilibrium result (4.9) stated in Theorem 4.1.1. This is indeed true, as in the usual approach: in our case, one needs to prove additionally that the approximate solution $\rho_N(t,\cdot)$ converges uniformly to $\rho(t,\cdot)$ in $\mathbb{T}$ (which does hold from Proposition 4.1.6), and that the solution $\rho(t,\cdot)$ is continuous. We have all in hands to conclude the proof of Theorem 4.1.1, this is detailed in [21, Section 3.3].

4.2 Stefan problems and free boundaries

The microscopic derivation of *Stefan problems* by means of hydrodynamic limits and the mathematical understanding of its atomic mechanisms is an arduous problem. On that subject we solved one crucial step in [22], written in collaboration with O. Blondel, C. Erignoux and M. Sasada.

The degenerate microscopic dynamics that we investigate is the one that has been introduced in [93] and called *facilitated exclusion process*. It is defined similarly as before, by means of a generator $\mathcal{L}_N$ of the form (4.1), but the local constraint is more restrictive: a particle can jump to the right (resp. left) empty neighbouring site if and only if it has a particle to its left (resp. right) neighbouring site (see Figure 4.2). More precisely, the constraint and the exclusion rule are encoded in the jump rates $c_{k,k+1}$ given as

$$c_{k,k+1}(\eta) = \eta(k-1)\eta(k)(1-\eta(k+1)) + (1-\eta(k))\eta(k+1)\eta(k+2). \tag{4.35}$$

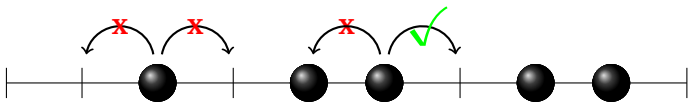

Figure 4.2: Allowed jumps are denoted by ✓. Forbidden jumps are denoted by x.

This dynamics is, as in the previous section, reversible. However, its invariant measures are not product, contrary to the vast majority of similar models. In order to describe them, we need to divide the state space into *transient* states, *absorbing* states and *ergodic* states:

- the ergodic component contains all the recurrent states for the process, which form an irreducible set, see Lemma 4.2.1 below;
- the absorbing states are the inactive ones;
- roughly speaking, depending on the initial number of particles in the system, some transient configurations (which we call the *good ones*) will lead to the ergodic component and the other transient configurations (which we call the *bad ones*) will be absorbed to an inactive state.

The precise definitions will be given right away in the next section. If the density of particles is bigger than $\frac{1}{2}$ the system is in the active state with a unique invariant measure, while all the invariant measures are superpositions of atoms on absorbing states if the density is less than $\frac{1}{2}$. The critical particle density is therefore $\frac{1}{2}$.

The critical behavior of conserved lattice gases has been studied numerically and analytically in the physics literature [83, 32, 8], in order to identify the universality classes of models displaying an active-absorbing phase transition. In particular, the facilitated exclusion process is not in the same universality class as directed percolation. Let us mention that, recently, the facilitated exclusion process appeared in its totally asymmetric version in [5], where the authors compare the behavior of its first particles with what happens in the totally asymmetric exclusion process (TASEP).

Our main result is that, in the active phase (where the ergodic component is ultimately reached), the macroscopic behavior of this microscopic dynamics, under periodic boundary conditions, is ruled by a nonlinear diffusion equation (Theorem 4.2.2 below). To apply the

usual entropy method, the existence of transient states is troublesome since, although invariant measures are supported only on the ergodic components, the process may stay in the transient states for some macroscopic time with positive probability. To guarantee this is not the case, we first show Theorem 4.2.3 below. More precisely: assume that the initial density is initially larger than $\frac{1}{2}$ and look at the microscopic system of size N at a macroscopic time $(\log N)^\alpha/N^2$ for some $\alpha > 0$; then with high probability it has already reached the ergodic component. This is the first main novelty of [22]. Moreover, the invariant measures are not product: when the density is close to $\frac{1}{2}$, these measures exhibit spatial correlations, and adapting the entropy method for these non-product grand canonical measures requires significant extra technical work.

As the nature of microscopic dynamics is different from the previous model of [59, 21], the macroscopic equation is also different from the porous medium equation, and we obtain the following macroscopic equation

$$\partial_t \rho = \partial^2_{uu}(-\rho^{-1}) = \partial_u(\rho^{-2}\partial_u \rho).$$

The equation is in the class of *fast diffusion equations*, which corresponds to the case with $m < 1$ in (4.2). Though the equation has a singularity at $\rho = 0$, it is not relevant for us since we always assume that the initial density is bigger than $\frac{1}{2}$ at any spatial point. Because of the phase transition described above, for density profiles below $\frac{1}{2}$ the system cannot be governed by this macroscopic equation, since it does not reflect the absorption phenomenon. In fact, for general density profiles, one expects that the macroscopic evolution should be the solution to a Stefan problem, with a free boundary between the active regions (with density higher than $\frac{1}{2}$) and the frozen ones. This property remains out of reach for the moment, we will come to it later in Chapter 5. It will require understanding the interplay between frozen regions with density lower than $\frac{1}{2}$ and active regions with higher density. The Stefan problem has been derived from microscopic dynamics in a few, less degenerate contexts [78, 61, 47]. We note that the same fast diffusion equation was derived recently from a non-degenerate zero-range process under a proper high density limit [67].

4.2.1 Statement of the main results

a) Classification of configurations

Let us first refine the classification of the configurations into transient/recurrent states as follows (this classification is fully justified in [22, Section 3]). Let $\mathcal{H}_N^m$ be the hyperplane of configurations with m particles, with $m \in \{0, ..., N\}$, namely:

$$\mathcal{H}_N^m = \Big\{\eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \sum_{k\in\mathbb{T}_N} \eta(k) = m\Big\}.$$

Then,

1. if $m \leqslant \frac{N}{2}$, some configurations in $\mathcal{H}_N^m$ are *blocked*, in the sense that no particle can jump because the constraint (4.35) is satisfied nowhere (they are *absorbing states* for the dynamics). Those are exactly the configurations in which all particles are isolated.
   The other configurations in $\mathcal{H}_N^m$ are not blocked but, starting from them, with probability one the process will arrive at a blocked configuration in a finite number of steps. We call them *transient bad* configurations. See Figure 4.3 below.
2. if $m > \frac{N}{2}$, the process will never reach an absorbing state (except in the trivial case $m = N$). In $\mathcal{H}_N^m$, there are configurations which are in the ergodic component (the recurrent states for the process, which in this case form an irreducible component); we call them *ergodic* configurations. They are the configurations in which empty sites are isolated.
   Starting from the other configurations, which are called *transient good* configurations, the process enters the ergodic component after a finite number of steps a.s. See Figure 4.3 below.

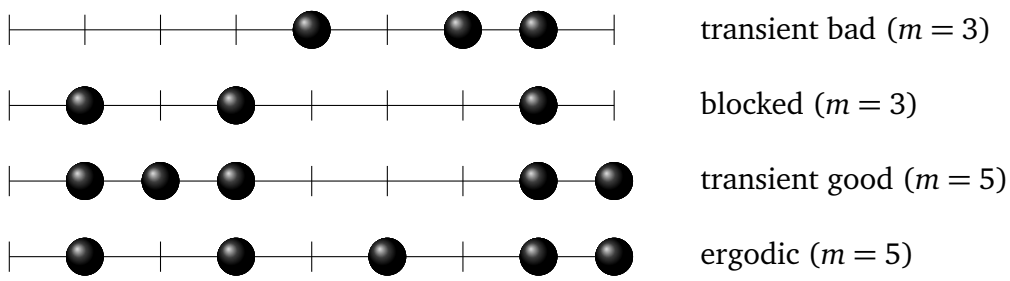

Figure 4.3: Four examples of configurations which belong to the different classes ($N = 9$).

Those observations lead to the following definition:

DEFINITION 4.2.1. *We denote by* $\mathcal{E}_N \subset \{0,1\}^{\mathbb{T}_N}$ *the set of* ergodic configurations on $\mathbb{T}_N$, *namely*

$$\mathcal{E}_N = \left\{ \eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \forall\, k \in \mathbb{T}_N,\ \big(\eta(k), \eta(k+1)\big) \neq (0,0) \text{ and } \sum_{k \in \mathbb{T}_N} \eta(k) > \frac{N}{2} \right\}. \tag{4.36}$$

For $m > \frac{N}{2}$ *we let* $\Omega_N^m = \mathcal{H}_N^m \cap \mathcal{E}_N$, *be the set of ergodic configurations on* $\mathbb{T}_N$ *which contain exactly* m *particles.*

We conclude this paragraph with the following result (easy to prove, see [22, Lemma 2.1]:

LEMMA 4.2.1. *For any* $N \geqslant 1$ *and any* $m > \frac{N}{2}$, Ω_N^m *is an irreducible component for the Markov process generated by* $\mathcal{L}_N$.

b) Invariant measures

The uniform measures on Ω_N^m, denoted below by π_N^m (with $m \in \{0, \dots, N\}$) are invariant for the Markov process induced by the infinitesimal generator $\mathcal{L}_N$: they indeed satisfy the *detailed*

balance condition: for any $k \in \mathbb{T}_N$ and $\eta \in \Omega_N^m$,

$$\pi_N^m(\eta)\eta(k-1)\eta(k)(1-\eta(k+1)) = \pi_N^m(\eta^{k,k+1})(1-\eta^{k,k+1}(k))\eta^{k,k+1}(k+1)\eta^{k,k+1}(k+2).$$

The marginals of π_N^m are fully characterized in [22, Lemma 6.1], and they permit to show the following: when $m/N \to \rho \in (\frac{1}{2}, 1)$, these measures locally converge to an infinite volume grand canonical measure π_ρ on $\{0,1\}^{\mathbb{Z}}$ for which an explicit formula can be derived. The measures π_ρ are translation invariant, but not product.

In the ergodic component, when the density of particles is larger than $\frac{1}{2}$, the invariant measures can be understood as follows: if $\rho \in (\frac{1}{2}, 1)$ parametrizes the density, then the invariant measure π_ρ puts a hole in some position with probability $1-\rho$, then puts after it $1+X$ particles, where X has geometric distribution with parameter $\frac{1-\rho}{\rho}$.

There are two fundamental properties of π_ρ which we prove in [22]:

1. For any density $\rho \in (\frac{1}{2}, 1)$, under π_ρ, the *correlations* between two boxes at a distance of order ℓ decay as $\exp(-C\ell)$. This is the purpose of [22, Lemma 6.5].
2. The *equivalence of ensembles*, linking canonical measures to grand-canonical measures as the system size goes to infinity, holds true. This is the content of [22, Corollary 6.10].

c) Statements

We are now ready to state the main results of [22]. Fix an initial smooth profile $\rho_0 : \mathbb{T} \to (\frac{1}{2}, 1]$, and consider the non-homogeneous product measure on $\{0,1\}^{\mathbb{T}_N}$ fitting ρ_0, defined as previously in (4.3), or in other words

$$\mu_N(\eta) = \prod_{k \in \mathbb{T}_N} \Big(\rho_0\big(\tfrac{k}{N}\big)\eta(k) + \big(1 - \rho_0\big(\tfrac{k}{N}\big)\big)(1-\eta(k)) \Big). \tag{4.37}$$

Let $\{\eta_t\}_{t \geqslant 0}$ denote the Markov process driven by the accelerated infinitesimal generator $N^2\mathcal{L}_N$ starting from the initial measure μ_N. Fix $T > 0$ and let $\mathbb{P}_{\mu_N}$ be as usual the probability measure on the Skorokhod path space $\mathcal{D}([0,T], \{0,1\}^{\mathbb{T}_N})$ and $\mathbb{E}_{\mu_N}$ be the corresponding expectation.

THEOREM 4.2.2 (Hydrodynamic limit). *For any $t \in [0,T]$, any $\delta > 0$ and any continuous test function $\varphi : \mathbb{T} \to \mathbb{R}$, we have*

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N}\left[\left| \frac{1}{N} \sum_{k \in \mathbb{T}_N} \varphi\big(\tfrac{k}{N}\big)\eta_t(k) - \int_{\mathbb{T}} \varphi(u)\rho(t,u)\mathrm{d}u \right| > \delta \right] = 0 \tag{4.38}$$

where $\rho(t,u)$ is the unique smooth solution of the hydrodynamic equation

$$\partial_t \rho = \Delta\Big(\frac{2\rho - 1}{\rho}\Big), \qquad \rho(0,\cdot) = \rho_0(\cdot) : \mathbb{T} \to (\tfrac{1}{2}, 1]. \tag{4.39}$$

REMARK *4.2.1.* The fact that there is a unique smooth solution to (4.39), provided that the initial profile satisfies $\rho_0 > \frac{1}{2}$, is quite standard in PDE theory, and can be found for instance in [99, Section 3.1]. The equation (4.39) belongs to the family of *quasilinear parabolic problems*

of the form $\partial_t \rho = \partial_u\big(\mathrm{A}(\rho, \partial_u \rho)\big)$, where, in our case, $\mathrm{A}(p,q) = q/p^2$. The classical theory, which gives smoothness of solutions and maximum principles, cannot be used for equation (4.39), because the latter is not uniformly parabolic[2]. However, the classical results can be used if the initial condition is *non-degenerate*. For instance, if ρ_0 satisfies $\frac{1}{2} + \varepsilon \leqslant \rho_0 \leqslant 1$, then one can apply the usual theory with such data: ρ actually never takes values in the critical region, and therefore one gets a classical solution for (4.39).

There are two main difficulties to prove Theorem 4.2.2. The first one lies in the fact that the dynamics is *a priori* not ergodic. We now state our second main result, which ensures that the accelerated system reaches its ergodic component at a macroscopic time t_{N} of order $(\log \mathrm{N})^{\alpha}/\mathrm{N}^2$ for some $\alpha > 0$. Therefore, for any macroscopic time $t > 0$ and for any N large enough such that $t_{\mathrm{N}} < t$, the configuration η_t belongs to the ergodic component with very high probability. This is the one of the main novelties of [22].

Theorem 4.2.3 (Transience time for the exclusion process with absorption). *Let us define* $\ell_{\mathrm{N}} = (\log \mathrm{N})^8$ *and* $t_{\mathrm{N}} = \ell_{\mathrm{N}}^4/\mathrm{N}^2$*, then we have*

$$\lim_{\mathrm{N}\to\infty} \mathbb{P}_{\mu_{\mathrm{N}}}\Big(\eta_{t_{\mathrm{N}}} \notin \mathcal{E}_{\mathrm{N}}\Big) = 0.$$

The second main difficulty to prove Theorem 4.2.2 comes from the nature of the invariant measures of the process, as explained above.

4.2.2 Sketch of the proof

a) Estimation of the transience time (Theorem 4.2.3)

In order to prove Theorem 4.2.3, we first use an already known *mapping*. One can map the exclusion (EX) dynamics to a zero-range (ZR) configuration as follows: with $\eta \in \mathcal{H}_{\mathrm{N}}^m$, $m < \mathrm{N}$, we associate a configuration $\omega \in \mathbb{N}^{\mathbb{T}_{\mathrm{N}-m}}$ as follows. Look for the first empty site to the left of or at site 1 and label it 1. Moving to the right, define $\omega(i)$ as the number of particles between the i-th and $(i+1)$-th empty sites in η. See an illustration Figure 4.4. Note that this mapping is not one-to-one: shifting η one site to the left may not change ω. In particular, the reverse mapping from ZR to EX is only defined up to the position of the empty site with label 1.

Using this mapping, the EX process $\{\eta_t\}_{t\geqslant 0}$ (generated by $\mathrm{N}^2\mathcal{L}_{\mathrm{N}}$) can be coupled with a ZR process $\{\omega_t\}_{t\geqslant 0}$ as follows: whenever a particle jumps in the process η_t, a particle in the corresponding pile in ω_t jumps in the same direction. In particular, a jump of a particle to an empty site is allowed in the EX process if and only if the corresponding pile in the ZR process has at least two particles. Then, one can easily check that $\{\omega_t\}_{t\geqslant 0}$ is a symmetric ZR process with the jump rate function: $g_k(\omega) = \mathbf{1}_{\{\omega(k)\geqslant 2\}}$, namely with infinitesimal generator $\mathrm{N}^2\mathcal{L}_{\mathrm{N}-m}^{\mathrm{ZR}}$

[2] We say that A is uniformly parabolic if there exist constants $0 < c_1 < c_2 < \infty$ such that $c_1 \leqslant \frac{\partial \mathrm{A}}{\partial q}(p,q) \leqslant c_2$, uniformly in (p,q).

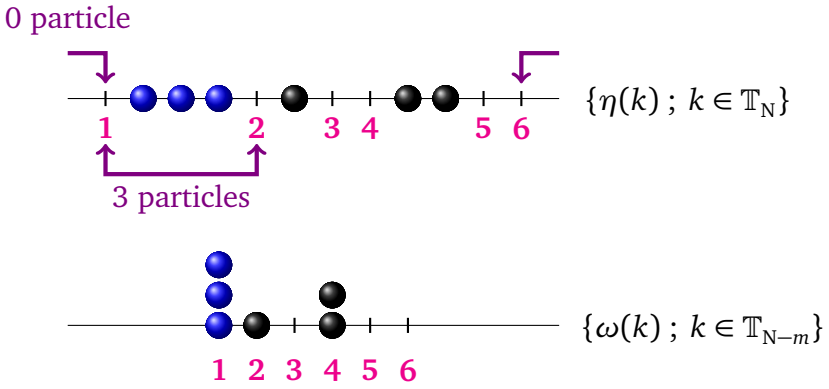

Figure 4.4: TOP: configuration in the exclusion dynamics.
BOTTOM: the associated zero-range configuration.

acting on functions $f : \mathbb{N}^{\mathbb{T}_{N-m}} \to \mathbb{R}$ as follows:

$$\mathcal{L}^{ZR}_{N-m} f(\omega) = \sum_{k\in\mathbb{T}_{N-m}} \sum_{z=\pm 1} \mathbf{1}_{\{\omega(k)\geqslant 2\}} \big(f(\omega^{k,k+z}) - f(\omega)\big), \tag{4.40}$$

where

$$\omega^{k,\ell}(j) = \begin{cases} \omega(k)-1; & j = k, \\ \omega(\ell)+1; & j = \ell, \\ \omega(j); & j \notin \{k,\ell\}. \end{cases}$$

Let us classify the possible states of this ZR process as we have already done for the EX process. In this setting, it is fairly clear that for $\omega \in \mathbb{N}^{\mathbb{T}_{N-m}}$ with m particles,

1. if $m \leqslant N-m$ (there are fewer particles than sites), either $\omega(k) \leqslant 1$ for all k (in which case the configuration is blocked), or this situation will be reached after a finite number of jumps a.s.
2. if $m > N-m$, by the pigeonhole principle there will always be at least one particle allowed to jump. The ergodic configurations are those where $\omega(k) \geqslant 1$ for all k, and the transient good the other ones, where there exists a site k such that $\omega(k) = 0$.

This translates immediately into the classification we claimed for the EX configurations[3]. Note that the invariant measures for the ZR process generated by $\mathcal{L}^{ZR}_{N-m}$ are well known, they are geometric product measures given by

$$\mu_\alpha(\omega) = \prod_k \frac{1}{1+\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{\omega(k)}, \qquad \alpha \geqslant 0. \tag{4.41}$$

Note however that the correspondence between invariant measures is not trivial. For only a few dynamics, we know the properties of the invariant measures seen from a tagged particle

[3]Note that, even though the reverse mapping from ZR to EX is only defined up to the position of the empty site with label 1, since the properties we consider are translation invariant we can safely transfer them from one setting to the other.

(*e.g.* for the simple exclusion process, they are product Bernoulli), however the setting here is more complicated.

Let us now go back to the estimation of the transience time: we want to use the fact that for all $t \geqslant 0$, η_t is in its ergodic component if and only if ω_t is in its ergodic component. Here are the main steps of the proof, detailed in [22, Section 4]:

1. For any integers $K > \ell$ and any $\delta > 0$ we define $B_{K,\ell}(\delta)$ as the set of *δ-regular ZR configurations*, namely the zero-range configurations on $\mathbb{T}_K$ such that in each box $\{j+1,\dots,j+\ell\}$, there are at least $(1+\delta)\ell$ particles, and those particles are not placed *abnormally* to the right of $\{j+1,\dots,j+\ell\}$ (in the sense that the cumulative distance of the particles to the first site in that box is not much bigger than it would be if $(1+\delta)\ell$ particles were spread evenly across the box).
2. We prove that, assuming the initial ZR configuration ω_0 to be δ-regular, the probability that the ZR process starting from ω_0 has not reached the ergodic component at a time of order ℓ^4 is exponentially small. This is the purpose of [22, Proposition 4.1].
3. Then, we prove that, while pairing the exclusion process and the zero-range process, if η_0 is started close to a smooth product measure as (4.37), then with high probability the associated zero-range configuration ω_0 is in $B_{K,\ell}(\delta)$ for some well chosen K, ℓ. This is the content of [22, Lemma 4.2].

The second point is the most involved, and requires several technical lemmas. The principle of the proof is the following: fix $k \in \mathbb{T}_K$. By waiting a long time (of order ℓ^4), we make sure that with high probability each particle initially present in $\{k+1,\dots,k+\ell\}$ will either

- have encountered an empty site in $\{k+1,\dots,k+\ell\}$, and gotten stuck there forever,
- or have left the box $\{k+1,\dots,k+\ell\}$ at some point.

At most ℓ particles fall into the first case, therefore, if the box initially contains at least $(1+\delta)\ell$ particles, $\delta\ell$ particles or more will have left the box $\{k+1,\dots,k+\ell\}$ at least once before $T_\ell = \ell^4$. In order for the site k to remain empty up until T_ℓ, all of these particles must have left by the other boundary $k+\ell$, and therefore, particles in $\{k+1,\dots,k+\ell\}$ would have performed an abnormally large number of steps to the right. The probability of the last case occurring decays faster than $\exp(-\ell^\varepsilon)$ for any $\varepsilon < \frac{1}{2}$.

b) Hydrodynamic limits (Theorem 4.2.2)

Our proof of the hydrodynamic limit relies on the classical *entropy method* (which is explained in details for instance in [71, Chapter 5]), but requires adaptations to solve the ergodicity issue, and to account for the non-product invariant measures. The main steps of the proof are given in [22, Section 5] in the form of two distinct results:

- the first one states that after reaching the ergodic component, the macroscopic density profile of the system is very close to the initial density profile ρ_0. This is [22, Lemma 5.1];
- the second result states that starting from the ergodic component, the hydrodynamic

limit holds. This is [22, Proposition 5.2]. Its proof uses the classical *Replacement Lemma*, which, because of the shape of our canonical and grand canonical measures, requires significant work. In particular, this is where the two fundamental properties of π_ρ – the correlations decay and the equivalence of ensemble – are crucial.

CHAPTER 5 For the future

5.1 Some of the current works in preparation

5.1.1 Moving boundaries

In the KCLG described in Section 4.2, the interfaces between these two phases propagate as particles from the supercritical phase ($\rho > \frac{1}{2}$) diffuse towards the subcritical phase ($\rho < \frac{1}{2}$). We expect that the macroscopic density profile evolves under the diffusive time scaling according to the Stefan problem

$$\partial_t \rho = \Delta\big(\mathrm{G}(\rho)\big) \qquad \text{where } \mathrm{G}(\rho) = \tfrac{2\rho-1}{\rho}\,\mathbf{1}_{\rho>\frac{1}{2}}. \tag{5.1}$$

In [22], as explained in Section 4.2, we have already treated the *liquid part* of the problem, *i.e.* when the density is initially larger than $\frac{1}{2}$ everywhere, and we have obtained a refined estimation of the time needed by the system to enter into the ergodic state.

In collaboration with O.Blondel and C. Erignoux, we are now considering the more complex case where the initial density can take arbitrary values. The proof of Funaki [47] to derive such a free boundary problem (5.1) from an exclusion-type dynamics relies on the decomposition of stationary, translation invariant, infinite volume measures. In our case, what we need is to we prove that any translation invariant infinite volume stationary measure is *decomposable* in the sense of [47, Lemma 4.1]. The answer is not obvious, because the invariant measures are not product. Our strategy consists in proving an equivalent of De Finetti's theorem [20, Section 35] for our model, namely: in the ergodic component, every invariant measure ν which is translation invariant can be written as a convex combination of the $\{\pi_\rho\}_{\rho>\frac{1}{2}}$. This is true if ν restricted on a finite box only depends on the number of particles in that box, as well as on the boundary values. For the simple exclusion process, this property holds thanks to the attractivity of the dynamics, which does not hold here. Other arguments are therefore needed.

Another approach would be to use the mapping with the ZR process (described in Section 4.2.2), the decomposition of the invariant measures has already been proved by Andjel in [3], and the techniques of Funaki may be used. Two new difficulties appear: first, one needs to write the corresponding macroscopic mapping to go back to the original Stefan problem (5.1); second, it is not straightforward to deduce a hydrodynamic limit for the EX process from the ZR through that relation, as noted in [48]. Let us assume that we are on a bounded domain. We

expect that in the diffusive space-time scaling, the empirical density profile for the ZR process converges towards the solution to a Stefan problem which writes as:

$$\partial_t \alpha = \Delta\big(\mathrm{F}(\alpha)\big) \qquad \text{where } \mathrm{F}(\alpha) = \tfrac{\alpha-1}{\alpha}\, \mathbf{1}_{\alpha>1}, \tag{5.2}$$

starting from some suitable initial profile α_0. The existence and uniqueness of weak solutions for this equation are known in the literature. Then, one can try to write a one-to-one mapping at the macroscopic level (between ρ and α): assume that there exists $x_0 < 1$ such that satisfies $\int_0^{x_0}(1+\alpha_0)(y)\mathrm{d}y = 1$, and define (at least formally)

$$\xi(t,x) = \int_0^x (1+\alpha(t,y))\,\mathrm{d}y, \qquad x \in [0,x_0]. \tag{5.3}$$

Then, formal computations show that ξ is a bijection from $[0,x_0]$ to $[0,1]$ and

$$\rho(t,u) = \frac{\alpha(t,\xi^{-1}(t,u))}{1+\alpha(t,\xi^{-1}(t,u))} \tag{5.4}$$

is solution to the Stefan problem (5.1). However PDE results are needed to make all the computations rigorous, especially because the solutions are defined in a weak sense.

5.1.2 Cumulants

We have seen in Section 2.2 that the non-equilibrium stationary state of a diffusive model can have an unexpected behaviour in the presence of several conservation laws. On the other hand, the investigation of current fluctuations of energy and particle in stationary states seems to give rise to similar universality classes than the ones predicted by the fluctuation hydrodynamics theory of Spohn [95]. In particular, numerics suggests that the cumulants of the large deviation function of current fluctuations can take several scaling forms depending on the universality class to which the multicomponent system belongs: recently formal computations have been done in [36] for two examples of multicomponent systems, using the FHT.

In collaboration with C. Bernardin, G. Ferré and G. Stoltz, we are currently considering the cumulant generating function of two-component systems. Very little is known, and that is why, instead of working with the one-dimensional chain of oscillators given in Section 2.1, we start with the *two-lane exclusion process* introduced in [91] for which the predictions have been done in the same paper. This model is remarkable since it belongs to stochastic lattice gases, but several universality classes arise for the same reason as in the chain of oscillators: local interaction but two conserved quantities which are nonlinearly related. In this system, particles are situated on two parallel chains, they cannot change lane, and they can jump to one of their nearest neighbour sites only if the latter is empty. The total particle number in each lane is conserved. The jump rate from a site in a given lane depends on the occupation numbers at sites in the other lane.

For one-dimensional diffusive systems, the cumulants of the current have been calculated in a few cases, using the macroscopic fluctuating theory (MFT) [18, 17, 4]. We plan to extend

the formal computations of [4], and obtain first the universal scaling form in the diffusive case, when both conserved quantity normally diffuse. Then, we will check on numerical simulations that our predictions are correct, which is far from being easy. Since cumulants are related to rare events, we will need to design clever algorithms, using some ideas taken from molecular simulation. Finally we will go to the superdiffusive cases, and use the FHT, as started in [36].

5.2 Other perspectives

5.2.1 Two-lane exclusion process

In the two-lane exclusion process briefly described in the previous section, can we obtain rigorously some of the predicted universality classes? For instance, using the same notations as in Section 2.1.3, can we rigorously derive ***(D,D)*** *and* $\boldsymbol{(D,\frac{3}{2}L)}$*?*

These two classes, which contain a diffusive peak, should indeed be studied first, since they have been proved for the harmonic chain already (recall Section 2.1.3). However the tools would certainly be different: the approaches to solve the harmonic case deeply rely on the linearity of the dynamics, which authorizes the use of powerful tools coming from the Fourier analysis. This is not the case of this model. Moreover, its dynamics is complex because:

- it is *non-gradient*, and therefore it should be treated with techniques developed for these situations: the non-gradient method of Varadhan [98, 88, 39] would be the suitable tool to derive the **(D,D)** class. One could start with the rigorous proof of the macroscopic equilibrium fluctuations as in [88], however in this case the spectral gap (which is one of the key ingredients) will probably not be uniform in the density. One could overcome this difficulty by cutoff arguments similar to the ones in [88, Section 8] ;
- the two conserved quantities in the two-lane exclusion process are really intricate. In [14], one of the essential features is that the *volume* acts as a fast variable for the *energy*, and the diffusion coefficient for the energy does not depend on it. Here, the densities in each lane have to be studied as a whole, and the system of equations is more difficult to close. However, some ideas of [14] could likely be adapted, in particular the introduction of the *two-dimensional correlation field* which mixes the two conserved quantities.

5.2.2 Gradient flow and microscopic interfaces

Can we use the gradient flow structure of the macroscopic laws in order to give a more precise analysis of the microscopic system?

Recall that in [21] (described in Section 4.1.2), we have proposed a nontrivial extension of the relative entropy method to degenerate intervals and singular solutions. More precisely, the degenerescence of the solution due to the presence of moving interfaces makes the relative entropy be generally infinite, and an approximation of the solution is needed to apply the original method. However, this program leads to significant technical obstacles, and may hardly

be extended to other degenerate parabolic equations. This tells us that the relative entropy may not be the most suitable tool in that case to derive hydrodynamic limits. At this point, one needs to understand better the mathematical structure of the hydrodynamic equation. In particular the PME (1.2) has many *gradient flow structures*, and it was shown by Otto [90] that the most natural interpretation from both physical and mathematical points of view is *via* the entropy functional $\mathcal{E}(\rho) = \int \rho^2$, and *not* the usual Boltzmann entropy $\int \rho \log \rho$. To be a little more precise, the gradient flow approach tells us that the solution $\rho_t(\cdot) = \rho(\cdot, t)$ to (4.4) is the only curve, in the space of probability measures $\mathcal{M}$ endowed with the Wasserstein distance, for which the nonnegative functional

$$\mathcal{E}(\rho_t) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^t \left|\nabla \mathcal{E}(\rho_s)\right|^2_{\mathrm{T}^\star_{\rho}\mathcal{M}} \,\mathrm{d}s + \frac{1}{2}\int_0^t \left|\dot{\rho}_s\right|^2_{\mathrm{T}_{\rho}\mathcal{M}} \,\mathrm{d}s$$

cancels. In the above, we used the formal notations $|\cdot|_{\mathrm{T}_{\rho}\mathcal{M}}$ and $|\cdot|_{\mathrm{T}^\star_{\rho}\mathcal{M}}$ in analogy with the Riemaniann setting. This formulation turns out to be very efficient to get asymptotic results. Therefore, one could try to define an equivalent of the entropy $\int \rho^2$ at the discrete level of the KCLG, which would be more efficient to prove hydrodynamic limits.

As we already mentioned in Section 1.3, the works [84, 86, 38] propose equivalent formulations of the gradient flow property for discrete Markov chains, and in [41] we have already used this discrete structure in order to recover the simplest hydrodynamic limit already well known: the heat equation derived from the simple exclusion process. More importantly, [38] gives a discrete gradient flow setting for the PME. However, the Markov chain considered in that article is not of the same form as the one we would get from the KCLG previously defined (in [38] it is already the *discretization* of the PME). This notably implies that the so-called *discrete continuity equation* cannot be written straightforwardly. Precisely, instead of having a discrete equation of the form

$$\dot{\rho}_t(k) = \sum_j \mathrm{Q}(k,j)\big(\rho_t^2(j) - \rho_t^2(k)\big)$$

(which is one of the conditions to apply [38]), in the KCLG defined above we get

$$\begin{aligned}\dot{\rho}_t(k) = \sum_j \mathrm{Q}(k,j)\big(&\rho_t(j-1)\rho_t(j) + \rho_t(j)\rho_t(j+1) - \rho_t(j-1)\rho_t(j+1) \\ &- \rho_t(k-1)\rho_t(k) + \rho_t(k)\rho_t(k+1) - \rho_t(k-1)\rho_t(k+1)\big).\end{aligned}$$

One can try to reproduce the arguments of [84, 86, 38] for this different continuity equation, and change slightly the metric. If this approach does not work, at least one would have understood which features to impose at the microscopic level in order to preserve the gradient flow structure that one can see at the macroscopic level, and maybe propose another mathematical model. Finally, provided with a microscopic model for which an equivalent of the entropy $\int \rho^2$ naturally appears, one could implement a generalized relative entropy method, using this new entropy which is more natural from the underlying gradient flow structure, and taking the inspiration from Yau's original method [100]. This would constitute a new tool to prove hydrodynamic limits.

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RÉSUMÉ

Cette habilitation rassemble une grande partie de mon travail de recherche effectué depuis la fin de mon doctorat. Mes recherches se sont principalement orientées vers la question suivante : comment dériver mathématiquement des équations aux dérivées partielles non-linéaires variées, toutes observées à notre échelle macroscopique, à partir d'une description microscopique des systèmes de particules qui composent notre milieu ? Ce problème exigeant nécessite de mettre en œuvre des limites d'échelle en temps long et en grand volume. Du point de vue mathématique, le principal défi consiste à prouver des théorèmes de convergence connus sous le nom de limites hydrodynamiques, qui permettent de valider les équations macroscopiques données par la physique. Des exemples typiques de problèmes de description micro/macro sont : la conduction de la chaleur, l'écoulement d'un fluide ou les transitions de phase d'un milieu hétérogène.

Avec différents co-auteurs, nous avons analysé plusieurs modèles mathématiques dont les buts sont multiples : comprendre les aspects microscopiques de la diffusion dite anormale ; décrire les effets de propagation dans des milieux multiphasés, ainsi que les interfaces grandissant de manière aléatoire. Deux familles de modèles unidimensionnels ont été privilégiées, ces modèles ont tous une composante stochastique et suivent une ou plusieurs lois de conservation. Ce sont (1) les gaz sur réseaux cinétiquement contraints, qui sont sujets à des restrictions dynamiques ; (2) les systèmes d'atomes Hamiltoniens perturbés par un bruit stochastique.

ABSTRACT

This manuscript reviews a large part of my research work since the end of my PhD thesis, which has been mostly directed towards the following demanding question: how can we derive various types of nonlinear partial differential equations (PDEs), observed at our macroscopic level, from the underlying microscopic particle systems via a suitably taken long time and large space scaling limit. The mathematical challenge consists in proving convergence theorems known as hydrodynamic limits, in order to recover the macroscopic PDEs given by physics. Typical examples of micro/macro description problems are heat conduction, fluid flow inside a physical medium or phase transitions in matter.

With my coauthors we have analyzed several mathematical models which aimed at: understanding the microscopic features of anomalous diffusion; describing propagation effects in multiphase media as well as randomly growing interfaces. Two classes of one-dimensional models are mainly considered, they all have a stochastic component and follow at least one conservation rule: (1) kinetically constrained lattice gases which are subject to dynamics restrictions ; (2) Hamiltonian systems of atoms perturbed by a stochastic noise.