On a probabilistic interpretation of the Keller-Segel parabolic-parabolic equations

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On a probabilistic interpretation of the Keller-Segel parabolic-parabolic equations

Sur une interprétation probabiliste des équations de Keller-Segel de type parabolique-parabolique

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Abstract

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density of a cell population and of the concentration of a chemical attractant.

This thesis is devoted to the study of the parabolic–parabolic Keller-Segel equations using probabilistic methods. To this aim, we give rise to a non linear stochastic differential equation of McKean-Vlasov type whose drift involves all the past of one dimensional time marginal distributions of the process in a singular way. These marginal distributions coupled with a suitable transformation of them are our probabilistic interpretation of a solution to the Keller Segel model. In terms of approximations by particle systems, an interesting and, to the best of our knowledge, new and challenging difficulty arises: each particle interacts with all the past of the other ones by means of a highly singular space-time kernel.

In the one-dimensional case, we prove that the parabolic-parabolic Keller-Segel system in the whole Euclidean space and the corresponding McKean-Vlasov stochastic differential equation are well-posed in well chosen space of solutions for any values of the parameters of the model. Then, we prove the well-posedness of the corresponding singularly interacting and non-Markovian stochastic particle system. Furthermore, we establish its propagation of chaos towards a unique mean-field limit whose time marginal distributions solve the one-dimensional parabolic-parabolic Keller-Segel model.

In the two-dimensional case there exists a possibility of a blow-up in finite time for the Keller-Segel system if some parameters of the model are large. Indeed, we prove the well-posedness of the mean field limit under some constraints on the parameters and initial datum. Under these constraints, we prove the well-posedness of the Keller-Segel model in the plane. To obtain this result, we combine PDE analysis and stochastic analysis techniques.

Finally, we propose a fully probabilistic numerical method for approximating the two-dimensional Keller-Segel model and survey our main numerical results.

Keywords: McKean-Vlasov stochastic processes; stochastic particle systems with singular non-Markovian interaction; probabilistic methods for PDEs; Keller-Segel PDE; chemotaxis models.

Résumé

En chimiotaxie, le modèle parabolique-parabolique classique de Keller-Segel en dimension d décrit l'évolution en temps de la densité d'une population de cellules et de la concentration d'un attracteur chimique.

Cette thèse porte sur l'étude des équations de Keller-Segel parabolique-parabolique par des méthodes probabilistes. Dans ce but, nous construisons une équation différentielle stochastique non linéaire au sens de McKean-Vlasov dont le coefficient dont la coefficient de dérive dépend, de manière singulière, de tout le passé des lois marginales en temps du processus. Ces lois marginales couplées avec une transformation judicieuse permettent d'interpréter les équations de Keller-Segel de manière probabiliste. En ce qui concerne l'approximation particulaire il faut surmonter une difficulté intéressante et, nous semble-t-il, originale et difficile: chaque particule interagit avec le passé de toutes les autres par l'intermédiaire d'un noyau espace-temps fortement singulier.

En dimension 1, quelles que soient les valeurs des paramètres de modèle, nous prouvons que les équations de Keller-Segel sont bien posées dans tout l'espace et qu'il en est de même pour l'équation différentielle stochastique de McKean-Vlasov correspondante. Ensuite, nous prouvons caractère bien posé du système associée des particules en interaction non markovien et singulière. Nous établissons aussi la propagation du chaos vers une unique limite champ moyen dont les lois marginales en temps résolvent le système Keller-Segel parabolique-parabolique.

En dimension 2, des paramètres de modèle trop grands peuvent conduire à une explosion en temps fini de la solution aux équations du Keller-Segel. De fait, nous montrons le caractère bien posé du processus non-linéaire au sens de McKean-Vlasov en imposant des contraintes sur les paramètres et données initiales. Pour obtenir ce résultat, nous combinons des techniques d'analyse d'équations aux dérivées partielles et d'analyse stochastique.

Finalement, nous proposons une méthode numérique totalement probabiliste pour approcher les solutions du système Keller-Segel bi-dimensionnel et nous présentons les principaux résultats de nos expérimentations numériques.

Mots clefs: processus stochastiques de McKean-Vlasov; particules stochastiques en interaction non markovien et singulière; methodes probabilistes pour les EDP; EDP de Keller-Segel; modèles de chimiotaxie.

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Chapter 1

On the Keller-Segel model for chemotaxis: From the literature to our main results

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\frac{1}{2}\nabla\rho - \chi\rho\nabla c)(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ \partial_t c(t,x) = \frac{1}{2}\triangle c(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0,\ x\in\mathbb{R}^d. \\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0(x). \end{cases}$$

The goal of this thesis is to propose a new probabilistic interpretation for this non-linear doubly parabolic system and analyze it from theoretical and numerical viewpoint.

In this introductory chapter we provide an overview of the literature concerning this model and our main results.

We start with biological phenomena aimed to be modeled by the Keller-Segel system: chemotaxis. In Section 1.1 we define it, revisit the historical aspect of its investigation and give some examples of biological processes governed by or involving chemotaxis.

Then, Section 1.2 explains the behaviour of cells when undergoing chemotaxis on a micro and a macro level. Afterwards, we review the pioneer work of Keller and Segel [46, 47, 48] who pose the above system of PDEs in its more general form.

Since it has been posed, the system is a subject of huge amount of PDE analysis literature. An interesting phenomenon emerging from it is the possibility of a blow-up in finite time. A selection of the PDE analysis results on the Keller-Segel system is given in Section 1.3.

Recently, probabilistic interpretations have appeared for mollified or parabolic-elliptic versions of the fully parabolic model. In Section 1.4 we review the state of the art for these stochastic approaches.

In Section 1.5 we present and discuss our own probabilistic interpretation: a McKean-Vlasov stochastic process whose drift involves all the past of one dimensional time marginals of the process in a singular way. These time marginals coupled with a suitable transformation of them are our candidate for a solution to the Keller-Segel system. In terms of approximations by particle systems, an interesting and, to the best of our knowledge, new and challenging difficulty arises: each particle interacts with all the past of the other ones by means of a highly singular space-time kernel. In this Section we also state our main results and summarize some of our numerical results.

1.1 Our biological motivations: Phenomena of chemotaxis

In order to give meaning to the notion of chemo-taxis, we will start from the suffix *taxis* (pl. *taxes*), an ancient Greek word for arrangement. Taxis represents oriented movement of a motile organism in response to a stimulus (e.g. light, temperature, food). The movement can be directed towards or away from the stimulus. In the first case, we have positive taxis and in the later negative taxis. It is important to emphasize that only the motile organisms are capable of performing such movements. Motile essentially means able to move by itself. For example, bacteria cells use structures called flagella to enable these movements. Taxes should not be confused with tropism and kinesis. These are another classes of movements in response to a stimulus. The first one represents the movements that include growth towards or away from the stimulus. The difference is that in taxes the organism must have motility and the exhibited movement is not growth, but rather a guided change of position. On the other hand, in kinesis, the presence of of stimulus influences the changes of velocity of the organism, but not its direction in movement.

Taxes are also classified by the type of stimulus governing them, which is indicated by a prefix. *Photo*-taxis is governed by light, *thermo*-taxis by temperature. If the presence of oxygen triggers the movements, we have *aero*-taxis. Finally, a chemical stimulus is responsible for *chemo*-taxis.

Since the end of 17th century and Leeuwenhoek's advances in the field of microbiology, scientists have been studying the movements of organisms. However, bacterial chemotaxis was discovered two centuries after by Engelmann [27] and Pfeffer [63, 64]. By Pfeffer's original definition, chemotaxis is defined as anything that causes the oriented movement of an organism or a cell relative to a chemical gradient. In his work, Pfeffer also gave the basis for assays on how to detect chemotaxis, i.e. the capillary method [63, 64]. Chemical that prompts positive chemotaxis was called the *chemo-attractant*, while chemical that causes the organism to flee away from the source was called *chemo-repellent*. Chemo-attractants usually represent favourable environment for the organism, e.g. food, while the chemo-repellents are noxious substances, such as poisons. One interesting consequence of positive chemotaxis is cell aggregation. The chemo-attractants produced by the fellow species increase self-attraction among the population and further stimulate cell aggregation [18].

The study of the phenomena of chemotaxis may be divided into two periods: before 1960's and after. As mentioned in [4], the work before 1960's was carried out in complex media and was of a quite subjective nature. The review of this period is given in [6, 80, 81]. In the second period, the first priority was to develop conditions for obtaining motility and chemotaxis in defined media [1, 5, 2, 3] . Then it was important to find quantitative methods that objectively detect chemotaxis [1, 77]. This work, mostly by Adler, altered the attention from phenomenological to quantitative research and initiated studies to reveal the molecular mechanism of bacterial chemotaxis. Afterwards, the number of groups studying bacterial chemotaxis has been continuously rising. Bacterial motility and chemotaxis have been studied most intensively in *Escherichia coli* and its close relative *Salmonella enterica serovar Typhimurium*. We refer to [26] for a very complete and thorough further reading, which deals not only with bacterial chemotaxis, but also with chemotaxis as a mean of cell-cell communication, chemotaxis in amoeba, blood cells, sperm cells and nervous system.

After such an extensional research in the field, natural question that poses itself is what the significance of chemotaxis is. It has been established that chemotaxis plays a role in some of the most important biological processes, not only for humans, but for almost all species.

Naturally, we start from the role of chemotaxis in reproduction, as it is the essential process for existance of life. It is firstly discovered in marine species [54] that chemotaxis is responsible of guided movement of spermatozoa to the egg during fertilization. The research spread to all species, from non-mammals to mammals. It has been established that for humans and some other mammals, chemotaxis besides the previous role in guiding, has a selective role as well. Namely, not all of the spermatozoa have the ability to fertilize the egg. The ones that do have it are chemotactically responsive. Chemotaxis is in charge for selecting them and then guiding towards the egg. For a full review on sperm chemotaxis we refer to Chapter 7 in [26].

Not only does the chemotaxis have a reproductive role, but it also appears in the embryonic phase once the fertilization is successfully completed. During the development of the embryo, cell migration has a crucial role in morphogenetic processes and formation of nervous system [35]. Many of these migration are caused by chemotaxis. The development and especially wiring of nervous system depends on the precise guidance of axonal growth cones to their targets. Mechanism underlying it is again chemotaxis [25].

Furthermore, we find its role in functioning of the immune system. Certainly, movement and quick response are essential when it come to the immune system. In order to threat an infection, the white blood cells need to migrate towards it. They are attracted by the change of chemical gradient that the infection produces [58].

So far, we have only seen the positive aspects of chemotactic movements. However, a negative aspect is the participation of chemotaxis in cancer metastasis and progression. Once the tumor had affected a certain tissue, cancer cells use chemotaxis to migrate towards the surrounding tissue and invade blood vessels [67].

An interesting role of chemotaxis can be found in agronomy and the use of bio-fertilizers. Namely, certain groups of bacteria in the rhizosphere region of soil positively influences plant growth. Bacteria successfully colonizes the rhizosphere thanks to chemotactic attraction from the root exudates of the plants [61].

We conclude this part with one fascinating way to use chemotaxis in medical purposes. Particularly, in construction of nanorobots for human drug delivery. The idea is to design autonomously moving artificial cells which would carry drugs and be capable of chemotactic movements. These movements would rely on artificial chemotaxis. This concept is described and analyzed in [50].

1.2 Modelling of chemotaxis and the Keller–Segel approach

As the biological research of the phenomenon grew and altered its interest towards experiments, the need for mathematical models for chemotaxis emerged. Mathematical models help in better understanding of experimental results and allow biologists to study different characteristics of bacterial systems without the need to intensively repeat the experiments. When one desires to mathematically model chemotaxis, first the goal and nature of the results should be clearly defined. That is to say, are we interested in the particular behaviour of one individual (cell, bacteria) of the population or of the whole population at once. This leads us to two main approaches when modelling chemotactic movements, the microscopic and macroscopic approaches, respectively.

As the microscopic models focus on the individual cell, it is important to understand the biological processes that are happening within it when the cell becomes chemotactically active. We will try to illustrate it on the example of *E. Coli*, as its chemotaxis is understood best. When there is no stimuli in its environment, *E. Coli* swims in a random walk. The random walk takes on a biased character, towards the attractant or away from the repellent, as soon as the presence of stimuli is sensed. The movement itself is a series of "runs" and "tumbles". Runs are movements following a (fairly) straight line, which are suddenly interrupted by a change in the direction, a tumble. When E. Coli exhibits positive chemotaxis, the number of tumbles decrease. The opposite happens with the negative chemotaxis. If there is a change of gradient in the extra-cellular environment, the bacterium is unable to detect it along its own length, because its size is too small. Instead, the cell is equipped with membrane receptors, which are able to distinguish very low attractant concentrations. Once the attractant is detected, the receptor passes the signal inside the cell. Thanks to the intra-cellular proteins, called Che proteins (from Chemotaxis proteins), a signaling cascade occurs and finally arrives to flagellar motors. Then, the flagella are rotated clockwise or counterclockwise, depending on the type of the stimulus. Clockwise rotation leads to tumbling and counterclockwise to runs. An important part of the process is also the adaptation, which includes resetting of receptors, as if they have not been stimulated at all. Furthermore, since the bacteria are able to sense a tiny change in gradients, they need to be able to amplify the signal (gain process).

The mathematical models for one cell try to represent above mentioned processes, individually or together. So far, none of the models was able to reproduce well all of them together. One of the reasons is that they all occur in different time scales. The models which do a good job in representing ligand binding and adaptation, can not represent well also the chemoreceptor sensitivity and gain and vice versa. For a review on these and many other processes and how they have been modeled in the literature, we refer to the thorough and comprehensive review by Tindall *et. al* [75].

Now, we will see how a population exhibiting chemotactic activity behaves on the example of slime molds. Slime molds are populations of amoebae that grow by cell division. The cells wander around their environment exhausting food supplies which they are able to find using chemotaxis. Once the nourishment is consumed, cells disperse uniformly around the area at their disposal. A while later, some of the cells begin emitting a signal that attracts other cells who start moving towards it and are triggered to emit the same attracting signal. The cells aggregate, forming a slug that may move, respond to chemical stimuli and detect food sources. Eventually, the slug produces fruiting bodies and releases spores in order to recommence the life cycle. The pioneer of biological research of slime molds was Bonner (see e.g. [9]). What is fascinating about slime molds even today, is that individually, they are very simple organisms that exhibit "intelligent" behaviour once they aggregate. In the study [74], the authors were even able to reproduce a map of Tokyo rail system once the different stimuli were put in the right places.

Motivated by describing the onset of slime mold aggregation using a macroscopic approach, Evelyn F. Keller and Lee A. Segel propose in [46] a model of four coupled parabolic equations.

Namely, the authors start from the individual properties of the cells in order to derive a model for the aggregation stage. Let $\rho(t, x)$ denote the density of the amoebae at point x in time t, $c(t, x)$

denotes the concentration of the chemo-attractant (acrasin), $\eta(t,x)$ denotes the concentration of the enzyme that degrades the chemo-attractant (acrasinase) and, finally, $\nu(t,x)$ denotes the concentration of a bio-chemical complex V formed by acrasin and arcasinase. The individual properties taken into account are the following:

1. The amoeba moves according to a random motion analogous to a diffusion that is biased towards the direction of the positive gradient of the attractant.
2. The acrasin is produced by the amoebae with rate $f(c)$.
3. The acrasinase is produced by the amoebae with rate $g(c,\eta)$.
4. The complex V dissociates into arcasinase and a degraded product (d.p.):

$$c + \eta \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} V \overset{k_2}{\rightarrow} \eta + \text{d.p.}$$

5. Acrasin, arcasinase and the complex V diffuse according to Fick's law.

In order to derive the equation for ρ, the authors use the mass balance equation and the fact that the flux of amobea mass is proportional to $\nabla\rho$ (by Fick's law) and ∇c (by Fourier's law). Birth and death are not taken into account. Thus,

$$\frac{\partial}{\partial t}\rho(t,x) = \nabla\cdot(D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c).$$

Here, D_1 represents the strength of the random movement and χ the impact of the chemo-attractant gradient to the flow of the population. The chemo-attractant diffuses according to Fick's law and its dynamics involves its production and consumption rates as described above,

$$\frac{\partial}{\partial t}c(t,x) = D_c\triangle c + f(c)\rho - k_1 c\eta + k_{-1}\nu.$$

The equations for η and ν are derived in the same way. The authors arrive to the following system:

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) = \nabla\cdot(D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c), \quad t>0, \quad x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}c(t,x) = D_c\triangle c + f(c)\rho - k_1 c\eta + k_{-1}\nu, \quad t>0, \quad x\in\mathbb{R}^d, \\ \frac{\partial}{\partial t}\eta(t,x) = D_\eta\triangle\eta + \rho g(c,\eta) - k_1 c\eta + (k_{-1}+k_2)\nu, \quad t>0, \quad x\in\mathbb{R}^n, \\ \frac{\partial}{\partial t}\nu(t,x) = D_\nu\triangle\nu + k_1 c\eta - (k_{-1}+k_2)\nu, \quad t>0, \quad x\in\mathbb{R}^d, \\ \rho(0,x)=\rho_0(x),\ c(0,x)=c_0,\ \eta(0,x)=\eta_0,\ \nu(0,x)=\nu_0, \quad x\in\mathbb{R}^d. \end{cases} \tag{1.1}$$

Here k_{-1}, k_1 and k_2 are positive constants.

Then, the authors argue that the aggregation occurs as, in some point of maturation, the individual properties of the cells change. Thus, a uniform distribution is no longer favorable and it becomes unstable. The objective is to see how such change in individual cells impacts the whole population, rather to explain why and how such change happens. In order to do so, the authors propose a simplified version of the latter system *"as it is useful for the sake of clarity to employ the simplest reasonable model"* [46, p. 403]. They assume that the bio-chemical complex V is in a steady state w.r.t. the chemical reaction: $k_1 c\eta - (k_{-1}+k_2)\nu = 0$ and that the total concentration of the free and bound degradant is constant: $\eta + \nu = \eta_0$. Thus, (1.1) transforms into the following system of non-linear parabolic equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho(t,x) = \nabla \cdot (D_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c), \quad t>0, \quad x \in \mathbb{R}^d, & (1.2a) \\ \dfrac{\partial}{\partial t}c(t,x) = D_c \triangle c + f(c)\rho - k(c)c, \quad t>0, \quad x \in \mathbb{R}^d, & (1.2b) \\ \rho(0,x) = \rho_0(x), \ c(0,x) = c_0, \quad x \in \mathbb{R}^d. \end{cases}$$

Then, the authors study how a small time dependent perturbation of the uniform configuration influences a linearized version of (1.2) for $d = 2$. They find conditions under which the uniform state is temporarily or definitely perturbed. The latter may be interpreted as the beginning of aggregation. Finally, analyzing these conditions, the conclusion is that a definite perturbation occurs as a result of: i) increase in the sensitivity of the population to a given acrasin gradient, ii) increase in the rate which cells produce the acrasin or iii) increase in the rate of acrasin production (f) due to high acrasin production. In other words, if the cells are too sensitive to a certain attractant or they start producing too much of it, we may expect an aggregation. This claim will often be revisited in this thesis.

The above work is followed by two more articles by the same authors [47, 48]. In [47], the chemotaxis of amoebae is modelled when the concentration of the acrasin c is assumed to be given. Equation (1.2a) is viewed as evolution of a probability density function and is derived as collective behaviour of individual cell behaviours, where $D_1(\rho,c) = \mu(c)$ and $\chi(\rho,c) = \rho\chi(c)$. In [48], the authors use the system (1.2) in $d = 1$ to reproduce the experimental results of Adler's capillary essays. They assume again the specific form of motility and sensitivity functions: $D_1(\rho,c) = \mu(c)$ and $\chi(\rho,c) = \rho\chi(c)$. In (1.2b), $k(c)$ is supposed to be zero and the cells no longer produce the chemo-attractant but consume it with the rate $f(c)$ (i.e. the sign in front of $f(c)\rho$ has changed). The goal was to observe the traveling bands of bacteria up to the capillary tube, as in the experimental case and to compare with the experimental data some quantitative properties (width and speed of the traveling bands). The comparison result were encouraging, but as the authors notice, what is more encouraging is that their model is capable of describing different assays of chemotaxis and that their framework may serve when describing other collective chemotactic phenomena.

Indeed, we deliberately used here the technical terms "ameboe", "acrasin", "acrasinase" in order to help the reader concretize this example of chemotactic activity. Once one understands the phenomenon behind it and the mathematical description of Keller and Segel, one could easily change these words with "cell population", "chemo-attractant" and "chemo-degradant", respectively and obtain a general model for chemotaxis. Nowadays, any model of the following form is called a Keller-Segel type model:

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) = \nabla \cdot (f_1(\rho,c)\nabla\rho - \chi(\rho,c)\nabla c) + f_2(\rho,c), \quad t>0, \quad x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t}c(t,x) = D_c \triangle c + f_3(c,\rho) - f_4(c,\rho)c, \quad t>0, \quad x \in \mathbb{R}^d, \\ \rho(0,x) = \rho_0(x), \ c(0,x) = c_0, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Here the function f_2 accounts for birth and death of the cell population. It is usually neglected assuming the phenomena occurs over a short period of time.

This thesis will be devoted to the so-called classical Keller–Segel model of parabolic-parabolic type given by

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\nabla \rho - \chi \rho \nabla c)(t,x), \quad t > 0,\ x \in \mathbb{R}^d, & \text{(1.4a)} \\ \alpha\ \partial_t c(t,x) = \triangle c(t,x) - \lambda c(t,x) + \rho(t,x), \quad t > 0,\ x \in \mathbb{R}^d, & \text{(1.4b)} \\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0(x), \quad x \in \mathbb{R}^d. \end{cases}$$

where $\lambda \geq 0$ and $\alpha, \chi > 0$. It corresponds to $f_1(\rho, c) \equiv \text{const}$, $\chi(\rho, c) = \chi\rho$, $f_2 \equiv 0$, $f_3(c, \rho) = \rho$ and $f_4(c, \rho) = \lambda$ in (1.3). This system is as well called the "minimal model" as it does not involve complicated functions for sensitivity of the population, production and decay of chemo-attractant but rather simple linear functions. Still, it is rich enough to describe the phenomena in question as we will see in the next section.

Notice that the first equation in (1.4) preserves total mass as long as the solutions are well defined. We will denote

$$M := \int_{\mathbb{R}^d} \rho_0(x)dx = \int_{\mathbb{R}^d} \rho(t,x)dx.$$

We also remark that when $\alpha = 0$, (1.4b) is an elliptic equation and the system may be decoupled using Green's functions. This is the so-called parabolic–elliptic version of the model. Even though this thesis is focused on the case $\alpha = 1$ (more general on $\alpha > 0$), we will see that the two cases are somehow inseparable since the techniques used to analyze the parabolic-elliptic model are the groundwork for the doubly parabolic model.

1.3 PDE analysis of the Keller-Segel system

As the Keller-Segel system is designed to model the onset of cell aggregation when triggered by chemical stimulus, it is no surprise that the solutions may blow-up in finite time. The definition of the blow-up in finite time for a solution (ρ, c) is the following : there exists a time $T_0 < \infty$ such that ρ_t converges to a measure not belonging to $L^1(\mathbb{R}^d)$ as $t \to T_0$. In general, the question of well-posedness of (1.4) is a subject of an extensive amount of PDE literature over the past almost 40 years. A very complete review of the results obtained until early 2000's can be found in Horstmann [41, 42]. Then, we suggest to the interested reader the review of Perthame [62] which after a theoretical review of the Keller-Segel system shows its connection with kinetic models for chemotaxis and the work of Hillen and Painter [39] reviewing results on different variations of (1.4).

The principal conclusion when investigating the literature about the Keller-Segel system is that whether we have global well-posedness or a blow up in finite time is highly correlated with the space dimension of the problem. In addition, various results obtained depend also on the prescribed initial and possible boundary conditions, type of the domain and value of parameter α.

Here we will summarize some of the results in the literature and will classify them in three groups: $d = 1$, $d = 2$ and $d \geq 3$.

The one-dimensional case

The well-posedness of (1.4) in $d = 1$ is the least elaborate case. It was previously studied by Osaki and Yagi [60] and Hillen and Potapov [40]. The conclusion is: The solution exists globally in time on bounded intervals with periodic or Neumann boundary conditions.

In [60] the authors analyze a more general model:

$$\begin{cases} \partial_t \rho(t,x) = a\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}(\rho \frac{\partial}{\partial x}\chi(c) \quad t>0,\ x\in I, \\ \partial_t c(t,x) = \frac{\partial^2 c}{\partial x^2} - \lambda c(t,x) + d\rho(t,x), \quad t>0,\ x\in I, \\ \rho(0,x)=\rho_0(x), \quad c(0,x)=c_0(x), \quad x\in I \\ \frac{\partial \rho}{\partial}(t,\alpha) = \frac{\partial \rho}{\partial}(t,\beta) = \frac{\partial c}{\partial}(t,\alpha) = \frac{\partial c}{\partial}(t,\beta) = 0, \quad t>0, \end{cases} \tag{1.5}$$

where $I=(\alpha,\beta)$. They assume χ is a smooth function on $(0,\infty)$, differentiable three times and that these derivatives satisfy certain estimates. The case $\chi(c)=\chi c$, $\chi>0$ corresponding to (1.4), is included in their assumptions. Supposing $\rho_0 \in L^2(I)\cap L^1(I)$, $c_0\in H^1(I)$ and $\inf_I c_0(x)>0$, they prove (1.5) admits a unique global solution belonging to

$$\rho \in \mathcal{C}([0,\infty);L^2(I))\cap \mathcal{C}^1((0,\infty);L^2(I))\cap \mathcal{C}((0,\infty);H^2_N(I)),$$
$$c \in \mathcal{C}([0,\infty);H^1(I))\cap \mathcal{C}^1((0,\infty);H^1(I))\cap \mathcal{C}((0,\infty);H^3_N(I)).$$

Here the subscript N emphasizes that the Neumann boundary condition is satisfied by functions belonging to $H^2_N(I)$ and $H^3_N(I)$. They prove such solution is a classical solution in the case of (1.4).

Their well-posedness proof is divided into two steps: first, they establish the existence of a unique local in time solution to (1.5). Second, they prove the following energy estimate:

$$\frac{\partial}{\partial t}\int_I \left(\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 + \left(\frac{\partial^2 c}{\partial x^2}\right)^2\right)dx + \int_I \left(\frac{a}{2}\left(\frac{\partial^3 \rho}{\partial x^3}\right)^2 + \frac{b}{2}\left(\frac{\partial^4 c}{\partial x^4}\right)^2\right)dx$$
$$+ \int_I \left(\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 + \lambda\left(\frac{\partial^4 c}{\partial x^4}\right)^2\right)dx \le p(\|\rho\|_{H^1} + \|c\|_{H^2}).$$

This helps them to extend the local solution to an arbitrary time horizon $T>0$.

The work in [40] concerns the classical model (1.4) on a bounded interval $(0,l)$ with either Neumann or periodic boundary conditions. The global well-posedness is obtained assuming that $\rho_0 \in L^\infty(I)\cap L^1(I)$ and $c_0\in W^{\sigma,p}(I)$, where p and σ belong to a particular set of parameters. This set is defined as follows: a tuple of parameters (σ,p,r,P,Q) is admissible if

$$1<\sigma<2, \quad \frac{1}{\sigma-1}<p<\infty, \quad \frac{2p}{\sigma p+1}<r<\frac{1}{\sigma-1}$$
$$1<P<1\frac{1}{p}, \quad \frac{1}{P}+\frac{1}{Q}=1, \quad \frac{1}{p}<\frac{Q}{r}<\frac{1}{p}+2.$$

The result again is obtained by globalizing a local solution obtained applying Banach's fixed point theorem. Then, this solution is turned into a global one by using the regularity of heat semi-group and the mild formulation of (1.4).

The two-dimensional case

In the parabolic-elliptic version of the system, i.e. when $\alpha=0$, one may decouple the equations by expressing c in terms of ρ using Green's functions. In this setting, the system exhibits a threshold behaviour: if $M\chi < 8\pi$ the solutions exists globally in time, if $M\chi > 8\pi$ every solution blows-up in finite time (see e.g. Blanchet *et. al* [8] and Nagai and Ogawa [56]). As for the profile of the singularity, Herrero and and Velázquez [37] prove existence of a radially symmetric solution on a disc with Neumann boundary condition that blows-up at the origin in finite time by acquiring a δ-function type singularity. This phenomenon is called in the literature the "*chemotactic collapse*". The condition in the threshold implies that in order to form a singularity, the total mass of the cell population needs to be large or the attraction of the chemical needs to be very strong. This is in accordance with the conclusions made by Keller and Segel in [46] when analyzing the instability of the system.

On the other hand, the parabolic-parabolic model (1.4) expresses a less straight-forward behaviour. It has been proved that when $M\chi < 8\pi$ one has global existence (see Calvez and Corrias [20], Mizogouchi [55]). However, in Biller *et. al* [7] the authors find an initial configuration of the system in which a global solution in some sense exists with $M\chi > 8\pi$. Then, Herrero and Velázquez [38] construct a radially symmetric solution on a disk that blows-up and develops δ-function type singularities. Finally, unique solution with any positive mass exists when the reverse diffusion of the chemoattractant is large enough (Corrias *et. al* [22]). Thus, in the case of parabolic-parabolic model, the value 8π can still be understood as a threshold, but in a different sense: under it there is global existence, over it there exists a solution that blows up.

In [20] the authors obtain the global existence when $M < 8\pi$ and $\alpha = \chi = 1$ assuming as well that

1. $\rho_0 \in L^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2, \log(1+|x|^2)dx)$ and $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$;
2. $c_0 \in H^1(\mathbb{R}^2)$ if $\lambda > 0$ or $c_0 \in L^1(\mathbb{R}^2)$ and $|\nabla c_0| \in L^2(\mathbb{R}^2)$ if $\alpha = 0$;
3. $\rho_0 c_0 \in L^1(\mathbb{R}^2)$.

Notice that the mass condition is equivalent to $M\chi < 8\pi$ for a given $\chi > 0$ by rescaling of (1.4). In the same sub-critical case, the global existence result is obtained in [55] assuming $\rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $c_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Both of these works use energy methods to prove the apriori estimates for the solutions of (1.4). Then, these estimates lead to existence of global solution in sub-critical case. The free energy functional associated to (1.4) is

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} \rho(t,x) logc(t,x)\ dx - \int_{\mathbb{R}^2} \rho(t,x)c(t,x)\ dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla c(t,x)|^2\ dx + \frac{\lambda}{2}\int_{\mathbb{R}^2} c^2(t,x)\ dx.$$

The technical computations exploit in [55] the Trudinger-Moser inequality while in [20] two alternatives are proposed: either to use the so-called Onfori inequality on the whole space or the Hardy-Littlewood-Sobolev inequality.

In addition, in [55] the critical case $M = 8\pi$ is treated. Under the assumptions $\rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\rho_0 \in L^1(\mathbb{R}^2, \log(1+|x|^2)dx)$, $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$ and $c_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, global existence for (1.4) is obtained.

On the other hand, in [22], global existence and uniqueness is obtained for any positive mass M under some restriction on the parameter $\alpha > 0$ and the initial datum. The authors are interested in the so-called integral solution to (1.4) that is a couple that satisfies

$$\begin{aligned}\rho(t,\cdot) &= G(t,\cdot) * \rho_0 - \sum_{i=1}^{2}\int_0^t \nabla_i G(t-s,\cdot) * (\rho(s,\cdot)\nabla_i c(t,\cdot))\,ds \\ c(t,\cdot) &= e^{-\frac{\lambda}{\alpha}t} G(\frac{t}{\alpha},\cdot) * c_0 + \int_0^t e^{-\frac{\lambda}{\alpha}(t-s)} G(\frac{t-s}{\alpha},\cdot) * \rho(s,\cdot)\,ds,\end{aligned} \tag{1.6}$$

where $G(t,x) = \frac{1}{4\pi t} e^{\frac{|x|^2}{4t}}$. This formulation is also known as mild form of (1.4) or the Duhamel's formula. It is supposed $\chi = 1$. The following theorem is proved:

Theorem 1.3.1 (Theorem 2.1 [22]). *Let $\alpha > 0$, $\lambda \geq 0$, $\rho_0 \in L^1(\mathbb{R}^2)$ and $c_0 \in H^1(\mathbb{R}^2)$. There exists $\delta = \delta(M,\alpha)$ and $T = T(M,\alpha)$ such that if $\|\nabla c_0\|_{L^2(\mathbb{R})} < \delta$ there exist an integral solution (1.6) of the Keller-Segel model with $\rho \in L^\infty((0,T); L^1(\mathbb{R}^2))$ and $|\nabla c \in L^\infty((0,T); L^2(\mathbb{R}^2))|$. Moreover, the total mass M is conserved and there exists a constant $C = C(\alpha)$ such that if $M < C(\alpha)$, the solution is global and*

$$\begin{aligned} t^{1-\frac{1}{p}} \|\rho(t,\cdot)\|_{L^p(\mathbb{R}^2)} &\leq C(M,\alpha), \quad t > 0, \\ t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c(t,\cdot)\|_{L^r(\mathbb{R}^2)} &\leq C(M,\alpha), \quad t > 0, \end{aligned}$$

for all $p \in [1,\infty]$ and $r \in [2,\infty]$.

In order to prove this result, the authors apply Banach's fixed point theorem iterating the formulation (1.6). In order to exhibit a contraction the condition on the initial datum emerges. In order to pass from local to global solution in time, the condition on the mass emerges. However, as the latter is of the form

$$C_1 \alpha^a \sqrt{M} + C_2 \alpha^b \|\nabla c_0\|_{L^2(\mathbb{R})} < 1,$$

for some constants $C_1, C_2, b > 0$ and $a < 0$, one can have M as large as one likes as soon as α is large enough as well. For the same reason, the smaller is α, the more restrictive is the condition on the mass. Once the existence is proved, uniqueness and positivity of solutions follow from the following theorem

Theorem 1.3.2 (Theorem 2.6 [22]). *Let $\alpha > 0$, $\lambda \geq 0$ and let $\rho_0^i \in L^1(\mathbb{R}^2)$ and $c_0^i \in H^1(\mathbb{R}^2)$, $i = 1,2$, be two initial data sufficiently small so that the corresponding solutions (ρ^i, c^i) of (1.6) are global. Then, for any $p \in [1,\infty]$ and $r \in [2,\infty]$, there exists $C = C(p,r) > 0$ independent of t, such that for $t > 0$ it holds*

$$\begin{aligned} & t^{1-\frac{1}{p}} \|\rho^1(t,\cdot) - \rho^2(t,\cdot)\|_{L^p(\mathbb{R}^2)} + t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c^1(t,\cdot) - \nabla c^2(t,\cdot)\|_{L^r(\mathbb{R}^2)} \\ & \leq C(\|\rho_0^1 - \rho_0^2\|_{L^p(\mathbb{R}^2)} + t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c_0^1 - \nabla c_0^2\|_{L^r(\mathbb{R}^2)}). \end{aligned}$$

The case $d \geq 3$

When $d \geq 3$ the total mass is no longer the relevant parameter for the well-posedness analysis, but rather the $L^{\frac{d}{2}+\varepsilon}(\mathbb{R}^2)$-norms where $\varepsilon \geq 0$.

In fact, for the parabolic–elliptic model, Corrias et. *al.* [24] assume that $\rho_0 \in L^p(\mathbb{R}^d)$ for any $1 \leq p < \infty$ and is non-negative. Then, they prove that there exists a constant $K_0(\chi, d)$ such that if $\|\rho_0\|_{L^{\frac{d}{2}}(\mathbb{R}^2)} \leq K_0$, then the elliptic model has a global weak solution that preserves the initial mass and satisfies some $L^p(\mathbb{R}^d)$-norm estimates. Then, they prove that the elliptic system has no global smooth solution with fast decay if the quantity $\left(\int_{\mathbb{R}^d} \rho_0(x)\, dx\right)^{\frac{d}{d-2}}$ is large. However, such a condition cannot be replaced by a condition on the magnitude of $L^{\frac{d}{2}}(\mathbb{R}^2)$-norm of ρ_0 as in the case of $d = 2$.

Corrias and Perthame [23] study the purely parabolic-parabolic case (1.4) with $\alpha = 1$. Assuming that $\rho_0 \in L^1(\mathbb{R}^d) \cap L^a(\mathbb{R}^d)$, where $\frac{d}{2} < a \leq d$ and $\nabla c_0 \in L^d(\mathbb{R}^d)$, they prove that if $\|\rho_0\|_{L^a(\mathbb{R}^d)} + \|\nabla c_0\|_{L^d(\mathbb{R}^d)} \leq C(d,a)$ the parabolic system has at least one weak and global positive solution satisfying a certain estimate. When proving the existence they work with the integral formulation (1.6) in dimension d and prove some a priori estimates. A rigorous derivation of such estimates of a regularized version of the integral equation gives in the limit a weak solution.

1.4 Our mathematical motivations: Singular McKean-Vlasov dynamics

Analyzing a non-linear parabolic PDE of the McKean-Vlasov type through the associated stochastic process became a classical topic in probability theory over the past 30 years. The idea is to see such a PDE as a Fokker-Planck equation for a time evolution of a probability measure that is a time marginal of a stochastic process. A simple example is the following equation

$$\frac{\partial}{\partial t}\mu_t = \triangle \mu_t - \nabla \cdot ((\phi * \mu_t)\mu_t),$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a given kernel and μ_0 an initial condition. Then, by Itô's formula, one can prove that the marginal distributions $(\mu_t)_{t\geq 0}$ of the solution to the following stochastic process

$$\begin{cases} dX_t = \left(\int \phi(X_t - y)\mu_t(dy)\right)dt + dW_t, \\ X_0 \sim u_0, X_t \sim \mu_t \end{cases}$$

satisfies the above PDE (see Chapter 2 for more details). As the goal of this thesis is to construct and analyze such a stochastic interpretation for the parabolic-parabolic Keller-Segel equations, we review here the current state of the art on this topic.

Recently, stochastic interpretations have been proposed for a simplified version of the model in $d = 2$, that is,

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\nabla \rho - \chi \rho \nabla c)(t,x), \quad t > 0,\ x \in \mathbb{R}^2, \\ -\triangle c(t,x) = \rho(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0. \end{cases} \tag{1.7}$$

This is the parabolic-elliptic model which corresponds to the case $\alpha = 0$ and $\lambda = 0$ in (1.4). These interpretations all rely on the fact that, in the case of (1.7), the equations for ρ_t and c_t can be decoupled and c_t can be explicited as the convolution of ρ_t and a logarithmic kernel. Thus, one obtains the following closed form of the above system:

$$\partial_t \rho(t,x) = \triangle \rho(t,x) - \chi \nabla \cdot ((k * \rho(t,\cdot))\rho(t,x)), \tag{1.8}$$

where $k(x) = -\frac{x}{2\pi|x|^2}$. Consequently, the corresponding stochastic process of McKean–Vlasov type whose ρ_t is the time marginal density involves the singular interaction kernel k. That is why, so far, only partial results are obtained and heavy techniques are used to get them.

Namely, the first stochastic interpretation of (1.7) is due to Haškovec and Schmeiser [36] who analyze a particle system with McKean–Vlasov interactions and Brownian noise. More precisely, as the ideal interaction kernel k is strongly singular, they introduce a kernel with a cut-off parameter and obtain the tightness of the particle probability distributions w.r.t. the cut-off parameter and the number of particles. They also obtain partial results in the direction of the propagation of chaos (rigorously defined in Chapter 2). Then, Godinho and Quiñinao [33] analyze the case where k is replaced by $\frac{-x}{2\pi|x|^{1+\alpha}}$ for some $\alpha \in (0,1)$. They prove the well-posedness of the corresponding particle system and propagation of chaos towards the limit equation.

More recently, Fournier and Jourdain [31] and Cattiaux and Pédèches [21] study to the following Mc-Kean-Vlasov stochastic equation related to (1.8):

$$\begin{cases} dX_t = \sqrt{2}dW_t + \chi(k * \rho(t,\cdot))(X_t)dt, \\ X_t \sim \rho_t, X_0 \sim \rho_0. \end{cases} \tag{1.9}$$

The connection between (1.9) and (1.8) is established by Itô's formula (see Chapter 2 for such a connection in a general setting). An habitual approach is to analyze the corresponding mean field model

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dW_t^i + \frac{\chi}{N}\sum_{i=1}^N k(X_t^{i,N} - X_t^{j,N})dt, \\ X_0^{i,N} \text{ i. i. d.} \sim \rho_0. \end{cases} \tag{1.10}$$

and prove that when $N \to \infty$, the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ weakly converges to the law of the process (1.9) (propagation of chaos). Due to the singular nature of k it is not obvious that system (1.10) is well defined. Nevertheless, Fournier and Jourdain [31] almost achieve this program in the subcritical case. Namely, when $\chi < 2\pi$, they obtain the well–posedness of the particle system. In addition, they obtain a consistency property which is weaker than the propagation of chaos. They also describe complex behaviors of the particle system in the sub and super critical cases. Cattiaux and Pédèches [21] obtain the well-posedness of this particle system without cut-off by using Dirichlet forms rather than pathwise approximation techniques. They leave the other stages of the program to some future work.

Theorem 1.4.1 (Theorems 5 and 6 [31]). *Let $N \geq 2$ and $\chi < \frac{2\pi N}{N-1}$. Assume ρ_0 has a finite moment of order 1. There exists a solution $(X_t^{i,N})_{t\geq 0, 1\leq i\leq N}$ to (1.10). In addition, the family $\{(X_t^{i,N})_{t\geq 0}, 1 \leq i \leq N\}$ is exchangeable and for any $\alpha \in (\frac{(N-1)\chi}{2\pi N}, 1)$ and any $T > 0$ one has*

$$\mathbb{E}\int_0^T \left|X_s^{1,N} - X_s^{2,N}\right|^{\alpha-2} ds \leq \frac{(2\sqrt{2}\mathbb{E}(1+|X_0^{1,N}|^2)^{1/2} + 4\sqrt{2}T)^\alpha}{\alpha(2\alpha - \frac{(N-1)\chi}{\pi N})}. \tag{1.11}$$

Next, suppose $\chi < 2\pi$. Then

i) The sequence $(\mu^N))_{N\geq 2}$ is tight.

ii) Any (possibly random) weak limit point μ of $(\mu^N)_{N\geq 2}$ is a.s. the law of a solution to the nonlinear SDE (1.9).

iii) In particular, one can find a subsequence N_k such that $(\mu_t^{N_k}))_{t\geq 0}$ goes in law, as $k \to \infty$, to some $(\mu_t)_{t\geq 0}$, which is a.s. a weak solution to (1.8).

The main tool in showing these results is that (1.11) apriori holds true. Thus, the authors start from a regularized version of (1.10) and are able to build their way up towards (1.8). In fact, thanks to (1.11) one is able to control the effect of the singularity of the kernel, i. e. one can show that the Lebesgue measure of the set of crossing times between particles is null, independently of the number of particles. Two main drawbacks of this result are that it holds in a very sub-critical case ($\chi < 2\pi$) and that it is not a propagation of chaos result, but rather a tightness/consistency result. The reason is that the uniqueness does not hold in the class of weak solutions the authors work in. Then, the next theorem ensures the existence of the particle system until 3-particles collide.

Theorem 1.4.2 (Theorem 7 [31])**.** *Let $\chi > 0$ and $N > \max\{2, \frac{\chi}{2\pi}\}$ be fixed. There exists a solution $(X_t^{i,N})_{0\leq t<\tau^N, 1\leq i\leq N}$ to (1.10) where*

$$\tau^N := \sup_{l\geq 1} \inf\{t \geq 0 : \exists i,j,k \text{ pairwise different such that } \\ |X_t^{i,N} - X_t^{j,N}| + |X_t^{i,N} - X_t^{k,N}| + |X_t^{j,N} - X_t^{k,N}| \leq \frac{1}{l}\}.$$

The family $(X_t^{i,N})_{0\leq t<\tau^N, 1\leq i\leq N}$ is exchangeable and for any $\alpha \in (\frac{\chi}{2\pi N}, 1)$, a.s. for any $t \in [0, \tau^N)$ one has

$$\int_0^t |X_s^{1,N} - X_s^{2,N}|^{\alpha-2}\, ds < \infty.$$

Finally, it holds that

i) $\tau^N = \infty$ a.s. if $\chi \leq 8\pi \frac{N-2}{N-1}$,

ii) $\tau^N < \infty$ a.s. if $\chi > 8\pi \frac{N-2}{N-1}$.

The main ingredient when proving the preceding result is to show that the process $R_t^I = \frac{1}{2}\sum |X_t^{i,N} - \bar{X}^I|^2$ where $I \subset \{1, \dots, N\}$ and $\bar{X}^I = \frac{1}{|I|}\sum_{i\in I} X_t^{i,N}$ behaves almost like a square of a Bessel process of dimension $(|I|-1)(2 - \frac{\chi|I|}{4\pi N})$. Then, the condition on χ ensures that for all $|I| \geq 3$ the dimension of the Bessel process is greater than 2. Thus, the process R_t^I never reaches zero and no collision involving three or more particles occur. However, the main difficulty lies in the above mentioned almost like square Bessel process behaviour of R^I: when $|I| = N$ it is exactly the square of a Bessel process, then by backward induction it is shown that some terms can be neglected and that square Bessel behaviour holds even when $|I| < N$.

Contrary to [31] for proving the existence part, Cattiaux and Pédèches [21] use Dirichlet forms. In fact, they prove that the form

$$\mathcal{E}(f,g) = \int_M < \nabla f, \nabla g >, \quad f,g \in C_0^\infty(M).$$

is regular and local (M is given below). The main result in [21] is the following theorem:

Theorem 1.4.3 (Theorem 1.2 [21])**.** *Let*
$M = \{$ there exists at most one pair $i \neq j$ such that $X^{i,N} = X^{j,N}\}$. Then

- *For $N \geq 4$ and $\chi < 8\pi \frac{N}{N-1}$, there exists a unique (in distribution) non explosive solution of (1.10), starting from any $x \in M$. Moreover, the process is strong Markov, lives in M and admits a symmetric σ- finite, invariant measure given by*

$$\mu(dX^1, \ldots, dX^N) = \prod_{1 \leq i < j \leq N} |X^i - X^j|^{-\frac{\chi}{N}} dX^1, \ldots, dX^N.$$

- *For $N \geq 2$ and $\chi > 8\pi$ the system (1.10) does not admit any global (in time) solution.*

- *For $N \geq 2$ and $\chi = 8\pi$, either the system (1.10) explodes or the N particles are glued in finite time.*

The techniques in [31] and [21] are based on the particular structure of the interaction kernel and on the fact the process they are constructing is strongly Markov. We will see that the latter will not be the case with our interpretation and thus, we will not be able to adapt their techniques in this thesis.

In the fully parabolic case of (1.4) ($\alpha > 0$), recently a probabilistic interpretation of a smoothed Keller-Segel alike system of parabolic type was developed. For a parabolic–parabolic version of the model with a smooth coupling between ρ_t and c_t, Budhiraja and Fan [17] study a particle system with a smooth time integrated kernel and prove it propagates chaos. Moreover, adding a forcing potential term to the model, under a suitable convexity assumption, they obtain uniform in time concentration inequalities for the particle system and uniform in time error estimates for a numerical approximation of the limit non-linear process. As our main focus is (1.4) in the case $\alpha > 0$ without any cut-off, we will neither enter in the details of this work nor in the work of Stevens [69] who as well deals with a smooth Keller-Segel system.

We conclude this chapter by reviewing some examples from the literature of McKean-Vlasov stochastic processes with singular interaction arising as probabilistic interpretations of non-linear Fokker-Planck equations. Osada [59] studies an SDE related to 2D-Navier-Stokes equation written in vorticity formulation. The interaction kernel is of the form $K(x) = \frac{x^t}{|x|^2}$. Jourdain and Méléard [44] study a non-linear diffusion with normal reflecting boundary conditions and a singularity that involves the Poisson kernel related to vortex equation. Fournier and Hauray [30] study the 3-d Landau equation where the kernel is of the form $k(x) = -2|x|^\gamma x$, for $\gamma \in (-2, 0)$. Calderoni and Pulvirenti [19] study the Burger's equation, where the interaction kernel is the δ-dirac function. Bossy and Talay [14] interpret the solution of the Burger's equation as a distribution function of a probability measure solving a PDE of Mc-Kean Vlasov type where the interaction kernel becomes the Heaviside function. Bossy *et. al.* [13] study the Lagrangian stochastic model where the interaction is given through a conditional expectation, while Bossy and Jabir [12] study the Lagrangian stochastic model with specular reflections on the boundary. Le Cavil *et. al.* [51] study the stochastic process and particle system related to a nonconservative McKean-Vlasov PDE with the coefficients depending of the marginal densities. Other types of singularities have bees studied in the case of particle systems with collisions related to Boltzman or Landau equations: see e.g. Guérin and Méléard [34] and Fournier [29].

1.5 Our probabilistic interpretation of the parabolic-parabolic Keller-Segel system and main results

1.5.1 Probabilistic interpretation

In order to build a stochastic interpretation of (1.4), we will in the sequel formally decouple the Keller-Segel system. From now on we set $\alpha = 1$.

Assume for a moment the function $c(t,x)$ is given and let us start from (1.4a):

$$\partial_t \rho(t,x) = \triangle \rho(t,x) - \nabla \cdot (\chi \rho(t,x) \nabla c(t,x)), \quad t > 0,\ x \in \mathbb{R}^d.$$

Then, by Itô's formula, the time marginal distributions $(\rho(t,\cdot))_{t\geq 0}$ of the process $(X_t)_{t\geq 0}$ solution to

$$\begin{cases} dX_t = \chi \nabla c(t, X_t) dt + \sqrt{2} dW_t, \quad t \geq 0 \\ X_0 \sim \rho_0 \end{cases} \tag{1.12}$$

satisfy (1.4a). We have already noticed that the integral (or Feynman-Kac) representation of the equation for c is

$$c(t,x) = e^{-\lambda t}(G(t,\cdot) * c_0)(x) + \int_0^t e^{-\lambda(t-s)}(G(t-s,\cdot) * \rho(s,\cdot))(x)\, ds, \tag{1.13}$$

where $G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$. Therefore, we can formally compute $\nabla c(t,x)$. Taking into account that we do not wish to derive the function ρ, one has

$$\nabla c(t,x) = e^{-\lambda t} \nabla ((G(t,\cdot) * c_0)(x)) + \int_0^t e^{-\lambda(t-s)} (\nabla G(t-s,\cdot) * \rho(s,\cdot))(x))\, ds.$$

Plugging the preceding equation into (1.12), one obtains the following McKean-Vlasov non-linear stochastic dynamics:

$$\begin{cases} dX_t = \chi e^{-\lambda t}(G(t,\cdot) * \nabla c_0)(X_t) dt + \chi \left\{ \int_0^t \int_{\mathbb{R}^d} K_{t-s}(X_t - y) \rho(s,y)\ dy\ ds \right\} dt + \sqrt{2} dW_t, \quad t \leq T, \\ X_0 \sim p_0,\ X_t \sim \rho(t,x) dx, \end{cases} \tag{1.14}$$

where $K_t(x) := e^{-\lambda t} \nabla G(t,x) = e^{-\lambda t} \frac{-x}{(4\pi)^{\frac{d}{2}} t^{\frac{d}{2}+1}} e^{-\frac{|x|^2}{4t}}$ and $T > 0$ is an arbitrary time horizon. Notice that $(X_t)_{t \leq T}$ is a d-dimensional stochastic process and that we impose that for any $t > 0$, the law of X_t is absolutely continuous w.r.t. Lebesgue's measure. The drift of (1.14) has two components: one that depends on the initial concentration and one that depends on the time marginals of the law of the process. What is unusual is that the interaction between the solution and its probability law happens not only in space, at each time t, but as well in time. That is, at each time $t > 0$ the drift involves all the time marginals up to time t. This sets (1.14) apart from the general setting of McKean-Vlasov processes (see e.g. Sznitman [72]). Another point we would like to insist on is the singular nature of the interaction kernel K. As the Gaussian density is derived in space, a singularity emerges in time and it is of order $\frac{1}{t^{\frac{d}{2}+1}}$. Remark as well that the limit $\lim_{(t,x)\to(0,0)} K_t(x)$ is not well defined. This singularity should be integrated in time and we expect that the convolution in space will somehow smooth it. Throughout the thesis, we will refer to the equations of the form (1.14) as "the non-linear SDEs with space and time interactions".

To conclude, our probabilistic interpretation of (1.4) is the non-linear stochastic equation (1.14) paired with the transformation (1.13) of the time marginal laws of (1.14). In order to come back to the Keller-Segel equations (1.4), one should follow the following program:

Step 1 Construct a (weak) solution to (1.14) and extract the family of densities $(\rho(t,\cdot))_{t\geq 0}$.

Step 2 Construct the family $(c(t,\cdot))_{t\geq 0}$ as a transformation of ρ as in (1.13).

Step 3 Prove the pair (ρ, c) satisfies (1.4).

Step 3 of the program requires that an adequate notion of solution is precised. The main question we tried to reply in this thesis is whether this program can be carried out in different spatial dimensions of the problem and under which conditions.

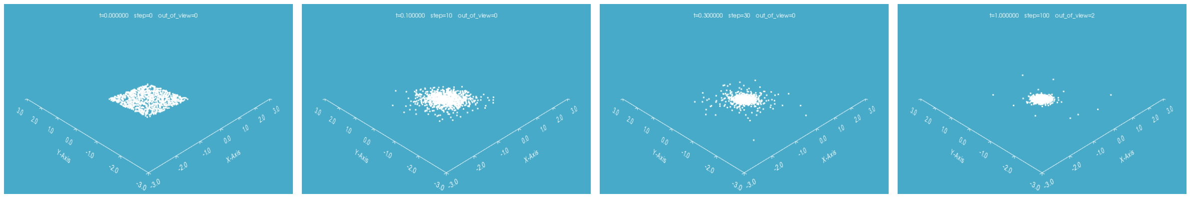
Another natural question is to associate to (1.14) the corresponding system of interacting particles. Namely, plugging the empirical measure of N particles in the place of the unknown law of the process in (1.14), one obtains the following system of stochastic equations:

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dW_t^i + \chi e^{-\lambda t}(\nabla c_0 * G(t,\cdot))(X_t^{i,N})dt + \\ \left\{\frac{\chi}{N}\sum_{j=1}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds \ \mathbb{1}_{\{X_t^{i,N}\neq X_t^{j,N}\}}\right\} dt, \quad t\leq T, \\ X_0^{i,N}, \text{ i.i.d. and independent of } W := (W^i, 1\leq i\leq N). \end{cases} \tag{1.15}$$

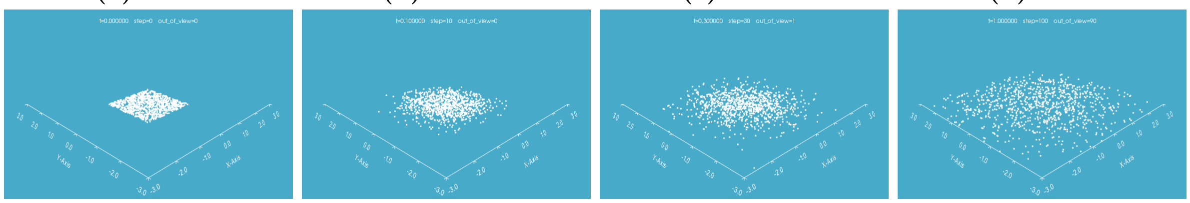
Here the W^i's are N independent standard d-dimensional Brownian motions and $X_0^{1,N}$ is distributed according to ρ_0. System (1.15) inherits from (1.14) that at each time $t>0$ each particle interacts in a singular way with the past of all the other particles. In fact, as soon as a particle at time t crosses the past of another particle, we do not know how to integrate the singularity in time. The only hope in that case is that the instant s in the past in which the encounter happens is far away from t. As $\lim_{(t,x)\to(0,0)} K_t(x)$ is not well defined, we must ensure that when $s\to t$, the integral $\int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds$ is well defined. That is why, first of all, we will not consider an interaction of a particle with itself. Then, we will set an interaction to zero every time $X_t^{i,N} = X_t^{j,N}$. That is why the indicator $\mathbb{1}_{\{X_t^{i,N}\neq X_t^{j,N}\}}$ is added to the dynamics. In order to justify it does not influence the dynamics, we should always make sure that the set $\{t\leq T, X_t^{i,N} = X_t^{j,N}, i\neq j\}$ has Lebesgue measure zero. The non-Markovian nature of the particle system makes it impossible to adapt the techniques used in the elliptic case [21, 31].

Many questions arise when one considers (1.15): Is it well-defined? Under which conditions? Does it propagate chaos? Does it exhibit agglomerations according to χ in the two-dimensional case? This thesis aims to reply to them and set a foundation for future works on (1.14) and (1.15).

Before passing to the main results of this thesis, we give an illustration of the behaviour of the particle system through a numerical simulation. In $d=2$ we apply the Euler scheme to (1.15). The particles are initially distributed according to the uniform distribution on the square $[-1,1]\times[-1,1]$. The initial concentration has been chosen to be a standard two dimensional Gaussian density. When χ is large the particles very quickly form an agglomeration in the center of the domain where the initial concentration attains its maximum. On the contrary, when χ is small the particles diffuse in the plain and the diffusion prevails the singular interaction of the particles. A typical result of such a simulation is given in Figure 1.1 (the pictures will be enlarged in our last chapter).



(a) $t = 0$ **(b)** $t = 0.1$ **(c)** $t = 0.3$ **(d)** $t = 1$



(e) $t = 0$ **(f)** $t = 0.1$ **(g)** $t = 0.3$ **(h)** $t = 1$

Figure 1.1: (a)-(e): $\chi = 50$; (f)-(j): $\chi = 1$. Euler scheme is applied to (1.15), with $N = 1000, d = 2$. Particles are initially distributed uniformly on $[-1, 1] \times [-1, 1]$. Initial concentration of the chemical is a centered Gaussian density. When χ is large an agglomeration of particles appear in the center of the domain, whilst when χ is small the particles diffuse.

1.5.2 Main results of the thesis

In this thesis we introduce and analyze a new probabilistic interpretation of the parabolic-parabolic Keller-Segel model without cut-off in the cases $d = 1$ and $d = 2$. Our first goal is to carry out the above defined program and validate our approach by getting new well-posedness results for (1.4) in the parabolic-parabolic case ($\alpha = 1$). Our second objective is to study the corresponding particle system.

We start with Chapter 2 that introduces the probabilistic tools and notions needed on a smoothed version of (1.14). Namely, as soon as there is some regularity on the interaction kernel, one can adapt the arguments in Sznitman [72] in order to obtain well-posedness and propagation of chaos for a McKean-Vlasov SDE with a time and space interaction. The connection of such an SDE and a non-linear parabolic PDE is established.

In Chapter 3 we study (1.14) in $d = 1$ and prove it is well defined and provides a unique solution for the Keller-Segel system in $d = 1$. This result is available as a preprint [73].

Chapter 4 proposes another way to deal with the one-dimensional McKean-Vlasov SDE and proves some Sobolev regularity results on time marginals of the law of the solution.

In Chapter 5 we deal with the one-dimensional particle system and prove it is well defined and it propagates chaos towards the process built in Chapter 3. This is a joint work with Jean-Francois Jabir [43].

The two-dimensional McKean-Vlasov SDE is studied in Chapter 6. After proving it is well-defined, we establish the connection with the two-dimensional Keller-Segel system.

Finally, Chapter 7 describes and studies a purely probabilistic method to approximate the solutions of the fully parabolic two dimensional Keller-Segel system. In addition, it gives some theoretical insights about the particle system in $d = 2$.

Let us summarize our main results.

The one-dimensional case

Our first main result is given in Chapter 3. It concerns the well-posedness of a non-linear one-dimensional stochastic differential equation (SDE) with time and space interaction. As our technique of analysis is not limited to the above kernel K, we consider the following McKean-Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \Big\{ \int_0^t (k_{t-s} * p_s)(X_t)ds \Big\} dt + dW_t, \quad t \le T, \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim p_0. \end{cases} \tag{1.16}$$

The set of hypothesis (H) assumed on the kernel k is given in Chapter 3 and among them the key one is $k \in L^1((0,T] \times \mathbb{R})$.

Due to the singular nature of the kernel, (1.16) cannot be analyzed by means of standard coupling methods or Wasserstein distance contractions as in Chapter 2. Both to construct local in time solutions and to go from local to global solutions, an important issue consists in properly defining the set of weak solutions. Namely, without any assumption on the initial density ρ_0, we need to add the following constraint in the classical definition of a weak solution to (1.16):

- The probability distribution $\mathbb{P} \circ X^{-1}$ has time marginal densities $(p_t,\ t \in [0,T])$ with respect to Lebesgue measure which satisfy

$$\forall 0 < t \le T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \le \frac{C_T}{\sqrt{t}}. \tag{1.17}$$

To prove that this constraint is satisfied in the limit of an iterative procedure (where the kernel is not cut off), the norms of the successive time marginal densities cannot be allowed to exponentially depend on the L^∞-norm of the successive corresponding drifts. They neither can be allowed to depend on Hölder-norms of the drifts. Therefore, we use an accurate estimate (with explicit constants) on densities of one-dimensional diffusion processes with bounded measurable drifts which is obtained by a stochastic technique rather than by PDE techniques (See Section 3.3). This strategy allows us to get uniform bounds on the sequence of drifts, which is essential to get existence and uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure, and to suitably paste local solutions when constructing the global solution.

Theorem (3.2.3)**.** *Let $T > 0$. Suppose that $p_0 \in L^1(\mathbb{R})$ is a probability density function and $b \in L^\infty([0,T] \times \mathbb{R})$ is continuous w.r.t. the space variable. Under the hypothesis (H), Eq. (1.16) admits a unique weak solution (in the above sense which includes (1.17)).*

The Hypothesis (H) is satisfied by the Keller-Segel kernel K. Thus, applying the above theorem to it, we extract the family of marginals ρ and complete **Step 1** from our program. Then, we perform the **Step 2** by considering the function c as transformation of ρ according to (1.13). Then, we are in the position to prove the well-posedness for the Keller-Segel system in $d = 1$. The precise notion of solution is given in Chapter 3. Our strategy consists in proving the time marginal distributions of the exhibited weak solutions satisfy the mild formulation (1.6) (for $d = 1$) of the system. To this end, we impose the condition (1.17) on the function ρ. Finally, the **Step 3** follows.

Corollary (3.2.6). *Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. Given any $\chi > 0$, $\lambda \geq 0$ and $T > 0$, the time marginals $\rho(t,x) \equiv p_t(x)$ of the probability distribution of the unique solution to Eq. (1.14) with $d = 1$ and the corresponding function $c(t,x)$ provide a global solution to (1.4) with $d = 1$ in some sense. Any other solution (ρ^1, c^1) with the same initial condition (ρ_0, c_0) satisfies $\|\rho^1(t,\cdot) - \rho(t,\cdot)\|_{L^1(\mathbb{R})} = 0$ and $\|\frac{\partial c^1}{\partial x}(t,\cdot) - \frac{\partial c}{\partial x}(t,\cdot)\|_{L^1(\mathbb{R})} = 0$ for every $0 \leq t \leq T$.*

This seems to notably improve the results in [60, 40].

Chapter 4 revisits the work done on the level of the non-linear process in Chapter 3, through a regularization procedure. Namely, we regularize the interaction kernel K and combines the results from Chapter 2 and Chapter 3 to prove the regularized equation in the limit (when the regularization parameter vanishes) satisfies (1.14) in $d = 1$. The goal then is to obtain the rate of convergence of the marginal laws of the solution to the regularized equation to the marginal laws of the solution to (1.14) in $d = 1$. In order to get this rate of convergence, we prove some Sobolev regularity results for the one-dimensional marginals of a stochastic process with bounded and measurable drift.

The objective of Chapter 5 is to analyze the particle system related to (1.14) in $d = 1$. As neither the linear part of the drift plays any role, nor the parameters of the equation, we set $\alpha = 1$, $\lambda = 0$, $\chi = 1$, and $c_0' \equiv 0$. We thus consider the following particle system:

$$\begin{cases} dX_t^{i,N} = \left\{ \frac{1}{N} \sum_{j=1, j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N}) ds \; \mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}} \right\} dt + dW_t^i, \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1 \leq i \leq N), \end{cases} \tag{1.18}$$

where the W^i's are N independent standard Brownian motions. Compared to the stochastic particle systems introduced for the parabolic–elliptic model, an interesting fact occurs: the difficulties arising from the singular interaction can now be resolved by using purely Brownian techniques rather than by using Bessel processes. The construction of a weak solution to (1.18) involves arguments used by Krylov and Röckner [49, Section 3] to construct a weak solution to SDEs with singular drifts. It relies on the Girsanov transform which removes all the drifts of (1.18). Our calculation is based on the fact that the kernel K is in $L^1(0,T; L^2(\mathbb{R}))$.

Theorem (5.2.1). *Given $0 < T < \infty$ and $N \in \mathbb{N}$, there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t; 0 \leq t \leq T), \mathbb{Q}^N, W, X^N)$ to the N-interacting particle system (1.18) that satisfies, for any $1 \leq i \leq N$,*

$$\mathbb{Q}^N \left(\int_0^T \left(\frac{1}{N} \sum_{j=1, j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N}) ds \mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}} \right)^2 dt < \infty \right) = 1.$$

Notice that in the above result, no additional condition on the initial law is necessary. Due to the singular nature of the kernel K, we need to introduce a partial Girsanov transform of the N-particle system in order to obtain uniform in N bounds for moments of the corresponding exponential martingale. We believe this trick may be useful when proving tightness and propagation of chaos for other particle systems with singular interactions.

Theorem (5.2.5). *Assume that the $X_0^{i,N}$'s are i.i.d. and that the initial distribution of $X_0^{1,N}$ has a density ρ_0. The empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ of (1.18) converges in the distribution sense, when $N \to \infty$, to the unique weak solution of (1.14).*

To the best of our knowledge, this is the first time in the literature that the parabolic-parabolic Keller-Segel system without cut-off is derived as a limit of a system of interacting stochastic particles, when the number of particles tends to infinity.

The two-dimensional case

In Chapter 6 we study (1.14) in $d = 2$. The increase of dimension leads to an increase in singularity of the kernel K. We start with explaining why L^∞-spaces are no longer a good choice for the drift and density of the to be constructed stochastic process. As a consequence, we turn to the L^p-spaces. We redefine the notion of a weak solution to our McKean-Vlasov SDE by including the following constraint:

- The probability distribution $\mathbb{P} \circ X^{-1}$ has time marginal densities $(p_t,\ t \in (0,T])$ with respect to Lebesgue measure which satisfy for any

$$\forall 1 < q < \infty\ \exists C_q > 0, \quad \sup_{t \le T} t^{1-\frac{1}{q}} \|p_t\|_{L^q(\mathbb{R}^2)} \le C_q. \tag{1.19}$$

To prove that this constraint is satisfied, we conveniently regularize the McKean-Vlasov SDE and apply the results from Chapter 2. Then we analyze the associated regularized mild equation and prove estimates of type (1.19) for the regularized densities. These estimates are uniform w.r.t. the regularizing parameter under a condition involving the parameter χ and the size of initial datum. Once such an estimate is obtained, we prove the convergence of martingale problems related to regularized dynamics towards the our NLSDE. We obtain the following theorem:

Theorem (6.2.3)**.** *Let $T > 0$ and suppose that X_0 has a probability density function p_0. Furthermore, assume that $c_0 \in H^1(\mathbb{R}^2)$. Then, Equation (1.14) in $d = 2$ admits a weak solution under the following condition*

$$A\chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B\sqrt{\chi} < 1, \tag{1.20}$$

where A and B are defined as in Proposition 6.3.7.

Extract the time marginals ρ of the constructed solution to (1.14) to complete **Step 1** from our program. Then, we perform the **Step 2** by considering the function c as transformation of ρ according to (1.13). Thanks to the estimates in (1.19), we obtain

$$\forall 2 \le r \le \infty\ \exists C_r > 0, \quad \sup_{t \le T} t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c_t\|_{L^r(\mathbb{R}^2)} \le C_r.$$

Then, we prove the well-posedness for the Keller-Segel system in $d = 2$. The precise notion of solution is given in Chapter 6. Again, we aim to satisfy the mild formulation (1.6) of the system and impose the condition (1.19) on the function ρ. Finally, the **Step 3** is a consequence of Theorem 6.2.3.

Corollary (6.2.5)**.** *Let ρ_0 a probability density function and $c_0 \in H^1(\mathbb{R}^2)$. Under the condition (1.20) the system (1.4) in $d = 2$ admits a unique solution in some sense.*

Concerning the particle system in $d = 2$, at the present, we do not have a mathematical answer to the question of its well-posedness. We cannot apply the techniques from Chapter 5 as for $q \ge 2$, the $L^1((0,T);L^q(\mathbb{R}^2))$–norm of the interaction kernel explodes. In Chapter 7 we give some theoretical insights about this problematic. In short, the increase in singularity together with the non-Markovian setting lead to strong difficulties when turning to Girsanov or trajectorial techniques. The well-posedness of the 2-d particle system without cut-off remains open for some of our future works. In this Chapter 7 we also analyze a probabilistic numerical method to approximate the system (1.4) in $d=2$ coming from our stochastic interpretation for it. We compare it with another numweical method proposed by Fatkullin [28] which combines stochastic simulations and PDE resolution.

Chapter 2

McKean-Vlasov equations with smooth time and space interaction

2.1 Introduction

In this chapter we study a McKean-Vlasov stochastic equation with space and time interaction as in (1.14) where the singular kernel K is replaced with a smooth kernel L. Under some assumptions on boundness and Lipschitz continuity in space for L, we prove that in such a setting one can modify the classical techniques in Sznitman [72] to obtain the existence of the solution and propagation of chaos for the corresponding particle system. Then, we explore the connections between such non-linear SDE and a non-linear parabolic equation. Namely, we derive the Fokker-Planck equation and its mild formulation for the marginal laws of the process. Thus, we see how the empirical measure of the associated particle system can be used to approximate the solution to a non-linear parabolic equation.

The purpose of this chapter is to illustrate that the singular interaction is the main difficulty in (1.14) despite its unusual form (integral in time and space). Moreover, on an example with regular interaction we wanted to show the main arguments behind the connections PDE-SDE and the so called particle methods for non-linear parabolic PDEs. In addition, we use the opportunity to define some classical notions of probability theory in this new setting that will be necessary to read this thesis (weak solutions, martingale problems, propagation of chaos).

The plan of the chapter is the following: In Section 2.2 we study the above mentioned NLSDE and prove its well-posedness and the propagation of chaos for the associated particle system. In Section we derive the associated Fokker-Planck equation, mild equation and some properties for the one dimensional time marginals of the process.

2.2 Non-linear stochastic equations with smooth time and space interactions

Let $T > 0$. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ equipped with a d–dimensional Brownian motion (W) and an $\mathcal{F}_0-$measurable random variable X_0, we study the stochastic equation

$$\begin{cases} dX_t = dW_t + \Big\{ \int_0^t \int_{\mathbb{R}^d} L(t-s, X_t - y)\mathbb{Q}_s(dy)\ ds \Big\} dt, \quad t \leq T, \\ \mathbb{Q}_s := \mathcal{L}(X_s), \quad X_0 \sim q_0, \end{cases} \tag{2.1}$$

where L maps $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^d$.

As the drift coefficient of (2.1) depends on the marginals of the unknown law of the process, we call it a non-linear stochastic equation (NLSDE) in the sense of McKean–Vlasov. A typical example of such equations is studied in Sznitman [72]. What differs Equation (2.1) from the setting in [72] is that the interaction with the law of the process happens both in time and space. Nevertheless, when the interaction kernel is sufficiently regular this does not induce any additional difficulty.

Let us define the notion of existence in law or weak solution for (2.1).

Definition 2.2.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is said to be a weak solution to the equation (2.1) up to time $T > 0$ if:*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.*
2. *The process $X := (X_t)_{t\in[0,T]}$ is $\mathbb{R}^d$-valued, continuous, and $(\mathcal{F}_t)$-adapted. In addition, the probability distribution of X_0 is q_0.*
3. *The process $W := (W_t)_{t\in[0,T]}$ is a d-dimensional $(\mathcal{F}_t)$-Brownian motion.*
4. *Denote by $(\mathbb{Q}_t,\ t \in [0,T])$ the time marginals of the probability distribution $\mathbb{P} \circ X^{-1}$. For all $t \in (0,T]$, one has that*
$$\mathbb{P}\left(\int_0^t \left|\int_0^s \int_{\mathbb{R}^d} L(s-u, X_s - y)\mathbb{Q}_u(dy)\ ds\right|\ dt < \infty\right) = 1.$$
5. *$\mathbb{P}$-a.s. the pair (X,W) satisfies (2.1).*

This is a classical definition of a weak solution (see e.g. [45]) to a stochastic equation. An equivalent formulation is given in terms of the associated martingale problem. In the case of linear SDEs this equivalence is explained in [45, Section 5.4]. The same arguments are valid for non-linear SDEs. Thus, we pose the non-linear martingale problem associated to (2.1).

Definition 2.2.2. *A probability measure $\mathbb{Q}$ on the canonical space $\mathcal{C}([0,T];\mathbb{R}^d)$ equipped with its canonical filtration and a canonical process (w_t) is a solution to the non-linear martingale problem (MP) if:*

(i) $\mathbb{Q}_0(dx) := \mathbb{Q} \circ w_0^{-1}(dx) = p_0(dx)$.

(ii) For any $t \in (0,T]$, denote $\mathbb{Q}_t(dx) := \mathbb{Q} \circ w_t^{-1}(dx)$. Then,
$$\int_0^T \int_{\mathbb{R}^d} \left|\int_0^t \int_{\mathbb{R}^d} L(t-s, x-y)\mathbb{Q}_s(dy)\ ds\right| \mathbb{Q}_t(dx)\ dt < \infty$$

(iii) For any $f \in C_K^2(\mathbb{R}^d)$ the process $(M_t)_{t\le T}$, defined as
$$M_t := f(w_t) - f(w_0) - \int_0^t \big[\frac{1}{2}\triangle f(w_u) + \nabla f(w_u) \cdot \int_0^u \int L(u-\tau, w_u - y)\mathbb{Q}_\tau(dy)\ d\tau\big]du$$
is a $\mathbb{Q}$-martingale.

Throughout this thesis, well-posedness of martingale problems will be our primary technique when proving existence in law for NLSDEs of type (2.1). However, we will see in this chapter that once the kernel L is regular enough, a fixed point kind of argument may be applied. Notice that in both formulations we have an integrability condition for the drift term (Definition 2.2.1–4., Definition 2.2.2–(ii)). In order to satisfy it, some additional assumptions on the interaction kernel or/and on the one-dimensional time marginals of the law of the process must be imposed. We will suppose throughout this section the following hypothesis:

Hypothesis (H0)**.** *The function $L : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$\forall (t, x) \in (0, T) \times \mathbb{R}^d, \quad |L(t, x)| \leq h_1(t),$$
$$\forall (t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d, \quad |L(t, x) - L(t, y)| \leq h_2(t)|x - y|,$$

where $h_i : (0, T) \to \mathbb{R}^+$ is such that there exists $D_T > 0$ such that for any $t \leq T$, one has $\int_0^t h_i(s)ds \leq D_T$.

Notice that the time interaction induces a slight change in (H0) with respect to what is assumed on the interaction kernel in [72]. We still assume the kernel is bounded and Lipshitz in space, but in order to treat the additional integral in time, we introduce the functions h_1 and h_2.

Let $C((0, T); \mathbb{R}^d)$ be a set of continuous $\mathbb{R}^d$-valued functions defined on $(0, T)$ and $\mathcal{P}_T$ be the set of probability measures on $C((0, T); \mathbb{R}^d)$. For a $\mathbb{Q} \in \mathcal{P}_T$ and $(t, x) \in (0, T) \times \mathbb{R}^d$ denote by

$$b(t, x; (\mathbb{Q}_s)_{s\leq t}) := \int_0^t \int_{\mathbb{R}^d} L(t - s, x - y)\mathbb{Q}_s(dy)\ ds.$$

In view of Hypothesis (H0), for a given $\mathbb{Q} \in \mathcal{P}_T$ one has that

$$\forall (t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d : \begin{cases} |b(t, x; (\mathbb{Q}_s)_{s\leq t})| \leq D_T, \\ |b(t, x; (\mathbb{Q}_s)_{s\leq t}) - b(t, y; (\mathbb{Q}_s)_{s\leq t})| \leq D_T|x - y|. \end{cases} \tag{2.2}$$

This will ensure that the above discussed integrability conditions are fulfilled. As the kernel associated to our Keller-Segel NLSDE does not have such nice properties, we will be prompt to search for weak solutions in the spaces of measures whose one-dimensional time marginals have some specific properties. Therefore, the above notion of weak solution will be redefined (cf. Chapters 3 and 6).

For a smooth interaction kernel, we will prove the following theorem:

Theorem 2.2.3. *Under the hypothesis (H0), Equation* (2.1) *admits a unique strong solution.*

Thanks to (2.2), to prove Theorem 2.2.3 one could adapt the fixed point argument in the proof of Theorem 1.1 in [72] adding the time interaction everywhere. In order to do so, let us introduce for two measures $m_1, m_2 \in \mathcal{P}_T$, their distance with Wasserstein metric, given by

$$D_{1,T}(m_1, m_2) = \inf_{\pi \in \Pi(m_1, m_2)} \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} \sup_{s\leq T} |w_s^1 - w_s^2| \wedge 1\ d\pi(w^1, w^2),$$

where $\Pi(m_1, m_2)$ is the set of all couplings of m_1 and m_2. In view of Villani [79, Cor. 6.13 and Thm. 6.18], Wasserstein distances metrize weak convergence and $(\mathcal{P}_T, D_{1,T})$ is a complete separable metric space. Two important properties of the Wasserstein metric follow from its definition. Firstly, for two random processes X^1 and X^2 with laws m_1 and m_2, respectively, one has

$$D_{1,T}(m_1, m_2) \leq \mathbb{E}\ [\sup_{s\leq T} |X^1_s - X^2_s| \wedge 1]. \tag{2.3}$$

Secondly, taking another finite time $t \leq T$ and repeating all the definitions with natural extensions of the same concepts from greater to smaller time, one obviously has

$$D_{1,t}(m_1, m_2) \leq D_{1,T}(m_1, m_2), \quad t \leq T. \tag{2.4}$$

Proof of Theorem 2.2.3. To prove Theorem 2.2.3, one should search for a fixed point of the map $\Phi : \mathcal{P}_T \to \mathcal{P}_T$ that to a given $m \in \mathcal{P}_T$ associates the law of the solution to the following SDE:

$$\begin{cases} dX_t = dW_t + b(t, X_t; (m_s)_{s\leq t})dt, \\ X_0 \sim p_0. \end{cases}$$

Notice that this equation is well-defined in strong sense thanks to (2.2) (see e.g. [45, Thm. 5.2.9]). To exhibit the fixed point, the following contraction inequality should be demonstrated for $m_1, m_2 \in \mathcal{P}_T$:

$$D_{1,t}(\Phi(m_1), \Phi(m_2)) \leq C_T \int_0^t D_{1,u}(m_1, m_2)du.$$

To prove the latter, follow the steps in [72]. Always use (H0) when dealing with the time interaction. Let $m_1, m_2 \in \mathcal{P}_T$. Associate to m_1 the law of the solution of

$$X^1_t = X_0 + W_t + \int_0^t \int_0^s b(u, X^1_u; (m_{1,r})_{r\leq u})\ du\ ds,$$

and to m_2 the law of the solution of:

$$X^2_t = X_0 + W_t + \int_0^t \int_0^s b(u, X^2_u; (m_{2,r})_{r\leq u})\ du\ ds.$$

Then,

$$\begin{aligned} |X^1_s - X^2_s| \leq \Big| \int_0^s \int_0^u \Big[\int_{C((0,T);\mathbb{R}^d} & L(u-\alpha, X^1_u - w_\alpha)dm_1(w) \\ & - \int_{C((0,T);\mathbb{R}^d} L(u-\alpha, X^2_u - w_\alpha)dm_2(w)\Big] d\alpha du\Big| =: F(s). \end{aligned}$$

Taking $\sup_{s\leq t}$ of the previous expression and an expectation on both sides, one has

$$\mathbb{E}\ [\sup_{s\leq t} |X^1_s - X^2_s|] \leq \mathbb{E}\ [\sup_{s\leq t} F(s)]. \tag{2.5}$$

In the following computations C_T is a constant that may change from line to line. Taking π to be any coupling of m_1 and m_2, it comes

$$F(s) = \Big| \int_0^s \int_0^u \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} \big[L(u-\alpha, X^1_u - w^1_\alpha) - L(u-\alpha, X^2_u - w^2_\alpha)\big] d\pi(w^1, w^2)\ d\alpha\ du\Big|.$$

In view of (H0), one has

$$|L(u-\alpha, X_u^1 - w_\alpha^1) - L(u-\alpha, X_u^2 - w_\alpha^2)| \leq (2h_1(u-\alpha) + h_2(u-\alpha))|X_u^1 - w_\alpha^1 - X_u^2 + w_\alpha^2| \wedge 1.$$

Therefore,

$$\mathbb{E}\ [\sup_{s\leq t} F(s)] \leq C\mathbb{E}\int_0^t \int_0^u (h_1(u-\alpha) + h_2(u-\alpha)) \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} \big[|X_u^1 - X_u^2| \wedge 1 + |w_\alpha^1 - w_\alpha^2| \wedge 1\big] d\pi(w^1, w^2)\ d\alpha\ du.$$

Then,

$$\mathbb{E}\ [\sup_{s\leq t} F(s)] \leq 2C_T\mathbb{E}\int_0^t |X_u^1 - X_u^2| \wedge 1\ du + \int_0^t \int_0^u (h_1(u-\alpha) + h_2(u-\alpha)) \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} |w_\alpha^1 - w_\alpha^2| \wedge 1\ d\pi(w^1, w^2) d\alpha\ du.$$

Using that $|X_u^1 - X_u^2| \leq \sup_{r\leq u}|X_r^1 - X_r^2|$ and $|w_\alpha^1 - w_\alpha^2| \leq \sup_{r\leq u}|w_r^1 - w_r^2|$ and applying Fubini's theorem, one obtains

$$\mathbb{E}\ [\sup_{s\leq t} F(s)] \leq C_T\big[\int_0^t \mathbb{E}[\sup_{r\leq u}|X_r^1 - X_r^2| \wedge 1]du + \int_0^t \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} \sup_{r\leq u}|w_r^1 - w_r^2| \wedge 1\ d\pi(w^1, w^2) du\big].$$

Coming back to (2.5), one gets

$$\mathbb{E}\ [\sup_{s\leq t}|X_s^1 - X_s^2|] \leq C_T\big[\int_0^t \mathbb{E}[\sup_{r\leq u}|X_r^1 - X_r^2| \wedge 1]du + \int_0^t \int_{C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)} \sup_{r\leq u}|w_r^1 - w_r^2| \wedge 1\ d\pi(w^1, w^2) du\big].$$

Taking an infimum over all couplings π of m_1 and m_2 of the above expression, leads to

$$\mathbb{E}\ [\sup_{s\leq t}|X_s^1 - X_s^2|] \leq C_T\int_0^t \mathbb{E}\ [\sup_{r\leq u}|X_r^1 - X_r^2| \wedge 1]du + C_T\int_0^t D_{1,u}(m_1, m_2)du.$$

Gronwall's lemma implies

$$\mathbb{E}\ [\sup_{s\leq t}|X_s^1 - X_s^2| \wedge 1] \leq C_T\int_0^t D_{1,u}(m_1, m_2)du. \tag{2.6}$$

As X^1 and X^2 have laws $\Phi(m_1)$ and $\Phi(m_2)$, respectively, the property (2.3) of the Waserstein distance together with (2.6), lead to the contraction inequality

$$D_{1,t}(\Phi(m_1), \Phi(m_2)) \leq C_T\int_0^t D_{1,u}(m_1, m_2)du.$$

Firstly, we can conclude the weak uniqueness. Namely, let m_1 and m_2 be the laws of two weak solutions to (2.1). Then, $\Phi(m_1) = m_1$ and $\Phi(m_2) = m_2$. In view of the above contraction inequality, one has

$$D_{1,T}(m_1, m_2) = D_{1,T}(\Phi(m_1), \Phi(m_2)) \leq C_T\int_0^T D_{1,u}(m_1, m_2)du.$$

By Gronwall's lemma, $D_{1,T}(m_1, m_2) = 0$.

Secondly, the strong uniqueness for (2.1) follows. Assume we have two strong solutions with the same notation as above. We have just seen that $D_{1,T}(m_1, m_2) = 0$. In view of (2.6) and (2.4), one has

$$\mathbb{E}\,[\sup_{s \leq T} |X_s^1 - X_s^2|] \leq C_T D_{1,T}(m_1, m_2) = 0.$$

This implies strong uniqueness.

Finally, by the standard contraction argument, one gets weak existence. Construct the sequence $\{m_k, k \in \mathbb{N}\} = \{\Phi^k(m), m \in \mathbb{N}\}$. Here m is any element of $\mathcal{P}_T$. The contraction inequality leads to

$$D_{1,T}(\Phi^{k+1}(m), \Phi^k(m)) \leq C_T \int_0^T D_{1,u}(\Phi^k(m), \Phi^{k-1}(m))du.$$

Iterating this expression, one has

$$D_{1,T}(\Phi^{k+1}(m), \Phi^k(m)) \leq \frac{C_T^k T^k}{k!} D_{1,T}(\Phi(m), m).$$

Since, $\frac{C_T^k T^k}{k!} \to 0$ as $k \to \infty$, the sequence $(m_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. As the space $\mathcal{P}_T$ is complete with respect to the Wasserstein metric, there exists a probability measure $\mathbb{Q}$ such that $m_k \overset{w}{\to} \mathbb{Q}$. By the construction of the sequence m_k, $\Phi(\mathbb{Q}) = \mathbb{Q}$.

The existence of a strong solution follows from the results of Yamada and Watanabe summarized in Chapter 5.3 of [45]. □

A natural discretization of (2.1) is obtained by plugging the empirical measure of N particles in the place of the law of the process. Like this, one obtains instead of one non-linear equation, a system of N dependent linear equations. This system is called in the literature the particle system associated to (2.1). It is defined on the product probability space $(\Omega^N, \mathcal{F}^{\otimes N}, \mathbb{P}^{\otimes N})$ filtered by the natural extension of the original filtration to the product space, and equipped with an N-dimensional Brownian motion adapted to it. It reads

$$\begin{cases} dX_t^{i,N} = dW_t^i + \left\{ \frac{1}{N} \sum_{j=1}^N \int_0^t L(t-s, X_t^{i,N} - X_s^{j,N})ds \right\} dt, \\ X_0^{i,N} \text{ i.i.d.} \sim p_0. \end{cases} \tag{2.7}$$

Notice that the particle system inherits from the NLSDE the unusual interaction in time by becoming non-Markovian. In each time $t > 0$ every particle interacts with all the past of all the other particles.

Theorem 2.2.4. *Under the hypothesis (H0), the particle system in* (2.7) *admits a unique strong solution.*

Proof. Notice that the drift of each particle is uniformly bounded according to (H0). Thus, by Novikov's condition one can use the Girsanov transform in order to construct a weak solution to the particle system (see e.g. [45, Prop. 5.3.6]).

As above, to finish the proof one should show that strong uniqueness holds. Let us drop the index N for simplicity. Let $X = (X^1, \dots, X^N)$ and $Y = (Y^1, \dots, Y^N)$ be two strong solutions to (2.7). Then, in view of (H0), for $i \le N$ and $t \le T$ one has

$$|X_t^i - Y_t^i| \le \int_0^t \frac{1}{N} \sum_{i=1}^N \int_0^s h_2(s-u)(|X_s^i - Y_s^i| + |X_u^j - Y_u^j|) du\, ds.$$

Notice that $|X_s^i - Y_s^i| + |X_u^j - Y_u^j| \le 2 \sup_{u \le s} \max_{1 \le k \le N} |X_u^k - Y_u^k|$. Thus,

$$|X_t^i - Y_t^i| \le C_T \int_0^t \sup_{u \le s} \max_{1 \le k \le N} |X_u^k - Y_u^k|\, ds.$$

Taking the maximum w.r.t. $1 \le i \le N$ and the supremum in $t \le T$, one gets

$$\sup_{t \le T} \max_{1 \le i \le N} |X_t^i - Y_t^i| \le C_T \int_0^t \sup_{u \le s} \max_{1 \le k \le N} |X_u^k - Y_u^k|\, ds.$$

Apply Gronwall's lemma to finish the proof. □

An intuitive question that now can be posed is what happens with the particle system once $N \to \infty$. Do we recover (2.1)? In which sense? In order to answer it let us introduce the notion of propagation of chaos.

Definition 2.2.5. *Let u_N a sequence of symmetric probabilities on $C((0,T);\mathbb{R}^d)^N$ and u a probability measure on $C((0,T);\mathbb{R}^d)$. u_N is u-chaotic, if for any $f_1, \dots, f_k \in C_b(C((0,T);\mathbb{R}^d))$, any $k \ge 1$:*

$$lim_{N \to \infty} \int_{C((0,T);\mathbb{R}^d)^N} f_1(x_1) \cdots f_k(x_k) u_N(dx_1 \dots dx_N) = \prod_{i=1}^k \int_{C((0,T);\mathbb{R}^d)} f_i(x) u(dx).$$

In view of [72, Prop. 2.2-i)], U^N is u-chaotic is equivalent to: $a)$ the same definition with $k = 2$; $b)$ the sequence of empirical measures $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^i}$ ($\mathcal{P}_T$-valued random variables), converges in law to the constant random variable u, where X^i are canonical coordinates on $C((0,T);\mathbb{R}^d)^N$.

Let us denote the law of the process in (2.1) and (2.7) by $\mathbb{Q}$ and $\mathbb{Q}^N$, respectively. The notion of propagation of chaos in this context tells us that if $\mathbb{Q}^N$ is $\mathbb{Q}$-chaotic, then the joint law of any k-tuple of particles ($k \ge 2$), converges, when the number of particles goes to infinity, to the product measure $\mathbb{Q}^{\times k}$. Equivalently, it means that the empirical measure of N particles converges in law to $\delta_{\mathbb{P}}$. This is the analogue to the law of large numbers in the context of a system of interacting stochastic particles.

In order to establish the propagation of chaos in the above case we define N i.i.d. copies of (2.1) on the same filtered probability space as the particle system,

$$\begin{cases} d\bar{X}_t^i = \sigma dW_t^i + b(t, \bar{X}_t^i)dt + \Big\{ \int_0^t \int L(t-s, \bar{X}_t^i - y) q_s(dy) ds \Big\} dt \\ q_s := \mathcal{L}(\bar{X}_s^i), \quad \bar{X}_0^i = X_0^{i,N}. \end{cases} \tag{2.8}$$

We will prove the following claim:

Theorem 2.2.6. *Under the hypothesis (H0), for any $i \geq 1$ and any $T > 0$:*

$$\sup_N \sqrt{N}\mathbb{E}[sup_{t\leq T}|X_t^{i,N} - \bar{X}_t^i|] < \infty.$$

Proof. We adapt the arguments in [72, Thm. 1.4]. For simplicity, we drop N in the notation of (2.7). In addition, C or C_T will denote constants that may change from line to line. One has

$$\mathbb{E}[\sup_{t\leq T}|X_t^i - \bar{X}_t^i|] \leq \mathbb{E}\int_0^T \Big|\int_0^s \frac{1}{N}\sum_{j=1}^N L(s-\alpha, X_s^i - X_\alpha^j) - \int L(s-\alpha, \bar{X}_s^i - y)p_\alpha(dy)d\alpha\Big|\, ds.$$

Let us note $\tilde{L}(t-s, x-x') = L(t-s, x-x') - \int L(t-s, x-y)p_s(dy)$. Notice that

$$\Big|\int_0^s \frac{1}{N}\sum_{j=1}^N L(s-\alpha, X_s^i - X_\alpha^j) - \int L(s-\alpha, \bar{X}_s^i - y)p_\alpha(dy)\, d\alpha\Big|$$
$$\leq \int_0^s \frac{1}{N}\sum_{j=1}^N \{|L(s-\alpha, X_s^i - X_\alpha^j) - L(s-\alpha, \bar{X}_s^i - X_\alpha^j)| + |L(s-\alpha, \bar{X}_s^i - X_\alpha^j) - L(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)|\}d\alpha$$
$$+ \Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|.$$

In view of (H0), one has

$$\mathbb{E}[\sup_{t\leq T}|X_t^i - \bar{X}_t^i|] \leq \int_0^T \int_0^s h_2(s-\alpha)\mathbb{E}|X_s^i - \bar{X}_s^i|d\alpha\, ds$$
$$+ \int_0^T \int_0^s h_2(s-\alpha)\frac{1}{N}\sum_{j=1}^N \mathbb{E}|X_\alpha^j - \bar{X}_\alpha^j|d\alpha\, ds + \int_0^T \mathbb{E}\Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|\, ds.$$

Summing the previous expression over i going from 1 to N and using that for $\alpha \leq s$, one has $|X_\alpha^j - \bar{X}_\alpha^j| \leq \sup_{r\leq s}|X_r^j - \bar{X}_r^j|$ and for $s \leq T$, one has $|X_s^j - \bar{X}_s^j| \leq \sup_{r\leq s}|X_r^j - \bar{X}_r^j|$, we get

$$\sum_{i=1}^N \mathbb{E}[\sup_{t\leq T}|X_t^i - \bar{X}_t^i|] \leq D_T\int_0^T \sum_{i=1}^N \mathbb{E}[\sup_{r\leq s}|X_r^j - \bar{X}_r^j|]ds +$$
$$+ D_T\int_0^T \sum_{j=1}^N \mathbb{E}[\sup_{r\leq s}|X_r^j - \bar{X}_r^j|]ds + C_T\int_0^T \mathbb{E}\Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|ds$$
$$= C_T\int_0^T \sum_{j=1}^N \mathbb{E}[\sup_{r\leq s}|X_r^j - \bar{X}_r^j|]ds + C_T\int_0^T \mathbb{E}\Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|ds.$$

Gronwall's lemma implies that

$$\sum_{i=1}^N \mathbb{E}[\sup_{t\leq T}|X_t^i - \bar{X}_t^i|] \leq C_T\int_0^T \mathbb{E}\Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|ds.$$

Fix an $1 \leq i \leq N$. By symmetry in law of (2.7) and (2.8), one has

$$\mathbb{E}[\sup_{t\leq T}|X_t^i - \bar{X}_t^i|] \leq C_T\int_0^T \mathbb{E}\Big|\frac{1}{N}\sum_{j=1}^N \int_0^s \tilde{L}(s-\alpha, \bar{X}_s^i - \bar{X}_\alpha^j)d\alpha\Big|\, ds.$$

After the Cauchy-Schwarz inequality, it comes

$$\mathbb{E}[\sup_{t\leq T}|X_t^i-\bar{X}_t^i|]\leq C_T\int_0^T\sqrt{\frac{\mathbb{E}}{N^2}\left(\sum_{j=1}^N\int_0^s\tilde{L}(s-\alpha,\bar{X}_s^i-\bar{X}_\alpha^j)d\alpha\right)^2}ds. \tag{2.9}$$

Notice that $\tilde{L}$ is centered,

$$\mathbb{E}[\tilde{L}(t-s,x-\bar{X}_s^i)]=\mathbb{E}[L(t-s,x-\bar{X}_s^i)]-\int L(t-s,x-y)\rho(s,dy)=0.$$

This together with the fact that $\bar{X}_\alpha^{j_1}$ and $\bar{X}_\alpha^{j_2}$ are independent for $j_1\neq j_2$, implies that the mixed terms of the squared sum are zero. For the other terms we use that L is bounded by h_1, and thus $\tilde{L}$ also is. One gets

$$\frac{\mathbb{E}}{N^2}\left(\sum_{j=1}^N\int_0^s\tilde{L}(s-\alpha,\bar{X}_s^i-\bar{X}_\alpha^j)d\alpha\right)^2\leq\frac{1}{N^2}\sum_{j=1}^N\mathbb{E}\left(\int_0^s\tilde{L}(s-\alpha,\bar{X}_s^i-\bar{X}_\alpha^j)d\alpha\right)^2\leq D_T^2\frac{1}{N}. \tag{2.10}$$

Combine (2.10) and (2.9) in order to conclude the proof. □

Notice that Theorem 2.2.6 implies $\mathbb{Q}^N$ is $\mathbb{Q}$-chaotic. Namely, denote by $\mathbb{Q}^{2,N}$ the law of the couple $(X^{1,N},X^{2,N})$. Then, weak convergence of the probability measure $\mathbb{Q}^{2,N}$ to the product measure $\mathbb{Q}\times\mathbb{Q}$ implies Definition 2.2.5 for $k=2$. $\mathbb{Q}^{2,N}$ and $\mathbb{Q}\times\mathbb{Q}$ belong to the space $\mathcal{P}_T^{(2)}$ of probability measures on $C((0,T);\mathbb{R}^d)\times C((0,T);\mathbb{R}^d)$. The definition of the Wasserstein metric can naturally be rewritten for such a space and will satisfy the analogues of (2.3) and (2.4). Denote the 1-Wasserstein metric on $\mathcal{P}_T^{(2)}$ by $D_{1,T}^{(2)}$. Thus,

$$D_{1,T}^{(2)}(\mathbb{Q}^{2,N},\mathbb{Q}\times\mathbb{Q})\leq\mathbb{E}[\sup_{s\leq T}|(X_s^{1,N},X_s^{2,N})-(\bar{X}_s^1,\bar{X}_s^2)|\wedge 1]\leq\mathbb{E}[\sup_{s\leq T}|X_s^{1,N}-\bar{X}_s^1|]+\mathbb{E}[\sup_{s\leq T}|X_s^{2,N}-\bar{X}_s^2|].$$

Applying Theorem 2.2.6 and letting $N\to\infty$, one gets $D_{1,T}^{(2)}(\mathbb{Q}^{2,N},\mathbb{Q}\times\mathbb{Q})\to 0$.

The propagation of chaos property enables one to conclude a numerical algorithm for approximating the law $\mathbb{Q}$. It tells us that when N is large enough the empirical measure of N particles behaves like the limit law. Thus, the empirical measure of these particles in, for example, time t will approximate the marginal $\mathbb{Q}_t$. The particles in (2.7) are themselves approximated by an Euler scheme. A natural question arising once such a numerical algorithm is constructed, is what the rates of convergence in the number of particles and time discretization step of the Euler scheme are. A review of such results for a NLSDE without time interaction can be found in Bossy [11]. In this thesis such question will not be treated. However, one can imagine that with a slight change of hypothesis on the interaction kernel (same as we did in (H0) w.r.t. [72]) one can obtain some of the results mentioned in [11]. This remains to be checked in some of our future works.

We conclude this section with the following remark:

Remark 2.2.7. *All the obtained results may be generalized to a stochastic process of the type*

$$\begin{cases} dX_t=\sigma(X_t)dW_t+\left\{\int_0^t\int_{\mathbb{R}^d}L(t-s,X_t-y)\mathbb{Q}_s(dy)\ ds\right\}dt+b(t,X_t)dt, \quad t\leq T,\\ \mathbb{Q}_s:=\mathcal{L}(X_s), \quad X_0\sim p_0,\end{cases} \tag{2.11}$$

where $\sigma:\mathbb{R}^d\to\mathbb{R}^d$ and $b:(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ are uniformly bounded on $(0,T)\times\mathbb{R}^d$ and Lipschitz functions in space with a uniform constant with respect to time. In that case the 2-Wasserstein metric should be used.

2.3 The associated McKean-Vlasov-Fokker-Planck equation

In the sequel $\mathbb{Q}$ will denote the law of the process X constructed in Theorem 2.2.3. We aim to establish a connection between the time marginals of $\mathbb{Q}$ and the following Fokker-Planck equation:

$$\begin{cases} \frac{\partial}{\partial t}\mu_t = \frac{1}{2}\triangle\mu_t - \nabla\cdot(b(t,\cdot;(\mu_s)_{s\leq t})\mu_t), \quad t\in(0,T), \\ \mu_{t=0} = q_0. \end{cases} \tag{2.12}$$

As the initial condition q_0 is a probability measure, from the probabilistic point of view, Equation (2.12) describes the time evolution of a probability measure μ_t. Let us define the notion of weak solution for (2.12).

Definition 2.3.1. *A measurable family $(\mu_t)_{t\leq T}$ of probability measures on $\mathbb{R}^d$ is a weak solution to (2.12) if for any $f\in C_b^2(\mathbb{R}^d)$ and any $t\in(0,T)$ one has*

$$\int_{\mathbb{R}^d} f(x)\mu_t(dx) = \int_{\mathbb{R}^d} f(x)q_0(dx) + \int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot b(s,x;\mathbb{Q})\mu_s(dx)\ ds + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\triangle f(x)\mu_s(dx)\ ds.$$

Hypothesis (H0) implies that everything makes sense in the preceding equation.

Proposition 2.3.2. *The family $(\mathbb{Q}_t)_{t\leq T}$ of probability measures on $\mathbb{R}^d$ has the following properties:*

i) For any $t\in(0,T]$, $\mathbb{Q}_t$ admits a probability density function q_t. In addition, $q_t\in L^p(\mathbb{R}^d)$ for any $1<p<\infty$ and

$$\forall 0<t\leq T: \quad \|q_t\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}(1-\frac{1}{p})}} e^{(\lambda'-\frac{1}{2})T\beta^2},$$

where $\beta := \sup_{t\leq T}\|b(t,\cdot;(\mathbb{Q}_r)_{r\leq t}\|$.

ii) $(q_t)_{t\leq T}$ is a weak solution to (2.12).

Proof. i) This result is directly implied by Méléard and Roelly [53, Lemma 1.1] as b is bounded. We write the proof in order to explicit the L^p-norm estimate in function of β and t. Let $t\in(0,T]$, $p>1$ and p' such that $\frac{1}{p}+\frac{1}{p'}=1$. For $f\in C_K^\infty(\mathbb{R}^d)$, define the linear functional

$$H_t(f) := \int_{\mathbb{R}^d} f(x)Q_t(dx) = \mathbb{E}(f(X_t)).$$

In view of Girsanov's theorem (e.g. [45, Chapter 3, Thm. 5.1]), one has

$$H_t(f) = \mathbb{E}(f(W_t+X_0)(Z_T)^{-1})),$$

where Z_T is the exponential martingale

$$Z_T := e^{-\int_0^T b(s,X_s;(\mathbb{Q}_r)_{r\leq s})\cdot dW_s - \frac{1}{2}\int_0^T |b(s,X_s;(\mathbb{Q}_r)_{r\leq s})|^2\ ds}.$$

Remember that (H0) provides the boundness of b and thus by Novikov's condition Z_T is a martingale. Now, choose $1 < q < p'$ and q' such that $\frac{1}{q} + \frac{1}{q'} = 1$. Hölder's inequality for $\lambda = \frac{p'}{q}$ and λ' such that $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$ leads to

$$|H_t(f)| \leq \left(\mathbb{E}(|f(X_0 + W_t)|^\lambda)\right)^{\frac{1}{\lambda}} \left(\mathbb{E}((Z_T)^{-\lambda'})\right)^{\frac{1}{\lambda'}} =: AB.$$

Applying Hölder's inequality for q and q',

$$A = \left(\int\int |f(x+y)|^\lambda g_t(y)\ dy\ q_0(dx)\right)^{\frac{1}{\lambda}} \leq \left(\int \left(\int |f(x+y)|^{p'}\ dy\right)^{\frac{1}{q}} \|g_t\|_{L^{q'}(\mathbb{R}^d)}\ q_0(dx)\right)^{\frac{1}{\lambda}}$$

$$= \left(\|f\|^{\frac{p'}{q}}_{L^{p'}(\mathbb{R}^d)} \|g_t\|_{L^{q'}(\mathbb{R}^d)}\right)^{\frac{q}{p'}} = \|f\|_{L^{p'}(\mathbb{R}^d)} \frac{C}{t^{\frac{d}{2}(1-\frac{1}{q'})\frac{q}{p'}}}.$$

Notice that

$$B^{\lambda'} = \mathbb{E}\Big(e^{\lambda' \int_0^T b(s,X_0+W_s;(\mathbb{Q}_r)_{r\leq s})\cdot dW_s - (\lambda')^2 \int_0^T |b(s,X_0+W_s;(\mathbb{Q}_r)_{r\leq s})|^2\ ds}$$
$$e^{((\lambda')^2 - \frac{\lambda'}{2}) \int_0^T |b(s,X_0+W_s;(\mathbb{Q}_r)_{r\leq s})|^2\ ds}\Big).$$

Apply the Cauchy-Schwarz inequality. It comes

$$B^{\lambda'} \leq e^{((\lambda')^2 - \frac{\lambda'}{2})T\beta^2}.$$

Therefore,

$$|H_t(f)| \leq \frac{C}{t^{\frac{d}{2}(1-\frac{1}{p})}} e^{(\lambda' - \frac{1}{2})T\beta^2} \|f\|_{L^{p'}(\mathbb{R}^d)}.$$

Then, H_t is a bounded linear functional defined on a dense subspace of $L^{p'}(\mathbb{R}^d)$. Therefore, it extends to a bounded linear functional on $L^{p'}(\mathbb{R}^d)$. By Riesz representation theorems (e.g. [15, Thm. 4.11 and 4.14]), there exists a unique $q_t \in L^p(\mathbb{R}^d)$ such that q_t is the probability density of $\mathbb{Q}_t$ and $\|q_t\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}(1-\frac{1}{p})}} e^{(\lambda'-\frac{1}{2})T\beta^2}$.

ii) Let $f \in C_b^2(\mathbb{R}^d)$. Apply Itô's formula on $f(X_t)$:

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s) \cdot b(s, X_s; (q_u)_{u\leq s})\ ds + \int_0^t \nabla f(X_s) \cdot\ dW_s + \frac{1}{2}\int_0^t \triangle f(X_s)\ ds.$$

Taking the expectation on both sides one gets the condition from Definition 2.3.1.

□

Another way to connect the process in (2.1) and the PDE (2.12) is to show that the family $(q_t)_{t\leq T}$ satisfies the following mild formulation of (2.12) in the sense of the distributions:

$$\mu_t = g_t * p_0 - \sum_{i=1}^d \int_0^t \nabla_i g_{t-s} * (b_i(s, \cdot; (\mu_u)_{u\leq t})\mu_s)\ ds. \tag{2.13}$$

Here a convolution involving a measure ν and a function f should be understood as $(f * \nu)(x) = \int f(x-y)\nu(dy)$.

Proposition 2.3.3. *The marginals $(q_t)_{t\in(0,T]}$ satisfy in the sense of the distributions the mild equation* (2.13)*.*

Proof. In order to derive (2.13) from (2.1) for $f \in C_b^2(\mathbb{R}^d)$ consider the Cauchy problem

$$\begin{cases} \frac{\partial G}{\partial s} + \frac{1}{2}\triangle G = 0, \quad 0 \le s < t, \quad x \in \mathbb{R}^d, \\ \lim_{s\to t^-} G(s,x) = f(x). \end{cases} \tag{2.14}$$

The function

$$G_{t,f}(s,x) = \int f(y) g_{t-s}(x-y)dy$$

is a smooth solution to (2.14) where g_t denotes the density of W_t. Applying Itô's formula to $G_{t,f}(t, X_t)$ we get

$$\begin{aligned} G_{t,f}(t,X_t) - G_{t,f}(0,X_0) &= \int_0^t \frac{\partial G_{t,f}}{\partial s}(s,X_s)ds + \int_0^t \nabla G_{t,f}(s,X_s)\cdot b(s,X_s;\mathbb{Q})\ ds \\ &\quad + \int_0^t \nabla G_{t,f}(s,X_s)\cdot dWs + \frac{1}{2}\int_0^t \triangle G_{t,f}(s,X_s)ds. \end{aligned}$$

In view of (2.2) and (2.14), we obtain

$$\mathbb{E}f(X_t) = \mathbb{E}G_{t,f}(0,X_0) + \int_0^t \mathbb{E}\left[\nabla G_{t,f}(s,X_s)\cdot b(s,X_s;\mathbb{Q})\right] ds =: I + II. \tag{2.15}$$

On the one hand one has

$$I = \int\int f(y)g_t(x-y)dy\ q_0(dx) = \int f(y)(g_t * q_0)(y)dy.$$

On the second hand one has

$$\begin{aligned} II &= \int_0^t \int \nabla_x \big[\int f(y) g_{t-s}(x-y)dy\big]\cdot b(s,x;(q_u)_{u\le s})q_s(x)\ dx\ ds \\ &= -\int_0^t \int\int f(y)\nabla g_{t-s}(y-x)dy\cdot b(s,x;(q_u)_{u\le s})q_s(x)\ dx\ ds \\ &= -\sum_{i=1}^d \int f(y)\int_0^t [\nabla_i g_{t-s} * (b_i(s,\cdot;(q_u)_{u\le s})q_s)](y)\ ds\ dy. \end{aligned}$$

Thus (2.15) can be written as

$$\int f(y)q_t(y)\,dy = \int f(y)(g_t * q_0)(y)\,dy - \sum_{i=1}^d \int f(y)\int_0^t [\nabla_i g_{t-s} * (b_i(s,\cdot;(q_u)_{u\le s})q_s)](y)\ ds\ dy,$$

which is the mild equation (2.13). □

Remark that by constructing the stochastic process, we have not just built a family of probability measures on $\mathbb{R}^d$ that is a solution to the Fokker-Planck equation. In fact we have built an object that belongs to much wider class, a probability measure $\mathbb{Q}$ on the space of trajectories $C([0,T];\mathbb{R}^d)$. In addition, the stochastic process can be seen as the time evolution of one individual in an infinite population following the dynamics in (2.12). Thus, a micro model for (2.12) is obtained. Then, a tool as Girsanov transform can provide us with additional information about the solution of the PDE, like in Proposition 2.3.2 - i). The reader will see another purely probabilistic technique to obtain $L^{\infty}(\mathbb{R}^d)$-norm estimates for the marginal densities in Section 3.3. Finally, one should not forget the particle system associated to the stochastic process and now to the PDE. The propagation of chaos property, proved in the previous chapter, tells us that the empirical measure of large number of particles converges towards the law of the stochastic process and by that to a solution of (2.12). Thus, a numerical method for approximating the PDE is obtained. This method is purely probabilistic and it is quite convenient since its complexity grows linearly with the dimension d and not exponentially as it is the case with the deterministic numerical methods for elliptic and parabolic PDE's.

Notice that all the results proven in this chapter are due to the regularity assumption (H0) on the interaction kernel L. It allowed us to adapt the classical proof of Sznitman to show well-posedness of our time and space interacting Mc-Kean Vlasov diffusion. It ensured the well-posedness and propagation of chaos of the associated particle system even though the setting in it is non-Markovian. Finally, it justified all the computations when interpreting the marginal laws of the process as a solution to a non-linear Fokker-Planck equation.

The interaction kernel associated to the Keller-Segel system does not enjoy the regularity properties supposed in this chapter and thus, the above arguments do not apply. A specific analysis needs to be developed to overcome the singularity of the kernel. In particular, the well-posedness of the mean field limit and of the system of particles, each one interacting with the whole past of all the other ones, needs specific and original techniques of analysis.

Chapter 3

The one-dimensional case: The non-linear stochastic equation

This chapter is written in collaboration with Denis Talay and it is available as a preprint [73].

3.1 Introduction

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\frac{1}{2}\nabla\rho - \chi\rho\nabla c)(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\ \partial_t c(t,x) = \frac{1}{2}\triangle c(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0,\ x\in\mathbb{R}^d. \\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0(x). \end{cases} \tag{3.1}$$

For theoretical results on this system of PDEs and applications to Biology see Chapter 1.

Recently, stochastic interpretations have been proposed for a simplified version of the model, that is, the parabolic-elliptic model which corresponds to the case $\alpha = 0$. These interpretations all rely on the fact that, in the parabolic-elliptic case, the equations for ρ_t and c_t can be decoupled and c_t can be explicated as the convolution of ρ_t and a logarithmic kernel. Consequently, the corresponding stochastic process of McKean–Vlasov type whose ρ_t is the time marginal density involves the singular interaction kernel $k(x) = -\frac{x}{2\pi|x|^2}$ (when $\lambda = 0$). This explains why, so far, only partial results are obtained and heavy techniques are used to get them. A review of the works by Haškovec and Schmeiser [36], Fournier and Jourdain [31] and Cattiaux and Pédèches [21] is given in Section 1.4.

Budhiraja and Fan [17] have studied a McKean–Vlasov SDE related to a parabolic–parabolic version of the model with a smooth coupling between ρ and c and a forcing potential term. Under a suitable convexity assumption on the additional term, they obtain uniform in time concentration inequalities for the corresponding particle system and uniform in time error estimates for a numerical approximation of the exact McKean–Vlasov process.

We here deal with the parabolic–parabolic system ($\alpha > 0$) without cut-off and study the McKean-Vlasov stochastic representation of the mild formulation of the equation satisfied by ρ_t. This representation involves a singular interaction kernel which is different from the one in the above mentioned approaches and does not seem to have been studied in the McKean-Vlasov non-linear SDE literature. The system reads

$$\begin{cases} dX_t = b^\sharp(t, X_t)dt + \left\{ \int_0^t (K_{t-s}^\sharp * p_s)(X_t)ds \right\} dt + dW_t, \quad t > 0, \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{3.2}$$

where $K_t^\sharp(x) := \chi e^{-\lambda t} \nabla (\frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}})$ and $b^\sharp(t,x) := \chi e^{-\lambda t} \nabla \mathbb{E} c_0(x + W_t)$. Here, $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ and X_0 is an $\mathbb{R}^d$-valued $\mathcal{F}_0-$measurable random variable. Notice that the formulation requires that the one dimensional time marginals of the law of the solution are absolutely continuous with respect to Lebesgue's measure and that the process interacts with all the past time marginals of its probability distribution through a functional involving a singular kernel.

The analysis of the well-posedness of this non-linear stochastic equation and the proof that p_s is a solution to (3.2) for any s are delicate, particularly in the multi-dimensional case when χ is large enough to induce solutions with blow-ups in finite time (see Chapter 6). As numerical simulations of the related particle system in Chapter 7 appear to be effective, it seems interesting to validate our approach in the one-dimensional case.

The objective of this chapter is to prove general existence and uniqueness results for both the deterministic system (3.1) and the stochastic dynamics (3.2) in $d = 1$. In Chapter 5 we show the well-posedness and propagation of chaos property of the corresponding particle system where each particle interacts with all the past of the other ones by means of a time integrated singular kernel.

In this one-dimensional framework the PDE (3.1) was previously studied by Osaki and Yagi [60] and Hillen and Potapov [40] in bounded intervals I with periodic boundary conditions while we here deal with the problem posed on the whole space $\mathbb{R}$. In [60] one assumes $\rho_0 \in L^2(I) \cap L^1(I)$, $c_0 \in H^1(I)$ and $\inf_I c_0(x) > 0$. In [40] one assumes $\rho_0 \in L^\infty(I) \cap L^1(I)$ and $c_0 \in W^{\sigma,p}(I)$, where p and σ belong to a particular set of parameters. Here, we only suppose that ρ_0 is in $L^1(\mathbb{R})$.

We emphasize that we do not limit ourselves to the specific kernel $K_t^\sharp(x)$ related to the Keller–Segel model. We below show that the stochastic differential equation of Keller-Segel type is well-posed for a whole class of time integrated singular kernels. Due to the singular nature of the kernel, the mean-field SDE cannot be analyzed by means of standard coupling methods or Wasserstein distance contractions as in Chapter 2. Both to construct local in time solutions and to go from local to global solutions, an important issue consists in properly defining the set of weak solutions. Namely, without any assumption on the initial density ρ_0, we need to introduce some constraints on the time marginal densities. To prove that these constraints are satisfied in the limit of an iterative procedure (where the kernel is not cut off), the norms of the successive time marginal densities cannot be allowed to exponentially depend on the L^∞-norm of the successive corresponding drifts. They neither can be allowed to depend on Hölder-norms of the drifts. Therefore, we use an accurate estimate (with explicit constants) on densities of one-dimensional diffusion processes with bounded measurable drifts which is obtained by a stochastic technique rather than the PDE techniques. This strategy allows us to get uniform bounds on the sequence of drifts, which is essential to get existence and uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure, and to suitably paste local solutions when constructing the global solution.

The chapter is organized as follows. In Section 3.2 we state our main results. In Section 3.3 we prove a preliminary estimate on the probability density of diffusion processes whose drift is only supposed Borel measurable and bounded. In Section 3.4 we study a non-linear McKean-Vlasov-Fokker-Planck equation. In Section 3.5 we prove the local existence and uniqueness of a solution to a non-linear stochastic differential equation more general than (3.2) (for $d = 1$). In Section 3.6 we get the global well-posedness of this equation. In Section 3.7 we apply the preceding result to the specific case of the one-dimensional parabolic–parabolic Keller-Segel model. The appendix section 3.8 concerns an explicit formula for the transition density of a particular diffusion. The appendix section 3.9 is a reminder on standard convolution inequalities (used in this Chapter and some of the following ones).

Notation. In all the chapter we denote by C_T, $C_T(b_0, p_0)$, etc., any constant which depends on T and the other specified parameters, but is uniform w.r.t. $t \in [0,T]$ and may change from line. In addition, for $1 \le p, q \le \infty$ the space $L^q((0,T); L^p(\mathbb{R}))$ denotes the space of functions $f : (0,T) \times \mathbb{R} \to \mathbb{R}$ such that $\int_0^T \|f(t,\cdot)\|^q_{L^p(\mathbb{R})} dt < \infty$.

3.2 Our main results

Our first main result concerns the well-posedness of a non-linear one-dimensional stochastic differential equation (SDE) with a non standard McKean–Vlasov interaction kernel which at each time t involves in a singular way all the time marginals up to time t of the probability distribution of the solution. As our technique of analysis is not limited to the above kernel $K^\sharp$, we consider the following McKean-Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \left\{ \int_0^t (K_{t-s} * p_s)(X_t)ds \right\} dt + dW_t, \quad t \le T, \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim p_0, \end{cases} \tag{3.3}$$

and in all the sequel we assume the following conditions on the interaction kernel.

Hypothesis (H)**.** *The function K defined on $\mathbb{R}^+ \times \mathbb{R}$ is such that for any $T > 0$:*

1. *For any $t > 0$, K_t is in $L^1(\mathbb{R})$.*
2. *For any $t > 0$ the function $K_t(x)$ is a bounded continuous function on $\mathbb{R}$.*
3. *The set of points $x \in \mathbb{R}$ such that $\lim_{t\to 0} K_t(x) < \infty$ has full Lebesgue measure.*
4. *For any $t > 0$, the function $f_1(t) := \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds$ is well defined and bounded on $[0,T]$.*
5. *For any $T > 0$ there exists C_T such that, for any probability density ϕ on $\mathbb{R}$,*

$$\sup_{(t,x)\in(0,T]\times\mathbb{R}} \int \phi(y) \|K_\cdot(x-y)\|_{L^1(0,t)} \, dy \le C_T.$$

6. *Finally,*

$$\sup_{0\le t\le T} \int_0^T \|K_{T+t-s}\|_{L^1(\mathbb{R})} \, \frac{1}{\sqrt{s}} \, ds \le C_T.$$

As emphasized in the introduction, the well-posedness of the system (3.3) cannot be obtained by applying known results in the literature.

Given $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and a family of densities $(p_t)_{t\leq T}$ we set

$$B(t,x;p) := \int_0^t (K_{t-s} * p_s)(x) ds. \tag{3.4}$$

We now define the notion of a weak solution to (3.3).

Definition 3.2.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is said to be a weak solution to the equation (3.3) up to time $T > 0$ if:*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.*
2. *The process $X := (X_t)_{t\in[0,T]}$ is real-valued, continuous, and $(\mathcal{F}_t)$-adapted. In addition, the probability distribution of X_0 has density p_0.*
3. *The process $W := (W_t)_{t\in[0,T]}$ is a one-dimensional $(\mathcal{F}_t)$-Brownian motion.*
4. *The probability distribution $\mathbb{P} \circ X^{-1}$ has time marginal densities $(p_t,\ t \in [0,T])$ with respect to Lebesgue measure which satisfy*
$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}. \tag{3.5}$$
5. *For all $t \in [0,T]$ and $x \in \mathbb{R}$, one has that $\int_0^t |b(s,x)|\ ds < \infty$.*
6. *$\mathbb{P}$-a.s. the pair (X,W) satisfies (3.3).*

Remark 3.2.2. *For any $T > 0$, Inequality (3.5) and Hypothesis (H-4) lead to*

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} |B(t,x,p)| \leq C_T.$$

The following theorem provides existence and uniqueness of the weak solution to (3.3).

Theorem 3.2.3. *Let $T > 0$. Suppose that $p_0 \in L^1(\mathbb{R})$ is a probability density function and $b \in L^\infty([0,T] \times \mathbb{R})$ is continuous w.r.t. the space variable. Under the hypothesis (H), Eq. (3.3) admits a unique weak solution in the sense of Definition 3.2.1.*

We finally state an easy result which is useful to prove the propagation of chaos in the case of Keller-Segel kernel (see Chapter 5):

Corollary 3.2.4. *In addition to the assumptions of Theorem 3.2.3 suppose the following hypothesis:*

H-7. for any $t > 0$, K_t is in $L^2(\mathbb{R})$ and the function $f_2(t) := \int_0^t \frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{s^{1/4}} ds$ is well defined and bounded on $[0,T]$.

Then, there exists a unique weak solution to (3.3) *in the sense of the Definition 3.2.1 modified as follows: Instead of* (3.5) *one imposes*

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}. \tag{3.6}$$

Our next result concerns the well-posedness of the one-dimensional parabolic-parabolic Keller-Segel model

$$\begin{cases} \frac{\partial \rho}{\partial t}(t,x) = \frac{\partial}{\partial x} \cdot (\frac{1}{2}\frac{\partial \rho}{\partial x} - \chi\rho\frac{\partial c}{\partial x})(t,x), \quad t>0, \quad x \in \mathbb{R}, & \text{(3.7a)} \\ \frac{\partial c}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0, \quad x\in\mathbb{R}, & \text{(3.7b)} \\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\chi > 0$ and $\lambda \geq 0$. As this system preserves the total mass, that is,

$$\forall t > 0, \quad \int_\Omega \rho(t,x)dx = \int_\Omega \rho_0(x)dx =: M,$$

the new functions $\tilde{\rho}(t,x) := \frac{\rho(t,x)}{M}$ and $\tilde{c}(t,x) := \frac{c(t,x)}{M}$ satisfy the system (3.7) with the new parameter $\tilde{\chi} := \chi M$. Therefore, w.l.o.g. we may and do thereafter assume that $M = 1$.

Denote by g_t the density of W_t. We define the notion of solution for the system (3.7):

Definition 3.2.5. *Given the functions ρ_0 and c_0, and the constants $\chi > 0$, $\lambda \geq 0$, $T > 0$, the pair (ρ, c) is said to be a solution to* (3.7) *if $\rho(t,\cdot)$ is a probability density function for every $0 \leq t \leq T$, c is in $L^\infty([0,T]; C_b^1(\mathbb{R}))$, one has $\|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}$ for any $t \in (0,T]$, and the following equality*

$$\rho(t,x) = g_t * \rho_0(x) - \chi \int_0^t \frac{\partial g_{t-s}}{\partial x} * (\frac{\partial c}{\partial x}(s,\cdot)\ \rho(s,\cdot))(x)\ ds \tag{3.8}$$

is satisfied in the sense of the distributions with

$$c(t,x) = e^{-\lambda t}(g(t,\cdot\ ) * c_0)(x) + \int_0^t e^{-\lambda s}(g_s * \rho(t-s,\cdot))(x)\ ds. \tag{3.9}$$

Notice that the function $c(t,x)$ defined by (3.9) is a mild solution to (3.7b). These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model for which sub-critical and critical regimes exist depending on the parameters of the model (see [22] and references therein). In the one-dimensional case there is no critical regime as shown by the following result.

Corollary 3.2.6. *Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. Given any $\chi > 0$, $\lambda \geq 0$ and $T > 0$, the time marginals $\rho(t,x) \equiv p_t(x)$ of the probability distribution of the unique solution to Eq.* (3.2) *with $d = 1$ and the corresponding function $c(t,x)$ provide a global solution to* (3.7) *in the sense of Definition 3.2.5. Any other solution (ρ^1, c^1) with the same initial condition (ρ_0, c_0) satisfies $\|\rho^1(t,\cdot) - \rho(t,\cdot)\|_{L^1(\mathbb{R})} = 0$ and $\|\frac{\partial c^1}{\partial x}(t,\cdot) - \frac{\partial c}{\partial x}(t,\cdot)\|_{L^1(\mathbb{R})} = 0$ for every $0 \leq t \leq T$.*

Remark 3.2.7. *From estimates in Section 3.3 we could deduce some additional regularity results which we do not need here: See Remark 3.3.3. In particular, if $\rho_0 \in L^\infty(\mathbb{R})$, then $\rho \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}))$. If $\rho_0 \in L^2(\mathbb{R})$, then $\rho \in L^\infty([0,T]; L^1 \cap L^2(\mathbb{R}))$ and $t^{1/4}\|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$; in addition, one can then easily find modifications of the hypotheses on the kernel K allowing to get unique weak solutions with constraints on $\|p_t\|_{L^2(\mathbb{R})}$. We prefer to only suppose that $\rho_0 \in L^1(\mathbb{R})$.*

3.3 Preliminary: A density estimate

In the sequel, we will get local solutions to (3.3) and extend them to global solutions by means of an iterative procedure. The L^∞-norms of the successive drifts are needed to be bounded from above uniformly w.r.t. the iteration step. Standard density estimates obtained by using Girsanov theorem or PDE analysis do not help to this purpose. The reason is that they involve constants which exponentially depend on the L^∞-norm (or even Hölder-norm) of the drifts. Namely, let $X^{(b)}$ be a process defined by

$$X_t^{(b)} = X_0 + \int_0^t b(s, X_s^{(b)})\, ds + W_t, \quad t \in [0,T], \tag{3.10}$$

where $X_0 = x$. Suppose that the drift $b(\cdot,\cdot)$ is measurable and uniformly bounded. Denote by $\beta := \sup_{t\in[0,T]} \|b(t,\cdot)\|_{L^\infty(\mathbb{R})}$ and by $p(t,x,y)$ the transition probability density of $X^{(b)}$. Formally, from the mild equation satisfied by p, one gets

$$\|p(t,x,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}} + C\beta \int_0^t \frac{\|p(s,x,\cdot)\|_{L^\infty(\mathbb{R})}}{\sqrt{t-s}}\, ds.$$

Then, a Singular Gronwall lemma leads to an estimate that depends exponentially of β. To avoid Gronwall's lemma, we could use the fact that, in view of [53, Prop. 1.1], for any $t > 0$ one has $p(t,x,\cdot) \in L^q(\mathbb{R})$ with $1 < q < \infty$. However, the proof is based on the Girsanov transform (see Section 2.3) and therefore the $L^q(\mathbb{R})$-norm of the density depends exponentially of β. Thus, if we would apply such an estimate in the mild equation instead of a Gronwall lemma as above, still we would not avoid the exponential dependence on β.

We therefore proceed by using an accurate pointwise estimate (with explicit constants) on densities of one-dimensional diffusions with bounded measurable drifts. Estimate (3.11) below is obtained by using a stochastic technique. Its drawback is that the map $y \mapsto p_y^\beta(t,x,y)$ is not a probability density function. However, it suffices to nicely bound the successive drifts of the Picard iterations as shown by Proposition 3.5.3.

To obtain $L^\infty(\mathbb{R})$ estimates for the transition probability density $p^{(b)}(t,x,y)$ of $X^{(b)}$ under the only assumption that the drift $b(t,x)$ is measurable and uniformly bounded we slightly extend the estimate proved in Qian and Zheng [66] for time homogeneous drift coefficients $b(x)$. We here propose a proof different from the original one. It avoids the use of densities of pinned diffusions and the claim that $p^{(b)}(t,x,y)$ is continuous w.r.t. all the variables which does not seem obvious to us. In our proof we adapt the method in Makhlouf [52], the main difference being that instead of the Wiener measure our reference measure is the probability distribution of the particular diffusion process X^β considered in [66] and defined by

$$X_t^\beta = X_0 + \beta \int_0^t \operatorname{sgn}(y - X_s^\beta)\, ds + W_t.$$

Theorem 3.3.1. *Let $X^{(b)}$ be the process defined in (3.10) with $X_0 = x$. Let $p_y^\beta(t,x,z)$ be the transition density of X^β. Assume $\beta := \sup_{t\in[0,T]} \|b(t,\cdot)\|_\infty < \infty$. Then for all $y \in \mathbb{R}$ and $t \in (0,T]$ it holds that*

$$p^{(b)}(t,x,y) \leq p_y^\beta(t,x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz. \tag{3.11}$$

Proof. Let $f \in \mathcal{C}_K^\infty(\mathbb{R})$ and fix $t \in (0,T]$. Consider the parabolic PDE driven by the infinitesimal generator of X^β:

$$\begin{cases} \frac{\partial u}{\partial t}(s,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(s,x) + \beta \mathrm{sgn}(y-x)\frac{\partial u}{\partial x}(s,x) = 0, \quad 0 \le s < t, \quad x \in \mathbb{R}, \\ u(t,x) = f(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.12}$$

In view of Veretennikov [78, Thm. 1] there exists a solution $u(s,x) \in W_p^{1,2}([0,t] \times R)$. Applying the Itô-Krylov formula to $u(s, X_s^\beta)$ we obtain that

$$u(s,x) = \int f(z) p_y^\beta(t-s,x,z)\ dz.$$

The formula (3.35) from our appendix allows us to differentiate under the integral sign:

$$\frac{\partial u}{\partial x}(s,x) = \int f(z) \frac{\partial p_y^\beta}{\partial x}(t-s,x,z)\ dz, \quad \forall 0 \le s < t \le T.$$

Fix $0 < \varepsilon < t$. Now apply the Itô-Krylov formula to $u(s, X_s^{(b)})$ for $0 \le s \le t-\varepsilon$ and use the PDE (3.12). It comes:

$$\mathbb{E}(u(t-\varepsilon, X_{t-\varepsilon}^{(b)})) = u(0,x) + \mathbb{E}\int_0^{t-\varepsilon} (b(s,X_s^{(b)}) - \beta \mathrm{sgn}(y - X_s^{(b)}))\frac{\partial u}{\partial x}(s, X_s^{(b)})\ ds.$$

In view of Corollary 3.8.2 in the appendix there exists a function $h \in L^1([0,t]\times\mathbb{R})$ such that

$$\forall 0 < s < t \le T,\ \forall y,z \in \mathbb{R},\ \ \mathbb{E}\left|\frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z)\right| \le C_{T,\beta,x,y} h(s,z). \tag{3.13}$$

Consequently,

$$\begin{aligned}\mathbb{E}(u(t-\varepsilon, X_{t-\varepsilon}^{(b)})) = &\int f(z) p_y^\beta(t,x,z)\ dz \\ &+ \int f(z) \int_0^{t-\varepsilon} \mathbb{E}\left\{(b(s,X_s^{(b)}) - \beta\mathrm{sgn}(y - X_s^{(b)}))\frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z)\right\}\ ds\ dz.\end{aligned}$$

Let now ϵ tend to 0. By Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\int f(z) p^{(b)}(t,x,z) dz = &\int f(z) p_y^\beta(t,x,z) dz \\ &+ \int f(z) \int_0^{t} \mathbb{E}\left\{(b(s,X_s^{(b)}) - \beta\mathrm{sgn}(y - X_s^{(b)}))\frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z)\right\}\ ds\ dz.\end{aligned}$$

Therefore the density $p^{(b)}$ satisfies:

$$p^{(b)}(t,x,z) = p_y^\beta(t,x,z) + \int_0^t \mathbb{E}\left\{(b(s,X_s^{(b)}) - \beta\mathrm{sgn}(y - X_s^{(b)}))\frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z)\right\}\ ds.$$

As noticed in [66], in view of Formula (3.36) from our appendix we have for any $x \in \mathbb{R}$

$$(b(s,x) - \beta\mathrm{sgn}(y-x))\frac{\partial}{\partial x} p_y^\beta(t-s,x,y) \le 0.$$

This leads us to choose $z = y$ in the preceding equality, which gives us

$$p^{(b)}(t,x,y) = p_y^\beta(t,x,y) + \int_0^t \mathbb{E}\left\{(b(s,X_s^{(b)}) - \beta \mathrm{sgn}(y - X_s^{(b)}))\frac{\partial p_y^\beta}{\partial x}(t-s,X_s^{(b)},y)\right\}\,ds,$$

from which

$$\forall t \le T, \quad p^{(b)}(t,x,y) \le p_y^\beta(t,x,y).$$

We finally use Qian and Zheng's explicit representation (see [66] and our appendix section 3.8).

□

Corollary 3.3.2. *Assume X_0 is distributed according to the probability density function p_0 on $\mathbb{R}$. Denote by $p(t,\cdot)$ the probability density of $X_t^{(b)}$. One has*

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi t}} + \beta. \tag{3.14}$$

Proof. In view of (3.11) we have

$$\begin{aligned}
p(t,y) &\le \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}}\,dz\,dx \\
&\le \frac{1}{\sqrt{2\pi t}} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}-\beta\sqrt{t}}^\infty (z+\beta\sqrt{t}) e^{-\frac{z^2}{2}}\,dz\,dx \\
&= \frac{1}{\sqrt{2\pi t}}(\int p_0(x) e^{-\frac{(|x-y|-\beta t)^2}{2t}} dx + \beta\sqrt{t} \int p_0(x) \int_{\frac{|x-y|}{\sqrt{t}}-\beta\sqrt{t}}^\infty e^{-\frac{z^2}{2}}\,dz\,dx) \\
&\le \frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}}\,dx + \beta.
\end{aligned}$$

□

Remark 3.3.3. *If $p_0 \in L^\infty(\mathbb{R})$, the above calculation shows that*

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \le 2\|p_0\|_{L^\infty(\mathbb{R})} + \beta.$$

If $p_0 \in L^q(\mathbb{R}), q > 1$, Hölder's inequality leads to

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \le \frac{\|p_0\|_{L^q(\mathbb{R})}}{\sqrt{2\pi t}}(\int e^{-q\frac{(|y-x|-\beta t)^2}{2t}} dx)^{1/q'} \le \frac{C_q t^{\frac{1}{2q'}}}{\sqrt{t}} = \frac{C_q}{t^{\frac{1}{2q}}}.$$

3.4 A non-linear McKean–Vlasov–Fokker–Planck equation

Proposition 3.4.1. *Let $T > 0$. Assume $p_0 \in L^1(\mathbb{R})$, $b \in L^\infty([0,T]\times\mathbb{R})$ and Hypothesis (H). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ be a weak solution to (3.3) until T. Then,*

1. *The marginals $(p_t)_{t\in[0,T]}$ satisfy in the sense of the distributions the mild equation*

$$\forall t \in (0,T], \quad p_t = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s(b(s,\cdot) + B(s,\cdot\,;p)))ds. \tag{3.15}$$

2. *Equation* (3.15) *admits at most one solution* $(p_t)_{t\in[0,T]}$ *which for any* $t \in [0,T]$ *belongs to* $L^1(\mathbb{R})$ *and satisfies* (3.5).

Proof. We successively prove (3.15) and the uniqueness of its solution in $L^1(\mathbb{R})$.

1. Now, for $f \in C_b^2(\mathbb{R})$ consider the Cauchy problem

$$\begin{cases} \frac{\partial G}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(s,x) = 0, \quad 0 \le s < t, \quad x \in \mathbb{R}, \\ \lim_{s\to t^-} G(s,x) = f(x). \end{cases} \tag{3.16}$$

The function

$$G_{t,f}(s,x) = \int f(y) g_{t-s}(x-y)dy$$

is a smooth solution to (3.16). Applying Itô's formula we get

$$\begin{aligned} G_{t,f}(t,X_t) - G_{t,f}(0,X_0) &= \int_0^t \frac{\partial G_{t,f}}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial G_{t,f}}{\partial x}(s,X_s)(b(s,X_s) + B(s,X_s;p))ds \\ &+ \int_0^t \frac{\partial G_{t,f}}{\partial x}(s,X_s)dWs + \frac{1}{2}\int_0^t \frac{\partial^2 G_{t,f}}{\partial x^2}(s,X_s)ds. \end{aligned}$$

Using (3.16) we obtain

$$\mathbb{E}f(X_t) = \mathbb{E}G_{t,f}(0,X_0) + \int_0^t \mathbb{E}\left[\frac{\partial G_{t,f}}{\partial x}(s,X_s)(b(s,X_s) + B(s,X_s;p))\right] ds =: I + II. \tag{3.17}$$

On the one hand, one has

$$I = \int\int f(y) g_t(y-x)dy\ p_0(x)dx = \int f(y)(g_t * p_0)(y)dy.$$

On the second hand, one has

$$\begin{aligned} II &= \int_0^t \int \frac{\partial}{\partial x}\big[\int f(y) g_{t-s}(x-y)dy\big](b(s,x) + B(s,x;p))p_s(x)dxds \\ &= \int_0^t \int\int f(y)\frac{\partial g_{t-s}}{\partial x}(x-y)dy(b(s,x) + B(s,x;p))p_s(x)dxds \\ &= -\int f(y)\int_0^t [\frac{\partial g_{t-s}}{\partial x} * ((b(s,\cdot) + B(s,\cdot;p))p_s)](y)dsdy. \end{aligned}$$

Thus (3.17) can be written as

$$\int f(y)p_t(y)dx = \int f(y)(g_t * p_0)(y)dy - \int f(y)\int_0^t [\frac{\partial g_{t-s}}{\partial x} * ((b(s,\cdot) + B(s,\cdot;p))p_s)](y)dsdy,$$

which is the mild equation (3.15).

2. Assume p_t^1 and p_t^2 are two mild solutions in the sense of the distributions to (3.15) which satisfy

$$\exists C>0, \forall t \in (0,T], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} + \|p_t^2\|_{L^\infty(\mathbb{R})} \le \frac{C_T}{\sqrt{t}}.$$

Then,

$$\begin{aligned}
\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &\le \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [B(s,\cdot\,;p^1)p_s^1 - B(s,\cdot\,;p^2)p_s^2)\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [b(s,\cdot\,)(p_s^1 - p_s^2)]\|_{L^1(\mathbb{R})} ds \\
&\le \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [(B(s,\cdot;p^1) - B(s,\cdot;p^2))p_s^1]\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [(p_s^1 - p_s^2)B(s,\cdot\,;p^2)]\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [b(s,\cdot\,)(p_s^1 - p_s^2)]\|_{L^1(\mathbb{R})} ds \\
&=: I + II + III.
\end{aligned}$$

As

$$\|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t-s}},$$

the convolution inequality (3.37) and Remark 3.2.2 lead to

$$II \le \int_0^t \|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})} \|(p_s^1 - p_s^2)B(s,\cdot;p^2)\|_{L^1(\mathbb{R})} ds \le C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

As b is bounded, we also have

$$|III| \le C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

We now turn to I. Notice that

$$\|B(s,\cdot;p^1) - B(s,\cdot;p^2)\|_{L^1(\mathbb{R})} \le \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau,$$

from which, since by hypothesis (p_t) satisfies (3.5), one has

$$\begin{aligned}
I &\le \int_0^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})} \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau \ ds \\
&\le \int_0^t \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} \int_\tau^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \ d\tau.
\end{aligned}$$

In addition, using Hypothesis (H-4),

$$\int_\tau^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \le \frac{1}{\sqrt{\tau}} \int_\tau^t \frac{1}{\sqrt{t-s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds = \frac{1}{\sqrt{\tau}} \int_0^{t-\tau} \frac{\|K_s\|_{L^1(\mathbb{R})}}{\sqrt{t-\tau-s}} ds \le \frac{C_T}{\sqrt{\tau}}.$$

It comes:

$$I \leq C_T \int_0^t \frac{\|p^1_\tau - p^2_\tau\|_{L^1(\mathbb{R})}}{\sqrt{\tau}} d\tau.$$

Gathering the preceding estimates we obtain

$$\|p^1_t - p^2_t\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p^1_s - p^2_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p^1_s - p^2_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Applying a Singular Gronwall Lemma (see Lemma 3.4.2 below), we conclude

$$\forall t \in (0,T], \quad \|p^1_t - p^2_t\|_{L^1(\mathbb{R})} = 0,$$

which ends the proof. □

In the above proof we have used the following result:

Lemma 3.4.2. *Let $(u(t))_{t\geq 0}$ be a non-negative bounded function such that for a given $T>0$, there exists a positive constant C_T such that for any $t \in (0,T]$:*

$$u(t) \leq C_T \int_0^t \frac{u(s)}{\sqrt{s}} ds + C_T \int_0^t \frac{u(s)}{\sqrt{t-s}} ds. \tag{3.18}$$

Then, $u(t) = 0$ for any $t \in (0,T]$.

Proof. The relation in (3.18) reduces to

$$u(t) \leq 2C_T \sqrt{t} \int_0^t \frac{u(s)}{\sqrt{s}\sqrt{t-s}} ds.$$

Iterating the preceding expression, one gets

$$u(t) \leq (2C_T)^2 \sqrt{t} \int_0^t \frac{\sqrt{s}}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{u(r)}{\sqrt{s-r}\sqrt{r}} dr\ ds.$$

Fubini's theorem leads to

$$u(t) \leq (2C_T)^2 \sqrt{t} \int_0^t \frac{u(r)}{\sqrt{r}} \int_0^s \frac{1}{\sqrt{t-s}\sqrt{s-r}} ds\ dr.$$

Using the definition of the β-function one arrives to

$$u(t) \leq (2C_T)^2 \sqrt{T} \beta(\frac{1}{2}, \frac{1}{2}) \int_0^t \frac{u(r)}{\sqrt{r}} dr.$$

Now, apply the classical Gronwall lemma to finish the proof. □

3.5 A local existence and uniqueness result for Equation (3.3)

Set

$$D(T) := \int_0^T \int_{\mathbb{R}} |K_t(x)| dx dt < \infty. \tag{3.19}$$

The main result in this section is the following theorem.

Theorem 3.5.1. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $p_0 \in L^1(\mathbb{R})$ and $b \in L^\infty((0,T_0)\times\mathbb{R})$ continuous w.r.t. space variable. Under Hypothesis (H), Equation (3.3) admits a unique weak solution up to T_0.*

Iterative procedure. Consider the following sequence of SDE's. For $k = 1$

$$\begin{cases} dX_t^1 = b(t, X_t^1)\ dt + \Big\{ \int_0^t (K_{t-s} * p_0)(X_t^1) ds \Big\} dt + dW_t, \\ X_0^1 \sim p_0. \end{cases} \tag{3.20}$$

Denote the drift of this equation by $b^1(t, x)$. Supposing that, in the step $k-1$, the one dimensional time marginals of the law of the solution have densities $(p_t^{k-1})_{t \geq 0}$, we define the drift in the step k as

$$b^k(t, x, p^{k-1}) = b(t, x) + B(t, x; p^{k-1}),$$

where B is as in (3.4). The corresponding SDE is

$$\begin{cases} dX_t^k = b^k(t, X_t^k, p^{k-1}) dt + dW_t, \\ X_0^k \sim p_0. \end{cases} \tag{3.21}$$

In order to prove the desired local existence and uniqueness result we set up the non-linear martingale problem related to (3.3).

Definition 3.5.2. *A probability measure $\mathbb{Q}$ on the canonical space $\mathcal{C}([0, T_0]; \mathbb{R})$ equipped with its canonical filtration and a canonical process (w_t) is a solution to the non-linear martingale problem $(MP(p_0, T_0, b))$ if:*

(i) $\mathbb{Q}_0 = p_0$.

(ii) For any $t \in (0, T_0]$, the one dimensional time marginals of $\mathbb{Q}$, denoted by $\mathbb{Q}_t$, have densities q_t w.r.t. Lebesgue measure on $\mathbb{R}$. In addition, they satisfy

$$\forall 0 < t \leq T_0, \quad \|q_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}}. \tag{3.22}$$

(iii) For any $f \in \mathcal{C}_K^2(\mathbb{R})$ the process $(M_t)_{t \leq T_0}$, defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t [\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y) dy d\tau)] du$$

is a $\mathbb{Q}$-martingale.

Notice that the arguments in Remark 3.2.2 justify that all the integrals in the definition of M_t are well defined.

We start with the analysis of Equations (3.20)-(3.21).

Proposition 3.5.3. *Same assumptions as in Theorem 3.5.1. Then, for any $k \geq 1$, Equations (3.20)-(3.21) Equations (3.20)-(3.21) admit unique weak solutions up to T_0. For $k \geq 1$, denote by $\mathbb{P}^k$ the law of $(X_t^k)_{t \leq T_0}$. Moreover, for $t \in (0, T_0]$, the time marginals $\mathbb{P}_t^k$ of $\mathbb{P}^k$ have densities p_t^k w.r.t. Lebesgue measure on $\mathbb{R}$. Setting $\beta^k = \sup_{t \leq T_0} \|b^k(t, \cdot, p^{k-1})\|_{L^\infty(\mathbb{R})}$ and $b_0 := \|b\|_{L^\infty([0,T_0] \times \mathbb{R})}$, one has*

$$\forall 0 < t \leq T_0, \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0, T_0)}{\sqrt{t}} \quad \textit{and} \quad \beta^k \leq C(b_0, T_0).$$

Finally, there exists a function $p^\infty \in L^\infty([0,T_0];L^1(\mathbb{R}))$ *such that*

$$\sup_{t\le T_0}\|p_t^k - p_t^\infty\|_{L^1(\mathbb{R})} \to 0, \ \text{as } k\to\infty.$$

Moreover,

$$\forall 0<t\le T_0, \quad \|p_t^\infty\|_{L^\infty(\mathbb{R})} \le \frac{C(b_0,T_0)}{\sqrt{t}}. \tag{3.23}$$

Proof. We proceed by induction.

Case $k=1$**.** In view of (H-5), one has $\beta^1 \le b_0 + C_{T_0}$. This implies that the equation (3.20) has a unique weak solution in $[0,T_0]$ with time marginal densities $(p_t^1)_{t\le T_0}$ which in view of Corollary 3.3.2 satisfy

$$\forall t\in(0,T_0], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi t}} + \beta^1.$$

Case $k>1$**.** Assume now that the equation for X^k has a unique weak solution and assume β^k is finite. In addition, suppose that the one dimensional time marginals satisfy

$$\forall t\in(0,T_0], \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi t}} + \beta^k.$$

In view of (H-4), the new drift satisfies

$$\begin{aligned}|b^{k+1}(t,x;p^k)| &\le b_0 + \int_0^t \|p_s^k\|_{L^\infty(\mathbb{R})}\|K_{t-s}\|_{L^1(\mathbb{R})}ds \le b_0 + \int_0^t (\frac{1}{\sqrt{2\pi s}} + \beta^k)\|K_{t-s}\|_{L^1(\mathbb{R})}ds \\ &\le b_0 + C_{T_0} + \beta^k D(T_0).\end{aligned}$$

Thus, we conclude that $\beta^{k+1} \le b_0 + C_{T_0} + \beta^k D(T_0)$. Therefore, there exists a unique weak solution to the equation for X^{k+1}. Furthermore, by Corollary 3.3.2:

$$\forall t\in(0,T_0], \quad \|p_t^{k+1}\|_{L^\infty(\mathbb{R})} \le \frac{C_{T_0}}{\sqrt{t}} + \beta^{k+1}.$$

Notice that

$$\forall k>1, \quad \beta^{k+1} \le b_0 + C_{T_0} + \beta^k D(T_0) \quad \text{and} \quad \beta^1 \le b_0 + C_{T_0}.$$

Thus, as $D(T_0)<1$, iterating the previous relation we have

$$\forall k\ge 1, \quad \beta^k \le \frac{b_0 + C_{T_0}}{1-D(T_0)} + b_0 + C_{T_0} \tag{3.24}$$

and

$$\|p_t^k\|_{L^\infty(\mathbb{R})} \le \frac{C_{T_0}}{\sqrt{t}} + \beta^k \le \frac{C_{T_0}}{\sqrt{t}} + \frac{b_0 + C_{T_0}}{1-D(T_0)} + b_0 + C_{T_0}. \tag{3.25}$$

Finally, it remains to prove that the sequence p^k converges in $L^\infty([0,T_0];L^1(\mathbb{R}))$. In order to do so, we will prove p^k is a Cauchy sequence. As the space is Banach's, the convergence will follow.

Applying the same procedure as in Section 3.4, one can derive the mild equation for $(p_t^k)_{t\in[0,T_0]}$. Thus, for every $k \geq 1$, the marginals $(p_t^k)_{t\in(0,T_0]}$ satisfy the mild equation

$$\forall t \in (0,T], \quad p_t^k = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s,\cdot,p^{k-1}))ds \tag{3.26}$$

in the sense of the distributions. Assume for a moment that we have proved that for any $0 < t \leq T_0$, one has

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C_{T_0} \int_0^t \frac{\|p_s^{k-1} - p_s^{k-2}\|_{L^1(\mathbb{R})}}{\sqrt{s}}\, ds. \tag{3.27}$$

Remember that $\int_0^t f(u_1)\cdots\int_0^{u_{k-1}} f(u_k)du_k \ \ldots du_1 = \frac{1}{k!}\left(\int_0^t f(u)du\right)^k$ for any positive integrable function f. Then, iterating (3.27) one gets,

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq 2\frac{(C_{T_0}\sqrt{t})^{k-1}}{(k-1)!}.$$

Therefore, $\sup_{t\leq T_0} \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \to 0$, as $k \to \infty$ as desired.

It remains to prove the inequality (3.27). In the sequel $C(T_0) > 0$ will denote a constant that depends on T_0 and may change from line to line. In view of (3.26), one has

$$\begin{aligned}
\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} &\leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s,\cdot,p^{k-1}) - p_s^{k-1} b^{k-1}(s,\cdot,p^{k-2}))\|_{L^1(\mathbb{R})}\, ds \\
&\leq \int_0^t \frac{1}{\sqrt{t-s}} \|b^{k-1}(s,\cdot,p^{k-2})(p_s^k - p_s^{k-1})\|_{L^1(\mathbb{R})}\, ds \\
&\quad + \int_0^t \frac{1}{\sqrt{t-s}} \|(b^k(s,\cdot,p^{k-1}) - b^{k-1}(s,\cdot,p^{k-2}))p_s^k\|_{L^1(\mathbb{R})}\, ds \\
&=: I + II.
\end{aligned} \tag{3.28}$$

According to (3.24), one has

$$I \leq C(T_0) \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}}\, ds.$$

According to (3.25), one has

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-u} * (p_u^{k-1} - p_u^{k-2})\|_{L^1(\mathbb{R})}\, du\, ds.$$

Convolution inequality (3.37) and Fubini-Tonelli's theorem lead to

$$II \leq C(T_0) \int_0^t \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-u}\|_{L^1(\mathbb{R})}\, ds\, du.$$

Apply the change of variables $t - s = s'$. It comes,

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_0^{t-u} \frac{1}{\sqrt{s'}} \|K_{t-u-s'}\|_{L^1(\mathbb{R})}\, ds'\, du.$$

According to (H-4) one has

$$II \leq C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \, du.$$

Coming back to (3.28) and using our above estimates on I and II, we obtain

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C(T_0) \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} \, ds + C(T_0) \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \, du.$$

We are in the situation

$$\Phi(t) := \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq A(t) + C \int_0^t \frac{\Phi(s)}{\sqrt{t-s}} \, ds,$$

where $A(t) \geq 0$ is a bounded increasing function. Iterate this relation and use the monotonicity of A. It comes

$$\Phi(t) \leq C_T A(t) + C^2 \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{\Phi(u)}{\sqrt{s-u}} \, du \, ds.$$

Apply Fubini's theorem to get

$$\Phi(t) \leq C_T A(t) + C^2 \int_0^t \Phi(u) \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}} \, ds \, du.$$

Notice that $\int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}} \, ds = \int_0^1 \frac{1}{\sqrt{1-x}\sqrt{x}} \, dx$. Now, apply Gronwall's lemma to get (3.27) and the convergence of p^k to p^∞.

In order to obtain (3.23), fix $t \in (0, T]$ and use (3.25) and the fact that the convergence in $L^1(\mathbb{R})$ implies the almost sure convergence of a subsequence. $\square$

Corollary 3.5.4. *Same assumptions as in Proposition 3.5.3. Assume that $(\mathbb{P}^k)_{k\geq 1}$ admits a weakly convergent subsequence $(\mathbb{P}^{n_k})_{k\geq 1}$. Denote its limit by $\mathbb{Q}$. Then for any $t \in (0, T_0]$, one has that $\mathbb{Q}_t(dx) = p_t^\infty(x)dx$, where p^∞ is constructed in Proposition 3.5.3.*

Proof. Let $f \in C_K^\infty(\mathbb{R})$. Then by weak convergence,

$$< f, \mathbb{Q}_t >= \lim_{k\to\infty} < f, p_t^{n_k} >=< f, p_t^\infty > + \lim_{k\to\infty} < f, p_t^{n_k} - p_t^\infty > .$$

In view of Proposition 3.5.3, one has

$$\lim_{k\to\infty} | < f, p_t^{n_k} - p_t^\infty > | \leq \|f\|_{L^\infty(\mathbb{R})} \lim_{k\to\infty} \|p_t^{n_k} - p_t^\infty\|_{L_1(\mathbb{R})} = 0.$$

Thus, $< f, \mathbb{Q}_t >=< f, p_t^\infty >$ which completes the proof. $\square$

Proposition 3.5.5. *Same assumptions as in Theorem (3.5.1). Then,*

1) *The family of probabilities $(\mathbb{P}^k)_{k>1}$ is tight.*

2) *Any weak limit $\mathbb{P}^\infty$ of a convergent subsequence of $(\mathbb{P}^k)_{k\geq 1}$ solves $(MP(p_0, T_0, b))$.*

Proof. In view of (3.24), we obviously have

$$\exists C_{T_0} > 0, \quad \sup_k \mathbb{E}|X_t^k - X_s^k|^4 \le C_{T_0}|t-s|^2, \quad \forall\, 0 \le s \le t \le T_0.$$

This is a sufficient condition for tightness (see e.g. [45, Chap.2, Pb.4.11]).

Let $(\mathbb{P}^{n_k})$ be a weakly convergent subsequence of $(\mathbb{P}^k)_{k\ge1}$ and let $\mathbb{P}^\infty$ denote its limit. Let us check that $\mathbb{P}^\infty$ solves the martingale problem $(MP(p_0, T_0, b))$. To simplify the notation, we below write $\mathbb{P}^k$ instead of $\mathbb{P}^{n_k}$ and $\bar{p}^{k-1}$ instead of p^{n_k-1}.

i) Each $\mathbb{P}_0^k$ has density p_0, and therefore $\mathbb{P}_0^\infty$ also has density p_0.

ii) Corollary 3.5.4 implies that the time marginals of $\mathbb{P}^\infty$ are absolutely continuous with respect to Lebesgue's measure and satisfy (3.22).

iii) Set

$$M_t := f(w_t) - f(w_0) - \int_0^t \big[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u,w_u) + \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u)d\tau)\big]du,$$

We have to prove

$$\mathbb{E}_{\mathbb{P}^\infty}[(M_t - M_s)\phi(w_{t_1},\dots,w_{t_N})] = 0, \quad \forall \phi \in C_b(\mathbb{R}^N) \text{ and } 0 \le t_1 < \cdots < t_N < s \le t \le T_0, N \ge 1.$$

The process

$$M_t^k := f(w_t) - f(x(0)) - \int_0^t \big[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)(b(u,w_u) + \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)d\tau\big)\big]du$$

is a martingale under $\mathbb{P}^k$. Therefore, it follows that

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(w_{t_1},\dots,w_{t_N})] \\
&= \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(w_t) - f(w_s))] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u)du] \\
&+ \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)b(u,w_u)du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\; d\tau\; du].
\end{aligned}$$

Since $(\mathbb{P}^k)$ weakly converges to $\mathbb{P}^\infty$, the first two terms on the r.h.s. obviously converge. Now, observe that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\; d\tau\; du] \\
&- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * p_\tau^{\infty})(w_u)\; d\tau\; du] \\
&= \big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\; d\tau\; du] \\
&- \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * p_\tau^{\infty})(w_u)\; d\tau\; du]\big) \\
&+ \big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * p_\tau^{\infty})(w_u)\; d\tau\; du] \\
&- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \frac{\partial f}{\partial x}(w_u)\int_0^u (K_{u-\tau} * p_\tau^{\infty})(w_u)\; d\tau\; du]\big) \\
&=: I + II.
\end{aligned}$$

Now, in view of (3.25) and the definition of $D(T)$ as in (3.19), one has

$$
\begin{aligned}
|I| &\leq \|\phi\|_{L^\infty(\mathbb{R})} \int_s^t \int_0^u \int |\frac{\partial f}{\partial x}(x)||(K_{u-\tau} * (\bar{p}_\tau^{k-1} - p_\tau^\infty))(x)|p_u^k(x) dx \; d\tau \; du \\
&\leq \|\phi\|_{L^\infty(\mathbb{R})} \|\frac{\partial f}{\partial x}\|_{L^\infty(\mathbb{R})} \int_s^t \frac{C_{T_0}}{\sqrt{u}} \int_0^u \|K_{u-\tau}\|_{L^1(\mathbb{R})} \|\bar{p}_\tau^{k-1} - p_\tau^\infty\|_{L^1(\mathbb{R})} d\tau \; du \\
&\leq C_{T_0} D(T_0) \|\phi\|_{L^\infty(\mathbb{R})} \|\frac{\partial f}{\partial x}\|_{L^\infty(\mathbb{R})} \sup_{r \leq T_0} \|\bar{p}_r^{k-1} - p_r^\infty\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

Proposition 3.5.3 implies that $I \to 0$ as $k \to \infty$.

Now, to prove that $II \to 0$, it suffices to prove that the functional $F : \mathcal{C}([0, T_0]; \mathbb{R}) \to \mathbb{R}$ defined by

$$
w_\cdot \mapsto \phi(w_{t_1}, \ldots, w_{t_N}) \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) \; dy \; d\tau \; du
$$

is continuous. Let (w^n) a sequence converging in $\mathcal{C}([0, T_0]; \mathbb{R})$ to w. Since ϕ is a continuous function, it suffices to show that

$$
\begin{aligned}
&\lim_{n \to \infty} \int_s^t \frac{\partial f}{\partial x}(w_u^n) \int_0^u \int K_{u-\tau}(w_u^n - y) p_\tau^\infty(y) \; dy \; d\tau \; du \\
&= \int_s^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) p_\tau^\infty(y) \; dy \; d\tau \; du.
\end{aligned}
\tag{3.29}
$$

For $(u, \tau) \in [s, t] \times [0, t]$, set

$$
h_{u,\tau}(x) := \mathbb{1}\{\tau < u\} \frac{\partial f}{\partial x}(x_u) \int K_{u-\tau}(x - y) p_\tau^\infty(y) dy.
$$

The hypothesis (H-2) implies the continuity of $h_{u,\tau}$ on $\mathbb{R}$. Furthermore,

$$
|h_{u,\tau}(x)| \leq C \mathbb{1}\{\tau < u\} \|p_\tau^\infty\|_{L^\infty(\mathbb{R})} \|K_{u-\tau}\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{\tau}} \mathbb{1}\{\tau < u\} \|K_{u-\tau}\|_{L^1(\mathbb{R})}.
$$

In view of (H-4), we apply the theorem of dominated convergence to conclude (3.29). This ends the proof.

□

Proof of Theorem 3.5.1: Proposition 3.5.5 implies the existence of a weak solution $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ to (3.3) up to time T_0. Thus, the marginals $\mathbb{P} \circ X_t^{-1} =: p_t$ satisfy $\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$, $t \in (0, T_0]$. In addition, as $|B(t, x; p)| \leq C(T_0)$, one has that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is the unique weak solution of the linear SDE

$$
d\tilde{X}_t = b(t, \tilde{X}_t) dt + B(t, \tilde{X}_t; p) dt + dW_t, \quad t \leq T_0. \tag{3.30}
$$

Now suppose that there exists another weak solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ to (3.3) up to T_0 and denote $\hat{\mathbb{P}} \circ \hat{X}_t^{-1}(dx) = \hat{p}_t(x) dx$. By Proposition 3.4.1 we have $\hat{p}_t = p_t$, for $t \leq T_0$. Therefore, $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ is a weak solution to (3.30), from which $\hat{\mathbb{P}} \circ \hat{X}^{-1} = \mathbb{P} \circ X^{-1}$.

3.6 Proof of Theorem 3.2.3: A global existence and uniqueness result for Equation (3.3)

We now construct a solution for an arbitrary time horizon $T > 0$. We will do it by restarting the equation after the time horizon T_0 fixed in the previous section. We start with $T = 2T_0$. Then, we will see how to generalize this procedure for an arbitrary $T > 0$.

Throughout this section, we denote by Ω_0 the canonical space $\mathcal{C}([0,T_0];\mathbb{R})$ and by $\mathcal{B}_0$ its Borel σ-field. We denote by $\mathbb{Q}^1$ the probability distribution of the unique weak solution to (3.3) up to T_0 constructed in the previous section.

3.6.1 Solution on $[0, 2T_0]$

Proposition 3.6.1. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $p_0 \in L^1(\mathbb{R})$ and let $b \in L^\infty([0,2T_0]\times\mathbb{R})$ be continuous w.r.t. the space variable. Under the hypothesis (H), Equation (3.3) admits a unique weak solution up to $2T_0$.*

We start with analyzing the dynamics of (3.3) after T_0 and informally explaining the construction of a solution between T_0 and $2T_0$. Assume, for a while, that Proposition 3.6.1 holds true. Denote the density of X_t by p^1_t, for $t \leq T_0$ and by p^2_t, for $t \in (T_0, 2T_0]$. Let $t \geq 0$. In view of Equation (3.3), we would have

$$X_{T_0+t} = X_{T_0} + \int_{T_0}^{T_0+t} b(s,X_s)\ ds + \int_{T_0}^{T_0+t}\int_0^s (K_{s-\theta} * p_\theta)(X_s)\ d\theta\ ds + W_{T_0+t} - W_{T_0}.$$

Observe that

$$\int_{T_0}^{T_0+t}\int_0^s (K_{s-\theta} * p_\theta)(X_s)\ d\theta\ ds = \int_{T_0}^{T_0+t}\int_0^{T_0} (K_{s-\theta} * p^1_\theta)(X_s) d\theta ds\ + \int_{T_0}^{T_0+t}\int_{T_0}^{s} (K_{s-\theta} * p^2_\theta)(X_s)\ ds\ dt$$
$$=: B_1 + B_2.$$

The change of variables $s - T_0 = s'$ leads to

$$B_1 = \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p^1_\theta)(X_{T_0+s'})\ d\theta\ ds'$$

and, in combination with $\theta - T_0 = \theta'$, to

$$B_2 = \int_0^t \int_{T_0}^{T_0+s'} (K_{T_0+s'-\theta} * p^2_\theta)(X_{T_0+s'}) d\theta ds' = \int_0^t \int_0^{s'} (K_{s'-\theta'} * p^2_{T_0+\theta'})(X_{T_0+s'})\ d\theta'\ ds'.$$

Now set $Y_t := X_{T_0+t}$ and $\tilde{p}_t(y) := p^2_{T_0+t}(y)$. Consider the new Brownian motion $\overline{W}_t := W_{T_0+t} - W_{T_0}$. It comes:

$$Y_t = Y_0 + \int_0^t b(s+T_0, Y_s) ds + \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p^1_\theta)(Y_s)\ d\theta\ ds + \int_0^t \int_0^s (K_{s'-\theta'} * \tilde{p}_\theta)(Y_s)\ d\theta\ ds + \overline{W}_t,$$

for $t \in [0, T_0]$. Setting

$$b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x) ds \quad \text{and} \quad \tilde{b}(t, x) := b(T_0 + t, x),$$

we have

$$\begin{cases} dY_t = \tilde{b}(t, Y_t)dt + b_1(t, Y_t, T_0)dt + \left\{ \int_0^t (K_{t-s} * \tilde{p}_s)(Y_t) ds \right\} dt + d\overline{W}_t, \ t \leq T_0, \\ Y_0 \sim p_{T_0}^1(y)dy, \ Y_s \sim \tilde{p}_s(y)dy. \end{cases} \tag{3.31}$$

To prove Proposition 3.6.1 we construct a weak solution to (3.31) on $[0, T_0]$ and suitably paste its probability distribution with $\mathbb{Q}^1$. We then prove that the so defined measure solves the desired non-linear martingale problem. Notice that the SDE (3.31) is of the same type as (3.3).

Lemma 3.6.2. *Same assumptions as in Proposition 3.6.1. Denote by p_t^1 the time marginals of $\mathbb{Q}^1$. Set $b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x) ds$ and $\tilde{b}(t, x) := b(T_0 + t, x)$. Then, Equation (3.31) admits a unique weak solution up to T_0.*

Proof. Let us check that we can apply Theorem 3.5.1 to (3.31).

Firstly, by construction the initial law $p_{T_0}^1$ of Y satisfies the assumption of Theorem 3.5.1. Secondly, the role of the additional drift b is now played by the sum of the two linear drifts, $\tilde{b}$ and b_1. By hypothesis, $\tilde{b}$ is bounded in $[0, T_0] \times \mathbb{R}$ and continuous in the space variable. Using (3.5) and (H-6) we conclude that b_1 is bounded uniformly in t and x since

$$|b_1(t, x, T_0)| \leq C_{T_0} \int_0^{T_0} \frac{\|K_{T_0+t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds < C_{T_0}.$$

To show that $b_1(t, x, T_0)$ is continuous w.r.t. x we use (H-2) and proceed as at the end of the proof of Proposition 3.5.5.

We now are in a position to apply Theorem 3.5.1: There exists a unique weak solution to (3.31) up to T_0.

□

Denote by $\mathbb{Q}^2$ the probability distribution of the process $(Y_t, t \leq T_0)$. Notice that $\mathbb{Q}^2$ is the solution to the martingale problem $(MP(p_{T_0}^1, T_0, \tilde{b} + b_1))$.

A new measure on $\mathcal{C}([0, 2T_0]; \mathbb{R})$. Let $\mathbb{Q}^1$, $\mathbb{Q}^2$ and (p_t^1) be as above. Let (p_t^2) denote the time marginal densities of $\mathbb{Q}^2$. In particular, $\mathbb{Q}_0^2 = \mathbb{Q}_{T_0}^1$, i.e. $p_0^2(z)dz = p_{T_0}^1(z)dz$. Define the mapping X^0 from Ω_0 to $\mathbb{R}$ as $X^0(w) := w_0$. Using [45, Thm.3.19, Chap.5] to justify the introduction of regular conditional probabilities, for each $y \in \mathbb{R}$ we define the probability measure $\mathbb{Q}_y^2$ on $(\Omega_0, \mathcal{B}_0)$ by

$$\forall A \in \mathcal{B}_0, \quad \mathbb{Q}_y^2(A) = \mathbb{P}^2(A | X^0 = y).$$

In particular,

$$\mathbb{Q}_y^2(w \in \Omega_0, w_0 = y) = 1.$$

We now set $\Omega := \mathcal{C}([0, 2T_0]; \mathbb{R})$. For $w^1_\cdot, w^2_\cdot \in \Omega_0$ we define the concatenation $w = w^1 \otimes_{T_0} w^2 \in \Omega$ of these two paths as the function in Ω defined by

$$\begin{cases} w_\theta = w^1_\theta, & 0 \le \theta \le T_0, \\ w_{\theta+T_0} = w^1_{T_0} + w^2_\theta - w^2_0, & 0 \le \theta \le t - T_0. \end{cases}$$

On the other hand, for a given path $w \in \Omega$, the two paths $w^1_\cdot, w^2_\cdot \in \Omega_0$ such that $w = w^1 \otimes_{T_0} w^2$ are

$$\begin{cases} w^1_\theta = w_\theta, & 0 \le \theta \le T_0, \\ w^2_\theta = w_{T_0+\theta}, & 0 \le \theta \le T_0. \end{cases}$$

We define the probability distribution $\mathbb{Q}$ on Ω equipped with its Borel σ–field as follows. For any continuous and bounded functional φ on Ω,

$$\mathbb{E}_{\mathbb{Q}}[\varphi] = \int_\Omega \varphi(w)\ \mathbb{Q}(dw) := \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2)\ \mathbb{Q}^2_y(dw^2)\ p^1_{T_0}(y)\ dy\ \mathbb{Q}^1(dw^1). \tag{3.32}$$

Notice that if φ acts only on the part of the path up to $t \le T_0$ of any $w_\cdot \in \Omega$, then

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_\theta)_{\theta\le t})] = \int_{\Omega_0} \varphi((w_\theta)_{\theta\le t})\ \mathbb{Q}^1(dx) = \mathbb{E}_{\mathbb{Q}^1}[\varphi((w_\theta)_{\theta\le t})]. \tag{3.33}$$

Proof of Proposition 3.6.1. Let us prove that the probability measure $\mathbb{Q}$ solves the non–linear martingale problem $(MP(p_0, 2T_0, b))$ on the canonical space $\mathcal{C}([0, 2T_0]; \mathbb{R})$.

i) By (3.33), it is obvious that $\mathbb{Q}_0 = \mathbb{Q}^1_0$. By construction, $\mathbb{Q}^1_0$ has density p_0.

ii) Next, let us characterize the one dimensional time marginals of $\mathbb{Q}$. For $f \in C_b(\mathbb{R})$ and $t \in [0, 2T_0]$, consider the functional $\varphi(w) = f(w_t)$, for any $x \in \mathcal{C}([0, 2T_0]; \mathbb{R})$. For $t \le T_0$, by (3.33),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\Omega_0} f(w_t)\ \mathbb{Q}^1(dx) = \int_{\mathbb{R}} f(z) p^1_t(z)\ dz.$$

Therefore, $\mathbb{Q}_t(dz) = p^1_t(z)dz$.

For $T_0 \le t \le 2T_0$, by (3.32),

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\Omega_0} \int_{\mathbb{R}} \int_{\Omega_0} f(w^2_{t-T_0})\ \mathbb{Q}^2_y(dw^2)\ p^1_{T_0}(y)\ dy\ \mathbb{Q}^1(dw^1) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) \mathbb{Q}^2_{y,t-T_0}(dz)\ p^1_{T_0}(y)\ dy.$$

By Fubini's theorem:

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \mathbb{Q}^2_{y,t-T_0}(dz)\ p^1_{T_0}(y)\ dy.$$

Since $\mathbb{Q}^2_0 = p^1_{T_0}$ we deduce

$$\mathbb{E}_{\mathbb{Q}}[\varphi(w)] = \int_{\mathbb{R}} f(z) p^2_{t-T_0}(z)\ dz,$$

which shows that $\mathbb{Q}_t(dz) = p^2_{t-T_0}(z)dz$.

Therefore, the one dimensional marginals of $\mathbb{Q}$ have densities q_t which, by construction, belong to $L^\infty(\mathbb{R})$ and satisfy $\|q_t\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{t}}$.

iii) It remains to show that for $f \in \mathcal{C}_K^2(\mathbb{R})$, the process M_t defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t \Big[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u) + \frac{\partial f}{\partial x}(w_u)\Big(b(u,w_u) + \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y)\, dy\, d\tau\Big)\Big] du$$

is a $\mathbb{Q}$-martingale, i.e. $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_s) = M_s$.

(a) Let $s \le t \le T_0$:
For any $n \in N$, any continuous bounded functional ϕ on $\mathbb{R}^n$, and any $t_1 \le \cdots \le t_n \le s \le t \le T_0$, by (3.33):

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \ldots, w_{t_n})(M_t - M_s)) = \mathbb{E}_{\mathbb{Q}^1}(\phi(w_{t_1}, \ldots, w_{t_n})(M_t - M_s)).$$

As $\mathbb{Q}^1$ solves $(MP(p_0, T_0, b))$ up to T_0,

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \ldots, w_{t_n})(M_t - M_s)) = 0.$$

(b) For $s \le T_0 \le t \le 2T_0$,

$$\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_s) = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_{T_0})|\mathcal{B}_s].$$

Let us prove that $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_{T_0}) = M_{T_0}$. Notice that

$$M_t - M_{T_0} = f(w_t) - f(w_{T_0}) - \int_{T_0}^t \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u) du - \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) b(u, w_u)\, du$$
$$- \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y)\, dy\, d\tau\, du.$$

Write the last integral as

$$\int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^u \int K_{u-\tau}(w_u - y) q_\tau(y)\, dy\, d\tau\, du$$
$$= \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_0^{T_0} \int K_{u-\tau}(w_u - y) p^1_\tau(y)\, dy\, d\tau\, du$$
$$+ \int_{T_0}^t \frac{\partial f}{\partial x}(w_u) \int_{T_0}^u \int K_{u-\tau}(w_u - y) p^2_{\tau - T_0}(y)\, dy\, d\tau\, du =: I_1 + I_2.$$

Now,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^{T_0} \int K_{u+T_0-\tau}(w_{u+T_0} - y) p^1_\tau(y)\, dy\, d\tau\, du.$$

For $w \in \Omega$ identify $w^1, w^2 \in \Omega_0$ such that $w = w^1 \otimes_{T_0} w^2$. Then,

$$I_1 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w^2_u) \int_0^{T_0} (K_{u+T_0-\tau} * p^1_\tau)(w^2_u)\, d\tau\, du = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w^2_u) b_1(u, w^2_u, T_0) du.$$

Proceeding as above,

$$I_2 = \int_0^{t-T_0} \frac{\partial f}{\partial x}(w_{u+T_0}) \int_0^u \int K_{u-\tau}(w_{u+T_0} - y) p^2_\tau(y)\, dy\, d\tau\, du$$
$$= \int_0^{t-T_0} \frac{\partial f}{\partial x}(w^2_u) \int_0^u \int K_{u-\tau}(w^2_u - y) p^2_\tau(y)\, dy\, d\tau\, du.$$

Similarly

$$\int_{T_0}^{t}\frac{\partial f}{\partial x}(w_u)b(u,w_u)du=\int_0^{t-T_0}\frac{\partial f}{\partial x}(w_{u+T_0})b(u+T_0,w_{u+T_0})du$$
$$=\int_0^{t-T_0}\frac{\partial f}{\partial x}(w_u^2)b(u+T_0,w_u^2)du=\int_0^{t-T_0}\frac{\partial f}{\partial x}(w_u^2)\tilde{b}(u,w_u^2)du.$$

It comes:

$$M_t-M_{T_0}=f(w_{t-T_0}^2)-f(w_0^2)-\int_0^{t-T_0}\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u^2)du-\int_0^{t-T_0}\frac{\partial f}{\partial x}(w_u^2)\tilde{b}(u,w_u^2)du$$
$$-\int_0^{t-T_0}\frac{\partial f}{\partial x}(w_u^2)\big[b_1(u,w_u^2,T_0)+\int_0^u\int K_{u-\tau}(w_u^2-y)p_\tau^2(y)\ dy\ d\tau\big]du=:F(w^2).$$

By definition of the measure $\mathbb{Q}$,

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1},\dots w_{t_n})(M_t-M_{T_0}))=$$
$$\int_{\Omega_0}\phi(w_{t_1}^1,\dots,w_{t_n}^1)\int_{\mathbb{R}}\int_{\Omega_0}F(w^2)\mathbb{Q}_y^2(dw^2)p_{T_0}^1(y)\ dy\ \mathbb{Q}^1(dw^1).$$

By the definition of $\mathbb{Q}^2$:

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1},\dots w_{t_n})(M_t-M_{T_0}))=\int_{\Omega_0}\phi(w_{t_1}^1,\dots,w_{t_n}^1)\int_{\Omega_0}F(w^2)\ \mathbb{Q}^2(dw^2)\ \mathbb{Q}^1(dw^1).$$

As $\mathbb{Q}^2$ solves $(MP(p_{T_0}^1,T_0,\tilde{b}+b_1))$, one has

$$\mathbb{E}_{\mathbb{Q}^2}(F)=\int_{\Omega_0}F(w^2)\ \mathbb{Q}^2(dw^2)=0.$$

Finally, we conclude that $\mathbb{E}_{\mathbb{Q}}(M_t|B_{T_0})=M_{T_0}$ and therefore $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_s)=M_s$ for all $s\le T_0\le t\le 2T_0$.

(c) For $T_0\le s\le t\le 2T_0$: we may rewrite the difference M_t-M_s in the same manner:

$$M_t-M_s=f(w_{t-T_0}^2)-f(w_{s-T_0}^2)-\int_{s-T_0}^{t-T_0}\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(w_u^2)du$$
$$-\int_{s-T_0}^{t-T_0}\frac{\partial f}{\partial x}(w_u^2)\left[b(u,w_u^2))+b_1(u,w_u^2,T_0)+\int_0^u\int K_{u-\tau}(w_{u+T_0}-y)p_\tau^2(y)\ dy\ d\tau\right]du$$
$$=:F(w^2).$$

Now, take $t_1\le\dots\le t_n<s$. Let us suppose that the first m are before T_0 and others after. We have that

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1},\dots w_{t_n})(M_t-M_s))=$$
$$\int_{\Omega_0}\int_{\mathbb{R}}\int_{\Omega_0}F(w_{t_1}^1,\dots,w_{t_m}^1,w_{t_{m+1}-T_0}^2,\dots,w_{t_n-T_0}^2)\varphi(w^2)\ \mathbb{Q}_y^2(dw^2)\ p_{T_0}^1(y)\ dy\ \mathbb{Q}^1(dw^1).$$

Since $\mathbb{Q}^2$ solves $(MP(p^1_{T_0}, T_0, \tilde{b} + b_1))$, one has that $\mathbb{E}_{\mathbb{Q}^2}(\phi'(w^2_{t'_1}, \dots w^2_{t'_n})F) = 0$ for any continuous bounded functional ϕ' on $\mathbb{R}^n$, any $n \in N$ and any $t'_1 \leq \dots \leq t'_n < s - T_0$. Taking $\phi'(w^2_{t'_1}, \dots w^2_{t'_n}) = \phi(w^1_{t_1}, \dots, w^1_{t_m}, w^2_{t_{m+1}-T_0}, \dots, w^2_{t_n-T_0})$ for a fixed x^1, we conclude that

$$\int_{\mathbb{R}} \int_{\Omega_0} \phi(w^1_{t_1}, \dots, w^1_{t_m}, w^2_{t_{m+1}-T_0}, \dots, w^2_{t_n-T_0}) \varphi(w^2)\ \mathbb{Q}^2_y(dw^2)\ p^1_{T_0}(y)\ dy = 0.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}(\phi(w_{t_1}, \dots w_{t_n})(M_t - M_s)) = 0.$$

Thus, $\mathbb{E}^{\mathbb{Q}}(M_t|\mathcal{B}_s) = M_s$ for $T_0 \leq s \leq t \leq 2T_0$.

To summarize the preceding, we have just shown the existence of a solution to $(MP(p_0, 2T_0, b))$. Finally, we proceed as in the proof of Theorem 3.5.1 to deduce the existence and uniqueness of a weak solution to (3.3) up to $2T_0$.

3.6.2 End of the proof of Theorem 3.2.3: construction of the global solution

Given any finite time horizon $T > T_0$, split the interval $[0, T]$ into $n = [\frac{T}{T_0}] + 1$ intervals of length not exceeding T_0 ($[\frac{T}{T_0}]$ denoting the integer part of $\frac{T}{T_0}$) and repeat n times the procedure used in the preceding subsection.

Remark 3.6.3. *Using similar arguments as above one can construct a solution to* (3.3) *when the initial condition p_0 is in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ or, respectively, $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. In these cases we use Remark 3.3.3 in the iterative procedure. Consequently, the weak solution is unique under the constraint that the one dimensional marginal densities $(p_t)_{t \leq T}$ belong to $L^\infty((0, T); L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$ or, respectively, satisfy*

$$\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{t^{1/4}}.$$

3.7 Application to the one-dimensional Keller–Segel model

In this section we prove Corollary 3.2.6. We start with checking that $K^\sharp$ satisfies Hypothesis (H). The condition (H-1) is satisfied since for $t > 0$ one has

$$\|K^\sharp_t\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t}} \int |z| e^{-\frac{z^2}{2}} dz.$$

From the definition of $K^\sharp$ it is clear that for $t > 0$, $K^\sharp_t$ is a bounded and continuous function on $\mathbb{R}$. The condition (H-3) is also obviously satisfied. As already noticed,

$$\|K^\sharp_{t-s}\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t-s}},$$

from which,

$$f_1(t) := \int_0^t \frac{\|K^\sharp_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds = C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} ds = C \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}} dx = C,$$

where C is a universal constant. Now let ϕ be a probability density on $\mathbb{R}$. For $(t,x) \in (0,T] \times \mathbb{R}$, one has

$$\begin{aligned}\int \phi(y) \|K^{\sharp}_{\cdot}(x-y)\|_{L^1(0,t)}\ dy &\leq C \int \phi(y)|x-y| \int_0^t \frac{1}{s^{3/2}} e^{-\frac{|x-y|^2}{2s}} ds\ dy \\ &= \int \phi(y)|x-y| \int_{\frac{|x-y|}{\sqrt{t}}}^{\infty} \frac{z^3}{|x-y|^3} e^{-\frac{z^2}{2}} \frac{|x-y|^2}{z^3} dz\ dy \\ &= \int \phi(y) \int_{\frac{|x-y|}{\sqrt{t}}} e^{-\frac{z^2}{2}} dz\ dy.\end{aligned}$$

This shows that (H-5) is satisfied. Finally, to prove (H-6) we notice that for every $t \in [0,T]$

$$\int_0^T \|K^{\sharp}_{T+t-s}\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}}\ ds \leq \int_0^T \frac{C}{\sqrt{T+t-s}\sqrt{s}} ds \leq C \int_0^T \frac{1}{\sqrt{T-s}\sqrt{s}} ds = C.$$

Therefore, in view of Theorem 3.2.3, Equation (3.2) with $d=1$ is well-posed.[1]

Denote by $\rho(t,x)$ the time marginals of the constructed probability distribution. Now, define the function c as in (3.9). In view of Inequality (3.6), for any $t \in (0,T]$ the function $c(t,\cdot)$ is well defined and bounded continuous. Let us show that $c \in L^\infty([0,T]; C^1_b(\mathbb{R}))$.

We have

$$\frac{\partial c}{\partial x}(t,x) = \frac{\partial}{\partial x}\left(e^{-\lambda t}\mathbb{E}(c_0(x+W_t)\right) + \frac{\partial}{\partial x}\left(\mathbb{E}\int_0^t e^{-\lambda s}\rho(t-s, x+W_s) ds\right).$$

Then observe that

$$\begin{aligned}\mathbb{E}\int_0^t e^{-\lambda s}\rho(t-s,x+W_s)ds &= \int_0^t e^{-\lambda s}\int \rho(t-s,x+y)\frac{1}{\sqrt{2\pi s}} e^{\frac{-y^2}{4s}}\ dy\ ds \\ &= \int_0^t e^{-\lambda(t-s)}\int \rho(s,y)\frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-(y-x)^2}{4(t-s)}}\ dy\ ds \\ &=: \int_0^t f(s,x) ds.\end{aligned}$$

As for any $0 < s < t$

$$\left|\frac{\partial}{\partial x}\frac{1}{\sqrt{t-s}} e^{\frac{-(y-x)^2}{2(t-s)}}\right| \leq \frac{|y-x|}{2(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} \leq \frac{C}{t-s},$$

we have

$$\frac{\partial f}{\partial x}(s,x) = e^{-\lambda(t-s)}\int \rho(s,y)\frac{y-x}{2\sqrt{2\pi}(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} dy.$$

Now, we repeat the same argument for $\frac{\partial}{\partial x}\int_0^t f(s,x)ds$. In order to justify the differentiation under the integral sign we notice that

$$\left|\frac{\partial f}{\partial x}(s,x)\right| \leq \frac{C_T}{\sqrt{(t-s)s}}.$$

[1] With similar calculations as for f_1, one easily checks that the function f_2 is bounded on any compact time interval. Thus, Corollary 3.2.4 applies as well as Theorem 3.2.3.

Gathering the preceding calculations we have obtained

$$\frac{\partial c}{\partial x}(t,x) = e^{-\lambda t}\mathbb{E}c_0'(x+W_t) + \int_0^t e^{-\lambda(t-s)} \int \rho_s(y) \frac{y-x}{\sqrt{2\pi}(t-s)^{3/2}} e^{\frac{-(y-x)^2}{2(t-s)}} dy\ ds.$$

Using the assumption on c_0 and Inequality (3.5), for any $t \in (0,T]$ one has

$$\|\frac{\partial c}{\partial x}(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|c_0'\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition, the preceding calculation and Lebesgue's Dominated Convergence Theorem show that $\frac{\partial c}{\partial x}(t,\cdot)$ is continuous on $\mathbb{R}$. We thus have obtained the desired property.

The above discussion shows that we are in a position to apply Proposition 3.4.1 with $b(t,x) \equiv \chi e^{-\lambda t}\mathbb{E}c_0'(x+W_t)$ and $B(t,x;\rho)$ defined as in (3.4) with $K \equiv K^\sharp$: the function $\rho(t,x)$ satisfies (3.8) in the sense of the distributions. Therefore, it is a solution to the Keller Segel system (3.7) in the sense of Definition 3.2.5. We now check the uniqueness of this solution.

Assume there exists another solution ρ^1 satisfying Definition 3.2.5 with the same initial condition as ρ. For notation convenience, in the calculation below we set $c_t(x) := c(t,x)$, $c_t^1(x) := c^1(t,x)$, $\rho_t(x) := \rho(t,x)$, and $\rho_t^1(x) := \rho^1(t,x)$.

Using Definition 3.2.5,

$$\begin{aligned}
&\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} \le \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * (\frac{\partial c_s^1}{\partial x}\rho_s^1 - \frac{\partial c_s}{\partial x}\rho_s)\|_{L^1(\mathbb{R})} ds \\
&\le \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * (\frac{\partial c^1}{\partial x}(\rho_s^1 - \rho_s))\|_{L^1(\mathbb{R})} ds + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * (\rho_s(\frac{\partial c^1}{\partial x} - \frac{\partial c_s}{\partial x}))\|_{L^1(\mathbb{R})} ds \\
&=: I + II.
\end{aligned}$$

Using standard convolution inequality (3.37) and $\|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})} \le \frac{C}{\sqrt{t-s}}$ we deduce:

$$I \le C\int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds \quad \text{and} \quad II \le C\int_0^t \frac{\|\frac{\partial c_s^1}{\partial x} - \frac{\partial c_s}{\partial x}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}\sqrt{s}} ds.$$

Therefore

$$\|\frac{\partial c_s^1}{\partial x} - \frac{\partial c_s}{\partial x}\|_{L^1(\mathbb{R})} \le \int_0^s \|(\rho_u^1 - \rho_u) * \frac{\partial g_{s-u}}{\partial x}\|_{L^1(\mathbb{R})} du \le C\int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}} du, \tag{3.34}$$

from which

$$\begin{aligned}
II &\le C\int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}}\ du\ ds \\
&\le C\int_0^t \|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{s}\sqrt{s-u}\sqrt{t-s}}\ ds\ du \le C_T \int_0^t \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{u}} du.
\end{aligned}$$

Gathering the preceding bounds for I and II we get

$$\|p_t^1 - p_t\|_{L^1(\mathbb{R})} \le C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Lemma 3.4.2 implies that $\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} = 0$ for every $t \le T$. In view of (3.34) we also have $\|\frac{\partial c_t^1}{\partial x} - \frac{\partial c_t}{\partial x}\|_{L^1(\mathbb{R})} = 0$. This completes the proof of Corollary 3.2.6.

3.8 Appendix A

We here propose a light simplification of the calculations in [66].

Proposition 3.8.1. *Let $y \in \mathbb{R}$ and let β be a constant. Denote by $p_y^\beta(t,x,z)$ the transition probability density (with respect to the Lebesgue measure) of the unique weak solution to*

$$X_t = x + \beta \int_0^t sgn(y - X_s)\ ds + W_t.$$

Then

$$\begin{aligned} p_y^\beta(t,x,z) &= \frac{1}{\sqrt{2\pi} t^{3/2}} \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z-y|)-\frac{\beta^2}{2}t} (\bar{y} + |z-y| + |y-x|) e^{-\frac{(\bar{y}+|z-y|+|y-x|)^2}{2t}}\, d\bar{y} \\ &+ \frac{1}{\sqrt{2\pi t}} e^{\beta(|y-x|-|z-y|)-\frac{\beta^2}{2}t} (e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(|z-y|+|y-x|)^2}{2t}}). \end{aligned} \tag{3.35}$$

In particular,

$$p_y^\beta(t,x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}}\ dz. \tag{3.36}$$

Proof. Let f be a bounded continuous function. The Girsanov transform leads to

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x+W_t) e^{\beta \int_0^t \operatorname{sgn}(y-x-W_s)dW_s - \frac{\beta^2}{2}t}).$$

Let L_t^a be the Brownian local time. By Tanaka's formula ([45], p. 205):

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

Therefore for $a = y - x$

$$\int_0^t \operatorname{sgn}(y - x - W_s) dW_s = |y-x| + L_t^a - |W_t - (y-x)|,$$

from which

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x+W_t) e^{\beta(|y-x| + L_t^{y-x} - |W_t-(y-x)|) - \frac{\beta^2}{2}t}).$$

recall that (W_t, L_t^a) has the following joint distribution (see [10, p.200,Eq.(1.3.8)]:

$$\begin{cases} \bar{y} > 0: \quad \mathbb{P}(W_t \in dz, L_t^a \in d\bar{y}) = \frac{1}{\sqrt{2\pi} t^{3/2}} (\bar{y} + |z-a| + |a|) e^{-\frac{(\bar{y}+|z-a|+|a|)^2}{2t}}\ d\bar{y}\ dz. \\ \mathbb{P}(W_t \in dz, L_t^a = 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|z-a|+|a|)^2}{2t}} dz. \end{cases}$$

It comes:

$$\begin{aligned} \mathbb{E}(f(X_t)) = \frac{1}{\sqrt{2\pi} t^{3/2}} \int_{\mathbb{R}} \int_0^\infty & f(x+z) e^{\beta(|y-x|+\bar{y}-|z-(y-x)|)-\frac{\beta^2}{2}t} (\bar{y} + |z-(y-x)| + |y-x|) \\ & e^{-\frac{(\bar{y}+|z-(y-x)|+|y-x|)^2}{2t}}\ d\bar{y}\ dz \\ + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} & f(x+z) e^{\beta(|y-x|-|z-(y-x)|)-\frac{\beta^2}{2}t} (e^{-\frac{z^2}{2t}} - e^{-\frac{(|z-(y-x)|+|y-x|)^2}{2t}})\ dz. \end{aligned}$$

The change of variables $x+z=z'$ leads to

$$\mathbb{E}(f(X_t)) = \frac{1}{\sqrt{2\pi}t^{3/2}}\int_{\mathbb{R}} f(z')\int_0^\infty e^{\beta(|y-x|+\bar{y}-|z'-y|)-\frac{\beta^2}{2}t}(\bar{y}+|z'-y)|+|y-x|)e^{-\frac{(\bar{y}+|z'-y|+|y-x|)^2}{2t}}\,d\bar{y}\,dz'$$
$$+\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} f(z')e^{\beta(|y-x|-|z'-y|)-\frac{\beta^2}{2}t}(e^{-\frac{(z'-x)^2}{2t}}-e^{-\frac{(|z'-y|+|y-x|)^2}{2t}})dz',$$

from which the desired result follows.

□

In the next corollary we use the same notation as in the proof of Theorem 3.3.1.

Corollary 3.8.2. *Let $0<s<t\le T$. Then for any $z,y\in\mathbb{R}$, there exists $C_{T,\beta,x,y}$ such that*

$$\mathbb{E}|\left(\frac{\partial}{\partial x}p_y^\beta\right)(t-s,X_s^{(b)},z)|\le C_{T,\beta,x,y}h(s,z),$$

where h belongs to $L^1([0,t]\times\mathbb{R})$.

Proof. By Girsanov's theorem, for some constant $C_{T,\beta}$ we have

$$\mathbb{E}\left|\left(\frac{\partial}{\partial x}p_y^\beta\right)(t-s,X_s^{(b)},z)\right|\le C_{T,\beta}\sqrt{\mathbb{E}\left|\left(\frac{\partial}{\partial x}p_y^\beta\right)(t-s,W_s^x,z)\right|^2}.$$

Observe that

$$\frac{\partial}{\partial\bar{x}}p_y^\beta(t-s,\bar{x},z)=\frac{\beta}{\sqrt{2\pi(t-s)}}e^{-2\beta|z-y|}e^{-\frac{(|z-y|+|y-\bar{x}|-\beta(t-s))^2}{2(t-s)}}\operatorname{sgn}(\bar{x}-y)$$
$$+\frac{\beta}{\sqrt{2\pi(t-s)}}e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)}e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}}\operatorname{sgn}(\bar{x}-y)$$
$$+\frac{z-\bar{x}}{2\pi(t-s)^{3/2}}e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)}e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}}.$$

The sum of the absolute values of the first two terms in the right-hand side is bounded from above by

$$\frac{\beta}{\sqrt{2\pi(t-s)}}e^{-2\beta|z-y|+\beta|y-\bar{x}|}.$$

Thus,

$$\mathbb{E}\left|\left(\frac{\partial}{\partial x}p_y^\beta\right)(t-s,X_s^{(b)},z)\right|\le\frac{C_{T,\beta}}{\sqrt{2\pi(t-s)}}\sqrt{\mathbb{E}e^{2\beta|y-W_s^x|}}$$
$$+\frac{C_{T,\beta}}{(t-s)^{3/2}}\sqrt{\mathbb{E}(|z-W_s^x|^2e^{2\beta|y-W_s^x|-\frac{(z-W_s^x)^2}{t-s}})}=:B+A.$$

Notice that

$$A\le\frac{C_{T,\beta}}{(t-s)^{3/2}}(\mathbb{E}[|z-W_s^x|^4e^{-2\frac{(z-W_s^x)^2}{t-s}}]\mathbb{E}[e^{4\beta|y-W_s^x|}])^{1/4}=:\frac{C_{T,\beta}}{(t-s)^{3/2}}(A_1A_2)^{1/4}.$$

Firstly, as there exists an $\alpha > 0$ such that $|a|^4 e^{-a^2} \leq C e^{-\alpha a^2}$, one has

$$A_1 \leq C(t-s)^2 \int e^{-\alpha \frac{(z-u)^2}{t-s}} g_s(u-x) du \leq \frac{(t-s)^{2+1/2}}{\sqrt{s+(t-s)/(2\alpha)}} e^{-\frac{(z-x)^2}{2(s+(t-s)/(2\alpha))}}.$$

Secondly,

$$A_2 = \int e^{4\beta|y-u|} g_s(u-x) du = e^{-4\beta y} \int_y^\infty e^{4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du + e^{4\beta y} \int_{-\infty}^y e^{-4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du$$
$$= e^{4\beta(x-y)} e^{8\beta^2 s} \int_y^\infty \frac{1}{\sqrt{s}} e^{-\frac{(u-x-4\beta s)^2}{2s}} du + e^{4\beta(y-x)} e^{8\beta^2 s} \int_{-\infty}^y \frac{1}{\sqrt{s}} e^{-\frac{(u-x+4\beta s)^2}{2s}} du \leq e^{8\beta^2 s} C_{\beta,x,y}.$$

Therefore,

$$A \leq C_{T,\beta,x,y} \frac{1}{(t-s)^{7/8}} g_{s+(t-s)/(2\alpha)}(z-x).$$

The term B is treated in the similar way as A_2. $\square$

3.9 Appendix B: A reminder on the standard convolution inequalities

We give here the two standard convolution inequalities in their general form, as they are used in the following chapters as well. The following is proven in Brezis [15, Thm. 4.15]:

Lemma 3.9.1 (The convolution inequality)**.** *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ with $l \leq p \leq \infty$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{q} - 1 \geq 0$. Then, $f * g \in L^p(\mathbb{R}^d)$ and*

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}. \tag{3.37}$$

The following is an extension of Lemma 3.37 and it is proven in [15, Thm. 4.33]:

Lemma 3.9.2 (The convolution inequality)**.** *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $l \leq p, q \leq \infty$. Then, $f * g \in L^r(\mathbb{R}^d)$ and*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}. \tag{3.38}$$

Chapter 4

The one-dimensional case: Regularization approach to the non-linear stochastic equation

4.1 Introduction

In this chapter we adopt another approach in proving the well-posedness of the stochastic differential equation related to Keller-Segel model in $d = 1$,

$$\begin{cases} dX_t = (c_0' * g_t)(X_t) + \Big\{ \int_0^t (K_{t-s} * p_s)(X_t) ds \Big\} dt + dW_t, \quad t > 0, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{4.1}$$

where $K_t(x) := \chi e^{-\lambda t} \frac{\partial}{\partial x} (\frac{1}{(2\pi t)^{1/2}} e^{-\frac{x^2}{2t}})$ and $b(t, x) := \chi e^{-\lambda t} \frac{\partial}{\partial x} \mathbb{E} c_0(x + W_t)$. Namely, we regularize the interaction kernel K and prove the regularized equation in the limit is (4.1). The goal of this mini-chapter is now to obtain the rate of convergence of the marginal laws of the solution to the regularized equation to the laws of X_t. This is an interesting question on its own when one deals with McKean-Vlasov dynamics through a regularization procedure and it will involve Sobolev regularity of a whole class of probability density functions. The latter will be obtained by the help of heat kernel estimates in Strook and Varadhan [70].

From now on we will, in addition to ρ_0 is a probability density function and $c_0 \in \mathcal{C}_b^1(\mathbb{R})$, suppose that $\rho_0 \in L^\infty(\mathbb{R})$. This will smoothen out in time the $L^\infty(\mathbb{R})$-norm estimates of ρ_t and enable us to get the rate of convergence. Namely, if $\rho_0 \in L^\infty(\mathbb{R})$, one has $\|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$ (see Remark 3.6.3). As seen in the previous chapter, the parameter $\chi > 0$ plays no role in the mathematical analysis of the problem. We will, thus, assume $\chi = 1$.

A convenient regularization: For an $\varepsilon > 0$, define $K_t^\varepsilon(x) := \frac{-x}{\sqrt{2\pi}(t+\varepsilon)^{\frac{3}{2}}} e^{-\frac{x^2}{2t}}$ and

$$\begin{cases} dX_t^\varepsilon = dW_t + (c_0' * g_t)(X_t^\varepsilon) + \int_0^t (K_{t-s}^\varepsilon * \rho_s^\varepsilon)(X_t^\varepsilon) \, ds \, dt, \quad t \leq T, \\ X_0^\varepsilon \sim \rho_0, \; X_t^\varepsilon \sim \rho_t^\varepsilon. \end{cases} \tag{4.2}$$

We denote the drift of (4.2) by $b^\varepsilon(t, x; \rho^\varepsilon)$ and by $b(t, x; \rho)$ the drift of (4.1). The well-posedness of (4.2) is due to Theorem 2.2.3. However, to get more information about the one-dimensional marginals of the law of this process, one needs to apply Theorem 3.2.3 and Remark 3.6.3. It is easy to check that K^ε satisfies Hypothesis (H). For example, notice that for any $1 \leq p < \infty$,

$$\|K_t^\varepsilon\|_{L^p(\mathbb{R})} \leq \frac{C_p}{t^{1-\frac{1}{2p}}},$$

where the constants C_p do not depend on the regularization parameter ε. Thus, assuming that $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $c_0 \in \mathcal{C}_b^1(\mathbb{R})$, one has the following estimates:

$$\forall 0 < t \leq T: \quad \|\rho_t^\varepsilon\|_{L^\infty(\mathbb{R})} \leq C, \quad \text{and} \quad \|b^\varepsilon(t, \cdot; \rho^\varepsilon)\| \leq C.$$

Thus, one has $\|\rho_t^\varepsilon\|_{L^2(\mathbb{R})} \leq C$. Since the L^1-norm of the regularized kernel does not depend on ε, neither do the above constants. This implies that for any sequence $\varepsilon_k \to 0$ as $k \to \infty$, one has the tightness of (X^{ε_k})'s w.r.t. $k \geq 1$. It is easy to check that the solution to the martingale problem corresponding to (4.2) converges to the one related to (4.1). Let $\mathbb{P}^\infty$ be a weak limit of a converging subsequence $(\mathbb{P}^k)$ of the laws of (X^{ε_k}). We will prove $\mathbb{P}^\infty$ solves the NLMP related to (4.1). We place ourselves in the context of Proposition 3.5.5 and adopt the notation from its proof to this setting. In order not to repeat ourselves, we will not check all the requirements here, but we will quickly review the most interesting details.

Define the functional $T_t(g)$ by

$$T_t(g) := \int g(y)\mathbb{P}_t^\infty(dy), \quad g \in C_K(\mathbb{R}).$$

By weak convergence we have

$$T_t(g) = \lim_{k\to\infty} \int g(y)p_t^k(y)dy,$$

and thus

$$|T_t(g)| \leq C\|g\|_{L^2(\mathbb{R})}.$$

Therefore, for each $0 < t \leq T_0$, T_t is a bounded linear functional on a dense subset of $L^2(\mathbb{R})$. Thus, T_t can be extended to a linear functional on $L^2(\mathbb{R})$. By Riesz-representation theorem (e.g. [15, Thm. 4.11 and 4.14]), there exists a unique $p_t^\infty \in L^2(\mathbb{R})$ such that

$$\forall 0 < t \leq T: \quad \|p_t^\infty\|_{L^2(\mathbb{R})} \leq C$$

and p_t^∞ is the probability density of $\mathbb{P}_t^\infty(dy)$.

In order to prove that

$$E_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(x(u))b^{\varepsilon_k}(u, x(u); \rho^{\varepsilon_k})\, du] \to \mathbb{E}_{\mathbb{P}}[\phi(\dots)\int_s^t f'(x(u))b(u, x(u); \rho)\, du], \text{ as } k \to \infty,$$

one decomposes their difference into:

$$I = E_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(x(u))b(u, x(u); \rho^{\varepsilon_k})\, du] - E_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(x(u))b(u, x(u); \rho)\, du]$$

and

$$II = E_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(x(u))b(u, x(u); \rho)\, du] - \mathbb{E}_{\mathbb{P}}[\phi(\dots)\int_s^t f'(x(u))b(u, x(u); \rho)\, du].$$

Convergence of II is due to the continuity of the functional inside the expectation. This has been already proven in the proof of Proposition 3.5.5. Then, convergence of I is obtained in two steps: firstly, one proves that for a fixed $u \in [s, t]$, one has that $|b^{\varepsilon_k}(u, x; \rho^{\varepsilon_k}) - b(u, x(u); \rho)| \to 0$, as $k \to \infty$, secondly one bounds this difference by an integrable function of u independent of k. The conclusion will follow by dominated convergence. The integrable bound comes from the density and kernel estimates. We now check the first step. Notice that

$$b(u,\cdot;\rho) - b^{\varepsilon_k}(u,\cdot;\rho^{\varepsilon_k}) = \int_0^s \left(K_{u-r} * (\rho_r - \rho_r^{\varepsilon_k}) + (K_{u-r} - K_{u-r}^{\varepsilon_k}) * \rho_r^{\varepsilon_k}\right) dr.$$

For $r < u$, K_{u-r} is a continuous and bounded function on $\mathbb{R}$. By weak convergence,

$$|K_{u-r} * (\rho_r - \rho_r^{\varepsilon_k})| \to 0, \text{ as } k \to \infty.$$

In addition,

$$|K_{u-r} * (\rho_r - \rho_r^{\varepsilon_k})| \le C\|K_{u-r}\|_{L^2(\mathbb{R})} \le \frac{C}{(u-r)^{\frac{1}{4}}}.$$

By dominated convergence, $|\int_0^s K_{u-r} * (\rho_r - \rho_r^{\varepsilon_k})\, du| \to 0$, as $k \to \infty$.
It remains to check that

$$\left|\int_0^s (K_{u-r} - K_{u-r}^{\varepsilon_k}) * \rho_r^{\varepsilon_k}\, dr\right| \to 0, \text{ as } k \to \infty.$$

Now, for $r < u$, $|K_{u-r}^{\varepsilon_k}(x) - K_{u-r}(x)| \to 0$, as $k \to \infty$. We can apply dominated convergence as the following bound is integrable in $(0,s) \times \mathbb{R}$:

$$|(K_{u-r}^{\varepsilon_k}(x(u)-y) - K_{u-r}(x(u)-y))\rho_r^{\varepsilon_k}(y)| \le C|K_{u-r}(x(u)-y)|.$$

Finally, this concludes the first step and as well the convergence of I. Therefore, the martingale problems converge and we obtain the existence of a solution to the NLMP related to (4.1). As uniqueness holds $\mathbb{P}^\infty$ is the law of the process X.

The plan is the following: in the Section 4.2, we prove the above mentioned Sobolev estimates and in the Section 4.3, we prove that the rate of convergence of ρ_t^ε towards ρ_t in $L^1(\mathbb{R})$-norm is of order $\sqrt{\varepsilon}$. Let $p > 1$. The following notation will be used:

$$\begin{aligned} L^p((0,T);W_p^1(\mathbb{R})) := \Big\{ & u \in L^p((0,T)\times\mathbb{R}) | \exists\, h \in L^p((0,T)\times\mathbb{R}) \text{ such that} \\ & \int_{(0,T)\times\mathbb{R}} u(t,x)\frac{\partial}{\partial x}\phi(t,x)dtdx = -\int_{(0,T)\times\mathbb{R}} h(t,x)\phi(t,x)dtdx, \forall \phi \in C_c^\infty((0,T)\times\mathbb{R})\Big\}. \end{aligned}$$

4.2 Sobolev regularity of a certain class of probability densities

In the sequel, we will prove the result about the speed of convergence of ρ_t^ε to ρ_t by analyzing the mild equations they satisfy. In order to find the rate of convergence of $b^\varepsilon(t,\cdot;\rho^\varepsilon)$ towards $b(t,\cdot,\rho)$, we will need some Sobolev regularity for ρ_t^ε and ρ_t. More generally, in this section we are interested in diffusion processes in $d = 1$ with bounded and measurable drift and a constant diffusion coefficient σ. Without loss of generality, we will assume $\sigma = 1$. Let $T > 0$, define

$$\begin{cases} dX_t = b(t,X_t)dt + dW_t, & t \le T, \\ X_0 \sim \rho_0 \end{cases} \tag{4.3}$$

Suppose that $\sup_{t\le T}\|b(t,\cdot)\|_\infty < \infty$ and $\rho_0 \in L^1 \cap L^\infty(\mathbb{R})$. In that case, Equation (4.3) admits a unique weak solution (see [45], p. 327). In addition, by Girsanov theorem, the one dimensional time marginals of the solution are absolutely continuous with respect to Lebesgue measure and since the drift is bounded they are as well uniformly bounded in time and space (see Section 3.3). Let us denote with ρ_t the density of X_t. Then $\rho := (\rho_t)_{0\leq t\leq T} \in L^p((0,T)\times\mathbb{R})$, for any $p \geq 1$.

One derives the mild equation satisfied by ρ_t as in Section 3.4,

$$\rho_t = g_t * \rho_0 - \int_0^t (\frac{\partial}{\partial x} g_{t-s} * (\rho_s b_s))ds, \tag{4.4}$$

where $g_t(x)$ denotes centered one dimensional Gaussian density with variance equal to t. We will prove the following theorem:

Theorem 4.2.1. *Let $1 < p < 2$ and $p' > 2$ its conjugate. Assume that $\rho_0 \in L^1 \cap L^\infty(\mathbb{R})$. Then, $\rho \in L^p((0,T); W^1_p(\mathbb{R}))$ and*

$$\|\frac{\partial}{\partial x}\rho\|_{L^p((0,T)\times\mathbb{R})} \leq C(t, b, \rho_0).$$

The following is the estimate in Strook and Varadhan [71, p. 315]:

Lemma 4.2.2. *Denote by S_d the set of $d\times d$ symmetric matrices. Let $c : [0,\infty) \to S_d$ be a measurable function for which there exists $0 < \lambda < \Lambda < \infty$ with the property*

$$\lambda|\theta|^2 \leq \langle\theta, c(t)\theta\rangle \leq \Lambda|\theta|^2, \quad t \geq 0 \quad \text{and} \quad \theta \in \mathbb{R}^d.$$

Extend c to $\mathbb{R}$ by taking $c(-s) = c(s), s \geq 0$ and set $C(s,t) = \int_s^t c(u)du$ for $s \geq t$. Define

$$g(s,x;t,y) = \mathbb{1}_{\{(t-s)\in(0,\infty)\}} \left[(2\pi)^d \ \det \ C(s,t)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\langle y-x, C(s,t)^{-1}(y-x)\rangle}{2}\right\}$$

on $(\mathbb{R}\times\mathbb{R}^d)^2$. For $f \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d)$, define

$$Gf(s,x) = \int_s^\infty \int_{\mathbb{R}^d} g(s,x;t,y) f(t,y)dy \ dt$$

and

$$G^\star f(t,y) = \int_{-\infty}^t \int_{\mathbb{R}^d} g(s,x;t,y) f(s,x)dx \ ds.$$

Then for all $1 < p < \infty$ and for all $1 \leq i,j \leq d$:

$$\left\|\frac{\partial^2 Gf}{\partial x_i \partial x_j}\right\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} \leq C_d(p,\lambda,\Lambda)\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}$$

and

$$\left\|\frac{\partial^2 G^\star f}{\partial y_i \partial y_j}\right\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} \leq C_d(p,\lambda,\Lambda)\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}.$$

Proof of Theorem 4.2.1. Take $f \in C_K^\infty((0,T)\times\mathbb{R})$. Define the linear functional

$$\tilde{T}_p(f) = \int_{(0,T)\times\mathbb{R}} \frac{\partial}{\partial x} f(t,x)\rho_t(x) \ dx \ dt.$$

Assume for a moment that

$$|T_p(f)| \leq C\|f\|_{L^{p'}((0,T)\times\mathbb{R})}. \tag{4.5}$$

Then, $\tilde{T}_p$ is a linear functional continuous for the $L^{p'}((0,T)\times\mathbb{R})$ norm, defined on a dense subspace of $L^{p'}((0,T)\times\mathbb{R})$. Therefore, it extends to a bounded linear functional T_p on $L^{p'}((0,T)\times\mathbb{R})$. By Riesz representation theorems (e.g. [15, Thm. 4.11 and 4.14]), there exists $h \in L^p((0,T)\times\mathbb{R})$ such that for any $f \in L^{p'}((0,T)\times\mathbb{R})$:

$$T_p(f) = \int_{(0,T)\times\mathbb{R}} f(t,x)h_t(x)\ dx\ dt.$$

Denote by $\frac{\partial}{\partial x}\rho_t(x) := -h(x)$. Then it holds for any $f \in \mathcal{C}_K^\infty((0,T)\times\mathbb{R})$:

$$\int_{(0,T)\times R} \frac{\partial}{\partial x} f(t,x)\rho_t(x)\ dx\ dt = -\int_{(0,T)\times R} f(t,x)\frac{\partial}{\partial x}\rho_t(x)\ dx\ dt. \tag{4.6}$$

In addition, $\|\frac{\partial}{\partial x}\rho\|_{L^p((0,T)\times\mathbb{R})} \leq C(t,b,p,p_0)$. As $\rho \in L^p((0,T)\times\mathbb{R})$, the theorem is proved.

It remains to prove the relation in (4.5). Let $f \in C_c^\infty((0,T)\times\mathbb{R})$. Multiply (4.4) by $\frac{\partial}{\partial x}f(t,x)$ and integrate over $(0,T)\times\mathbb{R}$:

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}} \frac{\partial}{\partial x} f(t,x)\rho_t(x)\ dx\ dt &= \int_{(0,T)\times\mathbb{R}} \frac{\partial}{\partial x} f(t,x)(g_t * \rho_0)(x)\ dx\ dt \\ &+ \int_{(0,T)\times\mathbb{R}} \frac{\partial}{\partial x} f(t,x)\int_0^t (\frac{\partial}{\partial x} g_{t-s} * (\rho_s b_s))(x)\ ds\ dx\ dt =: A + B.\end{aligned}$$

It comes down to controlling the terms A and B in terms of $\|f\|_{L^{p'}((0,T)\times\mathbb{R})}$.

Term A: We start by integrating by parts the space integral. Notice that by dominated convergence, one has $\frac{\partial}{\partial x}(g_t * \rho_0)(x) = (\frac{\partial}{\partial x}g_t * \rho_0)(x)$. Therefore,

$$A = -\int_0^T \int_{\mathbb{R}} f(t,x)(\frac{\partial}{\partial x}g_t * \rho_0)(x)\ dx\ dt.$$

Applying Hölder's inequality,

$$|A| \leq \|f\|_{L^{p'}((0,T)\times\mathbb{R})}\|\frac{\partial}{\partial x}g_t * \rho_0\|_{L^p((0,T)\times\mathbb{R})}.$$

Notice that

$$\|\frac{\partial}{\partial x}g_t * \rho_0\|^p_{L^p((0,T)\times\mathbb{R})} = \int_0^T \|\frac{\partial}{\partial x}g_t * \rho_0\|^p_{L^p(\mathbb{R})}dt.$$

As $\|\frac{\partial}{\partial x}g_t\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$ and in view of Convolution inequality (3.37), one has

$$\|\frac{\partial}{\partial x}g_t * \rho_0\|^p_{L^p((0,T)\times\mathbb{R})} \leq \int_0^T \frac{C\|\rho_0\|^p_{L^p(\mathbb{R})}}{t^{p/2}}dt.$$

For any $p > 1$, one has $\rho_0 \in L^p(\mathbb{R})$ since $\rho_0 \in L^1 \cap L^\infty(\mathbb{R})$. Moreover, since $1 < p < 2$ the preceding integral is well defined and we have that

$$\|\frac{\partial}{\partial x}g_t * \rho_0\|^p_{L^p((0,T)\times\mathbb{R})} \leq C(T,p,\|\rho_0\|_{L^p(\mathbb{R})}).$$

Therefore,

$$|A| \leq C(T,p,\|\rho_0\|_{L^p(\mathbb{R})})\|f\|_{L^{p'}((0,T)\times\mathbb{R})}.$$

Term B: This term reads

$$B = \int_0^T \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y) \rho_s(y) b(s,y)\ dy\ ds\ dx\ dt.$$

Here we will use the estimates of Lemma 4.2.2. To do so, we need to rewrite B in a convenient form. Firstly, we conclude below that Fubini's theorem applies to B so that we can change the order of integration as we like. Indeed, b and ρ are uniformly bounded, $\frac{\partial}{\partial x} g_{t-s}$ is integrable in space and time and $f \in \mathcal{C}_K^\infty((0,T)\times\mathbb{R})$. Thus,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left|\frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y) \rho_s(y) b(s,y)\right|\ dy\ ds\ dx\ dt \\
&\le C(b,\rho) \int_0^T \int_{\mathbb{R}} \left|\frac{\partial}{\partial x} f(t,x)\right| \int_0^t \frac{C}{\sqrt{t-s}} ds\ dx\ dt \le C(b,\rho)\sqrt{T} \int_0^T \int_{\mathbb{R}} \left|\frac{\partial}{\partial x} f(t,x)\right|\ dx\ dt < \infty.
\end{aligned}$$

Therefore, we rewrite B as

$$B = \int_0^T \int_{\mathbb{R}} \int_s^T \int_{\mathbb{R}} \frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y)\ dx\ dt\ \rho_s(y) b(s,y)\ dy\ ds.$$

We focus on the inner integrals. Integrate by parts in space and then change the variables:

$$\int_s^T \int_{\mathbb{R}} \frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y)\ dx\ dt = -\int_s^T \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} f(t,x+y) g_{t-s}(x)\ dx\ dt.$$

Notice that $\frac{\partial^2}{\partial x^2} f(t,x+y) = \frac{\partial^2}{\partial y^2} f(t,x+y)$ and since f is regular, the order of integration and derivation can be exchanged. It comes

$$\int_s^T \int_{\mathbb{R}} \frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y)\ dx\ dt = -\frac{\partial^2}{\partial y^2} \int_s^T \int_{\mathbb{R}} f(t,x+y) g_{t-s}(x)\ dx\ dt.$$

After another change of variables,

$$\int_s^T \int_{\mathbb{R}} \frac{\partial}{\partial x} f(t,x) \frac{\partial}{\partial x} g_{t-s}(x-y)\ dx\ dt = -\frac{\partial^2}{\partial y^2} \int_s^T \int_{\mathbb{R}} f(t,x) g_{t-s}(x-y)\ dx\ dt.$$

Therefore,

$$B = -\int_0^T \int_{\mathbb{R}} \rho_s(y) b(s,y) \frac{\partial^2}{\partial y^2} \left[\int_s^T \int_{\mathbb{R}} f(t,x) g_{t-s}(x-y)\ dx\ dt\right]\ dy\ ds.$$

Applying Hölder's inequality,

$$|B| \le C(b) \|\rho\|_{L^p((0,T)\times\mathbb{R})} \left\| \frac{\partial^2}{\partial y^2} \int_\cdot^T \int_{\mathbb{R}} f(t,x) g_{t-\cdot}(x-\cdot)\ dx\ dt \right\|_{L^{p'}((0,T)\times\mathbb{R})}.$$

Now, Lemma 4.2.2 provides a bound for the last term. The framework is the following. For $d=1$, we define $c(t) \equiv 1$. Then $C(t,s) = t-s, s \le t$. The function $g(s,x;t,y)$ in Lemma 4.2.2 is here $g(s,x;t,y) := \mathbb{1}\{s \le t\} g_{t-s}(y-x)$. Define $\tilde{f}(t,x) = \mathbb{1}\{0 \le t \le T\} f(t,x)$. Then,the functional G becomes

$$G\tilde{f}(s,y) = \int_s^\infty \int_{\mathbb{R}} g(s,y;t,x) \tilde{f}(s,y) dx dt = \int_s^T \int_{\mathbb{R}} f(t,x) g_{t-s}(x-y) dx dt.$$

Applying the estimate in Lemma 4.2.2,

$$\|\frac{\partial^2}{\partial y^2}G\tilde{f}\|_{L^{p'}((0,T)\times\mathbb{R})} \leq \|\frac{\partial^2}{\partial y^2}G\tilde{f}\|_{L^{p'}(\mathbb{R}\times\mathbb{R}))} \leq C\|\tilde{f}\|_{L^{p'}(\mathbb{R}\times\mathbb{R}))} = c\|f\|_{L^{p'}((0,T)\times\mathbb{R})}.$$

Finally, this implies

$$|B| \leq C(b,p)\|f\|_{L^{p'}((0,T)\times\mathbb{R})}.$$

Combining the estimates of terms A and B, we obtain (4.5). Thus, the theorem is proved. □

Remark 4.2.3. *The relation (4.6) holds for a wider class of functions f. Namely, by density arguments, it holds as well for $f \in L^{p'}((0,T);W_1^{p'}(\mathbb{R}))$.*

4.3 Rate of convergence

Theorem 4.3.1. *Let $1 < p < 2$, $T > 0$ and $\varepsilon > 0$. Then, for any $t \leq T$ one has*

$$\|\rho_t - \rho_t^\varepsilon\|_{L^p(\mathbb{R})} \leq C_T\sqrt{\varepsilon}.$$

Proof. Remember that by the iterative procedures applied to (4.1) and (4.2), one has that

1. Both drifts are uniformly bounded in time and space. Namely, $b, b^\varepsilon \in L^\infty((0,T)\times\mathbb{R})$.
2. By construction $\rho \in L^\infty((0,T);L^1(\mathbb{R})\cap L^\infty(\mathbb{R}))$ and $\rho^\varepsilon \in L^\infty((0,T);L^1(\mathbb{R})\cap L^\infty(\mathbb{R}))$. In addition, the estimate for ρ^ε is uniform in ε.

We write the mild equations satisfied by ρ and ρ^ε, respectively:

$$\rho_t = g_t * \rho_0 - \int_0^T \frac{\partial g_{t-s}}{\partial x} * (\rho_s b(s,\cdot;\rho))\, ds \text{ and } \rho_t^\varepsilon = g_t * \rho_0 - \int_0^T \frac{\partial g_{t-s}}{\partial x} * (\rho_s^\varepsilon b^\varepsilon(s,\cdot;\rho^\varepsilon))\, ds.$$

Notice that,

$$\|\rho_t - \rho_t^\varepsilon\|_{L^p(\mathbb{R})} \leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * ((\rho_s - \rho_s^\varepsilon)b(s,\cdot;\rho))\|_{L^p(\mathbb{R})}\, ds$$
$$+ \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * ((b(s,\cdot,\rho) - b^\varepsilon(s,\cdot;\rho^\varepsilon))\rho_s^\varepsilon)\|_{L^p(\mathbb{R})}\, ds =: A + B.$$

By the convolution inequality (3.37), one has

$$A \leq \|b\|_\infty \int_0^t \|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})}\|\rho_s - \rho_s^\varepsilon\|_{L^p(\mathbb{R})}\, ds \leq C_b \int \frac{\|\rho_s - \rho_s^\varepsilon\|_{L^p(\mathbb{R})}}{\sqrt{t-s}}\, ds.$$

The difference of the two drifts writes as

$$b(s,\cdot;\rho) - b^\varepsilon(s,\cdot;\rho^\varepsilon) = \int_0^s (K_{s-r} * (\rho_r - \rho_r^\varepsilon) + (K_{s-r} - K_{s-r}^\varepsilon) * \rho_r^\varepsilon)\, dr.$$

Thus, the term B decomposes to

$$B \leq \int_0^t \| \frac{\partial g_{t-s}}{\partial x} * (\rho_s^\varepsilon \int_0^s K_{s-r} * (\rho_r - \rho_r^\varepsilon)\, dr) \|_{L^p(\mathbb{R})}$$
$$+ \| \frac{\partial g_{t-s}}{\partial x} * (\rho_s^\varepsilon \int_0^s (K_{s-r} - K_{s-r}^\varepsilon) * \rho_r^\varepsilon\, dr) \|_{L^p(\mathbb{R})}\, ds =: B_1 + B_2.$$

We first treat B_1. Applying the convolution inequality (3.37) twice and the bounds on ρ_s^ε, one has

$$B_1 \leq C \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \|K_{s-r} * (\rho_r - \rho_r^\varepsilon)\|_{L^p(\mathbb{R})}\, dr\, ds$$
$$\leq C \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{1}{\sqrt{s-r}} \|\rho_r - \rho_r^\varepsilon\|_{L^p(\mathbb{R})}\, dr\, ds.$$

Applying Fubini's theorem,

$$B_1 \leq C \int_0^t \|\rho_r - \rho_r^\varepsilon\|_{L^p(\mathbb{R})} \int_r^t \frac{1}{\sqrt{t-s}\sqrt{s-r}}\, ds\, dr \leq C \int_0^t \|\rho_r - \rho_r^\varepsilon\|_{L^p(\mathbb{R})}\, dr.$$

Let us pass to B_2. This term will give us the rate of convergence. The convolution inequality and the bound on the density lead to

$$B_2 \leq C \int_0^t \frac{1}{\sqrt{t-s}} \| \int_0^s (K_{s-r} - K_{s-r}^\varepsilon) * \rho_r^\varepsilon\, dr \|_{L^p(\mathbb{R})}\, ds.$$

Set

$$F(x) := \int_0^t \int \frac{-(x-y)}{(s-r)^{3/2}} e^{-\frac{(x-y)^2}{2(s-r)}} \rho_r^\varepsilon(y) dy\, dr - \int_0^t \int \frac{-(x-y)}{(s-r+\varepsilon)^{3/2}} e^{-\frac{(x-y)^2}{2(s-r)}} \rho_r^\varepsilon(y) dy\, dr.$$

Notice that Theorem 4.2.1 applies to both ρ and ρ^ε. Moreover, as for a fixed $\varepsilon > 0$ the drift of (4.2) is in $\mathcal{C}_b^\infty(\mathbb{R})$, the density ρ_t^ε is differentiable everywhere (see e.g. Nualart [57]). Notice that,

$$\frac{-(x-y)}{(s-r)^{3/2}} e^{-\frac{(x-y)^2}{2(s-r)}} = C \frac{\partial}{\partial y} g_{s-r}(x-y) \text{ and } \frac{-(x-y)}{(s-r+\varepsilon)^{3/2}} e^{-\frac{(x-y)^2}{2(s-r)}} = C \left(\frac{s-r}{s-r+\varepsilon} \right)^{3/2} \frac{\partial}{\partial y} g_{s-r}(x-y).$$

After an integration by parts

$$F(x) = C \int_0^s \int \left(1 - \left(\frac{s-r}{s-r+\varepsilon} \right)^{3/2} \right) g_{s-r}(x-y) \frac{\partial}{\partial y} \rho_r^\varepsilon(y) dy\, dr.$$

Notice that

$$\left| 1 - \left(\frac{s-r}{s-r+\varepsilon} \right)^{3/2} \right| = \left| \frac{(s-r+\varepsilon)^{3/2} - (s-r+\varepsilon)\sqrt{s-r} + (s-r+\varepsilon)\sqrt{s-r} - (s-r)^{3/2}}{(s-r+\varepsilon)^{3/2}} \right|$$
$$= \left| \frac{\sqrt{s-r+\varepsilon} - \sqrt{s-r}}{\sqrt{s-r+\varepsilon}} + \frac{\varepsilon\sqrt{s-r}}{(s-r+\varepsilon)\sqrt{s-r+\varepsilon}} \right| \leq 2 \frac{\sqrt{\varepsilon}}{\sqrt{s-r}}.$$

Next, we are interested in the $L^p(\mathbb{R})$ norm of F. After the convolution inequality (3.37),

$$\|F\|_{L^p(\mathbb{R})} \leq C\sqrt{\varepsilon} \int_0^s \frac{1}{\sqrt{s-r}} \|g_{s-r} * \frac{\partial}{\partial x} \rho_t^\varepsilon\|_{L^p(\mathbb{R})}\, dr \leq C\sqrt{\varepsilon} \int_0^s \frac{1}{\sqrt{s-r}} \|\frac{\partial}{\partial x} \rho_t^\varepsilon\|_{L^p(\mathbb{R})}\, dr.$$

This implies that

$$B_2 \le C\sqrt{\varepsilon}\int_0^t \frac{1}{\sqrt{t-s}}\int_0^s \frac{1}{\sqrt{s-r}}\|\frac{\partial}{\partial x}\rho_t^\varepsilon\|_{L^p(\mathbb{R})}\ dr\ ds.$$

Fubini's theorem and Hölder's inequality lead to

$$\begin{aligned} B_2 &\le C\sqrt{\varepsilon}\int_0^t \|\frac{\partial}{\partial x}\rho_t^\varepsilon\|_{L^p(\mathbb{R})}\int_r^t \frac{1}{\sqrt{t-s}\sqrt{s-r}}\ ds\ dr = C\sqrt{\varepsilon}\int_0^t \|\frac{\partial}{\partial x}\rho_t^\varepsilon\|_{L^p(\mathbb{R})}\ dr \\ &\le C\sqrt{\varepsilon}t^{\frac{1}{p'}}\left(\int_0^t \|\frac{\partial}{\partial x}\rho_t^\varepsilon\|_{L^p(\mathbb{R})}^p\ dr\right)^{\frac{1}{p}}. \end{aligned}$$

In view of Theorem 4.2.1, one gets

$$B_2 \le C_T\sqrt{\varepsilon}.$$

The constant C_T depends on p_0, T, c_0', but not on ε. Finally, the term B is estimated by

$$B \le C\int_0^t \|\rho_r - \rho_r^\varepsilon\|_{L^p(\mathbb{R})}\ dr + C_T\sqrt{\varepsilon}.$$

The estimates on A and B together lead to

$$\|\rho_t - \rho_t^\varepsilon\|_{L^p(\mathbb{R})} \le C_T\int_0^t \frac{1}{\sqrt{t-s}}\|\rho_s - \rho_s^\varepsilon\|_{L^p(\mathbb{R})}ds + C_T\sqrt{\varepsilon}.$$

It remains to apply Gronwall's lemma in order to finish the proof. □

Chapter 5

The one-dimensional case: Particle system and propagation of chaos

This chapter is the subject of a paper [43] that appeared in *Electronic Communications of Probability*. It is a joint work with Jean Francois Jabir (HSE Moscow) and Denis Talay.

5.1 Introduction

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\frac{1}{2}\nabla\rho - \chi\rho\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\partial_t c(t,x) = \frac{1}{2}\triangle c - \lambda c + \rho, \quad t>0,\ x\in\mathbb{R}^d, \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), \end{cases} \tag{5.1}$$

for some parameters $\chi > 0$, $\lambda \geq 0$ and $\alpha \geq 0$. See Chapter 1 or Perthame [62] and references therein for theoretical results on this system of PDEs and applications to biology. When $\alpha = 0$, the system (5.1) is parabolic–elliptic, and when $\alpha = 1$ (or more generally, when $0 < \alpha \leq 1$), the system is parabolic–parabolic.

For the parabolic–elliptic version of the model with $d = 2$, the first stochastic interpretation of this system is due to Haškovec and Schmeiser [36] who analyze a particle system with McKean–Vlasov interactions and Brownian noise. More precisely, as the ideal interaction kernel should be strongly singular, they introduce a kernel with a cut-off parameter and obtain the tightness of the particle probability distributions w.r.t. the cut-off parameter and the number of particles. They also obtain partial results in the direction of the propagation of chaos. More recently, in the subcritical case, that is, when the parameter χ of the parabolic–elliptic model is small enough, Fournier and Jourdain [31] obtain the well–posedness of a particle system without cut-off. In addition, they obtain a consistency property which is weaker than the propagation of chaos. They also describe complex behaviors of the particle system in the sub and super critical cases. Cattiaux and Pédèches [21] obtain the well-posedness of this particle system without cut-off by using Dirichlet forms rather than pathwise approximation techniques.

For a parabolic–parabolic version of the model with a smooth coupling between ρ_t and c_t, Budhiraja and Fan [17] study a particle system with a smooth time integrated kernel and prove it propagates chaos. Moreover, adding a forcing potential term to the model, under a suitable convexity assumption, they obtain uniform in time concentration inequalities for the particle system and uniform in time error estimates for a numerical approximation of the limit non-linear process.

In Section 1.4 the reader may found more details on [36, 31, 21].

For the pure parabolic–parabolic model without cut-off or smoothing, in the one-dimensional case with $\alpha = 1$, we have proved in Chapter 3 the well-posedness of PDE (5.1) and of the following non-linear SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \Big\{\chi \int_0^t (K_{t-s} \star \rho_s)(X_t)ds\Big\}dt + dW_t, \quad t > 0, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{5.2}$$

where $K_t(x) := e^{-\lambda t}\frac{\partial}{\partial x}(\frac{1}{(2\pi t)^{1/2}}e^{-\frac{x^2}{2t}})$ and $b(t,x) = e^{-\lambda t}\frac{\partial}{\partial x}\mathbb{E}[c_0(x + W_t)]$.

Under the sole condition that the initial probability law $\mathcal{L}(X_0)$ has a density, it is shown that the law $\mathcal{L}(X)$ uniquely solves a non-linear martingale problem and its time marginals have densities. These densities coupled with a suitable transformation of them uniquely solve the one–dimensional parabolic–parabolic Keller–Segel system without cut-off. In Chapter 6 additional techniques are developed for the two-dimensional version of (5.2).

The objective of this chapter is to analyze the particle system related to (5.2). It inherits from the limit equation that at each time $t > 0$ each particle interacts in a singular way with the past of all the other particles. We prove that the particle system is well–posed and propagates chaos to the unique weak solution of (5.2). To the best of our knowledge, this is the first time in the literature that the parabolic-parabolic Keller-Segel system is derived as a limit of a system of interacting stochastic particles, when the number of particles tends to infinity. Compared to the stochastic particle systems introduced for the parabolic–elliptic model, an interesting fact occurs: the difficulties arising from the singular interaction can now be resolved by using purely Brownian techniques rather than by using Bessel processes. Due to the singular nature of the kernel K, we need to introduce a partial Girsanov transform of the N-particle system in order to obtain uniform in N bounds for moments of the corresponding exponential martingale. Our calculation is based on the fact that the kernel K is in $L^1(0, T; L^2(\mathbb{R}))$. Notice that in the case of the multi-dimensional Keller–Segel particle system the $L^1(0, T; L^2(\mathbb{R}^d))$-norm of the kernel is infinite, so these techniques can not be used in higher dimension. For more details see Chapter 7.

The chapter is organized as follows: In Section 5.2 we state our two main results and comment our methodology. In Section 5.3 and Appendix we prove technical lemmas. In Section 5.4 we prove our main results.

In all the chapter we denote by C any positive real number independent of N. Any time C will depend on N or any other parameter that will be explicitly written.

5.2 Main results

Our main results concern the well–posedness and propagation of chaos of

$$\begin{cases} dX_t^{i,N} = \Big\{\frac{1}{N}\sum_{j=1, j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds \ \mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}}\Big\}dt + dW_t^i, \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1 \leq i \leq N), \end{cases} \tag{5.3}$$

where $K_t(x) = \frac{-x}{\sqrt{2\pi}t^{3/2}}e^{-\frac{x^2}{2t}}$ and the W^i's are N independent standard Brownian motions. It corresponds to $\alpha = 1$, $\lambda = 0$, $\chi = 1$, and $c_0' \equiv 0$. It is easy to extend our methodology to (5.2) under the hypotheses made in Chapter 3.

Theorem 5.2.1. *Given $0 < T < \infty$ and $N \in \mathbb{N}$, there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t; 0 \le t \le T), \mathbb{Q}^N, W, X^N)$ to the N-interacting particle system (5.3) that satisfies, for any $1 \le i \le N$,*

$$\mathbb{Q}^N \left(\int_0^T \left(\frac{1}{N} \sum_{j=1, j\neq i}^{N} \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N}) ds \; \mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}} \right)^2 dt < \infty \right) = 1. \tag{5.4}$$

In view of Karatzas and Shreve [45, Chapter 5, Proposition 3.10], one has the following uniqueness result:

Corollary 5.2.2. *Weak uniqueness holds in the class of weak solutions satisfying (5.4).*

The construction of a weak solution to (5.3) involves arguments used by Krylov and Röckner [49, Section 3] to construct a weak solution to SDEs with singular drifts. It relies on the Girsanov transform which removes all the drifts of (5.3).

Remark 5.2.3. *Our construction shows that the law of the particle system is equivalent to Wiener's measure. Thus, a.s. the set $\{t \le T, X_t^{i,N} = X_t^{j,N}\}$ has Lebesgue measure zero.*

Our second main theorem concerns the propagation of chaos of the system (5.3). Before we proceed to its statement, we need to define the non-linear martingale problem (MPKS) associated to the non-linear SDE:

$$\begin{cases} dX_t = \Big\{ \int_0^t (K_{t-s} \star \rho_s)(X_t) ds \Big\} dt + dW_t, \quad t \le T, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx. \end{cases} \tag{5.5}$$

For any measurable space E we denote by $\mathcal{P}(E)$ the set of probability measures on E.

Definition 5.2.4. *$\mathbb{Q} \in \mathcal{P}(C[0,T]; \mathbb{R})$ is a solution to (MPKS) if:*

(i) $\mathbb{Q}_0(dx) = \rho_0(x)\, dx$;

(ii) *For any $t \in (0, T]$, the one dimensional time marginal $\mathbb{Q}_t$ of $\mathbb{Q}$ has a density ρ_t w.r.t. Lebesgue measure on $\mathbb{R}$ which belongs to $L^2(\mathbb{R})$ and satisfies*

$$\exists C_T, \quad \forall\, 0 < t \le T, \quad \|\rho_t\|_{L^2(\mathbb{R})} \le \frac{C_T}{t^{1/4}};$$

(iii) *Denoting by $(x(t); t \le T)$ the canonical process of $C([0,T]; \mathbb{R})$, we have: For any $f \in C_b^2(\mathbb{R})$, the process defined by*

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left(\left(\int_0^s \int K_{s-r}(x(s) - y)\rho_r(y) dy dr \right) f'(x(s)) + \frac{1}{2} f''(x(s)) \right) ds$$

is a $\mathbb{Q}$-martingale w.r.t. the canonical filtration.

In Chapter 3, we have proven that (MPKS) admits a unique solution and that a suitable notion of weak solution to (5.5) is equivalent to the notion of solution to (MPKS) (see Corollary 3.2.4).

Theorem 5.2.5. *Assume that the $X_0^{i,N}$'s are i.i.d. and that the initial distribution of $X_0^{1,N}$ has a density ρ_0. The empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ of (5.3) converges in the distribution sense, when $N \to \infty$, to the unique weak solution of (5.5).*

To prove the tightness and weak convergence of μ^N, we use a Girsanov transform which removes a fixed small number of the drifts of (5.3) rather than all the drifts. This trick, which may be useful for other singular interactions, allows us to get uniform w.r.t. N bounds for the needed Girsanov exponential martingales.

5.3 Preliminaries

On the path space define the functional F_t as

$$F_t(x,\widehat{x}) = \left(\int_0^t K_{t-s}(x_t - \widehat{x}_s)ds\ \mathbb{1}_{\{x_t \neq \widehat{x}_t\}}\right)^2, \tag{5.6}$$

where $(x,\widehat{x}) \in \mathcal{C}([0,T];\mathbb{R}) \times \mathcal{C}([0,T];\mathbb{R})$. The objective of this section is to show that $\int_0^T F_t(w,Y)\ dt$ has finite exponential moments when w is a Brownian motion and Y is a process independent of w. The following key property of the kernel K_t will be used:

$$\|K_t\|_{L^p(\mathbb{R})} = \left(C\int_0^\infty \frac{z^2}{t^{p-\frac{1}{2}}} e^{-\frac{pz^2}{2}}dz\right)^{\frac{1}{p}} = \frac{C_p}{t^{1-\frac{1}{2p}}},\ 1 \le p < \infty. \tag{5.7}$$

We will proceed as in the proof of the local Novikov Condition (see [45, Chapter 3, Corollary 5.14]) by localizing on small intervals of time.

Lemma 5.3.1. *Let $w := (w_t)$ be a $(\mathcal{G}_t)$-Brownian motion with an arbitrary initial condition μ_0 on some probability space equipped with a probability measure $\mathbb{P}$ and a filtration $(\mathcal{G}_t)$. There exists a universal real number $C_0 > 0$ such that*

$$\forall x \in \mathcal{C}([0,T];\mathbb{R}),\quad \forall 0 \le t_1 \le t_2 \le T,\quad \int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}}[F_t(w,x)]\ dt \le C_0\sqrt{T}\sqrt{t_2 - t_1}.$$

Proof. By the definition of F,

$$\int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} F_t(w,x)dt \le \int_{t_1}^{t_2}\int_0^t\int_0^t \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}}|K_{t-s}(w_t - x_s)K_{t-u}(w_t - x_u)|\ ds\ du\ dt. \tag{5.8}$$

Let $g_t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. In view of (5.7), one has

$$\sqrt{\int K_{t-s}^2(y + w_{t_1} - x_s))g_{t-t_1}(y)dy} \le C\frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{(t-t_1)^{1/4}} \le \frac{C}{(t-s)^{3/4}(t-t_1)^{1/4}}.$$

Here we used that the density of $w_t - w_{t_1}$ is bounded by $\frac{C}{\sqrt{t-t_1}}$. We repeat the above calculations replacing s with u. Coming back to (5.8), one has

$$\int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}}[F_t(w,x)]\,dt \leq \int_{t_1}^{t_2} \frac{C}{\sqrt{t-t_1}} \int_0^t \int_0^t \frac{1}{(t-s)^{3/4}(t-u)^{3/4}} dsdudt = \int_{t_1}^{t_2} \frac{C\sqrt{t}}{\sqrt{t-t_1}} dt$$
$$\leq C_0\sqrt{T}\sqrt{t_2-t_1}.$$

∎

Lemma 5.3.2. *Same assumptions as in Lemma 5.3.1. Let C_0 be as in Lemma 5.3.1. For any $\kappa > 0$, there exists $C(T,\kappa)$ independent of μ_0 such that, for any $0 \leq T_1 \leq T_2 \leq T$ satisfying $T_2 - T_1 < \frac{1}{C_0^2 T\kappa^2}$,*

$$\forall x \in \mathcal{C}([0,T];\mathbb{R}), \quad \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\exp\left\{\kappa \int_{T_1}^{T_2} F_t(w,x)dt\right\}\right] \leq C(T,\kappa).$$

Proof. We adapt the proof of Khasminskii's lemma in Simon [68]. Admit for a while we have shown that there exists a constant $C(\kappa,T)$ such that for any $M \in \mathbb{N}$

$$\sum_{k=1}^{M} \frac{\kappa^k}{k!} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k \leq C(T,\kappa), \tag{5.9}$$

provided that $T_2 - T_1 < \frac{1}{C_0^2 T\kappa^2}$. The desired result then follows from Fatou's lemma.

We now prove (5.9). By the tower property of conditional expectation,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k\right] = k! \int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2}\int_{t_{k-1}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[F_{t_1}(w,x)F_{t_2}(w,x)$$
$$\times \cdots \times F_{t_{k-1}}(w,x)\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}} F_{t_k}(w,x)\right)\Big]\,dt_k\,dt_{k-1}\cdots dt_2\,dt_1.$$

In view of Lemma 5.3.1,

$$\int_{t_{k-1}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}} F_{t_k}(w,x)\,dt_k \leq C_0\sqrt{T}\sqrt{T_2-t_{k-1}} \leq C_0\sqrt{T}\sqrt{T_2-T_1}.$$

Therefore, by Fubini's theorem,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k\right] \leq k! C_0\sqrt{T}\sqrt{T_2-T_1}\int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[F_{t_1}(w,x)F_{t_2}(w,x)$$
$$\times \cdots \times F_{t_{k-1}}(w,x)\Big]dt_{k-1}\cdots dt_2\,dt_1.$$

Now we repeatedly condition with respect to $\mathcal{G}_{t_{k-i}}$ $(i \in \{2,\ldots,k-1\})$ and combine Lemma 5.3.1 with Fubini's theorem. It comes:

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k \leq k!(C_0\sqrt{T}\sqrt{T_2-T_1})^{k-1}\int_{T_1}^{T_2}\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}[F_{t_1}(w,x)]dt_1 \leq k!(C_0\sqrt{T}\sqrt{T_2-T_1})^k.$$

Thus, (5.9) is satisfied provided that $T_2 - T_1 < \frac{1}{C_0^2 T\kappa^2}$. ∎

Proposition 5.3.3. *Let $T > 0$. Same assumptions as in Lemma 5.3.1. Suppose that the filtered probability space is rich enough to support a continuous process Y independent of w. For any $\alpha > 0$,*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha \int_0^T F_t(w,Y)dt\right\}\right] \leq C(T,\alpha),$$

where $C(T,\alpha)$ depends only on T and α, but does neither depend on the law $\mathcal{L}(Y)$ nor of μ_0.

Proof. Denote by $\mathbb{P}^Y := \mathbb{P} \circ Y^{-1}$. Observe that

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,Y)dt\right\} = \int_{\mathcal{C}([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} \mathbb{P}^Y(dx). \tag{5.10}$$

Set $\delta := \frac{1}{2C_0^2 T\alpha^2} \wedge T$, where C_0 is as in Lemma 5.3.1. Set $n := \left[\frac{T}{\delta}\right]$, where $\left[\frac{T}{\delta}\right]$ denotes the greatest integer less than or equal to $\frac{T}{\delta}$. Then,

$$\exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} = \prod_{m=0}^{n} \exp\left\{\alpha \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(w,x)\,dt\right\}.$$

Condition the right-hand side by $\mathcal{G}_{(T-\delta)\vee 0}$. Notice that δ is small enough to be in the setting of Lemma 5.3.2. Thus,

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} \leq C(T,\alpha)\mathbb{E}_{\mathbb{P}} \prod_{m=1}^{n} \exp\left\{\alpha \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(w,x)\,dt\right\}.$$

Successively, conditioning by $\mathcal{G}_{(T-(m+1))\vee 0}$ for $m \in \{1,2,\dots n\}$ and using Lemma 5.3.2,

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} \leq C^n(T,\alpha)\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^{(T-n\delta)\vee 0} F_t(w,x)dt\right\} \leq C(T,\alpha).$$

The proof is completed by plugging the preceding estimate into (5.10). □

5.4 Existence of the particle system and propagation of chaos

5.4.1 Existence: Proof of Theorem 5.2.1

We start from a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{W})$ on which are defined an N-dimensional Brownian motion $W = (W^1, \dots, W^N)$ and the random variables $X_0^{i,N}$(see (5.3)). Set

$$\bar{X}_t^{i,N} := X_0^{i,N} + W_t^i, \quad t \leq T$$

and $\bar{X} := (\bar{X}^{i,N}, 1 \leq i \leq N)$. For $x \in \mathcal{C}([0,T];\mathbb{R})^N$ denote

$$b_t^{i,N}(x) := \frac{1}{N}\int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds\; \mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}}$$

and the vector of all the drifts as $B_t^N(x) := (b_t^{1,N}(x), \dots, b_t^{N,N}(x))$. For a fixed $N \in \mathbb{N}$, consider

$$Z_T^N := \exp\left\{\int_0^T B_t^N(\bar{X}) \cdot dW_t - \frac{1}{2}\int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}.$$

To prove Theorem 5.2.1, it suffices to prove the following Novikov condition holds true (see e.g. [45, Chapter 3, Proposition 5.13]):

Proposition 5.4.1. *For any $T > 0$, $N \geq 1$, $\kappa > 0$, there exists $C(T, N, \kappa)$ such that*

$$\mathbb{E}_{\mathbb{W}}\left(\exp\left\{\kappa\int_0^T |B_t^N(\bar{X})|^2 dt\right\}\right) \leq C(T, N, \kappa). \tag{5.11}$$

Proof. Drop the index N for simplicity. Using the definition of B_t^N and Jensen's inequality one has

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}\right] \leq \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \int_0^T \kappa N F_t(\bar{X}^i, \bar{X}^j)\, dt\right\}\right],$$

where F_t is defined in (5.6). Applying one more time the Jensen's inequality, we deduce

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}\right] \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa N\int_0^T F_t(\bar{X}^i, \bar{X}^j)\, dt\right\}\right].$$

As the $\bar{X}^i$'s are independent Brownian motions, we are in a position to use Proposition 5.3.3. This concludes the proof.

□

5.4.2 Girsanov transform for $1 \leq k < N$ particles

In the proof of Theorem 5.2.1 we used (5.7) and a Girsanov transform. However, the right-hand side of (5.11) goes to infinity with N. Thus, Proposition 5.4.1 cannot be used to prove the tightness and propagation of chaos of the particle system. We instead define an intermediate particle system. Let us fix $1 \leq k < N$. Proceeding as in the proof of Theorem 5.2.1 one gets the existence of a weak solution on $[0, T]$ to

$$\begin{cases} d\widehat{X}_t^{l,N} = dW_t^l, \quad 1 \leq l \leq k, \\ d\widehat{X}_t^{i,N} = \left\{\frac{1}{N}\sum_{j=k+1}^N \int_0^t K_{t-s}(\widehat{X}_t^{i,N} - \widehat{X}_s^{j,N})\, ds\; \mathbb{1}_{\{\widehat{X}_t^{i,N} \neq \widehat{X}_t^{j,N}\}}\right\} dt + dW_t^i, \quad k+1 \leq i \leq N, \\ \widehat{X}_0^{i,N} \text{ i.i.d. and independent of } (W) := (W^i, 1 \leq i \leq N). \end{cases} \tag{5.12}$$

Below we set $\hat{X} := (\hat{X}^{i,N}, 1 \leq i \leq N)$ and we denote by $\mathbb{Q}^{k,N}$ the probability measure under which $W = (W^1, \dots, W^N)$ is an N-dimensional Brownian motion and $\hat{X}$ is well defined. Notice that $(\widehat{X}^{l,N}, 1 \leq l \leq k)$ is independent of $(\widehat{X}^{i,N}, k+1 \leq i \leq N)$ and that $(\widehat{X}^{i,N}, k+1 \leq i \leq N)$ interact in the same way as (5.3) without first k particles. We now study the exponential local martingale associated to the change of drift between (5.3) and (5.12). For $x \in C([0,T];\mathbb{R})^N$ set

$$\beta_t^{(k)}(x) := \Big(b_t^{1,N}(x), \dots, b_t^{k,N}(x), \frac{1}{N}\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{k+1} - x_s^i)ds \; \mathbb{1}_{\{x_t^{k+1} \neq x_t^i\}}, \dots,$$
$$\frac{1}{N}\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{N} - x_s^i)ds \; \mathbb{1}_{\{x_t^{N} \neq x_t^i\}}\Big).$$

In the sequel we will need uniform w.r.t N bounds for moments of

$$Z_T^{(k)} := \exp\left\{-\int_0^T \beta_t^{(k)}(\widehat{X}) \cdot dW_t - \frac{1}{2}\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\}. \tag{5.13}$$

Proposition 5.4.2. *For any $T > 0$, $\gamma > 0$ and $k \geq 1$ there exists $N_0 \geq k$ and $C(T,\gamma,k)$ s.t.*

$$\forall N \geq N_0, \quad \mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\gamma \int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\} \leq C(T,\gamma,k).$$

Proof. For $x \in \mathcal{C}([0,T];\mathbb{R})^N$, one has

$$|\beta_t^{(k)}(x)|^2 = \sum_{i=1}^k \left(\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(x_t^i - x_s^j)ds \; \mathbb{1}_{\{x_t^j \neq x_t^i\}}\right)^2$$
$$+ \frac{1}{N^2}\sum_{j=1}^{N-k}\left(\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{k+j} - x_s^i)ds \; \mathbb{1}_{\{x_t^{k+j} \neq x_t^i\}}\right)^2.$$

By Jensen's inequality,

$$|\beta_t^{(k)}(x)|^2 \leq \frac{1}{N}\sum_{i=1}^k \sum_{j=1}^N F_t(x^i, x^j) + \frac{k}{N^2}\sum_{j=1}^{N-k}\sum_{i=1}^k F_t(x^{k+j}, x^i),$$

where F_t is as in (5.6). For simplicity we below write $\mathbb{E}$ (respectively, $\hat{X}^i$) instead of $\mathbb{E}_{\mathbb{Q}^{k,N}}$ (respectively, $\hat{X}^{i,N}$). Observe that

$$\begin{aligned}
&\mathbb{E}\exp\left\{\gamma\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\} \\
&\leq \Big(\mathbb{E}\exp\Big\{\sum_{i=1}^k \frac{2\gamma}{N}\sum_{j=1}^N \int_0^T F_t(\widehat{X}^i, \widehat{X}^j)dt\Big\}\Big)^{\frac{1}{2}} \Big(\mathbb{E}\exp\Big\{\frac{2\gamma k}{N^2}\sum_{j=1}^{N-k}\sum_{i=1}^k \int_0^T F_t(\widehat{X}^{k+j}, \widehat{X}^i)\;dt\Big\}\Big)^{\frac{1}{2}} \\
&=: A^{\frac{1}{2}} B^{\frac{1}{2}}.
\end{aligned}$$

Now, Hölder's and Jensen's inequalities lead to

$$A \leq \Big(\prod_{i=1}^k \mathbb{E}\exp\Big\{2\gamma k \frac{1}{N}\sum_{j=1}^N \int_0^T F_t(\widehat{X}^i, \widehat{X}^j)dt\Big\}\Big)^{\frac{1}{k}} \leq \Big(\prod_{i=1}^k \frac{1}{N}\sum_{j=1}^N \mathbb{E}\exp\Big\{2\gamma k\int_0^T F_t(\widehat{X}^i, \widehat{X}^j)dt\Big\}\Big)^{\frac{1}{k}}.$$

In view of Proposition 5.3.3, one has

$$A \leq C(T,k,\gamma).$$

Again, combine Hölder's and Jensen's inequalities. It comes

$$B \leq \Big(\prod_{j=1}^{N-k} \frac{1}{k} \sum_{i=1}^{k} \mathbb{E} \exp\Big\{ \frac{2\gamma k^2}{N} \int_0^T F_t(\widehat{X}^{k+j}, \widehat{X}^i) dt \Big\} \Big)^{\frac{1}{N-k}}.$$

It now remains to prove that there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{N \geq N_0} \mathbb{E} \left[\exp\left\{ \frac{2\gamma k^2}{N} \int_0^T F_t(\hat{X}^{k+j}, \hat{X}^i) dt \right\} \right] \leq C(T, k, \gamma).$$

We postpone the proof of this inequality to the Appendix (see Proposition 5.5.1). □

5.4.3 Propagation of chaos : Proof of Theorem 5.2.5

Tightness

We start with showing the tightness of $\{\mu^N\}$ and of an auxiliary empirical measure which is needed in the sequel.

Lemma 5.4.3. *Let $\mathbb{Q}^N$ be as above. The sequence $\{\mu^N\}$ is tight under $\mathbb{Q}^N$. In addition, let $\nu^N := \frac{1}{N^4} \sum_{i,j,k,l=1}^N \delta_{X^{i,N}_\cdot, X^{j,N}_\cdot, X^{k,N}_\cdot, X^{l,N}_\cdot}$. The sequence $\{\nu^N\}$ is tight under $\mathbb{Q}^N$.*

Proof. The tightness of $\{\mu^N\}$, respectively $\{\nu^N\}$, results from the tightness of the intensity measure $\{\mathbb{E}_{\mathbb{Q}^N} \mu^N(\cdot)\}$, respectively$\{\mathbb{E}_{\mathbb{Q}^N} \nu^N(\cdot)\}$: See Sznitman [72, Prop. 2.2-ii]. By symmetry, in both cases it suffices to check the tightness of $\{\mathrm{Law}(X^{1,N})\}$. We aim to prove

$$\exists C_T > 0, \forall N \geq N_0, \quad \mathbb{E}_{\mathbb{Q}^N}[|X^{1,N}_t - X^{1,N}_s|^4] \leq C_T |t-s|^2, \quad 0 \leq s,t \leq T, \tag{5.14}$$

where N_0 is as in Proposition 5.4.2. Let $Z^{(1)}_T$ be as in (5.13). One has

$$\mathbb{E}_{\mathbb{Q}^N}[|X^{1,N}_t - X^{1,N}_s|^4] = \mathbb{E}_{\mathbb{Q}^{1,N}}[(Z^{(1)}_T)^{-1} |\widehat{X}^{1,N}_t - \widehat{X}^{1,N}_s|^4].$$

As $\widehat{X}^{1,N}$ is a one dimensional Brownian motion under $\mathbb{Q}^{1,N}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^N}[|X^{1,N}_t - X^{1,N}_s|^4] &\leq (\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z^{(1)}_T)^{-2}])^{1/2} (\mathbb{E}_{\mathbb{Q}^{1,N}}[|\widehat{X}^{1,N}_t - \widehat{X}^{1,N}_s|^8])^{1/2} \\ &\leq (\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z^{(1)}_T)^{-2}])^{1/2} C|t-s|^2. \end{aligned}$$

Observe that, for a Brownian motion $(W^\sharp)$ under $\mathbb{Q}^{1,N}$,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z^{(1)}_T)^{-2}] = \mathbb{E}_{\mathbb{Q}^{1,N}} \exp\left\{ 2 \int_0^T \beta^{(1)}_t(\hat{X}) \cdot dW^\sharp_t - \int_0^T |\beta^{(1)}_t(\hat{X})|^2 dt \right\}.$$

Adding and subtracting $3 \int_0^T |\beta^{(1)}_t(\hat{X})|^2 dt$ and applying again the Cauchy-Schwarz inequality,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z^{(1)}_T)^{-2}] \leq \left(\mathbb{E}_{\mathbb{Q}^{1,N}} \exp\left\{ 6 \int_0^T |\beta^{(1)}_t(\hat{X})|^2 dt \right\} \right)^{1/2}.$$

Applying Proposition 5.4.2 with $k = 1$ and $\gamma = 6$, we obtain the desired result. □

Convergence

For a space S we denote by $\mathcal{P}(S)$ the set of probability measures on it. To prove Theorem 5.2.5 we have to show that any limit point of $\{\mathrm{Law}(\mu^N)\}$ is $\delta_{\mathbb{Q}}$, where $\mathbb{Q}$ is the unique solution to (MPKS). Since the particles interact through an unbounded singular functional, we adapt the arguments in Bossy and Talay [14, Thm. 3.2].

Let $\phi \in \mathcal{C}_b(\mathbb{R}^p)$, $f \in \mathcal{C}_b^2(\mathbb{R})$, $0 < t_1 < \cdots < t_p \leq s < t \leq T$ and $m \in \mathcal{P}(C([0,T];\mathbb{R}))$. Set

$$G(m) := \int_{\mathcal{C}([0,T];\mathbb{R})^2} \phi(x^1_{t_1}, \dots, x^1_{t_p}) \Big(f(x^1_t) - f(x^1_s) \\ - \frac{1}{2}\int_s^t f''(x^1_u)du - \int_s^t f'(x^1_u) \mathbb{1}_{\{x^1_u \neq x^2_u\}} \int_0^u K_{u-\theta}(x^1_u - x^2_\theta) d\theta du \Big) dm(x^1) \otimes dm(x^2).$$

We start with showing that

$$\lim_{N\to\infty} \mathbb{E}[\big(G(\mu^N)\big)^2] = 0. \tag{5.15}$$

Observe that

$$G(\mu^N) = \frac{1}{N}\sum_{i=1}^N \phi(X^{i,N}_{t_1}, \dots, X^{i,N}_{t_p}) \Big(f(X^{i,N}_t) - f(X^{i,N}_s) - \frac{1}{2}\int_s^t f''(X^{i,N}_u)du \\ - \frac{1}{N}\sum_{j=1}^N \int_s^t f'(X^{i,N}_u) \mathbb{1}_{\{X^{i,N}_u \neq X^{j,N}_u\}} \int_0^u K_{u-\theta}(X^{i,N}_u - X^{j,N}_\theta) d\theta \; du \Big).$$

Apply Itô's formula to $\frac{1}{N}\sum_{i=1}^N (f(X^{i,N}_t) - f(X^{i,N}_s))$. It comes:

$$\mathbb{E}[\big(G(\mu^N)\big)^2] \leq \frac{C}{N^2}\mathbb{E}\Big(\sum_{i=1}^N \int_s^t f'(X^{i,N}_u) dW^i_u\Big)^2 \leq \frac{C}{N}.$$

Thus, (5.15) holds true.

Suppose for a while we have proven the following lemma:

Lemma 5.4.4. *Let $\Pi^\infty \in \mathcal{P}(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4))$ be a limit point of $\{law(\nu^N)\}$. Then*

$$\lim_{N\to\infty} \mathbb{E}[\big(G(\mu^N)\big)^2] = \int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \Big\{ \int_{\mathcal{C}([0,T];\mathbb{R})^4} \Big[f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t f''(x^1_u)du \\ - \int_s^t f'(x^1_u) \mathbb{1}_{\{x^1_u \neq x^2_u\}} \int_0^u K_{u-\theta}(x^1_u - x^2_\theta) d\theta \; du \Big] \times \phi(x^1_{t_1}, \dots, x^1_{t_p}) d\nu(x^1, \dots, x^4) \Big\}^2 d\Pi^\infty(\nu), \tag{5.16}$$

and

i) Any $\nu \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)$ belonging to the support of Π^∞ is a product measure: $\nu = \nu^1 \otimes \nu^1 \otimes \nu^1 \otimes \nu^1$.

ii) For any $t \in (0,T]$, the time marginal ν^1_t of ν^1 has a density ρ^1_t which satisfies

$$\exists C_T,\ \forall 0 < t \leq T, \quad \|\rho^1_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{\frac{1}{4}}}.$$

Then, (5.15) and (5.16) imply

$$\int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \Bigg\{ \int_{\mathcal{C}([0,T];\mathbb{R})^4} \Big[f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t f''(x^1_u)du \\ - \int_s^t f'(x^1_u)\mathbb{1}_{\{x^1_u \neq x^2_u\}} \int_0^u K_{u-\theta}(x^1_u - x^2_\theta)d\theta\, du \Big] \times \phi(x^1_{t_1},\dots,x^1_{t_p})d\nu(x^1,\dots,x^4) \Bigg\}^2 d\Pi^\infty(\nu) = 0.$$

Let $\nu \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)$ belong to the support of Π^∞. Then, parts $i)$ and $ii)$ of Lemma 5.4.4 lead to

$$\int_{\mathcal{C}([0,T];\mathbb{R})} \phi(x^1_{t_1},\dots,x^1_{t_p})\Big[f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t f''(x^1_u)du \\ - \int_s^t f'(x^1_u) \int_0^u \int K_{u-\theta}(x^1_u - y)\rho^1_\theta(y)dy\, d\theta\, du \Big] d\nu^1(x^1) = 0.$$

We deduce that ν^1 solves (MPKS) and thus that $\nu^1 = \mathbb{Q}$. As (MPKS) admits a unique solution, the support of Π^∞ reduces to $\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}$. Now, let ϕ a continuous and bounded functional on $\mathcal{C}([0,T];\mathbb{R})^4$ such that $\phi(x^1,x^2,x^3,x^4) = \varphi(x^1)$. One one side, $<\Pi^\infty,\phi> = <\delta_{\mathbb{Q}},\varphi>$. On the other side, by weak convergence and definition of ϕ, one has

$$<\Pi^\infty,\phi> = \lim_{N\to\infty} \mathbb{E} <\nu^N,\phi> = \lim_{N\to\infty} \mathbb{E} <\mu^N,\varphi>,$$

where convergent subsequences of ν^n and μ^n are not renamed. It follows that any limit point of Law(μ^N) is $\delta_{\mathbb{Q}}$, which ends the proof.

Proof of Lemma 5.4.4

Proof of (5.16): Step 1. Notice that

$$\mathbb{E}[(G(\mu^N))^2] = \frac{1}{N^2}\mathbb{E}\sum_{i,k=1}^N \Phi_2(X^{i,N},X^{k,N}) + \frac{1}{N^3}\mathbb{E}\sum_{i,k,l=1}^N \Phi_3(X^{i,N},X^{k,N},X^{l,N}) \\ + \frac{1}{N^3}\mathbb{E}\sum_{i,j,k=1}^N \Phi_3(X^{k,N},X^{i,N},X^{j,N}) + \frac{1}{N^4}\mathbb{E}\sum_{i,j,k,l=1}^N \Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}), \tag{5.17}$$

where

$$\Phi_2(X^{i,N},X^{k,N}) := \phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\ \phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p}) \\ \Big(f(X^{i,N}_t) - f(X^{i,N}_s) - \frac{1}{2}\int_s^t f''(X^{i,N}_u)du\Big)\Big(f(X^{k,N}_t) - f(X^{k,N}_s) - \frac{1}{2}\int_s^t f''(X^{k,N}_u)du\Big),$$

$$\Phi_3(X^{i,N},X^{k,N},X^{l,N}) := -\phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\ \phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p}) \\ \Big(f(X^{i,N}_t) - f(X^{i,N}_s) - \frac{1}{2}\int_s^t f''(X^{i,N}_{u_1})du_1\Big)\int_s^t f'(X^{k,N}_u)\mathbb{1}_{\{X^{k,N}_u \neq X^{l,N}_u\}}\int_0^u K_{u-\theta}(X^{k,N}_u - X^{l,N}_\theta)\ d\theta\ du,$$

and

$$\Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}) := \int_s^t \int_s^t \int_0^{u_1} \int_0^{u_2} \phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p}) f'(X^{i,N}_{u_1})$$
$$f'(X^{k,N}_{u_2}) K_{u_1-\theta_1}(X^{i,N}_{u_1}-X^{j,N}_{\theta_1}) K_{u_2-\theta_2}(X^{k,N}_{u_2}-X^{l,N}_{\theta_2}) \mathbb{1}_{\{X^{i,N}_{u_1}\neq X^{j,N}_{u_1}\}} \mathbb{1}_{\{X^{k,N}_{u_2}\neq X^{l,N}_{u_2}\}} d\theta_1\, d\theta_2\, du_1\, du_2.$$

Let C_N be the last term in the r.h.s. of (5.17). In Steps 2-4 below we prove that C_N converges as $N\to\infty$ and we identify its limit. Define the function F on $\mathbb{R}^{2p+6}$ as

$$F(x^1,\dots,x^{2p+6}) := f'(x^1)f'(x^3)\phi(x^7,\dots,x^{p+6})\phi(x^{p+7},\dots,x^{2p+6})$$
$$\times K_{u_1-\theta_1}(x^1-x^2)K_{u_2-\theta_2}(x^3-x^4)\mathbb{1}_{\{x^1\neq x^5\}}\mathbb{1}_{\{x^2\neq x^6\}}\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}.$$

We set $C_N = \int_s^t \int_s^t \int_0^{u_1} \int_0^{u_2} A_N\, d\theta_1\, d\theta_2\, du_1\, du_2$ with

$$A_N := \frac{1}{N^4}\sum_{i,j,k,l=1}^N \mathbb{E}(F(X^{i,N}_{u_1},X^{j,N}_{\theta_1},X^{k,N}_{u_2},X^{l,N}_{\theta_2},X^{j,N}_{u_1},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})).$$

We now aim to show that A_N converges pointwise (Step 2), that $|A_N|$ is bounded from above by an integrable function w.r.t. $d\theta_1\, d\theta_2\, du_1\, du_2$ (Step 3), and finally to identify the limit of C_N (Step 4).

Proof of (5.16): Step 2. Fix $u_1,u_2\in[s,t]$ and $\theta_1\in[0,u_1)$ and $\theta_2\in[0,u_2)$. Define τ^N as

$$\tau^N := \frac{1}{N^4}\sum_{i,j,k,l=1}^N \delta_{X^{i,N}_{u_1},X^{j,N}_{\theta 1},X^{k,N}_{u_2},X^{l,N}_{\theta_2},X^{j,N}_{u_1},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p}}.$$

Define the measure $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ on $\mathbb{R}^{2p+6}$ as follows: For any Borel set S in $\mathbb{R}^{2p+6}$

$$\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}(S) = \mathbb{E}(\tau^N(S)).$$

The convergence of $\{\mathrm{law}(\nu^N)\}$ implies the weak convergence of $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ to a measure on $\mathbb{R}^{2p+6}$ defined by

$$\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}(S) := \int_{\mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_{\mathcal{C}([0,T];\mathbb{R})^4} \mathbb{1}_S(x^1_{u_1},x^2_{\theta 1},x^3_{u_2},x^4_{\theta_2},x^2_{u_1},x^4_{u_2},x^1_{t_1},\dots,$$
$$x^1_{t_p},x^3_{t_1},\dots,x^3_{t_p})d\nu(x^1,x^2,x^3,x^4)d\Pi^\infty(\nu).$$

Let us show that $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ admits an L^2-density w.r.t. the Lebesgue measure on $\mathbb{R}^{2p+6}$. Let $h\in\mathcal{C}_K(\mathbb{R}^{2p+6})$. By weak convergence,

$$\left|< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h >\right|$$
$$= \left|\lim_{N\to\infty}\frac{1}{N^4}\sum_{i,j,k,l=1}^N \mathbb{E}h(X^{i,N}_{u_1},X^{j,N}_{\theta 1},X^{k,N}_{u_2},X^{l,N}_{\theta_2},X^{j,N}_{u_1},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})\right|.$$

When, in the preceding sum, at least two indices are equal, we bound the expectation by $\|h\|_\infty$. When $i \neq j \neq k \neq l$, we apply Girsanov's transform in Section 5.4.2 with four particles and Proposition 5.4.2. This procedure leads to

$$\left| < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h > \right| \leq \lim_{N\to\infty} \Big(\|h\|_\infty \frac{C}{N}$$
$$+ \frac{C_T}{N^4} \sum_{i\neq j\neq k\neq l} \Big(\mathbb{E} h^2(\hat{X}^{i,N}_{u_1}, \hat{X}^{j,N}_{\theta 1}, \hat{X}^{k,N}_{u_2}, \hat{X}^{l,N}_{\theta_2}, \hat{X}^{j,N}_{u_1}, \hat{X}^{l,N}_{u_2}, \hat{X}^{i,N}_{t_1}, \dots, \hat{X}^{i,N}_{t_p}, \hat{X}^{k,N}_{t_1}, \dots, \hat{X}^{k,N}_{t_p}) \Big)^{1/2} \Big).$$

As for fixed $i \neq j \neq k \neq l$, the processes $\hat{X}^{i,N}$, $\hat{X}^{j,N}$, $\hat{X}^{k,N}$ and $\hat{X}^{l,N}$ are independent Brownian motions, we have

$$\left| < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h > \right| \leq C_{u_1,u_2,\theta_1,\theta_2,t_1,\dots,t_p} \|h\|_{L^2(\mathbb{R}^{2p+6})}.$$

It follows from Riesz's representation theorem (e.g. [15, Thm. 4.11 and 4.14]) that $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ has a density w.r.t. Lebesgue's measure in $L^2(\mathbb{R}^{2p+6})$. Therefore, the functional F is continuous $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ - a.e. As for $u_1, u_2 \in [s,t]$ and $\theta_1 \in [0,u_1)$, $\theta_2 \in [0,u_2)$ F is bounded, by weak convergence one has

$$\lim_{N\to\infty} A_N =< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, F > .$$

Proof of (5.16): Step 3. By definition of F we may restrict ourselves to the case $i \neq j$ and $k \neq l$. Use the Girsanov transforms from Section 5.4.2 with $r \in \{2,3,4\}$ according to, respectively, $(i = k, j = l)$, $(i = k, j \neq l)$, $(i \neq k, j \neq l)$, etc. It comes:

$$A_N = \Big| \frac{1}{N^4} \sum_{i,j,k,l=1}^N \mathbb{E}_{\mathbb{Q}^{r,N}}((Z_T^{(r)})^{-1} F(\cdots)) \Big| \leq \frac{1}{N^4} \sum_{i,j,k,l=1}^N \Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)})^{-2} \Big)^{1/2} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots)) \Big)^{1/2}.$$

By Proposition 5.4.2, $\mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)})^{-2}$ can be bounded uniformly w.r.t. N (see the paragraph Tightness). As the functions f and ϕ are bounded we deduce

$$\sqrt{\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots))} \leq C \mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(K^2_{u_1-\theta_1}(\hat{X}^{i,N}_{u_1} - \hat{X}^{j,N}_{\theta_1}) K^2_{u_1-\theta_1}(\hat{X}^{k,N}_{u_2} - \hat{X}^{l,N}_{\theta_2})) \Big)^{1/2},$$

for $i \neq j$ and $k \neq l$. In view of (5.7), for any $0 < \theta < u < T$ we have

$$\Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(K^4_{u-\theta}(\hat{X}^{i,N}_u - \hat{X}^{j,N}_\theta)) \Big)^{1/4} \leq \frac{C}{u^{1/8}} \|K_{u-\theta}\|_{L^4(\mathbb{R})} \leq \frac{C}{u^{1/8}(u-\theta)^{7/8}}.$$

Therefore,

$$\big(\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots)) \big)^{1/2} \leq C \frac{\mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}}}{u_1^{1/8}(u_1-\theta_1)^{7/8} u_2^{1/8}(u_2-\theta_2)^{7/8}}.$$

We thus have obtained:

$$A_N \leq C \frac{\mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}}}{u_1^{1/8}(u_1-\theta_1)^{7/8} u_2^{1/8}(u_2-\theta_2)^{7/8}}.$$

We remark that the r.h.s. belongs to $L^1((0,T)^4)$.

Proof of (5.16): Step 4. Steps 2 and 3 allow us to conclude that

$$\lim_{N\to\infty} C_N = \int_s^t \int_s^t \int_s^t \int_s^t < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, F > \; d\theta_1\, d\theta_2\, du_1\, du_2.$$

By definition of $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ and F we thus have obtained that

$$\lim_{N\to\infty} C_N = \int_{\mathbb{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_s^t \int_s^t \int_{\mathcal{C}([0,T];\mathbb{R})^4} f'(x^1_{u_1}) f'(x^3_{u_2}) \phi(x^1_{t_1},\dots,x^1_{t_p}) \phi(x^3_{t_1},\dots,x^3_{t_p})$$
$$\times \int_0^{u_1} \int_0^{u_2} K_{u_1-\theta_1}(x^1_{u_1} - x^2_{\theta_1}) K_{u_2-\theta_2}(x^3_{u_2} - x^4_{\theta_2}) \mathbb{1}_{\{x^1_{u_1} \neq x^2_{u_1}\}} \mathbb{1}_{\{x^3_{u_2} \neq x^4_{u_2}\}}$$
$$d\nu(x^1,x^2,x^3,x^4)\; d\theta_1\; d\theta_2\; du_1\; du_2\; d\Pi^\infty(\nu).$$

A similar procedure is applied to the three other terms in the r.h.s. of (5.17). We identify their limits:

$$\lim_{N\to\infty} \frac{1}{N^2} \mathbb{E} \sum_{i,k=1}^N \Phi_2(X^{i,N}, X^{k,N}) = \int_{\mathbb{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_{\mathcal{C}([0,T];\mathbb{R})^4} \phi(x^1_{t_1},\dots,x^1_{t_p}) \phi(x^3_{t_1},\dots,x^3_{t_p})$$
$$\left(f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t f''(x^1_u)\; du \right) \left(f(x^3_t) - f(x^3_s) - \frac{1}{2}\int_s^t f''(x^3_u)\; du \right) d\nu(x^1,x^2,x^3,x^4)\, d\Pi^\infty(\nu),$$

$$\lim_{N\to\infty} \frac{1}{N^3} \mathbb{E} \sum_{i,k,l=1}^N \Phi_3(X^{i,N}, X^{k,N}, X^{l,N}) = -\int_{\mathbb{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_{\mathcal{C}([0,T];\mathbb{R})^4} \phi(x^1_{t_1},\dots,x^1_{t_p}) \phi(x^3_{t_1},\dots,x^3_{t_p})$$
$$\left(f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t f''(x^1_u)\; du \right) \int_s^t f'(x^3_u) \mathbb{1}_{\{x^3_u \neq x^4_u\}} \int_0^u K_{u-\theta}(x^3_u - x^4_\theta)\; du\; d\theta$$
$$d\nu(x^1,x^2,x^3,x^4)\; d\Pi^\infty(\nu),$$

and

$$\lim_{N\to\infty} \frac{1}{N^3} \mathbb{E} \sum_{k,i,j=1}^N \Phi_3(X^{k,N}, X^{i,N}, X^{j,N}) = -\int_{\mathbb{P}(\mathcal{C}([0,T];\mathbb{R})^4)} \int_{\mathcal{C}([0,T];\mathbb{R})^4} \phi(x^1_{t_1},\dots,x^1_{t_p}) \phi(x^3_{t_1},\dots,x^3_{t_p})$$
$$\left(f(x^3_t) - f(x^3_s) - \frac{1}{2}\int_s^t f''(x^3_u)\; du \right) \int_s^t f'(x^1_u) \mathbb{1}_{\{x^1_u \neq x^2_u\}} \int_0^u K_{u-\theta}(x^1_u - x^2_\theta)\; du\; d\theta$$
$$d\nu(x^1,x^2,x^3,x^4)\; d\Pi^\infty(\nu).$$

Once all the limits in the r.h.s. of (5.17) are obtained, we use the claim i) of Lemma 5.4.4 to obtain (5.16).

Proof of i) and ii). Now, we prove the claims i) and ii) of Lemma 5.4.4.

i) For any measure $\nu \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R})^4)$, denote its first marginal by ν^1. One easily gets Π^∞ a.e., $\nu = \nu^1 \otimes \nu^1 \otimes \nu^1 \otimes \nu^1$ (see [14, Lemma 3.3]).

ii) Take $h \in \mathcal{C}_K(\mathbb{R})$. Using similar arguments as in the above Step 1, for any $0 < t \leq T$ one has $\Pi^\infty(d\nu)$ a.e.,

$$< \nu_t^1, h >= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N} < \mu_t^N, h >= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N}(h(X_t^{1,N})) = \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^{1,N}}(Z_T^{(1)} h(W_t^{1,N}))$$
$$\leq \frac{C}{t^{1/4}} \|h\|_{L^2(\mathbb{R})}.$$

5.5 Appendix

Proposition 5.5.1. *Same assumptions as in Proposition 5.3.3. There exists $N_0 \in \mathbb{N}$ depending only on T and α, such that*

$$\sup_{N\geq N_0} \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{\alpha}{N}\int_0^T \left(\int_0^t K_{t-s}(Y_t - w_s)ds \mathbb{1}_{\{w_t\neq Y_t\}}\right)^2 dt\right\}\right] \leq C(T,\alpha).$$

Compared to the proof of Proposition 5.4.2, as w and Y exchanged places in the left-hand side, it is not so obvious to use the independence of Brownian increments. However, the weight $\frac{1}{N}$ enables us to skip the localization part (see Lemmas 5.3.1 and 5.3.2).

Proof. Fix $N \in \mathbb{N}$. Set $I := \int_0^T \left(\int_0^t K_{t-s}(Y_t - w_s)ds\right)^2 dt$. One has

$$\begin{aligned}
I^k &\leq C\left(\int_0^T \int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{(Y_t - w_s)^2}{(t-s)^{9/4}} e^{-\frac{(Y_t-w_s)^2}{t-s}} ds\ dt\right)^k \\
&\leq CT^{k/4}\left(\int_0^T \int_0^t \frac{(Y_t - w_s)^2}{(t-s)^{9/4}} e^{-\frac{(Y_t-w_s)^2}{t-s}} ds\ dt\right)^k.
\end{aligned}$$

For $0 \leq s < T$ and for $(\omega, \widehat{\omega}) \in C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$, define the functional H_s as

$$H_s(\omega, \widehat{\omega}) = \int_s^T \frac{(\omega_t - \widehat{\omega}_s)^2}{(t-s)^{9/4}} e^{-\frac{(\omega_t-\widehat{\omega}_s)^2}{t-s}} dt.$$

As the processes Y and w are independent,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(Y,w)ds\right)^k = \int_{C([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k \mathbb{P}^Y(dx).$$

As before we observe that, for any $x \in C([0,T];\mathbb{R})$,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k = k! \int_0^T \int_{s_1}^T \cdots \int_{s_{k-1}}^T \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{s_{k-1}}}(H_{s_1}(x,w)\dots H_{s_k}(x,w))\right) ds_k \dots ds_1.$$

Using again that $g_{s_k-s_{k-1}}(z) \leq \frac{1}{\sqrt{s_k-s_{k-1}}}$, one has

$$
\begin{aligned}
&\int_{s_{k-1}}^{T} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{s_{k-1}}} H_{s_k}(x,w)\ ds_k \\
&= \int_{s_{k-1}}^{T} \int_{s_k}^{T} \int \frac{(x_t - z - w_{s_{k-1}})^2}{(t-s_k)^{9/4}} e^{-\frac{(x_t-z-w_{s_{k-1}})^2}{t-s_k}} g_{s_k-s_{k-1}}(z)\ dz\ dt\ ds_k \\
&\leq \int_{s_{k-1}}^{T} \frac{1}{\sqrt{s_k - s_{k-1}}} \int_{s_k}^{T} \frac{1}{(t-s_k)^{3/4}} \int z^2 e^{-z^2}\ dz\ dt\ ds_k \leq CT^{1/4}\sqrt{T - s_{k-1}} \leq CT^{3/4}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,w) ds \right)^k \\
&\leq k! C T^{3/4} \int_0^T \int_{s_1}^T \cdots \int_{s_{k-2}}^T \mathbb{E}_{\mathbb{P}} \left(H_{s_1}(x,w) \ldots H_{s_{k-1}}(x,w) \right) ds_{k-1} \ldots ds_1.
\end{aligned}
$$

Repeat the previous procedure $k-2$ times. It comes:

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,w) ds \right)^k \leq k! C^{k-1} T^{3(k-1)/4} \int_0^T \mathbb{E}_{\mathbb{P}} \left(H_{s_1}(x,w) \right) ds_1 \\
&\leq k! C^{k-1} T^{3(k-1)/4} \int_0^T \frac{1}{\sqrt{s_1}} \int_{s_1}^T \frac{1}{(t-s_1)^{3/4}} \int z^2 e^{-z^2}\ dz\ dt\ ds_1 \leq k! C^k T^{\frac{3k}{4}}.
\end{aligned}
$$

This implies that for any $M \geq 1$,

$$
\mathbb{E}_{\mathbb{P}} \sum_{k=1}^{M} \frac{\alpha^k I^k}{N^k k!} \leq \sum_{k=1}^{M} \frac{\alpha^k C^k T^k}{N^k}.
$$

Choose N_0 large enough to have $\frac{\alpha}{N_0} CT < 1$. To conclude, we apply Fatou's lemma. □

Chapter 6

The two-dimensional case: The non-linear stochastic equation

6.1 Introduction

This chapter is devoted to the analysis of the Mc-Kean Vlasov non–linear SDE (1.14) with $d = 2$ and its connection with the two–dimensional Keller-Segel system. Its main contribution is in how to deal with singular interaction kernels that lead to a process whose law has small chances to be absolutely continuous w.r.t. Wiener's measure. In addition, our procedure leads to a new well-posedness result for the Keller-Segel system in $d = 2$.

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$, equipped with a 2-dimensional Brownian motion $(W_t)_{t\geq 0}$, consider the SDE

$$\begin{cases} dX_t = dW_t + b_0(t, X_t)dt + \chi\Big\{ \int_0^t e^{-\lambda(t-s)} \int K_{t-s}(X_t - y)p_s(y)dy\ ds\Big\}dt, t \leq T \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0, \end{cases} \tag{6.1}$$

where X_0 is an $\mathbb{R}^2$-valued $\mathcal{F}_0-$measurable random variable, g_t denotes the probability density of W_t and for $(t, x) \in (0, T] \times \mathbb{R}^2$

$$b_0(t, x) := \chi e^{-\lambda t}(\nabla c_0 * g_t)(x) \text{ and } K_t(x) := \nabla g_t(x) = -\frac{x}{2\pi t^2}e^{-\frac{|x|^2}{2t}}.$$

Here $|x|$ denotes the Euclidean norm. Notice that K_t is a two dimensional vector. We denote its coordinates by K_t^i with $i = 1, 2$ and

$$b^i(t, x; p) = b_0^i(t, X_t) + \chi \int_0^t e^{-\lambda(t-s)} \int K_{t-s}^i(X_t - y)p_s(y)dy\ ds\ dt.$$

The main difficulty when dealing with (6.1) is the singular nature of the kernel K. In Chapters 3 and 5 we overcame it thanks to the fact that the one–dimensional kernel $(K^\sharp)$ belonged to the space $L^1((0, T); L^1(\mathbb{R})) \cap L^1((0, T); L^2(\mathbb{R}))$ and by the help of precise $L^\infty(\mathbb{R})$-norm density estimates (see Section 3.3). When $d = 2$, the singularity in time of the kernel is stronger and therefore estimates such as (3.14) do not seem to remain true.

The following technical lemmas will be used throughout this chapter and actually show that for $q \geq 2$, the $L^1((0, T); L^q(\mathbb{R}^2))$–norm of K^i explodes. This was not the case in $d = 1$ and it is in this sense that the two–dimensional kernel is more singular.

Lemma 6.1.1. *Let $t > 0$ and $i \in \{1, 2\}$. Then, for any $1 \leq q < \infty$ one has*

$$\|K_t^i\|_{L^q(\mathbb{R}^2)} = \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{C_1(q)}{t^{\frac{3}{2}-\frac{1}{q}}}, \tag{6.2}$$

where

$$C_1(q) = \frac{2^{\frac{1}{q}-\frac{1}{2}}}{\pi^{1-\frac{1}{2q}} q^{\frac{1}{q}+\frac{1}{2}}} \left(\Gamma(\frac{q+1}{2})\right)^{\frac{1}{q}}.$$

Here $\Gamma(x)$ denotes the Gamma function: $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z}\, dz$.

Proof. Let $1 \leq q < \infty$. One has

$$\|K_t^i\|_{L^q(\mathbb{R}^2)} = \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{1}{2\pi t^2} \left(\int_{\mathbb{R}^2} |x_i|^q e^{-\frac{q|x|^2}{2t}} dx\right)^{\frac{1}{q}} = \frac{1}{2\pi t^2} \left(\int_{\mathbb{R}} e^{-q\frac{x^2}{2t}} dx \int_{\mathbb{R}} |x|^q e^{-q\frac{x^2}{2t}} dx\right)^{\frac{1}{q}}$$
$$= \frac{1}{2\pi t^2} \left(\frac{\sqrt{2\pi t}}{\sqrt{q}}\, 2 \int_0^\infty x^q e^{-q\frac{x^2}{2t}}\, dx\right)^{\frac{1}{q}}.$$

Apply the change of variables $\frac{qx^2}{2t} = y$. It comes

$$\|K_t^i\|_{L^q(\mathbb{R}^2)} = \|\nabla_i g_t\|_{L^q(\mathbb{R}^2)} = \frac{1}{2\pi t^2} \left(\frac{\sqrt{2\pi t}}{\sqrt{q}}\, 2 \left(\frac{2t}{q}\right)^{\frac{q-1}{2}} \int_0^\infty y^{\frac{q-1}{2}} e^{-y} \frac{t}{q}\, dy\right)^{\frac{1}{q}}$$
$$= \frac{1}{2\pi t^2} \left(\frac{2t}{q}\right)^{\frac{1}{q}+\frac{1}{2}} \pi^{\frac{1}{2q}} \left(\Gamma(\frac{q+1}{2})\right)^{\frac{1}{q}}.$$

This ends the proof. □

The change of variables $\frac{x}{\sqrt{t}} = z$ leads to

Lemma 6.1.2. *Let $t > 0$. Then, for any $1 \leq q < \infty$ one has*

$$\|g_t\|_{L^q(\mathbb{R}^2)} = \frac{C_2(q)}{t^{1-\frac{1}{q}}}, \tag{6.3}$$

where

$$C_2(q) = \frac{1}{(2\pi)^{1-\frac{1}{q}} q^{\frac{1}{q}}}.$$

The functions $C_1(q)$ and $C_2(q)$ will be used only when we need the explicit constants in a computation. In all other cases we will use notation C_q that may change from line to line.

Discussion on the $1d$-approach in the $2d$-setting The change of the space dimension has a significant impact on the techniques we used in Chapter 3 to prove the well-posedness of the NLSDE. In Chapter 3, we used Picard's iteration process to exhibit a weak solution. In each step the $L^\infty([0, T] \times \mathbb{R})$-norm of the drift and $L^1((0, T]; L^\infty(\mathbb{R}))$-norm of the marginal densities were controlled simultaneously. These controls were obtained thanks to a probabilistic method which exhibits sharp density estimates for a process whose drift is uniformly bounded in space and time (see Section 3.3). A generalisation to the multidimensional case of the results in Section 3.3 can be found in Qian *et al.* [65] in the case of time homogeneous drifts. There, the authors show that the estimate of the transition density of a d-dimensional stochastic process is a product of one-dimensional estimates provided that the Euclidean norm of the drift vector is uniformly bounded. With the arguments we used in $d = 1$, one can easily extend the results in [65] to time inhomogeneous drifts and get the following. Suppose that the drift $b(t, x)$ of a two-dimensional linear process (X^b) is bounded, i.e. $\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} |b(t, x)| \leq \beta$. Then the two-dimensional transition density of (X^b_t) satisfies

$$p_b(t, x, y) \leq \frac{1}{2\pi t} \prod_{i=1}^{2} \left(\int_{\frac{|x^i - y^i|}{\sqrt{t}}}^{\infty} z e^{-\frac{(z-\beta\sqrt{t})}{2}}\, dz \right).$$

If the initial condition is assumed to belong to $L^1(\mathbb{R}^2)$, the arguments used to prove Corollary 3.3.2 lead to

$$p^b_t(y) \leq \beta^2 + \frac{2\beta}{\sqrt{2\pi t}} + \frac{1}{2\pi t}.$$

This estimate is not integrable in time. Consequently, it is too crude to be applied in each step of the Picard's iteration procedure developed in Section 3.5. One could overcome this by imposing more regularity on the initial condition. For example, if $p_0 \in L^\infty(\mathbb{R}^2)$, one would get

$$p^b_t(y) \leq \beta^2 + \frac{2\beta}{\sqrt{2\pi t}} + C\|p_0\|_{L^\infty(\mathbb{R}^2)}.$$

Now, using the same notation as in Section 3.5, in view of (6.2) one would get the following relation for the drift bounds in the iteration process :

$$\beta_{k+1} = \chi\|\nabla c_0\|_\infty + C\chi(\beta_k^2\sqrt{T} + \beta_k + \|p_0\|_{L^\infty(\mathbb{R}^2)}\sqrt{T}).$$

Thus, when $d = 2$ one gets a quadratic relationship between the L^∞-norms of successive drifts, whereas the relationship was linear in the 1-d case (see Section 3.5). Therefore, in order to control β^k's uniformly in k, one should impose conditions on χ, $\|p_0\|_{L^\infty(\mathbb{R}^2)}$, $\|\nabla c_0\|_\infty$ and T. The easiest way to exhibit suitable conditions is to search for a positive zero of the polynomial

$$P(x) = C\chi\sqrt{T}x^2 + (C\chi - 1)x + C\chi\sqrt{T}\|p_0\|_{L^\infty(\mathbb{R}^2)} + \chi\|\nabla c_0\|_\infty.$$

This leads to the constraints

$$\chi < \frac{1}{C} \quad \text{and} \quad (C\chi - 1)^2 \geq 4C\chi\sqrt{T}(C\chi\sqrt{T}\|p_0\|_{L^\infty(\mathbb{R}^2)} + \chi\|\nabla c_0\|_{L^\infty(\mathbb{R}^2)}).$$

These constraints are equivalent to

$$\chi C + 2\chi\sqrt{C\sqrt{T}(C\sqrt{T}\|p_0\|_{L^\infty(\mathbb{R}^2)} + \|\nabla c_0\|_{L^\infty(\mathbb{R}^2)})} < 1. \tag{6.4}$$

As we do not want χ to depend on the time horizon, a condition on T should be imposed that depends on $\|\nabla c_0\|_\infty$ and $\|p_0\|_{L^\infty(\mathbb{R}^2)}$. This condition together with one on χ would suffice to get tightness and local in time weak solutions for (6.1) up to a small time T_1. Restarting this procedure after the imposed time horizon becomes tricky as the norm of the new initial condition increased, as well all as the constants involved in the condition (6.4). Thus, the new time horizon T_2 is much smaller than T_1 and iterating the procedure leads to a sequence $(T_k)_{k\geq1}$ such that

$\sum_k T_k < \infty$. We believe it is impossible to get global well-posedness by applying the above procedure. In addition, we do want a result as general as possible and no additional assumptions on p_0 except it being a probability density function.

All in all, the $L^\infty([0,T]\times\mathbb{R}^2)$ and $L^1((0,T];L^\infty(\mathbb{R}^2))$ seem not to be a good choice for a functional space for the drift and density of (6.1), respectively. The main reason is that the density estimates at our disposal seem to be too crude in $d=2$. It is thus necessary to change the approach. We have chosen to use the L^q-spaces.

Formal discussion on an adequate L^q-space for the drift and density functions In order to understand what kind of an $L^q(\mathbb{R}^2)$-estimate we can expect for the density p_t of X_t, we formally derive and analyze the mild equation for p_t:

$$p_t = g_t * p_0 - \sum_{i=1}^{2}\int_0^t \nabla_i g_{t-s} * \left(p_s\left(b_0^i(s,\cdot) + \chi\int_0^s e^{-\lambda(s-r)}K_{s-r}^i * p_r\, dr\right)\right)\, ds.$$

The term $\|g_t * p_0\|_{L^q(\mathbb{R}^2)}$ can give an idea of the behaviour of $\|p_t\|_{L^q(\mathbb{R}^2)}$. In view of the convolution inequality (3.37) and (6.3), one has

$$\|g_t * p_0\|_{L^q(\mathbb{R}^2)} \le \|g_t\|_{L^q(\mathbb{R}^2)}\|p_0\|_{L^1(\mathbb{R}^2)} = \frac{C_q}{t^{1-\frac{1}{q}}}.$$

This prompts us to assume for a moment that for every $1<q<\infty$ there exists $C_q>0$ such that

$$\sup_{t\le T} t^{1-\frac{1}{q}}\|p_t\|_{L^q(\mathbb{R}^2)} \le C_q. \tag{6.5}$$

Then, the non-linear part of the drift satisfies for $i\in\{1,2\}$

$$\left\|\int_0^t e^{-\lambda(t-s)}K_{t-s}^i * p_s\, ds\right\|_{L^q(\mathbb{R}^2)} \le \int_0^t \|K_{t-s}^i\|_{L^1(\mathbb{R}^2)}\|p_s\|_{L^q(\mathbb{R}^2)}\, ds \le C_q\int_0^t \frac{1}{\sqrt{t-s}s^{1-\frac{1}{q}}}\, ds.$$

The change of variables $\frac{s}{t}=u$ leads to

$$\left\|\int_0^t e^{-\lambda(t-s)}K_{t-s}^i * p_s\, ds\right\|_{L^q(\mathbb{R}^2)} \le \frac{C_q}{t^{\frac{1}{2}-\frac{1}{q}}}.$$

In order to have the same type of estimate for b_0^i one needs to suppose that ∇c_0 belongs to a suitable $L^r(\mathbb{R}^2)$ space for ∇c_0. In view of (6.3) and the standard convolution inequality (3.38),

$$\|b_0^i(t,\cdot)\|_{L^q(\mathbb{R}^2)} \le \|g_t\|_{L^m(\mathbb{R}^2)}\|\nabla c_0\|_{L^r(\mathbb{R}^2)} \le \frac{\|\nabla c_0\|_{L^r(\mathbb{R}^2)}}{t^{1-\frac{1}{m}}},$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$. Therefore, r should satisfy $\frac{1}{2}-\frac{1}{q}=\frac{1}{x}-\frac{1}{q}$. Thus, one should have $\nabla c_0\in L^2(\mathbb{R}^2)$. Notice that in order to apply the convolution inequality above we need $q\ge 2$.

We conclude that if $c_0\in H^1(\mathbb{R}^2)$ and if the marginals of X_t satisfy (6.5), then the $L^r(\mathbb{R}^2)$-norm of the drift $b(t,x;p)$ for all $r\ge 2$ satisfies

$$t^{\frac{1}{2}-\frac{1}{r}}\|b(t,\cdot;p)\|_{L^r(\mathbb{R}^2)} \le C_r. \tag{6.6}$$

The above discussion, motivates us to redefine the notion of a weak solution to our McKean-Vlasov SDE (NLSDE) in order to include the constraints of type (6.5). To prove these constraints are satisfied, we conveniently regularize the NLSDE and firstly apply the results from Chapter 2. Then we analyze the associated regularized mild equation and prove estimates of type (6.5) for the regularized densities. These estimates are uniform w.r.t. the regularizing parameter under a condition involving the parameter χ and the size of initial datum. Once such an estimate is obtained, we prove the convergence of martingale problems related to regularized dynamics towards the our NLSDE.

The plan of the chapter is the following: Main results are stated in Section 6.2. A convenient regularization is exhibited in Section 6.3 and the estimates on the time marginals of the regularized equation are obtained; Existence (resp. well-posedness) for the NLSDE (resp. Keller-Segel model) is proved in Section 6.4 (resp. Section 6.5); Uniqueness in law for the NLSDE is proved in Section 6.6.

6.2 Main results

Having, in mind the discussion about convenient functional spaces above, we define the notion of weak solution to (6.1).

Definition 6.2.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is said to be a weak solution to the equation (6.1) up to time $T > 0$ if:*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.*
2. *The process $X := (X_t)_{t\in[0,T]}$ is $\mathbb{R}^2$-valued, continuous, and $(\mathcal{F}_t)$-adapted. In addition, the probability distribution of X_0 has density p_0.*
3. *The process $W := (W_t)_{t\in[0,T]}$ is a two-dimensional $(\mathcal{F}_t)$-Brownian motion.*
4. *The probability distribution $\mathbb{P} \circ X^{-1}$ has time marginal densities $(p_t,\ t \in (0,T])$ with respect to Lebesgue measure which satisfy for any*
$$\forall 1 < q < \infty \ \exists C_q > 0, \quad \sup_{t\le T} t^{1-\frac{1}{q}} \|p_t\|_{L^q(\mathbb{R}^2)} \le C_q. \tag{6.7}$$
5. *For any $t \in [0,T]$ and $x \in \mathbb{R}^2$, one has $\int_0^t |b_0(s,x)|\ ds < \infty$.*
6. *$\mathbb{P}$-a.s. the pair (X, W) satisfies (6.1).*

Remark 6.2.2. *Notice that under the condition $c_0 \in H^1(\mathbb{R}^2)$ one gets applying Hölder's inequality and (6.3),*
$$\int_0^t |b_0(s,x)|\ ds \le C\|\nabla c_0\|_{L^2(\mathbb{R}^2)} \int_0^t \frac{1}{\sqrt{s}}\ ds.$$
Moreover (6.7) implies
$$\int_0^t \left| \int_0^s K_{s-u} * p_u(x)\ du \right|\ ds \le C \int_0^t \frac{1}{\sqrt{s}}\ ds.$$

The first objective of this chapter is to prove the following claim:

Theorem 6.2.3. *Let $T > 0$ and suppose that X_0 has a probability density function p_0. Furthermore, assume that $c_0 \in H^1(\mathbb{R}^2)$. Then, Equation (6.1) admits a weak solution under the following condition*

$$A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B\sqrt{\chi} < 1, \tag{6.8}$$

where A and B are defined as in Proposition 6.3.7 below.

We do not apply Picard's iteration since in each iteration step we will need a well-posedness result for a linear SDE whose drift satisfies (6.6). In view of Krylov and Röckner [49], the well-posedness follows from a finite $L^p((0,T);L^r(\mathbb{R}^2))$-norm of the drift with $\frac{1}{p}+\frac{1}{r}<\frac{1}{2}$. Unfortunately, the property in (6.6) will imply the opposite condition $\frac{1}{p}+\frac{1}{r}>\frac{1}{2}$ for the same norm to be finite. We do not see how to circumvent this without a cut-off. To prove Theorem 6.2.3 we will use a regularization method. The goal is to prove that the time marginals of the regularized version of (6.1) satisfy the property (6.7) with uniform constants with respect to the regularization parameter. Then, the tightness will follow thanks to (6.6) for $r = \infty$. It will remain, then, to solve the non-linear martingale problem related to (6.1).That is why we chose to only regularize instead of iterating and regularizing. The well-posedness of the regularized equation will follow from Chapter 2. In addition, the incompatibility of (6.6) and the condition in [49] makes us doubt that Girsanov transform techniques would work and that the law of (6.1) is absolutely continuous w.r.t. Wiener's measure.

The next objective is to use Theorem 6.2.3 to get a well-posedness of the Keller - Segel model in $d = 2$. The system reads

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) = \nabla\cdot(\dfrac{1}{2}\nabla\rho(t,x) - \chi\rho(t,x)\nabla c(t,x)), \quad t>0, \quad x\in\mathbb{R}^2, & (6.9a)\\ \dfrac{\partial c}{\partial t}(t,x) = \dfrac{1}{2}\triangle c(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0, \quad x\in\mathbb{R}, & (6.9b)\\ \rho(0,x) = \rho_0(x), \quad c(0,x) = c_0, \end{cases}$$

where $\chi > 0$ and $\lambda \geq 0$. The parameter χ is called the chemotactic sensitivity and, together with the total mass $M := \int \rho_0(x)\, dx$, plays an important role in the well-posedness theory for (6.9). Notice that the two diffusion coefficient are deliberately chosen to be equal to $\frac{1}{2}$ in order to have unit diffusion coefficient and standard Gaussian kernel in the formulation of (6.1).

As seen in Section 1.2, Keller-Segel system was constructed to model the onset of cell aggregation due to chemotactic behaviour of slime molds. Therefore, it is no wonder that critical regimes in which the solutions blow-up in finite time have been found in the literature. The definition of this phenomenon is the following:

$$\exists\ T_0 < \infty : \sup_{t\leq T_0}(\|\rho_t\|_{L^\infty(\mathbb{R}^2)} + \|c_t\|_{L^\infty(\mathbb{R}^2)}) = \infty.$$

Indeed, the question of global existence versus blow-up in $d = 2$ was extensively studied in the PDE literature. We have no intention to review all of it here, but rather mention some of the results. A very thorough review can be found in Horstmann [41].

In the parabolic-elliptic version of the system, i.e. when (6.9b) is in steady state, the behaviour of the system has been completely understood. There, the system exhibits the "threshold" behaviour: if $M\chi < 8\pi$ the solutions are global in time, if $M\chi > 8\pi$ every solution blows-up in finite time (see e.g. Blanchet *et. al* [8] and Nagai and Ogawa [56]).

On the other hand, the fully parabolic model (6.9) expresses a less straight-forward behaviour. It has been proved that when $M\chi < 8\pi$ one has global existence (see Calvez and Corrias [20], Mizogouchi [55]). However, in Biller *et. al* [7] the authors find an initial configuration of the system in which a global solution in some sense exists with $M\chi > 8\pi$. Finaly, Herrero and Velázquez [38] construct a radially symmetric solution on a disk that blows-up and develops δ-function type singularities. Finally, unique solution with any positive mass exists under some condition on the reverse diffusion coefficient of the chemoattractant and initial datum (Corrias *et. al* [22]). Thus, in the case of parabolic-parabolic model, the value 8π can still be understood as a threshold, but in a different sense: below it there is global existence, above it there exists a solution that blows up.

The new functions $\tilde{\rho}(t,x) := \frac{\rho(t,x)}{M}$ and $\tilde{c}(t,x) := \frac{c(t,x)}{M}$ satisfy the system (6.9) with the new parameter $\tilde{\chi} := \chi M$. Therefore, w.l.o.g. we may and do thereafter assume that $M = 1$. We consider the following notion of solution to (6.9):

Definition 6.2.4. *Given the functions ρ_0 and c_0, and the constants $\chi > 0$, $\lambda \geq 0$, $T > 0$, the pair (ρ, c) is said to be a solution to (6.9) if $\rho(t,\cdot)$ is a probability density function for every $0 \leq t \leq T$, one has*

$$\forall 1 < q < \infty \ \exists C_q > 0 : \quad \sup_{t \leq T} t^{1-\frac{1}{q}} \|\rho(t,\cdot)\|_{L^q(\mathbb{R}^2)} \leq C_q,$$

and the following equality

$$\rho(t,x) = g_t * \rho_0(x) + \chi \sum_{i=1}^{2} \int_0^t \nabla_i g_{t-s} * \nabla_i c(s,\cdot)\ \rho(s,\cdot))(x)\ ds \tag{6.10}$$

is satisfied in the sense of the distributions with

$$c(t,x) = e^{-\lambda t}(g(t,\cdot\ ) * c_0)(x) + \int_0^t e^{-\lambda s}(g_s * \rho(t-s,\cdot))(x)\ ds. \tag{6.11}$$

Notice that the function $c(t,x)$ defined by (6.11) is a mild solution to (6.9b). These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model (see [22] and references therein).

A consequence of Theorem 6.2.3 is the well-posedness of (6.9).

Corollary 6.2.5. *Let ρ_0 a probability density function and $c_0 \in H^1(\mathbb{R}^2)$. Under the condition (6.8) the system (6.9) admits a unique solution in the sense of Definition 6.2.4.*

In [20] the authors obtain the global existence in sub-critical case assuming:

i) $\rho_0 \in L^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2, \log(1+|x|^2)dx)$ and $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$;

ii) $c_0 \in H^1(\mathbb{R}^2)$ if $\lambda > 0$ or $c_0 \in L^1(\mathbb{R}^2)$ and $|\nabla c_0| \in L^2(\mathbb{R}^2)$ if $\lambda = 0$;

iii) $\rho_0\ c_0 \in L^1(\mathbb{R}^2)$.

We should emphasize that their sub-critical condition translates into $4\chi < 8\pi$ for (6.9) due to the additional diffusion coefficients in it. In the same sub-critical case, the global existence result is obtained in [55] assuming $\rho_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $c_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Our result does not assume any additional conditions other than that ρ_0 is a probability density function and $c_0 \in H^1(\mathbb{R}^2)$. The price to pay is the smallness condition (6.8) that not just involves the parameter χ, but the size of the initial datum as well.

Corollary 6.2.5 is very similar to the result in [22, Thm. 2.1]. Indeed, the assumptions on initial conditions are the same and as well the notion of solution. The objective is different in the sense that the goal of [22] was to exhibit global existence for (6.9) for any positive mass and $\chi = 1$ as long as the following two conditions are satisfied

C1: There exists $\delta = \delta(M, \alpha)$ such that $\|\nabla c_0\|_{L^2(\mathbb{R})} < \delta$,

C2: There exists $C(\alpha)$ such that $M < C(\alpha)$,

where α is the inverse diffusion coefficient of the chemo-attractant (see (1.4)). The condition on the total mass is similar to (6.8) on χ, but as C grows with α, one can have M as large as one likes as soon as α is large enough as well (see Section 1.3 for more details). In this chapter the objective is to get results for the classical K-S model ($\alpha = 1$) with respect to chemo-attractant sensitivity (and mass). When we assume the same in the framework of [22], we see that we have removed the assumption on the smallness of the initial datum (C1). The reason lies in our method: in [22] the Banach's fixed point is used to construct solutions locally in time (where C1 emerges) and then such solution is globalized (where C2 emerges). In our case only a condition of C2 type appears as, thanks to our regularization procedure, we directly construct a global solution. The well-posedness of the regularized equation comes from Chapter (2).

Finally, using the so-called transfer of uniqueness we prove the weak uniqueness for (6.1). Namely, we will use the results in Trevisan [76] to prove the following theorem:

Theorem 6.2.6. *Under a smallness condition on χ precised in Section 6.6, weak uniqueness in the sense of Definition 6.2.1 holds for Eq.* (6.1)*.*

6.3 Regularization

We define the regularized version of the interaction kernel K and the linear part of the drift as follows. For $\varepsilon > 0$ and $(t, x) \in (0, T) \times \mathbb{R}^2$ define

$$K_t^\varepsilon := -\frac{x}{2\pi(t+\varepsilon)^2} e^{-\frac{|x|^2}{2t}}, \quad g_t^\varepsilon(x) =: \frac{1}{2\pi(t+\varepsilon)} e^{-\frac{|x|^2}{2t}} \quad \text{and} \quad b_0^\varepsilon(t,x) := \chi e^{-\lambda t}(\nabla c_0 * g_t^\varepsilon)(x).$$

The regularized Mc-Kean-Vlasov equation reads

$$\begin{cases} dX_t^\varepsilon = dW_t + b_0^\varepsilon(t, X_t^\varepsilon)dt + \chi\Big\{ \int_0^t e^{-\lambda(t-s)} \int K_{t-s}^\varepsilon(X_t^\varepsilon - y)\mu_s^\varepsilon(dy)\, ds \Big\} dt, t \le T, \\ \mu_s^\varepsilon := \mathcal{L}(X_s^\varepsilon), \quad X_0^\varepsilon \sim p_0, \end{cases} \tag{6.12}$$

Set

$$b^\varepsilon(t, x; \mu^\varepsilon) := b_0^\varepsilon(t, x) + \chi \int_0^t e^{-\lambda(t-s)} \int K_{t-s}^\varepsilon(x - y)\mu_s^\varepsilon(dy)\, ds.$$

It is clear that there exists $C_\varepsilon > 0$ such that for any $t \in (0,T)$, one has

$$\forall x,y \in \mathbb{R}^2, \quad |b_0^\varepsilon(t,x) - b_0^\varepsilon(t,y)| + |K_t^\varepsilon(x) - K_t^\varepsilon(y)| \leq C_\varepsilon |x-y| \quad \text{and} \quad |b_0^\varepsilon(t,x)| + |K_t^\varepsilon(x)| \leq C_\varepsilon. \tag{6.13}$$

Notice that $C_\varepsilon \to \infty$ as $\varepsilon \to 0$. Similar computations as the ones to get (6.2) and (6.3) lead to the following estimates. For $t \in (0,T]$ and $1 \leq q < \infty$, one has

$$\|K_t^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)} \leq \frac{C_1(q)}{(t+\varepsilon)^{\frac{3}{2}-\frac{1}{q}}} \quad \text{and} \quad \|g_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq \frac{C_2(q)}{(t+\varepsilon)^{1-\frac{1}{q}}}. \tag{6.14}$$

Proposition 6.3.1. *Let $T > 0, \chi > 0$, $\nabla c_0 \in L^2(\mathbb{R}^2)$ and p_0 a probability density function on $\mathbb{R}^2$. Then, for any $\varepsilon > 0$, Equation (6.12) admits a unique strong solution. Moreover, the one dimensional time marginals of the law of the solution admit probability density functions, $(p_t^\varepsilon)_{t\leq T}$. In addition, for $t \in (0,T)$, p_t^ε satisfies the following mild equation in the sense of the distributions:*

$$p_t^\varepsilon = g_t * p_0 - \sum_{i=1}^{2} \int_0^t \nabla_i g_{t-s} * (p_s^\varepsilon b^{\varepsilon,i}(s,\cdot;p^\varepsilon))ds. \tag{6.15}$$

Proof. In view of (6.13) and Theorem 2.2.3, the strong solution to Equation (6.12) is uniquely well defined. In addition, as the drift term is bounded, we can apply Girsanov's transformation and conclude that the one dimensional time marginals of the law of the solution admit probability density functions. By classical arguments (see Chapter 2), one can prove that for $t \in (0,T)$, p_t^ε satisfies (6.15) in sense of the distributions. □

In the sequel, for $1 < q < \infty$, uniform in ε estimates on $\sup_{t\leq T} t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)}$ will be crucial. They will imply uniform in ε estimates on $\sup_{t\leq T} t^{\frac{1}{2}-\frac{1}{r}} \|b^{\varepsilon,i}(t,\cdot)\|_{L^r(\mathbb{R}^2)}$ for $2 \leq r \leq \infty$. In particular, for $i = 1,2$ and $t \in (0,T]$,

$$\|b^{\varepsilon,i}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\sqrt{t}}.$$

The latter will enable us to prove tightness of the probability laws of (X^ε).

6.3.1 Density estimates

For $0 < a,b < 1$, we denote

$$\beta(a,b) := \int_0^1 \frac{1}{u^a\,(1-u)^b} du. \tag{6.16}$$

Now we prove some technical lemmas that will be used throughout this chapter.

Lemma 6.3.2. *Let $t > 0$ and $0 < a,b < 1$. Then,*

$$\int_0^t \frac{1}{s^a(t-s)^b}\, ds = t^{1-(a+b)}\beta(a,b).$$

Proof. Observe that

$$\int_0^t \frac{1}{s^a(t-s)^b}\, ds = \frac{1}{t^{(a+b)}} \int_0^t \frac{1}{\left(\frac{s}{t}\right)^a \left(1-\frac{s}{t}\right)^b}\, ds.$$

The change of variables $\frac{s}{t} = u$ implies the result. □

Lemma 6.3.3. *Let $t > 0$. Then, the function $b_0^i(t,\cdot)$ is continuous on $\mathbb{R}^2$ and for $r \in [2,\infty]$, one has*

$$\|b_0^i(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{t^{\frac{1}{2}-\frac{1}{r}}}.$$

Proof. As $\nabla_i c_0$ is only in $L^2(\mathbb{R}^2)$ we can not apply the classical results of convolution with a continuous function. The continuity of $b_0^i(t,\cdot) = \chi \nabla_i c_0 * g_t$ is a direct consequence of [15, Ex. 4.30-3.] as for a $t > 0$ both g_t and $\nabla_i c_0$ belong to $L^2(\mathbb{R}^2)$. However, in this case, one can use the particular form of the functions involved in the convolution to prove the continuity. Let $x_n \to x$ in $\mathbb{R}^2$ as $n \to \infty$. To prove $g_t * \nabla_i c_0(x_n) \to g_t * \nabla_i c_0(x)$ we need to bound $|g_t(x_n - y)\nabla_i c_0(y)|$ with an $h(y) \in L^1(\mathbb{R}^2)$. As g_t is continuous we would then apply the dominated convergence theorem. Let $R > 0$. Then there exists $n_0 \ge 1$ such that for $n \ge n_0$ one has that $|x_n - x| \le R$. Then, by reverse triangular inequality one has that

$$e^{-\frac{|x_n-y|^2}{2t}} \le e^{-\frac{(|x_n-x|-|y-x|)^2}{2t}} \le e^{\frac{R^2}{2t}} e^{-\frac{(|y-x|-R)^2}{2t}}.$$

Thus, we define $h(y) = \nabla_i c_0(y) \frac{1}{2\pi t} e^{\frac{R^2}{2t}} e^{-\frac{(|y-x|-R)^2}{2t}}$ and conclude $h \in L^1(\mathbb{R}^2)$ by Cauchy-Schwarz inequality.

Let $q \ge 1$ be such that $\frac{1}{q} + \frac{1}{2} = 1 + \frac{1}{r}$. By the convolution inequality (3.38)

$$\|b_0^i(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \chi \|\nabla_i c_0\|_{L^2(\mathbb{R}^2)} \|g_t\|_{L^q(\mathbb{R}^2)}.$$

In view of estimates on $\|g_t\|_{L^q(\mathbb{R}^2)}$ and the relation above between r and q, one has

$$\|b_0^i(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \|g_t\|_{L^{\frac{2r}{r+2}}} \le \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{t^{1-(\frac{1}{r}+\frac{1}{2})}}.$$

□

Repeating the arguments as in the preceding proof, one gets

Lemma 6.3.4. *For $t > 0$ and $r \in [2,\infty]$ one has*

$$\|b_0^{\varepsilon,i}(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \chi \|\nabla c_0\|_{L^2(\mathbb{R}^2)} \frac{C_2(\frac{2r}{r+2})}{(t+\varepsilon)^{\frac{1}{2}-\frac{1}{r}}}.$$

Lemma 6.3.5. *Let p_0 a probability density function on $\mathbb{R}^2$ and $1 < q < \infty$. One has*

$$\limsup_{t\to 0} t^{1-\frac{1}{q}} \|g_t * p_0\|_{L^q(\mathbb{R}^2)} = 0.$$

Proof. The proof is a special case of Lemma 8 in [16]. Let $f \in C_K(\mathbb{R}^2)$. Using the standard convolution inequality (3.37), one has

$$\begin{aligned}
t^{1-\frac{1}{q}} \|g_t * p_0\|_{L^q(\mathbb{R}^2)} &\le t^{1-\frac{1}{q}} \|g_t * f\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}} \|g_t * (p_0 - f)\|_{L^q(\mathbb{R}^2)} \\
&\le t^{1-\frac{1}{q}} \|g_t\|_{L^1(\mathbb{R}^2)} \|f\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}} \|g_t\|_{L^q(\mathbb{R}^2)} \|p_0 - f\|_{L^1(\mathbb{R}^2)} \\
&\le \|f\|_{L^q(\mathbb{R}^2)} t^{1-\frac{1}{q}} + C\|p_0 - f\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

Thus,

$$\limsup_{t\to 0} t^{1-\frac{1}{q}}\|g_t * p_0\|_{L^q(\mathbb{R}^2)} \le C\|p_0 - f\|_{L^1(\mathbb{R}^2)}.$$

Since f is arbitrary, the r.h.s. can be arbitrarily small (see e.g. [15, Theorem 4.3]) . $\square$

Define

$$\mathcal{N}_q^\varepsilon(t) := \sup_{s\in(0,t)} s^{1-\frac{1}{q}}\|p_s^\varepsilon\|_{L^q(\mathbb{R}^2)}. \tag{6.17}$$

The following lemma provides a first estimate for $\mathcal{N}_q^\varepsilon(t)$ for a fixed $\varepsilon > 0$. This estimate is not the optimal one in ε, but it is necessary in order to be sure that all the quantities we work with are well defined. Also, it will be used in order to obtain the limit behaviour of $\mathcal{N}_q^\varepsilon(t)$ as $t \to 0$.

Lemma 6.3.6. *Let $0 < t \le T$ and $\varepsilon > 0$ fixed. For any $1 < q < \infty$, there exists $C_\varepsilon(T,\chi) > 0$ such that*

$$\mathcal{N}_q^\varepsilon(t) \le C_\varepsilon(T,\chi). \tag{6.18}$$

Moreover, one has

$$\lim_{t\to 0}\mathcal{N}_q^\varepsilon(t) = 0. \tag{6.19}$$

As K^ε is smooth, we can propose a simplified version of the arguments in [16, p. 285-286] for the proof of (6.19).

Proof. The drift of the regularized stochastic equation is bounded. Indeed, $|K_t^\varepsilon| \le \frac{C}{\varepsilon^{3/2}}$ and Lemma 6.3.4 imply

$$\|b^{\varepsilon,i}(t,\cdot;p^\varepsilon)\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{\sqrt{\varepsilon}} + \frac{Ct}{\varepsilon^{3/2}} =: C_\varepsilon(1+t).$$

For $1 < q < \infty$ and q' such that $\frac{1}{q} + \frac{1}{q'} = 1$ integrate (6.15) w.r.t. a test function $f \in L^{q'}(\mathbb{R}^2)$ and apply Hölder's inequality. It comes

$$\left|\int p_t^\varepsilon(x)f(x)dx\right| \le \|f\|_{L^{q'}(\mathbb{R}^2)}\left(\|g_t * p_0\|_{L^q(\mathbb{R}^2)} + \sum_{i=1}^2\int_0^t \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)}ds\right). \tag{6.20}$$

a) Assume $1 < q < 2$. The above drift bound and the convolution inequality (3.37) applied in (6.20), lead to

$$\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \le \|g_t * p_0\|_{L^q(\mathbb{R}^2)} + C_\varepsilon(1+t)\sum_{i=1}^2\int_0^t \|\nabla_i g_{t-s}\|_{L^q(\mathbb{R}^2)}\|p_s^\varepsilon\|_{L^1(\mathbb{R}^2)}ds.$$

In view of (6.2), we deduce that

$$\int_0^t \|\nabla_i g_{t-s}\|_{L^q(\mathbb{R}^2)}\|p_s^\varepsilon\|_{L^1(\mathbb{R}^2)}\, ds \le C_q\int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{q}}} = C_q t^{\frac{1}{q}-\frac{1}{2}}.$$

Thus,

$$t^{1-\frac{1}{q}}\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \le t^{1-1/q}\|g_t * p_0\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}+\frac{1}{q}-1/2}C_\varepsilon(1+t). \tag{6.21}$$

To get (6.18), in (6.21) use the convolution inequality (3.37) and that $\|g_t\|_{L^q(\mathbb{R}^2)} = \frac{C}{t^{1-\frac{1}{q}}}$. To get (6.19) use Lemma 6.3.5 for the first term of the r.h.s. of (6.21) and the fact that the second term tends to zero as $t \to 0$.

b) Let $q \geq 2$ and $\frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{2} + \frac{1}{2q}$. Then, $1 < p_1, p_2 < 2$ and $1 + \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. The convolution inequality (3.38) and the drift estimate applied in (6.20), lead to

$$\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq \|g_t * p_0\|_{L^q(\mathbb{R}^2)} + C_\varepsilon(1+T)\sum_{i=1}^{2}\int_0^t \|\nabla_i g_{t-s}\|_{L^{p_1}(\mathbb{R}^2)}\|p_s^\varepsilon\|_{L^{p_2}(\mathbb{R}^2)}ds.$$

In view of (6.2) and the result in a), one has

$$t^{1-\frac{1}{q}}\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq t^{1-\frac{1}{q}}\|g_t * p_0\|_{L^q(\mathbb{R}^2)} + C_\varepsilon(1+T)t^{1-\frac{1}{q}}\int_0^t \frac{C_q}{(t-s)^{\frac{3}{2}-\frac{1}{p_1}}}\frac{C(\varepsilon,T)}{s^{1-\frac{1}{p_2}}}ds.$$

Apply Lemma 6.3.2 and use the relation between the exponents. It comes:

$$t^{1-\frac{1}{q}}\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq t^{1-\frac{1}{q}}\|g_t * p_0\|_{L^q(\mathbb{R}^2)} + t^{1-\frac{1}{q}}\frac{C(\varepsilon,T)}{t^{\frac{1}{2}-\frac{1}{q}}}\beta(1-\frac{1}{p_2},\frac{3}{2}-\frac{1}{p_1}).$$

Repeating the last steps as in a), one can obtain the desired result.

□

The following proposition enables one to control $f_q^\varepsilon(t)$ for a fixed q and uniformly on small ε.

Proposition 6.3.7. *Let $T > 0$ and fix a $q \in (2,4)$. Then, there exists $C > 0$ such that for any $t \in (0,T]$, $f_q^\varepsilon(t)$ defined in (6.17) satisfies*

$$\forall 0 < \varepsilon < 1: \quad \mathcal{N}_q^\varepsilon(t) \leq C,$$

provided that

$$A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + B\sqrt{\chi} < 1 \tag{6.22}$$

where, C_1, C_2 and $\beta(\cdot,\cdot)$ being defined as in Lemmas 6.1.1, 6.1.2 and Eq. 6.16 respectively,

$$A = C_1(q')C_2(\frac{2q}{q+2})\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'}) \quad \textit{and} \quad B = 2\sqrt{C_2(q)C_1(q')C_1(1)\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'})\beta(1-\frac{1}{q},\frac{1}{2})}.$$

Proof. Let $q' > 1$ be such that $\frac{1}{q} + \frac{1}{q'} = 1$. Integrating (6.15) w.r.t. a test function $f \in L^{q'}(\mathbb{R}^2)$, one again starts from

$$\left|\int p_t^\varepsilon(x)f(x)dx\right| \leq \|f\|_{L^{q'}(\mathbb{R}^2)}\left(\|g_t * p_0\|_{L^q(\mathbb{R}^2)} + \sum_{i=1}^{2}\int_0^t \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)}ds\right). \tag{6.23}$$

Let us fix $i \in \{1,2\}$, $s < t$ and denote $A_s^i := \|\nabla_i g_{t-s} * (p_s^\varepsilon b_s^{\varepsilon,i})\|_{L^q(\mathbb{R}^2)}$. Observe that $\frac{1}{q'} + \frac{2}{q} = 1 + \frac{1}{q}$. Apply the convolution inequality (3.38) and then use (6.2). It comes

$$A_s^i \leq \|\nabla_i g_{t-s}\|_{L^{q'}(\mathbb{R}^2)}\|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \leq \frac{C_1(q')\|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)}s^{1-\frac{1}{q}}\|p_s^\varepsilon\|_{L^q(\mathbb{R}^2)}}{(t-s)^{\frac{3}{2}-\frac{1}{q'}}s^{1-\frac{1}{q}}} \leq C_1(q')\mathcal{N}_q^\varepsilon(t)\frac{\|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)}}{(t-s)^{\frac{3}{2}-\frac{1}{q'}}s^{1-\frac{1}{q}}}.$$

In view of Lemma 6.3.4, (6.14) and Lemma 6.3.2, we get

$$\begin{aligned}\|b_s^{\varepsilon,i}\|_{L^q(\mathbb{R}^2)} &\leq \frac{C_2(\frac{2q}{q+2})\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{(s+\varepsilon)^{\frac{1}{2}-\frac{1}{q}}} + \chi\int_0^s \|K_{s-u}^{i,\varepsilon}\|_{L^1(\mathbb{R}^2)}\|p_u^\varepsilon\|_{L^q(\mathbb{R}^2)}du \\ &\leq \frac{C_2(\frac{2q}{q+2})\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{s^{\frac{1}{2}-\frac{1}{q}}} + \chi C_1(1)\mathcal{N}_q^\varepsilon(t)\int_0^s \frac{1}{\sqrt{s-u}\; u^{1-\frac{1}{q}}}\,ds \\ &\leq \frac{C_2(\frac{2q}{q+2})\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + \chi C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1-\frac{1}{q},\frac{1}{2})}{s^{\frac{1}{2}-\frac{1}{q}}}.\end{aligned}$$

It comes

$$A_s^i \leq C_1(q')\chi\mathcal{N}_q^\varepsilon(t)\frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1-\frac{1}{q},\frac{1}{2})}{(t-s)^{\frac{3}{2}-\frac{1}{q'}}s^{\frac{3}{2}-\frac{2}{q}}}.$$

Plug this into (6.20). The condition $q \in (2,4)$ ensures that $\frac{3}{2}-\frac{2}{q} < 1$ and $\frac{3}{2}-\frac{1}{q'} < 1$. Thus, Lemma 6.3.2 leads to

$$\begin{aligned}&\left|\int p_t^\varepsilon(x)f(x)dx\right| \leq \|f\|_{L^{q'}(\mathbb{R}^2)}\Big(\|g_t * p_0\|_{L^q(\mathbb{R}^2)} \\ &+ 2C_1(q')\chi\mathcal{N}_q^\varepsilon(t)\frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1-\frac{1}{q},\frac{1}{2})}{t^{1-\frac{1}{q}}}\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'})\Big).\end{aligned}$$

Take $\sup_{\|f\|_{L^{q'}}=1}$ in the preceding inequality. It follows from the convolution inequality (3.37) and (6.3) that

$$\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)} \leq \frac{C_2(q)}{t^{1-\frac{1}{q}}} + 2C_1(q')\chi\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'})\mathcal{N}_q^\varepsilon(t)\frac{C_2(\frac{2q}{q+2})\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + C_1(1)\mathcal{N}_q^\varepsilon(t)\beta(1-\frac{1}{q},\frac{1}{2})}{t^{1-\frac{1}{q}}}.$$

Let us denote

$$K_1 := 2C_1(q')C_1(1)\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'})\beta(1-\frac{1}{q},\frac{1}{2}) \text{ and } K_2 := 2C_1(q')C_2(\frac{2q}{q+2})\beta(\frac{3}{2}-\frac{2}{q},\frac{3}{2}-\frac{1}{q'}).$$

After rearranging the terms,

$$0 \leq K_1\chi(\mathcal{N}_q^\varepsilon(t))^2 + (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)\mathcal{N}_q^\varepsilon(t) + C_2(q). \tag{6.24}$$

Under the assumptions

$$K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1 < 0 \text{ and } (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)^2 - 4K_1C_2(q)\chi > 0,$$

the polynomial function

$$P(z) = K_1\chi z^2 + (K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - 1)z + C_2(q)$$

admits two positive roots. In view of Lemma 6.3.6 and (6.24), one has that $\lim_{t\to 0}\mathcal{N}_q^\varepsilon(t) = 0$ and $P(\mathcal{N}_q^\varepsilon(t)) > 0$ for any $t \in [0,T]$. Necessarily, for any $t \in [0,T]$ $\mathcal{N}_q^\varepsilon(t)$ is bounded from above by the smaller root of the polynomial function $P(z)$. As the constants do not depend on T and ε, this estimate is uniform in time and does not depend on the regularization parameter.

Notice that the above condition is equivalent to

$$K_2\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} + 2\sqrt{K_1C_2(q)\chi} < 1.$$

Denote $A := K_2$ and $B := 2\sqrt{C_2(q)K_1}$ to finish the proof. □

Remark 6.3.8. *The upper bound C of $\mathcal{N}_q^\varepsilon(t)$ is given by*

$$C = \frac{1 - A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)} - \sqrt{(1 - A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)})^2 - B^2\chi}}{2K_1\chi}.$$

Now, we will continue analyzing $\mathcal{N}_r^\varepsilon(t)$, for different values of r. We will see that different arguments are used when $r < q$ and $r > q$. The result obtained for $r < q$ will be used to control $\|b_t^\varepsilon\|_{L^r(\mathbb{R}^2)}$, for $r \geq 2$.

Corollary 6.3.9. *Same assumptions as in Proposition 6.3.7. Then, for $1 < r < q$, it holds*

$$\forall\ 0 < \varepsilon < 1, \quad \mathcal{N}_r^\varepsilon(T) \leq C_r.$$

Proof. Let $1 < r < q$. Define $\theta := \frac{1-\frac{1}{r}}{1-\frac{1}{q}}$. Then, $\frac{1}{r} = 1 - \theta + \frac{\theta}{q}$. As $p_t^\varepsilon \in L^1(\mathbb{R}^2)$, "interpolation inequalities" (see [15, p. 93]) lead to

$$\|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \|p_t^\varepsilon\|_{L^1(\mathbb{R}^2)}^{1-\theta}\|p_t^\varepsilon\|_{L^q(\mathbb{R}^2)}^{\theta} \leq \frac{C^\theta}{t^{\theta(1-\frac{1}{q})}} =: \frac{C_r}{t^{1-\frac{1}{r}}}.$$

□

Corollary 6.3.10. *Same assumptions as in Proposition 6.3.7. Then, for $2 \leq r \leq \infty$,*

$$\forall\ 0 < \varepsilon < 1, \quad \|b_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C_r(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{t^{\frac{1}{2}-\frac{1}{r}}}$$

Proof. In view of Lemma 6.3.4, one has for $i \in \{1, 2\}$

$$\|b_t^{i,\varepsilon}\|_{L^r(\mathbb{R}^2)} \leq \frac{C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{t^{\frac{1}{2}-\frac{1}{r}}} + \chi\int_0^t \|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)}ds. \tag{6.25}$$

a) For $r \in [2, q)$, Corollary 6.3.9 immediately implies

$$\|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{\sqrt{t-s}s^{1-\frac{1}{r}}}.$$

b) For $q \leq r \leq \infty$, choose p_1 such that $\frac{1}{p_1} := 1 + \frac{1}{r} - \frac{1}{q}$. Notice that, as $2 < q \leq r$, it follows that $\frac{1}{2} < \frac{1}{p_1} \leq 1$. Applying the convolution inequality (3.38) and Corollary 6.3.9, one has

$$\|K_{t-s}^{\varepsilon,i} * p_s^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p_1}}s^{1-\frac{1}{q}}}.$$

To finish the proof, in both cases, one plugs the obtained estimates in (6.25) and applies Lemma 6.3.2. □

Corollary 6.3.11. *Same assumptions as in Proposition 6.3.7. Then, for $q < r < \infty$, one has*

$$\forall\ 0 < \varepsilon < 1, \quad \mathcal{N}_r^\varepsilon(T) \leq C_r.$$

Proof. Let $1 < q_1, q_2 < 2$ such that $\frac{1}{q_1} = \frac{1}{q_2} = \frac{1}{2} + \frac{1}{2r}$. Then, $1 + \frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Convolution inequality (3.38) leads to

$$\|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq \|g_t * p_0\|_{L^r(\mathbb{R}^2)} + \sum_{i=1}^{2} \int_0^t \|\nabla_i g_{t-s}\|_{L^{q_1}(\mathbb{R}^2)} \|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)}\, ds.$$

Let us apply Hölder's inequality for $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$ such that $\lambda_1 = \frac{q}{2}$,

$$\|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)} \leq \|p_s^\varepsilon\|_{L^{\lambda_1 q_2}(\mathbb{R}^2)} \|b_s^{\varepsilon,i}\|_{L^{\lambda_2 q_2}(\mathbb{R}^2)}.$$

Notice that $1 < \lambda_1 < 2$ since $2 < q < 4$ by hypothesis. Then, $\lambda_2 > 2$, thus $\lambda_2 q_2 > 2$. In addition, $\lambda_1 q_2 = \frac{q}{2} q_2 < q$. In view of Corollaries 6.3.9 and 6.3.10, one has

$$\|p_s^\varepsilon b_s^{\varepsilon,i}\|_{L^{q_2}(\mathbb{R}^2)} \leq \frac{C}{s^{1-\frac{1}{\lambda_1 q_2}+\frac{1}{2}-\frac{1}{\lambda_2 q_2}}} = \frac{C}{s^{\frac{3}{2}-\frac{1}{q_2}}}.$$

Therefore,

$$t^{1-\frac{1}{q}} \|p_t^\varepsilon\|_{L^r(\mathbb{R}^2)} \leq C + t^{1-\frac{1}{q}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{q_1}} s^{\frac{3}{2}-\frac{1}{q_2}}}\, ds.$$

Apply Lemma 6.3.2 to finish the proof. □

Notice that the choice of the constants A and B depends only on the initially chosen $q \in (2, 4)$. One may analyze the constants in Condition (6.22) in function of q to get an optimal condition on χ.

6.4 Proof of Theorem 6.2.3

6.4.1 Tightness

Proposition 6.4.1. *Let $T > 0$. Denote $\varepsilon_k = \frac{1}{k}$, for $k \in \mathbb{N}$. $\mathbb{P}^k$ denotes the law of the solutions to (6.12) regularized with ε_k. If the initial law p_0 is a probability density, $\nabla c_0 \in L^2(\mathbb{R}^2)$ and $\chi > 0$ are such that Condition (6.22) is satisfied, then the probability laws $(\mathbb{P}^k)_{k\geq 1}$ are tight in $C([0, T]; \mathbb{R}^2)$ w.r.t. $k \in \mathbb{N}$.*

Proof. For $m > 2$ and $0 < s < t \leq T$, notice that

$$\mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|^m \leq \mathbb{E}\left(\left(\int_s^t b^{\varepsilon,1}(u, X_u^\varepsilon) du\right)^2 + \left(\int_s^t b^{\varepsilon,2}(u, X_u^\varepsilon) du\right)^2\right)^{\frac{m}{2}} + \mathbb{E}|W_t - W_s|^m.$$

In view of the drift estimate for $r = \infty$ in Corollary 6.3.10, one has

$$\mathbb{E}|X_t^\varepsilon - X_s^\varepsilon|^m \leq \left(2\int_s^t \frac{C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})}{\sqrt{u}} du\right)^m + C(t-s)^{\frac{m}{2}} \leq C(\chi, \|\nabla c_0\|_{L^2(\mathbb{R}^2)})(t-s)^{\frac{m}{2}}.$$

Kolmogorov's criterion implies tightness. □

6.4.2 Existence

In order to prove the existence of a weak solution, we will prove that the following non-linear martingale problem related to (6.1) admits a solution under the hypothesis of Theorem 6.2.3.

Definition 6.4.2. *A probability measure $\mathbb{Q}$ on the canonical space $\mathcal{C}([0,T];\mathbb{R}^2)$ equipped with its canonical filtration and a canonical process (w_t) is a solution to the non-linear martingale problem (MP) if:*

(i) $\mathbb{Q}_0 = p_0$.

(ii) For any $t \in (0,T]$, the one dimensional time marginals of $\mathbb{Q}$, denoted by $\mathbb{Q}_t$, have densities q_t w.r.t. Lebesgue measure on $\mathbb{R}$. In addition, they satisfy

$$\forall r \in (1,\infty)\ \exists C>0: \quad \sup_{t\in(0,T)} t^{1-\frac{1}{r}} \|q_t\|_{L^r(\mathbb{R}^2)} \leq C.$$

(iii) For any $f \in C_K^2(\mathbb{R}^2)$ the process $(M_t)_{t\leq T}$, defined as

$$M_t := f(w_t)-f(w_0)-\int_0^t [\frac{1}{2}\triangle f(w_u)+\nabla f(w_u)\cdot(b_0(u,w_u)+\chi\int_0^u\int K_{u-\tau}(w_u-y)q_\tau(y)dyd\tau)]du$$

is a $\mathbb{Q}$-martingale.

In view of Proposition 6.4.1, there exists a weakly convergent subsequence of $(\mathbb{P}^k)_{k\geq1}$ that we will still denote by $(\mathbb{P}^k)_{k\geq1}$. Denote its limit by $\mathbb{P}^\infty$. Let us prove that $\mathbb{P}^\infty$ solves the martingale problem (MP).

i) Each $\mathbb{P}_0^k$ has density p_0, and therefore $\mathbb{P}_0^\infty$ also has density p_0.

ii) Define the functional $\Lambda_t(\varphi)$ by

$$\Lambda_t(\varphi) := \int_{\mathbb{R}^2} \varphi(y)\mathbb{P}_t^\infty(dy), \quad \varphi \in C_K(\mathbb{R}^2).$$

By weak convergence we have

$$\Lambda_t(\varphi) = \lim_{k\to\infty}\int \varphi(y)p_t^k(y)dy,$$

and thus for any $1<r<\infty$ and its conjugate r', in view of Proposition 6.3.7 and Corollaries 6.3.9 and 6.3.11 one has

$$|\Lambda_t(\varphi)| \leq \frac{C}{t^{1-\frac{1}{r}}}\|\varphi\|_{L^{r'}(\mathbb{R}^2)}.$$

Therefore, for each $0<t\leq T$, Λ_t is a bounded linear functional on a dense subset of $L^{r'}(\mathbb{R}^2)$. Thus, Λ_t can be extended to a linear functional on $L^{r'}(\mathbb{R}^2)$. By Riesz-representation theorem (e.g. [15, Thm. 4.11 and 4.14]), there exists a unique $p_t^\infty \in L^r(\mathbb{R}^2)$ such that $\|p_t^\infty\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{t^{1-\frac{1}{r}}}$ and p_t^∞ is the probability density of $\mathbb{P}_t^\infty(dy)$.

iii) Set

$$
\begin{aligned}
&M_t^\infty := f(w_t) - f(w_0)\\
&- \int_0^t [\triangle f(w_u) + \nabla f(w_u)\cdot (b_0(u,w_u) + \frac{\chi}{8\pi}\int_0^u e^{-\lambda(u-\tau)}\int K_{u-\tau}(w_u-y)p_\tau^\infty(y)\ dy\ d\tau)]du.
\end{aligned}
$$

In order to prove that $(M_t^\infty)_{t\le T}$ is a $\mathbb{P}^\infty$ martingale, we will check that for any $N\ge 1$, $0\le t_1<\dots<t_N<s\le t\le T$ and any $\phi\in C_b((\mathbb{R}^2)^N)$, one has

$$
\mathbb{E}_{\mathbb{P}^\infty}[(M_t^\infty - M_s^\infty)\phi(w_{t_1},\dots,w_{t_N})] = 0. \tag{6.26}
$$

As $\mathbb{P}^k$ solves the non–linear martingale problem related to (6.12) with $\varepsilon_k=\frac{1}{k}$, one has

$$
\begin{aligned}
M_t^k := f(w_t) - f(x(0)) - \chi\int_0^t [\triangle f(w_u) + \nabla f(w_u)\cdot (b_0^{\varepsilon_k}(u,w_u)\\
+ \frac{\chi}{8\pi}\int_0^u e^{-\lambda(u-\tau)}(K_{u-\tau}^{\varepsilon_k} * p_\tau^k)(w_u)d\tau)]du
\end{aligned}
$$

is a martingale under $\mathbb{P}^k$. Thus,

$$
\begin{aligned}
0 = \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(w_{t_1},\dots,w_{t_N})] = \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(w_t)-f(w_s))]\\
+\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \triangle f(w_u)du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0^{\varepsilon_k}(u,w_u)du]\\
+\frac{\chi}{8\pi}\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u e^{-\lambda(u-\tau)}(K_{u-\tau}^{\varepsilon_k} * p_\tau^k)(w_u)d\tau du].
\end{aligned}
$$

Since $(\mathbb{P}^k)$ weakly converges to $\mathbb{P}^\infty$, the first two terms on the r.h.s. converge to their analogues in (6.26). It remains to check the convergence of the last two terms. We will analyze separately the parts coming from the linear and non-linear drifts.

Linear part Notice that for $t>0$ and $x\in\mathbb{R}^2$

$$
|b_0^{\varepsilon_k}(t,x) - b_0(t,x)| \le C\frac{\chi e^{-\lambda t}\varepsilon_k}{t(t+\varepsilon_k)}\left|\int_{\mathbb{R}^2}\nabla c_0(x-y)e^{-\frac{|y|^2}{2t}}\ dy\right| \le \frac{\varepsilon_k\|\nabla c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}(t+\varepsilon_k)}.
$$

Thus, $\|b_0^{\varepsilon_k}(t,\cdot) - b_0(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\to 0, k\to\infty$ and from Lemmas 6.3.3 and 6.3.4 we have

$$
\|b_0^{\varepsilon_k}(t,\cdot) - b_0(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{\sqrt{t}}.
$$

Similarly, for $t>0$ and $r>2$,

$$
\|b_0^{\varepsilon_k}(t,\cdot) - b_0(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \|\nabla c_0\|_{L^2(\mathbb{R}^2)}\frac{\varepsilon_k}{t(t+\varepsilon_k)}Ct^{\frac{1}{r}+\frac{1}{2}}.
$$

Therefore,

$$
\|b_0^{\varepsilon_k}(t,\cdot) - b_0(t,\cdot)\|_{L^r(\mathbb{R}^2)} \to 0, k\to\infty \tag{6.27}
$$

and from Lemmas 6.3.3 and 6.3.4 we have

$$
\|b_0^{\varepsilon_k}(t,\cdot) - b_0(t,\cdot)\|_{L^r(\mathbb{R}^2)} \le \frac{C}{t^{\frac{1}{2}-\frac{1}{r}}}. \tag{6.28}
$$

Now, observe that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0^{\varepsilon_k}(u,w_u)du] - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0(u,w_u)du] \\
&= (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0^{\varepsilon_k}(u,w_u)du] - \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0(u,w_u)du]) \\
&+ (\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0(u,w_u)du] - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot b_0(u,w_u)du]) =: I_k + II_k.
\end{aligned}$$

We start from II_k. Define for $x \in C([0,T];\mathbb{R}^2)$ the functional

$$F(x) := \phi(x_{t_1},\dots,x_{t_N})\int_s^t \nabla f(x_u)\cdot b_0(u,x_u)du.$$

In view of Lemma 6.3.3, for $u>0$ and $i=1,2$, the function $b_0^i(u,\cdot)$ is bounded and continuous on $\mathbb{R}^2$ and one has $\|b_0^i(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{\sqrt{t}}$. By dominated convergence one gets that $F(\cdot)$ is continuous. In addition, $F(\cdot)$ is bounded on $C([0,T];\mathbb{R}^2)$. Thus, by weak convergence, $II_k \to 0$, as $k\to\infty$.

We turn to I_k:

$$|I_k| \le \|\phi\|_\infty \int_s^t \sum_{i=1}^2 \int_{\mathbb{R}^2} |\nabla_i f(z)(b_0^{\varepsilon_k,i}(u,z) - b_0^i(u,z))| p_u^k(z)\ dz\ ds.$$

Apply the Hölder's inequality for $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$ such that $1<\lambda<2$. In view of Corollary 6.3.9, one has

$$|I_k| \le \|\phi\|_\infty \|\nabla f\|_\infty \int_s^t \frac{C}{u^{1-\frac{1}{\lambda}}} \sum_{i=1}^2 \|b_0^{\varepsilon_k,i}(u,\cdot) - b_0^i(u,\cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} du.$$

In view of (6.27), $\|b_0^{\varepsilon_k,i}(u,\cdot) - b_0^i(u,\cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} \to 0$ as $k\to\infty$. In addition (6.28) leads to

$$\frac{C}{u^{1-\frac{1}{\lambda}}} \sum_{i=1}^2 \|b_0^{\varepsilon_k,i}(u,\cdot) - b_0^i(u,\cdot)\|_{L^{\lambda'}(\mathbb{R}^2)} \le \frac{C}{u^{\frac{1}{\lambda'}+\frac{1}{2}-\frac{1}{\lambda'}}}.$$

By dominated convergence, $I_k \to 0$, as $k\to\infty$.

Non-linear part Let us first analyze the difference of the two drifts. Fix $0<s<t$, $x\in\mathbb{R}^2$ and $i \in 1,2$. Notice that

$$|K_{t-s}^{\varepsilon_k,i}(x) - K_{t-s}^i(x)| \le \frac{(t-s)\varepsilon_k + \varepsilon_k^2}{(t-s)^2(t-s+\varepsilon_k)^2}|x^i| e^{\frac{|x|^2}{2(t-s)}}.$$

Thus for any $x\in\mathbb{R}^2$ and any $0<s<t$, we have that $|K_{t-s}^{\varepsilon_k,i}(x) - K_{t-s}^i(x)| \to 0, k\to\infty$. After integration, for any $1<r<2$ one has

$$\|K_{t-s}^{\varepsilon_k,i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \le C_r \frac{(t-s)\varepsilon_k + \varepsilon_k^2}{(t-s)^2(t-s+\varepsilon_k)^2}(t-s)^{\frac{1}{2}+\frac{1}{r}}.$$

Therefore, for any $0<s<t$, one gets

$$\|K_{t-s}^{\varepsilon_k,i} - K_{t-s}^i\|_{L^r(\mathbb{R}^2)} \to 0, k\to\infty. \tag{6.29}$$

In addition, (6.2) and (6.14) lead to

$$\|K^{\varepsilon_k,i}_{t-s} - K^i_{t-s}\|_{L^r(\mathbb{R}^2)} \leq \|K^{\varepsilon_k,i}_{t-s}\|_{L^r(\mathbb{R}^2)} + \|K^i_{t-s}\|_{L^r(\mathbb{R}^2)} \leq \frac{C_r}{(t-s)^{\frac{3}{2}-\frac{1}{r}}}. \tag{6.30}$$

For $t > 0$, $x \in \mathbb{R}^2$ and $i \in 1, 2$, one has

$$\left|\int_0^t (K^{\varepsilon_k,i}_{t-s} * p^k_s)(x)ds - \int_0^t (K^i_{t-s} * p^\infty_s)(x)ds\right| \leq \left|\int_0^t (K^{\varepsilon_k,i}_{t-s} * p^k_s)(x)ds - \int_0^t (K^i_{t-s} * p^k_s)(x)ds\right|$$
$$+ \left|\int_0^t (K^i_{t-s} * p^k_s)(x)ds - \int_0^t (K^i_{t-s} * p^\infty_s)(x)ds\right| =: A_k + B_k.$$

We start from B_k. For $s < t$ and $i = 1, 2$, the kernel $K^i_{t-s}(\cdot)$ is a continuous and bounded function on $\mathbb{R}^2$. Thus, by weak convergence we have that $\lim_{k\to\infty}(K^i_{t-s} * p^k_s)(x) = (K^i_{t-s} * p^\infty_s)(x)$. In addition, for $r > 2$ Hölder's inequality, part *ii)* and Proposition 6.3.7 lead to

$$|(K^i_{t-s} * p^k_s)(x) - (K^i_{t-s} * p^\infty_s)(x)| \leq \frac{C_r}{(t-s)^{\frac{3}{2}-\frac{1}{r'}} s^{1-\frac{1}{r}}}.$$

As the bound is integrable in $(0, t)$, the dominated convergence theorem implies that $B_k \to 0$, as $k \to \infty$.

In A_k we apply the Hölder's inequality with $1 < r < 2$ and the density bounds from Corollary 6.3.9. It comes

$$|A_k| \leq \int_0^t \|K^{\varepsilon_k,i}_{t-s} - K^i_{t-s}\|_{L^{r'}(\mathbb{R}^2)} \frac{C_r}{s^{1-\frac{1}{r}}}\, ds.$$

In view of (6.29) and (6.30), one can apply the dominated convergence. Thus, $A_k \to 0$, as $k \to \infty$. Finally, we obtain

$$\lim_{k\to\infty}\left|\int_0^t (K^{\varepsilon_k,i}_{t-s} * p^k_s)(x)ds - \int_0^t (K^i_{t-s} * p^\infty_s)(x)ds\right| = 0. \tag{6.31}$$

As in the linear part, we decompose

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K^{\varepsilon_k}_{u-\tau} * p^k_\tau)(w_u)d\tau du] \\
&- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K_{u-\tau} * p^\infty_\tau)(w_u)d\tau du] \\
&\leq \Big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K^{\varepsilon_k}_{u-\tau} * p^k_\tau)(w_u)d\tau du] \\
&- \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K_{u-\tau} * p^\infty_\tau)(w_u)d\tau du]\Big) \\
&+ \Big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K_{u-\tau} * p^\infty_\tau)(w_u)d\tau du] \\
&- \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t \nabla f(w_u)\cdot\int_0^u (K_{u-\tau} * p^\infty_\tau)(w_u)d\tau du]\Big) \\
&=: C_k + D_k.
\end{aligned}$$

Start from D_k. Similarly to the linear part, we need the boundness and continuity of the functional

$$H(x) := \phi(x(t_1),\dots,x(t_N))\int_s^t \nabla f(x(u))\cdot\int_0^u (K_{u-\tau}*p_\tau^\infty)(x(u))d\tau du, \quad x\in\mathcal{C}([0,T];\mathbb{R}^2).$$

The continuity comes from the fact that the kernel is a continuous function on $\mathbb{R}^2$ whenever $\tau<u$. Namely, if $x_n\in\mathcal{C}([0,T];\mathbb{R}^2)$ converges to $x\in\mathcal{C}([0,T];\mathbb{R}^2)$, then $K^i_{u-\tau}(x_n(u)-y)\to K^i_{u-\tau}(x(u)-y)$. In addition $|K^i_{u-\tau}(x_n(u)-y)p_\tau^\infty(y)|\le \frac{C}{(u-\tau)^{3/2}}p_\tau^\infty(y)$, for $i\in\{1,2\}$, as $n\to\infty$. Thus, by dominated convergence, for $\tau<u$ one has

$$K^i_{u-\tau}*p_\tau^\infty(x_n(u))\to K^i_{u-\tau}*p_\tau^\infty(x(u)),\ n\to\infty.$$

For $\frac{1}{r}+\frac{1}{r'}=1$ such that $r>2$ apply Hölder's inequality and after the estimate in *ii*). It comes

$$|K^i_{u-\tau}*p_\tau^\infty(x_n(u))|\le \frac{C_r}{(u-\tau)^{\frac{3}{2}-\frac{1}{r'}}\tau^{1-\frac{1}{r}}}.$$

By dominated convergence,

$$\int_0^u (K^i_{u-\tau}*p_\tau^\infty)(x_n(u))d\tau \to \int_0^u (K^i_{u-\tau}*p_\tau^\infty)(x(u))d\tau, n\to\infty.$$

Moreover, in view of Lemma 6.3.2, one has

$$\left|\nabla f(x_n(u))\cdot\int_0^u K_{u-\tau}*p_\tau^\infty(x_n(u))d\tau\right|\le C\|\nabla f\|_\infty \frac{\beta(1-\frac{1}{r},\frac{3}{2}-\frac{1}{r'})}{\sqrt{u}}.$$

Finally, after one more application of dominated convergence the continuity of the functional H follows. This procedure obviously implies H is a bounded functional on $\mathcal{C}([0,T];\mathbb{R}^2)$. Thus, by weak convergence, D_k converges to zero.

We turn to C_k. Let us just for this part denote by $b^i(u,z):=\int_0^u K^i_{u-\tau}*p_\tau^\infty(z)d\tau$ and $b^{k,i}(u,z):=\int_0^u K^{\varepsilon_k,i}_{u-\tau}*p_\tau^\infty(z)d\tau$. Notice that

$$|C_k|\le\|\phi\|_\infty\int_s^t\sum_{i=1}^2\int_{\mathbb{R}^2}|\nabla_i f(z)(b^{k,i}(u,z)-b^i(u,z))|p_u^k(z)dz.$$

After Hölder inequality for $\frac{1}{r}+\frac{1}{r'}=1$ such that $r>2$, one has

$$|C_k|\le\|\phi\|_\infty\int_s^t\frac{C}{u^{1-\frac{1}{r'}}}\sum_{i=1}^2\left(\int|\nabla_i f(z)|^r|b^{k,i}(u,z)-b^i(u,z)|^r dz\right)^{1/r}du.$$

Let $u>0$. In view of (6.31), $|b^{k,i}(u,z)-b^i(u,z)|^r\to 0$ as $k\to\infty$. Now, we do not omit $|\nabla_i f(z)|^q$ as in the linear part. Instead, we use it in order to integrate in space with respect to drift bounds. Namely, for $u>0$ and $i=1,2$, we have seen that $|b^{k,i}(u,\cdot)|+|b^i(u,\cdot)|\le\frac{C}{\sqrt{u}}$. Thus,

$$|\nabla_i f(z)|^r|b^{k,i}(u,z)-b^i(u,z)|^r\le\frac{C}{u^{\frac{r}{2}}}|\nabla_i f(z)|^r.$$

By dominated convergence,

$$\|\nabla_i f(\cdot)(b^{k,i}(u,\cdot)-b^i(u,\cdot))\|_{L^r(\mathbb{R}^2)}\to 0,\ k\to\infty.$$

Using that $\|b^{k,i}(u,\cdot)\|_{L^r(\mathbb{R}^2)} + \|b^i(u,\cdot))\|_{L^r(\mathbb{R}^2)} \leq \frac{C}{u^{\frac{1}{2}-\frac{1}{r}}}$, one gets

$$\frac{1}{u^{1-\frac{1}{r'}}}\|\nabla_i f(\cdot)(b^{k,i}(u,\cdot) - b^i(u,\cdot))\|_{L^r(\mathbb{R}^2)} \leq \|\nabla_i f(\cdot)\|_\infty \frac{C}{u^{\frac{1}{2}-\frac{1}{r}+1-\frac{1}{r'}}}$$

Thus, by dominated convergence, we get that $C_k \to 0$, as $k \to \infty$.

As all the terms converge, we get that (6.26) holds true. Thus, the process $(M_t^\infty)_{t\leq T}$ is a $\mathbb{P}^\infty$ martingale.

6.5 Application to the two-dimensional Keller-Segel model

In this section we prove Corollary 6.2.5. The parameter λ does not play any role in the above results. Therefore, we will assume here $\lambda = 0$. It is easy to extend the following arguments for $\lambda > 0$.

Denote by $\rho(t,\cdot) \equiv p_t(x)$ the time marginals of the probability distribution constructed in Theorem 6.2.3. As such, ρ satisfies for any $1 \leq q < \infty$,

$$\sup_{t\leq T} t^{1-\frac{1}{q}}\|\rho(t,\cdot)\|_{L^q(\mathbb{R}^2)} \leq C_q.$$

The corresponding drift function satisfies for any $1 \leq r \leq \infty$,

$$t^{\frac{1}{2}-\frac{1}{r}}\|b(t,\cdot;\rho)\|_{L^r(\mathbb{R}^2)} \leq C_r.$$

Following the arguments in Proposition 2.3.3 one may derive the mild equation for $\rho(t,\cdot)$. The above estimates ensure that everything is well defined. Thus, one arrives to the following: for any $f \in C_K^\infty(\mathbb{R}^2)$ and any $t \in (0,T]$,

$$\int f(y)\rho(t,y)\,dy = \int f(y)(g_t * \rho_0)(y)dy$$
$$-\sum_{i=1}^{2}\int f(y)\int_0^t [\nabla_i g_{t-s} * (b^i(s,\cdot;\rho)\rho(s,\cdot))](y)\;ds\;dy.$$

Thus ρ satisfies in the sense of the distributions

$$\rho(t,\cdot) = g_t * \rho_0 - \sum_{i=1}^{2}\int_0^t \nabla_i g_{t-s} * (b^i(s,\cdot;\rho))\rho(s,\cdot))\,ds. \tag{6.32}$$

Now, define the function $c(t,x)$ as

$$c(t,x) := (g(t,\cdot) * c_0)(x) + \int_0^t \rho(t-s,\cdot) * g(s,\cdot)(x)\;ds.$$

Thanks to the density estimates $c(t,x)$ is well defined for all $x \in \mathbb{R}^2$ as soon as $t > 0$. Indeed,

$$|c(t,x)| \leq \frac{\|c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}} + C\int_0^t \|\rho(t-s,\cdot)\|_{L^2(\mathbb{R}^2)}\|g_s\|_{L^2(\mathbb{R}^2)}\,ds \leq \frac{\|c_0\|_{L^2(\mathbb{R}^2)}}{\sqrt{t}} + C\beta(\frac{1}{2},\frac{1}{2}).$$

It is obvious that $c(t,\cdot) \in L^2(\mathbb{R}^2)$. Thanks to the density estimates and the fact that g_t is strongly derivable as soon as $t > 0$, $c(t,x)$ is derivable in any point x and

$$\frac{\partial}{\partial x_i} c(t,x) = \nabla_i (g(t,\cdot) * c_0)(x) + \int_0^t (\rho_{t-s} * \nabla_i g(s,\cdot))(x)\ ds.$$

The fact that $c_0 \in H^1(\mathbb{R}^2)$ enables us to write $\nabla_i(g(t,\cdot) * c_0) = (g(t,\cdot) * \nabla_i c_0)$. Now, remark that $\chi \frac{\partial}{\partial x_i} c(t,x)$ is exactly the drift in (6.32). Thus, the couple (ρ, c) satisfies Definition 6.2.4.

Assume there exists another couple (ρ^1, c^1) satisfying Definition 6.2.4 with the above initial conditions (ρ_0, c_0). As such, they satisfy

$$\forall\ 1 \leq q < \infty\ \exists\ C > 0: \quad \sup_{t \leq T} t^{1-\frac{1}{q}} \|\rho_t^1\|_{L^q(\mathbb{R}^2)} \leq C$$

and

$$\forall\ 2 \leq r \leq \infty\ \exists\ C > 0: \quad \sup_{t \leq T} t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c_t^1\|_{L^r(\mathbb{R}^2)} \leq C.$$

We are in the position to apply [22, Thm. 2.6] and conclude that for a $1 \leq q < \infty$ and a $2 \leq r \leq \infty$, there exists a constant $C(q,r)$ not depending on time such that for $t > 0$ it holds

$$t^{1-\frac{1}{q}} \|\rho_t - \rho_t^1\|_{L^q(\mathbb{R}^2)} + t^{\frac{1}{2}-\frac{1}{r}} \|\nabla c_t - \nabla c_t^1\|_{L^r(\mathbb{R}^2)} = 0$$

6.6 Weak uniqueness for the non-linear process: Proof of Theorem 6.2.6

In this section we come back to the non-linear process (6.1) and prove Theorem 6.2.6. As the parameter λ does not play any role, we will assume here $\lambda = 0$. It is easy to extend the following arguments for $\lambda > 0$. In addition, the parameter χ already satisfies the requirement (6.22).

In the preceding section we proved that under the condition (6.22), the one-dimensional time marginals of any weak solution to (6.1) are the solution to (6.9) in the sense of Definition 6.2.4. Thus, they are uniquely determined as the function $(\rho(t,\cdot))_{t \leq T}$ from the previous section. We define the linearized process

$$\begin{cases} d\tilde{X}_t = b_0(t,\tilde{X}_t)dt + \chi \int_0^t K_{t-s} * \rho(s,\cdot)(\tilde{X}_t)\ ds\ dt + dW_t, \\ \tilde{X}_0 \sim \rho_0. \end{cases} \tag{6.33}$$

We will denote in this section

$$b(t,x) := b_0(t,x)dt + \chi \int_0^t K_{t-s} * \rho(s,\cdot)(x)\ ds.$$

By definition, one has

$$\forall r \in [2,\infty]\ \exists C: \quad \sup_{t \leq T} t^{\frac{1}{2}-\frac{1}{r}} \|b(t,\cdot)\|_{L^r(\mathbb{R}^2)} \leq C. \tag{6.34}$$

Let us define the notion of solution to (6.33).

Definition 6.6.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), \tilde{X}, W)$ is said to be a weak solution to the equation (6.33) up to time $T > 0$ if:*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.*
2. *The process $\tilde{X} := (\tilde{X}_t)_{t\in[0,T]}$ is $\mathbb{R}^2$-valued, continuous, and $(\mathcal{F}_t)$-adapted. In addition, the probability distribution of $\tilde{X}_0$ has density ρ_0.*
3. *The process $W := (W_t)_{t\in[0,T]}$ is a two-dimensional $(\mathcal{F}_t)$-Brownian motion.*
4. *The probability distribution $\mathbb{P} \circ \tilde{X}^{-1}$ has time marginal densities $(\tilde{p}_t,\ t \in (0,T])$ with respect to Lebesgue measure which satisfy*

$$\forall\ 1 < q < \infty\ \exists\ C_q > 0\ \forall 0 < t \le T, \quad t^{1-\frac{1}{q}} \|\tilde{p}_t\|_{L^q(\mathbb{R}^2)} \le C_q. \tag{6.35}$$

5. *For any $t \in [0,T]$ and $x \in \mathbb{R}^2$, one has that $\int_0^t |b_0(s,x)|\, ds < \infty$.*
6. *$\mathbb{P}$-a.s. the pair $(\tilde{X}, W)$ satisfies (6.33).*

It is clear that any solution to (6.1) in the sense of Definition 6.2.1 is a solution to (6.33) in the sense the preceding definition. Therefore, if we prove uniqueness of the weak solution in the sense of Definition 6.6.1 to (6.33), we will have the uniqueness of the solution in the sense of Definition 6.2.1 to (6.1).

In order to do so, we will use the so-called transfer of uniqueness proved in Trevisan [76]. The goal is to use the [76, Lemma 2.12] in the sense $i)$ implies $ii)$. This result is stated in the sequel once all the objects appearing in it are introduced. Firstly, let us define the mild equation associated to the laws $(\tilde{p}_t)_{t\le T}$ in the sense of distributions.

$$\tilde{p}_t = g_t * \rho_0 - \sum_{i=1}^{2} \int_0^t \nabla_i g_{t-s} * (b(s,\cdot)\tilde{p}_s)\, ds. \tag{6.36}$$

We define the space $\mathcal{R}_{[0,T]}$ as follows

$$\mathcal{R}_{[0,T]} := \{(\nu_t)_{t\le T} : \begin{cases} 1.\ \nu_0 = \rho_0 \\ 2.\ \nu_t \text{ is a probability density function} \\ 3.\ \forall 1 < q < \infty,\ \forall 0 < t \le T : \quad t^{1-\frac{1}{q}} \|\nu_t\|_{L^q(\mathbb{R}^2)} < \infty \text{ and } \nu_t \text{ satisfies (6.36).} \end{cases}$$

We prove it admits a unique solution under a condition precised in the proof.

Lemma 6.6.2. *Equation (6.36) admits a unique solution in the space $\mathcal{R}_{[0,T]}$ provided χ is small enough.*

Proof. Let us suppose there exist two families of densities $(\tilde{p}_t^1)_{t\le T}$ and $(\tilde{p}_t^2)_{t\le T}$ satisfying (6.7) and

(6.35). We will prove $\sup_{t\leq T}\|\tilde{p}^1_t-\tilde{p}^2_t\|_{L^1(\mathbb{R}^2)}=0$. Notice that

$$\begin{aligned}\|\tilde{p}^1_t-\tilde{p}^2_t\|_{L^1(\mathbb{R}^2)} &\leq \sum_{i=1}^2\int_0^t \|\nabla_i g_{t-s}*((b_0^i(s,\cdot+b^i(s,\cdot;\rho))(\tilde{p}^1_s-\tilde{p}^2_s))\|_{L^1(\mathbb{R}^2)}\,ds\\ &\leq \sum_{i=1}^2\int_0^t \|\nabla_i g_{t-s}\|_{L^1(\mathbb{R}^2)}\|b_0^i(s,\cdot)+b^i(s,\cdot;\rho)\|_{L^\infty(\mathbb{R}^2)}\|\tilde{p}^1_s-\tilde{p}^2_s\|_{L^1(\mathbb{R}^2)}\,ds \qquad (6.37)\\ &\leq \sup_{s\leq t}\|\tilde{p}^1_s-\tilde{p}^2_s\|_{L^1(\mathbb{R}^2)}\sum_{i=1}^2\int_0^t\frac{C_1(1)}{\sqrt{t-s}}\|b_0^i(s,\cdot)+b^i(s,\cdot;\rho)\|_{L^\infty(\mathbb{R}^2)}\,ds.\end{aligned}$$

In view of Lemma 6.3.3, for any $0<s\leq T$, one has

$$\|b_0^i(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\leq\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)(\mathbb{R}^2)}\frac{C_2(2)}{\sqrt{s}}. \qquad (6.38)$$

Let $q\in(2,4)$ as in Proposition 6.3.7. According to the definition of ρ, one has $\sup_{s\leq T}s^{1-\frac{1}{q}}\|\rho_s\|_{L^q(\mathbb{R}^2)}\leq C(\chi)$, where $C(\chi)$ is given in Remark 6.3.8. Apply Hölder's inequality, Lemma 6.11 and this estimate on ρ to obtain for any $0<s\leq T$ the following

$$\|b^i(s,\cdot;\rho)\|_{L^\infty(\mathbb{R}^2)}\leq\chi C_1(\frac{q}{q-1})C(\chi)\frac{\beta(1-\frac{1}{q},\frac{3}{2}-\frac{1}{q'})}{\sqrt{s}}. \qquad (6.39)$$

Plug (6.38) and (6.39) in (6.37). It comes

$$\begin{aligned}&\|\tilde{p}^1_t-\tilde{p}^2_t\|_{L^1(\mathbb{R}^2)}\\ &\leq 2\sup_{s\leq t}\|\tilde{p}^1_s-\tilde{p}^2_s\|_{L^1(\mathbb{R}^2)}\beta(\frac{1}{2},\frac{1}{2})C_1(1)\left(\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)(\mathbb{R}^2)}C_2(2)+\chi C_1(\frac{q}{q-1})C(\chi)\beta(1-\frac{1}{q},\frac{3}{2}-\frac{1}{q'})\right).\end{aligned}$$

Thus, $\sup_{t\leq T}\|\tilde{p}^1_t-\tilde{p}^2_t\|_{L^1(\mathbb{R}^2)}=0$, provided that

$$H(\chi):=2\beta(\frac{1}{2},\frac{1}{2})C_1(1)\left(\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)(\mathbb{R}^2)}C_2(2)+\chi C_1(\frac{q}{q-1})C(\chi)\beta(1-\frac{1}{q},\frac{3}{2}-\frac{1}{q'})\right)<1. \qquad (6.40)$$

Remember that χ already satisfies (6.22). In view of Remark 6.3.8, one has

$$\chi C(\chi)=\frac{1-A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)}-\sqrt{(1-A\chi\|\nabla c_0\|_{L^2(\mathbb{R}^2)})^2-B^2\chi}}{2K_1}.$$

Since $\chi C(\chi)\to 0$ as $\chi\to 0$, we have $H(\chi)\to 0$ as $\chi\to 0$. Thus it is possible to choose χ small enough in order to satisfy in the same time (6.22) and (6.40). $\square$

Now, notice that for $0<s\leq t\leq T$ one has

$$\tilde{p}_t=g_{t-s}*(g_s*\rho_0)-\sum_{i=1}^2\int_0^s g_{t-s}*(\nabla_i g_{s-u}*(b(u,\cdot)\tilde{p}_u))\,du-\sum_{i=1}^2\int_s^t\nabla_i g_{t-u}*(b(u,\cdot)\tilde{p}_u)\,du.$$

Therefore

$$\tilde{p}_t=g_{t-s}*\tilde{p}_s-\sum_{i=1}^2\int_s^t\nabla_i g_{t-u}*(b(u,\cdot)\tilde{p}_u)\,du. \qquad (6.41)$$

From here, for a $\nu \in \mathcal{R}_{[0,s]}$ we define

$$p^{\nu}_{s,t} = g_{t-s} * \nu_s - \sum_{i=1}^{2} \int_s^t \nabla_i g_{t-u} * (b(u,\cdot)p^{\nu}_{s,u})\, du. \tag{6.42}$$

Now we define for any $0 \le s \le T$ the space

$$\mathcal{R}_{[s,T]} := \{(p^{\nu}_{s,t})_{s\le t\le T} : \begin{cases} 1.\ \nu \in \mathcal{R}_{[0,s]} \\ 2.\ \forall 0 \le t \le T: \quad p_{s,t} \text{ is a probability density function} \\ 3.\ \forall 1 < q < \infty,\ \forall s \le t \le T: \quad (t-s)^{1-\frac{1}{q}} \|p^{\nu}_{s,t}\|_{L^q(\mathbb{R}^2)} < \infty \\ 4.\ (p^{\nu}_{s,t})_{s\le t\le T} \text{ satisfies (6.42)}. \end{cases}$$

We will prove the following lemma for the classes $(\mathcal{R}_{[s,T]})_{0\le s\le T}$.

Lemma 6.6.3. *For any $0 \le s \le T$, the following two properties are satisfied:*

Property 1: Let $(p^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$ and let $(q^{\nu}_{s,t})_{s\le t\le T}$ a family of probability measures that satisfies (6.42) and is such that $q^{\nu}_{s,t} \le Cp^{\nu}_{s,t}$ for $t \in [s,T]$. Then, $(q^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$.

Property 2: Let $r \le s$ and $(q^{\nu}_{r,t})_{r\le t\le T} \in \mathcal{R}_{[r,T]}$. Then, the restriction $(q^{\nu}_{r,t})_{s\le t\le T}$ belongs to $\mathcal{R}_{[s,T]}$.

Proof. **Property 1:** let $s \in [0,T]$, $(p^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$ and let $(q^{\nu}_{s,t})_{s\le t\le T}$ a family of probability measures that satisfies (6.42) and is such that $q^{\nu}_{s,t} \le Cp^{\nu}_{s,t}$ for $t \in [s,T]$. We should prove that $(q^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$. As for $t \in [s,T]$, we have $q^{\nu}_{s,t} \le p^{\nu}_{s,t}$ then for a test function $f \in C_K(\mathbb{R}^2)$ one has

$$|\int f(x) q^{\nu}_{s,t}(dx)| \le |\int f(x) p^{\nu}_{s,t}(x) dx|.$$

Let $q > 1$ and $q' > 1$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. As $(p^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$, one has

$$|\int f(x) q^{\nu}_{s,t}(dx)| \le \|f\|_{L^{q'}(\mathbb{R}^2)} \|p^{\nu}_{s,t}\|_{L^q(\mathbb{R}^2)} \le \frac{C}{(t-s)^{1-\frac{1}{q}}} \|f\|_{L^{q'}(\mathbb{R}^2)}.$$

By Riesz representation theorem, $q^{\nu}_{s,t}$ is absolutely continuous with respect to Lebesgue's measure. We still denote its probability density by $q^{\nu}_{s,t}$ and conclude

$$\|q^{\nu}_{s,t}\|_{L^q(\mathbb{R}^2)} \le \frac{C}{(t-s)^{1-\frac{1}{q}}}.$$

Therefore, $(q^{\nu}_{s,t})_{s\le t\le T} \in \mathcal{R}_{[s,T]}$.

Property 2: Let $r \le s$ and $(q^{\nu}_{r,t})_{r\le t\le T} \in \mathcal{R}_{[r,T]}$. We should prove that the restriction $(q^{\nu}_{r,t})_{s\le t\le T}$ belongs to $\mathcal{R}_{[s,T]}$. Let $t \ge s$. Notice that

$$q^{\nu}_{r,t} = g_{t-s} * (g_{s-r} * \nu_r) - \sum_{i=1}^{2} g_{t-s} * \int_r^s \nabla_i g_{s-u} * (b(u,\cdot) q^{\nu}_{r,t})\ du - \sum_{i=1}^{2} \int_s^t \nabla_i g_{t-u} * (b(u,\cdot) q^{\nu}_{r,t})\ du.$$

Therefore, for $t \in [s,T]$ one has

$$q^{\nu}_{r,t} = g_{t-s} * q^{\nu}_{r,s} - \sum_{i=1}^{2} \int_s^t \nabla_i g_{t-u} * (b(u,\cdot) q^{\nu}_{r,t})\ du.$$

In addition, for $t \in [s, T]$ and $r \leq s$, one has

$$(t-s)^{1-\frac{1}{m}} \|q^{\nu}_{r,t}\|_{L^m(\mathbb{R}^2)} \leq (t-r)^{1-\frac{1}{m}} \|q^{\nu}_{r,t}\|_{L^m(\mathbb{R}^2)} \leq C.$$

Thus the restriction $(q^{\nu}_{r,t})_{s \leq t \leq T}$ belongs to $\mathcal{R}_{[s,T]}$. $\square$

We are ready to state the result [76, Lemma 2.12] in our framework:

Lemma 6.6.4. *As $\mathcal{R} := (\mathcal{R}_{[s,T]})_{0 \leq s \leq T}$ satisfies the properties in Lemma 6.6.3, the following conditions are equivalent:*

i) for every $s \in [0, T]$ and $\bar{\nu} \in \mathcal{R}_{[0,s]}$, there exists at most one $\nu \in \mathcal{R}_{[s,T]}$ with $\nu_s = \bar{\nu}_s$.

ii) for every $s \in [0, T]$, if $\mathbb{Q}^1$ and $\mathbb{Q}^2$ are the laws of two weak solutions to (6.33) starting from s with $\mathbb{Q}^1_s = \mathbb{Q}^2_s$, then $\mathbb{Q}^1 = \mathbb{Q}^2$.

To apply the preceding lemma in the sense i) implies ii) for $s = 0$, it remains to check that for a fixed $\nu \in \mathcal{R}_{[0,s]}$ the equation (6.42) admits a unique solution in $\mathcal{R}_{[s,T]}$. In order to do so, repeat the same as in the proof of Lemma 6.6.2 to get the uniqueness of (6.42). As the constants do not depend on t, T, one gets the same condition on χ for the uniqueness. We, thus, conclude the uniqueness in law for (6.33) holds.

Chapter 7

The two-dimensional case: Particle system and numerical simulations

The numerical simulations in this chapter were achieved in collaboration with Victor Martin-Lac, research engineer in team Tosca, Inria from September 2017 to June 2018. They concern a probabilistic numerical method designed to solve the 2-d-Keller-Segel system.

7.1 Introduction

The regularization method applied in Chapter 6 leads to a following particle approximation of (6.1): For $N \in \mathbb{N}$ and $\varepsilon > 0$,

$$\begin{cases} dX_t^{i,N,\varepsilon} = dW_t^i + b_0^\varepsilon(t, X_t^{i,N\varepsilon})dt + \chi\Big\{\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}^\varepsilon(X_t^{i,N,\varepsilon} - X_s^{j,N,\varepsilon})ds\Big\}dt, \\ X_0^{i,N} \text{ i.i.d.} \sim p_0. \end{cases} \tag{7.1}$$

where $(W^i)_{i\leq N}$ are standard 2-dimensional independent Brownian motions. In view of (6.13) and Theorem 2.2.4, System (7.1) admits a unique strong solution. Then, according to Theorem 2.2.6 for a fixed $\varepsilon > 0$, the particle system propagates chaos towards (6.12). Thus, the empirical measure $\mu^{N,\varepsilon} := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N,\varepsilon}}$ converges in law towards the law $\mathbb{P}^\varepsilon$ of the regularized process in (6.12) when $N \to \infty$. Then, in view of Chapter (6), the law $\mathbb{P}^\varepsilon$ converges to the law of the non linear process in (6.1) when $\varepsilon \to 0$. Thus, for a large N, a small ε and a $t > 0$, the empirical measure $\mu_t^{N,\varepsilon}$ is a good approximation of the marginal density p_t of X_t. Thus, applying the Euler scheme to (7.1), we can construct a numerical approximation for the function p_t.

A natural question concerns the behavior of the particle system (7.1) in the limit $\varepsilon = 0$. In other words, is the following particle system well defined?

$$\begin{cases} dX_t^{i,N} = dW_t^i + b_0(t, X_t^{i,N})dt + \chi\Big\{\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(X_t^{i,N,} - X_s^{j,N,})ds\Big\}dt, \\ X_0^{i,N} \text{ i.i.d.} \sim p_0. \end{cases} \tag{7.2}$$

At the present, we do not have a mathematical answer to this question. This chapter is devoted to some theoretical comments and some numerical simulations.

The plan is the following: we first see why the techniques used in Chapter 5 do not give results on (7.2). Then, we analyze a purely probabilistic method to discretize the Keller-Segel system in $d = 2$ coming from our probabilistic interpretation. Finally, we compare it with a probabilistic-deterministic method recently proposed by Fatkullin [28].

7.2 Theoretical insights: Extending the techniques from $d = 1$

We start from a probability space $(\Omega, \mathcal{F}, \mathbb{W})$ and the driftless system

$$\begin{cases} d\bar{X}_t^{i,N} = dW_t^i, \quad t \leq T \\ \bar{X}_0^i \text{ i.i.d.} \sim p_0, \end{cases} \tag{7.3}$$

Following the arguments in Chapter 5, we would like to add the drift terms in (7.2) using the Girsanov transformation. To do so, for $x \in C([0,T]; (\mathbb{R}^2)^N)$ we denote by $b_t^i(x)$ the drift term of the i–th particle and we aim to get the Novikov condition for the drift vector $B_t^N(x) = (b_t^1, \dots, b_t^N)$. The quantity of interest is

$$\mathbb{E} \exp\left\{\kappa \int_0^T |B_t^N(\bar{X})|^2 \, dt\right\},$$

for $\kappa > 0$. As in Chapter 5, we develop the exponential in a sum

$$\mathbb{E} \exp\left\{\kappa \int_0^T |B_t^N(\bar{X})|^2 \, dt\right\} = \mathbb{E} \sum_{k=1}^{\infty} \frac{\kappa^k}{k!} \left(\int_0^T |B_t^N(\bar{X})|^2 \, dt\right)^k. \tag{7.4}$$

7.2.1 No Khasminskii's lemma procedure

Let us assume $b \equiv 0$ and $p_0 = \delta_0$. For $k = 1$ in (7.4) one of the terms we should control is

$$A := \mathbb{E} \int_0^T \left(\int_0^t K_{t-s}^1(W_t^1 - W_s^2) \, ds\right)^2 \, dt.$$

We will often use the following standard formula for an integral of two one-dimensional Gaussian densities:

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-m_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-m_2)^2}{2\sigma_2^2}} \, dx = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_2^2}} e^{-\frac{(m_1-m_2)^2}{2(\sigma_1^2+\sigma_2^2)}}. \tag{7.5}$$

In addition, we denote by g^{1d} when we want to emphasize that we have the one-dimensional Gaussian density. Now, notice that

$$\begin{aligned} A &= 2\int_0^T \int_0^t \int_s^t \mathbb{E}[K_{t-s}^1(W_t^1 - W_s^2) K_{t-u}^1(W_t^1 - W_u^2)] \, du \, ds \, dt \\ &= 2\int_0^T \int_0^t \int_s^t \int_{\mathbb{R}^2} g_t(z) \int_{\mathbb{R}^2} g_s(y) K_{t-s}^1(z-y) \int_{\mathbb{R}^2} K_{t-u}^1(z-x-y) g_{u-s}(x) \, dx \, dy \, dz \, du \, ds \, dt. \end{aligned}$$

Observe that $K_{t-u}^1(z-x-y) = \frac{\partial}{\partial z_1} g_{t-u}(z-x-y)$. Use (7.5) and compute the integral on $\mathbb{R}^2$ as a product of integrals on $\mathbb{R}$. It comes

$$\int_{\mathbb{R}^2} K_{t-u}^1(z-x-y) g_{u-s}(x) \, dx = \frac{\partial}{\partial z_1} g_{t-s}(z-y) = K_{t-s}^1(z-y).$$

Thus

$$A = 2\int_0^T \int_0^t (t-s) \int_{\mathbb{R}^2} g_t(z) \int_{\mathbb{R}^2} g_s(y) (K_{t-s}^1(z-y))^2 \, dy \, dz \, ds \, dt.$$

Use again the same formula to integrate w.r.t. y_2 and z_2. It comes

$$\int_{\mathbb{R}} g_t^{1d}(z_2) \int_{\mathbb{R}} g_{\frac{t-s}{2}}^{1d}(z_2 - y_2) g_s^{1d}(y_2)\ dy_2\ dz_2 = C \int_{\mathbb{R}} g_t^{1d}(z_2) g_{\frac{t+s}{2}}^{1d}(z_2)\ dz_2 = \frac{C}{\sqrt{3t+s}}.$$

Thus,

$$A = C \int_0^T \int_0^t \frac{(t-s)^{3/2}}{\sqrt{3t+s}} \int_{\mathbb{R}} g_t^{1d}(z_1) \int_{\mathbb{R}} g_s^{1d}(y_1) \frac{(z_1-y_1)^2}{(t-s)^4} e^{-\frac{(z_1-y_1)^2}{t-s}}\ dy_1\ dz_1\ ds\ dt.$$

The change of variables $\frac{z_1-y_1}{\sqrt{t-s}} = y$ and Fubini's theorem lead to

$$\begin{aligned}
A &= C \int_0^T \int_0^t \frac{(t-s)^{3/2}}{\sqrt{3t+s}} \int_{\mathbb{R}} g_t^{1d}(z_1) \int_{\mathbb{R}} g_s^{1d}(z_1 - y\sqrt{t-s}) \frac{y^2(t-s)}{(t-s)^4} e^{-y^2} \sqrt{t-s}\ dy\ dz_1\ ds\ dt \\
&= C \int_0^T \int_0^t \frac{1}{\sqrt{3t+s}(t-s)} \int_{\mathbb{R}} y^2 e^{-y^2} \int_{\mathbb{R}} g_t^{1d}(z_1) g_s^{1d}(z_1 - y\sqrt{t-s})\ dz_1\ dy\ ds\ dt \\
&= C \int_0^T \int_0^t \frac{1}{\sqrt{3t+s}(t-s)} \int_{\mathbb{R}} y^2 e^{-y^2} \frac{1}{\sqrt{t+s}} e^{-\frac{(t-s)y^2}{2(t+s)}}\ dy\ ds\ dt \\
&= C \int_0^T \int_0^t \frac{1}{\sqrt{3t+s}(t-s)} \int_{\mathbb{R}} y^2 e^{-\frac{(3t+s)y^2}{2(t+s)}}\ dy\ ds\ dt.
\end{aligned}$$

The singularity when $s \to t$ is not integrable. We conclude that $A = \infty$. Thus, it is not possible to obtain the Novikov's condition if the initial law is a Dirac measure.

Remark 7.2.1. *The above computations do not change when adding an initial condition to the Brownian motion.*

7.2.2 Fernique's theorem does not apply

In [32], Friz and Oberhauser show a generalised version of Fernique's theorem which implies the Novikov condition.

Theorem 7.2.2 (Thm. 2 [32])**.** *Let (E, H, μ) be an abstract Wiener space. Assume $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$ is a measurable map and $N \subset E$ a null set and c some positive constant such that for any $x \notin N$ one has*

- $|f(x)| < \infty$,
- $\forall h \in H : \quad |f(x)| \leq c(|f(x-h)| + \sigma |h|_H)$.

Then,

$$\int \exp\{\eta |f(x)|^2\}\ \mu(dx) < \infty \text{ if } \eta < \frac{1}{2c^2\sigma^2}.$$

Here σ is defined as

$$\sigma := \sup_{\xi \in E^\star, |\xi|_{E^\star}=1} \left(\int <\xi, x>^2\ \mu(dx) \right)^{\frac{1}{2}} < \infty.$$

Let H be the standard Cameron-Martin space. In order to apply their Theorem 2, one should define for $x \in C([0,T];(\mathbb{R}^2)^N)$

$$f(x) := \sqrt{\int_0^T |B_t^N(x)|^2 \, dt}.$$

Then it should be proved that

- f is finite $\mathbb{W}$-a.e.
- f is a pseudo norm, i.e. for any $h \in H$

$$|f(x)| \leq c(|f(x-h)| + \sigma |h|_H),$$

  where $|h|_H = \sqrt{\int_0^T (\dot{h}(s))^2 \, ds}$.

Both conditions are problematic: we do not know how to prove that f is finite and the exponential in the definition of the drift disables us to bound $|f(x-h+h)|$ with a linear combination of $|f(x-h)|$ and $|h|_H$.

7.2.3 Main difficulties

At the present, we are still working on the well-posedness of (7.2). What makes this job difficult is, as seen above, the singular nature of the interaction kernel. The increase of dimension lead to an increase in time singularity which can no longer be tamed by using Brownian techniques. This makes us doubt that the laws of the particles are actually absolutely continuous with respect to Wiener's measure, while its one dimensional marginals should be.

Thus, an idea might be to find a reference process different than (7.3) and then use the Girsanov transformation. One choice for the reference system is the system containing only the linear part of the drift, i.e.

$$\begin{cases} d\bar{X}_t^{i,N} = dW_t^i + b_0(t, \bar{X}_t^{i,N})dt, \quad t \leq T \\ \bar{X}_0^i \text{ i.i.d.} \sim p_0. \end{cases} \tag{7.6}$$

Using the regularization techniques from Chapter 6, one can prove that System (7.6) is well defined under some condition on the size of χ and $\|\nabla c_0\|_{L^2(\mathbb{R})}$. In addition, under these conditions the laws p_t^i of $\bar{X}_t^{i,N}$ satisfy

$$\forall 1 < q < \infty \; \exists C_q > 0, \quad \sup_{t \leq T} t^{1-\frac{1}{q}} \|p_t^i\|_{L^q(\mathbb{R}^2)} \leq C_q.$$

Unfortunately, such property is not powerful enough to control the time singularity of the kernel and will not improve the computations done in Subsection 7.2.1. Until present, we have not found a suitable reference particle system.

A completely different approach could be to start from the regularized system (7.1) and for a fixed N try to get tightness of $(X^{1,N,\varepsilon}, \; \varepsilon > 0)$. Then, take a limit point and prove it satisfies (7.2). The usual criterion of tightness we used in this thesis, leads us to the following quantity:

$$\mathbb{E}\left(\int_s^t \frac{\chi}{N} \sum_{j=2}^N \int_0^u K_{u-\theta}^{\varepsilon,i}(X_u^{1,N,\varepsilon} - X_\theta^{j,N,\varepsilon}) \, d\theta \, du \right)^m.$$

For an $m \in N$ and $m > 1$, we could follow the Khasminskii's argument as in Chapter 5 to control this quantity. However, we will need the joint distribution of $(X_u^{1,N,\varepsilon}, X_\theta^{j,N,\varepsilon})$ for some $u \neq \theta$. The non-markovian nature of the system prevents us of having a representation for marginal densities of the laws of the particles. Thus, it is not clear how to get some estimates of the law of the above written couple that could help us in integrating the singularity. Another way to get tightness might be to follow the arguments of Fournier and Jourdain [31]. It would come done to control uniformly in $\varepsilon > 0$ the following quantity quantity

$$\mathbb{E}\frac{\chi}{N}\sum_{j=2}^{N}\int_0^T \left|\int_0^u K_{u-\theta}^{\varepsilon,i}(X_u^{1,N,\varepsilon} - X_\theta^{j,N,\varepsilon})\ d\theta\right|^{2-\alpha} du,$$

where $\alpha \in (0,1)$ is to be chosen. Again, we are not sure how to proceed once the quantity of interest is identified as we do not have information about the joint laws $(X_u^{1,N,\varepsilon}, X_\theta^{j,N,\varepsilon})$ for some $u \neq \theta$. Another idea would be to apply a functional Itô's formula in order to control the above quantity. We have not tried this option yet.

The question of well-posedness of (7.2) without cut-off remains open for our future work.

7.3 Our probabilistic numerical method

For a fixed time horizon $0 < T < \infty$, we choose $\triangle t > 0$ and $n \in \mathbb{N}$ such that $n\triangle t = T$. In the sequel, we propose a discrete approximation $(\bar{X}_{k\triangle t})_{1\le k\le n} := (\bar{X}_{k\triangle t}^{1,N}, \ldots, \bar{X}_{k\triangle t}^{N,N})_{1\le k\le n}$ of (7.2). Then, we use it to construct a discretization $(\bar{\rho}, \bar{c})$ of a solution (ρ, c) to (6.9).

For a given probability measure p_0 on $\mathbb{R}^2$ we assume $(\bar{X}_0^{i,N})_{1\le i\le N}$ are independent identically distributed according to p_0. We suppose the initial concentration $c_0 \in H^1(\mathbb{R}^2)$ is given and that in each point $x \in \mathbb{R}^2$ we can compute $\nabla c_0 * g_t(x)$. For $1 \le i \le N$ and $1 \le k \le n$, we apply the Euler scheme on (7.2). One gets

$$\bar{X}_{(k+1)\triangle t}^{i,N} = \bar{X}_{k\triangle t}^{i,N} + \triangle t\ b_0(k\triangle t, \bar{X}_{k\triangle t}^{i,N}) + \triangle t\ \chi \sum_{j=1,j\neq i}^{N} V_{k\triangle t}^{i,j} + (W_{(k+1)\triangle t}^i - W_{k\triangle t}^i),$$

where $V_{k\triangle t}^{i,j} = \int_0^{k\triangle t} e^{-\lambda(k\triangle t - s)} K_{k\triangle t - s}(\bar{X}_{k\triangle t}^{i,N} - X_s^{j,N})\ ds$. One way to discretize $V_{k\triangle t}^{i,j}$ is to use the values $\bar{X}_0^{j,N}, \ldots, \bar{X}_{k\triangle t}^{j,N}$ and Riemann sums. This is, of course, one of many possible choices when discretizing this integral, but disputable when the integral is singular. Nevertheless, we set

$$\tilde{V}_{k\triangle t}^{i,j} = \sum_{l=0}^{k-1} \triangle t\ e^{-\lambda(k-l)\triangle t} K_{(k-l)\triangle t}(\bar{X}_{k\triangle t}^{i,N} - \bar{X}_{l\triangle t}^{j,N}).$$

Finally, we obtain the following discrete approximation of the particle system (7.2):

$$\begin{cases} \bar{X}_{(k+1)\triangle t}^{i,N} = \bar{X}_{k\triangle t}^{i,N} + \triangle t\ b_0(k\triangle t, \bar{X}_{k\triangle t}^{i,N}) + \triangle t\ \chi \sum_{j=1,j\neq i}^{N} \tilde{V}_{k\triangle t}^{i,j} + (W_{(k+1)\triangle t}^i - W_{k\triangle t}^i) \\ \bar{X}_0^{i,N} \text{i. i. d.} \sim p_0; \quad \tilde{V}_{k\triangle t}^{i,j} = \sum_{l=0}^{k-1} \triangle t\ K_{(k-l)\triangle t}(\bar{X}_{k\triangle t}^{i,N} - \bar{X}_{l\triangle t}^{j,N}) \end{cases} \tag{7.7}$$

Notice that each $\bar{X}_{k\triangle t}^{i,N}$ is a two dimensional vector. The system (7.7) can be simulated easily. First, one obtains N independent realizations of the distribution p_0: $x_0^1, \ldots, x_0^N$ and initializes

$\bar{X}_0^{i,N} = x_0^i$. In each time step of the discretization, we simultaneously calculate both coordinates of $\bar{X}_{k\triangle t}^{i,N}$, simulating two independent realizations of standard Gaussian law for the Brownian increment. The non-Markovian nature of (7.2) manifests itself in terms of the complexity of the above scheme. In order to compute $(\bar{X}_T^{i,N})_{1\le i\le N}$, on one hand, one needs to simulate $O(\frac{N}{\triangle t})$ realizations of a standard Gaussian random variable. On the other hand the kernel K needs to be evaluated $O(\frac{N^2}{\triangle t^2})$ times. Thus, once we choose one of the classical methods to simulate Gaussian random variables, asymptotically (when $N \to \infty$ and $\triangle t \to 0$) the complexity of the interaction term prevails and we have that the scheme is exponentially complex both in number of particles and time step.

Now, we construct the time discretization $(\bar{\rho}(k\triangle t,\cdot), \bar{c}(k\triangle t,\cdot))_{1\le k\le n}$. As $\rho(t,\cdot)$ is the density of the random variable X_t and the particle system is its discretization, the empirical measure $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ is an approximation for $\rho(t,\cdot)$. Having this in mind and the mild equation satisfied by the function $c(t,\cdot)$, we set

$$\begin{cases} \bar{\rho}(k\triangle t, x) = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_t^{i,N}}(x), \\ \bar{c}(k\triangle t, x) = g_t * c_0(x) + \frac{1}{N}\sum_{i=1}^N \sum_{l=1}^{k-1} \triangle t \; g_{(k-l)\triangle t}(x - \bar{X}_{l\triangle t}^{i,N}). \end{cases} \tag{7.8}$$

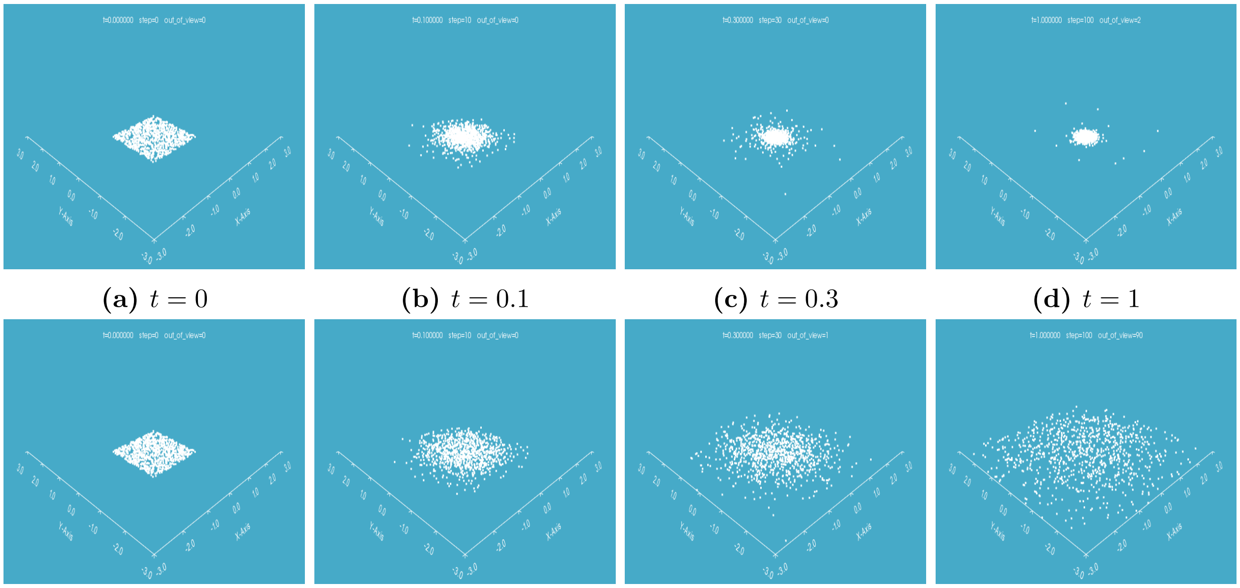
Then one should choose the points in space in which $(\bar{\rho}, \bar{c})$ are evaluated. One way to do so is to see what is the domain on which the system (7.7) evolves up to time T and then discretize it in a grid with the steps $(\triangle x^1, \triangle x^2)$. Notice that at each point x_m of the grid the complexity of computing $\tilde{c}(T, x_m)$ is of order $O(\frac{N}{\triangle t^2})$.

We propose, thus, the couple $(\bar{\rho}, \bar{c})$ to be a numerical approximation of a solution to the Keller-Segel system in dimension two. In order to justify the convergence of our scheme to it as $N \to \infty$ and $\triangle t \to 0$, one should prove that the particle system (7.2) is well defined and that the propagation of chaos towards the law of the process (6.1) holds for it. Then, one should show that the scheme (7.7) converges. As we do not intend to prove all this, our numerical method should not be seen in a rigorous way. As mentioned in the introduction, all this can be proven in the regularized case (see Eq. (6.12)).

Nevertheless, the numerical simulations seem to be effective as they are able to capture two different behaviours of the solutions according to the size of the parameter χ. Namely, when χ is large we observe an agglomeration of particles as the simulation advances. This induces a formation of singularity in our approximations for ρ and c. When χ is small, the particles diffuse and no singularity is detected (see Figure 7.1).

7.4 Comparison with another particle method

As two-dimensional Keller-Segel model may develop singularities, a probabilistic numerical method seems to be more adapted than purely deterministic methods (such as finite elements or Galerkin methods). The reason is that standard numerical schemes develop instabilities around the singularities. In addition, these schemes can not carry on the computation once a singularity has been formed. On the other hand, as seen in the preceding section, in stochastic particle approaches singularities manifest themselves as agglomeration of particles. As such, they are not harmful from a computational point of view. Thus, the simulation can continue after a formation



(a) $t = 0$ **(b)** $t = 0.1$ **(c)** $t = 0.3$ **(d)** $t = 1$

(e) $t = 0$ **(f)** $t = 0.1$ **(g)** $t = 0.3$ **(h)** $t = 1$

Figure 7.1: (a)-(e): $\chi = 50$; (f)-(j): $\chi = 1$. Euler scheme is applied to (1.15), with $N = 1000, d = 2$. Particles are initially distributed uniformly on $[-1, 1] \times [-1, 1]$. Initial concentration of the chemical is a centered Gaussian density. When χ is large an agglomeration of particles appear in the center of the domain, whilst when χ is small the particles diffuse.

of a singularity and we can have an idea what might happen in the system once such irregularity is developed.

Probabilistic-Deterministic Particle Method of I. Fatkullin

The above remarks are noticed by Fatkullin in [28] who develops a probabilistic-deterministic particle method (PDPM) for approximating (6.9) in a rectangular domain with Neumann no-flux boundary condition. Namely, he applies a particle method for (6.9a) and a classical numerical method for (6.9b) simultaneously. Let $\triangle\tau$ be the time step of the particle simulation and $\triangle t$ of concentration simulation, where the grid size is $\triangle x$. The density function is reconstructed from N independent realizations, obtained by applying the Euler scheme, of a linear stochastic process related to the Fokker-Planck equation (6.9a):

$$Y_{(k+1)\triangle\tau} = Y_{k\triangle\tau} + \triangle\tau\ \nabla c(Y_{k\triangle\tau}) + \sqrt{\triangle\tau}N(0, 1).$$

By reflecting the particles escaping the space domain the no-flux boundary condition is enforced. The ∇c is obtained by bilinear interpolation of values CX and CY obtained from the concentration field $C_{i,j}$:

$$CX_{i,j} = \frac{1}{2\triangle x}[C_{i+1,j}(t) - C_{i-1,j}(t)], \quad CY = \frac{1}{2\triangle x}[C_{i,j+1}(t) - C_{i,j-1}(t)].$$

The particle field $P_{i,j}$ is obtained from the particles positions by distributing a part of its weight to four nearest neighbouring grid points. In order to get the concentration field $C_{i,j}$ the author applies the implicit second order finite-difference scheme:

$$\frac{1}{\triangle t}[C_{i,j}(t+\triangle t)-C_{i,j}(t)=\frac{1}{\triangle x^2}D^{(2)}_{i,j}C(t+\triangle t)-\lambda^2 C_{i,j}(t+\triangle t)+P_{i,j}(t),$$

where $D^{(2)}$ is the second order difference operator. Adaptive time step for the Euler scheme is used in the vicinity of singularities in order to avoid artificial oscillations. For each particle, if needed, the time step $\triangle t$ is divided into sub-intervals of size $\triangle\tau$ so that $|\nabla c(Y_{k\triangle\tau})|<\triangle x$.

The boundary problem

This mix of probabilistic and deterministic techniques is quite efficient and even though it uses an implicit scheme for the concentration function, it is much faster than (7.8) as there is no interaction between the particles.

However, its main drawback is the artificial boundary condition imposed on the particle evolution. In fact, the particles are forced to stay in the domain which introduces a bias towards it. Namely, an agglomeration of particles, once it is formed tends to migrate to the corner of the domain. Fatkullin notices this in the case of the elliptic model when analyzing formation and interaction of singularities. He argues that the singularities form and interact in the following way: the agglomerations may form in different areas of the domain, then if close enough they attract one another and eventually merge. Finally, they travel towards the boundary and concentrate around the corners of the domain.

Simulating the PDPM in the parabolic-parabolic case, we notice the same phenomenon. Particle agglomerations are attracted by each other and the boundary. We argue this event is artificial by comparing PDPM with the purely probabilistic particle method (PPPM) in (7.7). In PPPM, we do not introduce any constraint on the particle positions, but only decide the domain of visualization for ρ and c once we see where most of the particles evolve up to a given time horizon.

As the position of an agglomeration obviously induces a peak in the approximation for the density and concentration functions, we will compare the particle evolution of PPPM and PDPM in the case a singularity is developed. We will be interested in formation of agglomerations, their positions and time needed for them to develop.

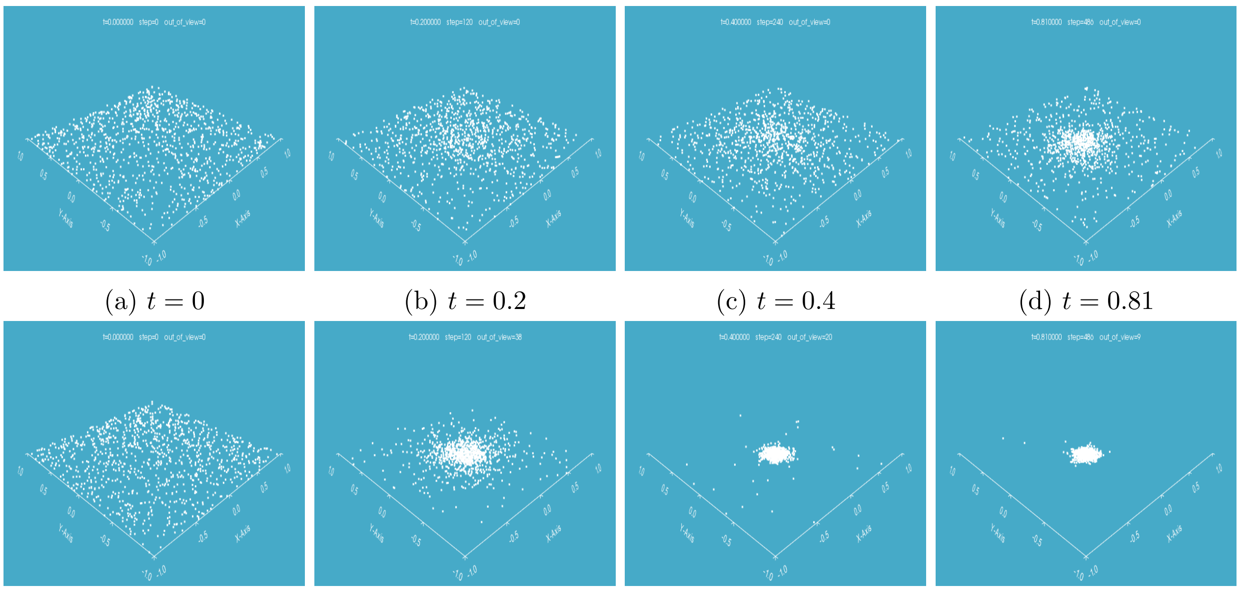
Numerical results

We simulate $N=1000$ particles using PPPM and PDPM. For both simulations we choose the following parameters: $\alpha=1$, $\chi=50$, $\lambda=1$, $T=3$, $\triangle t=0.00167$, $n=1800$. Initially, the particles are uniformly distributed on the square $[0,1]\times[0,1]$, and the initial concentration is the standard Gaussian two-dimensional density. In PDPM we have chosen the domain to be also equal to $[0,1]\times[0,1]$ with $\triangle x_1=\triangle x_2=0.067$. The particles are left to evolve according to PPPM and PDPM.

The following graphs are a selection of various simulations run with different values of the numerical parameters. The goal is to analyze qualitatively the graphs. Thus, even if with finer discretization parameters the simulation is more accurate, qualitatively the same phenomena occur always.

An agglomeration is visible in both methods around the point $(0, 0)$, although it appears sooner in PPPM (Figure 7.2). Proceeding in time both of the agglomerations are well formed, but the one in PDPM starts displacing from its initial position (Figure 7.3). Until T no changes occur in PPPM, while the agglomeration in PDPM gets glued on the boundary (Figure 7.4).

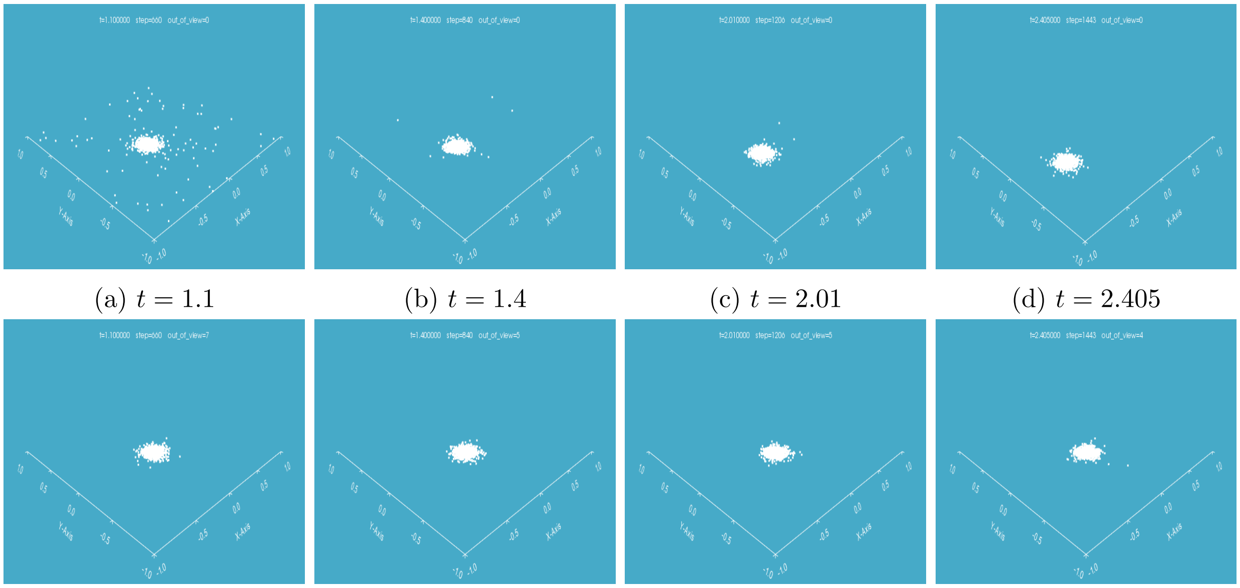
In order to be sure that this is not due to a badly chosen set of discretization parameters, we repeat the PDPM simulation for a finer grid in time and space. Namely, we choose $T = 20$, $\triangle t = 0.001$, $\triangle x = \triangle y = 0.002$ and all the other parameters as above. The results are given in Figure 7.5. As in the parabolic-elliptic simulations in [28], the agglomeration ends up in the corner of the domain.



(a) $t = 0$ (b) $t = 0.2$ (c) $t = 0.4$ (d) $t = 0.81$

(e) $t = 0$ (f) $t = 0.2$ (g) $t = 0.4$ (h) $t = 0.81$

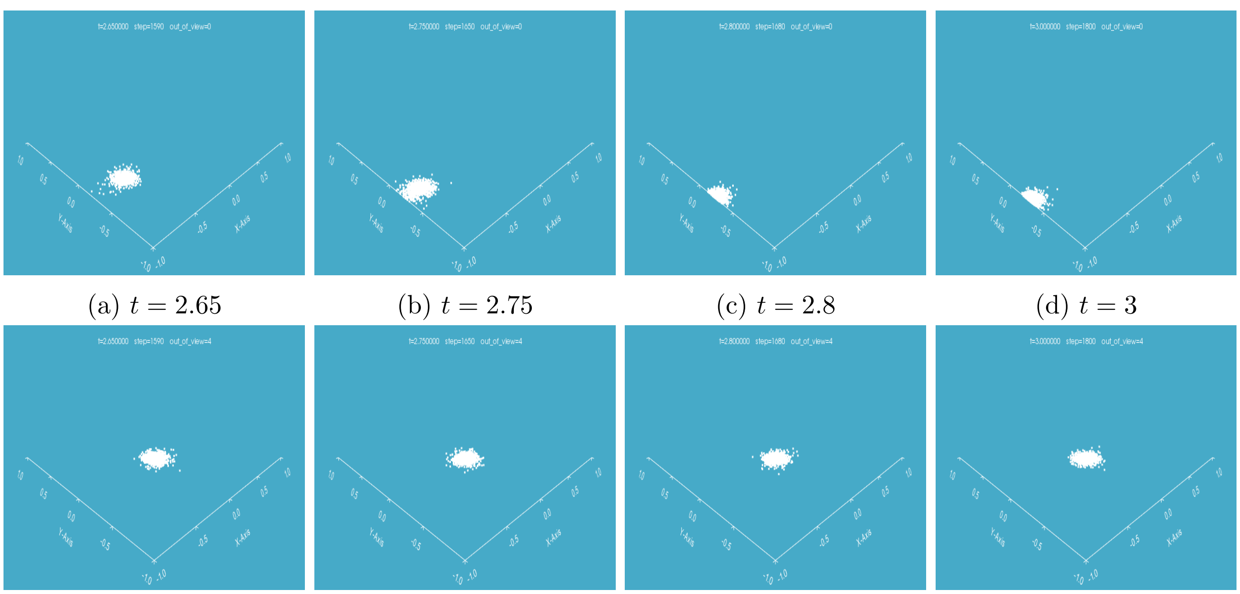
Figure 7.2: (a)-(d): PDPM; (e)-(h): PPPM.



(a) $t = 1.1$ (b) $t = 1.4$ (c) $t = 2.01$ (d) $t = 2.405$

(e) $t = 1.1$ (f) $t = 1.4$ (g) $t = 2.01$ (h) $t = 2.405$

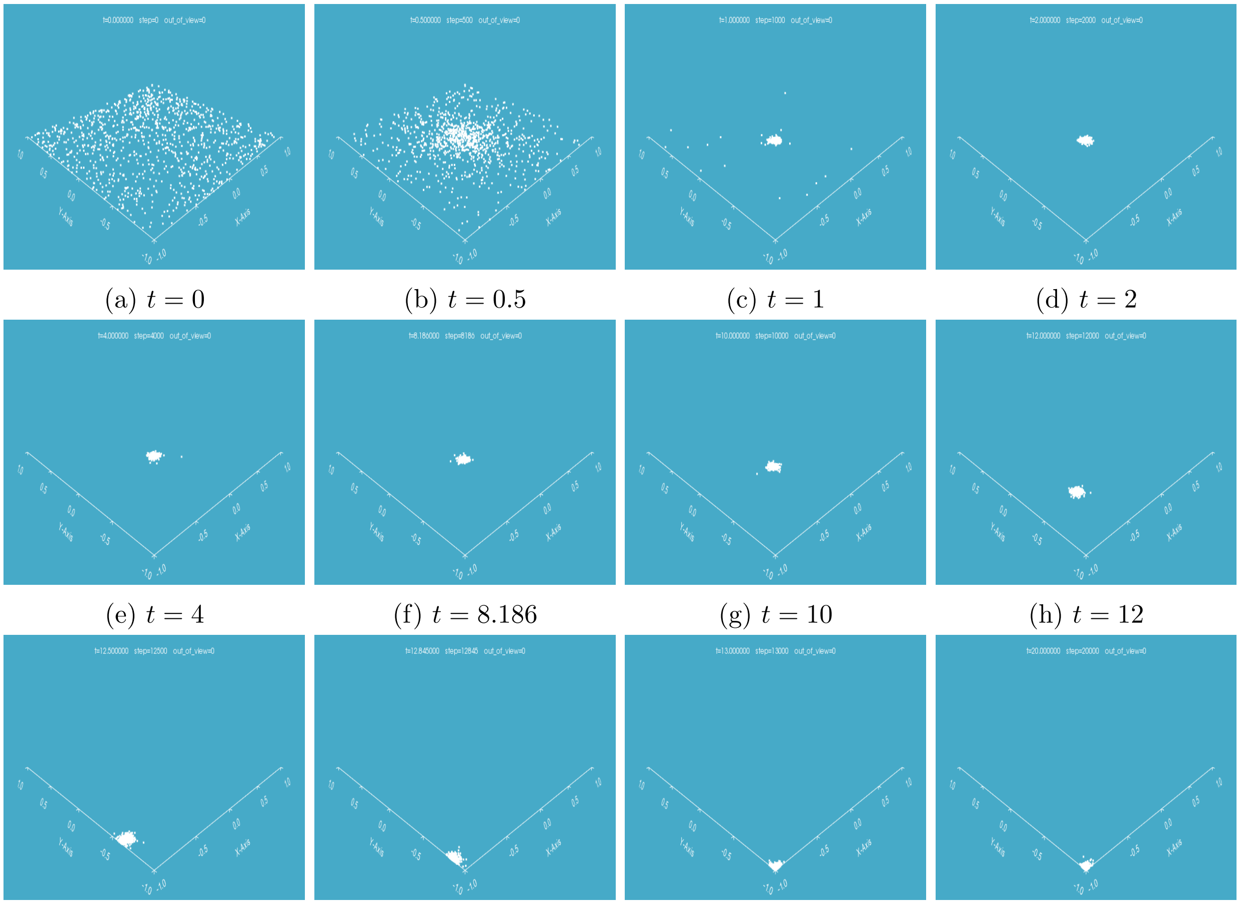
Figure 7.3: The continuation of the evoulution in Figure 7.2–(a)-(e): PDPM; (f)-(j): PPPM.



(a) $t = 2.65$ (b) $t = 2.75$ (c) $t = 2.8$ (d) $t = 3$

(e) $t = 2.65$ (f) $t = 2.75$ (g) $t = 2.8$ (h) $t = 3$

Figure 7.4: The continuation of the evoulution in Figure 7.3–(a)-(e): PDPM; (f)-(j): PPPM.



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 2$

(e) $t = 4$ (f) $t = 8.186$ (g) $t = 10$ (h) $t = 12$

(i) $t = 12.5$ (j) $t = 12.845$ (k) $t = 13$ (l) $t = 20$

Figure 7.5: PDPM: long time and finer grid.

If the domain is chosen far away from the initial conditions, then we can expect that no attraction by the border happens. Indeed, if we choose the domain to be $[-3, 3] \times [-3, 3]$ and the initial conditions as above, defined on the sub–domain $[-1, 1] \times [-1, 1]$, we do not observe the above phenomenon up to large time horizons. In this setting it was easy to find a convenient domain as we know the initial law and the PPPM tells us where most of the particles should be. However, if the initial state is more complicated, one may have to choose a very big domain in order not to influence the computations by its border. Then, most of the domain will most of the time be empty. However, one will have to choose the space grid according to the size of domain. That will imply more elements on the grid and useless computations in the zones where particles do not pass their time. This will slow down the algorithm.

Conclusion:

a) The Fatkullin method is fast, but requires to a priori know where the particles will aggregate which is impossible in practice. It also requires to properly define artificial boundary conditions (the no flux condition may be violated by the true solution at the chosen artificial boundary).

b) The purely stochastic particle method allows one to solve the Keller-Segel equation without any artificial boundary condition. However, the method needs some improvement as its its main drawback is the computational time coming from the interaction of every particle with the past of all the other particles.

Conclusion

7.5 Perspectives: How to improve our method?

The principal issue of our numerical scheme is computational time. The interaction between particles involves their past and as we have chosen to use the Riemann sums to approximate the integral $\int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})\, ds$, we have to store all the positions:

$$\tilde{V}^{i,j}_{k\triangle t} = \sum_{l=0}^{k-1} \triangle t \; e^{-\lambda(k-l)\triangle t} K_{(k-l)\triangle t}(\bar{X}^{i,N}_{k\triangle t} - \bar{X}^{j,N}_{l\triangle t}).$$

In addition, as we advance in time (k becomes larger), the computation of the interaction term involves more elements and it is slower and slower. Just to give an idea, computing position of 1000 particles with 600 time steps takes around three hours on a personal computer. On the other hand, PDPM does the same job in two seconds (see Figure 7.6).

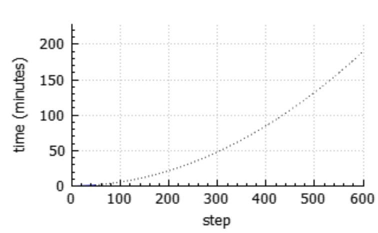


Figure 7.6: Computational time of PPPM for $N = 1000$ and $n = 600$. The curve is provided by our program during one of the simulations with the above parameters.

Some possible points to improve the computation time:

1. **Neglect contributions far away in time:** As soon as the current step in time (k) becomes large enough, one may neglect the contribution of an interaction that happens "far away in time". In fact, our conjecture is is that most of the magnitude of $\tilde{V}^{i,j}_{k\triangle t}$ is due to the last elements of the sum . The idea is to find a criterion that would discard some of the first ones or use a default value for them as soon as k is large enough. However, the threshold for k is random and thus needs a careful analysis.

2. **Neglect contributions far away in space:** If two particles are far away in space (even in different times), the kernel is dominated by the exponential and thus such a contribution is minimal. The idea is to find a criterion that neglects the contribution of $\bar{X}^{j,N}_{l\triangle t}$ to the drift of $\bar{X}^{i,N}_{k\triangle t}$ if $|\bar{X}^{j,N}_{l\triangle t} - \bar{X}^{i,N}_{k\triangle t}|$ is large enough ($l \leq k$).

Some possible points to improve the accuracy of the algorithm:

1. **Computations around zero:** The particles are initialized in such a way that when the time horizon is very small we do not see their interaction at all. The idea would be to completely change some of the first time steps of the computations in order to better capture the interaction between particles.

2. **Singular integral approximation:** As soon as two particles are very close at an instant of time t, the integral $\int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})\, ds$ may become singular. In that case, Riemann sums are not an appropriate way to discretize such an integral. It is necessary to diagnose what the critical distance between the positions of two particles at time t is under which numerical problems appear (instabilities, large values, etc.). It will also be necessary to find an effective approximation method for critical cases of small (strictly positive) distances, or to be reassured that this configuration appears very rarely during simulations.

Bibliography

[1] Adler, J. Chemotaxis in bacteria. *Science 153*, 3737 (1966), 708–716.

[2] Adler, J. Chemoreceptors in bacteria. *Science 166*, 3913 (1969), 1588–1597.

[3] Adler, J. A method for measuring chemotaxis and use of the method to determine optimum conditions for chemotaxis by escherichia coli. *Microbiology 74*, 1 (1973), 77–91.

[4] Adler, J. Chemotaxis in bacteria. *Annual Review of Biochemistry 44*, 1 (1975), 341–356.

[5] Adler, J., and Templeton, B. The effect of environmental conditions on the motility of escherichia coli. *Microbiology 46*, 2 (1967), 175–184.

[6] Berg, H. C. Chemotaxis in bacteria. *Annual Review of Biophysics and Bioengineering 4*, 1 (1975), 119–136. PMID: 1098551.

[7] Biler, P., Corrias, L., and Dolbeault, J. Large mass self-similar solutions of the parabolic-parabolic Keller-Segel model of chemotaxis. *J. Math. Biol. 63*, 1 (2011), 1–32.

[8] Blanchet, A., Dolbeault, J., and Perthame, B. t. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations* (2006), No. 44, 32.

[9] Bonner, J. *The Cellular Slime Molds.* Princeton Legacy Library. Princeton University Press, 2015.

[10] Borodin, A. N., and Salminen, P. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 1996.

[11] Bossy, M. Some stochastic particle methods for nonlinear parabolic PDEs. In *GRIP—Research Group on Particle Interactions*, vol. 15 of *ESAIM Proc.* EDP Sci., Les Ulis, 2005, pp. 18–57.

[12] Bossy, M., and Jabir, J.-F. Particle approximation for Lagrangian stochastic models with specular boundary condition. *Electron. Commun. Probab. 23* (2018), Paper No. 15, 14.

[13] Bossy, M., Jabir, J.-F., and Talay, D. On conditional McKean Lagrangian stochastic models. *Probab. Theory Related Fields 151*, 1-2 (2011), 319–351.

[14] Bossy, M., and Talay, D. Convergence rate for the approximation of the limit law of weakly interacting particles: application to the Burgers equation. *Ann. Appl. Probab. 6*, 3 (1996), 818–861.

[15] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[16] Brezis, H. m., and Cazenave, T. A nonlinear heat equation with singular initial data. *J. Anal. Math. 68* (1996), 277–304.

[17] Budhiraja, A., and Fan, W.-T. Uniform in time interacting particle approximations for nonlinear equations of Patlak-Keller-Segel type. *Electron. J. Probab. 22* (2017), Paper No. 8, 37.

[18] Budrene, E. O.; Berg, H. C. Dynamics of formation of symmetrical patterns by chemotactic bacteria. *Nature 376* (1995).

[19] Calderoni, P., and Pulvirenti, M. Propagation of chaos for Burgers' equation. *Ann. Inst. H. Poincaré Sect. A (N.S.) 39*, 1 (1983), 85–97.

[20] Calvez, V., and Corrias, L. The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$. *Commun. Math. Sci. 6*, 2 (2008), 417–447.

[21] Cattiaux, P., and Pédèches, L. The 2-D stochastic Keller-Segel particle model: existence and uniqueness. *ALEA Lat. Am. J. Probab. Math. Stat. 13*, 1 (2016), 447–463.

[22] Corrias, L., Escobedo, M., and Matos, J. Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller-Segel system in the plane. *J. Differential Equations 257*, 6 (2014), 1840–1878.

[23] Corrias, L., and Perthame, B. Critical space for the parabolic-parabolic Keller-Segel model in $\mathbb{R}^d$. *C. R. Math. Acad. Sci. Paris 342*, 10 (2006), 745–750.

[24] Corrias, L., Perthame, B., and Zaag, H. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. *Milan J. Math. 72* (2004), 1–28.

[25] Dickson, B. J. Molecular mechanisms of axon guidance. *Science 298*, 5600 (2002), 1959–1964.

[26] Eisenbach, M. *Chemotaxis*, 1 ed. Imperial College Press, 2004.

[27] Engelmann, T. W. Neue methode zur untersuchung der sauerstoffausscheidung pflanzlicher und thierischer organismen. *Archiv für die gesamte Physiologie des Menschen und der Tiere 25*, 1 (1881), 285–292.

[28] Fatkullin, I. A study of blow-ups in the Keller-Segel model of chemotaxis. *Nonlinearity 26*, 1 (2013), 81–94.

[29] Fournier, N. Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition. *Ann. Appl. Probab. 25*, 2 (2015), 860–897.

[30] Fournier, N., and Hauray, M. Propagation of chaos for the Landau equation with moderately soft potentials. *Ann. Probab. 44*, 6 (2016), 3581–3660.

[31] Fournier, N., and Jourdain, B. Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab. 27*, 5 (2017), 2807–2861.

[32] Friz, P., and Oberhauser, H. A generalized Fernique theorem and applications. *Proc. Amer. Math. Soc. 138*, 10 (2010), 3679–3688.

[33] Godinho, D., and Quiñinao, C. Propagation of chaos for a subcritical keller–segel model. *Ann. Inst. H. Poincaré Probab. Statist. 51*, 3 (08 2015), 965–992.

[34] Guérin, H., and Méléard, S. Convergence from Boltzmann to Landau processes with soft potential and particle approximations. *J. Statist. Phys. 111*, 3-4 (2003), 931–966.

[35] Hatten M.E., H. N. Neurogenesis and migration. In *Fundamentals of neuroscience*, Z. M, Ed. Academic Press, New York, 1998, pp. 451–479.

[36] Haškovec, J., and Schmeiser, C. Convergence of a stochastic particle approximation for measure solutions of the 2D Keller-Segel system. *Comm. Partial Differential Equations 36*, 6 (2011), 940–960.

[37] Herrero, M. A., and Velázquez, J. J. L. Singularity patterns in a chemotaxis model. *Math. Ann. 306*, 3 (1996), 583–623.

[38] Herrero, M. A., and Velázquez, J. J. L. A blow-up mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24*, 4 (1997), 633–683 (1998).

[39] Hillen, T., and Painter, K. J. A user's guide to PDE models for chemotaxis. *J. Math. Biol. 58*, 1-2 (2009), 183–217.

[40] Hillen, T., and Potapov, A. The one-dimensional chemotaxis model: global existence and asymptotic profile. *Math. Methods Appl. Sci. 27*, 15 (2004), 1783–1801.

[41] Horstmann, D. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein. 105*, 3 (2003), 103–165.

[42] Horstmann, D. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein. 106*, 2 (2004), 51–69.

[43] Jabir, J.-F., Talay, D., and Tomašević, M. Mean–field limit of a particle approximation of the one-dimensional parabolic–parabolic Keller-Segel model without smoothing. *Electronic Communications in probability 23*, 84 (2018), 1–14.

[44] Jourdain, B., and Méléard, S. Probabilistic interpretation and particle method for vortex equations with Neumann's boundary condition. *Proc. Edinb. Math. Soc. (2) 47*, 3 (2004), 597–624.

[45] Karatzas, I., and Shreve, S. E. *Brownian motion and stochastic calculus*, second ed., vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[46] Keller, E. F., and Segel, L. A. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology 26*, 3 (1970), 399 – 415.

[47] Keller, E. F., and Segel, L. A. Model for chemotaxis. *Journal of Theoretical Biology 30*, 2 (1971), 225 – 234.

[48] Keller, E. F., and Segel, L. A. Traveling bands of chemotactic bacteria: A theoretical analysis. *Journal of Theoretical Biology 30*, 2 (1971), 235 – 248.

[49] Krylov, N. V., and Röckner, M. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields 131*, 2 (2005), 154–196.

[50] Lagzi, I. Chemical robotics — chemotactic drug carriers. *Central European Journal of Medicine 8*, 4 (2013), 377–382.

[51] Le Cavil, A., Oudjane, N., and Russo, F. Probabilistic representation of a class of non-conservative nonlinear partial differential equations. *ALEA Lat. Am. J. Probab. Math. Stat. 13*, 2 (2016), 1189–1233.

[52] Makhlouf, A. Representation and Gaussian bounds for the density of Brownian motion with random drift. *Commun. Stoch. Anal. 10*, 2 (2016), 151–162.

[53] Meleard, S., and Roelly-Coppoletta, S. A propagation of chaos result for a system of particles with moderate interaction. *Stochastic Processes and their Applications 26* (1987), 317 – 332.

[54] Miller, R. Sperm chemo-orientation in the metazoa. In *Biology of Fertilization*, C. Metz and A. Monroy, Eds. Academic Press, New York, 1985, p. 275–337.

[55] Mizoguchi, N. Global existence for the Cauchy problem of the parabolic-parabolic Keller-Segel system on the plane. *Calc. Var. Partial Differential Equations 48*, 3-4 (2013), 491–505.

[56] Nagai, T., and Ogawa, T. Global existence of solutions to a parabolic-elliptic system of drift-diffusion type in $\mathbf{R}^2$. *Funkcial. Ekvac. 59*, 1 (2016), 67–112.

[57] Nualart, D. *Malliavin calculus and its applications*, vol. 110 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2009.

[58] Nuzzi, Paul A. Lokuta, M. A., and Huttenlocher, A. *Adhesion Protein Protocols*. Humana Press, Totowa, NJ, 2007, ch. Analysis of Neutrophil Chemotaxis, pp. 23–35.

[59] Osada, H. A stochastic differential equation arising from the vortex problem. *Proc. Japan Acad. Ser. A Math. Sci. 61*, 10 (1985), 333–336.

[60] Osaki, K., and Yagi, A. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcial. Ekvac. 44*, 3 (2001), 441–469.

[61] Pedraza, R. O., Mentel, M. I., Ragout, A. L., Xiqui, M. L., Segundo, D. M., and Baca, B. E. Plant growth-promoting bacteria: The role of chemotaxis in the association azospirillum brasilense-plant. In *Chemotaxis: Types, Clinical Significance, and Mathematical Models*, T. C. Williams, Ed. Nova Science Pub Inc, 2011, ch. 2, pp. 53–83.

[62] Perthame, B. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math. 49*, 6 (2004), 539–564.

[63] Pfeffer, W. Lokomotorische richtungsbewegungen durch chemische reize. *Untersuch. aus d. Botan. Inst. Tübingen 1* (1884), 363–482.

[64] Pfeffer, W. Ueber chemotaktische bewegungen von bakterien, flagellaten und volvocineen. *Untersuch. aus d. Botan. Inst. Tübingen 2* (1888), 582–661.

[65] Qian, Z., Russo, F., and Zheng, W. Comparison theorem and estimates for transition probability densities of diffusion processes. *Probab. Theory Related Fields 127*, 3 (2003), 388–406.

[66] Qian, Z., and Zheng, W. Sharp bounds for transition probability densities of a class of diffusions. *C. R. Math. Acad. Sci. Paris 335*, 11 (2002), 953–957.

[67] Roussos, E. T., Condeelis, J. S., and Patsialou, A. Chemotaxis in cancer. *Nat Rev Cancer 11*, 8 (Aug 2011), 573–587.

[68] Simon, B. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.) 7*, 3 (1982), 447–526.

[69] Stevens, A. The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems. *SIAM J. Appl. Math. 61*, 1 (2000), 183–212.

[70] Stroock, D. W., and Varadhan, S. R. S. Diffusion processes with continuous coefficients. I. *Comm. Pure Appl. Math. 22* (1969), 345–400.

[71] Stroock, D. W., and Varadhan, S. R. S. *Multidimensional diffusion processes*, vol. 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1979.

[72] Sznitman, A.-S. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, vol. 1464 of *Lecture Notes in Math.* Springer, Berlin, 1991, pp. 165–251.

[73] Talay, D., and Tomašević, M. A new stochastic interpretation of Keller-Segel equations: the 1-D case. *arXiv:1712.10254v3*.

[74] Tero, A., Takagi, S., Saigusa, T., Ito, K., Bebber, D. P., Fricker, M. D., Yumiki, K., Kobayashi, R., and Nakagaki, T. Rules for biologically inspired adaptive network design. *Science 327*, 5964 (2010), 439–442.

[75] Tindall, M. J., Porter, S. L., Maini, P. K., Gaglia, G., and Armitage, J. P. Overview of mathematical approaches used to model bacterial chemotaxis I: The single cell. *Bulletin of Mathematical Biology 70*, 6 (2008), 1525–1569.

[76] Trevisan, D. Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. *Electron. J. Probab. 21* (2016), Paper No. 22, 41.

[77] Tso, W.-W., and Adler, J. Negative chemotaxis in escherichia coli. *J Bacteriol 118*, 2 (1974), 560–576.

[78] Veretennikov, A. Y. Parabolic equations and itô's stochastic equations with coefficients discontinuous in the time variable. *Mathematical notes of the Academy of Sciences of the USSR 31*, 4 (1982), 278–283.

[79] Villani, C. *The Wasserstein distances.* Springer Berlin Heidelberg, Berlin, Heidelberg, 2009, pp. 93–111.

[80] Weibull, C. Movement. In *The bacteria*, C. Gunsalus and R. Y. Stanier, Eds., vol. 1. Academic Press, New York, 1960, pp. 153–205.

[81] Ziegler, H. In *Encyclopedia of Plant Physiology*, W. Ruhland, Ed., vol. 17:II. Springer, Berlin, 1962, pp. 484–532.