# Asymptotic models for acoustic, elastic and electromagnetic media. Corner and edge asymptotics for elliptic systems 

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informatics mathematics

## Outline

(1) Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests
(3) Perspectives


## Asymptotic conditions for elasto-acoustic coupling

- Motivations : High Performance Computing for Geophysics Applications
- Numerical Models :
- To reproduce earthquakes
- To take into account the propagation of acoustic and seismic waves

- Contributions : Thin layer models for the propagation of seismic waves and acoustic waves


## Configuration and Approach

- Configuration : The medium of interest is a Land Area surrounded by a Fluid Zone


The thickness of the Fluid Zone is negligible w.r. to the wavelength

- Difficulty : To apply a FEM on a mesh with thin cells in the Fluid Zone and much larger outside
- Approach : An Asymptotic Method. To derive an Equivalent Boundary Condition to "replace" the thin layer


## Land Area

## References

$\square$ A. Bendali, K. Lemrabet (SIAM J. Appl. Math. 56(6): 1664-1693, 1996) The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation
國 V. P. (ESAIM: Math. Model. and Numer. Anal. 48 (5),1431-1449, 2014) Equivalent Boundary Conditions for an Elasto-Acoustic Problem set in a Domain with a Thin Layer
$\square$ J. DiAZ, V. P. (Math. Models Methods Appl. Sci. 26(08):1531-1566, 2016) Equivalent Robin Boundary Conditions for Acoustic and Elastic MediaM. Bonnet, A. Burel, M. Duruflé, P. Joly (ESAIM: Math. Model. and Numer. Anal. 50 (1), 43-75, 2016)
Effective transmission conditions for thin-layer transmission problems in elastodynamics. The case of a planar layer model

## Corner asymptotics for the magnetic potential

## Statement of the problem

The magnetic potential $\mathcal{A}$ solves the 2D eddy-current problem ( $\mathcal{P}$ )

$$
\left\{\begin{aligned}
-\Delta \mathcal{A}^{-}+\mathrm{i} \kappa \mu_{0} \sigma \mathcal{A}^{-} & =0 & & \text { in } \Omega_{-} \\
-\Delta \mathcal{A}^{+} & =\mu_{0} J & & \text { in } \Omega_{+} \\
\mathcal{A}^{+} & =0 & & \text { on } \Gamma
\end{aligned}\right.
$$


with continuous Dirichlet and Neumann traces across $\sum$.

## $\Omega_{-}$: a conducting body ( $\sigma>0$ )

$\Omega_{+}$: a dielectric medium
$\omega \in(0,2 \pi)$ : the angle of the corner $\mathbf{c}$
$J$ : a smooth data, vanishing near the corner c.

## Corner Asymptotics

## Remark

There exists a unique solution $\mathcal{A}$ in $\mathrm{H}_{0}^{1}(\Omega)$ to problem $(\mathcal{P})$. Moreover, $\mathcal{A}$ belongs to $\mathrm{H}^{\frac{5}{2}-\varepsilon}(\Omega)$ for any $\varepsilon>0$. In particular, $\mathcal{A}$ belongs to $\mathcal{C}^{1}(\bar{\Omega})$.

In general $\mathcal{A}$ does not belong to $\mathcal{C}^{2}(\bar{\Omega})$ but $\mathcal{A}$ possesses a corner asymptotic expansion near cinvolving

- The singular functions: belong to the kernel of the considered operator in $\mathbb{R}^{2}$
- The singular coefficients: the calculation requires the knowledge of dual singular functions.

Contributions : To make explicit the Corner Asymptotics of $\mathcal{A}$ near $\mathbf{c}$.

## Motivation : The eddy-current phenomenon

- Limit problem $\left(\mathcal{P}_{0}\right)$ in large frequency/high conductivity :

$$
\left\{\begin{aligned}
-\Delta \mathcal{A}_{0}^{+} & =\mu_{0} J \quad \text { in } \quad \Omega_{+} \\
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- If $\omega \in(0, \pi)$, Problem $\left(\mathcal{P}_{0}\right)$ has non $\mathcal{C}^{1}$ singularities. Strongest singularity : $\mathfrak{s}_{\alpha}=r^{\alpha} \sin \alpha\left(\theta-\frac{\omega}{2}\right)$ with $\alpha=\frac{\pi}{2 \pi-\omega}$.


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- Problem $(\mathcal{P})$ has only $\mathcal{C}^{1}$ singularities!


## Motivation : Multi-scale analysis for eddy-currents

Buret et al (IEEE Trans. on Mag. 48, 2 (2012))Eddy currents and corner singularities

## Motivation : Multi-scale analysis for eddy-currents

目
Buret et al (IEEE Trans. on Mag. 48, 2 (2012))
Eddy currents and corner singularities

- When $\delta=\sqrt{2 /\left(\kappa \mu_{0} \sigma\right)} \ll 1$, there holds

$$
\mathcal{A}_{\delta} \approx \mathcal{A}_{0, \mathrm{reg}}+\psi \mathbf{a} \delta^{\alpha} K_{\alpha}(. / \delta)+\boldsymbol{R}_{\delta}
$$

Here $K_{\alpha}$ is a profile defined in $\mathbb{R}^{2}$ such that

$$
-\Delta K_{\alpha}+2 i K_{\alpha} \mathbb{1}_{\Omega_{-}}=0 \quad \text { near } \mathbf{c}, \quad \text { and } \quad K_{\alpha} \sim \mathfrak{s}_{\alpha} \quad \text { as }|X| \rightarrow \infty .
$$

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$$

$\Longrightarrow$ The knowledge of the singularities of $-\Delta+i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}$is "crucial".

## Application : A modified impedance boundary condition

## Radial behavior of the surface impedance

$$
\frac{1}{i \omega \mu_{0}} z_{\delta}:=\frac{\mathcal{A}_{\delta}}{\partial_{n} \mathcal{A}_{\delta}} \simeq \delta \frac{K_{\alpha}(\cdot / \delta)}{\left(\partial_{n} K_{\alpha}\right)(\cdot / \delta)} \quad \text { near } \mathbf{c}
$$




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## A Model Problem

Problem ( $\mathbf{P}_{\varepsilon}$ ) set in a smooth domain $\Omega^{\varepsilon}=\Omega_{\mathrm{s}} \cup\left\lceil\cup \Omega_{\mathrm{f}}^{\varepsilon}\right.$ :

$$
\begin{array}{ll}
\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u})+\omega^{2} \rho \mathbf{u}=\mathbf{f} & \text { in } \Omega_{\mathrm{s}} \\
\Delta \mathrm{p}+\left(\frac{\omega}{c_{f}}\right)^{2} \mathrm{p}=0 & \text { in } \Omega_{\mathrm{f}}^{\varepsilon} \\
\partial_{\mathbf{n}} \mathrm{p}=\rho_{f} \omega^{2} \mathbf{u} \cdot \mathbf{n} & \text { on } \Gamma \\
\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n}=-\mathrm{pn} & \text { on } \Gamma \\
\mathrm{p}=0 & \text { on } \Gamma^{\varepsilon}
\end{array}
$$



Issue : Equivalent Boundary Conditions for $\mathbf{u}_{\varepsilon}$ as $\varepsilon \rightarrow 0$ ?

## Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u})=\underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

## Assumption

(i) $\underline{\underline{C}}(x)=\left(C_{i j k l}(x)\right)$ is symmetric : $C_{i j k l}=C_{j i k l}=C_{k l i j}$
(ii) $C_{i j k l}(x)$ are real valued smooth functions, up to $\Gamma$
(iii) The tensor $\underline{\underline{C}}(x)$ is positive

## Framework

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(i) $\underline{\underline{C}}(x)=\left(C_{i j k l}(x)\right)$ is symmetric : $C_{i j k l}=C_{j i k l}=C_{k l i j}$
(ii) $\bar{C}_{i j k l}(x)$ are real valued smooth functions, up to $\Gamma$
(iii) The tensor $\underline{\underline{C}}(x)$ is positive

## Assumption

The angular frequency $\omega$ is not an eigenfrequency of the problem

$$
\begin{cases}\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u})+\omega^{2} \rho \mathbf{u}=0 & \text { in } \Omega_{\mathrm{s}} \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n}=0 & \text { on } \Gamma\end{cases}
$$

## Methodology

(1) To derive an Asymptotic Expansion for $\left(\mathbf{u}_{\varepsilon}, \mathrm{p}_{\varepsilon}\right)$ when $\varepsilon \rightarrow 0$

$$
\begin{gathered}
\mathbf{u}_{\varepsilon}(x)=\mathbf{u}_{0}(x)+\varepsilon \mathbf{u}_{1}(x)+\varepsilon^{2} \mathbf{u}_{2}(x)+\cdots \\
\mathfrak{p}_{\varepsilon}(x)=\mathfrak{p}_{0}\left(y_{\alpha}, \frac{s}{\varepsilon}\right)+\varepsilon \mathfrak{p}_{1}\left(y_{\alpha}, \frac{s}{\varepsilon}\right)+\varepsilon^{2} \mathfrak{p}_{2}\left(y_{\alpha}, \frac{s}{\varepsilon}\right)+\cdots
\end{gathered}
$$

(2) Equivalent Conditions of order $k+1 \in \mathbb{N}$ : To identify a simpler problem satisfied by

$$
\mathbf{u}_{k, \varepsilon}:=\mathbf{u}_{0}+\varepsilon \mathbf{u}_{1}+\varepsilon^{2} \mathbf{u}_{2}+\cdots+\varepsilon^{k} \mathbf{u}_{k} \text { up to } \mathcal{O}\left(\varepsilon^{k+1}\right)
$$

(0) To prove stability and convergence results for Equivalent models

## Step 1 : Multiscale Expansion - First terms

(1)

$$
\mathfrak{p}_{0}=0
$$

(2)

$$
\begin{cases}\nabla \cdot \underline{\sigma}\left(\mathbf{u}_{0}\right)+\omega^{2} \rho \mathbf{u}_{0}=\mathbf{f} & \text { in } \Omega_{\mathrm{s}} \\ \mathbf{T}\left(\mathbf{u}_{0}\right)=0 & \text { on } \Gamma\end{cases}
$$

(3)

$$
\mathfrak{p}_{1}(., S)=(S-1) \rho_{f} \omega^{2} \mathbf{u}_{0} \cdot \mathbf{n}, \quad S \in(0,1)
$$

(4)

$$
\begin{cases}\nabla \cdot \underline{\sigma}\left(\mathbf{u}_{1}\right)+\omega^{2} \rho \mathbf{u}_{1}=0 & \text { in } \Omega_{\mathrm{s}} \\ \mathbf{T}\left(\mathbf{u}_{1}\right)=\rho_{f} \omega^{2} \mathbf{u}_{0} \cdot \mathbf{n} \mathbf{n} & \text { on } \Gamma\end{cases}
$$

(5)

$$
\mathfrak{p}_{2}(., S)=-\left(S^{2}-1\right) \mathcal{H} \rho \omega^{2} \mathbf{u}_{0} \cdot \mathbf{n}+(S-1) \rho \omega^{2} \mathbf{u}_{1} \cdot \mathbf{n}
$$

## Step 2 : Equivalent Conditions on 「

We identify a family of simpler problems

$$
\left(\mathbf{P}_{\varepsilon}^{k}\right) \begin{cases}\nabla \cdot \underline{\underline{\sigma}}\left(\mathbf{u}_{\varepsilon}^{k}\right)+\omega^{2} \rho \mathbf{u}_{\varepsilon}^{k}=\mathbf{f} & \text { in } \Omega_{s} \\ \mathbf{T}\left(\mathbf{u}_{\varepsilon}^{k}\right)+\mathrm{B}_{k, \varepsilon}\left(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}\right) \mathbf{n}=0 & \text { on } \Gamma\end{cases}
$$

(1) Order 1:

$$
\mathbf{T}\left(\mathbf{u}_{0}\right)=0
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$$

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$$

(2) Order 2 :

$$
\mathbf{T}\left(\mathbf{u}_{\varepsilon}^{1}\right)-\varepsilon \omega^{2} \rho_{f} \mathbf{u}_{\varepsilon}^{1} \cdot \mathbf{n} \mathbf{n}=0
$$

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$$

(3) Order 3:

$$
\mathbf{T}\left(\mathbf{u}_{\varepsilon}^{2}\right)-\varepsilon \omega^{2} \rho_{f}(1-\varepsilon \mathcal{H}) \mathbf{u}_{\varepsilon}^{2} \cdot \mathbf{n} \mathbf{n}=0
$$

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$$

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$$

(3) Order 3:

$$
\mathbf{T}\left(\mathbf{u}_{\varepsilon}^{2}\right)-\varepsilon \omega^{2} \rho_{f}(1-\varepsilon \mathcal{H}) \mathbf{u}_{\varepsilon}^{2} \cdot \mathbf{n} \mathbf{n}=0
$$

(4) Order 4:

$$
\mathbf{T}\left(\mathbf{u}_{\varepsilon}^{3}\right)-\varepsilon \omega^{2} \rho\left(1-\varepsilon \mathcal{H}+\frac{\varepsilon^{2}}{3}\left[\Delta_{\Gamma}+\kappa^{2} \mathbb{I}+4 \mathcal{H}^{2}-\mathcal{K}\right]\right) \mathbf{u}_{\varepsilon}^{3} \cdot \mathbf{n} \mathbf{n}=0
$$

## Step 3 : Stability and Convergence results

## Theorem

There exists $\varepsilon_{0}>0$ s.t. for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$, Problem $\left(\mathbf{P}_{\varepsilon}^{k}\right)$ with data $\mathbf{f} \in \mathrm{L}^{2}\left(\Omega_{\mathbf{s}}\right)$ has a unique solution $\mathbf{u}_{\varepsilon}^{k} \in \mathrm{~V}_{k}$ and

- Stability

$$
\left\|\mathbf{u}_{\varepsilon}^{k}\right\|_{1, \Omega_{\mathrm{s}}} \leqslant C\|\mathbf{f}\|_{0, \Omega_{\mathrm{s}}}
$$

- Convergence

$$
\left\|\mathbf{u}_{\varepsilon}-\mathbf{u}_{\varepsilon}^{k}\right\|_{1, \Omega_{\mathrm{s}}} \leqslant C \varepsilon^{k+1}
$$

$\mathrm{V}_{k}=\mathrm{H}^{1}\left(\Omega_{\mathrm{s}}\right)$ when $k=0,1,2$
$\mathrm{V}_{k}=\left\{\mathbf{u} \in \mathrm{H}^{1}\left(\Omega_{\mathrm{s}}\right)|\mathbf{u} \cdot \mathbf{n}|_{\Gamma} \in \mathrm{H}^{1}(\Gamma)\right\}$ when $k=3$

## Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$
\mathbf{r}_{\varepsilon}^{N}:=\mathbf{u}_{\varepsilon}-\sum_{n=0}^{N} \varepsilon^{n} \mathbf{u}_{n} \text { in } \Omega_{\mathrm{s}}, \text { and } r_{\varepsilon}^{N}:=\mathrm{p}_{\varepsilon}-\sum_{n=0}^{N} \varepsilon^{n} \mathrm{p}_{n} \text { in } \Omega_{\mathrm{f}}^{\varepsilon} \text {. }
$$

Evaluation of the right hand sides in

$$
\left\{\begin{array}{lll}
\Delta r_{\varepsilon}^{N}+\kappa^{2} r_{\varepsilon}^{N} & =f_{\varepsilon} & \text { in } \Omega_{\mathrm{f}}^{\varepsilon} \\
\nabla \cdot \underline{\underline{\sigma}}\left(\mathbf{r}_{\varepsilon}^{N}\right)+\omega^{2} \rho \mathbf{r}_{\varepsilon}^{N} & =0 & \text { in } \Omega_{\mathrm{s}} \\
\partial_{\mathrm{n}} r_{\varepsilon}^{N}-\rho_{\mathrm{f}} \omega^{2} \mathbf{r}_{\varepsilon}^{N} \cdot \boldsymbol{n} & =g_{\varepsilon} & \text { on } \Gamma \\
\mathrm{T}\left(\mathrm{r}_{\varepsilon}^{N}\right)+r_{\varepsilon}^{N} \mathbf{n} & =0 & \text { on } \Gamma \\
r_{\varepsilon}^{N} & =0 & \text { on } \Gamma^{\varepsilon} .
\end{array}\right.
$$

The RHS are explicit : $f_{\varepsilon}=\mathcal{O}\left(\varepsilon^{N-\frac{1}{2}}\right)$ in $\Omega_{\mathrm{f}}^{\varepsilon}, \quad g_{\varepsilon}=\mathcal{O}\left(\varepsilon^{N}\right)$ on $\Gamma$.

$$
\text { Optimal Estimates : }\left\|\mathbf{r}_{\varepsilon}^{N}\right\|_{1, \Omega_{\mathbf{s}}}+\sqrt{\varepsilon}\left\|r_{\varepsilon}^{N}\right\|_{1, \Omega_{\mathrm{f}}^{\varepsilon}} \leqslant C_{N} \varepsilon^{N+1}
$$

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## Numerical Simulations

## Framework

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_{\mathrm{s}}=\mathrm{D}(0 ; 0.01)$
- $\omega=1.5 \times 10^{6}$
- Source on $\Gamma(f=0): p_{i}(\mathbf{x})=\exp (i \omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d}=(1,0)$
- $\mathbb{P}_{3}$-finite elements (Lagrange) in the Library Hou10ni
- $\Omega_{\mathrm{s}}$ : Isotropic elastic material with Lamé coefficients :

$$
\mu=26.32 \times 10^{9} \text { and } \lambda=51.08 \times 10^{9} .
$$

- $c=1500 \mathrm{~m} \cdot \mathrm{~s}^{-1}, \quad \rho_{f}=1000 \mathrm{~kg} \cdot \mathrm{~m}^{-3}, \quad \rho=2700 \mathrm{~kg} \cdot \mathrm{~m}^{-3}$.


## Numerical Simulations

## Qualitative Comparisons

$$
\varepsilon=0.001
$$


$\left|\Re \mathbf{u}_{\varepsilon}\right|$
$\mathbf{u}_{\varepsilon}$ : analytical solution

$\left|\Re \mathbf{u}_{\varepsilon}^{2}\right|$
$\mathbf{u}_{\varepsilon}^{2}$ (model of order 3) : IPDGM

## Numerical Simulations

## Qualitative Comparisons

$$
\varepsilon=0.0001
$$


$\left|\Re \mathbf{u}_{\boldsymbol{\varepsilon}}\right|$
$\mathbf{u}_{\varepsilon}$ : analytical solution

$\left|\Re \mathbf{u}_{\varepsilon}^{2}\right|$
$\mathbf{u}_{\varepsilon}^{2}$ (model of order 3) : IPDGM

## Convergence of the models

For $k \in\{0,1,2\}$, we plot the $\mathrm{L}^{2}$-error $\left\|\mathbf{u}_{\varepsilon}-\mathbf{u}_{\varepsilon}^{k}\right\|_{0, \Omega_{\mathrm{s}}}$ w.r.t. $\varepsilon$ $\mathbf{u}_{\varepsilon}$ : analytical solution
$\mathbf{u}_{\varepsilon}^{k}$ : analytical solution or numerical solution (IPDGM)


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## Harmonic Maxwell Equations

Take a thin smooth closed sheet $\Omega_{o}^{\varepsilon}$, with high conductivity $\sigma_{o}^{\varepsilon}=\varepsilon^{-2} \tilde{\sigma}$, and surrounded by conductor bodies $\Omega_{-}^{\varepsilon}$ and $\Omega_{+}^{\varepsilon}$.

Maxwell equations :

$$
\begin{aligned}
& \operatorname{curl} \mathbf{E}^{\varepsilon}-i \omega \underline{\mu}^{\varepsilon} \mathbf{H}^{\varepsilon}=0 \\
& \operatorname{curl} \mathbf{H}^{\varepsilon}+\left(i \omega \underline{\epsilon}^{\varepsilon}-\underline{\sigma}^{\varepsilon}\right) \mathbf{E}^{\varepsilon}=\mathbf{j}
\end{aligned}
$$


$\underline{\mu}^{\varepsilon}=\left\{\begin{array}{ll}\mu_{-}, & \text {in } \Omega_{-}^{\varepsilon}, \\ \mu_{o}, & \text { in } \Omega_{o}^{\varepsilon}, \\ \mu_{+}, & \text {in } \Omega_{+}^{\varepsilon},\end{array} \quad \underline{\epsilon}^{\varepsilon}=\left\{\begin{array}{ll}\epsilon_{-}, & \text {in } \Omega_{-}^{\varepsilon}, \\ \epsilon_{o}, & \text { in } \Omega_{o}^{\varepsilon}, \\ \epsilon_{+}, & \text {in } \Omega_{+}^{\varepsilon},\end{array} \quad \underline{\sigma}^{\varepsilon}= \begin{cases}\sigma_{-}, & \text {in } \Omega_{-}^{\varepsilon}, \\ \sigma_{o}^{\varepsilon}=\varepsilon^{-2} \tilde{\sigma}, & \text { in } \Omega_{o}^{\varepsilon}, \\ \sigma_{+}, & \text {in } \Omega_{+}^{\varepsilon} .\end{cases}\right.\right.$

## References


K. Schmidt, S. Tordeux (ESAIM: Math. Model. and Numer. Anal. 45(6), pp.1115-1140, 2011)
High order transmission conditions for thin conductive sheets in magneto-quasistatics.
K. Schmidt, A. Chernov (SIAM J. Appl. Math. 73(6): 1980-2003, 2013) A unified analysis of transmission conditions for thin conducting sheets in the time-harmonic eddy current model.

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A unified analysis of transmission conditions for thin conducting sheets in the time-harmonic eddy current model.
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V. P., K. Schmidt, M. Duruflé (SIAM J. Appl. Math. 76(3), 1031-1052, 2016)

Equivalent Transmission Conditions for the time-harmonic Maxwell equations in 3D for a Medium with a Highly Conductive Thin Sheet

## Second Order Impedance Transmission Conditions

$$
\binom{\left[\mathbf{E}_{\varepsilon}^{1} \times \mathbf{n}\right]_{\Gamma}}{\left\{\mathbf{E}_{\varepsilon}^{1} \times \mathbf{n}\right\}_{\Gamma}}=i \omega \varepsilon\left(\begin{array}{ll}
\mathrm{L}_{1} & \mathrm{~L}_{3}  \tag{1a}\\
\mathrm{~L}_{3} & \mathrm{~L}_{2}
\end{array}\right)\binom{\left\{\mathbf{n} \times\left(\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n}\right)\right\}_{\Gamma}}{\left[\mathbf{n} \times\left(\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n}\right)\right]_{\Gamma}}
$$

$\mathrm{L}_{i}$ are differential operators given by

$$
\mathrm{L}_{i}=A_{i} \text { curl }_{\Gamma} \text { curl }_{\Gamma}-B_{i} \mathbb{I}, \quad i=1,2,3,
$$

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\end{array}\right)\binom{\left\{\mathbf{n} \times\left(\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n}\right)\right\}_{\Gamma}}{\left[\mathbf{n} \times\left(\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n}\right)\right]_{\Gamma}}
$$

$\mathrm{L}_{i}$ are differential operators given by

$$
\mathrm{L}_{i}=A_{i} \text { curl }_{\Gamma} \text { curl }_{\Gamma}-B_{i} \mathbb{I}, \quad i=1,2,3,
$$

in which $A_{i}, B_{i}$ are constants defined by

$$
\begin{array}{ll}
A_{1}=-\omega^{-2}\left\{\left(\epsilon+i \frac{\sigma}{\omega}\right)^{-1}\right\}_{\Gamma}, & B_{1}=2 \frac{\mu_{0}}{\gamma} \tanh \left(\frac{\gamma}{2}\right)-\{\mu\}_{\Gamma} \\
A_{2}=\frac{A_{1}}{4}, & B_{2}=\frac{\mu_{0}}{2 \gamma} \operatorname{coth}\left(\frac{\gamma}{2}\right)-\frac{\{\mu\}_{\Gamma}}{4}  \tag{2}\\
A_{3}=-\frac{1}{4} \omega^{-2}\left[\left(\epsilon+i \frac{\sigma}{\omega}\right)^{-1}\right]_{\Gamma}, & B_{3}=-\frac{1}{4}[\mu]_{\Gamma},
\end{array}
$$

and

$$
\gamma=\exp \left(\frac{3 i \pi}{4}\right) \sqrt{\omega \mu_{o} \tilde{\sigma}} .
$$

## Impedance transmission conditions

## Robustness versus the conductivity parameter



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(3) Perspectives


## Problem $(\mathcal{P})$ with Angular Conductor

## Main difficulty

$$
\left\{\begin{aligned}
-\Delta \mathcal{A}^{-}+\mathrm{i} \kappa \mu_{0} \sigma \mathcal{A}^{-} & =0 & & \text { in } \Omega_{-} \\
-\Delta \mathcal{A}^{+} & =\mu_{0} J & & \text { in } \Omega_{+} \\
\mathcal{A}^{+} & =0 & & \text { on } \Gamma
\end{aligned}\right.
$$



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\end{aligned}\right.
$$



The considered operator is

$$
-\Delta+4 i \zeta^{2} \mathbb{1}_{\Omega_{-}}= \begin{cases}-\Delta+i \kappa \mu_{0} \sigma & \text { in } \Omega_{-} \\ -\Delta & \text { in } \Omega_{+}\end{cases}
$$

The singularities are generated by the term $i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}$.

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$$

The singularities are generated by the term $i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}$.
$\Longrightarrow$ the derivation of the Corner Asymptotics is not obvious.

## Our Reference

M. Dauge et al (Mathematical Methods in the Applied Sciences. 2014, 37 (13), pp.1924-1955)

Corner asymptotics of the magnetic potential in the eddy-current model

- $\mathcal{A}$ possesses a corner asymptotic expansion

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathbb{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta),
$$

$(r, \theta)$ : polar coordinates centered at $\mathbf{c}$
$\mathfrak{S}^{k, p}$ : primal singular functions
$\Lambda^{\kappa, p}$ : singular coefficients

- We provide a constructive procedure to determine the primal and the dual singularities
- We generalize the method of moments and we introduce the method of quasi-dual functions to determine $\Lambda^{k, p}$


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## Laplace Operator

- In the case when $\zeta=0$, we consider the solution $\mathcal{A}$ to

$$
\left\{\begin{aligned}
-\Delta \mathcal{A} & =\mu_{0} J \text { in } \Omega \\
\mathcal{A} & =0 \text { on } \Gamma
\end{aligned}\right.
$$

- $\mathcal{A}$ admits the Taylor expansion at $\mathbf{c}$ :

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0}+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \underbrace{r^{k} \cos (k \theta-p \pi / 2)}_{=\mathfrak{s}^{k, p}(r, \theta)}
$$

$\mathfrak{s}^{k, p}$ : harmonic polynomials

## The Dual Function Method

盏 V.G. Maz'Ya, B.A. Plamenevskil (Amer. Math. Soc. Trans. (2) '84) On the coefficients (...) in domains with conical points
We use dual harmonic functions :

$$
\mathfrak{k}^{k, p}(r, \theta)= \begin{cases}-\frac{1}{2 \pi} \log r, & \text { if } k=0, p=0 \\ \frac{1}{2 k \pi} r^{-k} \cos (k \theta-p \pi / 2), & \text { if } k \geqslant 1, p=0,1\end{cases}
$$

## Proposition

Let $\mathcal{A}$ be the solution to the Laplace equation. Then

$$
\mathcal{J}_{R}\left(\mathfrak{k}^{0,0}, \mathcal{A}\right)=\Lambda^{0,0} \quad \text { and } \quad \mathcal{J}_{R}\left(\mathfrak{k}^{k, p}, \mathcal{A}\right)=\Lambda^{k, p}, \quad k \geqslant 1, p=0,1
$$

Here, for $R>0, \quad \mathcal{J}_{R}(K, A)=\int_{r=R}\left(K \partial_{r} A-\partial_{r} K A\right) R \mathrm{~d} \theta$.

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(3) Perspectives


## The singularities

The singularities $\mathfrak{U}$ of $\mathscr{L}_{\zeta}=-\Delta+4 i \zeta^{2} \mathbb{1}_{\mathcal{S}_{-}}$:

$$
\begin{cases}-\Delta \mathfrak{U}+4 \mathrm{i} \zeta^{2} \mathfrak{U}=0 & \text { in } \mathcal{S}_{-} \\ -\Delta \mathfrak{U}=0 & \text { in } \mathcal{S}_{+}\end{cases}
$$

with continuous Dirichlet and Neumann traces across $\mathcal{G}$.


## Description of the singularities

國 V.A. Kondratev (Trudy Moskov. Mat. Obšč. '67)
Boundary value problems for elliptic equations in domains with conical or angular points

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\mathfrak{U}=\underbrace{\mathfrak{u}_{0}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{u}_{j}}_{\text {shadow }}
$$

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$$
\mathfrak{U}=\underbrace{\mathfrak{u}_{0}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{u}_{j}}_{\text {shadow }}
$$

- We solve

$$
\begin{aligned}
& \Delta \mathfrak{u}_{0}=0, \quad \Delta \mathfrak{u}_{1}=4 \mathfrak{u}_{0} \mathbb{1}_{\mathcal{S}_{-}}, \quad \ldots, \quad \Delta \mathfrak{u}_{j}=4 \mathfrak{u}_{j-1} \mathbb{1}_{\mathcal{S}_{-}}, \\
& \mathfrak{u}_{j} \in \mathrm{~S}^{\lambda}=\operatorname{Span}\left\{r^{\lambda} \log ^{9} r \Phi(\theta), \quad q \in \mathbb{N}, \quad \Phi \in \mathcal{C}^{1}(\mathbb{T}), \Phi^{ \pm} \in \mathcal{C}^{\infty}\left(\overline{\mathbb{T}}_{ \pm}\right)\right\} \\
& \text {Here, } \lambda \in \mathbb{Z}, \mathbb{T}=\mathbb{R} /(2 \pi \mathbb{Z}) \text { and } \mathbb{T}_{-}=(-\omega / 2, \omega / 2)
\end{aligned}
$$

## Existence of the shadows

## Lemma

Let $\lambda \in \mathbb{Z}$ and $\mathfrak{f} \in \mathrm{T}^{\lambda-2}$. Then, there exists $\mathfrak{u} \in \mathrm{S}^{\lambda}$ such that $\Delta \mathfrak{u}=\mathfrak{f}$.
Moreover

$$
\begin{aligned}
& \text { (i) If } \lambda \in \mathbb{Z} \backslash\{0\} \text {, } \operatorname{deg} \mathfrak{u} \leqslant \operatorname{deg} \mathfrak{f}+1 \text {, } \\
& \text { (ii) If } \lambda=0, \operatorname{deg} \mathfrak{u} \leqslant \operatorname{deg} \mathfrak{f}+2 \text {. }
\end{aligned}
$$

Here,

$$
\mathrm{T}^{\lambda}=\operatorname{Span}\left\{r^{\lambda} \log ^{q} r \Psi(\theta), \quad q \in \mathbb{N}, \quad \Psi \in L^{2}(\mathbb{T}), \Psi^{ \pm} \in \mathcal{C}^{\infty}\left(\overline{\mathbb{T}}_{ \pm}\right)\right\}
$$

The degree of $\mathfrak{g}$ is its degree as polynomial of $\log r$.

## Proof of Lemma

M. Dauge Elliptic Boundary Value Problems in Corner Domains Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Berlin 1988
We introduce the Mellin symbol $\mathscr{M}$ associated with $\Delta$

$$
\mathscr{M}(\mu)=\left(\mu^{2}+\partial_{\theta}^{2}\right), \theta \in \mathbb{T} .
$$

Setting

$$
\mathfrak{f}=\sum_{q=0}^{\operatorname{deg} \mathfrak{f}} r^{\lambda-2} \log ^{q} r \Psi_{q} \in T^{\lambda-2},
$$

and

$$
\mathfrak{u}=\operatorname{Res}_{\mu=\lambda}\left\{r^{\mu} \mathscr{M}(\mu)^{-1} \sum_{q=0}^{\operatorname{deg} \mathfrak{f}} \frac{q!\Psi_{q}}{(\mu-\lambda)^{q+1}}\right\} \in S^{\lambda}
$$

we check that $\Delta \mathfrak{u}=\mathfrak{f}$.

## Primal and Dual Singularities

- Primal singular functions $\mathfrak{S}$ belong to $H^{1}$ in any bounded neighborhood $\mathcal{B}$ of $\mathbf{c}$.
- Dual singular functions $\mathfrak{K}$ do not belong to $H^{1}$ in such a neighborhood $\mathcal{B}$.

The functions $\mathfrak{K}$ are needed for the determination of the coefficients $\Lambda$ involved in the asymptotics.

## A basis for Primal Singularities

For $(k, p)=(0,0)$ and for $(k, p) \in \mathbb{N}^{*} \times\{0,1\}$,

$$
\mathfrak{S}^{k, p}=\underbrace{\mathfrak{s}^{k, p}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{s}_{j}^{k, p}}_{\text {shadow }} .
$$

## Lemma

For any $j \geqslant 1$, there exists $\mathfrak{s}_{j}^{k, p} \in \mathrm{~S}^{k+2 j}$, with deg $\mathfrak{s}_{j}^{k, p} \leqslant j$, satisfying

$$
\Delta \mathfrak{s}_{j}^{k, p}=4 \mathfrak{s}_{j-1}^{k, p} \mathbb{1}_{\mathcal{S}_{-}} .
$$

Application: $\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta)$

## A basis for Dual Singularities

For $(k, p)=(0,0)$ and for $(k, p) \in \mathbb{N}^{*} \times\{0,1\}$,

$$
\mathfrak{K}^{k, p}=\underbrace{\mathfrak{f}^{k, p}}_{\text {leading part }}+\sum_{i \geqslant 1}\left(i \zeta^{2}\right)^{j} \underbrace{\mathfrak{k}_{j}^{k, p}}_{\text {shadow }} .
$$

## Lemma

For any $j \geqslant 1$, there exists $\mathfrak{k}_{j}^{k, p} \in \mathrm{~S}^{-k+2 j}$, with deg $\mathfrak{k}_{j}^{k, p} \leqslant j+1$, satisfying

$$
\Delta \mathfrak{e}_{j}^{k, p}=4 \mathfrak{e}_{j-1}^{k, p} 1_{\mathcal{S}_{-}} .
$$

## The Quasi-Dual Function Method

- M. Costabel et al (SIAM J. Math. Anal. 35-5, 1177-1202, 2004)

A quasidual function method for extracting edge stress intensity functions
S. ShANNON et al (Int. J. Fract. 181: 25-50, 2013)

Extracting generalized edge flux intensity functions by the quasidual function method along circular 3-D edges

- We use quasi-dual functions $\mathfrak{K}_{m}^{k, p}$ :

$$
\mathfrak{K}_{m}^{k, p}=\mathfrak{k}^{k, p}+\sum_{j=1}^{m}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{k}_{j}^{k, p}}_{\text {shadow }}
$$

## The Quasi-Dual Function Method

## Extraction of coefficients

## Theorem

Let $\mathcal{A}$ be the solution to problem $(\mathcal{P})$. Let $k \in \mathbb{N}$ and $p \in\{0,1\}$ ( $p=0$ if $k=0$ ). Then for all $m$ such that $2 m+2>k$,
$\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{k, p}, \mathcal{A}\right) \underset{R \rightarrow 0}{=} \Lambda^{k, p}+\sum_{\ell=1}^{[k / 2]} \mathcal{J}^{k, p ; k-2 \ell, p} \Lambda^{k-2 \ell, p}+\mathcal{O}\left(R^{-k} R_{0}^{2 m+2} \log R\right)$,
where $R_{0}=\zeta R(1+\sqrt{|\log R|})$.

## The Quasi-Dual Function Method

## Examples

- For $k=0$

$$
\Lambda^{0,0} \underset{R \rightarrow 0}{=} \mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)+\mathcal{O}\left(R_{0}^{2+2 m} \log R\right)
$$

- For $k=1$

$$
\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_{R}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)+\mathcal{O}\left(R^{-1} R_{0}^{2 m+2}\right)
$$

- For $k=2$, we need $m \geqslant 1$

$$
\Lambda^{2,0} \underset{R \rightarrow 0}{=} \mathcal{J}_{R}\left(\mathfrak{K}_{1}^{2,0}, \mathcal{A}\right)-\mathcal{J}^{2,0 ; 0,0} \Lambda^{0,0}+\mathcal{O}\left(R^{-2} R_{0}^{4} \log R\right)
$$

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## Framework

- $\Omega$ : disk of radius 50 mm
- Conducting sector : $\omega=\pi / 4$
- $\zeta=1 /(5 \sqrt{2}) \mathrm{mm}^{-1}$
- Source : $\mathcal{A}^{+}=\frac{|\theta|}{2 \pi}$ on $\partial \Omega$



## Finite Element Solution

We plot the real part and the imaginary part of the FE solution.



## Accuracy for the computation of $\Lambda^{0,0}$ as a function of $R$

- There holds

$$
\Lambda_{R \rightarrow 0}^{0,0}=\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)+\mathcal{O}\left(R_{0}^{2+2 m} \log R\right)
$$

- We plot $\left|\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)-\mathcal{A}\right|_{\mathrm{c}}\left|/|\mathcal{A}|_{\mathrm{c}}\right|$ as a function of $R(m=0,1)$.



## Accuracy for the computation of $\Lambda^{1,0}$ as a function of $R$

- Recall that

$$
\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_{R}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)+\mathcal{O}\left(R^{-1} R_{0}^{2 m+2}\right) .
$$

- We plot $\left|\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)-\mathcal{J}_{R_{\mathrm{s}}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\right| /\left|\mathcal{J}_{R_{\mathrm{s}}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\right|$.



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## Perspectives

## Electromagnetism

- To justify and to extend the multi-scale expansion for the corner problem

$$
\mathcal{A}_{\delta} \approx \mathcal{A}_{0, \text { reg }}+\psi \mathbf{a} \delta^{\alpha} K_{\alpha}(. / \delta)+R_{\delta}
$$

- To tackle the problem in the (linear) ferromagnetic case :

$$
\mu_{r}=\mu_{-} / \mu_{+} \gg 1
$$

- To apply the method of matched asymptotic expansions for 3D scattering problems by small obstacles (PhD thesis of J. Labat)


## Perspectives

## Elasto-acoustic coupling

- To couple the layer of water with the atmosphere

- To derive Equivalent Conditions when a high-order ABC is imposed on top of the water
- To extend Equivalent Conditions to transient problems


## Thank you for your attention

