## Asymptotic models for acoustic, elastic and electromagnetic media. Corner and edge asymptotics for elliptic systems

## VICTOR PÉRON

Université de Pau et des Pays de l'Adour Team MAGIQUE 3D, INRIA Bordeaux Sud-Ouest

Habilitation degree defense

Pau, December 6, 2017





Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

#### Asymptotic conditions for elasto-acoustic coupling

- Motivations : High Performance Computing for Geophysics Applications
- Numerical Models :
  - To reproduce earthquakes
  - To take into account the propagation of acoustic and seismic waves



• **Contributions** : Thin layer models for the propagation of seismic waves and acoustic waves

Corner Asymptotics

## **Configuration and Approach**

• **Configuration** : The medium of interest is a Land Area surrounded by a Fluid Zone



The thickness of the Fluid Zone is negligible w.r. to the wavelength

- **Difficulty** : To apply a FEM on a mesh with thin cells in the Fluid Zone and much larger outside
- **Approach** : An Asymptotic Method. To derive an Equivalent Boundary Condition to "replace" the thin layer



#### References

- A. BENDALI, K. LEMRABET (SIAM J. Appl. Math. 56(6): 1664–1693, 1996) The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation
- V. P. (ESAIM: Math. Model. and Numer. Anal. 48 (5),1431–1449, 2014)
   Equivalent Boundary Conditions for an Elasto-Acoustic Problem set in a Domain with a Thin Layer
- J. DIAZ, V. P. (Math. Models Methods Appl. Sci. 26(08):1531–1566, 2016) Equivalent Robin Boundary Conditions for Acoustic and Elastic Media
  - M. BONNET, A. BUREL, M. DURUFLÉ, P. JOLY (ESAIM: Math. Model. and Numer. Anal. 50 (1), 43–75, 2016)

Effective transmission conditions for thin-layer transmission problems in elastodynamics. The case of a planar layer model

Corner Asymptotics

Perspectives

#### Corner asymptotics for the magnetic potential Statement of the problem

The magnetic potential  ${\mathcal A}$  solves the 2D eddy-current problem  $({\mathcal P})$ 

with continuous Dirichlet and Neumann traces across  $\Sigma$ .

 $\Omega_-$  : a conducting body ( $\sigma >$  0)

 $\Omega_+$  : a dielectric medium

 $\omega \in (0,2\pi)$  : the angle of the corner  ${f c}$ 

J: a smooth data, vanishing near the corner **c**.

## **Corner Asymptotics**

#### Remark

There exists a unique solution  $\mathcal{A}$  in  $\mathrm{H}^{1}_{0}(\Omega)$  to problem  $(\mathcal{P})$ . Moreover,  $\mathcal{A}$  belongs to  $\mathrm{H}^{\frac{5}{2}-\varepsilon}(\Omega)$  for any  $\varepsilon > 0$ . In particular,  $\mathcal{A}$  belongs to  $\mathcal{C}^{1}(\overline{\Omega})$ .

In general  $\mathcal{A}$  does not belong to  $\mathcal{C}^2(\overline{\Omega})$  but  $\mathcal{A}$  possesses a corner asymptotic expansion near **c** involving

• The singular functions :

belong to the kernel of the considered operator in  $\mathbb{R}^2$ 

• The singular coefficients :

the calculation requires the knowledge of dual singular functions.

**Contributions** : To make explicit the Corner Asymptotics of  $\mathcal{A}$  near **c**.

Corner Asymptotics

Perspectives

## Motivation : The eddy-current phenomenon

• Limit problem ( $\mathcal{P}_0$ ) in large frequency/high conductivity :

$$\left\{ egin{array}{ll} -\Delta \mathcal{A}_0^+ = \mu_0 J & ext{ in } \Omega_+ \ \mathcal{A}_0^+ = 0 & ext{ on } \partial \Omega_+ \end{array} 
ight.$$



Corner Asymptotics

Perspectives

## Motivation : The eddy-current phenomenon

• Limit problem ( $\mathcal{P}_0$ ) in large frequency/high conductivity :

$$\left\{egin{array}{ll} -\Delta \mathcal{A}_0^+ = \mu_0 J & ext{ in } \Omega_+ \ \mathcal{A}_0^+ = 0 & ext{ on } \partial \Omega_+ \end{array}
ight.$$



• If  $\omega \in (0, \pi)$ , Problem  $(\mathcal{P}_0)$  has non  $\mathcal{C}^1$  singularities. Strongest singularity :  $\mathfrak{s}_{\alpha} = r^{\alpha} \sin \alpha (\theta - \frac{\omega}{2})$  with  $\alpha = \frac{\pi}{2\pi - \omega}$ .

Corner Asymptotics

Perspectives

## Motivation : The eddy-current phenomenon

• Limit problem ( $\mathcal{P}_0$ ) in large frequency/high conductivity :

$$\left\{egin{array}{ll} -\Delta \mathcal{A}_0^+ = \mu_0 J & ext{ in } \Omega_+ \ \mathcal{A}_0^+ = 0 & ext{ on } \partial \Omega_+ \end{array}
ight.$$



- If  $\omega \in (0, \pi)$ , Problem  $(\mathcal{P}_0)$  has non  $\mathcal{C}^1$  singularities. Strongest singularity :  $\mathfrak{s}_{\alpha} = r^{\alpha} \sin \alpha (\theta - \frac{\omega}{2})$  with  $\alpha = \frac{\pi}{2\pi - \omega}$ .
- Problem  $(\mathcal{P})$  has only  $\mathcal{C}^1$  singularities!

Corner Asymptotics

Perspectives

#### Motivation : Multi-scale analysis for eddy-currents

BURET et al (IEEE Trans. on Mag. 48, 2 (2012))

Eddy currents and corner singularities

Corner Asymptotics

#### Motivation : Multi-scale analysis for eddy-currents

BURET et al (IEEE Trans. on Mag. 48, 2 (2012)) Eddy currents and corner singularities

• When  $\delta = \sqrt{2/(\kappa\mu_0\sigma)} \ll$  1, there holds

$$\mathcal{A}_{\delta} pprox \mathcal{A}_{0,\mathrm{reg}} + \psi \, \mathsf{a} \, \delta^{lpha} oldsymbol{\mathcal{K}}_{oldsymbol{lpha}}(./\delta) + \mathsf{R}_{\delta} \; .$$

Here  $K_{\alpha}$  is a *profile* defined in  $\mathbb{R}^2$  such that

 $-\Delta K_{\alpha} + 2iK_{\alpha} \mathbb{1}_{\Omega_{-}} = 0 \quad \text{near } \mathbf{c}, \quad \text{and} \quad K_{\alpha} \sim \mathfrak{s}_{\alpha} \quad \text{as } |X| \to \infty \,.$ 

Corner Asymptotics

#### Motivation : Multi-scale analysis for eddy-currents

BURET et al (IEEE Trans. on Mag. 48, 2 (2012)) Eddy currents and corner singularities

• When  $\delta = \sqrt{2/(\kappa\mu_0\sigma)} \ll$  1, there holds

$$\mathcal{A}_{\delta} pprox \mathcal{A}_{0,\mathrm{reg}} + \psi \, \mathsf{a} \, \delta^{lpha} oldsymbol{\mathcal{K}}_{oldsymbol{lpha}}(./\delta) + \mathsf{R}_{\delta} \; .$$

Here  $K_{\alpha}$  is a *profile* defined in  $\mathbb{R}^2$  such that

 $-\Delta K_{\alpha} + 2iK_{\alpha} \mathbb{1}_{\Omega_{-}} = 0 \quad \text{near } \mathbf{c}, \quad \text{and} \quad K_{\alpha} \sim \mathfrak{s}_{\alpha} \quad \text{as } |X| \to \infty \,.$ 

Corner Asymptotics

Perspectives

#### Motivation : Multi-scale analysis for eddy-currents

BURET et al (IEEE Trans. on Mag. 48, 2 (2012)) Eddy currents and corner singularities

• When  $\delta = \sqrt{2/(\kappa\mu_0\sigma)} \ll$  1, there holds

 $\mathcal{A}_{\delta} pprox \mathcal{A}_{0,\mathrm{reg}} + \psi \, \mathsf{a} \, \delta^{lpha} \mathit{K}_{m{lpha}}(./\delta) + \mathit{R}_{\delta} \; .$ 

Here  $K_{\alpha}$  is a *profile* defined in  $\mathbb{R}^2$  such that

 $-\Delta \mathsf{K}_{\alpha} + 2\mathsf{i}\mathsf{K}_{\alpha}\, \mathbbm{1}_{\Omega_{-}} = 0 \quad \text{near } \mathbf{c}, \quad \text{and} \quad \mathsf{K}_{\alpha} \sim \mathfrak{s}_{\alpha} \quad \text{as } |X| \to \infty \,.$ 

 $\implies$  The knowledge of the singularities of  $-\Delta + i\kappa\mu_0\sigma\, \mathbbm{1}_{\Omega_-}$  is "crucial".

Corner Asymptotics

## Application : A modified impedance boundary condition

Radial behavior of the surface impedance

$$\frac{1}{i\omega\mu_0} Z_{\delta} := \frac{\mathcal{A}_{\delta}}{\partial_n \mathcal{A}_{\delta}} \simeq \delta \frac{\mathcal{K}_{\alpha}(\cdot/\delta)}{(\partial_n \mathcal{K}_{\alpha})(\cdot/\delta)} \quad \text{near } \mathbf{c}$$



Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

Perspectives

#### **A Model Problem**

Problem  $(\mathbf{P}_{\varepsilon})$  set in a smooth domain  $\Omega^{\varepsilon} = \Omega_{s} \cup \Gamma \cup \Omega_{f}^{\varepsilon}$ :



**Issue** : Equivalent Boundary Conditions for  $\mathbf{u}_{\varepsilon}$  as  $\varepsilon \to 0$ ?

Corner Asymptotics

Perspectives

## Framework

Hooke's law : 
$$\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$$

#### Assumption

(i)  $\underline{\underline{C}}(x) = (C_{ijkl}(x))$  is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$ (ii)  $\overline{C}_{ijkl}(x)$  are real valued smooth functions, up to  $\Gamma$ (iii) The tensor  $\underline{\underline{C}}(x)$  is positive

Corner Asymptotics

Perspectives

## Framework

Hooke's law : 
$$\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$$

#### Assumption

(i) 
$$\underline{C}(x) = (C_{ijkl}(x))$$
 is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$   
(ii)  $\overline{C_{ijkl}}(x)$  are real valued smooth functions, up to  $\Gamma$   
(iii) The tensor  $\underline{C}(x)$  is positive

#### Assumption

The angular frequency  $\omega$  is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{ in } \Omega_s \\ \underline{\sigma}(\mathbf{u}) \, \mathbf{n} = 0 & \text{ on } \Gamma \end{cases}$$

Corner Asymptotics

Perspectives

## Methodology

• To derive an Asymptotic Expansion for  $({f u}_{m arepsilon},{f p}_{m arepsilon})$  when m arepsilon o 0

$$\mathbf{u}_{\varepsilon}(x) = \mathbf{u}_{0}(x) + \varepsilon \mathbf{u}_{1}(x) + \varepsilon^{2} \mathbf{u}_{2}(x) + \cdots$$
$$\mathbf{p}_{\varepsilon}(x) = \mathbf{p}_{0}(y_{\alpha}, \frac{s}{\varepsilon}) + \varepsilon \mathbf{p}_{1}(y_{\alpha}, \frac{s}{\varepsilon}) + \varepsilon^{2} \mathbf{p}_{2}(y_{\alpha}, \frac{s}{\varepsilon}) + \cdots$$

② Equivalent Conditions of order k + 1 ∈ N : To identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k$$
 up to  $\mathcal{O}(\varepsilon^{k+1})$ 

To prove stability and convergence results for Equivalent models

Corner Asymptotics

Perspectives

## Step 1 : Multiscale Expansion - First terms

0  $\mathfrak{p}_0 = 0$ 2  $\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on } \Gamma \end{cases}$ 3  $\mathfrak{p}_1(.,S) = (S-1)\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}, \quad S \in (0,1)$ **(4)**  $\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{ in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{ on } \Gamma \end{cases}$ 6  $\mathfrak{p}_2(.,S) = -(S^2 - 1)\mathcal{H}\rho\omega^2 \mathbf{u}_0 \cdot \mathbf{n} + (S - 1)\rho\omega^2 \mathbf{u}_1 \cdot \mathbf{n}$ 

Corner Asymptotics

Perspectives

## Step 2 : Equivalent Conditions on Γ

We identify a family of simpler problems

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_{\varepsilon}^{k}) + \omega^{2} \rho \, \mathbf{u}_{\varepsilon}^{k} = \mathbf{f} & \text{in } \Omega_{s} \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{0} & \text{on } \Gamma \end{cases}$$

Order 1 :

 $\boldsymbol{T}(\boldsymbol{u}_0)=\boldsymbol{0}$ 

Corner Asymptotics

Perspectives

## Step 2 : Equivalent Conditions on Γ

We identify a family of simpler problems

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_{\varepsilon}^{k}) + \omega^{2} \rho \, \mathbf{u}_{\varepsilon}^{k} = \mathbf{f} & \text{in } \Omega_{\varepsilon} \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{0} & \text{on } \Gamma \end{cases}$$

Order 1 :

 $\boldsymbol{T}(\boldsymbol{u}_0)=\boldsymbol{0}$ 

Order 2 :

$$\mathbf{T}(\mathbf{u}_{\boldsymbol{\varepsilon}}^{1}) - \boldsymbol{\varepsilon}\omega^{2}\rho_{f}\mathbf{u}_{\boldsymbol{\varepsilon}}^{1}\cdot\mathbf{n}\,\mathbf{n} = \mathbf{0}$$

Corner Asymptotics

Perspectives

## Step 2 : Equivalent Conditions on Γ

We identify a family of simpler problems

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_{\varepsilon}^{k}) + \omega^{2} \rho \, \mathbf{u}_{\varepsilon}^{k} = \mathbf{f} & \text{in } \Omega_{\varepsilon} \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{0} & \text{on } \Gamma \end{cases}$$

Order 1 :

 $\boldsymbol{T}(\boldsymbol{u}_0)=\boldsymbol{0}$ 

Order 2 :

$$\mathbf{T}(\mathbf{u}_{m{arepsilon}}^1) - m{arepsilon}\omega^2 
ho_f \mathbf{u}_{m{arepsilon}}^1 \cdot \mathbf{n} \, \mathbf{n} = 0$$

Order 3 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^{2}) - \boldsymbol{\varepsilon}\omega^{2}
ho_{f}\left(1 - \boldsymbol{\varepsilon}\mathcal{H}\right)\mathbf{u}_{\varepsilon}^{2}\cdot\mathbf{n}\,\mathbf{n} = 0$$

Corner Asymptotics

Perspectives

## Step 2 : Equivalent Conditions on Γ

We identify a family of simpler problems

$$(\mathbf{P}_{\varepsilon}^{k}) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_{\varepsilon}^{k}) + \omega^{2} \rho \, \mathbf{u}_{\varepsilon}^{k} = \mathbf{f} & \text{in } \Omega_{s} \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^{k}) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_{\varepsilon}^{k} \cdot \mathbf{n}) \, \mathbf{n} = \mathbf{0} & \text{on } \Gamma \end{cases}$$

Order 1 :

 $\mathbf{T}(\mathbf{u}_0) = 0$ 

Order 2 :

$$\mathbf{T}(\mathbf{u}_{oldsymbol{arepsilon}}^1) - oldsymbol{arepsilon} \omega^2 
ho_f \mathbf{u}_{oldsymbol{arepsilon}}^1 \cdot \mathbf{n} \, \mathbf{n} = 0$$

Order 3 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^{2}) - \boldsymbol{\varepsilon}\omega^{2}
ho_{f}\left(1 - \boldsymbol{\varepsilon}\mathcal{H}\right)\mathbf{u}_{\varepsilon}^{2}\cdot\mathbf{n}\,\mathbf{n} = 0$$

Order 4 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^{3}) - \boldsymbol{\varepsilon}\omega^{2}\rho\left(1 - \boldsymbol{\varepsilon}\mathcal{H} + \frac{\boldsymbol{\varepsilon}^{2}}{3}\left[\Delta_{\Gamma} + \kappa^{2}\mathbb{I} + 4\mathcal{H}^{2} - \mathcal{K}\right]\right)\mathbf{u}_{\varepsilon}^{3} \cdot \mathbf{n} \mathbf{n} = 0$$

Corner Asymptotics

## **Step 3 : Stability and Convergence results**

#### Theorem

There exists  $\varepsilon_0 > 0$  s.t. for all  $\varepsilon \in (0, \varepsilon_0)$ , Problem ( $\mathbf{P}_{\varepsilon}^k$ ) with data  $\mathbf{f} \in L^2(\Omega_s)$  has a unique solution  $\mathbf{u}_{\varepsilon}^k \in V_k$  and

Stability

 $\|\mathbf{u}_{\boldsymbol{\varepsilon}}^{k}\|_{1,\Omega_{\mathrm{s}}}\leqslant C\|\mathbf{f}\|_{0,\Omega_{\mathrm{s}}}$ 

• Convergence

$$\|\mathbf{u}_{\boldsymbol{\varepsilon}}-\mathbf{u}_{\boldsymbol{\varepsilon}}^{k}\|_{1,\Omega_{s}}\leqslant C\boldsymbol{\varepsilon}^{k+1}$$

$$\begin{split} \mathrm{V}_k &= \mathrm{H}^1(\Omega_{\mathsf{s}}) \text{ when } k = 0, 1, 2 \\ \mathrm{V}_k &= \{ \mathbf{u} \in \mathrm{H}^1(\Omega_{\mathsf{s}}) \mid \ \mathbf{u} \cdot \mathbf{n} |_{\Gamma} \in \mathrm{H}^1(\Gamma) \} \text{ when } k = 3 \end{split}$$

Corner Asymptotics

Perspectives

#### Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_{\boldsymbol{\varepsilon}}^{N} := \mathbf{u}_{\boldsymbol{\varepsilon}} - \sum_{n=0}^{N} \boldsymbol{\varepsilon}^{n} \mathbf{u}_{n} \quad \text{in} \quad \Omega_{\mathrm{s}} \;, \quad \text{and} \quad r_{\boldsymbol{\varepsilon}}^{N} := \mathbf{p}_{\boldsymbol{\varepsilon}} - \sum_{n=0}^{N} \boldsymbol{\varepsilon}^{n} \mathbf{p}_{n} \quad \text{in} \quad \Omega_{\mathrm{f}}^{\boldsymbol{\varepsilon}} \;.$$

Evaluation of the right hand sides in

$$\begin{cases} \Delta r_{\varepsilon}^{N} + \kappa^{2} r_{\varepsilon}^{N} = f_{\varepsilon} & \text{in } \Omega_{f}^{\varepsilon} \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_{\varepsilon}^{N}) + \omega^{2} \rho \mathbf{r}_{\varepsilon}^{N} = 0 & \text{in } \Omega_{s} \\ \partial_{\mathbf{n}} r_{\varepsilon}^{N} - \rho_{f} \omega^{2} \mathbf{r}_{\varepsilon}^{N} \cdot \mathbf{n} = g_{\varepsilon} & \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_{\varepsilon}^{N}) + r_{\varepsilon}^{N} \mathbf{n} = 0 & \text{on } \Gamma \\ r_{\varepsilon}^{N} = 0 & \text{on } \Gamma^{\varepsilon} .\end{cases}$$

The RHS are explicit :  $f_{\varepsilon} = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$  in  $\Omega_{f}^{\varepsilon}$ ,  $g_{\varepsilon} = \mathcal{O}(\varepsilon^{N})$  on  $\Gamma$ .

Optimal Estimates :  $\|\mathbf{r}_{\varepsilon}^{N}\|_{1,\Omega_{s}} + \sqrt{\varepsilon} \|r_{\varepsilon}^{N}\|_{1,\Omega_{t}^{\varepsilon}} \leq C_{N} \varepsilon^{N+1}$ 

Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

## Numerical Simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain :  $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on  $\Gamma$  (f = 0) : p\_i(x) = exp(i\omega x \cdot d) with d = (1,0)
- $\bullet \ \mathbb{P}_3\text{-finite elements}$  (Lagrange) in the Library Hou10ni
- $\Omega_{\text{s}}$  : Isotropic elastic material with Lamé coefficients :

$$\mu =$$
 26.32  $imes$  10<sup>9</sup> and  $\lambda =$  51.08  $imes$  10<sup>9</sup> .

•  $c = 1500 \text{ m.s}^{-1}$ ,  $\rho_f = 1000 \text{ kg.m}^{-3}$ ,  $\rho = 2700 \text{ kg.m}^{-3}$ .

Corner Asymptotics

Perspectives

#### Numerical Simulations Qualitative Comparisons

*ε* = 0.001;



Corner Asymptotics

Perspectives

#### Numerical Simulations Qualitative Comparisons

*ε* = 0.0001;



Corner Asymptotics

Perspectives

## **Convergence of the models**

For 
$$k \in \{0, 1, 2\}$$
, we plot the L<sup>2</sup>-error  $\|\mathbf{u}_{\varepsilon} - \mathbf{u}_{\varepsilon}^{k}\|_{0,\Omega_{s}}$  w.r.t.  $\varepsilon$   
 $\mathbf{u}_{\varepsilon}$ : analytical solution  $\mathbf{u}_{\varepsilon}^{k}$ : analytical solution or numerical solution (IRDGM)

 $\mathbf{u}_{\boldsymbol{\varepsilon}}^{\kappa}$ : analytical solution or numerical solution (IPDGM)



Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations
- Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model
  - Laplace Operator
  - The Problem with Angular Conductor
  - Numerical Tests

#### 3 Perspectives

Corner Asymptotics

Perspectives

#### **Harmonic Maxwell Equations**

Take a thin smooth closed sheet  $\Omega_o^{\varepsilon}$ , with high conductivity  $\sigma_o^{\varepsilon} = \varepsilon^{-2} \tilde{\sigma}$ , and surrounded by conductor bodies  $\Omega_{-}^{\varepsilon}$  and  $\Omega_{+}^{\varepsilon}$ .



## References

K. SCHMIDT, S. TORDEUX (ESAIM: Math. Model. and Numer. Anal. 45(6), pp.1115–1140, 2011) High order transmission conditions for thin conductive sheets in magneto-guasistatics.

K. SCHMIDT, A. CHERNOV (SIAM J. Appl. Math. 73(6): 1980–2003, 2013) A unified analysis of transmission conditions for thin conducting sheets in the time-harmonic eddy current model.

#### References

- K. SCHMIDT, S. TORDEUX (ESAIM: Math. Model. and Numer. Anal. 45(6), pp.1115–1140, 2011)
  - High order transmission conditions for thin conductive sheets in magneto-quasistatics.
- K. SCHMIDT, A. CHERNOV (SIAM J. Appl. Math. 73(6): 1980–2003, 2013) A unified analysis of transmission conditions for thin conducting sheets in the time-harmonic eddy current model.

V. P., K. SCHMIDT, M. DURUFLÉ (SIAM J. Appl. Math. 76(3), 1031–1052, 2016)

Equivalent Transmission Conditions for the time-harmonic Maxwell equations in 3D for a Medium with a Highly Conductive Thin Sheet

Corner Asymptotics

## Second Order Impedance Transmission Conditions

$$\begin{pmatrix} [\mathbf{E}_{\boldsymbol{\varepsilon}}^{1} \times \mathbf{n}]_{\Gamma} \\ \{\mathbf{E}_{\boldsymbol{\varepsilon}}^{1} \times \mathbf{n}\}_{\Gamma} \end{pmatrix} = i\omega\boldsymbol{\varepsilon} \begin{pmatrix} L_{1} & L_{3} \\ L_{3} & L_{2} \end{pmatrix} \begin{pmatrix} \{\mathbf{n} \times (\mathbf{H}_{\boldsymbol{\varepsilon}}^{1} \times \mathbf{n})\}_{\Gamma} \\ [\mathbf{n} \times (\mathbf{H}_{\boldsymbol{\varepsilon}}^{1} \times \mathbf{n})]_{\Gamma} \end{pmatrix}$$
(1a)

 $L_i$  are differential operators given by

$$L_i = A_i \operatorname{curl}_{\Gamma} \operatorname{curl}_{\Gamma} - B_i \mathbb{I}, \quad i = 1, 2, 3,$$

Corner Asymptotics

#### Second Order Impedance Transmission Conditions

$$\begin{pmatrix} [\mathbf{E}_{\varepsilon}^{1} \times \mathbf{n}]_{\Gamma} \\ \{\mathbf{E}_{\varepsilon}^{1} \times \mathbf{n}\}_{\Gamma} \end{pmatrix} = i\omega\varepsilon \begin{pmatrix} L_{1} & L_{3} \\ L_{3} & L_{2} \end{pmatrix} \begin{pmatrix} \{\mathbf{n} \times (\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n})\}_{\Gamma} \\ [\mathbf{n} \times (\mathbf{H}_{\varepsilon}^{1} \times \mathbf{n})]_{\Gamma} \end{pmatrix}$$
(1a)

 $L_i$  are differential operators given by

$$\mathbf{L}_i = A_i \, \operatorname{\mathbf{curl}}_{\Gamma} \operatorname{\mathbf{curl}}_{\Gamma} - B_i \mathbb{I} \,, \qquad i = 1, 2, 3,$$

in which  $A_i$ ,  $B_i$  are constants defined by

$$A_{1} = -\omega^{-2} \left\{ \left(\epsilon + i\frac{\sigma}{\omega}\right)^{-1} \right\}_{\Gamma}, \quad B_{1} = 2\frac{\mu_{o}}{\gamma} \tanh\left(\frac{\gamma}{2}\right) - \left\{\mu\right\}_{\Gamma}, \\ A_{2} = \frac{A_{1}}{4}, \qquad B_{2} = \frac{\mu_{o}}{2\gamma} \coth\left(\frac{\gamma}{2}\right) - \frac{\left\{\mu\right\}_{\Gamma}}{4}, \quad (2) \\ A_{3} = -\frac{1}{4}\omega^{-2} \left[ \left(\epsilon + i\frac{\sigma}{\omega}\right)^{-1} \right]_{\Gamma}, \quad B_{3} = -\frac{1}{4} \left[\mu\right]_{\Gamma},$$

and

$$\gamma = \exp{(rac{3i\pi}{4})} \sqrt{\omega \mu_{o} ilde{\sigma}} \; .$$

Corner Asymptotics

Perspectives

## Impedance transmission conditions

Robustness versus the conductivity parameter



Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

Perspectives

# Problem $(\mathcal{P})$ with Angular Conductor Main difficulty

$$\left\{ \begin{array}{ccc} -\Delta \mathcal{A}^- + i\kappa \mu_0 \sigma \mathcal{A}^- = 0 & \text{ in } \ \Omega_- \\ & -\Delta \mathcal{A}^+ = \mu_0 J & \text{ in } \ \Omega_+ \\ & \mathcal{A}^+ = 0 & \text{ on } \ \Gamma \end{array} \right.$$



Corner Asymptotics

Perspectives

# Problem $(\mathcal{P})$ with Angular Conductor Main difficulty

The considered operator is

$$-\Delta + 4i\zeta^2 \, \mathbbm{1}_{\Omega_-} = egin{cases} -\Delta + i\kappa\mu_0\sigma & ext{in }\Omega_-, \ -\Delta & ext{in }\Omega_+. \end{cases}$$

The singularities are generated by the term  $i\kappa\mu_0\sigma \mathbbm{1}_{\Omega_-}$ .

Corner Asymptotics

Perspectives

## Problem $(\mathcal{P})$ with Angular Conductor Main difficulty

$$\begin{cases} -\Delta \mathcal{A}^{-} + i\kappa\mu_{0}\sigma\mathcal{A}^{-} = 0 & \text{in } \Omega_{-} \\ & -\Delta \mathcal{A}^{+} = \mu_{0}J & \text{in } \Omega_{+} \\ & \mathcal{A}^{+} = 0 & \text{on } \Gamma \end{cases}$$



The considered operator is

$$-\Delta + 4i\zeta^2 \, \mathbbm{1}_{\Omega_-} = \left\{ egin{array}{cc} -\Delta + i\kappa\mu_0\sigma & ext{in }\Omega_-, \ -\Delta & ext{in }\Omega_+. \end{array} 
ight.$$

The *singularities* are generated by the term  $i\kappa\mu_0\sigma \mathbbm{1}_{\Omega_-}$ .

 $\implies$  the derivation of the Corner Asymptotics is not obvious.

#### **Our Reference**

M. DAUGE *et al* (Mathematical Methods in the Applied Sciences. 2014, 37 (13), pp.1924-1955)

Corner asymptotics of the magnetic potential in the eddy-current model

•  $\mathcal{A}$  possesses a corner asymptotic expansion

$$\mathcal{A}(r, heta) \underset{r o 0}{\sim} \Lambda^{0,0}\mathfrak{S}^{0,0}(r, heta) + \sum_{k \ge 1} \sum_{p \in \{0,1\}} \Lambda^{k,p}\mathfrak{S}^{k,p}(r, heta),$$

- $(r, \theta)$ : polar coordinates centered at **c**  $\mathfrak{S}^{k,p}$ : primal singular functions
- $\Lambda^{k,p}$  : singular coefficients
- We provide a constructive procedure to determine the primal and the dual singularities
- We generalize the method of moments and we introduce the method of quasi-dual functions to determine Λ<sup>k,p</sup>

Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

## **Laplace Operator**

• In the case when  $\zeta=$  0, we consider the solution  ${\cal A}$  to

$$\left\{egin{array}{ll} -\Delta \mathcal{A}=\mu_0 J ext{ in } \Omega \ \mathcal{A}=0 ext{ on } \Gamma \end{array}
ight.$$

•  $\mathcal{A}$  admits the Taylor expansion at  $\mathbf{c}$  :

$$\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} + \sum_{k \ge 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \underbrace{r^k \cos(k\theta - p\pi/2)}_{=\mathfrak{s}^{k,p}(r,\theta)}$$

 $\mathfrak{s}^{k,p}$  : harmonic polynomials

Corner Asymptotics

## **The Dual Function Method**

V.G. MAZ'YA, B.A. PLAMENEVSKII (Amer. Math. Soc. Trans. (2) '84) *On the coefficients (...) in domains with conical points* We use dual harmonic functions :

$$\mathfrak{k}^{k,p}(r,\theta) = \begin{cases} -\frac{1}{2\pi} \log r, & \text{if } k = 0, \ p = 0, \\ \frac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \ge 1, \ p = 0, 1 \end{cases}$$

#### Proposition

Let  $\mathcal{A}$  be the solution to the Laplace equation. Then

$$\mathcal{J}_R(\mathfrak{k}^{0,0},\mathcal{A})=\Lambda^{0,0}$$
 and  $\mathcal{J}_R(\mathfrak{k}^{k,\rho},\mathcal{A})=\Lambda^{k,\rho},\ k\geqslant 1,\ \rho=0,1.$ 

Here, for 
$$R > 0$$
,  $\mathcal{J}_R(K, A) = \int_{r=R} (K \partial_r A - \partial_r K A) R d\theta$ .

Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

Perspectives

## The singularities

The singularities 
$${\mathfrak U}$$
 of  ${\mathscr L}_\zeta = -\Delta + 4{
m i}\zeta^2\, {\mathbb I}_{{\mathcal S}_-}$  :

$$\begin{cases} -\Delta\mathfrak{U}+4\mathrm{i}\zeta^2\mathfrak{U}=0 & \text{ in }\mathcal{S}_-,\\ -\Delta\mathfrak{U}=0 & \text{ in }\mathcal{S}_+, \end{cases}$$

with continuous Dirichlet and Neumann traces across  $\mathcal{G}$ .



Equivalent Conditions

Corner Asymptotics

## Description of the singularities

V.A. KONDRATEV (Trudy Moskov. Mat. Obšč. '67)

Boundary value problems for elliptic equations in domains with conical or angular points

۲

Equivalent Conditions

Corner Asymptotics

Perspectives

#### Description of the singularities

V.A. KONDRATEV (Trudy Moskov. Mat. Obšč. '67)

Boundary value problems for elliptic equations in domains with conical or angular points

 $\mathfrak{U} = \underbrace{\mathfrak{u}_{0}}_{\textit{leading part}} + \sum_{j \ge 1} (\mathrm{i}\zeta^{2})^{j} \underbrace{\mathfrak{u}_{j}}_{\textit{shadow}},$ 

۲

Equivalent Conditions

Corner Asymptotics

#### Description of the singularities

V.A. KONDRATEV (Trudy Moskov. Mat. Obšč. '67)

Boundary value problems for elliptic equations in domains with conical or angular points

 $\mathfrak{U} = \underbrace{\mathfrak{u}_{0}}_{\text{leading part}} + \sum_{j \ge 1} (\mathrm{i}\zeta^{2})^{j} \underbrace{\mathfrak{u}_{j}}_{\text{shadow}},$ 

We solve

$$\Delta \mathfrak{u}_0 = 0, \quad \Delta \mathfrak{u}_1 = 4\mathfrak{u}_0 \, \mathbb{1}_{\mathcal{S}_-}, \quad \dots, \quad \Delta \mathfrak{u}_j = 4\mathfrak{u}_{j-1} \, \mathbb{1}_{\mathcal{S}_-},$$

$$\mathfrak{u}_j \in \mathrm{S}^{\lambda} = \mathrm{Span}\left\{r^{\lambda}\log^q r\,\Phi( heta), \quad q \in \mathbb{N}, \ \Phi \in \mathcal{C}^1(\mathbb{T}), \ \Phi^{\pm} \in \mathcal{C}^{\infty}(\overline{\mathbb{T}}_{\pm})
ight\}$$
  
Here,  $\lambda \in \mathbb{Z}$ ,  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$  and  $\mathbb{T}_{-} = (-\omega/2, \, \omega/2)$ 

## **Existence of the shadows**

#### Lemma

Let 
$$\lambda \in \mathbb{Z}$$
 and  $\mathfrak{f} \in T^{\lambda-2}$ . Then, there exists  $\mathfrak{u} \in S^{\lambda}$  such that  $\Delta \mathfrak{u} = \mathfrak{f}$ .  
Moreover  
(i) If  $\lambda \in \mathbb{Z} \setminus \{0\}$ , deg  $\mathfrak{u} \leq \deg \mathfrak{f} + 1$ ,

(ii) If  $\lambda = 0$ , deg  $\mathfrak{u} \leqslant \deg \mathfrak{f} + 2$ .

Here,

$$\mathsf{T}^\lambda = \operatorname{Span}\left\{ r^\lambda \log^q r \, \Psi( heta), \quad q \in \mathbb{N}, \; \, \Psi \in {\mathcal L}^2(\mathbb{T}), \; \Psi^\pm \in \mathcal{C}^\infty(\overline{\mathbb{T}}_\pm) 
ight\}.$$

The degree of  $\mathfrak{g}$  is its degree as polynomial of log r.

Corner Asymptotics

#### **Proof of Lemma**

 M. Dauge Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Berlin 1988
 We introduce the Mellin symbol *M* associated with Δ

$$\mathscr{M}(\mu) = (\mu^2 + \partial_{\theta}^2), \ \theta \in \mathbb{T}.$$

Setting

$$\mathfrak{f} = \sum_{q=0}^{\deg \mathfrak{f}} r^{\lambda-2} \log^q r \, \Psi_q \in \mathbf{T}^{\lambda-2} \,,$$

and

$$\mathfrak{u} = \mathop{\mathrm{Res}}_{\mu=\lambda} \, \left\{ r^{\mu} \mathscr{M}(\mu)^{-1} \sum_{q=0}^{\deg \mathfrak{f}} \frac{q! \, \Psi_q}{(\mu-\lambda)^{q+1}} \right\} \in \mathbf{S}^{\lambda} \, ,$$

we check that  $\Delta \mathfrak{u} = \mathfrak{f}$ .

Corner Asymptotics

## **Primal and Dual Singularities**

- *Primal singular functions*  $\mathfrak{S}$  belong to  $H^1$  in any bounded neighborhood  $\mathcal{B}$  of **c**.
- Dual singular functions  $\mathfrak{K}$  do not belong to  $H^1$  in such a neighborhood  $\mathcal{B}$ .

The functions  $\Re$  are needed for the determination of the coefficients  $\Lambda$  involved in the asymptotics.

Corner Asymptotics

.

Perspectives

#### A basis for Primal Singularities

For 
$$(k,p)=(0,0)$$
 and for  $(k,p)\in\mathbb{N}^* imes\{0,1\},$ 

$$\mathfrak{S}^{k,p} = \underbrace{\mathfrak{s}^{k,p}}_{\text{leading part}} + \sum_{j \ge 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{s}^{k,p}_j}_{\text{shadow}}$$

#### Lemma

For any  $j \ge 1$ , there exists  $\mathfrak{s}_j^{k,p} \in \mathbb{S}^{k+2j}$ , with deg  $\mathfrak{s}_j^{k,p} \le j$ , satisfying

$$\Delta \mathfrak{s}_{j}^{\kappa,\rho} = 4\mathfrak{s}_{j-1}^{\kappa,\rho} \mathbbm{1}_{\mathcal{S}_{-}}.$$

Application : 
$$\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r,\theta) + \sum_{k \ge 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r,\theta)$$

Corner Asymptotics

.

Perspectives

## A basis for Dual Singularities

For 
$$(k, p) = (0, 0)$$
 and for  $(k, p) \in \mathbb{N}^* \times \{0, 1\}$ ,

$$\mathfrak{K}^{k,p} = \underbrace{\mathfrak{t}^{k,p}}_{\textit{leading part}} + \sum_{j \ge 1} (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{t}^{k,p}_j}_{\textit{shadow}}$$

#### Lemma

For any  $j \ge 1$ , there exists  $\mathfrak{k}_j^{k,\rho} \in S^{-k+2j}$ , with deg  $\mathfrak{k}_j^{k,\rho} \le j+1$ , satisfying

$$\Delta \mathfrak{k}_{j}^{k,p} = 4\mathfrak{k}_{j-1}^{k,p} \, \mathbb{1}_{\mathcal{S}_{-}}$$

Corner Asymptotics

## **The Quasi-Dual Function Method**

- M. COSTABEL *et al* (SIAM J. Math. Anal. 35-5, 1177–1202, 2004) A quasidual function method for extracting edge stress intensity functions
  - S. SHANNON et al (Int. J. Fract. 181: 25–50, 2013) Extracting generalized edge flux intensity functions by the quasidual function method along circular 3-D edges
- We use quasi-dual functions  $\Re_m^{k,p}$ :

$$\mathfrak{K}_m^{k,p} = \mathfrak{k}^{k,p} + \sum_{j=1}^m (\mathrm{i}\zeta^2)^j \underbrace{\mathfrak{k}_j^{k,p}}_{shadow}$$

Corner Asymptotics

#### The Quasi-Dual Function Method Extraction of coefficients

#### Theorem

Let  $\mathcal{A}$  be the solution to problem  $(\mathcal{P})$ . Let  $k \in \mathbb{N}$  and  $p \in \{0, 1\}$  (p = 0 if k = 0). Then for all m such that 2m + 2 > k,

$$\mathcal{J}_{R}(\mathcal{R}_{m}^{k,p},\mathcal{A}) \underset{R \to 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{[k/2]} \mathcal{J}^{k,p;\,k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}(R^{-k}R_{0}^{2m+2}\log R),$$
where  $R_{0} = \zeta R \left(1 + \sqrt{|\log R|}\right)$ .

Corner Asymptotics

## **The Quasi-Dual Function Method**

#### **Examples**

• For *k* = 0

$$\Lambda^{0,0} \underset{R \to 0}{=} \mathcal{J}_{R}(\mathfrak{K}^{0,0}_{m}, \mathcal{A}) + \mathcal{O}(R_{0}^{2+2m} \log R)$$

$$\Lambda^{1,0} \underset{R \to 0}{=} \mathcal{J}_R(\mathfrak{K}^{1,0}_m, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2})$$

• For k = 2, we need  $m \ge 1$ 

$$\Lambda^{2,0} =_{R \to 0} \mathcal{J}_{R}(\mathfrak{K}^{2,0}_{1}, \mathcal{A}) - \mathcal{J}^{2,0;\,0,0} \Lambda^{0,0} + \mathcal{O}(R^{-2} R_{0}^{4} \log R)$$

Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests

#### 3 Perspectives

Corner Asymptotics

Perspectives

## Framework

- $\Omega$  : disk of radius 50 mm
- Conducting sector :  $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2})~\mathrm{mm^{-1}}$
- Source :  $\mathcal{A}^+ = rac{|\theta|}{2\pi}$  on  $\partial \Omega$



Corner Asymptotics

Perspectives

## **Finite Element Solution**

We plot the real part and the imaginary part of the FE solution.



Corner Asymptotics

## Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

• There holds

$$\Lambda^{0,0} \underset{R \to 0}{=} \mathcal{J}_R(\mathfrak{K}^{0,0}_m, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R) \,.$$

• We plot  $|\mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) - \mathcal{A}|_c|/|\mathcal{A}|_c|$  as a function of  $R \ (m = 0, 1)$ .



Corner Asymptotics

## Accuracy for the computation of $\Lambda^{1,0}$ as a function of R

Recall that

$$\Lambda^{1,0} \underset{R \to 0}{=} \mathcal{J}_R(\mathfrak{K}^{1,0}_m, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2})$$

• We plot  $|\mathcal{J}_R(\mathfrak{K}_m^{1,0},\mathcal{A}) - \mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0},\mathcal{A})| / |\mathcal{J}_{R_s}(\mathfrak{K}_1^{1,0},\mathcal{A})|.$ 



Corner Asymptotics

## **Outline**

#### Equivalent Conditions for Elasto-Acoustics and for Electromagnetism

- Equivalent Boundary Conditions for Acoustic and Elastic Media
- Numerical Tests
- Impedance Transmission Conditions for Maxwell Equations

## 2 Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model

- Laplace Operator
- The Problem with Angular Conductor
- Numerical Tests



Corner Asymptotics

#### Perspectives Electromagnetism

• To justify and to extend the multi-scale expansion for the corner problem

$$\mathcal{A}_{\delta} pprox \mathcal{A}_{0,\mathrm{reg}} + \psi \, a \, \delta^{lpha} \mathit{K}_{\alpha}(./\delta) + \mathit{R}_{\delta} \; .$$

- To tackle the problem in the (linear) ferromagnetic case :  $\mu_{\rm r}=\mu_-/\mu_+\gg {\rm 1}$
- To apply the method of matched asymptotic expansions for 3D scattering problems by small obstacles (PhD thesis of J. Labat)

Corner Asymptotics

#### Perspectives Elasto-acoustic coupling

• To couple the layer of water with the atmosphere



- To derive Equivalent Conditions when a high-order ABC is imposed on top of the water
- To extend Equivalent Conditions to transient problems

## Thank you for your attention