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sous la direction de M. Luigi SANTOCANALE

Titre:

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Algebra and Combinatorics of Parity Games

Walid BELKHIR

Resumé

Les jeux de parité sont la représentation combinatoire de la théorie des infimums, suprimums, et du plus petit point fixe et du plus grand point fixe sur les treillis complets. En gros, le formalisme des jeux de parité peut être considéré comme un μ-calcul sur les treillis complets. Les hiérarchies et le pouvoir expressif sont un thème centrale dans la théorie des points fixes. La première partie de cette thèse est consacrée à l'étude du problème de la hiérarchie des variables sur le μ-calcul des treillis. Des travaux antérieurs sur ce problème dans le cas du μ-calcul propositionnel modal ont dégagé une mesure de complexité des graphes: c'est *l'enchevêtrement*. Le dernier est la partie combinatoire de la hiérarchie des variables. La deuxième partie de cette thèse est consacrée à l'étude de l'enchevêtrement dans le contexte de la théorie des graphes, indépendemment de son origine dans la théorie des points fixes. Plusieurs résultats seront démontrés dans cette direction, tels que la reconnaissance des graphes d'enchevêtrement borné, la décomposition arborescente de tels graphes, et la fermeture par mineurs.

Mots-Clés: μ-calculs, μ-treillis, sémantique des jeux, décomposition de graphes.

Abstract

Parity games are the combinatorial description of the theory of binary infimums, and supremums, and of the least and greatest fixed points over complete lattices. Roughly speaking, the parity games formalism may be understood as a μ-calculus over complete lattices. Hierarchies and logical expressiveness are at the core of fixed-point theory. The first part of this thesis is devoted to consider the variable hierarchy problem within the lattice μ-calculus. Earlier works on this problem within the propositional modal μ-calculus have derived a complexity measure, the so called *entanglement*. The latter is the combinatorial counterpart of the variable hierarchy. The second part of this thesis is devoted to consider the entanglement as a pure graph theoretic measure, independently of its origins in fixed-point theory. Further

results will be proved in this direction, such that the recognization of graphs of bounded entanglement, the tree decomposition of these graphs, and the closure under minors.
Keywords: μ-calculi, μ-lattices, game semantics, graph decomposition.

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Chapter 0

Introduction

0.1 Fixed point theory as algebraic system

The formalism of the fixed points, or the μ-calculi, is essentially an algebra of monotonic functions over complete lattices. This is justified by the well known Tarski Theorem [Tar55] stating that the monotonic functions over complete lattices have got a least and a greatest fixed point. The algebraic presentation of the theory of fixed points as well as the related properties have been developed in [Niw85, BÉ93]. In [Niw85], the author defined the notion of *μ-algebra* out of a given Σ-algebra: the terms, or briefly the μ-terms, are built up using the operators of the signature of the Σ-algebra as well as the least and greatest fixed point operators. Then, these terms are interpreted in a complete lattice. This algebraic approach has been initiated with the *modal μ-calculus*[Pra81, Koz83]. The latter is a logic that enriches Hennessy-Milner logic with the least and greatest fixed point operators. The modal μ-calculus has shown its use as a specification language of the temporal properties of concurrent and reactive systems [Sti01]. The expressive power - computational complexity ratio of the modal μ-calculus is acceptable.

In the same spirit, the theory of μ-lattices [San02c] was constructed as the μ-algebra of the Σ-algebra of the lattices. The μ-lattices are a quasi variety if the least and greatest fixed points are definable by means of equations and

implications between equations.

Roughly speaking, a μ-calculus in the abstract sense of [AN01][§2] is a set of syntactic entities and a set of formal operations. The latter consists in the fixed point operators μ and ν and the substitution operation. These syntactic entities come with and intended interpretation over a class of complete lattices. Each entity t is interpreted as a monotonic mapping from $t^{ar(t)}$ to L, where $ar(t)$ is the arity of t, that corresponds to the free variables of t, and L is a complete lattice. The entities $\mu x.t$ and $\nu x.t$ of a μ-calculus are interpreted as the least and greatest parametrized fixed points of the interpretation of t. The substitution is interpreted by means of the functional composition.

Many syntactic entities can be structured to have a shape of a μ-calculus. For instance, this happened to infinite words [AN01, §5], automata (on words, infinite words, trees, ...) [AN01, §7], and parity games [AS03, SA05]. The lattice, on which the entities are interpreted, is usually the power set algebra. The interpretation of an automaton, viewed as an entity of the μ-calculus, is the language (of words, infinite words, trees, ...) which it accepts.

0.2 Hierarchies

The problem of hierarchies in μ-calculi, is at the core of this thesis. We begin by introducing the logical origins of this problem 80 years back.

0.2.1 Hierarchies in logic

The problem of the hierarchies was roughly addressed in the 1930's after a negative result of Church 1937 stating that: *the problem of satisfiability of first order logic (FO) is undecidable*. Consequently, the satisfiability problem transfers into a classification one: *which are the classes of FO formulae for which the problem of satisfiability is decidable?* This problem questions implicitly the use of the logical quantifiers $\exists$ and $\forall$, one would imagine several restrictions on the use of $\exists$ and $\forall$. For instance one can bound the alternation between $\exists$ and $\forall$, the number of quantifies, and the number of (bound)

variables .... A series of positive results follow, for all the fragment of FO cited bellow, the problem of satisfiability is decidable:

- the monadic calculus class (Löwenheim 1915, Kalmár 1929), i.e. the a predicate calculus where each predicat has just one variable,
- the $\exists^*\forall^*$ prefix class (Bernays-Schönfinkel-Ramsey, 1928-1932), i.e. the formulae of the form

  $$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \; \phi(\ldots, x_i, \ldots, y_j, \ldots)$$

  where ϕ is a formulae without quantifiers, and
- the $\exists^*\forall\exists^*$ prefix class (Gödel, Kalmár, Schüte, 1932-1934)

The common feature of the two latter classes relies on the restriction on the alternation between $\exists$ and $\forall$.

Another kind of restriction on the logical quantifiers consists in restricting the *number of bound variables*. This amounts to consider the class FO^k of first order formulae of at most k bound variables, allowing these variables to be reused any number of times. For instance, the formula that expresses

the existence of a path of length 2008 *in a given directed graph*

is a formula in FO^2, and may be written as:

$$\begin{aligned}
&\exists x \exists y (Succ(x,y) \wedge \\
&\qquad \exists x (Succ(y,x) \wedge \\
&\qquad\qquad \exists y (Succ(x,y) \wedge \ldots \\
&\qquad\qquad\qquad \ldots \exists x (Succ(y,x)) \ldots)))
\end{aligned}$$

A classical result of Mortimer [Mor75] shows that the class FO^2 is decidable for satisfiability, and it admits the finite model property. His proof implies that a formula ϕ in FO^2 has a model of $2^{2^{O(|\phi|)}}$ elements, and therefore the satisfiability problem is a problem in the class 2-NEXPTIME. However, the satisfiability problem is undecidable for the class FO^k where $k \geq 3$.

0.2.2 Hierarchies in μ-calculi

The fixed point operators μ and ν are similar to logical quantifiers $\exists$ and $\forall$. As consequence we can classify the μ-formulae into classes according to: (i) the alternation between μ and ν, (ii) the number of application of μ and ν, and (iii) the number of bound variables. This gives rise respectively to the three hierarchies: the *alternation depth hierarchy* [Niw86, Len96, Bra98, Arn99, San02a, SA05], the *star height* hierarchy [Egg63, BC84], and the *variable hierarchy* [Imm95, BGL07, BS08, BS].

The alternation hierarchy

The alternation depth of μ-terms is the most known and well studied measure [Niw86, Len96, Bra98, Arn99, San02a, SA05]. Intuitively, it measures the nested depth between μ and ν:

$$\mu X_1\ \nu Y_1\ \mu X_2\ \nu Y_2 \dots \mu X_n\ \nu Y_n\ \phi(\dots, X_i, \dots, Y_i, \dots)$$

As a consequence, it is possible to construct a hierarchy of μ-terms according to the alternation depth measure, giving rise to the so called *alternation hierarchy*. We recall its the algebraic definition as given in [Niw86]: the class $\Sigma_0 = \Pi_0$ is the class of μ-terms without application of fixed point operators μ and ν, the class Σ_{n+1} (resp. Π_{n+1}) is the closure of the two classes Σ_n and Π_n under the substitution (i.e. the composition) and the least fixed point operator (resp. the greatest fixed point operator). The class $Comp(\Sigma_n, \Pi_n)$ is the closure of Σ_n and Π_n under the composition operation. These classes are ordered par inclusions as depicted in Figure 0.2.1. However, a quite different definition of the alternation hierarchy has been suggested in [EL86]. There, the alternation depth of a μ-term is defined top down. The alternation hierarchy[1] collapses in the case of the μ-calculus of infinite words [Par81] as well as in the case of the modal μ-calculus over the class of transitive and

[1] Indeed we mean the semantic hierarchy, since the syntactic hierarchy is obviously strict.

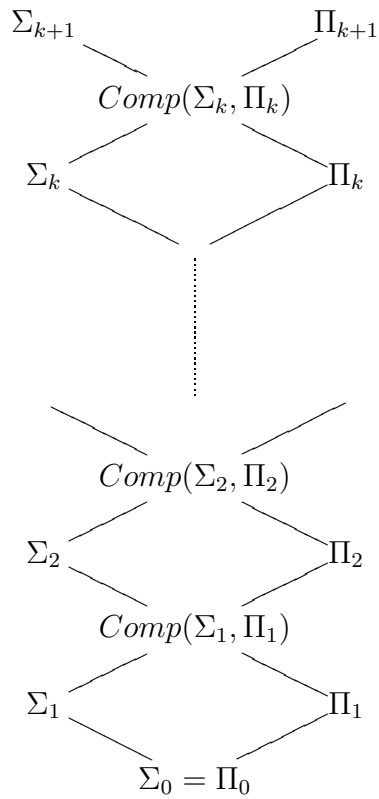


Figure 0.2.1: The alternation-depth hierarchy

symmetric systems, transitive and reflexive systems, and transitive systems [AF]. However, this hierarchy is strict for the modal μ-calculus in general [Bra98] and over the class of reflexive systems [AF].

The star height hierarchy

Another source of complexity of μ-terms is the nested depth of the application the fixed-point operators. This is the star-height. Consequently, a hierarchy of μ-terms can be constructed according to their star height.

The star height problem was first asked in formal language theory, and it consists in answering weather all regular languages can be expressed using

regular expressions of bounded star height. The solution is referenced in [Egg63] where examples of regular languages of arbitrary star height are constructed. The star height problem for regular trees was solved in [BC84], the latter are finite or infinite trees with only finite many distinct subtrees. It is worth to note that the study of the star height problem [Egg63, BC84], the latter is a an algebraic problem in nature, transfers into a pure graph theoretic problem. The minimal star height needed in regular expressions is captured by a digraph invariant which counts the number of embedded cycles, that is, it is the *feed back number* introduced in [Egg63], or the *rank* introduced in [BC84]. Theses invariants are the same, whereas the rank is formalized in a more intuitive way.

The variable hierarchy

A refinement of the star height problem consists in asking whether every μ-term t is equivalent to a μ-term t' where the number of bound variables in t' is bounded. In some cases the satisfiability problem of formulae of bounded variables can be done in polynomial time [DKV02].
Similarly to the previous works on the star height, the study of the variable hierarchy problem of modal μ-calculus [BGL07] requires a quite different complexity measure, the so called *entanglement* [BG05]. Roughly speaking, the entanglement of a μ-term computes the minimum number of fixed point variables of this term up to α-conversion. Whenever the μ-terms are viewed as graphs, the entanglement formalizes the intuitive notion of the nested depth of cycles. In Chapter 5, the entanglement will constitute our main tool for solving the variable hierarchy problem of the lattice μ-calculus. Moreover, the relation between the star height hierarchy and the variable hierarchy will be clarified in Chapter 3.

0.3 Logic and games

Game theory suggests a fruitful approach to essential issues in logic. The starting point was marked by understanding the logical quantifiers by means of two antagonist players, and considering the debate as a sort of game [Hen61, Hin73].
The idea consists in associating to an assertion F a game between two players, traditionally called Eva ($\exists$) and Adam ($\forall$). Intuitively, Eva's aim is to defend the assertion F, and Adam attacks F. To be more precise, the situation is not symmetric: the player $\exists$ pretends that F is *always* true, whereas the player $\forall$ pretends that F is *sometimes* false.

From the game theory point of view, the games related to logic are not identical. They differ according to the number of players, win conditions, what kind of information is at the hand of players ....
In general there are two players, Eva ($\exists$) and Adam ($\forall$), the play is potentially infinite. The result of play may be the win or the loss[2]. There are no probabilities attached to actions or choices. It is impossible to formalize at once all kinds of games, but we try to give a synthetic definition of the notion of game and the related concepts.

The players play by choosing elements of a given *domain* Ω. According to their choices or actions, the players construct a sentence:

$$s_0, s_1, s_2, \ldots$$

of the elements of Ω. These elements may be the states of a structure, the subformulae of a given formula, ... The infinite elements of Ω are called the *play*. The finite element of Ω are called the *positions*. The latter record the state of the play at each moment. There is a function τ that specifies who plays from a given position; if $\tau(s) = \exists$, this tells that at the moment the play reaches the position s, then the player $\exists$ exhausts his move. A similar definition of

[2] or even the draw in particular cases, see section 4.2.

the above one is given if $\tau(s) = \forall$. The rules of the game determine two sets $W_\exists$ and $W_\forall$ which contain the positions and the plays satisfying the following properties: if a position s is in $W_\exists$ then any play that begins by s is in $W_\exists$. There is no play that belongs to both $W_\exists$ and $W_\forall$. We say that the player $\exists$ *wins* a play π if π belongs to $W_\exists$ and likewise with $\forall$. If a position s, which is in the prefix of π, belongs to $W_\exists$, then we say that the player $\exists$ wins already at s.

The definition of games established above expects the game to continue to infinity even a player has already won at some position. Many games have the property that, in an infinite play, one of the players has already won at some position. A stronger position is that there is some number n such that for every play, one of the player has already won on the nth position. If $n = 1$ the game is said to be *memoryless*.

A *strategy* for a player is the set of rules that tells how a player should play depending on his opponent choices early in the game. A strategy for a player $\forall$ is a function that associates at each position s, where $\tau(s) = \forall$, an element π of the domain Ω when the play reaches s. A strategy, for a player, is *winning* if any play that respects this strategy is a win for this player independently of his opponent choices. Clearly, at most one player has a winning strategy. Otherwise, each player can play according to his winning strategy against his opponent, and every player should win, contradicting the fact that $W_\exists$ and $W_\forall$ are disjoint.
In many games, the notion of the winning strategy for player $\exists$ corresponds to a logical meaning. The later can be defined without using games. For instance take the notion of the *proof*, *bi-simulation*, ...
We discuss the main categories of games and their connection to logic.

I. **Games for the definition of the truth** [Hen61, Hin73].
   The idea is to associate a game definition to the formulae $\phi(x_1, \ldots, x_n)$ being true at $a_1, \ldots, a_n$. For instance consider the formula

$$\forall x_0\ \exists y_0\ \forall x_1\ \exists y_1 \ldots\ \psi(x_0, y_0, x_1, y_1, \ldots),$$

and consider the following game. The player $\forall$ chooses an object a_0 for x_0, and then the player $\exists$ chooses b_0 for y_0, then $\forall$ chooses a_1 for x_1 and soon. A play is winning for $\exists$ if and only if the formulae $\psi(a_0, b_0, a_1, b_1, \dots)$ is true. The starting formula is true if and only if the player $\exists$ has a winning strategy in this game. Later, Hintikka exploited the same kind of ideas for the interpretation of the conjunction and disjunction. A conjunction of the form $\phi \wedge \psi$ may be considered as a sort of universal quantifier that states that " every one of the formulae ϕ and ψ is valid". Therefore in the game associated to the formulae $\phi \wedge \psi$, denoted by $\mathcal{G}(\phi \wedge \psi)$, it is the player $\forall$ who decides to continue in the game $\mathcal{G}(\phi)$ or in $\mathcal{G}(\psi)$. In a similar way, the disjunction becomes like a existential quantifier and consequently, it is the player $\exists$ who decides the continuation of the play. While dealing with a formula of type $\exists x \phi(x)$, a possible improvement consists in allowing to the player $\exists$ to choose a possible a_1 and proceeds in $\phi(a_1)$ and later he is allowed to come back and suggests a second choice a_2 and proceeds in $\phi(a_2)$. However, he is not allowed to come back infinitely often. In this case, a winning strategy is no longer a naive computation of truth. The notion of winning strategy is more general than that of the truth.

Hintika has suggested a game definition for the negation, every game has a *dual* game that differs from the former by the fact that the two players $\exists$ and $\forall$ are transposed.

A remarkable link between valid first order logic formulae and the specification of net work protocols was established in [KL07], the games associated to valid formulae – by allowing the sort of come back choices for $\exists$, discussed above – allows to specify, for instance, the acquitment of packets and the composition of protocols.

II. **Games for modal logics [Par85, RR95]** One of the features of modal logics consists in taking into account a *structure*. A structure $\mathcal{A}$ is the data of a set $\mathcal{S}$ of elements or states, a binary relation $\mathcal{R}$ on $\mathcal{S}$,

(the elements of $\mathcal{R}$ are called the *transitions*), and a family $P_1, \ldots, P_n$ of subsets of $\mathcal{S}$. Given a formulae ϕ, the two players $\exists$ and $\forall$ play a game $\mathcal{G} := (\mathcal{A}, \phi, s_0)$ on $\mathcal{A}$. The game starts at some initial state s_0 of $\mathcal{S}$. The game proceeds and the players follow the instructions encoded in the formula ϕ. The instructions determine who would move and the win conditions.

If ϕ is P_i, then the player $\exists$ wins at s if s belongs to P_i, otherwise the player $\forall$ wins. The game associated to the formulae $\phi \wedge \psi$, $\phi \vee \psi$ and $\neg\phi$ are defined in a similar of that of Hintikka games cited above: from a formulae of the form $\phi \wedge \psi$ it is $\forall$ who has the turn ...

If the formula is of form $\Box\phi$, then the player $\forall$ chooses a transition from s to some state t (i.e. $(s,t) \in \mathcal{S}$), and the play continues from the state t according to the instructions of the formula ϕ. The rules for the formula $\Diamond\phi$ are similar to those of $\Box\phi$ apart that it is the player $\exists$ who moves. Finally, the formula ϕ is true at the state s in $\mathcal{A}$ if the player $\exists$ has a winning strategy in the game $\mathcal{G} = (\mathcal{A}, \phi, s)$ described so far.

These games for modal logic resembles those of Hintikka for first order logic. Indeed there are many generalization of modal logic including temporal logics, dynamic logic and modal μ-calculus, and so the related games may differ. For instance, we mention Hennsy-Milner logic used for the description of the behaviour of reactive systems, where the transitions are labeled by *actions*. Each action corresponds to the action that the system performs to change the state. It is clear that the semantics over the labeled transitions is more interesting than that of the simple notion of truth.

III. **The back and fourth games to compare the structures [Fra53, Ehr61]** Tarski formulated the notion of two structures $\mathcal{A}$ and $\mathcal{B}$ being equivalent in the sense that the first order formulae which hold in $\mathcal{A}$ hold as well in $\mathcal{B}$. The formulation in term of games was established

in [Ehr61]. In the back and forth game two structures $\mathcal{A}$ and $\mathcal{B}$ are considered, there are two players called the *spoiler* and the *duplicator*, but we shall name them $\forall$ and $\exists$ respectively. In this game, every move of a player is followed by a move of his opponent. $\forall$ chooses an element of one of the structures, and $\exists$ chooses an element of the other structure. After n moves, two sequences are built up, the first one is built up from $\mathcal{A}$, and the second one is built up from $\mathcal{B}$:

$$(a_1, \ldots, a_n \; ; \; b_1, \ldots, b_n)$$

This position is winning for player $\forall$ if and only if a given formula holds at $(a_1, \ldots, a_n)$ in the structure $\mathcal{A}$ but it does not hold at $(b_1, \ldots, b_n)$ in $\mathcal{B}$, or inversely it holds in $\mathcal{B}$ and not in $\mathcal{A}$.
The player $\exists$ wins in $\langle \mathcal{A}, \mathcal{B} \rangle$ if and only if every initial segment is not winning for player $\forall$. In a similar way we can define the game $\langle \mathcal{A}, \mathcal{B} \rangle_m$ consisting of just m moves. The two structures are said to be *back and forth* equivalent if the player $\exists$ has a winning strategy in the game $\langle \mathcal{A}, \mathcal{B} \rangle$, and they are said to be m-equivalent if the player $\exists$ has a winning strategy in the game $\langle \mathcal{A}, \mathcal{B} \rangle_m$. Clearly $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if they are m-equivalent for each m.

It is worth to note that a winning strategy for $\exists$ in $\langle \mathcal{A}, \mathcal{B} \rangle_m$ gives rise to a first order logic formula ϕ which is satisfied at $\mathcal{A}$ and $\mathcal{B}$ where the *alternation between the logical quantifiers* $\exists$ *and* $\forall$ *in the formula* ϕ *is at most* m. One can imagine that the players have got certain number of tokens, and during the game, the number of tokens on left structure $\mathcal{A}$ equals the number of tokens on the right structure $\mathcal{B}$. During the game, the players are allowed to replace the tokens which have been already placed. The winning conditions for the player $\exists$ are the same as before apart that we consider only the positions which are marked by tokens. The existence of a winning strategy for player $\exists$ in this game – where the number of tokens is p on each side – implies that the two structures

admit the same formulae where the *number of bound variables is at most* p, [Imm95]. The bound variables can be reused, the same bound variable may occur many times.

The back and forth games does not make precise assumptions on the formula in question, the latter being infinite or not. These game would be applied also on finite structures, as well as on infinite ones. A main feature of back and forth games is they usefulness in comparing the expressive power of logics.

The back and forth games can be adapted in a natural way to the Kripke structures. One of the players chooses a state s of the structure $\mathcal{A}$ and the other player chooses a state t from $\mathcal{B}$. As before, the players $\exists$ and $\forall$ exhaust their moves alternatively. Once the player $\exists$ has to move he chooses a state from either $\mathcal{A}$ or $\mathcal{B}$ and then the player $\forall$ chooses a state from the other structure. Besides the structures, a move is always performed via a transition from the current state. If – at any moment – one of the players moves to a state s' in $\mathcal{A}$ and the other player moves to a state t' of $\mathcal{B}$, and the formula in question holds at least in s' or in t', then the player $\forall$ looses. Similarly, the player $\exists$ looses when there is no possible transition available to him. If the two players begins playing from a state s of $\mathcal{A}$ and a state t of $\mathcal{B}$, then one cam prove that the player $\exists$ has a winning strategy if and only if the same modal formula holds at once on $\mathcal{A}$ form s and on $\mathcal{B}$ from t.

In computer science an essential notion is that of the *bissimulation*. We can consider the two structures $\mathcal{A}$ and $\mathcal{B}$ as systems and we ask whether the two systems behave in the same manner step by step with respect to their environment. This amounts to consider a binary relation R that links the states of $\mathcal{A}$ to the states of $\mathcal{B}$. Hence, a winning strategy in the back and forth game $\langle \mathcal{A}, \mathcal{B} \rangle$ for player $\exists$ witnesses the existence of such a bissimulation. And conversely, the bissimulation R may be used for player $\exists$ as a winning strategy in that game.

IV. **The games for the communication, the dialog and interactive computation [Bla92, AJ94, AM99, Joy97]**

Lorenzen in his *dialectic games* specifies the rules of a game in such a way the player $\exists$ has a winning strategy if and only if the formula in question is a theorem of intuitionist logic.

At the beginning of the 1990s many researchers looked for the games that stand for Linear logic [Gir87] in the same way that the Lorenzen games stand for intuitionist logic. Blass [Bla92] suggested such games for affine logic (allowing weakening) and later Abramsky et al. [AJ94] suggested the games corresponding to the multiplicative fragment of linear logic. Abramsky et al.'s games model the interaction between systems in a distributed and asynchronous mode.

A quite similar approach was pursued by A. Joyal [Joy97]: the free lattices as well as the associated games may be of use to stand the foundation of the communication theory. We means by the associated games the solution of Whitman of the word problem [Whi41, Whi42], see section 1.3. To each algebraic term A, that represents an element of the free lattice, is associated a game G_A of two players σ_A and π_A depicted as follows:

$$\boxed{(\sigma_A : G : \pi_A)}$$

To each pair (A, B) of terms is associated a game $\langle G_A, G_B \rangle$ consisting of two boards G_A, G_B and three players the *Mediator* M against two opponents σ_A and π_B. The game $\langle G_A, G_B \rangle$ is pictured as follows:

$$\boxed{\sigma_A : G_A : \quad M \quad : G_B : \pi_B}$$

A strategy for Mediator in the game $\langle G_A, G_B \rangle$ is considered as a protocol which ensures the communication between σ_A and π_B in an asynchronous way. The Mediator has a winning strategy in $\langle G_A, G_B \rangle$ if and

only if $A \leq B$ holds in every lattice where the terms A and B are interpreted. The transitivity of the relation $\leq$ is a consequence of the fact that the winning strategies (for Mediator) do *compose*.

Giorgi Japaridze [Jap03] suggested a computational logic for the study of the notion of calculus. Basically, the syntax of this logic is that of first order logic and some connectors of linear logic. Its semantic in terms of games is similar in spirit to the traditional game semantics. A winning strategy for player $\exists$ is a sort of computing machine.

0.4 The lattice μ-calculus L_μ

In this thesis, the main formalism under study is the lattice μ-calculus $\mathbb{L}_\mu$. The latter extends the lattice terms with the least and greatest fixed point operators μ and ν. The syntax of the terms of $\mathbb{L}_\mu$ is given by the following grammar:

$$t \;=\; x \mid \top \mid \bot \mid t \wedge t \mid t \vee t \mid \mu x.t_x \mid \nu x.t_x,$$

where x belongs to a countable set X of free variables. The semantics of the μ-terms is given over a complete lattice L. The interpretation of $\top$ (resp. of $\bot$) would be the supremum element of L (resp. the infimum element of L). The interpretation of $\wedge$ (resp. of $\vee$) would be the greatest lower bound (resp. the least upper bound) – of the related μ-terms – in L. Finally, the interpretation of μ (of ν) would be the least (resp. the greatest) fixed point – of the related μ-term – in L. The existence and the uniqueness of the fixed points is ensured by Tarski Theorem [Tar55]. The μ-terms have a natural translation into a sort of combinatorial objects: *the parity games with draws positions*. The combinatorial counterpart suggests a fruitful and effective framework when dealing with fixed point theory. This already happened to automata and modal μ-calculus formulas [JW95].

0.4.1 Parity games with draws positions

A parity game is a 2-player game played between player σ (verifier) and player π (falsifier) on a finite directed graph. In fact, each position belongs either to player σ or to player π or is declared as a draw position[3]. The vertices of the graph are understood as the positions of the play, and the edges are understood as the moves. Furthermore, integer priorities are assigned to vertices. During the play, the players form a path (finite or not) in the graph. In a finite play, the player who can not move looses. Clearly, if the play reaches a final draw position, then the game is declared as a draw. To determine the winner in an infinite play we check the parity of the maximal priority that occurs infinitely often: if it is even then player σ wins, otherwise player π wins.

The translation of μ-terms into parity games is given in the traditional way: the $\wedge$ operator corresponds to the player π, the join operator corresponds to the player σ, and a free variable corresponds to a draw position. The μ-operator corresponds to *finite* looping. For instance consider the μ-term:

$$t = \nu_x.(z \wedge \mu_y.(\top \vee (y \wedge x))),$$

the game associated to t is depicted in Figure 0.4.1.

0.4.2 The motivations for the study of the lattice μ-calculus

Our motivations for the study of the lattice μ-calculus $\mathbb{L}_\mu$ may be summarized in four main points:

I. The first motivation is that the parity games formalism constitutes itself a verification tool. The priorities assigned to the positions express different levels of fairness. On the other hand, a standard result in automatic verification states that the problem of the model checking of

[3]To be more precise, a draw position would be a vertex without successors.

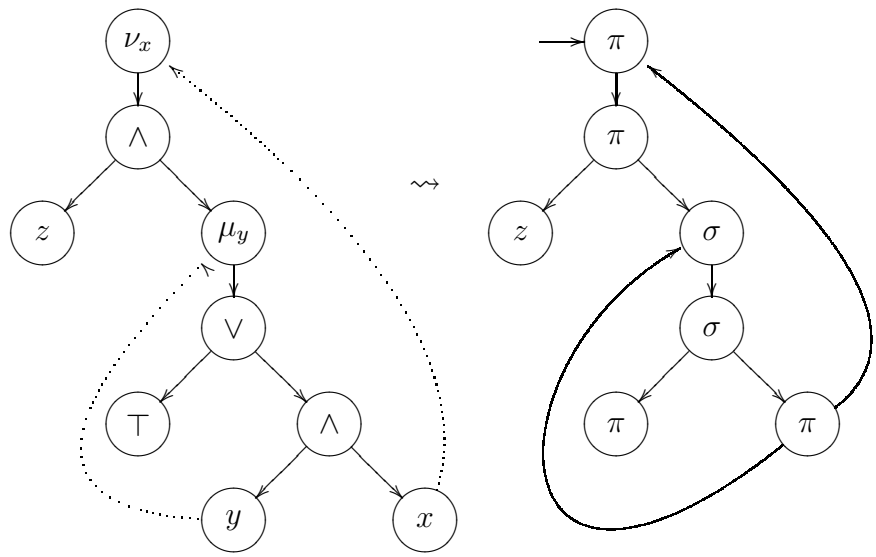


Figure 0.4.1: The μ-term t and the associated game.

modal μ-calculus reduces in a linear times to the problem of solving parity games i.e. the computation of winning positions for player σ [EL86]. It follows that developing fast algorithms for solving parity games gives rise to efficient aided verification tools. However, whether parity games may be solved in polynomial time remains an open question despite the effort of the community [Jur00a, Obd03, Obd07, Sch07].

II. The second point consists in the fact that games suggest an appropriate mathematical model of the interactive computation and the semantics of programming languages [Joy77, NYY92, AJ94, HO00]. In particular, A. Joyal suggested to develop a theory of communication by means of algebraic objects such as the free lattices [Joy97] and the free bicomplete categories [Joy95]. In this approach a term of the free lattice

(which is built up using $\wedge$ and $\vee$) is considered as a game G between $\wedge$ and $\vee$, respectively. The game G is a sort of *synchronous canal* of communication between π and σ; it is a game of perfect information. Each player can only move if his opponent has exhausted all his moves, there is no kind of parallel moves. Given two games G and H, a winning strategy for the Mediator in $\langle G, H\rangle$ witnesses the existence of a good *assynchronous* protocol allowing the communication between the player σ of G with the player π of H. The main feature of modeling the interactive computation with the μ-lattices, rather with the lattices, is the potential presence of infinite play. The underlying graph of a μ-term is no longer acyclic, but it may contains cycles allowing to modeling potential infinite behaviour. In this case, the winning strategy for Mediator is translated to the fact that if a dead lock occurs in the system, when using the protocol, then either the left user of G or the right user of H is responsible for it. In particular there is no infinite execution of the protocol itself without allowing the two users to communicate with each other. Figure 0.4.2, suggests that the channels G and H would be telephone wires and the protocol M an operating system. Indeed, the system operates in an asynchronous fashion. Besides it allows the left user to cooperate with the right one, the operating system may favor to communicate with one user and leaves the other user on hold.

III. The third motivation consists in the particular status of the lattice μ-calculus with respect to proof theory. The game $\langle G, H\rangle$ is a sort of a sequent $G \vdash H$. A winning strategy for Mediator in the game $\langle G, H\rangle$ represents a *cut free circular proof* of the sequent $G \vdash H$ [San02b]. The key ideas of proof theory, and in particular the cut elimination and the η-expansion, in their game theoretic shape i.e. the composition of winning strategies and the copycat strategy, have been very useful to solve problems arising from fix point theory. For instance, we cite the alternation hierarchy of the μ-lattices [San02a], the ambiguous classes of this hierarchy [AS03], and the variable hierarchy of the lattice μ-calculus

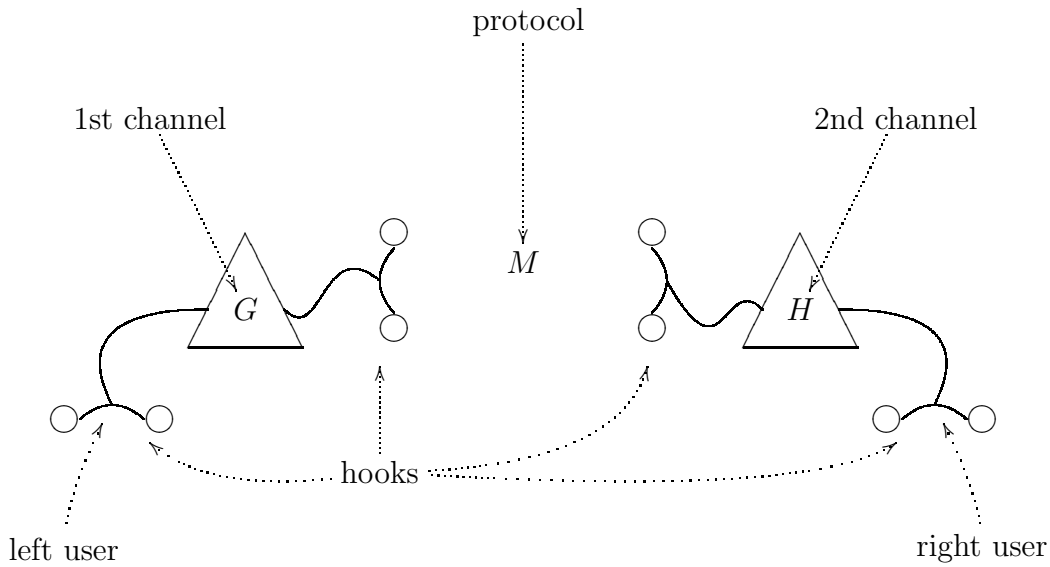


Figure 0.4.2: Games as channels, strategies as protocols.

[BS08, BS].

IV. The fourth motivation consists in the ability of the parity games to represent in a natural way the inductive and co-inductive types [San01]. Intuitively a induction type is a type built up using a finite iteration of constructors. A co-inductive type may be built up using an infinite iteration of constructors. A finite list of elements is an example of inductive type. An infinite list of elements is an example of co-inductive type. Note that the inductive and co-inductive constructors may be nested in the same way that the fixed point operators μ and ν are nested. A typical example of a data type where the induction and co-induction are nested is the locally finite graph. A graph is locally finite if it is infinite but each vertex has a finite number of successors. A main problem consists in ensuring that a type defined by means of the inductive and co-inductive constructors in not empty. The parity games suggest

a solution to this problem. Checking weather such a type is not empty reduces to compute the set of winning positions for player σ in the related parity game, and to ensure that the starting position belongs to this set.

0.5 Graph theoretic measures and entanglement

The entanglement is at once a logical and combinatorial complexity measure. On the one hand, it is a good indicator of the descriptive complexity of fixed point logics, it is the combinatorial counterpart of the variable hierarchy. On the other hand, it measures to what extent cycles are intertwined in a digraph. Its study in graph theory is still in an experimental stage, little is known about its algorithmic and graph theoretic properties [BG05, BS07].

As a matter of fact, the main result of [BG05] states that: *the parity games, for which the underlying graph has bounded entanglement, may be solved in polynomial time.* This result calls for the problem whether a graph has entanglement k. In other words, this calls for the fundamental problem of recognizing the graphs of bounded complexity measure. The second part of this thesis is devoted to the study of entanglement in the context of graph theory rather than fixed point theory. The problems such that the tree decomposition, the computation of excluded minors within entanglement are challenging.

We recall that the tree decomposition is a fundamental notion of algorithmic graph theory. The intuition is to ask to what degree a graph is close to a tree. The tree-likeness concept allows the generalization of all the tree properties to general graphs. Furthermore, the tree decomposition paradigm has shown its use in different domains:

I. the design of polynomial algorithms, the graph problems are easier to solve when they operate on trees,

II. the study of graphs characterized by a set of *excluded patterns*, a classical example is the characterization of Kuratowski of planar graphs [Kur30], and

III. the decidability problem of second order logic.

Many NP-hard problems can be solved in polynomial time on a particular class of graphs: the *well structured graphs*. These are the graphs which can be constructed by means of atomic graphs and gluing operations. An example of this approach is the construction of the graphs which does not contain the complete graph K_5 as a minor[4] by means of the gluing of planar graphs and a particular graph on 8 vertices, known as the graph of Möbius ladder [Wag37]. The study of series parallel graphs gave rise to the notion of *tree width*.
The second approach was pursued by Robertson and Seymour in their works on the graph minor theorem [RS86, RS90, RS03, RS04], referenced as Wagner conjecture [Wag70]. This theorem states that in an infinite family of graphs, there exists a graph which a minor of another one. As a consequence, any family of graph which is closed under minors, such that the planar graphs, can be characterized by a *finite* set of excluded minors. A key concept used in Robertson and Seymour proof is the decomposition of bounded width.
The third approach consists in looking for a characterization of the class of graphs for which the satisfiability problem of second order logic is decidable. It was conjectured in [See91] that such graphs are close to trees. The proof remains an open question.

We recall some of the standard tree decompositions in graph theory:

- the decomposition of graphs into 2-connected[5] components [Die05, §3], it is the most simple decomposition. It consists in considering the max-

[4] A graph H is a minor of G if H can be obtained from G by a series of edge deletion and edge contraction operations.

[5] A graph is k-connected if it remains connected after a removal of any $k-1$ vertices.

imal subgraphs which do not contain articulation points[6]. It is one of the elementary results of graph theory to show that the 2-connected components have a tree like structure. This kind of decomposition is the starting point in the characterization of undirected graphs of entanglement 2, see Chapter 7 and section 8.6,

- *Tutte's decomposition* of 2-connected graphs into cycles and 3-connected components [Tut66, §11]. Tutte's decomposition will be our main tool for the study of undirected graphs of entanglement 3, see Chapter 8,

- the decomposition of graphs into k-connected components [Hoh92]. This is a sort of generalization of Tutte's decomposition to higher connectivity. This decomposition is no longer canonical. However, it suggests a hopeful direction for the study of undirected graphs of entanglement k,

- *Ear decomposition* [BJG01, §7.2]. It is a non canonical decomposition of directed graphs in terms of directed cycles. Ear decomposition would be of use to study the entanglement in the directed setting, and in particular the directed graphs of entanglement 2,

- the tree decomposition associated to the complexity measures such that the directed tree width, the DAG-width [JRST01, BDHK06, Obd06]. It is a kind of refinement of the tree decomposition given by Robertson and Seymour [RS90]. In general, these measures are defined in two equivalent ways. The first one is game theoretic in nature and is given in terms of a variant of Robber and Cops games. The second one is given by means of a variant of the standard tree decomposition.

  Up to now, the entanglement is not completely associated to a natural tree decomposition. This might be justified by the absence of *monotonic* strategies for Cops in the entanglement games. We mean

[6] An articulation point is a vertex whose removal increases the number of connected components.

by the word monotonic the fact that any vertex of the graph is visited at most once by Robber. For instance monotonic strategies exist for the case of tree width, directed tree width and DAG-width [JRST01, BDHK06, Obd06]. There, natural tree decompositions are given out of the monotonic strategies.

0.6 Main results

We summarize the main results of this thesis.

The first kind of results proves that the variable hierarchy for the lattice μ-calculus, and hence for the games μ-calculus, is infinite, Theorem 5.7.8. Our proof requires many ingredients. The equivalence between μ-terms (i.e. parity games withs draws) is captured by Opponents-Mediator games. We shall construct, for arbitrary $n \in \mathbb{N}$, a μ-term (i.e. a game) for which the number of bound variables (i.e. the entanglement) is n. These games are called *strongly synchronizing* and strengthen the synchronizing games considered in [San02a]. Besides, every game equivalent to a strongly synchronizing one is related with it by a *weak simulation* with the *star property*. Finally, we shall prove that the weak simulation with the star property preserves the entanglement, up to a constant. This result will be exposed in Chapter 5, and was the subject of the publication [BS08]. A long and improved version was submitted to the journal Annals of Pure and Applied Logic [BS].
Moreover, the relation between the star height hierarchy and the variable hierarchy is established in Chapter 3. There we shall prove, under some assumption, that the variable hierarchy is a refinement of the star height hierarchy, i.e. the non collapse of the former implies the con collapse of the latter, Theorem 3.5.1. Our proof relies on the combinatorial characterization of the two hierarchies by means of the rank and the entanglement, respectively.

The second result consists in giving a combinatorial and algebraic characterization of the class ζ_2 of undirected graphs of entanglement at most 2.

On the one hand we shall characterize the graphs in ζ_2 by means of excluded subgraphs. This already gives the set of excluded minors that characterizes the class ζ_2. On the other hand, based on the combinatorial characterization, we shall give an algebraic construction of memberships of ζ_2 in terms of a set of small pieces, called the molecules, and an algebraic operators that glues two molecules on a prescribed set of vertices. The algebraic construction provides a tree decomposition of graphs in ζ_2. Finally, we shall give a linear time algorithm that decides memberships of ζ_2. This result is given in Chapter 7 and was the subject of the publication [BS07].

In the third kind of results, we continue the investigation of the structure of undirected graphs of entanglement 3 by trying to lift the approach considered with entanglement 2 to entanglement 3. Our main tool will be Tutte's decomposition paradigm that decomposes 2-connected graphs into cycles and 3-connected components. We shall give some necessary conditions on Tutte's tree to be a tree decomposition of a 2-connected graph of entanglement 3. These conditions deals with three features of the tree: *(i)* conditions on the structure of the 3-connected components, they are the 3-molecules that generalize the molecules considered in the previous result, *(ii)* conditions on the manner by which these components are glued together, a sort of interface between these components, and *(iii)* an upper bound on the diameter of the tree is given. This result is discussed in Chapter 8.
Besides this, we shall prove a fundamental result on the undirected entanglement: the class of undirected graphs of entanglement at most k is closed under minors, Theorem 6.4.2. Further results on the algorithmic properties of the entanglement are discussed in Chapter 6.

0.7 Organization of the thesis

This thesis is organized in 8 Chapters:

Chapter 1 introduces the necessary background on lattice theory, includ-

ing the central notions of complete lattice, free lattice and μ-lattice. Besides, we shall review the definition of least and greatest fixed point operators μ and ν, and their elementary properties.

Chapter 2 recalls the elementary definitions and facts of graph theory. There, the definition of digraphs, paths, tree with back edges, and the k-connectivity is given.

Chapter 3 introduces the notion of μ-calculus in its abstract sense as well as the three hierarchies: the alternation depth hierarchy, the star height hierarchy and the variable hierarchy. There, we shall prove, under some assumptions, that the variable hierarchy is a refinement of the star height hierarchy. We mean that the non collapse of the former implies the non collapse of the later. Our proof relies on the game theoretic characterization of the two hierarchies.

Chapter 4 introduces the lattice μ-calculus, its syntax and semantics. A syntactic preorder on lattice μ-terms that characterizes the semantic one is given by means of games and strategies as defined in [San02c]. Besides, a communication model grounded on the lattice μ-calculus is discussed.

Chapter 5 is devoted to the first main result of this thesis. There, we shall prove that the variable hierarchy of the lattice μ-calculus is infinite.

Chapter 6 is devoted to one of the basic properties of entanglement. There, we shall prove that the class of undirected graphs of entanglement at most k, for arbitrary fixed $k \in \mathbb{N}$, is closed under minors. Some algorithmic properties of entanglement are discussed.

Chapter 7 is devoted to the second main result of this thesis. We shall provide two characterizations of the class ζ_2 of undirected graphs of entanglement at most 2. The first one is in terms of forbidden subgraphs, and the second one is in terms of algebraic construction by means of small pieces, called the molecules. A linear time algorithm to decide memberships of the class ζ_2 is given.

Chapter 8 is devoted to the third main result of this thesis. Based on Tutte's tree, we shall give some necessary conditions the Tutte's tree to be a

tree decomposition of a 2-connected graph of entanglement 3. In this chapter, the entanglement is studied with respect to the two standard notions of graph theory: the n-connectivity and the n-cyclicity.

Chapter 1

Preliminaries on Lattices and Fixed-points

The fixed points calculi rely on the famous Knaster-Tarski fixed-point theorem [Tar55] stating that monotonic functions over a complete lattice have a least and greatest fixed-point. To understand the central role of complete lattices if fixed point theorems, we review the basic concepts of the lattice theory including the central notion of free lattice and Whitman's solution to the word problem. Then, we recall the definition of the least and greatest fixed point operators μ and ν, and their elementary properties. With such a background in hand, we end the chapter with the definition of the notion of μ-lattice.
The basic concepts of lattices and ordered sets can be found in [DP90]. Further details on the topic can be found in the monographs [Bir40, Grä98, FJN95, CLM07]. The material on the calculi of fixed points can be found in [Niw85], [AN01], [DP90, §8], and [BÉ93]. The solution of the word problem is presented in Whitman's classical papers [Whi41, Whi42]. The μ-lattices have been introduced in [San00, San02c]

Key words. Complete lattices, extremal fixed points, free objects.

1.1 Lattices

1.1.1 Posets

A *partially ordered set*, (or *poset*) formalizes the intuitive concept of ordering of the elements of a set.

Definition 1.1.1. A *partial order* over a set P is a binary relation $\leq \subseteq P \times P$ which is reflexive, antisymmetric, and transitive; that is for every $a, b, c \in P$, we have that:

1. $a \leq a$ (reflexivity),
2. if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry),
3. if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

A *poset* is a pair $\langle P, \leq \rangle$, where P is a set and $\leq$ is a partial order over P. ■

1.1.2 Lattices as posets

Definition 1.1.2. Let $\langle P, \leq \rangle$ be a poset, and let $X \subseteq P$. An element $e \in P$ is an *upper bound* of X if $x \leq e$ for all $x \in X$. Similarly, an element $e \in P$ is a *lower bound* of X if $e \leq x$ for all $x \in X$. ■

Definition 1.1.3. Let $\langle P, \leq \rangle$ be a poset, and let $X \subseteq P$.
An element $e \in P$ is the *least upper bound* of X if it is the least element in the set of upper bounds of X, i.e. if the following conditions hold:

- $\forall x \in X$, $x \leq e$,
- if e' is such that $\forall x \in X, x \leq e'$, then $e \leq e'$.

Similarly, the *greatest lower bound* of X (if it exists) is the element $e \in E$ such that:

- $\forall x \in X, e \leq x$,
- if e' is such that $\forall x \in X$, $e' \leq x$, then $e' \leq e$.

■

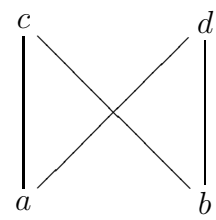


Figure 1.1.1: A poset which is not a lattice

Comments on the definition and further notations.

Clearly, if $X \subseteq P$ has a least upper bound, then this element is *unique*, and we shall denote it by $supX$. Dually, if X has a least upper bound, then this element is *unique*, and we shall denote it by $infX$.
The least upper bound of X is also called the *supremum* of X; the greatest lower bound of X is called the *infimum* of X.

Top and bottom. In definition 1.1.3 of $supX$ and $infX$ we have two extremal cases: when X is P and when X is empty.
The element $supP$, if it exists, is denoted by $\top$, read as "top ". Dually, the element $infP$, if it exists is denoted by $\perp$, read as "bottom".
Now let X be the empty set, then every element $e \in P$ satisfies (trivially) $x \leq e$ for all $x \in X$. Hence, $\{\, e \in P \mid x \leq e \text{ for every } x \in X \,\} = P$, therefore $sup\emptyset$ exists if and only if P has a bottom element, and in this case $sup\emptyset = \perp$. By duality, we get that $inf\emptyset$ exists if and only if P has a top element, and in this case $inf\emptyset = \top$.

The supremum and the infimum of a set need not exist in general. For example consider the set $\{\, a, b, c, d \,\}$ equipped with the orderings $a \leq c,\ a \leq d,\ b \leq c,\ b \leq d$ (depicted in Figure 1.1.1). Observe that the subset $\{\, c, d \,\}$ has no upper bound and the subset $\{\, a, b \,\}$ has two upper bounds, which are c and d, but none of them is a least upper bound.

Definition 1.1.4. A poset $\langle L, \leq \rangle$ is a *lattice* if and only if $sup\{\, a, b \,\}$ and $inf\{\, a, b \,\}$ exist, for all $a, b \in L$. ■

It may happen that a lattice $\langle L, \leq \rangle$ has both top and bottom elements $\top$ and $\bot$, in this case the lattice is called *bounded*.

To make the definition of a lattice more convenient, we point out that an equivalent definition is as follows:

A poset $\langle L, \leq \rangle$ is a lattice if and only if $supH$ and $infH$ exist, for each finite non empty subset H of L.

The proof of is left as an exercise, or see [Grä98, §1], for instance.

Complete Lattices.

Definition 1.1.5. A poset $\langle L, \leq \rangle$ is a *complete lattice* if for any subset (possibly infinite) X of L we have that $supX$ and $infX$ exist[1]. ■

Example 1.1.6. Let E be any set, and let $\mathcal{P}(E)$ be its powerset, ordered by inclusion. Then, $\langle \mathcal{P}(E), \subseteq \rangle$ is a complete lattice; $sup\{ X, Y \} = X \cup Y$, $inf\{ X, Y \} = X \cap Y$, $\top = E$ and $\bot = \emptyset$.

1.1.3 Lattices as algebraic structures

We introduced lattices as ordered sets with special properties, Definition 1.1.4. Next, we show that lattices may be viewed as algebraic structures. The motivation is simple: if we view order theoretic lattices as algebras then we can apply all concepts and results of equational logic and its model theory, that is, what is usually known as universal algebra [Grä79, BS81, Wec92].

Definition 1.1.7. An algebraic structure $\langle L, \vee, \wedge \rangle$, consisting of a set L and two binary operations $\vee$ and $\wedge$ on L, is a *lattice* if the following conditions hold for every $a, b, c \in L$:

- Commutative laws:

$$a \vee b = b \vee a, \qquad a \wedge b = b \wedge a. \tag{L1}$$

[1] It is enough to consider either the existence of $supX$ or $infX$.

- Associative laws:

$$a \vee (b \vee c) = (a \vee b) \vee c, \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c. \tag{L2}$$

- Idempotency laws:

$$a \vee a = a, \qquad a \wedge a = a \tag{L3}$$

- Absorption laws:

$$a \vee (a \wedge b) = a, \qquad a \wedge (a \vee b) = a. \tag{L4}$$

■

The binary operators $\wedge$ and $\vee$ are read as *meet* and *join*, respectively.

Connection between the two definitions.

An order-theoretic lattice $\langle L, \leq \rangle$, as given in definition 1.1.4, gives rise to an algebraic lattice $\langle L, \wedge, \vee \rangle$, as given in definition 1.1.7, and conversely. The following Theorem describes how to pass from a poset to an algebraic structure and conversely, moreover, it states that a lattice as a poset and a lattice as an algebraic structure are "equivalent".

Theorem 1.1.8.

(i) Let the poset $\mathfrak{L} = \langle L, \leq \rangle$ be a lattice (as defined in Definition 1.1.4). Define

$$a \wedge b := inf\{\, a, b \,\},$$
$$a \vee b := sup\{\, a, b \,\}.$$

Then the algebraic structure $\mathfrak{L}^{Alg} = \langle L, \wedge, \vee \rangle$ is a lattice (as defined in Definition 1.1.7).

(ii) Let the algebraic structure $\langle L, \wedge, \vee \rangle$ be a lattice. Define

$$a \leq b \text{ if and only if } a \wedge b = a.$$

Then $\mathfrak{L}^{Pos} = \langle L, \leq \rangle$ is a poset, and moreover $\mathfrak{L}^{Pos}$ is a lattice.

(iii) If the poset $\mathfrak{L} = \langle L, \leq \rangle$ is a lattice, then $(\mathfrak{L}^{Alg})^{Pos} = \mathfrak{L}$.

(iv) If the algebraic structure $\mathfrak{L} = \langle L, \wedge, \vee \rangle$ is a lattice, then $(\mathfrak{L}^{Pos})^{Alg} = \mathfrak{L}$

■

This standard theorem is well know in lattice theory, its proof can be found in many lattice monographs, see for instance [Grä98, §1] or [DP90, §2]. Since the two definitions of a lattice are equivalent, we may freely invoke aspects of either definition, that is definition 1.1.4 or 1.1.7, in such a way that suits the purpose at hand.

Remark 1.1.9. Recall that a lattice $\langle L, \leq \rangle$ is said to be bounded if the top and the bottom elements $\top, \bot$ exist. When thinking of L as an algebraic structure $\langle L, \wedge, \vee \rangle$, it is convenient to see the element $\top$ as the neutral element of the monoïd $\langle L, \wedge \rangle$ and to see $\bot$ as the neutral element of the monoïd $\langle L, \vee \rangle$. We say that L has a *one* if there exists $1 \in L$ such that $a \wedge 1 = a$, for all $a \in L$. Dually, L has a *zero* if there exists $0 \in L$ such that $a \vee 0 = a$, for all $a \in L$. The main observation is that a lattice $\langle L, \leq \rangle$ has a top if and only if $\langle L, \wedge, \vee \rangle$ has a one, and in this case $\top = 1$. A dual statement holds for $\bot$ and 0. Similarly, a lattice $\langle L, \wedge, \vee \rangle$ having both 1 and 0 is called also *bounded*. ■

Distributive lattices.

The standard interpretation of the propositional modal μ-calculus [Koz83] is over the Boolean algebra that arises from transition systems. Boolean algebras have a strong property: the two operators $\wedge$ and $\vee$ distribute over each other in the following sense:

$$\forall a, b, c \in L, \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \tag{D1}$$

$$\forall a, b, c \in L, \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{D2}$$

The distributivity laws (D1) and (D2) can be defined for lattices as well, a lattice $\langle L, \wedge, \vee \rangle$ is *distributive* if it enjoys the laws (D1) and (D2).

Proposition 1.1.10. *In a lattice $\langle L, \wedge, \vee \rangle$, the two laws (D1) and (D2) are equivalent.*

Proof. The aim of the proof is to illustrate the use of typical algebraic arguments. The proof of the same Proposition in not straightforward if L is viewed as poset.
(D1) $\implies$ (D2).

$$\begin{aligned}
(a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) && \text{(D1)} \\
&= a \wedge ((a \vee c) \wedge (b \vee c)) && \text{(L4) and (D1)} \\
&= (a \wedge (a \vee c)) \wedge (b \vee c) && \text{(L2)} \\
&= a \wedge (b \vee c) && \text{(L4)}
\end{aligned}$$

The proof of (D2) $\implies$ (D1) is obtained out of the above one by exchanging the operators $\wedge$ and $\vee$. □

Unlike the propositional modal μ-calculus, the interpretation of the lattice μ-calculus - which is the formalism under study - over the class of distributive lattices makes it trivial. To put right, we shall consider the standard and classical interpretation over the class of complete lattices.

Lattice homomorphism.

Definition 1.1.11. Given two lattices $\langle L, \wedge_L, \vee_L \rangle$ and $\langle M, \wedge_M, \vee_M \rangle$, a lattice homomorphism is a function $f : L \longrightarrow M$ such that:

$$f(a \wedge_L b) = f(a) \wedge_M f(b), \quad \text{and}$$
$$f(a \vee_L b) = f(a) \vee_M f(b)$$

for every $a, b \in L$. When lattices are equipped with extra structure, then the homomorphism should respects the extra structure as well. In particular, a morphism f between two bounded lattices L and M should also enjoy the following property:

$$f(\top_L) = f(\top_M), \quad \text{and}$$
$$f(\bot_L) = f(\bot_M)$$

■

1.2 Free lattices

The aim now is to consider the concept of the most general lattice, that is a lattice constructed out of a set of generators and satisfies the most general relations. The concept of the general lattice is also known as the *universal lattice*, that is a lattice that satisfies a kind of *universal* properties. Before giving the formal definitions we emphasize the motivation for studying such properties. The details of a given construction may be too concrete and even awkward, but if such a construction satisfies a universal property then we can leave all these details in the background and all what we need to know is already contracted in the universal property. Therefore, the proofs are short if they deal with the universal property rather than the concrete details.

The most general lattice is formalized with the concept of *free lattice*.

Definition 1.2.1. Let F be a lattice and $X \subseteq F$. We say that F is *freely generated* by X if, given any map $f : X \longrightarrow L$ from X to some arbitrary lattice L, then there exists a unique[2] lattice homomorphism $\tilde{f} : F \to L$ such that $\tilde{f}(x) = f(x)$, for each $x \in X$. Categorically speaking, if we name η the inclusion $X \subseteq F$, then the freeness of F may be rephrased by the existence and uniqueness of the homomorphism $\tilde{f}$ making the following diagram to

[2]This is the universal property.

commute:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "f"'] & F \arrow[dl, "\tilde{f}"'] \\ L & \end{tikzcd}$$

■

In algebra, the creation of free objects proceeds in two steps. The first step is to consider the collection of all possible *words* or *terms*, formed from a set of alphabet or generators. The second step consists in imposing a set of equivalence relations upon the terms, where these relations are those defining the algebraic object in question. The free object then consists of the set of equivalence classes. This kind of construction is known also as the Lindenbaum algebra [3] of an equational theory.

We shall show next, that the concept of free lattice may be viewed as the Lindenbaum algebra of the equational theory of lattices, if the latter is axiomtized by means of the equations (L1) – (L4).

Given a set X of generators, the construction of a free lattice over X, consists first in considering $T(X)$ the set of syntactic terms over X given by the following grammar

$$t := \quad x \mid \top \mid \bot \mid t \wedge t \mid t \vee t \tag{1.1}$$

$T(X)$ is an algebra with the two binary operators. The set of variables of a term t will be denoted by X_t.

Secondly, we consider interpretation of terms in $T(X)$ over a lattice L is given by the function

$$\| \, . \, \| : T(X) \longrightarrow \bigcup_{n \geq 0} L^{(L^n)}$$

[3] A Lindenbaum algebra of an equational theory TH is the equivalence classes of terms t of the theory TH with respect to the equivalence relation $\sim$, defined as $t_1 \sim t_2$ iff t_1 and t_2 are logically equivalent, in the sense that t_1 is derivable from t_2 in TH and t_2 is derivable from t_1, too.

where L^X is the X-fold product lattice of L with itself, and the interpretation of the term $t \in T(X)$ is given by the function $\|t\|^L : L^{X_t} \longrightarrow L$, in the following way:

- If $t = x$, then $\|t\|^L(v) = v(x)$.
- If $t = \top$ (resp. $t = \bot$), then $\|t\|^L$ is interpreted as the constant function to $\top_L$ (the supremum of L), (resp. to $\bot_L$, the infimum of L),
- If $t = t_1 \wedge t_2$, then $\|t\|^L(v) = \|t_1\|^L(v|t_1) \wedge_L \|t_2\|^L(v|t_2)$, where $v|t_i$ is the restriction of v to X_{t_i}.
- If $t = t_1 \vee t_2$, then $\|t\|^L(v) = \|t_1\|^L(v|t_1) \vee_L \|t_2\|^L(v|t_2)$.

If s and t are terms and L is a lattice, then we say that the identity $s = t$ holds in L if $\|s\|^L = \|t\|^L$, where $\|s\|^L, \|t\|^L$ are viewed as functions. We say that $s \sim t$ if and only if $\|s\|^L = \|t\|^L$ holds in every lattice L.

It is standard to check that $T(X)/\sim$ is a lattice freely generated by X, where each element is identified by its equivalent class. The free lattice $F(X)$ is general, in the sense that $s \sim t$ holds in $F(X)$ if and only if $\|s\|^L = \|t\|^L$ holds in any lattice L. The former construction is a standard realization of free algebras.

In universal algebras, a natural question consists in answering whether any two terms represents the same element of the algebra or not. This is known as the *word problem*.

1.3 Whitman's solution of the word problem

In [Whi42], Whitman solved the word problem of free lattices. Then, he showed that each element of the free lattice has a shortest representation, known as the *canonical form*[4]. The key ingredient of Whitman's solution is

[4]The canonical form enjoys the property of uniqueness up to commutativity and associativity.

the following Theorem.

Theorem 1.3.1. [Whi42]. In a free lattice $F(X)$ we have that

$$\begin{array}{rcll}
x_i \leq x_j & \text{iff} & i = j, & \text{where } x_i, x_j \in X, \\
\bigwedge s_i \leq x & \text{iff} & s_j \leq x, & \text{for some } j, \\
x \leq \bigvee t_i & \text{iff} & x \leq t_j, & \text{for some } j, \\
\bigvee s_i \leq t & \text{iff} & s_j \leq t, & \text{for every } j, \\
s \leq \bigwedge t_i & \text{iff} & s \leq t_j, & \text{for every } j, \\
\bigwedge s_i \leq \bigvee t_i & \text{iff} & \bigwedge s_i \leq t_j, & \text{for some } j, \\
 & & \text{or } s_j \leq \bigvee t_i, & \text{for some } j.
\end{array}$$

■

1.4 Game interpretation of Whitman's solution

Whitman's solution to the word problem on free lattices gives rise to an effective algorithm. We shall adopt a *game theoretic* approach to comprehend the construction of this algorithm. Given two terms s, t we shall construct a game $\langle s, t\rangle$ with the property that $s \leq t$ if and only if some player, the Mediator, has a winning strategy in the game $\langle s, t\rangle$.

There are two players in $\langle s, t\rangle$: the Mediator, and the Opponents. The positions of this games are pairs (s', t') of subterms of s and t. This game is played on the left and right subterms at the same time. The Mediator is playing with $\wedge$ on s and with $\vee$ on t. The Opponents are playing with $\vee$ on s and with $\wedge$ on t. Mediator's aim is to prove that $s \leq t$, and the aim of Opponents is to contradicts this. That is, Mediator's aim is to reach a final position of the form (x_i, x_j), where $i = j$, and Opponents' aim is to reach a final position of the form (x_i, x_j) where $i \neq j$. The formal definition of the game $\langle s, t\rangle$ follows.

Definition 1.4.1. Given two terms s and t the game $\langle s, t\rangle$ is defined by:

- its position are pairs (s', t') of subterms of s and t,
- the initial position is (s, t),
- the final position (x, y) : is winning for Mediator iff $x = y$, where $x, y \in X$.
- $(x, \bot)$: is losing for Mediator.
  $(x, t_1 \vee t_2)$: Mediator chooses $i \in \{1, 2\}$ and moves to (x, t_i),
- $(\top, x)$ is losing for Mediator.
  $(s_1 \wedge s_2, x)$: Mediator chooses $i \in \{1, 2\}$ and moves to (s_i, x),
- $(s_1 \wedge s_2, t_3 \vee t_4)$
  Mediator chooses $i \in \{1, 2\}$ and moves to $(s_i, t_3 \vee t_4)$,
  or chooses $j \in \{3, 4\}$ and moves to $(s_1 \wedge s_2, t_j)$,
- $(s, t_1 \wedge t_2)$: Opponents choose $i \in \{1, 2\}$ and move to (s, t_i),
  $(s_1 \vee s_2, t)$: Opponents choose $i \in \{1, 2\}$ and move to (s_i, t).

■

Proposition 1.4.2. *Let s, t be two terms. Then, $s \leq t$ if and only if Mediator has a winning strategy in the game $\langle s, t\rangle$.*

The proof of this Proposition is straightforward, we have just rephrased the statement of Theorem 1.3.1 in terms of games and strategies. Deciding whether two terms s and t satisfy $s \sim t$ amounts to check whether Mediator has a winning strategy in both games $\langle s, t\rangle$ and $\langle t, s\rangle$, which can be done in $O(|s|.|t|)$ time, where $|s|$ is the number of subterms [5] of s.

[5] $|s|$ is just the number of vertices of s, if the latter is viewed as a graph.

1.5 Fixed-points

Let $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ be two posets. A function $f : P \longrightarrow Q$ is said to be *monotonic*, or to be *order preserving*, if $p \leq q$ implies $f(p) \leq f(q)$.

Definition 1.5.1. Let L be a lattice and $f : L \longrightarrow L$ be a monotonic mapping .
The *least prefix-point* of f, whenever it exists, is an element $\mu.f \in L$, such that:

(i) $f(\mu.f) \leq \mu.f$,

(ii) if $f(l) \leq l$, then $\mu.f \leq l$

Dually, the *greatest post-fixed point* of f, whenever it exists, is an element $\nu.f \in L$, such that:

(i) $\nu.f \leq f(\nu.f)$,

(ii) if $l \leq f(l)$, then $l \leq \nu.f$

■

Remark 1.5.2.

- Observe that, if the least prefix-point (as well as the greatest postfix-point) exists, then it is *unique*.
- Throughout this thesis, whenever we want to emphasize that the function f depends on the variable x, then we shall write $\mu x.f(x)$ and $\nu x.f(x)$ for $\mu.f$ and $\nu.f$, respectively.

■

Lemma 1.5.3. Let L be a lattice and $f : L \longrightarrow L$, then

$$\mu.f = inf\{\, l \mid f(l) \leq l \,\},$$
$$\nu.f = sup\{\, l \mid l \leq f(l) \,\},$$

■

The following Theorem is known as Knaster-Tarski Theorem [Tar55].

Theorem 1.5.4. Let L be a complete lattice and $f : L \longrightarrow L$ be a monotonic function. Then, both μf and νf exist. ■

Tarski stated this Theorem in [Tar55], and so the theorem is often named as *Tarski's fixed point theorem*. However, Knaster established earlier the result in [Kna27] for the special case when L is the lattice of the powerset algebra.
Later, Davis proved in [Dav55] the converse of Knaster-Tarski theorem, providing a characterization of complete lattices by means of fixed-points. Davis's theorem follows.

Theorem 1.5.5. Let L be a lattice. If any monotonic function $f : L \longrightarrow L$ has a fixed-point, then the lattice L is complete. ■

1.6 μ-Lattices

A lattice L is a μ-lattice is a lattice for which every *unary polynomial* $p(x)$ has got a least prefix-point as well as the greatest post-fixed point[6]. We mean by the phrase "polynomial" the functions built up using the variables, the constants $\top$ and $\bot$, and the lattices operators $\wedge$ and $\vee$. A polynomial is unary if it is defined for all its arguments apart one.
The μ-lattices have been defined in [San02c] as the quasivariety[7] of lattices if the operators μ and ν are axiomatized by means of the identity (i) and the implication (ii) of Definition 1.5.1.

Consider the set of terms T_μ generated by the following grammar:

$$t \;=\; x \mid \top \mid \bot \mid t \wedge t \mid t \vee t \mid \mu x.t_x \mid \nu x.t_x \tag{1.2}$$

[6]Observe that polynomials are used to characterize μ-lattices in the same way the arbitrary monotonic functions are used to characterize complete lattices, Theorems 1.5.4 and 1.5.5

[7]A quasivariety of lattices is a class K that satisfies identities of the form $s_i = t_i$ and implications between identities, which are of the form $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies s = t$.

Given a lattice L, we define the interpretation of terms in $T(X)$ over a lattice L as the partial[8] function

$$\| \ . \ \| : T_\mu \longrightarrow \bigcup_{n \geq 0} L^{(L^n)}$$

where L^X is again the X-fold product lattice of L with itself, and the interpretation of the term $t \in T_\mu$ is given by the function $\|t\|^L : L^{X_t} \longrightarrow L$, as given before for the terms in $T(X)$, and moreover:

- Let $t = \mu x.t_x$. Assume that $\|t_x\|^L$ is defined. If there exists the least prefix-point of the unary function ϕ defined by

  $$\phi(l) = \|t_x\|(v^l), \text{ for each } v \in L^{X_t}$$

  where $v^l(y) = v(y)$ if $y \neq x$ and $v^l(x) = l$. Then we define

  $$\|t\|^L(v) = \mu.\phi$$

  Otherwise $\|t\|$ is undefined.

- The interpretation of $t = \nu x.t_x$ is obtained from the above one by substituting each symbol μ with the symbol ν, and the phrase *least prefix-point* with the phrase *greatest postfix-point*.

Definition 1.6.1. A lattice L is a μ-lattice if the interpretation $\| \ . \ \| : T_\mu \longrightarrow \bigcup_{n \geq 0} L^{(L^n)}$ is a total function. ■

For instance, complete lattices, finite lattices and distributive lattices are all μ-lattices.

[8] The interpretation whould be a toatal fucntion if L is a complete lattice, due to Theorem 1.5.4, or if L is a μ-lattice.

Free μ-lattices.

Definition 1.6.2. Let L_1, L_2 be two μ-lattices. A monotonic function $f : L_1 \longrightarrow L_2$ is a *morphism* of μ-lattices if for every $t \in T_\mu$ the following hold:

$$f \circ (\|t\|^{L_1}) = \|t\|^{L_2} \circ f^{X_t}$$

Where f^X is the X-fold product function of f with itself.
The statement would be rephrased by making, for every $t \in T_\mu$, the following diagram to commute:

$$\begin{tikzcd}
L_1^{X_t} \arrow[r, "\|t\|"] \arrow[d, "f^{X_t}"'] & L_1 \arrow[d, "f"] \\
L_2^{X_t} \arrow[r, "\|t\|"] & L_2
\end{tikzcd}$$

■

The notion of free μ-lattice is given in a similar way of that of free lattices, Definition1.2.1

Definition 1.6.3. Let F be a μ-lattice and $X \subseteq F$. We say that F is *freely generated* by X if, given any map $f : X \longrightarrow L$ from X to some arbitrary μ-lattice L, then there exists a unique homomorphism of μ-lattices $\tilde{f} : F \to L$ such that $\tilde{f}(x) = f(x)$, for each $x \in X$. ■

The word problem for μ-lattices was solved in [San02c]. Given two μ-terms s and t there was defined a game $\langle s, t\rangle$ that generalises the Mediator-Opponents games for lattices, Definition 1.4.1, and has the important completeness and soundness property. That is, Mediator has a winning strategy in the game $\langle s, t\rangle$ if and only if $\|s\|^L \leq \|t\|^L$ holds in every complete lattice L [9]. As before, there are two players in the game $\langle s, t\rangle$, the Mediator and

[9] Or equivalently in every μ-lattice L.

the Opponents. The positions of $\langle s, t\rangle$ are pairs of subterms of s and t. The Mediator is playing with $\wedge$ and ν on s and with $\vee$ and μ on t. The Opponents is playing with *join* and μ on s and with $\wedge$ and ν on t.

Before the definition of the game $\langle s, t\rangle$ we change slightly the definition of the subterms of a μ-term. If s is a μ-term, then the set of subterms of s, denoted by $sub(s)$, is defined as follows, if s is a constant or a free variable then $sub(s) = \{ s \}$, if $s = \{ s_1 \ op \ s_2 \}$ where $op \in \{ \wedge, \vee \}$ the $sub(s) = \{ s \} \cup sub(s_1) \cup sub(s_2)$, if s is a bound variable then, $sub(s) = sub(\theta x.t)$, where $\theta x.t$ is the unique term where the occurrence of x is under the scope of θ, and if t is of the form $\theta x.t$ then $sub(s) = \{ s \} \cup sub(t)$.

Definition 1.6.4. Let s, t be two μ-terms. The game $\langle s, t\rangle$ is defined in a similar way of the Mediator-Opponents game on lattice terms given in Definition 1.4.1, moreover Opponents play with μ on s and with ν on t, i.e. from a position of the form $(\mu x.s', t)$, Opponents move to (s', t), and from a position of the form $(s, \nu x.t')$ Opponents move to (s, t').
The winner in a finite play is given as in Definition 1.4.1.
An infinite play is winning for Mediator if and only if *(i)* the extreme fixed point variables generated infinitely often in s is a μ-variable, or *(ii)* the extreme fixed point variable generated infinitely often on t is a ν-variable. ■

Theorem 1.6.5. [San02c]. Let s, t be two μ-terms, then Mediator has a winning strategy in $\langle s, t\rangle$ if and only if $\|s\|^L \leq \|t\|^L$ holds in every complete lattice L. ■

If s, t are μ-terms, then let us define $s \sim t$ if and only if Mediator has a winning strategy in both games $\langle s, t\rangle$ and $\langle t, s\rangle$. The following Theorem is the main result of [San02c].

Theorem 1.6.6. The set $T_\mu \backslash \sim$ is a free μ-lattice. ■

Chapter 2

Preliminaries on Graphs

This chapter gives an introduction to most of the terminology and notation about graphs used later in this thesis. Most of the terminology and notation are standard.

2.1 Directed graphs, paths, subgraph ...

Definition 2.1.1. A *directed graph* (or shortly a *digraph*) $G = (V_G, E_G)$ is a set (finite or not) of vertices and a binary relation $E_G \subseteq V_G \times V_G$. The set E_G is called the edges of G. An edge of the form (v, v) is called a *self loop*. ■

A graph with vertex set V is said to be a graph *on* V. The number of vertices of G is denoted by $|V_G|$, and the number of its edges is denoted by $|E_G|$. The *successors* of a vertex v is the set $\{ w \in V_G \mid (v, w) \in E_G \}$. The *predecessors* of v is the set $\{ w \in V_G \mid (w, v) \in E_G \}$. All the graphs in this thesis are finite.

Definition 2.1.2. A graph $G = (V_G, E_G)$ is *undirected* or *symmetric* if the relation E_G is symmetric, that is whenever $(v, v') \in E_G$ then $(v', v) \in E_G$. ■

If G is undirected, then an edge of G is denoted by vw. In this case v and w are said to be *adjacent* or *neighbours*. If all the vertices of G are

pairwise adjacent then G is *complete*. A complete graph on n vertices is a K_n. Sometimes we call the K_n an *n-clique*.

Definition 2.1.3. A (undirected) graph G is *bipartite* if its vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2. ■

Definition 2.1.4. If G is a digraph, then a *path* in G is a sequence of the form $\pi = g_0 g_1 \dots g_n$ such that $(g_i, g_{i+1}) \in E_G$ for $0 \leq i < n$. A path is *simple* if $g_i \neq g_j$ for $i, j \in \{0, \dots, n\}$ and $i \neq j$. The integer n is the *length* of π, g_0 is the *source* of π, and g_n is the *target* of π . The path π is a *cycle* if $g_0 = g_n$. ■

Definition 2.1.5. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two digraphs. We say that G and H are *isomorphic* if there exists a bijection $\phi : V_G \to V_H$ with $(v, w) \in E_G$ if and only if $(\phi(v), \phi(w)) \in E_H$, for all $v, w \in V_G$. The map ϕ is called and *isomorphism*. ■

We do not distinguish between isomorphic digraphs. For example we consider that there exists just one graph on a single vertex, and there are three digraphs on two vertices.

Definition 2.1.6. Let G, H be two digraphs. If $V_G \subseteq V_H$ and $E_G \subseteq E_H$ then we say that G is a *subgraph* of H and write $G \subseteq H$. If $G \subseteq H$ and $G \neq H$, then we say that G is a *proper subgraph* of H.
If $G \subseteq H$ and G contains *all* the edges (v, w) of H with $v, w \in W$ then G is an *induced subgraph* of H; we say that G is the induced subgraph of H by W and write $G = H[W]$. ■

2.2 Connectivity

Definition 2.2.1. A digraph G is *connected* if for every vertices v_1, v_2 in V_G, there exists a path from v_1 to v_2 *or* there exists a path from v_2 to v_1. ■

Definition 2.2.2. A digraph G is *strongly connected* if for every vertices v_1, v_2 in V_G, there exists a path from v_1 to v_2 *and* there exists also a path from v_2 to v_1. ■

2.3 Trees and trees with back-edges

Definition 2.3.1. A pointed digraph $\langle V, E, v_0 \rangle$ of root v_0, is a *tree* if for each $v \in V$ there exists a unique path from v_0 to v. Let $v_1, v_2 \in V$, if there exists a path from v_1 to v_2 then v_1 is called an *ancestor* of v_2, and v_2 is called a *descendant* of v_1. ■

Definition 2.3.2. A *tree with back-edges* is a tuple $\mathcal{T} = \langle V, T, v_0, B \rangle$ such that $\langle V, T, v_0 \rangle$ is a tree, and $B \subseteq V \times V$ is a second set of edges such that if $(x, y) \in B$ then y is an ancestor of x in the tree $\langle V, T, v_0 \rangle$. We shall refer to edges in T as tree edges and to edges in B as back edges. We say that $r \in V$ is a return of $\mathcal{T}$ if there exists $x \in V$ such that $(x, r) \in B$. ■

If the graph G is equipped with extra data then the notion of isomorphism must respects this data. In particular, if G and h are rooted trees, then an isomorphism $\phi : G \to H$ must sends the root of G to the root of H.

A digraph is said to be a tree with back edges if it is isomorphic to some tree with back-edges.

Part I

Hierarchies and Expressiveness

Chapter 3

Hierarchies in μ-Calculi

We recall the rudiments of the notion of μ-calculus in its abstract sense as defined by Arnold and Niwinski [AN01, §2]. Then, we introduce the three hierarchies related to μ-calculi: the alternation depth hierarchy, the star height hierarchy, and the variable hierarchy. We shall discuss the relation between the star height and the variable hierarchy. Under some restrictions, we shall prove that the variable hierarchy is refinement of the star-height hierarchy, meaning that the non-collapse of the variable hierarchy implies the non-collapse of the star height hierarchy. The proof relies on the combinatorial characterization of the two hierarchies.

The most known and well studied measure of the complexity of the modal μ-calculus is the *alternation depth* of its formulae, that is, the number of alternations between the least and greatest fix point constructors. As a consequence, it is possible to construct a hierarchy of classes of formulae according to the alternation depth measure. The first level of this hierarchy contains for example PDL, CTL ... We mean that one can encode PDL and CTL formulae with modal μ-formulae of alternation depth one. This is not the case of Parikh's Game Logic GL [PP03], since GL intersects with all the classes of the hierarchy [Ber03]. As a consequence, it is not possible to use the alternation depth hierarchy to prove that GL is less expressive than L_μ. Another option was pursed in [BGL07] by encoding GL formulae with μ-formulae of just

two fixed point variables. This encoding gave rise to an orthogonal hierarchy, that is the hierarchy of L_μ formulae induced by the number of fixed point variables. The non collapse of the variable hierarchy [BGL02, BL05, Ber05] allowed to separate GL from L_μ.
Historically, the variable hierarchy problem was asked by Immerman and Poizat [Imm95] in the context of back and forth games $\langle A, B\rangle$. The aim was of deciding whether a logical formulae of fixed number of bound variables is able or not to distinguish the two structures A and B. Or, in other words, they were asking whether the two structures A and B are models of the same logic formulae where the number of bound variables is bounded.

The third hierarchy is the *star height hierarchy*: the formulae are classified into levels of a hierarchy according to the nested depth of the application of the same fixed point operator (or the iteration operator). The star-height problem was first asked in formal language theory and consists in answering whether all regular languages can be expressed using regular expressions of bounded star height. This question have been answered by Eggan in [Egg63], where he gave examples of regular languages of star height n for every $n \in \mathbb{N}$. The star height problem was asked later for regular trees [BC84], the latter are finite or infinite trees with only finite many distinct subtrees, up to isomorphism. Regular trees form the free iteration theory and they might be written by means of iterative theory expressions. These expressions use an iteration operator, denoted $\dagger$, which is in the case of matrix iteration theories is interdefinable with the Kleene's $*$ operator [BÉ93, §9].
The star height problem can be asked in a general way for iteration theories [BÉ93], where the *dagger* $\dagger$ operator is considered, as well as for μ-calculi, if just one fixed operator among $\{\mu, \nu\}$ is considered.
It is worth to note that the study of the star height hierarchy [Egg63, BC84], which is a typical algebraic problem, transfers into graph theoretic problem. In other words, the minimal star height needed is captured by a graph invariant that counts the number of embedded cycles i.e. *the feed back number* considered in [Egg63], or the *rank* considered in [BC84]. The feed back num-

ber and the rank are essentially the same. A similar approach was followed in [Ber05], the study of the variable hierarchy gives rise to a quite different graph invariant, the so called the *entanglement*.

This chapter is devoted to compare, under some assumptions, the star height hierarchy and the variables hierarchy. We shall discuss at the end of this Chapter how the strictness of the variable hierarchy would be of help to deduce the strictness of the star height hierarchy. The key Lemma is that the combinatorial measure characterizing the variable hierarchy (the entanglement) is lower than the combinatorial measure characterizing the star height hierarchy (the rank).

3.1 The μ-calculi: syntax and semantics

Roughly speaking, a μ-calculus in the abstract sense of [AN01] is a set of syntactic entities and a set of formal operations. The latter consists in the fixed point operators μ and ν and the substitution operation. These syntactic entities come with and intended interpretation over a class of complete lattices. Each entity t is interpreted as a monotonic mapping from $t^{ar(t)}$ to L, where $ar(t)$ is the arity of t, that corresponds the free variables of t, and L is a complete lattice. The entities $\mu x.t$ and $\nu x.t$ of a μ-calculus are interpreted as the least and greatest parametrized fixed points of the interpretation of t. The substitution is interpreted by means of the functional composition.

Many syntactic entities can be structured to have a shape of a μ-calculus. For instance, this happened to infinite words [AN01, §5], automata [AN01, §7], and parity games [AS03, SA05]. The interpretation of an automaton, viewed as an entity of the μ-calculus, is the language which it accepts.

Let E be a set of objects or entities and let Var be a fixed countable set of variables. The variables in Var will be denoted by $x, y, z, \ldots$
A mapping $\rho : Var \to E$ is called a *substitution*.
If ρ is a substitution into some set E, x a variable, and e and element of E, we denote by $\rho\{\, e/x \,\}$ the substitution ρ' defined by $\rho'(x) = e$ and $\rho'(y) = \rho(y)$

if $y \neq x$. More generally, if $x_1, \ldots, x_n$ are *distinct* variables and if $e_1, \ldots, e_n$ are elements of E, then

$$\rho\{\, e_1/x_1, \ldots, e_n/x_n \,\}$$

is the substitution ρ' defined by

$$\rho'(y) = \begin{cases} e_i & \text{if } y \in \{\, x_1, \ldots, x_n \,\}, \\ \rho(y) & \text{if } y \notin \{\, x_1, \ldots, x_n \,\} \end{cases}$$

Definition 3.1.1. A μ-calculus is a tuple $\langle T, id, ar, comp, \mu, \nu \rangle$, where

- T is an arbitrary set, its elements are the *objects* of the μ-calculus.
- id is a mapping from Var to T. We denote by $\hat{x}$ the element $id(x)$ in T.
- ar is a mapping associating to each $t \in T$ a subset of Var called the *arity* of t. If $x \in ar(t)$, we say that x occurs *free* in t, and the elements of $ar(t)$ are called the *free variables* of t.
- $comp$ is a mapping associating a term $comp(t, \rho)$ with any term t and any substitution ρ; we shall write also $t[\rho]$.
- μ and ν are two mappings from $Var \times T$ to T, the value of the mapping θ on x and t is written $\theta x.t$, for $\theta = \mu, \nu$.

Moreover, we assume the following axioms:

1. $ar(\hat{x}) = \{\, x \,\}$,
2. $ar(t[\rho]) = ar'(t, \rho)$, where $ar'(t, \rho) = \bigcup_{y \in ar(t)} ar(\rho(y))$,
3. $ar(\theta x.t) = ar(t) \setminus \{\, x \,\}$,
4. $\hat{x}[\rho] = \rho(x)$, for $x \in Var$,
5. $t[\rho] = t[\rho']$ if $\rho | ar(t) = \rho' | ar(t)$,

6. $(t[\rho])[\pi] = t[\rho \star \pi]$, where $\rho \star \pi$ is the substitution defined by $\rho \star \pi(x) = \rho(x)[\pi]$,

7. if $ar'(\theta x.t, \rho) \neq Var$, there exists a variable $y \notin ar'(\theta.t, \rho)$ (possibly equal to x), such that $(\theta x.t)[\rho] = \theta y(t[\rho\{\, \hat{y} \backslash x \,\}])$.

■

Axiom 7 explains how to perform substitution in presence of bound variables using a suitable renaming of variables: this is called the α-conversion in the λ-calculus. Let $t = \mu x.f(x, y)$ and let ρ be a substitution that substitutes $h(x)$ for y and keeps x unchanged. A naive syntactic substitution gives $\mu x.f(x, h(x))$ which is not the intended one. The reason is that the free occurrence of x in $h(x)$ becomes in the scope of the quantifier μx and becomes bound. However, we write $\mu z.f(z, y)$ instead of $\mu x.f(x, y)$ and now we can safely apply the substitution, getting $\mu z.f(z, h(x))$.

Semantics

Let $\langle T, id, ar, comp, \mu, \nu \rangle$ a μ-calculus. A *μ-interpretation* of T is a pair (L, I) where L is a complete lattice and I is a function that with each μ-term t associates a monotonic mapping $t : L^{ar(t)} \to L$ such that the substitution is interpreted as the functional composition and $\mu x.t$ (resp. $\nu x.t$) is interpreted as the least (greatest) parametrized fixed point of the interpretation of t.

3.1.1 The vectorial μ-calculus, system of equations

A fixed point of a function $f : L \to L$ is the solution of the equation $f(x) = x$. In general, we can consider a function for which the argument ranges over a product of lattices, say $L_1 \times \cdots \times L_n$. In this case, the function f may be identified with a tuple of functions $\langle f_1, \ldots, f_n \rangle$ where f_i is the function for which the argument ranges over L_i, defined as $f_i = \pi_i \circ f$ where π_i is the projection of $L_1 \times \cdots \times L_n$ on L_i. We say that f is monotonic if each f_i is monotonic, for $i = 1, \ldots, n$. The computation of fixed points on a product

of lattices may be viewed as solving *system of equations*. Let x_i a variable that ranges over L_i. A system of equation is a sequence

$$\begin{aligned}
&x_1 =_\theta f_1(x_1, x_2, \ldots, x_n, x),\\
&\vdots\\
&x_i =_\theta f_1(x_1, x_2, \ldots, x_n, x),\\
&\vdots\\
&x_n =_\theta f_n(x_1, x_2, \ldots, x_n, x),
\end{aligned}$$

where $\theta = \mu, \nu$.

We can consider a sequence of systems of equations, where each system is solved for the least or the greatest solution. This leads to the notion of *vectorial* fixed point terms. The latter can be cooperated with the notion of μ-calculus given in Section 3.1. The idea is that a vectorial μ-term will be presented as a vector of ordinary (i.e. scalar) μ-terms. Then, the axioms of the scalar μ-calculus are adapted to the vectorial μ-calculus. The formal definitions can be found in [AN01, §2.7].

3.1.2 The linking lemma: Bekič principle

The vectorial fixed points are, in general, more concise and intuitive, than the scalar ones. However, the vectorial formalism and the scalar one have the same expressive power. This is a consequence of a key Lemma, known as the *Bekič principle*.

Proposition 3.1.2. *[Bekič principle, [c84]] Let L_1 and L_2 be complete lattices and $f_1 : L_1 \times L_2 \to L_1$, $f_2 : L_1 \times L_2 \to L_2$ be monotonic mappings w.r.t. their arguments. Let $g_1(y) = \theta x.f_1(x, y)$, and let $(a, b) = \theta(x, y).(f_1(x, y), f_2(x, y))$. Then $b = \theta y.f_2(g_1(y), y)$ and $a = g_1(b)$.*

Generalized Bekič principle

The Bekič principle stated above may be generalized to any system of

equations. A solution of a system of equations may be obtained by means of the following transformations. First, we compute the solution of the first equation, and then we substitute this solution in the remaining equations. After this, we solve the next equation and we substitute again its solution in the remaining equation. By iterating this process we obtain step by step the solution of the desired fixed points. This is known also as the *Gauss elimination principle*, its proof can be found in [AN01, §1.4]

3.2 The alternation-depth hierarchy

We begin by recalling the most known and the most deeply analyzed hierarchy in μ-calculi. The least and greatest fixed point operators μ and ν behaves like logical quantifiers $\exists$ and $\forall$. Their use can be nested in a non naive way. An interesting question arises: does the nested use of least and greatest fixed point operators increases the expressive power of the language? In the next we shall recall a formal hierarchy that measures the complexity of objects by means of number of alternations between μ and ν.

3.2.1 Clones in μ-calculi

Definition 3.2.1. Let $\mathcal{T}_\mu$ be a μ-calculus and T_μ be the objects of $\mathcal{T}_\mu$, i.e. the first item of Definition 3.1.1. A subset $C \in T_\mu$ is a called a *clone* if it contains $id(Var)$ and is closed under composition, that is if $t \in C$ and if ρ is a substitution such that $\rho(y) \in C$ for any $y \in ar(t)$ then $t[\rho] \in C$.
Moreover, a clone C is a *μ-clone* if it is additionally closed under the μ-operation, that is if t in C then $\mu x.t$ is also in C. Similar definition for *ν-clones*.
Finally, C is called a *fixed-point clone* if it is both a μ-clone and a ν-clone, that is, if it is closed under both μ and ν. ■

Observe that, if $T' \in T_\mu$, then there exists a least clone containing T', we shall denote it by $Comp(T')$, the latter being the closure under the compo-

sition operator. Similarly, there exists a least μ-clone, a least ν-clone, and a least fixed-point clone containing T', we shall denote them respectively by $\mu(T'), \nu(T')$ and $fix(T')$.

Definition 3.2.2. Let $\mathcal{T}_\mu$ be a μ-calculus and T_μ be the objects of $\mathcal{T}_\mu$. Given $T' \subseteq T_\mu$ we define a hierarchy of the elements of T_μ with respect to T' by:

$$\Sigma_0(T') = \Pi_0(T') = Comp(T')$$

and, for $k < \omega$,

$$\begin{aligned} \Sigma_{k+1}(T') &= \mu(\Pi_k(T')) \\ \Pi_{k+1}(T') &= \nu(\Sigma_k(T')) \end{aligned}$$

■

The alternation hierarchy considered above is obviously strict since the μ-terms are considered from the syntactic point of view. However, this in not the case if we consider the alternation hierarchy for the interpretation of the μ-terms. In the latter case, we ask for the minimal number of alternations needed in any μ-terms t' such that the interpretation of t and that of t' are the same.

3.3 The star height hierarchy

The star height problem was asked in formal language theory and consists in answering whether all regular languages can be expressed using regular expressions of limited star height, i.e. with a limited nesting depth of Kleene stars $*$. This question have been answered by Eggan in [Egg63], he gave examples of regular languages of star height n for every $n \in \mathbb{N}$. The star height problem can be asked in a general way for iteration theories, where the *dagger* $\dagger$ operator is considered, as well as for μ-calculi. In this section we focus on the combinatorial part of the star height: the solution of the star height problem requires a digraph complexity measure called the *rank*.

The variable hierarchy problem requires also for another digraph complexity measure, that will be discussed in Section 3.4. The comparison between the two hierarchies will be given in terms of the comparison of their related digraph measures.

Definition 3.3.1. Let t be a μ-term. The star height of t is defined as follows:

$$h(t) = \begin{cases} 0 & \text{if } t \in \Pi_0 \cap \Sigma_0 \\ Max\{\, h(t'), h(\rho(x_1)), \ldots, h(\rho(x_n)) \,\} & \text{if } t = comp(t', \rho) \text{ where } x_i \in ar(t') \\ 1 + h(t') & \text{if } t = \theta x.t \text{ where } \theta = \mu, \nu \end{cases}$$

■

3.3.1 The *rank*: a digraph measure for the star height

In [Egg63] Eggan defined a complexity measure on graphs, called the *feed back number*, that captures the minimal star height. Intuitively, the feed back number describes the complexity of a digraph in terms of its cycle structure. This measure have been formulated in a more natural way by Courcelle et al. in [BC84], and they rename it the *rank*. There, they solved the star height problem for regular trees, and showed that the minimal star height is exactly the rank of the minimal graph of the tree.

We recall, Definition 2.2.2, that a digraph G is *strongly connected* if for each two vertices $v_1, v_2 \in V_G$ there exists a path in G from v_1 to v_2. A *strongly connected component* of G is a maximal strongly connected subgraph of G. We shall write $scc(v)$ for the strongly connected component of G that contains the vertex v. A strongly connected component is *trivial* if it reduces to a single vertex without loops. We shall write $SCC(G)$ for the set of the non trivial strongly connected components of G. We define a transitive relation $\prec$ on $SCC(G)$ as follows: $G' \prec_G G''$ if and only if $G' \neq G''$ and there is a path in G from G' to G''. If $G_1 \in SCC(G)$ then let $G_1^{\succ} = \{\, G' \in SCC(G) \mid G_1 \prec G' \,\}$.

Definition 3.3.2. The *rank* of a digraph G is defined as follows:

- if $SCC(G) = \emptyset$, then $r(G) = 0$,

- if $SCC(G) = \{ G \}$, then $r(G) = 1 + Min\{ r(G \setminus v) \mid v \in V_G) \}$,
- otherwise, $r(G) = Max\{ r(G') \mid G' \in SCC(G) \}$.

■

3.3.2 Thief and Cops games for the rank

To establish the relation between the rank and the entanglement, in a first step, we shall rephrase the definition of the rank in terms of games and strategies in the most direct way.

Definition 3.3.3. The rank game $\mathscr{R}(G, k), k \geq 0$ played alternatively between a Thief and Cops on the digraph G is defined as follows.

- Its positions are of the form (G', P, n) where $0 \leq n \leq k$, G' is a subgraph of G, and $P \in \{ Thief, Cops \}$ such that
  - the starting position is $(G, Thief, k)$,
  - if $SCC(G) = \emptyset$ or $n = 0$ then the play halts.
- If $SCC(G) = \{ G_1, \ldots, G_l \}$, (possibly $l = 1$) then Thief chooses some G_i and moves from $(G, Thief, n)$ to $(G_i, Cops, n)$.
- If the position is $(G, Cops, n)$ then[1] Cops chooses $v \in V_G$ and moves to $(G \setminus v, Thief, n - 1)$,
- Thief wins a play if and only if its final[2] position (G', P, n) is such that $SCC(G) \neq \emptyset$ and $n = 0$.

We define $\mathscr{R}(G)$ to be the minimum k such that Cops have a winning strategy in the rank game $\mathscr{R}(G, k)$ ■

Proposition 3.3.4. *Let G be a digraph, then $\mathscr{R}(G)$ equals $r(G)$.*

[1] Observe that in this case G is strongly connected.
[2] Observe that there is no infinite play.

Proof. We have just rephrased the definition of the rank by means of games and strategies following the game theoretic tradition. That is, Cops play the role of the *minimizer* and Thief plays the role of the *maximizer*. □

Now, in order to compare the rank with the entanglement in an easy way, we shall give a useful variant of rank games. The idea is that, whenever $SCC(G) = \{ G_1, \ldots, G_l \}$ and Thief moves from $(G, Thief, n)$ to $(G_i, Cops, n)$ then he is allowed later, and at any moment, to *come back* and move to $(G_j, Cops, n)$ where $G_i, G_j \in SCC(G)$ and $G_i \prec G_j$.

Definition 3.3.5. Let G be a digraph and $k \geq 0$. We define the rank game with *come back* $\mathscr{R}^B(G, k)$ between Thief and Cops on the digraph G as follows:

- Its positions are of the form (G', P, L, n) where $0 \leq n \leq k$, G' is a subgraph of G, $P \in \{ Thief, Cops \}$, and L is a set of quadruplet of the form (G, P, L, n) such that
  - the starting position is $(G, Thief, \emptyset, k)$,
  - if ($SCC(G) = \emptyset$ or $n = 0$) and $L = \emptyset$ then the play halts.

  If $SCC(G) = \{ G_1, \ldots, G_l \}$, (possibly $l = 1$) then Thief has two kind of moves:

- he chooses some $G_i \in SCC(G)$ and moves from

$$(G, Thief, L, n) \to (G_i, Cops, G_i^{\succ} \times (Cops, L, n) \cup L, n) \qquad \text{(forward move)}$$

- or he moves from

$$(G', Thief, L, n) \to B \text{ where } B \in L \qquad \text{(come back move)}$$

If the position is $(G, Cops, n)$ then[3] Cops chooses $v \in V_G$ and moves to $(G \setminus v, Thief, n-1)$,

[3]Observe that in this case G is strongly connected.

Thief wins a play if and only its final position (G', P, n) is such that $SCC(G) \neq \emptyset$ and $n = 0$.

We define $\mathscr{R}^B(G)$ to be the minimum k such that Cops have a winning strategy in the rank game $\mathscr{R}^B(G, k)$ ■

Fact 3.3.6. There is no infinite play in the game $\mathscr{R}^B(G, k)$. ■

Proof. We can construct a parity game $\mathcal{P}(G, k)$ that mimics the game $\mathscr{R}^B(G, k)$ with the following properties:

(i) its positions are the positions of $\mathscr{R}^B(G, k)$,

(ii) its underlying graph is a tree with back edges,

(iii) a priority function ω affects a number to its positions in such a way if two sons $s_1 = (G_1, Cops, L, n)$ and $s_2 = (G_2, Cops, L, n)$ satisfies $G_1 \prec G_2$ then $\omega(s_1) < \omega(s_2)$,

(iv) its forward edges are the forward moves of $\mathscr{R}^B(G, k)$,

(v) if there is a come back move from s to t then $\mathcal{P}(G, k)$ contains a back edge from s to the father of t,

(vi) a position of $\mathcal{P}(G, k)$ is labeled by σ (resp. by π) if Cops (resp. Thief) move from it.

The outcome of the play in $\mathcal{P}(G, k)$ is similar to that of parity games apart that (1) π wins if he is able to reach a final position with the desired properties provided in Definition 3.3.5, and (2) if π moves from a position p to q through a back edge, meaning that there is a unique simple path $pp_1 \ldots p_n q$ in $\mathcal{P}(G, k)$, then π must move from p to p' such that $\omega(p_1) < \omega(p')$. The latter condition ensures that the play does not proceed on the cycles of $\mathcal{P}(G, k)$. Therefore there is no infinite play in $\mathscr{R}^B(G, k)$. □

Lemma 3.3.7. Let G be a digraph, then Cops have a winning strategy in $\mathscr{R}^B(G, k)$ if and only if they have a winning strategy in $\mathscr{R}(G, k)$. Therefore $r(G) = \mathscr{R}(G) = \mathscr{R}^B(G)$. ■

Proof. First, if Thief has a winning strategy in $\mathscr{R}(G,k)$ then he also has a winning strategy in $\mathscr{R}^B(G,k)$ i.e. the latter being without using come back moves.
Second, if Thief has a winning strategy in $\mathscr{R}^B(G,k)$ i.e. with come back move of the form $(G', Thief, L, n) \to B$ then he was able to moves before to B.
This would be more easy to see if we consider the game $\mathcal{P}(G,k)$ described in the proof of Fact 3.3.6. A winning strategy for Thief in $\mathcal{P}(G,k)$ that contains back edges would be transformed into a winning strategy for him that does not contain back edges. □

3.4 The variable hierarchy

The variable hierarchy problem was introduced by Immerman and Poizat [Imm95] in the context of back and forth games $\langle A, B\rangle$. The aim was of deciding whether the logical formulae of fixed number of variables are able or not to distinguish the two structures A and B. The idea is that each player has got a fixed number p of tokens and during the play he mark the position with a token in such a way the number of tokens on both sides is equal. During the game each player can indeed replace the same token. The outcome of the play is as the standard back and forth game apart that only the positions which are marked by tokens are considered. The existence of a winning strategy for the prover with p tokens ensures that the two structures are models of formulae of at most p variables.
Later, the variable hierarchy problem was asked for the propositional modal μ-calculus [BGL02, BL05, Ber05] in order to answer the open question whether Parikh Game Logic [PP03] is a strict subset of modal μ-calculus. Parikh question was answered affirmatively, Game logic is less expressive than μ-calculus, in [BL05] as a consequence of two facts: *(i)* Game Logic is embeddable in the two variable fragment of modal μ-calculus, and *(ii)* the hierarchy of modal μ-calculus – made up according to the number of fixed point variables in μ-formulae – does not collapse.

Another consequence of Berwanger's et. all results is the formalization of Immermann's and Poizat's token games. In other words Berwanger's et. all precised that the fact that the minimum number of variables roughly needed in a μ-formula transfers into a complexity measure on the underlying graph of the formula. This complexity measure is known as the *entanglement* and it turns out to be the main tool used in analyzing the variables hierarchy problem in μ-calculi. Roughly speaking, the entanglement is the combinatorial part of the variable hierarchy. A major consequence is that the comparison between the star height hierarchy and the variable hierarchy transfers, under some assumptions, into a comparison of the their related combinatorial parts, i.e. between the rank and the entanglement.

3.4.1 Entanglement

The entanglement of a finite digraph G, denoted $\mathscr{E}(G)$, was defined in [BG05] by means of some games $\mathscr{E}(G,k)$, $k = 0, \ldots, |V_G|$. The game $\mathscr{E}(G,k)$ is played on the graph G by Thief against Cops, a team of k cops. The rules are as follows. Initially all the cops are placed outside the graph, Thief selects and occupies an initial vertex of G. After Thief's move, Cops may do nothing, may place a cop from outside the graph onto the vertex currently occupied by Thief, may move a cop already on the graph to the current vertex. In turn Thief must choose an edge outgoing from the current vertex whose target is not already occupied by some cop and move there. If no such edge exists, then Thief is caught and Cops win. Thief wins if he is never caught. The entanglement of G is the least $k \in N$ such that k cops have a strategy to catch the thief on G. It will be useful to formalize these notions.

Definition 3.4.1. The entanglement game $\mathscr{E}(G,k)$ of a digraph G is defined by:

- Its positions are of the form (v, C, P), where $v \in V_G$, $C \subseteq V_G$ and $|C| \leq k, \quad P \in \{Cops, Thief\}$.
- Initially Thief chooses $v_0 \in V_G$ and moves to $(v_0, \emptyset, Cops)$.

- Cops can move from $(v, C, Cops)$ to $(v, C', Thief)$ where C' can be
  - C : Cops skip,
  - $C \cup \{ v \}$: Cops add a new Cop on the current position,
  - $(C \setminus \{ x \}) \cup \{ v \}$: Cops move a placed Cop to the current position.
- Thief can move from $(v, C, Thief)$ to $(v', C, Cops)$ if $(v, v') \in E_G$ and $v' \notin C$.

Every finite play is a win for Cops, and every infinite play is a win for Thief. ■

$\mathscr{E}(G)$, the entanglement of G, is the minimum $k \in \{ 0, \ldots, |V_G| \}$ such that Cops have a winning strategy in $\mathscr{E}(G, k)$.

The following Proposition provides a useful variant of entanglement games.

Proposition 3.4.2. *Let $\widetilde{\mathscr{E}}(G, k)$ be the game played as the game $\mathscr{E}(G, k)$ apart that Cops is allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

- $(g, C, Cops) \to (g, C', Thief)$ *(generalized skip move),*
- $(g, C, Cops) \to (g, C' \cup \{ g \}, Thief)$ *(generalized replace move),*

where in both cases $C' \subseteq C$. Then Cops has a winning strategy in $\mathscr{E}(G, k)$ if and only if he has a winning strategy in $\widetilde{\mathscr{E}}(G, k)$.

Proof. Since every Cops' move in the game $\mathscr{E}(G, k)$ is a Cops' move in the game $\widetilde{\mathscr{E}}(G, k)$, and since there is no new kind of moves for Thief in the game $\widetilde{\mathscr{E}}(G, k)$, then a Cops' winning strategy in $\mathscr{E}(G, k)$ can be used to let Cops win in $\widetilde{\mathscr{E}}(G, k)$.

In the other direction, a winning strategy for Cops in $\widetilde{\mathscr{E}}(G, k)$ gives rise to a winning strategy for Cops in $\mathscr{E}(G, k)$ as follows.

Each position (g, C, P) of $\mathscr{E}(G, k)$ is matched by a position (g, C^-, P) of $\widetilde{\mathscr{E}}(G, k)$ such that $C^- \subseteq C$. A Thief's move $(g, C, Thief) \to (g', C, Cops)$

in $\mathscr{E}(G,k)$ shall be simulated by the move $(g, C^-, Thief) \to (g', C^-, Cops)$ in $\widetilde{\mathscr{E}}(G,k)$. Note that Thief can perform such a move, since if $g' \in C^-$ then already $g' \in C$.

Assume that the position $(g, C_0, Cops)$ of $\mathscr{E}(G,k)$ is matched by the position $(g, C_0^-, Cops)$ of $\widetilde{\mathscr{E}}(G,k)$. From $(g, C_0^-, Cops)$, Cops' winning strategy may suggest two kinds of moves.

It may suggest a generalized skip $(g, C_0^-, Cops) \to (g, C_1^-, Thief)$ with $C_1^- \subseteq C_0^-$. In this case, Cops skip on from the related position $(g, C_0, Cops)$, so that the new position $(g, C_0, Thief)$ is matched by $(g, C_1^-, Thief)$.

Otherwise, Cops' winning strategy in $\widetilde{\mathscr{E}}(G,k)$ may suggest a generalized replace move $(g, C_0^-, Cops) \to (g, C_1^- \cup \{ g \}, Thief)$. If $|C_0| < k$, then Cops perform the add move $(g, C_0, Cops) \to (g, C_0 \cup \{ g \}, Thief)$. Notice that $C_1^- \cup \{ g \} \subseteq C_0^- \cup \{ g \} \subseteq C_0 \cup \{ g \}$. If $|C_0| = k$, then observe that $C_0 \setminus C_1^-$ is not empty: we have $C_1^- \subseteq C_0^- \subseteq C_0$ and $|C_1^-| < k$, since $g \notin C_1^-$ and $|C_1^- \cup \{ g \}| \leq k$. Consequently we can pick $x \in C_0 \setminus C_1^-$ such that $x \neq g$, since $g \notin C_0$. Therefore Cops simulate the move $(g, C_0^-, Cops) \to (g, C_1^- \cup \{ g \}, Thief)$ of $\widetilde{\mathscr{E}}(G,k)$ with the replace move $(g, C_0, Cops) \to (g, C_0 \setminus \{ x \} \cup \{ g \}, Thief)$ on $\mathscr{E}(G,k)$. Notice again that the invariant $C_1^- \cup \{ g \} \subseteq C_0 \setminus \{ x \} \cup \{ g \}$ is maintained. $\square$

3.4.2 An ad hoc variant of entanglement games

The game theoretic definition of the entanglement, Definition 3.4.1 and even the variant given in Proposition 3.4.2, refers to some rules which are not very close to the rules of the rank games with come back which characterize the rank, Definition 3.3.5. And hence we can not establish the relation between the rank and the entanglement in an easy way. Therefore we shall give an equivalent variant of entanglement games, denoted $\mathscr{E}^V(G,k)$, with the property that its rules are close to those of the rank games.

First we explain informally the new features of this game w.r.t the games for entanglement. In the game $\mathscr{E}^V(G,k)$ Cops are allowed to skip, add a cop, replace a cop, retire a number of cops, and moreover we would like that they

can put cop on a vertex situated anywhere in the graph. However, the latter move is not allowed by entanglement rules. In order to make it possible, Cops should keep in reserve a set Vir of *virtual* cops for this purpose: whenever Cops decide to put a cop on an arbitrary vertex w then they should reserve a cop for this purpose and this cop can not be used until Thief visits vertex w. And at this moment, the virtual cop must be placed on w.

Definition 3.4.3. The game $\mathscr{E}^V(G,k)$ is defined as the entanglement game $\widetilde{\mathscr{E}}(G,k)$ apart that its positions are of the form (v, C, Vir, P) where $Vir \subseteq V_G$ and $|C \cup Vir| \leq k$. Besides the old Cop's move[4], the latter act on the set C, Cops can move from $(v, C, Vir, Cops)$ to $(v, C', Vir', Thief)$ such that:

- if $v \in Vir$ then Cops must update $C' = C \cup \{ v \}$ and $Vir' = Vir \setminus \{ v \}$,
- if $v \notin Vir$ then Cops may update $Vir' = Vir \cup \{ w \}$ where $w \in V_G$,
- if $v \notin Vir$ then Cops may update $Vir' = Vir \setminus A$ where $A \subseteq Vir$.

■

Lemma 3.4.4. Let G be a digraph. Cops have a winning strategy in $\mathscr{E}^V(G,k)$ if and only if they have a winning strategy in $\widetilde{\mathscr{E}}(G,k)$. ■

Proof. First, a winning strategy for Cops in $\widetilde{\mathscr{E}}(G,k)$ is still winning for Cops in $\mathscr{E}^V(G,k)$, the latter does not refer to virtual cops. In This case every position (v, C, Vir, P) in $\mathscr{E}^V(G,k)$ is matched with the position $(v, C, Cops)$ in $\mathscr{E}(G,k)$ where $Vir = \emptyset$.
Second, a Cops' winning strategy in $\mathscr{E}^V(G,k)$ is mapped to a Cops' winning strategy in $\widetilde{\mathscr{E}}(G,k)$ as follows. Every position (v, C, P) of $\widetilde{\mathscr{E}}(G,k)$ is matched with the position (v, C, Vir, P) of $\mathscr{E}^V(G,k)$.
A Thief's move from v to v' in $\widetilde{\mathscr{E}}(G,k)$ is simulated by the same move from v to v' in $\mathscr{E}^V(G,k)$, indeed this simulation is possible since Thief is allowed to cross virtual cops in $\mathscr{E}^V(G,k)$.

[4] Which are the skip, the add and the generalized replace.

Assume that the position $(v, C, Cops)$ is matched with the position $(v, C, Vir, Cops)$ and consider a Cop's move $M = (v, C, Vir, Cops) \to (v, C', Vir', Thief)$ in $\mathscr{E}^V(G, k)$. The move M is simulated in $\widetilde{\mathscr{E}}(G, k)$ by the move $(v, C, Cops) \to (v, C', Thief)$. □

3.5 The star height hierarchy versus the variable hierarchy

Theorem 3.5.1. Let G be a digraph. The entanglement of G is lower or equal to the rank of G. ■

Proof. To prove that $\mathscr{E}(G) \leq r(G)$ it is enough to prove $\mathscr{E}^V(G) \leq \mathscr{R}^B(G)$. Because Lemma 3.4.4 shows that $\mathscr{E}(G) = \mathscr{E}^V(G)$ and Lemma 3.3.7 shows that $r(G) = \mathscr{R}^B(G)$. Let $k = \mathscr{E}^V(G)$, we shall construct a winning strategy for Cops in the game $\mathscr{E}^V(G, k)$ out of a Cops' winning strategy in $\mathscr{E}^V(G, k)$. Every position of the form (v, C, Vir, P) in $\mathscr{E}^V(G, k)$ is matched with a position of the form (G', P, L, n) such that if the strongly connected component in the subgraph $G \setminus (C \cup Vir)$ which contains v, denoted by $scc(v)$, is not trivial then $scc(v) = G'$.

Let us consider a Thief's move in $\mathscr{E}^V(G, k)$ of the form

$$M = (v, C, Vir, Thief) \to (w, C, Vir, Cops).$$

If $scc(w)$ is trivial, then Cops just skip in $\mathscr{R}^B(G, k)$. Observe that either Thief will reach a vertex without no successors (where he loses), or he enters a non trivial strongly connected component. Otherwise i.e. $scc(w)$ is not trivial, the move M is simulated in $\mathscr{E}^V(G, k)$ according to w:

- $w \in V_{G'}$ then M is simulated by the move $(G', Thief, L, n) \to (G'', Cops, L, n)$ where G'' is the strongly connected component of G' containing w,
- $w \notin V_{G'}$, then the move M is simulated by the come back move $(G', Thief, L, n) \to (G^-, Cops, L', m)$ where $(G^-, Cops, L', m) \in L$ and $w \in V_{G^-}$.

A Cops' move of the form

$$N = (G', Cops, L, n) \rightarrow (G' \setminus w, Thief, L, n-1)$$

in $\mathscr{R}^B(G,k)$ is simulated in $\mathscr{E}^V(G,k)$ according to the nature of the position $(G', Cops, L, n)$.

- if the previous position $(G'', Thief, L'', n)$ that precedes $(G', Cops, L, n)$ was in the same strongly connected component i.e. $G' \subset G''$ (note that $L = L''$), then the move N is simulated either by $(v, C, Vir, Cops) \rightarrow (v, C \cup \{ v \}, Vir, Thief)$ if $v = w$, or by $(v, C, Vir, Cops) \rightarrow (v, C, Vir \cup \{ w \}, Thief)$ otherwise.

- if the position $\gamma = (G', Cops, L, n)$ comes from a come back move $(G'', Thief, L'', m) \rightarrow (G', Cops, L, n)$ then N is simulated by

  $$(v, C, Vir, Cops) \rightarrow (v, C \setminus (C \cap V_{G^-}), Vir \setminus (Vir \cap V_{G^-}), Thief).$$

  Let us define G^-. There exists just one position γ^- such that (i) γ^- has the same predecessor of γ in $\mathcal{P}(G,k)$, the latter being defined in the proof of Fact 3.3.6, and (ii) the position $(G'', Thief, L'', m)$ has been reached from the position γ^-. We define G^- to be the graph associated to γ^-, i.e. γ^- is of the form $(G^-, Cops, L^-, n^-)$.

□

3.5.1 Discussion

In Chapter 5 we shall prove that the variable hierarchy for the lattices μ-calculus, $\mathbb{L}_\mu$, is infinite. Let us argue that this result implies that the star height for $\mathbb{L}_\mu$ is also infinite. The proof of this result proceeds essentially into two steps : (i) the construction of *hard* formulae of arbitrary entanglement, and (ii) any formulae equivalent to a hard one have the same entanglement. To argue that the star height of $\mathbb{L}_\mu$ is also infinite, it suffices to construct hard formulae for which the entanglement equals the rank, which is the case

for the games that we shall construct in section 5.7. One the one hand, it follows from (ii) that any equivalent formulae to a hard one has the same entanglement, on the other hand it follows from Proposition 3.5.1 that the latter formulae has rank greater than the entanglement. This shows that the formulae constructed in such a way are also hard with respect to the star height. Therefore, the star height hierarchy is also infinite. This kind of arguments can be exported in the general case whenever one would like to prove the strictness of star height hierarchy out of the strictness of the variable hierarchy. It suffices to construct hard objects for which the entanglement equals the rank.

Chapter 4

The Lattice μ-Calculus

The lattice μ-calculus $\mathbb{L}_\mu$, is a set of particular syntactic objects that comes with an intended interpretation over some class of lattices. The intuition is that the lattice μ-calculus characterizes the μ-lattices in the same way the arbitrary monotonic functions characterize complete lattices[1]. The syntactic objects are built up out of the signature $\top, \bot, \wedge, \vee, \mu, \nu$, the latter would be interpreted respectively as the supremum, infimum, meet, join, the least and greatest fixed point over a lattice.

If we leave in the background the separation between the syntactic objects and the desired interpretation, we feel that the main aim of $\mathbb{L}_\mu$ is to *speak* about the fixed points in lattices.

When (the syntactic objects of) $\mathbb{L}_\mu$ is interpreted over a given lattice L, we require that such an interpretation would be significant and does not make $\mathbb{L}_\mu$ trivial, i.e. *(i)* on the one hand we want that the required fixed-points exist. This is a shared requirement with the models of μ-Σ-algebra [Niw85], *(ii)* on the other hand we want to keep the most feature of $\mathbb{L}_\mu$ i.e. the μ and ν. Logically speaking, we would not like that the interpretation disappears the quantifiers μ and ν. For instance, if the lattice L is distributive, then

[1] A complete lattice L has the property that each monotonic function from L to L has a fixed point. The converse of this result has been proved in [Dav55], thus establishing a characterization of complete lattices in terms of monotonic functions.

every term is semantically equivalent to an alternation free[2] one.

The standard and historical interpretation consists in taking L as a complete lattice. This is a consequence of Knaster-Tarski Theorem, since the lattice operators are monotonic.

Besides its primary theoretical purpose to describe complete lattices, $\mathbb{L}_\mu$ stands the foundation of the theory of communication as suggested by Joyal [Joy97], moreover it extends this theory allowing potentially infinite behaviours. This feature is discussed at the end of this Chapter.

In this Chapter we introduce $\mathbb{L}_\mu$: its syntax and standard semantics over complete lattices, then we provide a natural translation of $\mathbb{L}_\mu$ terms into a kind of combinatorial objects: the parity games with draws. These combinatorial objects will be our working tools in this Chapter and the following one. More precisely, the main algebraic concepts, such as the preoder relation, the variable hierarchy problems will be usefully formulated in terms of games and strategies. Finally, we recall the definition of the preorder on $\mathbb{L}_\mu$ given in terms of games and winning strategies.

4.1 The lattice μ-calculus $\mathbb{L}_\mu$: its syntax and semantics

We introduce the lattice μ-calculus, denoted $\mathbb{L}_\mu$, and its standard semantics over complete lattices. One may consider its semantics over μ-lattices as well.

Syntax of $\mathbb{L}_\mu$. The syntax of lattice μ-terms is given by the following grammar:

$$t \;=\; x \mid \top \mid \bot \mid t \wedge t \mid t \vee t \mid \mu x.t_x \mid \nu x.t_x$$

where x ranges over a countable set X of variables.

Semantics of $\mathbb{L}_\mu$ over complete lattices.

[2] A term t is alternation free if is of the form $\mu \ldots \mu\, t'$ or $\nu \ldots \nu\, t'$.

If t is a μ-term, then we denote by X_t the set of free variables of t. Given a complete lattice L, we define the interpretation of a μ-term t as the total function [3]

$$\| \ . \ \| : T_\mu \longrightarrow \bigcup_{n \geq 0} L^{(L^n)}$$

where the interpretation $\|t\|^L : L^{X_t} \longrightarrow L$ of a μ-term t is [4], is given inductively by:

- If $t = x$, then $\|t\|^L(v) = v(x)$.
- If $t = \top$ (resp. $t = \bot$), then $\|t\|^L$ is interpreted as the constant function to $\bigvee L$, i.e. the supremum of L (resp. to $\bigwedge L$, i.e. the infimum of L).
- If $t = t_1 \wedge t_2$, then $\|t\|^L(v) = \|t_1\|^L(v|t_1) \wedge_L \|t_2\|^L(v|t_2)$, where $\wedge_L$ denotes the *greatest lower bound* and $v|t_i$ is the restriction of v to X_{t_i}.
- If $t = t_1 \vee t_2$, then $\|t\|^L(v) = \|t_1\|^L(v|t_1) \vee_L \|t_2\|^L(v|t_2)$, where $\vee_L$ denotes the *least upper bound*,
- If $t = \mu x.t_x$, then let

  $$\phi(l) = \|t_x\|(v^l), \text{ for each } v \in L^{X_t}$$

  where $v^l(y) = v(y)$ if $y \neq x$ and $v^l(x) = l$. Then we define

  $$\|t\|^L(v) = \mu.\phi$$

- The semantics of $t = \nu x.t_x$ is obtained from the above one by substituting each symbol μ with the symbol ν, and the phrase *least prefix-point* with the phrase *greatest postfix-point*.

[3] This is a consequence of Knaster-Tarski theorem, the complete lattice model ensure the existence of the desired fixed points. Observe if L is an arbitrary lattice then the fixed points need not exist in general.

[4] Recall that L^X is the X-fold product lattice of L with itself.

4.2 Labeled parity games with draws

Lattice μ-terms have a natural translation into a kind of 2-players games: the *labeled parity games with draws*. We first define these games, then we provide such translation.

Definition 4.2.1. A *labeled parity game with draws* is a tuple $G = \langle Pos_E^G, Pos_A^G, Pos_D^G, M^G, \rho^G, p_\star^G, \lambda^G \rangle$ where:

- $Pos_E^G, Pos_A^G, Pos_D^G$ are finite pairwise disjoint sets of positions (Eva's positions, Adam's positions, and draw positions),
- M^G, the set of moves, is a subset of $(Pos_E^G \cup Pos_A^G) \times (Pos_E^G \cup Pos_A^G \cup Pos_D^G)$,
- ρ^G is a rank function from $(Pos_E^G \cup Pos_A^G)$ to $\mathbb{N}$.
- $p_\star^G \in Pos_E^G \cup Pos_A^G \cup Pos_D^G$ is the initial position.
- $\lambda^G : Pos_D^G \to X$ is a labelling of draw positions with variables of a countable set.

■

These data define a game between player Eva and player Adam starting from the initial position. The outcome of a finite play is determined according to the normal play condition: a player who cannot move loses. It can also be a draw, if a position in Pos_D^G is reached.[5] The outcome of an infinite play $\{ (g_k, g_{k+1}) \in M^G \}_{k \geq 0}$ is determined by means of the rank function ρ^G as follows: it is a win for Eva iff the maximum of the set $\{ i \in \mathbb{N} \mid \exists$ infinitely many k s.t. $\rho^G(g_k) = i \}$ is even. To simplify the notation, we shall use $Pos_{E,A}^G$ for the set $Pos_E^G \cup Pos_A^G$ and use similar notations such as $Pos_{E,D}^G$, etc. We let $Max^G = \max \rho^G(Pos_{E,A}^G)$ if the set $Pos_{E,A}^G$ is not empty, and $Max^G = -1$ otherwise. We denote by (G, g) the game that differs from

[5] Observe that there are no possible moves from a position in Pos_D^G.

G only on the starting position, i.e. $p_\star^{(G,g)} = g$, and similarly we write (G, g) to mean that the play has reached position g. With $\mathcal{G}$ we shall denote the collection of all labeled parity games; as no confusion will arise, we will call a labeled parity game with simply "game".

4.3 Translation of μ-terms into games.

The translation $\gamma : \mathbb{L}_\mu \to \mathcal{G}$ is defined inductively:

- If $t = x$, then $\gamma(t) = \hat{x}$, where $\hat{x}$ be the game with just one final draw position of zero rank and labeled with variable x.
- If $t = \top$, (resp. $t = \bot$) then $\gamma(t)$ is the game with just one initial position that belongs to Adam (resp. to Eva) and of zero rank.
- Let $t = t_1 \wedge t_2$. If $G_i = \gamma(t_i), i = 1, 2$, then the game $\gamma(t)$ is obtained out of the games[6] G_1 and G_2 by letting

$$\begin{aligned} Pos^{\gamma(t)}_{E,A,D} &= Pos^{G_1}_{E,A,D} \cup Pos^{G_2}_{E,A,D} \cup \{\, p_\star^{\gamma(t)} \,\} \text{ and} \\ M^{\gamma(t)} &= M^{G_1} \cup M^{G_2} \cup \{\, (p_\star^{\gamma(t)}, p_\star^{G_1}),\ (p_\star^{\gamma(t)}, p_\star^{G_2}) \,\}, \end{aligned}$$

  moreover $p_\star^{\gamma(t)} \in Pos^{\gamma(t)}_A$ and $\rho^{\gamma(t)}(p_\star^{\gamma(t)}) = 0$.
- Similarly, we define $\gamma(t)$ for $t = t_1 \vee t_2$ as in the previous case apart that $p_\star^{\gamma(t)} \in Pos^{\gamma(t)}_E$.
- Let $t = \theta x.t_x, \theta \in \{\, \mu, \nu \,\}$. Assume that $G_x = \gamma(t_x)$. Let $Pos_x = \{\, g \in Pos^{G_x}_D \mid \lambda^{G_x}(g) = x \,\}$ and $Pred_x = \{\, g \in Pos^{G_x}_{E,A} \mid (g, g') \in M^{G_x} \text{ and } g' \in Pos_x \,\}$. Then,
  - $Pos^{\gamma(t)}_E = Pos^{G_x}_E \cup \{\, p_\star^{\gamma(t)} \,\}$ if $\theta = \mu$ and $Pos^{\gamma(t)}_E = Pos^{G_x}_E$ if $\theta = \nu$.
  - $Pos^{\gamma(t)}_A = Pos^{G_x}_A$ if $\theta = \mu$ and $Pos^{\gamma(t)}_A = Pos^{G_x}_A \cup \{\, p_\star^{\gamma(t)} \,\}$ if $\theta = \nu$.
  - $Pos^{\gamma(t)}_D = Pos^{G_x}_D \setminus Pos_x$.

[6] We assume that the set of positions of G_1 and of G_2 are disjoint.

- $M^{\gamma(t)} = (M^{G_x} \cap (Pos^{\gamma(t)}_{E,A} \times Pos^{\gamma(t)}_{E,A,D})) \cup \{ (g, p^{\gamma(t)}_{\star}) \mid g \in Pred_x \}$.
- $\rho^{\gamma(t)}$ is the extension of ρ^{G_x} to $p^{\gamma(t)}_{\star}$ as follows:
  - If $\theta = \mu$, then $\rho^{\gamma(t)}(p^{\gamma(t)}_{\star}) = Max^{G_x}$ if Max^{G_x} is odd, and $\rho^{\gamma(t)}(p^{\gamma(t)}_{\star}) = Max^{G_x} + 1$ if Max^{G_x} is even.
  - If $\theta = \nu$, then $\rho^{\gamma(t)}(p^{\gamma(t)}_{\star}) = Max^{G_x}$ if Max^{G_x} is even, and $\rho^{\gamma(t)}(p^{\gamma(t)}_{\star}) = Max^{G_x} + 1$ if Max^{G_x} is odd.

Remark 4.3.1. We emphasize that, if t is a μ-term, then $\gamma(t)$ is game with the two following properties:

(i) (the underlying graph of) $\gamma(t)$ is a tree with back-edges, see Definition 2.3.2,

(ii) for every return r of $\gamma(t)$, r has just one successor.

■

One may also consider an alternative presentation of $\mathbb{L}_\mu$, the latter being given in terms of with simultaneous fixed points, i.e. instead of considering an unary fixed point function, we may consider a system of equations of form $\{ x_i =_{\theta_i} f_i(x_{i,1}, \ldots, x_{i,n}) \}_{i \in I}$ where $\theta_i \in \{ \mu, \nu \}$. The simultaneous fixed points have already been considered in [And94, Sei96]. The reason for choosing this presentation is that the system of equations' formalism provides a compact presentation, the latter being more intuitive and easy to comprehend. The syntactic procedure that constructs a canonical solution of a system of equations by means of scalar μ-terms is known as the *Gauss elimination principle* and shows the use of Bekič principle, see section 3.1.2. From these considerations we can argue that the two presentations are semantically equivalent.

As we have done before when we translated the (scalar) μ-terms into games, we can also translate systems of equations into games for which the underlying structure is no longer a tree with back-edges but a graph. The translation of algebraic objects into combinatorial ones is a useful approach

in μ-calculi, many algebraic concepts are better understood within their combinatorial counterpart. For instance, see [JW95] where a natural translation from modal μ-calculus formulae into μ-automata is given and vis-versa. Later, many authors have adopted this approach [GTW02, Ven06]. Summerising the parity games with draws stand for the lattice μ-calculus as μ-automata stand for modal μ-calculus, so we can write:

$$\text{modal } \mu\text{-calculus / } \mu\text{-automata} \sim \text{lattice } \mu\text{-calculus / parity games}$$

4.4 The preorder on games

In order to describe a preorder on the class $\mathcal{G}$, we shall define a new game $\langle G, H\rangle$ for a pair of games G and H in $\mathcal{G}$. This is not a pointed parity game with draws as defined in the previous section; to emphasize this fact, the two players will be named Mediator and Opponents instead of Eva and Adam. The games $\langle G, H\rangle$ are basically the Mediator-Opponents games over the μ-terms in L_μ already introduced in Definition 1.6.4. More precisely, instead of considering a Mediator-Opponents game of the form $\langle s, t\rangle$, where s, t are μ-terms in L_μ, we shall adopt a more combinatorial approach and consider the Mediator-Opponents game $\langle \gamma(s), \gamma(t)\rangle$.

Before formally defining the game $\langle G, H\rangle$, we recall its informal description and explanation. Mediator's goal is to prove that the relation $\|G\| \leq \|H\|$ holds in any complete lattice; Opponent's goal is to show that this relation does not hold. For example, if $G = \bigvee_{i\in I} G_i$ has the shape of a join and $H = \bigwedge_{j\in J} H_j$ has the shape of a meet, then this is an Opponent's position: Mediator should be prepared to prove $\|G_i\| \leq \|H_j\|$ for any pair of indexes i and j; thus Opponent should find a pair of indexes (i, j) and show that $\|G_i\| \not\leq \|H_j\|$. If $G = \bigwedge_{i\in I} G_i$ is a meet and $H = \bigvee_{j\in J} H_j$ is a join, then this is a Mediator's position: Mediator should find either an i and show that $\|G_i\| \leq \|H\|$ or a j and show that $\|G\| \leq \|H_j\|$; Opponent should be prepared to disprove any such relation.[7]

[7] These moves suffice to Mediator to reach his goal, as the relation $\leq$ that we shall

Thus the game is played on the two boards, simultaneously. At a first approximation, a position of $\langle G, H\rangle$ is a pair of positions from G and H. Since we code meets as Adam's positions and joins as Eva's positions, Mediator is playing with Adam on G and with Eva on H; Opponent is playing with Eva on G and with Adam on H. Thus a pair (g, h) in $Pos_A^G \times Pos_E^H$ clearly belongs to Mediator and a pair (g, h) in $Pos_E^G \times Pos_A^H$ clearly belongs to Opponent.

Definition 4.4.1. The game $\langle G, H\rangle$ is defined as follows:

- The set of Mediator's positions is

  $Pos_A^G \times Pos_{E,D}^H \cup Pos_{A,D}^G \times Pos_E^H \cup \{ (g, h) \in Pos_D^G \times Pos_D^H \mid \lambda^G(g) \neq \lambda^H(h) \}$
  and the set of Opponents' positions is $Pos_E^G \times Pos_{E,A,D}^H \cup Pos_{E,A,D}^G \times Pos_A^H$
  $\cup \{ (g, h) \in Pos_D^G \times Pos_D^H \mid \lambda^G(g) = \lambda^H(h) \}$

- Moves of $\langle G, H\rangle$ are either left moves $(g, h) \to (g', h)$, where $(g, g') \in M^G$, or right moves $(g, h) \to (g, h')$, where $(h, h') \in M^H$; however the Opponents can play only with *Eva* on G or with *Adam* on H.

- A finite play is a loss for the player who can not move. An infinite play γ is a win for Mediator if and only if its left projection $\pi_G(\gamma)$ is a win for Adam, or its right projection $\pi_H(\gamma)$ is a win for Eva.

■

Definition 4.4.2. If G and H belong to $\mathcal{G}$, then we declare that $G \leq H$ if and only if Mediator has a winning strategy in the game $\langle G, H\rangle$ starting from position $(p_\star^G, p_\star^H)$. ■

The following is the reason to consider such a syntactic relation:

define turns out to be transitive. This fact is analogous to a cut-elimination theorem and to Whitman's conditions characterizing free lattices [FJN95].

Theorem 4.4.3. [San02c] The relation $\leq$ is sound and complete with respect to the interpretation in any complete lattice, i.e. $\gamma(t_1) \leq \gamma(t_1)$ if and only if $\|t_1\|^L \leq \|t_2\|^L$ holds in every complete lattice L. ■

In the sequel, we shall write $G \sim H$ to mean that $G \leq H$ and $H \leq G$. For other properties of the relation $\leq$, see for example Proposition 2.5 of [AS03] or [San02c]. This is the main result of [San02c]

Theorem 4.4.4. The quotient $\mathcal{G}/\sim$ is a free μ-lattice. ■

It was proved in [San02c] that $G \leq G$ by exibing the *copycat* strategy in the game $\langle G, G\rangle$: from a position (g, g), it is Opponents' turn to move either on the left or on the right board. When they stop moving, Mediator will have the ability to copy all the moves played by the Opponents so far from the other board until the play reaches the position (g', g').
There it was also proved that if $G \leq H$ and $H \leq K$ then $G \leq K$, a sort of cut-elimination Theorem. This result was achieved by describing a game $\langle G, H, K\rangle$ with the following properties: *(i)* given two winning strategies R on $\langle G, H\rangle$, and S on $\langle H, K\rangle$ there is a winning strategy $R\|S$ on $\langle G, H, K\rangle$, that is the composition of the strategies R and S, *(ii)* given a winning strategy T on $\langle G, H, K\rangle$, there exists a winning strategy $T_{\backslash H}$ on $\langle G, K\rangle$.

Definition 4.4.5. Positions of the game $\langle G, H, K\rangle$ are triples $(g, h, k) \in Pos^G_{A,E,D} \times Pos^H_{A,E,D} \times Pos^K_{A,E,D}$ such that

- the set of Mediator's positions is

  $Pos^G_A \times Pos^H_{A,E,D} \times Pos^K_{E,D} \cup Pos^G_{A,D} \times Pos^H_{A,E,D} \times Pos^K_E \cup \mathcal{L}(M)$,
  and the set of Opponents' positions is
  $Pos^G_E \times Pos^H_{A,E,D} \times Pos^K_{E,A,D} \cup Pos^G_{E,A,D} \times Pos^H_{A,E,D} \times Pos^K_A \cup \mathcal{L}(O)$,
  where $\mathcal{L}(M), \mathcal{L}(O) \subseteq Pos^G_D \times Pos^H_{A,E,D} \times Pos^K_D$ are positions of Mediator and Opponents, respectively, defined as follows. Whenever $(g, h, k) \in Pos^G_D \times Pos^H_{A,E,D} \times Pos^K_D$, then if $h \in Pos^H_{E,A}$, then the position (g, h, k) belongs to Mediator, otherwise, i.e. $h \in Pos^H_D$, then the final position (g, h, k) belongs to Opponents if and only if $\lambda^G(g) = \lambda^H(h) = \lambda^K(k)$.

- Moves of $\langle G, H, K\rangle$ are either left moves $(g, h, k) \to (g', h, k)$ where $(g, g') \in M^G$ or central moves $(g, h, k) \to (g, h', k)$, where $(h, h') \in M^H$, or right moves $(g, h, k) \to (g, h, k')$, where $(k, k') \in M^K$; however the Opponents can play only with Eva on G or with Adam on K.

- As usual, a finite play is a loss for the player who cannot move. An infinite play γ is a win for Mediators if and only if $\pi_G(\gamma)$ is a win for Adam on G, or $\pi_K(\gamma)$ is a win for Eva on K.

■

4.5 Computational interpretation

We show that the lattice μ-calculus with its canonical preorder stands the basic of the interactive computation in the sense of Joyal.
A game G may be understood as a synchronous channel of communication available to the two users: Eva en Adam. The game G is a game with perfect information, and it is synchronous in the sense that Eva can not exhaust her move before Adam has exhausted his moves, and conversely. No kind of simultaneous moves is allowed. The moves of Mediator in the game $\langle G, H\rangle$ may be understood as an protocol allowing the user Eva of the left channel G to communicate with the user Adam of the right channel H in an assynchronous way. This is a consequence of the fact that Mediator plays with Adam on G and with Eva on H, and hence the remaining positions belong to the environment, i.e. the users. Moreover this kind of communication is qualified to be assynchronous in the sense that Mediator, when it is his turn, can privilege one side on the other one, implying that a user on one side is waiting whereas the other user on the other channel is exhausting his moves.

In Joyal games on lattice terms the duration of the session of the communication is "finite". However, if the communication is modeled by means of the $\mathbb{L}_\mu$ terms, then the duration of the session is potentially infinite, since $\mathbb{L}_\mu$ terms may contain cycles. The infinite running of the protocol has also

a nice computational interpretation. Recall that Mediator wins in $\langle G, H\rangle$ if and only if he wins with Adam on G or with Eva on H. Winning with Adam on G means that the highest return (i.e. the closest to the root of G) which occurs infinitely often belongs to Eva. Also winning with Eva on H means that the highest return that occurs infinitely often in H belongs to Adam. Therefore, Mediator wins in $\langle G, H\rangle$ means that a situation where the protocol is executed infinitely often without a concrete evolution of both users can not happen. From these considerations, a winning strategy for Mediator in the game $\langle G, H\rangle$ witnesses that the asynchronous communication between the two users of the channel is without *deadlock*.

Chapter 5

The Variable Hierarchy for The Lattice μ-Calculus is Infinite

Abstract. *The* variable hierarchy problem *asks whether every μ-term t is equivalent to a μ-term t' where the number of fixed-point variables in t' is bounded by a constant. In this chapter, we prove that the variable hierarchy of the* lattice μ-calculus *– whose standard interpretation is over the class of all complete lattices– is infinite, meaning that such a constant does not exist if the μ-terms are built up using the basic lattice operators as well as the least and the greatest fixed point operators. The proof relies on the description of the lattice μ-calculus by means of games and strategies. From the computational point of view, this negative result implies that one can not – in general – simplify the processes (i.e. the μ-terms) if their behaviour is specified with the lattice μ-calculus language.*

5.1 Introduction

Hierarchies and logical expressiveness issues are at the core of fixed-point theory [BÉ93, AN01]. These are the alternation depth hierarchy [Niw86, Len96, Bra98, Arn99, San02a, SA05], the star height [Egg63, BC84], and its refinement: the variable hierarchy [BGL07, BS08].

Recent work by Berwanger et al. [BGL07] proves that the expressive power of the modal μ-calculus [Koz83] increases with the number of fixed point variables. By introducing the *variable hierarchy* and showing that it does not collapse, they manage to separate the μ-calculus from dynamic game logic [PP03]. Their work may also be appreciated for the new research paths disclosed to the theory of fixed-points [AN01, BÉ93]. The variable hierarchy may be defined for every μ-calculus and for iteration theories as well, since one fixed-point operator is enough to define it. Thus, the question whether the variable hierarchy for a μ-calculus is strict is at least as fundamental as considering its alternation-depth hierarchy. In this chapter we answer this question for the *lattice μ-calculus over complete lattices*.

The *variable hierarchy problem* of $\mathbb{L}_\mu$ asks whether every μ-term t is equivalent to a μ-term t' where the number of fixed-point variables in t' is bounded by a constant. This amounts to consider the levels $\mathcal{T}_n$ of the variable hierarchy defined by:

$$\mathcal{T}_n = \{\, t \in \mathbb{L}_\mu \mid t \sim s \text{ for some } s \in \mathbb{L}_\mu \text{ s.t. } nbr(s) \leq n \,\}$$

where $\sim$ is the standard semantic equivalence of the μ-terms over complete lattices and $nbr(s)$ is the number of bound variables in s. Then, we ask whether the hierarchy made up of levels $\mathcal{T}_n$ collapses: is there a constant k such that $\mathcal{T}_n = \mathcal{T}_k$ for every $n \geq k$?

As a first step we shall provide a combinatorial refinement of the starting problem: on the one hand we consider the combinatorial representation of μ-terms by means of parity games, on the other hand to compute the minimal number of bound variables needed in a μ-term, up to α-conversion, two digraph complexity measures are needed: the *entanglement* and the *feedback*. The origin of these measures lies on the pebble games of Immerman and Poizat [Imm95] introduced to measure the expressive power of logical quantifiers in terms of the number of bound variables.

The feedback of a vertex v of a tree with back edges is the number of ancestors of v that are the target of a back edge whose source is a descendant

of v. The feedback of a tree with back edges is the maximum feedback of its vertices. The entanglement of a digraph G, denoted $\mathscr{E}(G)$, may be defined as follows: *it is the minimum feedback of its finite unravellings into a tree with back edges*. These measures are tied to the logic as follows. A μ-term may be represented as a tree with back-edges, the feedback of which corresponds to the minimum number of fixed point variables needed in the μ-term, up to α-conversion. Also, one may consider terms of a vectorial μ-calculus, i.e. systems of equations, and these roughly speaking are graphs. The step that constructs a canonical solution of a system of equations by means of μ-terms amounts to the construction of a finite unravelling of the graph. In view of these considerations, asking whether the hierarchy made of levels $\mathcal{T}_n$ collapses or not amounts to consider the same question for the levels $\mathcal{L}_n$ defined by:

$$\mathcal{L}_n = \{\, G \in \mathcal{G} \mid G \sim H \text{ for some } H \in \mathcal{G} \text{ s.t. } \mathscr{E}(H) \leq n \,\}.$$

Here $\mathcal{G}$ is the collection of parity games with draws positions and $\sim$ denotes the semantic equivalence.

We answer this question negatively, there is no such constant. We shall construct, for each $n \geq 1$, a parity game G_n with two properties:

(i) G_n has entanglement n, showing that G_n belongs to $\mathcal{L}_n$,

(ii) G_n is semantically equivalent to no game in $\mathcal{L}_{n-3}$.

Thus, we prove that the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$, $n \geq 3$, are strict.

The games G_n may be considered as a *generalization* of Whitman polynomials [FJN95, §1.5] used to show that free lattices are not complete. To carry on, we strengthen the notion of *synchronizing game*[1] from [San02a]. By playing with the η-expansion – i.e. the copycat strategy – and the cut-elimination – i.e. composition of strategies – we prove that the syntactical structure of a game H, which is semantically equivalent to a *strongly* synchronizing game

[1] A synchronizing game has the property that there exists just one winning strategy for Mediator in $\langle G, G \rangle$, the copycat strategy.

G, resembles that of G: every move (edge) in G can be simulated by a non empty finite sequence of moves (a path) of H; if two paths simulating distinct edges do intersect, then the edges do intersect as well. We formalize such situation within the notion of $\star$-*weak simulation*. The relevant result is that $\star$-weak simulation preserves entanglement up to a constant, that is, if there is a $\star$-weak simulation of G by H, then

$$\mathscr{E}(G) - 2 \leq \mathscr{E}(H) \qquad \text{(i.e. Theorem 5.5.8)}$$

Once we have proven that the games G_n are strongly synchronizing, then we can deduce that

$$n - 2 = \mathscr{E}(G_n) - 2 \leq \mathscr{E}(H)$$

for every game H semantically equivalent to G_n.

This Chapter is organized as follows. In Section 6.3, we firstly recall the definition of entanglement and we reduce the variable hierarchy problem of the lattice μ-calculus into a problem on the entanglement of parity games; then we define the $\star$-weak simulation between graphs that allows to compare their entanglements. In Section 5.6, we define strongly synchronizing games and we shall prove their *hardness* w.r.t the variable hierarchy, in particular every equivalent game to a strongly synchronizing one is related with it by a $\star$-weak simulation. In Section 5.7, we construct strongly synchronizing games of arbitrary entanglement. We sum up the discussion in our main result, Theorem 5.7.8.

5.2 Notation, preliminary definitions, and elementary facts

We provide the terminology and notation used in this Chapter. Most of them have already been given in Chapter 2. If G is a graph, then a path in G is a

sequence of the form $\pi = g_0 g_1 \dots g_n$ such that $(g_i, g_{i+1}) \in E_G$ for $0 \leq i < n$. A path is *simple* if $g_i \neq g_j$ for $i, j \in \{ 0, \dots, n \}$ and $i \neq j$. The integer n is the length of π, g_0 is the source of π, noted $\delta_0 \pi = g_0$, and g_n is the target of π, noted $\delta_1 \pi = g_n$. We denote by $\Pi^+(G)$ the set of simple non empty (i.e. of length greater than 0) paths in G.

If $\mathcal{T}$ is a tree with back edges, then a path in $\mathcal{T}$ can be factored as

$$\pi = \pi_1 * \dots * \pi_n * \tau$$

where each factor π_i is a sequence of tree edges followed by a back edge, and τ does not contain back edges. Such factorization is uniquely determined by the occurrences of back edges in π. For $i > 0$, let r_i be the return at the end of the factor π_i. Let also r_0 be the source of π. Let the b-length of π be the number of back edges in π. i.e. $r_i = \delta_1 \pi_i$.

Lemma 5.2.1. If π is a simple path of b-length n, then r_n is the vertex closest to the root visited by π. Hence, if a simple path π lies in the subtree of its source, then it is a tree path. ■

Proof. It is enough to observe that, for each i, r_i is the highest vertex visited by π_i. To this goal, if $\pi_i = d_i * b_i$, where d_i is a tree path and b_i is a back-edge, then either r_i belongs to d_i or it is an ancestor of the source of d_i. The first case is excluded by π_i being simple. □

Definition 5.2.2. A *cover* or *unravelling* of a (finite) directed graph H is a (finite) graph K together with a surjective graph morphism $\psi : K \longrightarrow H$ such that for each $v \in V_K$, the correspondence sending k to $\psi(k)$ restricts to a bijection from $\{ k \in V_K \mid (v, k) \in E_K \}$ to $\{ h \in V_H \mid (\psi(v), h) \in E_H \}$. ■

The notion of cover of pointed digraphs is obtained from the previous by replacing the surjectivity constraint by the condition that ψ preserves the root of the pointed digraphs.

We extend the definition of the notion of cover (unraveling) of graphs into that of games in the obvious way.

Definition 5.2.3. Let $G, H \in \mathcal{G}$. A *cover* or *unravelling* of a game G is a game H together with a surjective morphism ψ from positions of H to positions of G such that for each $h_1 \in Pos^H_{E,A}$, the correspondence sending h_2 to $\psi(h_2)$ restricts to a bijection from $\{\, h_2 \in Pos^H_{E,A,D} \mid (h_1, h_2) \in M^H \,\}$ to $\{\, g \in Pos^G_{E,A,D} \mid (\psi(v), g) \in Pos^G_{E,A,D} \,\}$, moreover

- $\psi(p^H_\star) = p^G_\star$,
- if $h \in Pos^H_E$ (resp. $h \in Pos^H_A$) then $\psi(h) \in Pos^G_E$ (resp. $\psi(h) \in Pos^G_A$) and $\rho^H(h) = \rho^G(\psi(h))$
- if $h \in Pos^H_D$ then $\psi(h) \in Pos^G_D$ and $\lambda^H(h) = \lambda^G(\psi(h))$.

■

Lemma 5.2.4. Let $G, H \in \mathcal{G}$. If H is a cover of G then $G \sim H$. ■

Proof. By playing according to the copy-cat strategy. □

We extend the notion of the isomorphism of graphs into the isomorphism of games as follows.

Definition 5.2.5. Two games $G, H \in \mathcal{G}$ are isomorphic if there is a bijection

$$\beta : Pos^H_{E,A,D} \longrightarrow Pos^G_{E,A,D}$$

such that:
$(h_1, h_2) \in M^H$ if and only if $(\beta(h_1), \beta(h_2)) \in M^G$ and moreover:

- $\beta(p^H_\star) = p^G_\star$,
- if $h \in Pos^H_E$ (resp. $h \in Pos^H_A$) then $\beta(h) \in Pos^G_E$ (resp. $\beta(h) \in Pos^G_A$) and $\rho^H(h) = \rho^G(\beta(h))$
- if $h \in Pos^H_D$ then $\beta(h) \in Pos^G_D$ and $\lambda^H(h) = \lambda^G(\beta(h))$.

■

5.3 Entanglement

In order to compute the minimum number of bound variables required in a vectorial μ-term, the digraph measure *entanglement* is needed. Its definition is as follows: the *entanglement* of a digraph G is *the minimum feedback of the finite unravellings of G into a tree with back edges*. In [BG05], the entanglement of G has been characterized by means of a game $\mathscr{E}(G,k)$, $k = 0, \ldots, |V_G|$, played by Thief against Cops, a team of k cops.

Definition 5.3.1. [i.e. Definition 3.4.1] The entanglement game $\mathscr{E}(G,k)$ of a digraph G is defined by:

- Its positions are of the form (v, C, P), where $v \in V_G$, $C \subseteq V_G$ and $|C| \leq k, \quad P \in \{Cops, Thief\}$.
- Initially Thief chooses $v_0 \in V_G$ and moves to $(v_0, \emptyset, Cops)$.
- Cops can move from $(v, C, Cops)$ to $(v, C', Thief)$ where C' can be
  - C : Cops skip,
  - $C \cup \{ v \}$: Cops add a new Cop on the current position,
  - $(C \setminus \{ x \}) \cup \{ v \}$: Cops move a placed Cop to the current position.
- Thief can move from $(v, C, Thief)$ to $(v', C, Cops)$ if $(v, v') \in E_G$ and $v' \notin C$.

Every finite play is a win for Cops, and every infinite play is a win for Thief. ■

The following will constitute our working definition of entanglement: *$\mathscr{E}(G)$, the entanglement of G, is the minimum $k \in \{ 0, \ldots, |V_G| \}$ such that Cops have a winning strategy in $\mathscr{E}(G,k)$.*
The following proposition provides a useful variant of entanglement games.

Proposition 5.3.2. *[i.e. Proposition 3.4.2] Let $\widetilde{\mathscr{E}}(G,k)$ be the game played as the game $\mathscr{E}(G,k)$ apart that Cops is allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

- $(g, C, Cops) \to (g, C', Thief)$ *(generalized skip move),*
- $(g, C, Cops) \to (g, C' \cup \{ g \}, Thief)$ *(generalized replace move),*

where in both cases $C' \subseteq C$. Then Cops has a winning strategy in $\mathscr{E}(G, k)$ if and only if he has a winning strategy in $\widetilde{\mathscr{E}}(G, k)$.

Proof. See the proof of the Proposition 3.4.2. □

Along this chapter when we refer to the entanglement or the feedback of a game $G \in \mathcal{G}$ we mean the *underlying graph* of G, and we shall denote it by $\overline{G}$.

5.4 A combinatorial refinement of the variable hierarchy problem.

The aim now is to show that the variable hierarchy problem of $\mathbb{L}_\mu$ transfers into a problem of entanglement of the underlying graph of labeled parity games with draws. The key observation consists in the two following Lemmas. Lemma 5.4.1 may be understood as the combinatorial version of the Bekič Principle Proposition 3.1.2, furthermore if G contains many ranks, then the property *(i)* of this Lemma becomes $fb(\overline{K}) \geq \mathscr{E}(\overline{G})$.

Lemma 5.4.1. Let $G \in \mathcal{G}$ with $Max^G = 0$. Then there exists a fair game[2] $K \in \mathcal{F}$ with $Max^K = 0$ such that: *(i)* $fb(\overline{K}) = \mathscr{E}(\overline{G})$ and *(ii)* $G \sim K$. ■

Proof. Let $G \in \mathcal{G}$ be as desired, we proceed in two steps: first, we exhibit a game H such that H is a tree with back edges, $Max^H = 0$, $fb(\overline{H}) = \mathscr{E}(\overline{G})$ and $H \sim G$. Secondly, out of H we construct a fair game K with the desired properties.
The first step consists in taking $\overline{H}$ (as a graph) as a cover of $\overline{G}$ with the desired feed back, and assigning to $\overline{H}$ the desired rank, labeling, ... Therefore,

[2] A game is fair if it verifies properties *(i)* and *(ii)* of Remark 4.3.1.

H is a cover of G and hence $H \sim G$ by Lemma 5.2.4.
The second step consists in inserting a new vertex $\dot{r}$ between each return r of H and its successors in such a way that $\dot{r}$ inherits all the successors of r as follows.
Without loss of generality we can assume that H does not contain self loops, otherwise we can substitute every loop $\{ (r, r) \}$ by a new 2-cycle $\{ (r, r'), (r', r) \}$. Let $\mathcal{R}_H$ be the set of returns of H and $Succ(g) = \{ s \mid (g, s) \in M^H \}$, then define the game K out of the game H as:

- $Pos_E^K = Pos_E^H \cup \{ \dot{r} \mid r \in \mathcal{R}_H \text{ and } r \in Pos_E^H \}$,
- $Pos_A^K = Pos_A^H \cup \{ \dot{r} \mid r \in \mathcal{R}_H \text{ and } r \in Pos_A^H \}$,
- $Pos_D^K = Pos_D^H$,
- $$\begin{aligned} M^K = \ & (M^H \ \setminus \ \{ (r, s) \in M^H \text{ and } r \in \mathcal{R}_H \}) \\ & \cup \{ (r, \dot{r}) \text{ s.t. } r \in \mathcal{R}_H \} \\ & \cup \{ (\dot{r}, s) \text{ s.t. } s \in Succ(r) \} \end{aligned}$$
- ρ^K assigns a zero rank to all positions (apart the draws ones), and
- $p_\star^K = p_\star^H$.

Clearly K is a fair game by construction. Besides this, we have that $K \sim H$, and $fb(\overline{K}) = fb(\overline{H})$. Summarising, we have that $fb(\overline{K}) = fb(\overline{H}) = \mathscr{E}(\overline{G})$ and $K \sim H \sim G$ and K is fair, as desired.

□

Lemma 5.4.2. Let $K \in \mathcal{F}$ such that $fb(\overline{K}) = k$. Then there exists a μ-term t such that: *(i)* the number of fixed point variables in t is k, and *(ii)* $\gamma(t) = K$. ■

Proof. The construction of t depends on the following Claim. The proof of this Claim is provided below inside Lemma 5.5.5.

Claim 5.4.3*:* Let H be a tree with back edges of feed back k. Let $\mathcal{R}$ be the set of returns of H and if $h \in V_H$ then define $Ancet(h)$ as the set of returns which are above h and pointed by a back-edge from the descendants of h. Then there exists a total labeling

$$l : \mathcal{R} \longrightarrow \{ 1, \ldots, k \}$$

with the following property:

$$\text{for all } h \in V_H, \text{ if } r_1, r_2 \in Ancet(h) \text{ then } l(r_1) \neq l(r_2) \tag{5.1}$$

■

More precisely, the labeling of the returns described above specifies which fixed point variable should be associated to each return without warring about the variable capture anomaly.
Firstly, we define a function $\tau : \mathcal{F} \longrightarrow \mathbb{L}_\mu$, that translates a fair game into a μ-term of the desired number of bound variables. The translation is by induction on the complexity of G, denoted $\mathbb{C}(G)$, which is a pair (p, q) where p is the number of the returns of G and q is number of its positions; these pairs are ordered by the lexicographic ordering :

$$(p_1, q_1) < (p_2, q_2) \quad \text{iff} \quad (p_1 < p_2) \quad \text{or} \quad (p_1 = p_2 \text{ and } q_1 < q_2)$$

To this goal, assume that l is a labeling of (the underlying graph of) G that satisfies Property (5.1), then we define the translation τ in the following way.

- If G is a draw position labeled by the variable x, then $\tau(G) = x$,
- If G is a final position that belongs to Eva (resp. to Adam), then $\tau(G) = \bot$ (resp. $\tau(G) = \top$),
- If the root of G, denoted r_G, is not a return then let $\{ G_i \}_{i \in I}$ be the set of the connected components of $G \setminus \{ r_G \}$, then clearly each G_i is a proper subgame of G, hence $\mathbb{C}(G_i) < \mathbb{C}(G)$, $i \in I$. Therefore, by

applying the induction hypothesis, each $\tau(G_i)$ is defined and hence we let

$$\tau(G) = \begin{cases} \bigwedge_{i\in I} \tau(G_i) & \text{if } r_G \in Pos_A^G \\ \\ \bigvee_{i\in I} \tau(G_i) & \text{if } r_G \in Pos_E^G, \end{cases}$$

- If r_G is a return, then define the **predecessor game** G^- as follows:
  - $Pos_{E,A}^{G^-} = Pos_{E,A}^G \setminus \{ r_G \}$,
  - $Pos_D^{G^-} = Pos_D^G \cup \{ s_i, i = 1, \ldots, k \}$, where $k = |Pred(r_G)|$ where $Pred(r_G)$ are the predecessors of r_G.
  - $$\begin{aligned} M^{G^-} = & \ (M^G \setminus \{ (g, r_G), (r_G, g) \in M^G \}) \\ & \cup \{ (p_i, s_i), p_i \in Pred(r_G) \} \end{aligned}$$
  - ρ^{G^-} is restriction of ρ^G to the positions of G^-,
  - $p_\star^{G^-}$ is the (unique) successor of r_G, and
  - λ^{G^-} is the extension of λ^G to $\{ s_i \}_{i=1,\ldots,k}$ as follows: $\lambda^{G^-}(s_i) = l(r_G)$, recall that l is just the labeling defined above.

  Clearly the number of returns of G^- is strictly less than the number of returns of G, even the number of positions of G^- may be greater than that of G, therefore $\mathbb{C}(G^-) < \mathbb{C}(G)$. Hence, by applying the induction hypothesis, $\tau(G^-)$ is defined and therefore we let:

$$\tau(G) = \begin{cases} \mu x_{l(r_G)}.\tau(G^-) & \text{if } r_G \in Pos_E^G \\ \nu x_{l(r_G)}.\tau(G^-) & \text{if } r_G \in Pos_A^G \end{cases}$$

Here, we assume that the variables are taken from a set $\{ x_i \}_{i=1,\ldots,k}$. Clearly, the number of fixed point variables of the μ-term $\tau(G)$ equals k, i.e. the feed back of $\overline{G}$; proving that $\tau(G)$ satisfies the property *(i)* of the Lemma.

Secondly, let us prove that $\tau(G)$ satisfies the property *(ii)*. In other words we shall show that our construction is sound. This amounts to prove that, for every fair game $G \in \mathcal{F}$ we have that

$$\gamma(\tau(G)) = G \tag{5.2}$$

up to game isomorphism, see Definition 5.2.5.

Lemma 5.4.4. To prove the equation (5.2), it is enough to prove that for every μ-term $t \in \mathbb{L}_\mu$, we have that

$$\tau(\gamma(t)) =_\alpha t \tag{5.3}$$

where $t_1 =_\alpha t_2$ means that the μ-terms t_1, t_2 are equal up to α-conversion. ■

Proof. (of Lemma 5.4.4) Let us prove that (5.3) $\implies$ (5.2). Assume that $\tau(\gamma(t)) =_\alpha t$ and $\gamma(t) = G$.
Since $\gamma(t) = G$, then by applying $\gamma \circ \tau$ to both sides of this equation we get:

$$\gamma(\underbrace{\tau(\gamma(t))}_{=_\alpha t}) = \gamma(\tau(G)) \tag{a}$$

But, by assumption we have $\tau(\gamma(t)) =_\alpha t$, hence by applying γ to both sides of this equation and according to the following Fact

Fact 5.4.5. Let $t_1, t_2 \in \mathbb{L}_\mu$, such that $t_1 =_\alpha t_2$, then $\gamma(t_1) = \gamma(t_2)$, up to game isomorphism. ■

we get

$$\begin{aligned} \gamma(\tau(\gamma(t))) &= \gamma(t) \\ &= G \text{ (by assumption)} \end{aligned} \tag{b}$$

From the equations (a) and (b), we get $\gamma(\tau(G)) = G$, as desired. This ends the proof of Lemma 5.4.4 □

We continue the proof of Lemma 5.4.2 by proving the equation (5.3).

The mapping $\tau \circ \gamma$ may be considered as renaming of the bound variables of the μ-term, this renaming is obtained out of the labeling l described above: if we consider a μ-term of the form $\theta x.t(x)$ then assume that $\gamma(\theta x.t(x)) = G$ is a game rooted at r, then the labeling l would assign the label $l(r)$ to r, hence $\tau(G)$ (i.e. $\tau(\gamma(\theta x.t(x))))$ will be a μ-term of form $\theta x_{l(r)}.t(x_{l(r)})$.
From these considerations we get :

$$\tau \circ \gamma(\theta x.t(x)) = [\theta x.t(x)]_{x \to x_{l(r)}}$$

where $[\theta x.t(x)]_{x \to y}$ is the term $\theta y.t(y \backslash x)$ and $t(y \backslash x)$ is the usual substitution for x by y in t.

The aim now is to show that the renaming $x \to x_{l(r)}$ does not give arise to *the variable capture anomaly.*

Definition 5.4.6. Let $\theta x.t(x)$ be a μ-term we say that the renaming $x \to y$ induces a variable capture in $\theta x.t(x)$ if whenever an occurrence of x is free in $t(x)$ then its new occurrence (y) becomes bound in $t(y \backslash x)$ ■

Let $\theta x.t(x)$ be a μ-term, and let r be the root of the game $\gamma(\theta x.t(x))$. Now, if there is a variable capture in $\theta x.t(x)$ induced by the renaming $x \to x_{l(r)}$, meaning that there is an occurrence of x which is free in $t(x)$ that becomes bound in $t(x_{l(r)} \backslash x)$, then this implies that there is a sub-term of $t(x)$, say t', of the form $t' = \theta x_{l(r)}.t(x_{l(r)})$, such that the game associated to t' is rooted at r' and $l(r') = l(r)$. Moreover, there is a tree path from r to r' in $\gamma(\theta x.t(x))$ and there is also a path from r' to r (composed of a tree path and a back edge), meaning that $r \in Ancet(r')$ and hence according to the labeling l we must have $l(r) \neq l(r')$, this is a contradiction. This ends the proof of Lemma 5.4.2. □

Let us consider a system of equation $S \in \Sigma^1$ (i.e. defined using just the μ operator), and its related game $\gamma(S)$, hence $Max^{\gamma(S)} = 0$. By Lemma 5.4.1, it follows that there exists a fair game K such that $K \sim \gamma(S)$ and $fb(\overline{K}) =$

$\mathscr{E}(\overline{\gamma(S)})$. By Lemma 5.4.2, we can associate to K a μ-term $t \in \mathbb{L}_\mu$ such that $\gamma(t) = K$ and the number of bound variables of t is $fb(\overline{K}) = \mathscr{E}(\overline{\gamma(S)})$. From these considerations, it follows that asking weather the levels $\mathcal{T}_n$ of hierarchy defined by:

$$\mathcal{T}_n = \{\, t \in \mathbb{L}_\mu \mid t \sim s \text{ for some } s \in \mathbb{L}_\mu \text{ s.t. } nbr(s) \leq n \,\}$$

collapse or not amounts to asking the same question for the levels $\mathcal{L}_n$ of hierarchy defined as follows:

$$\mathcal{L}_n = \{\, G \in \mathcal{G} \mid G \sim H \text{ for some } H \in \mathcal{G} \text{ s.t. } \mathscr{E}(\overline{H}) \leq n \,\}.$$

5.5 $\star$-Weak simulations

We define in the following a relation between graphs, called $\star$-*weak simulation*, that shall be of use in comparing entanglements. Intuitively, there is a weak simulation of a graph G by H if every edge of G is simulated by a non empty finite path of H. Observe now that, in such a situation, if the simulating paths do not intersect, except that in their endpoints, then H contains a subgraph obtained from G by stretching edges into non empty paths. This property implies the relation $\mathscr{E}(G) = \mathscr{E}(H)$. However, for the weak simulations that arise when considering a game H which is semantically equivalent to a strongly synchronizing game G, see Section 5.6, only a weaker property holds: if the simulating paths do intersect, then the edges being simulated intersect in some of their endpoints. We call a weak simulation with this property a $\star$-weak simulation. The weaker property suffices to prove the comparison stated at the end of this Section, Theorem 5.5.8.

Definition 5.5.1. A *weak simulation* (R, ς) of G by H is a binary relation $R \subseteq V_G \times V_H$ that comes with a partial function $\varsigma : V_G \times V_G \times V_H \longrightarrow \Pi^+(H)$, such that:
– R is surjective, i.e. for every $g \in V_G$ there exists $h \in V_H$ such that gRh,
– R is functional, i.e. if $g_i R h$ for $i = 1, 2$, then $g_1 = g_2$,

– if gRh and $g \to g'$, then $\varsigma(g, g', h)$ is defined and $h' = \delta_1\varsigma(g, g', h)$ is such that $g'Rh'$. ■

Next we study conditions under which existence of a weak simulation of G by H implies that $\mathscr{E}(G)$ is some lower bound of $\mathscr{E}(H)$. To this goal, let us abuse of notation and write $h \in \varsigma(g, g', h_0)$ if $\varsigma(g, g', h_0) = h_0h_1 \dots h_n$ and, for some $i \in \{ 0, \dots, n \}$, we have $h = h_i$. If $G = (V_G, E_G)$ is a directed graph then its undirected version $S(G) = (V_G, E_{S(G)})$ is the undirected graph such that $g, g' \in E_{S(G)}$ iff $(g, g') \in E_G$ or $(g', g) \in E_G$. We say that G has *girth at least* k if G does not contain loops, $(g, g') \in E_G$ implies $(g', g) \notin E_G$, and the shortest cycle in $S(G)$ has length at least k.

Definition 5.5.2. We say that a weak simulation (R, ς) of G by H is a $\star$-weak simulation (or that it has the $\star$-property) if G has girth at least 4, and if $(g, g'), (\tilde{g}, \tilde{g}')$ are distinct edges of G and $h \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$, then $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$. ■

We explain next this property. Given (R, ς), consider

$$C(h) = \{ (g, g') \in E_G \mid \exists h_0 \text{ s.t. } h \in \varsigma(g, g', h_0) \} .$$

Lemma 5.5.3. Let (R, ς) be a $\star$-weak simulation of G by H. If $C(h)$ is not empty, then there exists an element $c(h) \in V_G$ such that for each $(g, g') \in C(h)$ either $c(h) = g$ or $c(h) = g'$. If moreover $|C(h)| \geq 2$, then this element is unique. ■

Proof. Clearly the condition holds if $|C(h)| \leq 2$, by definition 5.5.2. Let us suppose that $|C(h)| \geq 3$.

Fix two undirected edges $\{ c(h), g_1 \}, \{ c(h), g_2 \}$ in the undirected version of $C(h)$. Consider a third undirected edge $\{ \tilde{g}_1, \tilde{g}_2 \} \in C(h)$, so that $|\{ \tilde{g}_1, \tilde{g}_2 \} \cup \{ c(h), g_1 \}| = 3$, and similarly $|\{ \tilde{g}_1, \tilde{g}_2 \} \cup \{ c(h), g_2 \}| = 3$.[3] If $c(h) \notin \{ \tilde{g}_1, \tilde{g}_2 \}$, then $\{ \tilde{g}_1, \tilde{g}_2 \} = \{ g_1, g_2 \}$, thus creating an undirected 3-cycle and contradicting the condition on the girth of G. □

[3] Observe that the condition on the cardinality implies that we cannot have $(g_1, g_2), (g_2, g_1) \in C(h)$. Thus, the requirement that G has no directed cycles of length 2 is somewhat superfluous.

That is, $C(h)$ considered as an undirected graph, is a star. Since $c(h)$ is unique whenever $|C(h)| \geq 2$, then $c(h)$ is a partial function which is defined for all h with $|C(h)| \geq 2$. This allows to define a partial function $f : V_H \longrightarrow V_G$, which is defined for every h for which $C(h) \neq \emptyset$, as follows:

$$f(h) = \begin{cases} c(h), & |C(h)| \geq 2\,, \\ g, & \text{if } C(h) = \{\,(g, g')\,\} \text{ and } h \text{ has no predecessor in } H\,, \\ g', & \text{if } C(h) = \{\,(g, g')\,\} \text{ and } h \text{ has a predecessor in } H\,. \end{cases} \tag{5.4}$$

Let us remark that if $h \in \varsigma(g, g', h_0)$, then $f(h) \in \{\, g, g' \,\}$. If gRh and h has no predecessor, then $f(h) = g$. Also, if h' is the target of $\varsigma(g, g', h_0)$ and g' has a successor, then $f(h') = g'$.

Lemma 5.5.4. If (R, ς) is a $\star$-weak simulation of G by H and $\rho : K \longrightarrow H$ is an unravelling of H, then there exists a $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$ of G by K. ■

Proof. We construct the $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$, where $\tilde{R} \subseteq V_G \times V_K$, as follows

$$g\tilde{R}k \iff gR\rho(k)$$

We consider first $\tilde{R}$ and we prove it to be surjective and functional. Since for each $g \in V_G$ there exists $h \in V_H$ such that gRh and since ρ is surjective, then there exists $k \in V_K$ such that $h = \rho(k)$, and hence $gR\rho(k)$, thus $g\tilde{R}k$. Therefore $\tilde{R}$ is surjective.
If $g_i\tilde{R}k$, $i = 1, 2$, then $g_iR\rho(k)$. Since R is functional, then $g_1 = g_2$. Therefore $\tilde{R}$ is functional.
We exhibit $\tilde{\varsigma}$ as follows. If $g\tilde{R}k_0$ and $g \to g'$, then, we take $\tilde{\varsigma}(g, g', k_0) = k_0, \ldots, k_n$, such that $\varsigma(g, g', \rho(k_0)) = \rho(k_0), \ldots, \rho(k_n)$. Note that the path $k_0, \ldots, k_n$ is unique. Therefore, $(\tilde{R}, \tilde{\varsigma})$ is a weak simulation.

Finally, whenever $(g, g'), (\tilde{g}, \tilde{g}')$ are distinct edges of G and $k_i \in \tilde{\varsigma}(g, g', k_0) \cap \tilde{\varsigma}(\tilde{g}, \tilde{g}', k_0)$, then $\rho(k_i) \in \varsigma(g, g', \rho(k_0)) \cap \varsigma(\tilde{g}, \tilde{g}', \rho(k_0))$. Since (R, ς) has the $\star$-property, we get $|\{\, g, g', \tilde{g}, \tilde{g}' \,\}| = 3$. It follows that $(\tilde{R}, \tilde{\varsigma})$ has the $\star$-property.

□

If H is a tree with back edges, rooted at h_0, then we say that a winning strategy for Cops in the game $\mathscr{E}(H,k)$ from position $(h_0, \emptyset, Cops)$ is ***rigid*** if *every time Thief has to move from a position of form* $(v, C, Thief)$ *then for every back edge* $(v,u) \in B_H$ *we have* $u \in C$.

Lemma 5.5.5. Let H be a tree with back edges, rooted at h_0, of feedback k, then Cops have a rigid winning strategy in the game $\mathscr{E}(H,k)$ from the position $(h_0, \emptyset, Cops)$. ■

Proof. The basic observation that allows to construct a rigid strategy is based on the following Claim, Lemma 12 of [BG05]

Claim 5.5.6*:* Let H be a tree with back edges of feed back k. Let $\mathcal{R}$ be the set of returns of H and if $h \in V_H$ then define $Ancet(h)$ as the set of returns which are above h and pointed by a back-edge from the descendants of h. Then there exists a total labeling

$$l : \mathcal{R} \longrightarrow \{\, 1, \ldots, k \,\}$$

with the following property:

$$\text{for all } h \in V_H, \text{ if } r_1, r_2 \in Ancet(h) \text{ then } l(r_1) \neq l(r_2)$$

■

Proof. (of the Claim) It is enough to describe a labeling that satisfies the following property:

$$\text{for all } r \in \mathcal{R}, \text{ if } r_1, r_2 \in Ancet(r) \text{ then } l(r_1) \neq l(r_2) \tag{5.5}$$

We describe a labeling of the returns of H that satisfies the property (5.5). On the one hand, for every return $r \in \mathcal{R}$ we have $|Ancet(r)| \leq k$, because the feed back of H is k. On the other hand we have $r \in Ancet(r)$. Therefore, we have the following invariant $\forall r \in \mathcal{R},\ |Ancet(r) \setminus \{\, r \,\}| \leq k-1$. Assume that the returns $r_j \in Ancet(r) \setminus \{\, r \,\}$ are labeled with $l(r_1), \ldots, l(r_{k-1})$, then clearly there is at least one label which we can assign to r such that no conflict arises. This ends the proof of the Claim. □

Let H be a tree with back edges rooted at h_0 with $fb(H) = k$, then according to the Claim there exists a labeling l of the returns of H that satisfies the property (5.5). We shall argue that this labeling may be interpreted as a rigid winning strategy for Cops in $\mathscr{E}(H, k)$ in the following way:

(i) every cop is denoted by a unique number,

(ii) at every return r visited by Thief, send the cop number $l(r)$ to r, and

(iii) Cops just skip on the vertices in $V_H \setminus \mathcal{R}$.

In other words if Cops move to $(h, C_H, Cop) \to (h, C'_H, Thief)$ then

$$C'_H = \begin{cases} C_H & \text{if } h \notin \mathcal{R}, \\ (C_H \setminus \{ r \}) \cup \{ h \} & \text{if } h \in \mathcal{R} \text{ and } \exists r \in C_H \text{ s.t. } l(r) = l(h), \\ C_H \cup \{ h \} & \text{otherwise.} \end{cases} \tag{REPLACE}$$

Let us prove now that the above Cops strategy is rigid, i.e. if Thief has to move from some $(h, C, Thief)$ then for every back edge $(h, r) \in B_H$ we have $r \in C$. Towards a contradiction: assume that there exists $(h, C, Thief)$ and a back edge $(h, r) \in B_H$ such that $r \notin C$. On the one hand, since r is a return then – by definition of the strategy – when r has been visited for the first time then there would be a cop placed there. On the other hand, since $r \notin C$, then this cop has been removed at some return r' that lies on the tree path from r to h, that is, on r' Cops have played the replace move

$$(r', C_{r'}, Cops) \to (r', (C_{r'} \setminus \{ r \}) \cup \{ r' \})$$

This implies $l(r') = l(r)$, according to the invariant (REPLACE). This is a contradiction because $r', r \in Ancet(r)$ and therefore $l(r') \neq l(r)$. $\square$

Remark 5.5.7. Let us remark that, by using a rigid strategy,

(i) every path chosen by Thief in H is a tree path,

(ii) if the position in $\mathscr{E}(H,k)$ is of the form $(h, C, Thief)$, and $h' \neq h$ is in the subtree of h, then the unique tree path from h to h' does contain no cops, apart possibly for the vertex h.

■

The following Theorem establishes the desired connection between $\star$-weak simulations and entanglement.

Theorem 5.5.8. If (R, ς) is a $\star$-weak simulation of G by H, then $\mathscr{E}(G) \leq \mathscr{E}(H) + 2$. ■

Proof. Let $k = \mathscr{E}(H)$. We shall define first a strategy for Cops in the game $\widetilde{\mathscr{E}}(G, k+2)$. In a second time, we shall prove that this strategy is a *winning* strategy for Cops.

Let us consider Thief's first move in $\widetilde{\mathscr{E}}(G, k+2)$. This move picks $g \in G$ leading to the position $(g, \emptyset, Cops)$ of $\widetilde{\mathscr{E}}(G, k+2)$. Cops answer by occupying the current position, i.e. he moves to $(g, \{\, g \,\}, Thief)$. After this move, Cops also choose a tree with back edges of feedback k to which H unravel, $\pi : \mathcal{T}(H) \longrightarrow H$, such that the root h_0 of $\mathcal{T}(H)$ satisfies $gR\pi(h_0)$. We can also suppose that h_0 is not a return, thus it has no predecessor. According to Lemma 5.5.4 we can lift the $\star$-weak simulation (R, ς) to a $\star$-weak simulation $(\tilde{R}, \tilde{\varsigma})$ of G by $\mathcal{T}(H)$. In other words, we can suppose from now on that H itself is a tree with back edges of feedback k rooted at h_0 and, moreover, that gRh_0.

From this point on, Cops use a memory to choose how to place cops in the game $\widetilde{\mathscr{E}}(G, k+2)$. To each Thief's position $(g, C_G, Thief)$ in $\widetilde{\mathscr{E}}(G, k+2)$ we associate a data structure (the memory) consisting of a triple $M(g, C, Thief) = (p, c, h)$, where $c, h \in V_H$ and $p \in V_H \cup \{\, \perp \,\}$ (we assume that $\perp \notin V_H$). Moreover c is an ancestor of h in the tree and, whenever $p \neq \perp$, p is an ancestor of c as well.

Intuitively, we are matching the play in $\widetilde{\mathscr{E}}(G, k+2)$ with a play in $\mathscr{E}(H, k)$, started at the root h_0 and played by Cops according to a rigid strategy, Lemma 5.5.5. Thus c is the vertex of H currently occupied by Thief in the

game $\mathscr{E}(H,k)$.[4] Instead of recalling all the play (that is, the history of all the positions played so far), we need to record the last position played in $\mathscr{E}(H,k)$: this is p, which is undefined when the play begins. Cops on G are positioned on the images of Cops on H by the function f defined in (5.4). Moreover, Cops eagerly occupy the last two vertices visited on G. Thief's moves on G are going to be simulated by sequences of Thief's moves on H, using the $\star$-weak simulation (R,ς). In order to make this possible, a simulation of the form $\varsigma(\tilde{g},g,\tilde{h})$ must be halted before its target h; the current position c is such halt-point. This implies that the simulation of $g \to g'$ by (R,ς) and the sequence of moves in H matching Thief's move on G are sligthly out of phase. To cope with that, Cops must guess in advance what might happen in the rest of the simulation and this is why the put cops on the current and previous positions in G. We also need to record h, the target of the previous simulation into the memory.

The previous considerations are formalized by requiring the following conditions to hold. To make sense of them, let us say that $f(\{\,p\,\}) = f(p)$ if $p \in V_H$ and that $f(\{\,p\,\}) = \emptyset$ if $p = \bot$. In the last two conditions we require that $p \neq \bot$.

- $C_G = f(C_H(c)) \cup f(\{\,p\,\}) \cup \{\,g\,\}\,,$ (COPS)
- $f(c) = g,$ and $f(h') \in f(\{\,p\,\}) \cup \{\,g\,\},$ whenever
  h' lies on the tree path from c to $h\,,$ (TAIL)
- $f(p) \to g\,, f(p)R\tilde{h}$ for some $\tilde{h} \in V_H, c \in \varsigma(f(p),g,\tilde{h}),$
  and h is the target of $\varsigma(f(p),g,\tilde{h})\,,$ (HEAD)
- on the tree path from p to c,
  c is the only vertex s.t. $f(c) = g\,.$ (HALT)

Since h_0 has no predecessors, then gRh_0 implies $f(h_0) = g$. Thus, at

[4]More precisely we are associating to the position $(g, C_G, Thief)$ of $\mathscr{E}(G,k+2)$ the position $(c, C_H, Thief)$ in $\mathscr{E}(H,k)$, where C_H is determined as $C_H = C_H(c)$ as in Remark 5.5.7.

the beginning, the memory is set to $(\perp, h_0, h_0)$ and conditions (COPS) and (TAIL) hold.

Consider now a Thief's move of the form $(g, C_G, Thief) \to (g', C_G, Cops)$, where $g' \notin C_G$. If g' has no successor, then Cops simply skip, thus reaching a winning position. Let us assume that g' has a successor, and write $\varsigma(g, g', h) = hh_1 \dots h_n$, $n \geq 1$; observe that $f(h_n) = g'$. If for some $i = 1, \dots, n$ h_i is not in the subtree of c, then the strategy halts, Cops abandon the game and looses. Otherwise, all the path $\pi = c \dots hh_1 \dots h_n$ lies in the subtree of c. By eliminating cycles from π, we obtain a simple path σ, of source c and target h_n, which entirely lies in the subtree of c. By Lemma 5.2.1, σ is the tree path from c to h_n. An explicit description of σ is as follows: we can write σ as the compose $\sigma_0 \star \sigma_1$, where the target of σ_0 and source of σ_1 is the vertex of $\varsigma(g, g', h)$ which is closest to the root h_0; moreover σ_0 is a prefix of the tree path from c to h, and σ_1 is a postfix of the path $\varsigma(g, g', h)$.

We cut σ as follows: we let c' be the first vertex on this path such that $f(c') = g'$. Thief's move $g \to g'$ on G is therefore simulated by Thief's moves from c to c' on H. This is possible since every vertex lies in the subtree of c and thus it has not yet been explored. Cops consequently occupy the returns on this path, thus modifying C_H to $C'_H = C_H(c') = (C_H \setminus X) \uplus Y$, where $X \subseteq C_H$ and Y is a set of at most k vertexes containing the last returns visited on the path from c to c'.

After the simulation on H, Cops move to $(g', C'_G, Thief)$ in $\widetilde{\mathscr{E}}(G, k+2)$, where $C'_G = f(C'_H) \cup \{ g, g' \}$. Let us verify that this is an allowed move according to the rules of the game. We remark that $f(Y) \subseteq f(\{ p \}) \cup \{ g, g' \}$ and therefore

$$\begin{aligned} C'_G &= f(C_H \setminus X) \cup f(Y) \cup \{ g, g' \} \\ &= (f(C_H \setminus X) \cup (f(Y) \setminus \{ g' \}) \cup \{ g \}) \cup \{ g' \} \\ &= A \cup \{ g' \}, \end{aligned}$$

where $A = f(C_H \setminus X) \cup (f(Y) \setminus \{ g' \}) \cup \{ g \} \subseteq f(C_H) \cup f(\{ p \}) \cup \{ g \} = C_G$. After the simulation Cops also update the memory to $M(g', C'_G, Thief) = (c, c', h_n)$. Since $f(c) = g$, then condition (COPS) clearly holds. Also, $f(c) =$

$g \to g'$, gRh and h_n is the target of $\varsigma(f(c), g', h)$. We have also that $c' \in \sigma_1$ and hence $c' \in \varsigma(f(c), g', h)$, since otherwise $c' \in \sigma_0$ and $f(c') \in \{ f(p), g \}$, contradicting $f(c') = g'$ and the condition on the girth of G. Thus condition (HEAD) holds as well. Also, condition (HALT) holds, since by construction c' is the first vertex on the tree path from c to h such that $f(c') = g'$. Let us verify that condition (TAIL) holds: by construction $f(c') = g'$, and the path from c' to h_n is a postfix of $\varsigma(g, g', h)$, and hence $f(h') \in \{ g, g' \}$ if h' lies on this tree path.

Let us now prove that the strategy is winning. If Cops never abandon, then an infinite play in $\widetilde{\mathscr{E}}(G, k+2)$ would give rise to an infinite play in $\mathscr{E}(H, k)$, a contradiction. Thus, let us prove that Cops will never abandon. To this goal we need to argue that when Thief plays the move $g \to g'$ on G, then the simulation $\varsigma(g, g', h) = hh_1 \dots h_n$ lies in the subtree of c. If this is not the case, let i be the first index such that h_i is not in the subtree of c. Therefore h_i is a return and, by the assumptions on H and on rigid strategy, $h_i \in C_H(c)$. Since $h_i \in \varsigma(g, g', h)$, $f(h_i) \in \{ g, g' \}$. Observe, however that we cannot have $f(h_i) = g'$, otherwise $g' \in f(C_H(c)) \subseteq C_G$. We deduce that $f(h_i) = g$ and that $g \in f(C_H) \subseteq C_G$.

Since $C_G \neq \bot$, then $(g, C_G, Thief)$ is not the initial position of the play, so that, if $M(g, C_G, Thief) = (p, c, h)$, then $p \neq \bot$. Let us now consider the last two moves of the play before reaching position $(g, C_G, Thief)$. These are of the form $(f(p), \tilde{C}_G, Thief) \to (g, \tilde{C}_G, Cops) \to (g, C_G, Thief)$, and have been played according to this strategy. Since $g \notin \tilde{C}_G$, it follows that the Cop on h_i has been dropped on H during the previous round of the strategy, simulating the move $f(p) \to g$ on G by the tree path from p to c. This is however in contradiction with condition (HALT), stating that c is the only vertex h on the tree path from p to c such that $f(h) = c$. $\square$

5.6 Strongly synchronizing games

In this section we define *strongly synchronizing* games, a generalization of synchronizing games introduced in [San02a]. We shall show that, for every game H equivalent to a strongly synchronizing game G, there is a $\star$-weak simulation of G by H.[5]

Let us say that $G \in \mathcal{G}$ is *bipartite* if $M^G \subseteq Pos_E^G \times Pos_{A,D}^G \cup Pos_A^G \times Pos_{E,D}^G$.

Definition 5.6.1. A game G is *strongly synchronizing* iff its is bipartite, it has girth strictly greater than 4 and, for every pair of positions g, k, the following conditions hold:

1. if $(G, g) \sim (G, k)$ then $g = k$.
2. if $(G, g) \leq (G, k)$ and $(G, k) \not\leq (G, g)$, then $k \in Pos_E^G$ and $(k, g) \in M^G$, or $g \in Pos_A^G$ and $(g, k) \in M^G$.

■

A consequence of the previous definition is the following Lemma.

Lemma 5.6.2. If G is strongly synchronizing, then the only winning strategy in the game $\langle G, G\rangle$ is the copycat strategy. Thus strongly synchronizing games are synchronizing as defined in [San02a]. ■

Proof. Let us consider a position $g \in Pos_E^G$, and let us analyze the position (g, g) of $\langle G, G\rangle$. Let us suppose that $(g, g') \in M^G$ and consider the possible Mediator's answers to the Opponents' move $(g, g) \to (g', g)$.

Mediator cannot answer $(g', g) \to (g'', g)$, since then the relation $(G, g'') \leq (G, g)$ implies that either $g'' = g$ (hence having a cycle of length 2 in G), or that there is an undirected edge between g'' and g, thus creating a length 3 cycle.

Similarly Mediator cannot answer $(g', g) \to (g', \tilde{g})$ with $g' \neq \tilde{g}$. Again, this would create a length 3 cycle in the undirected version of G. □

[5] In the sequel, we shall not distinguish between a game and its underlying graph.

We list next some useful properties of strongly synchronizing games.

Lemma 5.6.3. Let G be a strongly synchronizing and let $(g, g'), (\tilde{g}, \tilde{g}') \in M^G$ be distinct.

1. If $(G, g) \sim \hat{x}$ then $g \in Pos_D^G$ and $\lambda(g) = x$.
2. If $g, \tilde{g} \in Pos_E^G$ and, for some game H and $h \in Pos^H$, we have

$$(G, g') \leq (H, h) \leq (G, g) \text{ and}$$
$$(G, \tilde{g}') \leq (H, h) \leq (G, \tilde{g}),$$

   then $g = \tilde{g}$ or $g' = \tilde{g}'$, and $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.
3. If $g \in Pos_E^G$ and $\tilde{g} \in Pos_A^G$ and, for some H and $h \in Pos^H$, we have

$$(G, g') \leq (H, h) \leq (G, g) \text{ and}$$
$$(G, \tilde{g}) \leq (H, h) \leq (G, \tilde{g}'),$$

   then $g = \tilde{g}'$ or $g' = \tilde{g}$, and $|\{ g, g', \tilde{g}, \tilde{g}' \}| = 3$.

■

Proof. 1. Let χ_G be the set of free variables of G. First, we have the following claim.

Claim 5.6.4*:* If $(G, g) \sim \hat{x}$, then $x \in \chi_G$. ■

Proof. On the one hand, if $x \notin \chi_G$ then $G[x/\top] \sim G[x/\bot]$. One the other hand, $G[x/\top] \sim \hat{x}[x/\top] \sim \top$ and $G[x/\bot] \sim \hat{x}[x/\bot] \sim \bot$, thus $\bot = \top$. This ends the proof of the claim. □

If g has a successor, then the winning strategy in $\langle G, \hat{x}, G \rangle$ will suggest for example to play $(g, p_\star^{\hat{x}}, g) \to (g', p_\star^{\hat{x}}, g) \to (g', p_\star^{\hat{x}}, g')$, for some $(g, g') \in M^G$. Therefore $(G, g) \sim \hat{x} \sim (G, g')$, contradicting the fact that G is strongly synchronizing. Thus g has no successor, and clearly $g \in Pos_D^G$ and $\lambda^G(g) = x$, according to the claim.

2. We derive first $(G, g') \leq (G, \tilde{g})$ and $(G, \tilde{g}') \leq (G, g)$ and observe that each inequality is strict, because the game is bipartite. Therefore from item 2 of Definition 5.6.1 we have a diagram of the form

$$\begin{tikzcd} g \arrow[r, dash, ">"] \arrow[d] & \tilde{g}' \\ g' \arrow[r, dash, "<"] & \tilde{g} \arrow[u] \end{tikzcd}$$

   that is we have an undirected edge bewteen g and $\tilde{g}'$, and an undirected edge between g' and $\tilde{g}$.

   If $g \neq \tilde{g}$ and $g' \neq \tilde{g}'$, then the above diagram gives rise to an undirected cycle of length 4, which cannot happen.

3. As before, we derive $(G, \tilde{g}) \leq (G, g)$ and $(G, g') \leq (G, \tilde{g}')$ and moreover $(G, \tilde{g}) < (G, g)$ and $(G, g') < (G, \tilde{g}')$, since g and $\tilde{g}$ belong to opposite players. Therefore from item 2 of definition 5.6.1 we obtain a diagram of the form

$$\begin{tikzcd} g \arrow[r, dash, ">"] \arrow[d] & \tilde{g} \arrow[d] \\ g' \arrow[r, dash, "<"] & \tilde{g}' \end{tikzcd}$$

   If $g \neq \tilde{g}'$ and $g' \neq \tilde{g}$, then the above diagram gives rise to an undirected cycle of length 4, which cannot happen.

□

We are ready to state the main result of this section.

Proposition 5.6.5. *Let G be a strongly synchronizing game, and let $H \in \mathcal{G}$ be such that $G \leq H \leq G$, then there is a $\star$-weak simulation of G by H.*

Proof. Let S, S' be two winning strategies for Mediator in $\langle G, H \rangle$ and $\langle H, G \rangle$, respectively. Let $T = S||S'$ be the composal strategy in $\langle G, H, G \rangle$. Define

$$\begin{aligned} gRh \text{ iff } & (g, h, g) \text{ is a position of } T \\ & \text{and } g, h \text{ belong to the same player.} \end{aligned}$$

We consider first R and prove that it is functional and surjective. If $g_i R h, i = 1, 2$ then (g_1, h, g_1) and (g_2, h, g_2) are positions of T, hence $(G, g_1) \leq (H, h) \leq (G, g_1)$ and $(G, g_2) \leq (H, h) \leq (G, g_2)$, consequently $(G, g_1) \sim (G, g_2)$ implies $g_1 = g_2$, by definition 5.6.1. For surjectivity, we can assume that (a) all the positions of G are reachable from the initial position $p^G_\star$, (b) $p^G_\star$ and $p^H_\star$ belong to the same player (by possibly adding to H a new initial position leading to the old one). Since $T_{\backslash H}$ is the copycat strategy, given $g \in Pos^G_{E,A,D}$, from the initial position $(p^G_\star, p^H_\star, p^G_\star)$ of $\langle G, H, G\rangle$, the Opponents have the ability to reach a position of the form (g, h, g). The explicit construction of the function ς will show that h can be chosen to belong to the same player as g.

We construct now the function ς so that (R, ς) is a weak simulation. If gRh and $(g, g') \in M^G$, then we construct $\pi = h, \ldots, h'$ such that $g'Rh'$. Since G is bipartite, then $h \neq h'$ and π is nonempty. We let $\varsigma(g, g', h)$ be a reduction of π to a nonempty simple path.

We assume $(g, h) \in (Pos^G_E, Pos^H_E)$, the case $(g, h) \in (Pos^G_A, Pos^H_A)$ is dual. From position (g, h, g) it is Opponent's turn to move on the left, they choose a move $(g, g') \in M^G$. Since G is bipartite, we have either $g' \in Pos^G_D$ or $g' \in Pos^G_A$.

Case (i). If $g' \in Pos^G_D$ then the strategy T suggests playing a finite path on H, $(g', h, g) \to^* (g', h^*, g)$, possibly of zero length, and then it will suggest to play on the external right board. An infinite path played only on H cannot arise, since T is a winning strategy and such an infinite path is not a win for Mediator. Since $T_{\backslash H}$ is the copycat strategy, T suggests the only move $(g', h^*, g) \to (g', h^*, g')$. From this position T suggests playing a path on H leading to a final draw position $h_f \in Pos^H_D$ as follows $(g', h^*, g') \to^* (g', h_f, g')$, such that $\lambda^G(g') = \lambda^H(h_f)$, therefore $g'Rh_f$.

Case (ii). If $g' \in Pos^G_A$ then from position (g', h, g) it is Mediator's turn to move. We claim that T will suggest playing a nonempty finite path $(g', h, g) \to^+ (g', h', g)$ on the central board H, where $h' \in Pos^H_A$, and then suggests the move $(g', h', g) \to (g', h', g')$. Let $\tilde{h} \in Pos^H_{A,E,D}$ be such that the position $(g', \tilde{h}, g)$ has been reached from (g', h, g), through a (possibly empty) sequence

of central moves, by playing with T. Then T cannot suggest a move on the left board $(g', \tilde{h}, g) \to (g'', \tilde{h}, g)$, since $T_{\backslash H}$ is the copycat strategy. Also, if $\tilde{h} \in Pos_E^H$, T cannot suggest a move on the right board $(g', \tilde{h}, g) \to (g', \tilde{h}, \tilde{g})$. The reason is that $T = S||S'$, and the position $(\tilde{h}, g)$ of $\langle H, G\rangle$ does not allow a Mediator's move on the right board. Thus a sequence of central moves on H is suggested by T and, as mentioned above, this sequence cannot be infinite. We claim that its endpoint $h' \in Pos_A^H$. We already argued that $h' \notin Pos_E^H$, let us argue that $h' \notin Pos_D^H$. If this were the case, then strategy T suggests the only move $(g', h', g) \to (g', h_n, g')$, hence $(G, g') \sim (H, h')$. By Lemma 5.6.3.1, we get $g' \in Pos_D^G$, contradicting $g' \in Pos_A^G$.

This proves that (R, ς) is a weak simulation. We prove next that (R, ς) has the $\star$-property, thus assume that $h^* \in \varsigma(g, g', h_0), \varsigma(\tilde{g}, \tilde{g}', \tilde{h}_0)$. Let us suppose first that $g, \tilde{g} \in Pos_E^H$. By looking at the construction of these paths, we observe that the two sequences of moves

$$(g, h_0, g) \to (g', h_0, g) \to^* (g', h^*, g) \to^* (g', h_n, g) \to (g', h_n, g')\,,$$
$$(\tilde{g}, \tilde{h}_0, \tilde{g}) \to (\tilde{g}', \tilde{h}_0, \tilde{g}) \to^* (\tilde{g}', h^*, \tilde{g}) \to^* (\tilde{g}', \tilde{h}_m, \tilde{g}) \to (\tilde{g}', \tilde{h}_m, \tilde{g}')\,,$$

may be played in the game $\langle G, H, G\rangle$, according to the winning strategy $T = S||S'$. We have therefore that $(G, g') \leq (H, h^*) \leq (G, g)$ and $(G, \tilde{g}') \leq (H, h^*) \leq (G, \tilde{g})$.[6] Consequently $|\{\, g, g', \tilde{g}, \tilde{g}' \,\}| = 3$, by Lemma 5.6.3.2. If $g \in Pos_E^G$ and $\tilde{g} \in Pos_A^G$, a similar argument shows that the positions (g', h^*, g) and $(\tilde{g}, h^*, \tilde{g}')$ may be reached with T and hence $(G, g') \leq (H, h^*) \leq (G, g)$ and $(G, \tilde{g}) \leq (H, h^*) \leq (G, \tilde{g}')$. Lemma 5.6.3.3 implies then $|\{\, g, g', \tilde{g}, \tilde{g}' \,\}| = 3$. Finally, the cases $(g, \tilde{g}) \in \{\, (Pos_A^G, Pos_A^G), (Pos_A^G, Pos_E^G) \,\}$ are handled by duality. This completes the proof of Proposition 5.6.5. $\square$

[6]Similar inequalites may be derived even if $h^* \in Pos_D^H$. In this case the moves in the central board may be interleaved with the move on the right board.

5.7 Construction of strongly synchronizing games

In this section we complete the hierarchy theorem by constructing strongly synchronizing games of arbitrary entanglement according to the following recipe: given a digraph G we shall construct a game $\mathscr{P}(G) \in \mathcal{G}$ such that $\mathscr{E}(G) = \mathscr{E}(\mathscr{P}(G))$ and $\mathscr{P}(G)$ is strongly synchronizing. If $H \in \mathcal{G}$ and $H \sim \mathscr{P}(G)$, then by the result of the previous Section threre exists a $\star$-weak simulation of $\mathscr{P}(G)$ by H. This in turn implies, by Theorem 5.5.8, that the entanglement of H is at least $\mathscr{E}(G) - 2$.

To this goal, we need the following definition.

Definition 5.7.1. Let I be a finite set of indices. For $i, j \in I$ and $l \geq 1$, the *chain* of $\mathscr{A}_{i,j,l}$ is the graph defined as follows:

- The set of vertices is

$$\{\, v_{i,0,0}, v_{i,j,k}, v_{j,0,0} \mid k = 1, \ldots, l \,\} \cup \{\, w_{i,0,0}, w_{i,j,k}, w_{j,0,0} \mid k = 1, \ldots, l \,\}\,.$$

- The set of edges is

$$\{\, v_{i,0,0} \to v_{i,j,1}, v_{i,j,k} \to v_{i,j,k+1}, v_{i,j,k} \to v_{j,0,0} \mid k = 1, \ldots, l-1 \,\}$$
$$\cup\ \{\, v_{i,0,0} \to w_{i,0,0}, v_{i,j,k} \to w_{i,j,k}, v_{j,0,0} \to w_{j,0,0} \mid k = 1, \ldots, l \,\}\,.$$

■

For instance, the chain $\mathscr{A}_{i,j,5}$ appears in Figure 5.7.1.

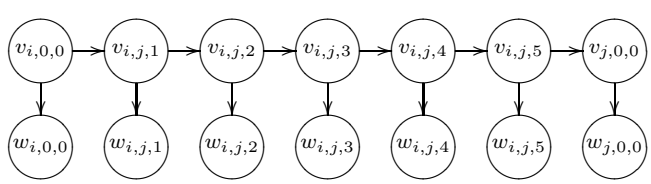


Figure 5.7.1: The chain $\mathscr{A}_{i,j,5}$.

Next, for a directed graph G we construct a game $\mathscr{P}_l(G)$. To avoid additional notations, we shall assume that G is strongly connected and not reduced to a point, that the set of vertices of G is the indexing set I.

Definition 5.7.2. Let $l \geq 5$ be an odd number, and let G be a directed graph. The game $\mathscr{P}_l(G) \in \mathcal{G}$ is defined as follows:

- The underlying graph of positions and moves of $\mathscr{P}_l(G)$ is obtained from G by identifying the vertex $i \in I$ with the vertex $v_{i,0,0}$ and substituting every edge $(i,j) \in E_G$ with a chain $\mathscr{A}_{i,j,l}$. That is:

$$Pos_{A,E,D}^{\mathscr{P}_l(G)} = \bigcup \{ V_{\mathscr{A}_{i,j,l}} \mid (i,j) \in E_G \},$$
$$M^{\mathscr{P}_l(G)} = \bigcup \{ E_{\mathscr{A}_{i,j,l}} \mid (i,j) \in E_G \}.$$

- The assignment of players to positions is as follows:

$$Pos_E^{\mathscr{P}_l(G)} = \{ v_{i,0,0} \mid i \in I \} \cup \{ v_{i,j,k} \mid (i,j) \in E_G \text{ and } k \bmod 2 = 0 \}$$
$$Pos_A^{\mathscr{P}_l(G)} = \{ v_{i,j,k} \mid (i,j) \in E_G \text{ and } k \bmod 2 = 1 \}$$
$$Pos_D^{\mathscr{P}_l(G)} = \{ w_{i,0,0} \mid i \in I \} \cup \{ w_{i,j,k} \mid (i,j) \in E_G \text{ and } k = 1, \ldots, l \}$$

  Let X be a countable set of variables that includes the set $\{ x_{i,j,l} \mid i,j \in I \text{ and } l = 0, \ldots, k \} \cup \{ x_i \mid i \in I \}$. The labelling of draw positions, $\lambda^{\mathscr{P}_l(G)} : Pos_D^{\mathscr{P}_l(G)} \longrightarrow X$, sends sends $w_{i,j,l}$ to $x_{i,j,l}$.

- Finally, the rank function $\rho^{\mathscr{P}_l(G)}$ assigns a constant odd rank to all positions (apart the draws ones).

■

Let K_2^+ be the complete directed graph on the vertices $I = \{ 0, 1 \}$. Then the game $\mathscr{P}_5(K_2^+)$ appears in Figure 5.7.2.

We state next the main facts about the game $\mathscr{P}_l(G)$: it is clear that $\mathscr{P}_l(G)$ is bipartite and the girth of $\mathscr{P}_l(G)$ is at least $l+1 \geq 6$. Moreover the entanglement is preserved.

Proposition 5.7.3. *The entanglement of the game $\mathscr{P}_l(G)$ equals that of the digraph $\mathscr{E}(G)$.*

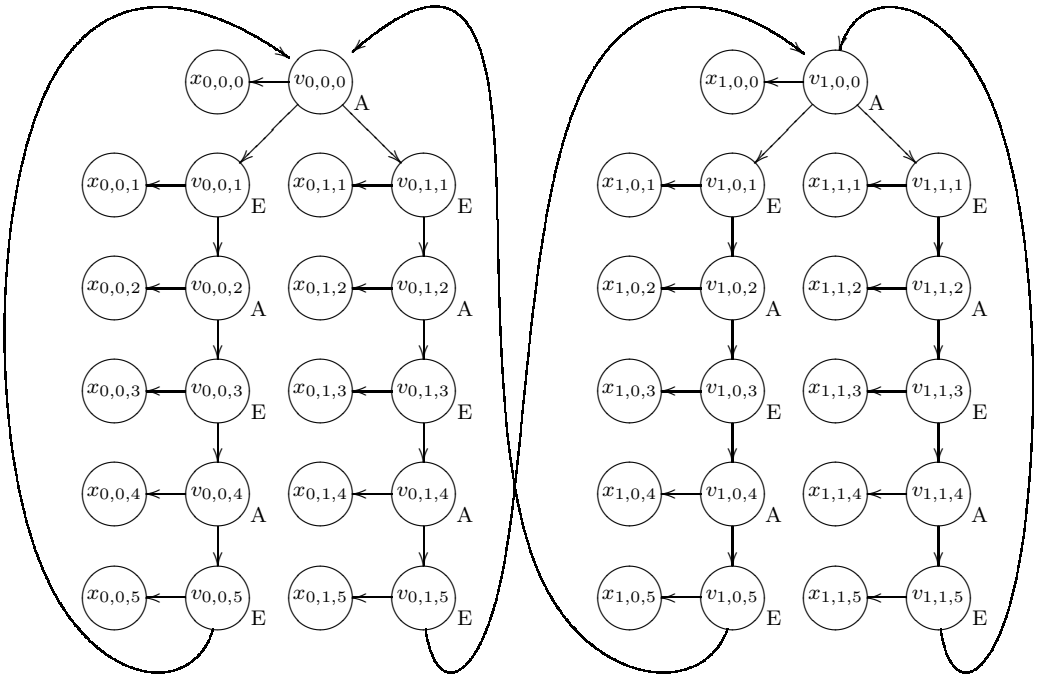


Figure 5.7.2: The game $\mathscr{P}_5(K_2^+)$

Proof. We shall prove a stronger statement: in a digraph, substituting an edge by a *directed acyclic graph* preserves the entanglement. Formally, given a graph G of vertices $\{ v_i, i \in I \}$ and a set of directed acyclic graphs $\{ D_{i,j} \}$, each dag $D_{i,j}$ comes with a source vertex v_i and a target vertex v_j. We define H out of G by sustituting each edge $v_i \to v_j$ of G by the dag $D_{i,j}$. It turns out that $\mathscr{E}(G) = \mathscr{E}(H)$.

First, we prove the inequality $\mathscr{E}(G) \leq \mathscr{E}(H)$. To this goal, we define a total function $f : V_H \longrightarrow V_G$ that sends the internal vertices $v \in D_{i,j} \setminus V_G$ of

each dag to its target, and its restiction on V_G is the identity:

$$f(v) = \begin{cases} v_j\,, & \text{if } v \in V_{D_{i,j}} \setminus v_i \\ v\,, & \text{otherwise}\,. \end{cases}$$

Let $k = \mathscr{E}(H)$, we shall show that every Cops' winning strategy in the game $\mathscr{E}(H,k)$ can be mapped to a Cops' winning strategy in $\widetilde{\mathscr{E}}(G,k)$.[7]To this goal, every position (g, C_G, P) of $\widetilde{\mathscr{E}}(G,k)$ is matched with a position (h, C_H, P) where $P \in \{\,Thief, Cops\,\}$, $f(h) = g$, and $f(C_H) = C_G$. Assume that $(v_i, C_G, Thief)$ is matched with $(v_i, C_H, Thief)$, hence thief's move $(v_i, C_G, Thief) \to (v_j, C_G, Cops)$ in G is simulated by a sequence of moves in H, i.e. a path $v_i h_1 \dots h_n v_j$. This simulation might be interleaved with Cops' moves in $\mathscr{E}(H,k)$ giving rise to a sequence of moves

$$(v_i, C_H, Thief) \to \cdots \to (v_j, C'_H, Cops)\,.$$

Cops' moves in this sequence as well as the last move in $\mathscr{E}(H,k)$ after the sequence, $(v_j, C'_H, Cops) \to (v_j, C''_H, Thief)$, are matched back to $\widetilde{\mathscr{E}}(G,k)$ with the move $(v_j, C_G, Cops) \to (v_j, f(C''_H), Thief)$.

Let us verify that the latter move is allowed w.r.t. to the rules of the game $\widetilde{\mathscr{E}}(G,k)$. To this end, observe that

$$C''_H = (C_H \setminus A) \cup B\,,$$

where $A \subseteq C_H$ and $B \subseteq V_{D_{i,j}} \setminus \{\,v_i\,\}$ and hence

$$\begin{aligned} f(C''_H) &= f(C_H \setminus A) \cup f(B) \\ &= (f(C_H) \setminus A') \cup f(B) \qquad\qquad \text{where } A' = f(C_H) \setminus f(C_H \setminus A) \\ &= (C_G \setminus A') \cup f(B)\,. \end{aligned}$$

Therefore $(v_j, C'_H, Cops) \to (v_j, C''_H, Thief)$ is a legal move, as $f(B) \subseteq v_j$.

[7]Recall that Cops in $\widetilde{\mathscr{E}}(G,k)$ are allowed to retire cops that have already been placed on G.

Finally, let us argue about the inequality $\mathscr{E}(H) \leq \mathscr{E}(G)$. Let $k = \mathscr{E}(G)$, we construct a winning strategy for Cops in $\mathscr{E}(H,k)$ out of Cops' winning strategy in $\mathscr{E}(G,k)$ as follows. In the game $\mathscr{E}(H,k)$, Cops skip on the vertices $V_H \setminus V_G$ of H, and hence Thief will either arrive to a vertex without a successor (where he looses) or he will arrive to some vertex $v \in V_G$, because the subgraph induced by $V_H \setminus V_G$ is a dag. Moreover, cops on the vertices $V_G \subseteq V_H$ are placed exactly as if in the game $\mathscr{E}(H,k)$. □

We come next to the problem of showing that the graphs $\mathscr{P}_l(G)$ are strongly synchronizing. We tackle this problem through some Lemmas.

Lemma 5.7.4. If $(\mathscr{P}_l(G), w_{i,j,k}) \leq (\mathscr{P}_l(G), g)$ then either $g = w_{i,j,k}$ or $g \in Pos_E^{\mathscr{P}_l(G)}$ and $g = v_{i,j,k}$. ■

Proof. We split the proof into several cases, according to the player that owns the position g.
Case (i). $g \in Pos_D^{\mathscr{P}_l(G)}$ Thus $g = w_{i',j',k'}$, then we need to have $(i,j,k) = (i',j',k')$ since, otherwise, the games $\hat{x}_{i,j,k}$ and $\hat{x}_{i',j',k'}$ are incomparable.

We let therefore $g = v_{i',j',k'}$.
Case (ii). $g \in Pos_A^{\mathscr{P}_l(G)}$ and $(i,j,k) \neq (i',j',k')$ Opponents can choose to move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$, the latter being a lost position for Mediator.
Case (iii). $g \in Pos_A^{\mathscr{P}_l(G)}$ and $(i,j,k) = (i',j',k')$ Opponents can choose to move $(w_{i,j,k}, v_{i,j,k}) \to (w_{i,j,k}, v_{i',j',k'})$ with $(i,j,k) \neq (i',j',k')$. From this position Mediator cannot move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$, nor can move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i'',j''',k''})$, since the girth of $\mathscr{P}_l(G)$, being at least 6, implies that $(i,j,k) \neq (i'',j'',k'')$ and $v_{i'',j'',k''} \in Pos_A^{\mathscr{P}_l(G)}$, falling back into case (ii).
Case (iv). $g \in Pos_E^{\mathscr{P}_l(G)}$ and $(i,j,k) \neq (i',j',k')$ Then Mediator cannot move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, w_{i',j',k'})$. He cannot either move $(w_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i'',j'',k''})$ since $v_{i'',j'',k''} \in Pos_A^{\mathscr{P}_l(G)}$, thus falling back either into case (ii), or into case (iii).

Therefore, the only possibility is that $g \in Pos_E^{\mathscr{P}_l(G)}$ and $(i,j,k) = (i',j',k')$. □

Dualizing the previous proof we obtain:

Lemma 5.7.5. If $(\mathscr{P}_l(G), g) \leq (\mathscr{P}_l(G), w_{i,j,k})$ then either $g = w_{i,j,k}$ or $g \in Pos_A^{\mathscr{P}_l(G)}$ and $g = v_{i,j,k}$. ■

Lemma 5.7.6. If $(\mathscr{P}_l(G), v_{i,j,k}) \leq (\mathscr{P}_l(G), v_{i',j',k'})$ and $v_{i,j,k} \neq v_{i',j',k'}$, then either $v_{i,j,k} \in Pos_A^{\mathscr{P}_l(G)}$ and $(v_{i,j,k}, v_{i',j',k'}) \in M^{\mathscr{P}_l(G)}$, or $v_{i',j',k'} \in Pos_E^{\mathscr{P}_l(G)}$ and $(v_{i',j',k'}, v_{i,j,k}) \in M^{\mathscr{P}_l(G)}$. ■

Proof. Let us suppose that $v_{i,j,k} \in Pos_A^{\mathscr{P}_l(G)}$. We remark that $v_{i',j',k'} \notin Pos_D^{\mathscr{P}_l(G)}$, and thus we split the proof into two cases.
Case (i). $v_{i',j',k'} \in Pos_A^{\mathscr{P}_l(G)}$ Then Opponents can move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, w_{i',j',k'})$ and this is a lost position for Mediator by Lemma 5.7.5.
Case (ii). $v_{i',j',k'} \in Pos_E^{\mathscr{P}_l(G)}$ Mediator has two kinds of moves. He can choose to move to a "variable", that is, to move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, w_{i',j',k'})$ or $(v_{i,j,k}, v_{i',j',k'}) \to (w_{i,j,k}, v_{i',j',k'})$. These moves, however, lead to lost positions, by Lemmas 5.7.4 and 5.7.5. Therefore, if the position $(v_{i,j,k}, v_{i',j',k'})$ is winning, then he can only move $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i,j,k}, v_{i'',j'',k''})$ or $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i'',j'',k''}, v_{i',j',k'})$. In the first case, if the position $(v_{i,j,k}, v_{i'',j'',k''})$ is winning, then $(i,j,k) = (i'',j'',k'')$ by case (i); hence $(v_{i',j',k'}, v_{i,j,k}) \in M^{\mathscr{P}_l(G)}$. In the second case, if Mediator moves to a winning position $(v_{i,j,k}, v_{i',j',k'}) \to (v_{i'',j'',k''}, v_{i',j',k'})$, then $(i',j',k') = (i'',j'',k'')$ by the dual of case (i) and hence $(v_{i,j,k}, v_{i',j',k'}) \in M^{\mathscr{P}_l(G)}$. □

We come to our main goal.

Proposition 5.7.7. *The games $\mathscr{P}_l(G)$ are strongly synchronizing.*

Proof. Let us prove first that $(\mathscr{P}_l(G), g) \sim (\mathscr{P}_l(G), \tilde{g})$ implies $g = \tilde{g}$. Thus we assume that $(\mathscr{P}_l(G), g) \sim (\mathscr{P}_l(G), \tilde{g})$, we split the proof that $g = \tilde{g}$ into three cases, according to the player of g.

Case (i). $g \in Pos_D^{\mathscr{P}_l(G)}$ Thus let $g = w_{i,j,k}$. If $g \neq \tilde{g}$, then Lemma 5.7.4 implies that $\tilde{g} = v_{i,j,k}$ with $\tilde{g} \in Pos_E^{\mathscr{P}_l(G)}$. Similarly Lemma 5.7.5 implies that $\tilde{g} = v_{i,j,k}$ with $\tilde{g} \in Pos_A^{\mathscr{P}_l(G)}$. Thus we reach a contradiction, and therefore $g = \tilde{g}$.
Case (ii). $g = v_{i,j,k} \in Pos_E^{\mathscr{P}_l(G)}$ Then $(\mathscr{P}_l(G), w_{i,j,k}) < (\mathscr{P}_l(G), g) \sim (\mathscr{P}_l(G), \tilde{g})$ and therefore $\tilde{g} = v_{i,j,k}$ by Lemma 5.7.4.
Case (iii). $g = v_{i,j,k} \in Pos_A^{\mathscr{P}_l(G)}$ Then $(G, \tilde{g}) \sim (G, g) < (G, w_{i,j,k})$ and therefore $\tilde{g} = v_{i,j,k}$ by Lemma 5.7.5.

Let us now prove that $(\mathscr{P}_l(G), g) \leq (\mathscr{P}_l(G), \tilde{g})$ and $g \neq \tilde{g}$ implies $\tilde{g} \in Pos_E^{\mathscr{P}_l(G)}$ and $(\tilde{g}, g) \in M^{\mathscr{P}_l(G)}$ or $g \in Pos_E^{\mathscr{P}_l(G)}$ and $(g, \tilde{g}) \in M^{\mathscr{P}_l(G)}$. This is the case if $g \in Pos_D^{\mathscr{P}_l(G)}$ or $\tilde{g} \in Pos_D^{\mathscr{P}_l(G)}$, by Lemmas 5.7.4 and 5.7.5. If both $g, \tilde{g} \in Pos_{E,A}^{\mathscr{P}_l(G)}$, then the statement follows from Lemma 5.7.6. □

We are now ready to state the main achievement of this Chapter.

Theorem 5.7.8. For $n \geq 3$, the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$ are strict. Therefore the variable hierarchy for the games μ-calculus is infinite. ■

For each $n \geq 0$, let G_n be a graph such that $\mathscr{E}(G_n) = n$. Then, by Proposition 5.7.3, the game $\mathscr{P}_l(G) \in \mathcal{G}$, $l \geq 5$ odd, is such that $\mathscr{E}(\mathscr{P}_l(G_n)) = \mathscr{E}(G_n) = n$, i.e. $\mathscr{P}_l(G_n) \in \mathcal{L}_n$. Also, since $\mathscr{P}_l(G_n)$ is strongly synchronizing, if $H \sim \mathscr{P}_l(G_n)$, then there exists a $\star$-weak simulation of $\mathscr{P}_l(G_n)$ by H. It follows by Theorem 5.5.8 that $n - 2 \leq \mathscr{E}(H)$. Therefore $\mathscr{P}_l(G) \notin \mathcal{L}_{n-3}$.

Part II

Entanglement in Graph Theory

Chapter 6

Graph Theoretic and Algorithmic Aspects of Entanglement

This Chapter is devoted to one of the basic properties of undirected entanglement. We prove that the class of undirected graphs of entanglement at most k, for arbitrary fixed $k \in \mathbb{N}$, is closed under taking minors. Some algorithmic properties of entanglement are discussed.

6.1 Introduction

A minor of a graph H describes the structure of H in a more general way than than the subgraph does. A graph G is *minor* [Die05, §1.7] of a graph H if G can be obtained from H by successive application of the following operations on it:

(a) delete an edge,

(b) contract an edge,

(c) delete an isolated vertex.

Wagner formulated a deep conjecture in [Wag70] which states that *in any infinite set of (finite) graphs, one of its members is a minor of another.* This conjecture has been proved by Robertson and Seymour in the Graph Minors series; their proof is completed in [RS04].

A class $\mathscr{C}$ of graphs is minor closed if is closed under taking minors, that is whenever $H \in \mathscr{C}$ then for each minor G of H we have that $G \in \mathscr{C}$. Let us see some examples of classes of graphs which are minor closed. The class of graphs being cycle free, i.e being a forest. The class of planar graphs, i.e. the graphs which can be drawn in the plan such that any two edges do not intersect. A similar notion is *knotless embeddable*: these are the graphs which can be embedded in 3-space in such a way no two cycles are linked and no cycle is knotted. This class is also minor closed. Several topological properties of graphs are minor closed, for instance being embeddable in a fixed surface.

Every minor closed class $\mathscr{C}$ of graphs can be described by specifying the minimal set of all the minors which are not in $\mathscr{C}$, these minors are called the *excluded minors for* $\mathscr{C}$. A important consequence of Graph Minors' Theorem, is the following result : *for every minor closed class of graphs, the set of excluded minors is finite.*

Some examples of finite excluded minors were already known for specific classes of graphs before the Robertson–Seymour theorem was proved. For example, the excluded minors for the class of all forests is the set containing only the cycle with three vertices. An excluded set for the class of paths is the set containing only the tree with four vertices, one of which has degree 3. Kuratowski proved in [Kur30] that a graph G is planar if and only if it does not contain the complete graph K_5 neither the bipartite graph $K_{3,3}$ as a minor. In other words, he states that the set $\{ K_5, K_{3,3} \}$ is the set of excluded minors for the class of all planar graphs.

Robertson and Seymour also proved in [RS86] that for every fixed graph M there is an algorithm that has time complexity $O(n^3)$ and that and that for a given graph G of vertices n decides if M is a minor of G. This yields an

other interesting result: *for every minor closed class $\mathscr{C}$ of graph there exists a $O(n^3)$ time algorithm for testing menbership in $\mathscr{C}$.*
Something is hidden in O: first, the constants are huge; and second, they depend on the list of excluded minors. Graph Minors Theorem guaranties only the finiteness of the list of excluded minors, this list may be very large and even it is not easy to find it in general, the theorem does not describe how to obtain it.

In this Chapter we prove that, for every $k \in \mathbb{N}$, the class of undirected graphs of entanglement at most k, is minor closed.

All the graphs in this Chapter are simple (without multi-edges), finite, and undirected.

Wagner conjecture was published in [Wag37]. Proofs of Wagner conjecture are usually technical and complicated, they are documented in the series of Graph Minors papers of Robertson and Seymour. Among these papers, the following are in a direct relation with this Chapter: [RS86, RS04]. A concise reference on the topic is [Die05].

6.2 Formal definitions

Intuitively, a contraction of an edge in a graph is to "slide" the vertices of an edge together until they coincide, formally:

Definition 6.2.1. Given a graph G, the contraction of an edge $\{ a, b \}$ is the graph $\partial^z_{a,b} G = (V', G')$ such that $V' = V_G \setminus \{ a, b \} \cup \{ z \}$ and $E' = E_G \setminus \{ \{ x, a \}, \{ x, b \},\ x \in V_G \} \cup \{ \{ x, z \}$ s.t. $x \in V_G, \{ x, a \} \in E_G$ or $\{ x, b \} \in E_G \}$. ■

Given a graph H and an edge e, *edge-deletion* results in a graph $H \setminus e$ with the same vertex set as H and the edge set $E_H \setminus \{e\}$.

The neighbours of a vertex v in a graph G are denoted by $\mathcal{N}(v) := \{ v' \mid vv' \in E_G \}$.

Definition 6.2.2. A class $\mathscr{C}$ is minor closed if whenever $H \in \mathscr{C}$ and G is a minor of H, then $G \in \mathscr{C}$. ■

The following is the Graph Minors theorem, [RS04].

Theorem 6.2.3. Let $G_1, G_2, \ldots$ be an infinite list of (finite) graphs. Then, for every i there exists $j > i$ such that G_i is a minor of G_j. ■

Corollary 6.2.4. Let $\mathscr{C}$ be minor closed class of graphs and let $\mathscr{F}$ be the set of excluded minors characterizing the class $\mathscr{C}$, in the sense that $G \in \mathscr{C}$ if and only if there is no $F \in \mathscr{F}$ such that F is a minor of G.
Then, $\mathscr{F}$ is finite. ■

6.3 Entanglement

To make this Chapter self contained, let us recall the Thief-Cops games defining entanglement. The entanglement of G is characterized by means of a game $\mathscr{E}(G, k)$, $k = 0, \ldots, |V_G|$, played by a Thief against Cops, a team of k cops.

Definition 6.3.1. (i.e. Definition 3.4.1) The entanglement game $\mathscr{E}(G, k)$ of a digraph G is defined by:

- Its positions are of the form (v, C, P), where $v \in V_G$, $C \subseteq V_G$ and $|C| \leq k, \quad P \in \{Cops, Thief\}$.
- Initially Thief chooses $v_0 \in V_G$ and moves to $(v_0, \emptyset, Cops)$.
- Cops can move from $(v, C, Cops)$ to $(v, C', Thief)$ where C' can be
  - C : Cops skip,
  - $C \cup \{ v \}$: Cops add a new Cop on the current position,
  - $(C \setminus \{ x \}) \cup \{ v \}$: Cops move a placed Cop to the current position.

- Thief can move from $(v, C, Thief)$ to $(v', C, Cops)$ if $(v, v') \in E_G$ and $v' \notin C$.

Every finite play is a win for Cops, and every infinite play is a win for Thief. ■

The entanglement of G, denoted by $\mathscr{E}(G)$, is the minimum $k \in \{0, \ldots, |V_G|\}$ such that Cops have a winning strategy in $\mathscr{E}(G, k)$.

Let us recall again a useful variant of entanglement games.

Proposition 6.3.2. *(i.e. Proposition 3.4.2) Let $\widetilde{\mathscr{E}}(G, k)$ be the game played as the game $\mathscr{E}(G, k)$ apart that Cops are allowed to retire a number of cops placed on the graph. That is, Cops moves are of the form*

- $(g, C, Cops) \to (g, C', Thief)$ *(generalized skip move),*
- $(g, C, Cops) \to (g, C' \cup \{g\}, Thief)$ *(generalized replace move),*

where in both cases $C' \subseteq C$. Then Cops have a winning strategy in $\mathscr{E}(G, k)$ if and only if they have a winning strategy in $\widetilde{\mathscr{E}}(G, k)$.

6.4 Closure under minor of undirected entanglement

In this section we prove the main result of this Chapter, Theorem 6.4.2.

Lemma 6.4.1. If G is a subgraph of H then $\mathscr{E}(G) \leq \mathscr{E}(H)$. ■

Proof. Let $k = \mathscr{E}(G)$, then clearly, if Thief has a winning strategy in $\mathscr{E}(G, k)$ then he can use it to win in $\mathscr{E}(H, k)$ by restricting his moves on G. □

Theorem 6.4.2. For arbitrary fixed $n \in \mathbb{N}$, the class of undirected graphs of entanglement at most n, is minor closed, that is if G is minor of H then $\mathscr{E}(G) \leq \mathscr{E}(H)$. ■

Proof. If G is obtained from H by edge-deletion then the statement obviously holds by Lemma 6.4.1. Otherwise, if $G = \partial^z_{ab} H$, for some $ab \in E_H$, then this allows to define a total function $f : V_H \longrightarrow V_G$ as follows:

$$f(v) = \begin{cases} z & \text{if } v \in \{ a, b \}, \\ v & \text{otherwise.} \end{cases}$$

Let $k = \mathscr{E}(H)$, using the function f we shall construct a Cops' winning strategy in the game $\widetilde{\mathscr{E}}(G, k)$, Definition 6.3.2, out of a Cops' winning strategy in $\mathscr{E}(H, k)$. To this goal, every position (g, C_G, P) of $\widetilde{\mathscr{E}}(G, k)$ is matched with the position (h, C_H, P) of $\mathscr{E}(H, k)$, where $P \in \{ Thief, Cops \}$, such that the following invariants hold.

- $g = f(h)$ and $C_G = f(C_H)$, (COPS)
- $if\ g = z($ hence $h \in \{ a, b \})$ and $P = Thief$, then
  $z \in C_G$ and $h \in C_H$; moreover $|C_H \cap \{ a, b \}| = 1$. (THIEF-Z)

The invariant (THIEF-Z) may be understood as follows: whenever Thief will move from z then z must be occupied by a cop. At this moment, in $\mathscr{E}(H, k)$, either a or b must be occupied by a cop but not both.

We simulate every Thief's move

$$M_G = (v, C_G, Thief) \rightarrow (w, C_G, Cops)$$

of $\widetilde{\mathscr{E}}(G, k)$ either by a move or a sequence of moves in $\mathscr{E}(H, k)$ according to the *locality* of Thief's move M_G:

1. If M_G is *outside* z, i.e. $v, w \neq z$ then in this case M_G is simulated by the same move in $\mathscr{E}(H, k)$.

2. If M_G is *entering* to z, i.e. $w = z$ and $vw \in E_G$. Assume $v \in \mathcal{N}(a)$ [1]. In this case, the move M_G is simulated by a *finite iteration* of Thief between a and b until Cops put a cop on a or b, and then the simulation

[1] The case $v \in \mathcal{N}(b) \setminus \mathcal{N}(a)$ is similar; ($\mathcal{N}(v)$ are just the neighbors of v).

is halted. That is, the move M_G is simulated by the *finite alternating* sequence $M_H^\star$ of moves that is the following sequence apart the last move:

$$\begin{aligned} M_H^\star = (v, C_H, Thief) \to (a, C_H, Cops) &\to (a, C_H, Thief) \to (b, C_H, Cops) \\ &\to (b, C_H, Thief) \to (a, C_H, Cops) \\ &\to \dots \\ &\to (x, C_H, Thief) \to (y, C_H, Cops) \\ &\quad \text{"}M_H^\star \text{ ends here"} \\ &\to (y, C'_H, Thief) \end{aligned}$$

Such that $\{x, y\} = \{a, b\}$ and $C'_H \neq C_H$. Clearly $y \in C'_H$. Observe that this sequence is possible i.e. $b \notin C_H$, because if $b \in C_H$ then it follows by the invariant (THIEF-Z) that $f(b) = z \in f(C_H) = C_G$, that is $z \in C_G$, which can not happen. The special case of Thief's first move to z is simulated by a similar finite alternating sequence of moves between a and b, apart that $C_H = C_G = \emptyset$.

3. If M_G is *leaving* z, i.e. $v = z$ and $vw \in E_G$. Assume that the position $(z, C_G, Thief)$ is matched with $(a, C_H, Thief)$. Recall that $z \in C_G$ and $a \in C_H$, by the invariant (THIEF-Z).

   3.1 If $w \in \mathcal{N}(a)$ then the move M_G is simulated by the same move of $\mathscr{E}(H, k)$.

   3.2 If $w \in \mathcal{N}(b) \backslash \mathcal{N}(a)$, then the move M_G is simulated by the following sequence of moves:

   $$(a, C_H, Thief) \to (b, C_H, Cops) \to (b, C'_H, Thief) \to (w, C'_H, Cops).$$

   This sequence is possible, i.e. $b \notin C_H$ because already $a \in C_H$, therefore $b \notin C_H$, by the invariant (THIEF-Z). At this point, the ending position of M_G – that is the position $(w, C_G, Cops)$ – is

matched with the position $(w, C'_H, Cops)$ of $\mathscr{E}(H,k)$, we emphasize that Cops' next move to $(w, C'_H, Cops) \to (w, C''_H, Thief)$ in $\mathscr{E}(H,k)$ should be mapped to the move

$$(w, C_G, Cops) \to (w, f(C''_H), Thief)$$

in $\widetilde{\mathscr{E}}(G,k)$, and the main technical part is to prove that the latter move respects the rules of the game .

A Cop's move in $\mathscr{E}(H,k)$ is mapped to a Cop's move in $\widetilde{\mathscr{E}}(G,k)$ as follows. Assume that the position $(g, C_G, Cops)$ of $\widetilde{\mathscr{E}}(G,k)$ is matched with the position $(h, C_H, Cops)$ of $\mathscr{E}(H,k)$ and moreover Cops have moved to

$$(h, C_H, Cops) \to (h, C'_H, Thief) \tag{6.1}$$

Therefore Cops in $\widetilde{\mathscr{E}}(G,k)$ should move to

$$(g, C_G, Cops) \to (g, f(C'_H), Thief) \tag{6.2}$$

the aim is prove that this move is legal w.r.t the rules of the game $\widetilde{\mathscr{E}}(G,k)$. We distinguish three cases according to the manner for which g has been reached by Thief in $\widetilde{\mathscr{E}}(G,k)$ in the previous round of simulation .

1. If g has been reached by an *outside move*, hence $g \neq z, g = h$ (g is the vertex considered in the equation 6.2, and h is considered in the equation 6.1). In this case, C'_H may be written: $C'_H = (C_H \setminus A) \cup B$, where $\emptyset \subseteq B \subseteq \{\, g \,\}$ and $|A| \leq 1$. (To be more precise we have $|A| \leq |B|$.) Therefore

$$\begin{aligned} f(C'_H) &= [f(C_H \setminus A)] \cup f(B) \\ &= \begin{cases} f(C_H) \cup f(B) & \text{if } a, b \in C_H \text{ and } A \subseteq \{\, a, b \,\}, \\ (f(C_H) \setminus f(A)) \cup f(B) & \text{otherwise} \end{cases} \end{aligned}$$

It is easy to see that this is a legal move.

2. If g has been reached by an *entering move*, hence $g = z$ and $h \in \{ a, b \}$ (again g is the vertex considered in the equation 6.2, and h is considered in the equation 6.1). In this case $z \notin C_G$ and therefore $a, b \notin C_H$. We shall argue that the move $(z, C_G, Cops) \to (z, f(C'_H), Thief)$ respects the rules of the game. Assume that $h = a$. In this case C'_H is of the form

$$C'_H = (C_H \setminus A) \cup B$$

where $0 \leq |A| \leq 1$ with $a, b \notin A$ and $\emptyset \subseteq B \subseteq \{ a \}$, therefore

$$\begin{aligned} f(C'_H) &= f[(C_H \setminus A) \cup B] \\ &= [f(C_H) \setminus f(A)] \cup f(B) \end{aligned}$$

Observe that $z \notin f(A)$ and $\emptyset \subseteq f(B) \subseteq \{ z \}$. Hence the move in question respects the rules of the game.

3. If g has been reached by a *leaving move*, hence $h = g$ and $hz \in E_G$. In this case $z \in C_G$ and either $a \in C_H$ or $b \in C_H$ but not both, by the invariant (THIEF-Z). We distinguish 2 cases:

   Case (i). If h has been reached by a single Thief's move in $\mathscr{E}(H, k)$, then one can check easily that every Cop's move from position $(h, C_H, Cops)$ in $\mathscr{E}(H, k)$ is mapped to the same move from $(h, C_G, Cops)$ in $\widetilde{\mathscr{E}}(G, k)$.

   Case (ii). If h has been reached by a sequence of moves in $\mathscr{E}(H, k)$. Let us go back to the previous round of the play. The previous move in $\widetilde{\mathscr{E}}(G, k)$ was indeed of the form

$$(z, C_G, Thief) \to (h, C_G, Cops)$$

   and its related simulation moves in $\mathscr{E}(H, k)$ are of the form

$$(a, C_H^{-1}, Thief) \to (b, C_H^{-1}, Cops) \to (b, C_H, Thief) \to (h, C_H, Cops)$$

   In $\mathscr{E}(H, k)$, if Cops move to $(h, C_H, Cops) \to (h, C'_H, Thief)$ then this move is obviously mapped to Cops move $(h, C_G, Cops) \to (h, f(C'_H), Thief)$

in $\widetilde{\mathscr{E}}(G,k)$. To this goal, note that $C'_H = (C'_H \setminus A) \cup B$ where $\emptyset \subseteq B \subseteq \{b, w\}$ and $A \subseteq V_H$ with $0 \leq |A| \leq 2$, let us compute $C'_G = f(C'_H)$ in terms of C_G:

$$\begin{aligned} f(C'_H) &= [f(C_H^{-1} \setminus A)] \cup f(B) \\ &= [\big(f(C_H^{-1}) \setminus f(A)\big) \cup Z] \cup f(B) \end{aligned}$$

where $\emptyset \subseteq Z \subseteq \{z\}$ and $\emptyset \subseteq f(B) = B' \subseteq \{z, w\}$, therefore

$$\begin{aligned} f(C'_H) &= [f(C_H^{-1}) \setminus f(A)] \cup (Z \cup B') \\ &= (C_G \setminus f(A)) \cup B'' \end{aligned}$$

where still $\emptyset \subseteq B'' = Z \cup B' \subseteq \{z, w\}$. Recall that $z \in C_G$ and hence the move in question respects the rules of the game.

Finally, the invariants are preserved by construction. □

A similar Proposition of the following one concerning the tree-width (instead of the entanglement) has been proved in [RS86].

Proposition 6.4.3. *If G is a direct minor of H then $\mathscr{E}(H) - 1 \leq \mathscr{E}(G)$*

Proof. We need the following Claim.

Claim 6.4.4*: To prove that $\mathscr{E}(H) - 1 \leq \mathscr{E}(G)$ it suffices to prove that $\mathscr{E}(H \setminus v) \leq \mathscr{E}(G)$, for some $v \in V_H$.* ■

Proof. Assume that $\mathscr{E}(H \setminus v) \leq \mathscr{E}(G)$, and let $k = \mathscr{E}(G)$. This implies that if Cops have a winning strategy in $\mathscr{E}(G,k)$ then they have a winning strategy S_1 in $\mathscr{E}(H \setminus v, k)$. Out of the winning strategy S_1 they can construct a winning strategy in $\mathscr{E}(H, k+1)$ as follows: if Thief moves in vertices in $V_H \setminus v$ then play with S_1, and if Thief goes to v then put the $(k+1)^{\text{th}}$ cop on v and never move it. This ends the proof of the Claim. □

If G is obtained from H by deleting some edge ab, then observe that $H \setminus a$ is a subgraph of G, therefore from Lemma 6.4.1 we get $\mathscr{E}(H \setminus a) \leq \mathscr{E}(G)$. We conclude – according to the Claim – that $\mathscr{E}(H) - 1 \leq \mathscr{E}(G)$. If G is obtained from H by contracting some edge, then the proof is similar to the above one. □

The following Corollary provides a good indication in the research of the set of excluded minors of graphs of bounded entanglement.

Corollary 6.4.5. Let $\mathscr{F}_k$ be the minimal excluded minors for the class of graphs of entanglement at most k. Then, graphs in $\mathscr{F}_k$ have exactly entanglement $k+1$. ■

In the following Chapter we shall prove the following Theorem:

Theorem 6.4.6. The five graphs pictured in Figure 6.4.1 are the minimal and complete excluded minors characterizing the class of undirected graphs of entanglement at most 2. ■

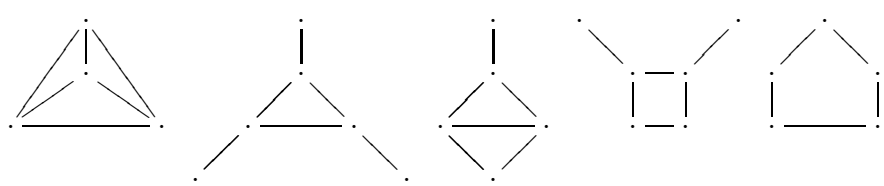


Figure 6.4.1: The minimal and complete excluded minors characterizing the undirected graphs of entanglement at most 2.

6.5 Algorithmic Properties

It is not difficult to argue that there exist polynomial time algorithms that, for fixed $k \geq 0$ decide on input G whether $\mathscr{E}(G) \leq k$. Such an algorithm constructs the game $\mathscr{E}(G, k)$ whose size is polynomial in $|V_G|$ and $|E_G|$, since

k is fixed. Since the game $\mathscr{E}(G,k)$ is clopen, i.e. it is a parity game of depth 1, it is well known [Jur00b] that such game can be solved in linear time w.r.t. the size of the graph underlying $\mathscr{E}(G,k)$.

In [BG05] the authors proved that $\mathscr{E}(G) = 0$ if and only if it is G is acyclic, and that $\mathscr{E}(G) \leq 1$ if and only if each strongly connected component of G has a vertex whose removal makes the component acyclic. Using these results it was argued that deciding whether a graph has entanglement at most 1 is a problem in NLOGSPACE.

Since the class of undirected graphs of entanglement at most k, for arbitrary fixed k, is minor closed Theorem 6.4.2, then by Roberston and Seymour Theorem it follows that testing weather an undirected graph has entanglement at most k can be checked in cubic time.

6.6 Conclusion

The set of forbidden minors is relatively large in general, and the main challenge consists in finding a compact representation of the forbidden minors. For the case of the symmetric graphs of entanglement at most 2, this will be done in the next Chapter.

Chapter 7

Undirected Graphs of Entanglement 2

We begin studying the structure of graphs of bounded entanglement. In this Chapter, we give a combinatorial and algebraic characterization of the class ζ_2 of undirected graphs of entanglement at most 2. On the one hand we characterize the graphs in ζ_2 by means of excluded subgraphs. This allows to provide the minimal set of excluded minors that characterizes the class ζ_2. On the other hand, based on the combinatorial characterization, we give an algebraic construction of memberships of ζ_2 in terms of a set of small pieces, called the molecules, and an algebraic operators that glues two molecules on a prescribed set of vertices. The algebraic construction provides a tree decomposition of graphs in ζ_2. Finally, we shall give a linear time algorithm that decides memberships of ζ_2.

7.1 Introduction

The *tree decomposition* is a rich concept that has shown its use in different fields: the design of polynomial algorithms, the study of the class of digraphs characterized by means of *forbidden patterns*, and the characterization of digraphs for which *monadic second order logic* is decidable.

Many NP-hard problems can be solved in polynomial time for particular classes of graphs i.e. the *well structured graphs*. These are the graphs built up out of finite basic graphs and finite *gluing* operations.
The second direction was pursed by Robertson and Seymour in their works on the Graph Minors' Theorem [RS04]. The latter states that every family of graphs closed under taking minors – such that the planar graphs – is characterized by a finite set of forbidden minors. The key concept used to prove this Theorem is the notion of *tree-width* [RS86, RS90]. Intuitively, the tree-width measures how a graph is close to a tree.
The third direction characterizes the class of digraphs for which the satisfiability of a given monadic second order logic formula is decidable. It was conjectured in [See91] that the graphs with such property are close to trees, and the prove of the conjecture still remains an open problem.

Unlike the complexity measures presented in the literature such as tree-width [JRST01], directed tree-width [JRST01], dag-width [BDHK06, Obd06], ..., entanglement is not associated yet to a tree decomposition. This has challenged us to consider the tree decomposition problem of undirected graphs of bounded entanglement as a starting point.

The second challenge comes from the model checking problem of the Propositional Modal μ-Calculus – hereby noted $\mathbb{L}_\mu^M$– : the main achievement of [BG05] states that parity games whose underlying graphs have bounded entanglement can be solved in polynomial time. This is a relevant result for the matter of verification, since model checking $\mathbb{L}_\mu^M$ is reducible in linear time to the problem of deciding the winner of a parity game. Berwanger's result calls for the problem of *deciding whether a graph has entanglement at most* k, a problem which we address in this Chapter. When settled, we can try to exploit the main result of [BG05], for example by designing algorithms to model check $\mathbb{L}_\mu^M$ that may perform well in practice. Using generic techniques we have shown in Section 6.5 that, for fixed k, deciding whether a graph has entanglement at most k is a problem in the class P. The algorithms solving

these problems can be combined to show that deciding the entanglement of a graph is in the class EXPTIME. As a consequence of the minor closure property, Theorem 6.4.2, deciding whether an undirected graph has entanglement at most k, is a problem in $O(n^3)$ [RS86].

Let us mention on the way that a problem that we indirectly address is that of solving parity games on undirected graphs. These games can be solved in linear time if Eva's and Adam's moves alternate. Yet, the complexity of the problem is not known if consecutive moves of the same player are allowed.

In this Chapter we shall provide two main characterizations of the class $\mathcal{U}_2$. One of them proceeds by forbidden subgraphs: an undirected graph belongs to $\mathcal{U}_2$ if and only if it does not contain

1. a simple cycle of length strictly greater than 4,
2. a length 3 simple cycle whose vertices have all degree 3, and
3. a length 4 simple cycle with two adjacent vertices of degree 3.

A second characterization constructs the class $\mathcal{U}_2$ from a class of atomic graphs, called the *molecules*, and an operation, the *legal* 1-*Sum*, that glues together two graphs along a prescribed pair of vertices. The second characterization provides a tree decomposition of graphs in $\mathcal{U}_2$: the biconnected components are the molecules; they form a tree with their glue points. The latter becoming the articulation points of the graph.
Based on the second characterization we obtain a linear time algorithm that decides if an undirected graph belongs to the class $\mathcal{U}_2$. The two characterizations may be appreciated on their own, independently of the algorithm they give rise.
Finally we shall formulate the first characterization in terms of the minimal forbidden minors characterizing the class $\mathcal{U}_2$.

As a matter of fact, some of the properties we shall encounter have already been under focus: the combinatorial characterization exhibits surprising analogies with the class of House-Hole-Domino free graphs, see [JO88,

CD03], a sort of generalization of graphs admitting a perfect elimination ordering. These graphs arise as the result of looking for wider notions of ordering for graphs that still ensure nice computational properties. On the other hand, the algebraic characterization recalls the well known fact that graphs of fixed arbitrary tree-width may be constructed by means of an algebra of pushouts and relabelings [Cou90]. The algebra of legal 1-Sum suggests that, for entanglement, it might be possible to develop an analogous generic algebraic framework. It also points to standard graph theoretic ideas, such as n-connectivity and cyclicity, as the natural tools by which to analyze entanglement. This approach, that takes the n-connectivity and cyclicity as a starting point to study entanglement, will be pursed in Chapter 8. As a consequence, there we shall provide a simple and elegant proof of the main results of the current Chapter.

7.2 Entanglement games

To tackle this Chapter, we assume that the reader is familiar with the notion of entanglement and its game theoretic definition, see Definition 3.4.1.

While wondering for a characterization of graphs of entanglement at most 2, we observed that such a question has a clear answer for *undirected* graphs. To deal with this kind of graphs, we recall that an undirected edge uv is just a pair (u, v), (v, u) of directed edges. We can use the results of [BG05] to give characterizations of undirected graphs of entanglement at most 1. To this goal, let us consider:

Definition 7.2.1. For $n \geq 0$ define the n-star of center x_0, noted $\varsigma^n_{x_0}$, to be the undirected graph (V, E) where $V = \{ x_0, a_1, ..., a_n \}$ and $E = \{ x_0a_1, ..., x_0a_n \}$. More generally, say that a graph is a star if it is isomorphic to some $\varsigma^n_{x_0}$. ■

Then we can easily deduce:

Proposition 7.2.2. *If G is an undirected graph, then $\mathscr{E}(G) = 0$ if and only if $E_G = \emptyset$, and $\mathscr{E}(G) \leq 1$ if and only if G is a disjoint union of stars.*

To end this section we recall a Lemma that later will be used often. We remark that its scope does not restrict to undirected graphs.

Lemma 7.2.3. (i.e. Lemma 6.4.1) If G is a subgraph of H then $\mathscr{E}(G) \leq \mathscr{E}(H)$. ■

7.3 Molecules, 1-Sum, and the class ζ_2

In this section we introduce a class of graphs and prove that the graphs in this class have entanglement at most 2. It will be the goal of thex next sections to prove that these are all the graphs of entanglement at most 2.

Definition 7.3.1. A *molecule* $\theta^{\varepsilon,n}_{a,b}$, where $\varepsilon \in \{0,1\}$ and $n \geq 0$, is the undirected graph (V,E) with $V = \{a, b, c_1, ..., c_n\}$ and

$$E = \begin{cases} \{ac_1, ..., ac_n, bc_1, ..., bc_n\}, & \varepsilon = 0, \\ \{ab, ac_1, ..., ac_n, bc_1, ..., bc_n\}, & \varepsilon = 1. \end{cases}$$

The *glue points* of a molecule $\theta^{\varepsilon,n}_{a,b}$ are a, b. Its *dead points* are $c_1, \ldots, c_n$. ■

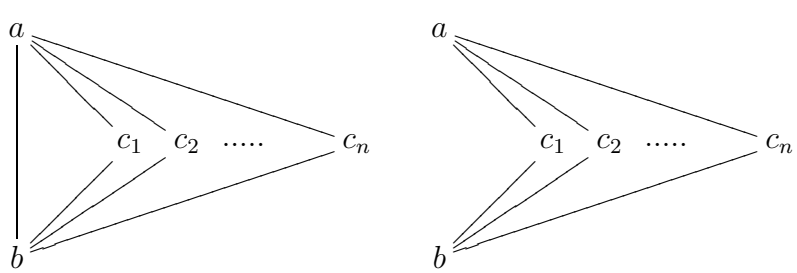


Figure 7.3.1: Molecules $\theta^{1,n}_{a,b}$ and $\theta^{0,n}_{a,b}$.

It is not difficult to prove that molecules have entanglement at most 2.

Definition 7.3.2. Let G_1 and G_2 be two undirected graphs with $V_{G_1} \cap V_{G_2} = \emptyset$, let $a_1 \in V_{G_1}$ and $a_2 \in V_{G_2}$. The 1-*Sum* of G_1 and G_2 on vertices a_1 and a_2, denoted $G_1 \bigoplus^z_{a_1,a_2} G_2$, is the graph G defined as follows:

$$\begin{aligned} V_G &= (V_{G_1} \setminus \{ a_1 \}) \cup (V_{G_2} \setminus \{ a_2 \}) \cup \{ z \}, \text{ where } z \notin V_{G_1} \cup V_{G_2}, \\ E_G &= \{ x_1 y_1 \in E_{G_1} \mid a_1 \notin \{ x_1, y_1 \} \} \cup \{ x_2 y_2 \in E_{G_2} \mid a_2 \notin \{ x_2, y_2 \} \} \\ &\quad \cup \{ xz \mid xa_1 \in E_{G_1} \text{ or } xa_2 \in E_{G_2} \}. \end{aligned}$$

■

Remark 7.3.3. We remark that $\bigoplus$ is a coproduct in the category of pointed undirected graphs and, for this reason, this operation is commutative and associative up to isomorphism. The graph η, whose set of vertices is a singleton, is a neutral element. ■

As we have observed, a molecule is an undirected graph coming with a distinguished set of vertices, its glue points. Let us call a pair (G, Gl) with $Gl \subseteq V_G$ a *glue graph*. For glue graphs we can define what it means that a 1-Sum is legal.

Definition 7.3.4. If G_1, G_2 are glue graphs, then we say that $G_1 \bigoplus^z_{a,b} G_2$ is a *legal* 1-Sum if $a \in Gl_{G_1}$ and $b \in Gl_{G_2}$. We shall then use the notation $G_1 \overline{\bigoplus}^z_{a,b} G_2$ and define

$$Gl_{G_1 \overline{\bigoplus}^z_{a,b} G_2} = (Gl_{G_1} \setminus \{ a \}) \cup (Gl_{G_2} \setminus \{ b \}) \cup \{ z \},$$

so that $G_1 \overline{\bigoplus}^z_{a,b} G_2$ is a glue graph. ■

Observe that the graph η can be made into a unit for the legal 1-Sum by letting $Gl_\eta = V_\eta$. Even if the operation $\overline{\bigoplus}$ is well defined only after the choice of the two glue points that are going to be collapsed, it should be clear what it means that a family of glue graphs is closed under the legal 1-Sum.

Definition 7.3.5. We let ζ_2 be the least class of glue graphs containing the molecules, the unit η, and closed under the legal 1-Sum operator and graph isomorphisms. ■

We need to make precise some notation and terminology. Firstly we shall abuse of notation and write

$$G = H \overline{\bigoplus}_v K$$

to mean that there exist subgraphs H, K of G such that $v \in Gl_G \cap V_H \cap V_K$ and G is isomorphic to the legal 1-Sum $H \overline{\bigoplus}^z_{v,v} K$. Notice that if H and K are distinct from η, then v is an articulation point of G. Second, we shall say that a graph G belongs to ζ_2 to mean that there exists a subset $Gl \subseteq V_G$ such that the glue graph (G, Gl) belongs to ζ_2. We can now state the main result of this section.

Proposition 7.3.6. *If G belongs to the class ζ_2, then $\mathscr{E}(G) \leq 2$.*

Proof. Observe that, given a molecule $\theta^{\varepsilon,n}_{a,b}$ occurring in an algebraic expression for G, we can rearrange the summands of the algebraic expression to write

$$G = L \overline{\bigoplus}_a \theta^{\varepsilon,n}_{a,b} \overline{\bigoplus}_b R \tag{7.1}$$

where $L, R \in \zeta_2$. A Cops winning strategy in the game $\mathscr{E}(G, 2)$ is summarized as follows. If Thief occupies some vertex of the molecule $\theta^{\varepsilon,n}_{a,b}$, Cops will place its two cops on a and b, in some order. By doing that, Cops will force Thief to move (i) on the left component L, in which case Cops can reuse the cop on b on L, (ii) on the molecule $\theta^{\varepsilon,n}_{a,b}$, in which case Thief will be caught in a dead point of the molecule, (iii) on the right component R, in which case Cops can reuse the cop on a on R.

Cops can recursively use the same strategy in $\mathscr{E}(L, 2)$ and $\mathscr{E}(R, 2)$. The recursion terminates as soon as in the expression (7.1) for G we have $L = R = \eta$.

□

The reader will have noticed similarities between the strategy proposed here and the strategy needed in [BG05] to argue that undirected trees have

entanglement at most 2. As a matter of fact, graphs in ζ_2 have an underlying tree structure.

We shall now give a preliminary characterization of class ζ_2.

Definition 7.3.7. Let G be a gluing graph. We define the derived graph of G, noted $\partial(G)$, as follows: its vertices $V(\partial G)$ are the glue points of G, and we let $ab \in E(\partial G)$ if either $ab \in E(G)$ or there exists $x \in V(G) \setminus Gl(G)$ such that $ax, xb \in E(G)$. ■

Proposition 7.3.8. *A gluing graph G is in ζ_2 if and only if ∂G is a forest, and moreover each $x \in V(G) \setminus Gl(G)$ has exactly two neighbors, which moreover are glue points.*

Proof. It is easy to verify that the condition is necessary, since it holds if G is a molecule or the unit graph η. The conditions is also preserved under formation of legal 1-Sum.

Therefore, let us suppose that ∂G is a forest and moreover each $x \in V(G) \setminus Gl(G)$ has exactly two neighbors, which moreover are glue points.

Let a be a leaf of the forest. If the degree of a in the forest is 0, then $(V(G) \setminus \{ a \}, V)$ is again a graph in the same class. Otherwise let $b \in V(G)$ such that ab is an edge in ∂G. Let us consider the subgraph of G induced by the vertices $\{ a \} \cup aE \cup \{ b \}$: clearly this graph is isomorphic to some molecule $\theta^{n,i}_{a,b}$. Also let G' be the glue graph induced by $V(G) \setminus aE \cup \{ b \}$, then $\delta G'$ is the forest $\partial G' \setminus \{ a \}$. Moreover we have

$$G = \theta^{n,i}_{a,b} \overline{\bigoplus_{b}} G' .$$

Since by the inductive hypothesis $G' \in \zeta_2$, we conclude that $G \in \zeta_2$. □

7.4 Combinatorial properties

The goal of this section is to setup the tools for the characterization Theorem 7.5.3. We deduce some combinatorial properties of undirected graphs of

entanglement at most 2. To this goal, let us say that a simple cycle is long it its length is strictly greater than 4, and say otherwise that it is short. Also, let us call a simple cycle of length 3 (resp. 4) a triangle (resp. square).

Proposition 7.4.1. *An undirected graph G such that $\mathscr{E}(G) \leq 2$ satisfies the following conditions:*

- *a simple Cycle of G is Short,* (CS)
- *a triangle of G has at least one vertex of degree 2,* (No-3C)
- *a square of G cannot have two adjacent vertices of degree strictly greater than 2.* (No-AC)

Condition (No-3C) excludes the graph scheme on the left of figure 7.4.1, made up of a triangle and 3 distinct 1-Sum. Notice that the vertices x, y, z in the figure might not be distinct. Condition (No-AC) excludes the graph scheme on the right of figure 7.4.1, made up of a square and two Adjacent 1-Sums. Even for this scheme x, y might not be distinct. We shall see with

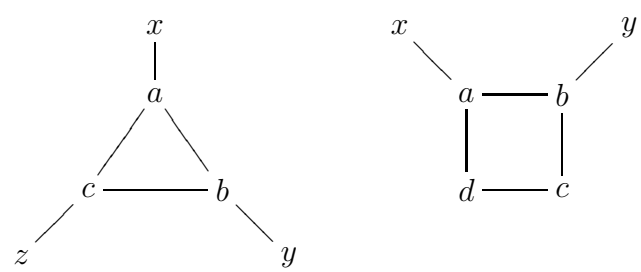


Figure 7.4.1: The graphs $3C$ and AC.

Theorem 7.5.3 that these properties completely characterize the class of undirected graphs of entanglement at most 2. Proposition 7.4.1 is an immediate consequence of Lemma 7.2.3 and of the following Lemmas 7.4.2, 7.4.3, 7.4.4.

For $n \geq 0$ let P_n be the path with n vertices and $n-1$ edges: $V_{P_n} = \{0, ..., n-1\}$ and $ij \in E_{P_n}$ iff $|i-j| = 1$. For $n \geq 3$, let C_n be the cycle

with n vertices and edges: $V_{C_n} = \{ 0, ..., n-1 \}$ and $ij \in E_{C_n}$ iff $|i-j| \equiv 1$ mod n.

Lemma 7.4.2. If $n \geq 5$ then $\mathcal{E}(C_n) \geq 3$. ■

Proof. To describe a winning strategy for Thief in the game $\mathscr{E}(C_n, 2)$ consider that the removal of one or two vertices from C_n transforms such graph into a disjoint union $P_i + P_j$ with $i + j \geq n - 2 \geq 3$: notice in particular that $i \geq 2$ or $j \geq 2$. In a position of the form $(v, C, Thief)$ with $v \in C$, Thief moves to a component P_i with $i \geq 2$. From a position of the form $(v, C, Thief)$ with $v \notin C$, v in some component P_i, and $i \geq 2$, Thief moves to some other vertex in the same component. This strategy can be iterated infinitely often, showing that Thief will never be caught. □

Lemma 7.4.3. Let $3C$ be the graph on the left of figure 7.4.1.We have $\mathscr{E}(3C) \geq 3$. ■

Proof. A winning strategy for Thief in the game $\mathscr{E}(3C, 2)$ is as follows. By moving on a, b, c, Thief can force Cops to put two cops there, say for example on a and b. Thief can then escape to c and iterate moves on the edge cz to force Cops to move one cop on one end of this edge. From a position of the form $(c, C, Thief)$ with $c \in C$, Thief moves to a free vertex among a, b. From a position of the form $(z, C, Thief)$ with $c \notin C$ Thief moves to c and forces again Cops to occupy two vertices among a, b, c. Up to a renaming of vertices, such a strategy can be iterated infinitely often, showing that Thief will never be caught.

Observe that the proof does not depend on x, y, z being distinct.

□

Lemma 7.4.4. Let AC be the graph on the right of figure 7.4.1.We have $\mathscr{E}(AC) \geq 3$. ■

Proof. By moving on a, b, c, d, Thief can force Cops to put two cops either on a, c or on b, d: let us say a, c. Thief can then escape to b and iterate moves

on the edge *by* to force Cops to move one cop on one end of this edge. From a position of the form $(b, C, Thief)$ with $b \in C$, Thief moves to a free vertex among a, c. From a position of the form $(y, C, Thief)$ with $b \notin C$ Thief moves to b and forces again Cops to occupy either a, c or b, d. Up to a renaming of vertices, such a strategy can be iterated infinitely often, showing that Thief will never be caught. Again, we observe that the strategy does not depend on x, y being distinct. □

We end this section by pointing out that $\mathscr{E}(C_n) = \mathscr{E}(3C) = \mathscr{E}(AC) = 3$ $(n \geq 5)$.

7.5 Characterization of entanglement at most 2

In this section we accomplish the characterization of the class of undirected graphs of entanglement at most 2: we prove that this class coincides with ζ_2.

The following Lemma is the key observation by which the induction works in the proof of Proposition 7.5.2. It is worth, before stating it, to recall the difference between $\bigoplus$, the 1-Sum of two ordinary undirected graphs, and $\overline{\bigoplus}$, the legal 1-Sum of two glue graphs.

Lemma 7.5.1. Let G be an undirected graph satisfying (No-3C) and (No-AC). If $G = \theta^{\varepsilon,n}_{v,b} \bigoplus_b H$ and $H \in \zeta_2$, then there is a subset $Gl' \subseteq V_G$ such that (H, Gl') is a glue graph in ζ_2, $b \in Gl'$, and moreover G is the result of the legal 1-Sum $G = \theta^{\varepsilon,n}_{v,b} \overline{\bigoplus}_b (H, Gl')$. Consequently, $G \in \zeta_2$, with v a glue point of G. ■

Proof. Since the graph H is in ζ_2, it already comes with a set of glue points Gl_H. Hence, if $b \in Gl_H$ then we simply let $Gl' = Gl_H$.

Otherwise b is a dead point of a certain molecule $\theta^{\delta,m}_{c,d}$ of H, so that we can write

$$H = H_c \overline{\bigoplus}_c \theta^{\delta,m}_{c,d} \overline{\bigoplus}_d H_d \,. \tag{7.2}$$

Observe also that $m \geq 1$ since b is a dead point of the molecule $\theta^{\delta,m}_{c,d}$.

If $\theta^{\varepsilon,n}_{v,b}$ is disconnected, i.e. if $\varepsilon = n = 0$, then the result is obvious, since then

$$\theta^{\varepsilon,n}_{v,b} \bigoplus_b \theta^{\delta,m}_{c,d} \overline{\bigoplus}_c H_c \overline{\bigoplus}_d H_c = \theta^{\varepsilon,n}_{v,b} \overline{\bigoplus}^b_{b,c} \theta^{\delta,m}_{c,d} \overline{\bigoplus}_c H_c \overline{\bigoplus}_d H_c \,.$$

Otherwise, if $\varepsilon + n > 0$, then $\deg_G(b) \geq 3$, since b has two neighbors in the molecule $\theta^{\delta,m}_{c,d}$ and at least one neighbor in the molecule $\theta^{\varepsilon,n}_{v,b}$.

If $\delta = 0$ then $1 \leq m \leq 2$: condition (No-AC) implies that if $m \geq 2$ then $\deg_G(c) = \deg_G(d) = 2$. If $m = 1$, then we can use the equality

$$\theta^{0,1}_{c,d} = \theta^{1,0}_{c,b} \overline{\bigoplus}_b \theta^{1,0}_{b,d}$$

to add b to the set of glue points of H. If $m = 2$, then $\deg_G(c) = \deg_G(d) = 2$. Let e be the unique dead point of $\theta^{\delta,m}_{c,d} = \theta^{0,2}_{c,d}$ which is distinct from b. We claim that we can replace the pair c, d with b, e in the set of glue points of H. As a matter of fact, in the algebraic expression (7.2) the legal 1-Sum are disjoint. This means that we can write

$$H_c = H_{c'} \overline{\bigoplus}_{c'} \theta^{0,0}_{c',c} \qquad\qquad H_d = \theta^{0,0}_{d,d'} \overline{\bigoplus}_{d'} H_{d'} \,,$$

and consequently we can also write

$$\begin{aligned} H &= H_c \overline{\bigoplus}_c \theta^{\delta,m}_{c,d} \overline{\bigoplus}_d H_d \\ &= H_{c'} \overline{\bigoplus}_{c'} \theta^{0,0}_{c',c} \overline{\bigoplus}_c \theta^{\delta,m}_{c,d} \overline{\bigoplus}_d \theta^{0,0}_{d,d'} \bigoplus_{d'} H_{d'} \\ &= H_{c'} \overline{\bigoplus}_{c'} \theta^{0,0}_{c',b} \overline{\bigoplus}_b \theta^{\delta,m}_{b,e} \overline{\bigoplus}_d \theta^{0,0}_{e,d'} \overline{\bigoplus}_{d'} H_{d'} \,. \end{aligned}$$

If $\delta = 1$, then $m = 1$, since condition (No-3C) implies either $\deg(c) = 2$ or $\deg(d) = 2$, hence $m = 1$. Let us suppose that $\deg(d) = 2$. We claim that we can replace d with b in the set of glue points of H.

In the algebraic expression (7.2) the legal 1-Sum on d are disjoint: that is we can write

$$H_d = \theta^{0,0}_{d,d'} \overline{\bigoplus}_{d'} H_{d'} \,.$$

Consequently we can write

$$\begin{aligned} H &= H_c \overline{\bigoplus}_c \theta^{\delta,m}_{c,d} \overline{\bigoplus}_d H_d \\ &= H_c \overline{\bigoplus}_c \theta^{\delta,m}_{c,d} \overline{\bigoplus}_d \theta^{0,0}_{d,d'} \overline{\bigoplus}_{d'} H_{d'} \\ &= H_c \overline{\bigoplus}_c \theta^{\delta,m}_{c,b} \overline{\bigoplus}_d \theta^{0,0}_{b,d'} \overline{\bigoplus}_{d'} H_{d'} \,. \end{aligned}$$

□

Proposition 7.5.2. *If G is an undirected graph satisfying* (CS), (No-3C), *and* (No-AC), *then $G \in \zeta_2$.*

Proof. The proof is by induction on $|V_G|$. Clearly the Proposition holds if $|V_G| = 1$, in which case $G = \eta \in \zeta_2$. Let us suppose the Proposition holds for all graphs H such that $|V_H| < |V_G|$.

If all the vertices in G have degree less than or equal to 2, then G is a disjoint union of paths and cycles of length at most 4. Clearly such a graph belongs to ζ_2. Otherwise, let v_0 be a vertex such that $\deg_G(v_0) \geq 3$ and consider the connected components G_ℓ, $\ell = 1, \ldots, h$, of the graph $G \setminus \{ v_0 \}$. Let $G^{v_0}_\ell$ be the subgraph of G induced by $V_{G_\ell} \cup \{ v_0 \}$. We shall show that this graph is of the form

$$G^{v_0}_\ell = \theta^{\varepsilon,m}_{v_0,v_1} \bigoplus_{v_1} H \,, \tag{7.3}$$

for some $\varepsilon \in \{ 0, 1 \}$, $m \geq 0$, and a graph $H \in \zeta_2$.

Clearly, if G_ℓ is already a connected component of G, then $G_\ell \in \zeta_2$ by the inductive hypothesis. We can pick any $v_1 \in V_{G_\ell}$ and argue that formula (7.3) holds with $m = \varepsilon = 0$, $H = G_\ell$.

Otherwise, let $\mathcal{N}_\ell = \{ a_1, ..., a_n \}$, $n \geq 1$, be the set of vertices of $G^{v_0}_\ell$ at distance 1 from v_0. We claim that either the subgraph of G_ℓ induced by $\mathcal{N}_\ell$, noted $\mathcal{N}_{G_\ell}$, is a star or there exists a unique $v_1 \in G_\ell$ at distance 1 from $\mathcal{N}_\ell$, and moreover the subgraph of G_ℓ induced by $\mathcal{N}_\ell \cup \{ v_1 \}$ is a star. In both cases, a vertex of such a star which is not the center has degree 2 in G.

(i) If $E_{\mathcal{N}_{G_\ell}} \neq \emptyset$, then $\mathcal{N}_{G_\ell}$ is a star. Let us suppose that $a_1 a_2 \in E_{G_\ell}$. Since G_ℓ is connected, if $a_k \in \mathcal{N}_\ell \setminus \{ a_1, a_2 \}$ then there exists a path from a_k to both

a_1 and a_2. Condition (CS) implies that either $a_1a_k \in E_{G_\ell}$, or $a_ka_2 \in E_{G_\ell}$. If $x_0 \in V_{G_\ell} \setminus \{ a_2 \}$ then there cannot be a simple path $a_k \dots x_0 \dots a_1$ otherwise $v_0a_k \dots x_0 \dots a_1a_2v_0$ is a long cycle. Therefore, a simple path from a_k to a_1 is of the form a_ka_1 or $a_ka_2a_1$. By condition (No-3C) it is not the case that $a_ka_1, a_ka_2 \in E_{G_\ell}$, otherwise $\{ v_0, a_1, a_2, a_k \}$ is a clique of cardinality 4. Finally, if $a_ka_1 \in E_{G_\ell}$ and $a_l \in \mathcal{N}_\ell \setminus \{ a_1, a_2, a_k \}$, then $a_la_1 \in E_{G_\ell}$ as well, by condition (CS), otherwise $v_0a_ka_1a_2a_lv_0$ is a long cycle. Therefore, if $|\mathcal{N}_\ell| > 2$, then $\mathcal{N}_{G_\ell}$ is a star with a prescribed center, which we can assume to be a_1. Since $\deg_G(v_0) \geq 3$, by condition (No-3C) only a_1 among vertices in $\mathcal{N}_\ell$ may have degree greater than 2. Otherwise $|\mathcal{N}_\ell| = 2$ and again at most one among a_i, $i = 1, 2$, has $\deg_G(a_i) > 2$. Again, we can assume that $\deg_G(a_2) = 2$. We deduce that the subgraph of $G_\ell^{v_0}$ induced by $\{ v_0 \} \cup \mathcal{N}_\ell$ is of the form $\theta^{1,n-1}_{v_0,a_1}$.

(ii) If $E_{\mathcal{N}_{G_\ell}} = \emptyset$, then we distinguish two cases. If $|\mathcal{N}_\ell| = 1$, then the subgraph of $G_\ell^{v_0}$ induced by $\{ v_0 \} \cup \mathcal{N}_\ell$ is $\theta^{1,0}_{v_0,a_1}$. Otherwise, if $|\mathcal{N}_\ell| \geq 2$, between any two distinct vertices in $\mathcal{N}_\ell$ there must exist a path in G_ℓ, since G_ℓ is connected. By condition (CS), if $a_i \dots x_{i,j} \dots a_j$ is a simple path from a_i to a_j with $x_{i,j} \in V_{G_\ell} \setminus \mathcal{N}_\ell$, then $a_ix_{i,j}, a_jx_{i,j} \in E_{G_\ell}$. Also (CS) implies that, for fixed i, $x_{i,k} = x_{i,j}$ if $k \neq j$, otherwise $v_0a_kx_{i,k}a_ix_{i,j}a_jv_0$ is a long cycle. We can also assume that $x_{i,j} = x_{j,i}$, and therefore $x_{i,j} = x_{i,k} = x_{l,k}$ whenever $i \neq j$ and $l \neq k$. Thus we can write $x_{i,j} = v_1$ for a unique v_1 at distance 2 from v_0. Since $|\mathcal{N}_\ell| \geq 2$ and $\deg_G(v_0) \geq 3$, condition (No-AC) implies that $\deg_G(a_i) = 2$ for $i = 1, \dots, n$. We have shown that in this case the subgraph of $G_\ell^{v_0}$ induced by $\mathcal{N}_\ell \cup \{ v_0, v_1 \}$ is a molecule $\theta^{0,n}_{v_0,v_1}$, with $n \geq 2$.

Until now we have shown that (7.3) holds with H a graph of entanglement at most 2. Since for such a graph $|V_H| < |V_G|$, the induction hypothesis implies $H \in \zeta_2$. Lemma 7.5.1 in turn implies that $G_\ell^{v_0} \in \zeta_2$, with v_0 a glue point of $G_\ell^{v_0}$. Finally we can use

$$G = G_1^{v_0} \overline{\bigoplus}_{v_0} G_2^{v_0} \overline{\bigoplus}_{v_0} \dots \overline{\bigoplus}_{v_0} G_h^{v_0} ,$$

to deduce that $G \in \zeta_2$.

$\square$

We can now state our main achievement.

Theorem 7.5.3. For a finite undirected graph G, the following are equivalent:

1. G has entanglement at most 2,
2. G satisfies conditions (CS), (No-3C), (No-AC),
3. G belongs to the class ζ_2.

■

Proof. As a matter of fact, we have shown in the previous section that 1 implies 2, in this section that 2 implies 3, and in section 7.3 that 3 implies 1. □

7.6 Forbidden minors characterization

We have proved in Chapter 6, Theorem 6.4.2 that the class of undirected graphs of entanglement $\leq k$ is closed under taking minors, and in this section we formulate the characterization of the class $\mathcal{U}_2$ in terms of minimal forbidden minors. We emphasize that graphs in the set of minimal minors characterizing entanglement at most k might contain articulation points, this makes the size of the set relatively large.

Theorem 7.6.1. The five graphs in Figure 7.6.1 are the minimal and complete forbidden minors characterizing the class $\mathcal{U}_2$. ■

Proof. We prove the minimality by inspection: all the five graphs has entanglement $= 3$, and any edge contraction or deletion results a graph of entanglement $= 2$.
We prove that this set is complete by observing that the graphs M_1, M_2 and M_3 in the Figure are exactly the graphs $3C$ of Figure 7.4.1. The graph M_4 is obtained from the graphs $4C$ of Figure 7.4.1 by distinction of vertices x, y.

Finally the graph M_5 is obtained out of the graphs $4C$ by identifying vertices x, y and deleting the edge ab. Note also that M_5 is the 5-cycle, that is the minimal forbidden long cycle w.r.t edge contraction. □

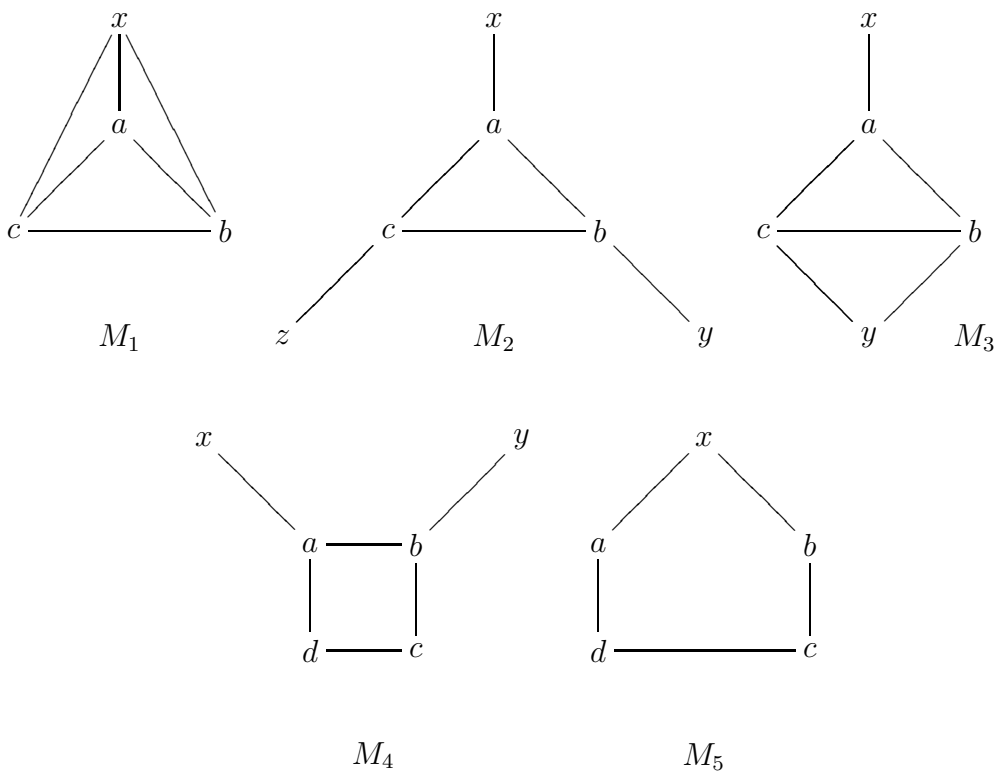


Figure 7.6.1: The five minimal forbidden minors characterizing entanglement at most 2.

7.7 A linear time algorithm

In this section we present a linear time algorithm that decides whether an undirected graph G has entanglement at most 2. We would like to thank the

anonymous referee for pointing to us the ideas and tools needed to transform the algebraic characterization of Section 7.3 into a linear time algorithm.

Let us recall that, for $G = (V, E)$ and $v \in V$, v is an *articulation point* of G iff there exist distinct $v_0, v_1 \in V \setminus \{ v \}$ such that every path from v_0 to v_1 visits v. Equivalently, v is an *articulation point* iff the subgraph of G induced by $V \setminus \{ v \}$ is disconnected. The graph G is *biconnected* if it does not contain articulation points. A subset of vertices $V' \subseteq V$ is biconnected iff the subgraph induced by V' is biconnected. A *biconnected component* of G is biconnected subset $C \subseteq V$ such that if $C \subseteq V'$ and V' is biconnected then $C = V$. The *superstructure* of G is the graph F_G defined as follows. Its set of vertices is the disjoint union $V_{F_G} = \mathcal{A}(G) \uplus \mathcal{C}(G)$, where

$$\mathcal{A}(G) = \{ a \in V \mid a \text{ is an articulation point of G} \},$$
$$\mathcal{C}(G) = \{ C \subseteq V \mid C \text{ is a biconnected component of } G \},$$

and its set of edges is of the form

$$E_{F_G} = \{ aC \mid a \in \mathcal{A}(G), C \in \mathcal{C}(G), \text{ and } a \in C \}.$$

It is well known that F_G is a forest and that Depth-First-Search techniques may be used to compute the superstructure F_G in time $O(|V| + |E|)$, see [CLR90, §23-2]. Observe also that this implies that $\sum_{C \in \mathcal{C}(G)} |C| = O(|V| + |E|)$. This relation that may also be derived considering that biconnecetd components do not share common edges, so that $|V_{F_G}| = O(|V| + |E|)$ and $|E_{F_G}| = O(|V| + |E|)$ since F_G is a forest. We have therefore

$$\begin{aligned}\sum_{C \in \mathcal{C}(G)} |C| &= |V \setminus \mathcal{A}(G)| + \sum_{a \in \mathcal{A}(G)} |\{ C \in \mathcal{C}(G) \mid a \in C \}| \\ &= |V \setminus \mathcal{A}(G)| + |E_{F_G}| = O(|V| + |E|).\end{aligned}$$

The algorithm ENTANGLEMENT-TWO relies on the following considerations. If a graph G belongs to the class ζ_2, then it has an algebraic expression explaining how to construct it using molecules as building blocks and legal

1-Sum as operation. We can assume that in this expression the molecule $\theta^{0,1}_{a,b}$ does not appear, since each such occurrence may be replaced by the 1-Sum $\theta^{1,0}_{a,x} \overline{\bigoplus}_x \theta^{1,0}_{x,b}$. W.r.t. this normalized expression, if G is connected then its articulation points are exactly those glue points v of G that appears in the algebraic expression as subscripts of some legal 1-Sum $\overline{\bigoplus}_v$; the molecules are the biconnected components of G.

The algorithm computes the articulation points and the biconnected components of G – that is, its superstructure – and afterwards it checks that each biconnected component together with its articulation points is a molecule.

```
ENTANGLEMENT-TWO(G)
// Input an undirected graph G, accept if G ∈ ζ2
if |E| ≥ 3|V| then reject
foreach v ∈ V do deg(v) := |vE|
let F_G = (A(G) ⊎ G(G), E_{F_G}) be the superstructure of G
foreach C ∈ A(G) do
  if not IS-MOLECULE(C, { a ∈ A(G) | a ∈ C }) then reject
accept
```

For a biconnected component together with a set of candidate glue points to be a molecule we need of course these candidates to be at most 2. Also, every vertex whose degree in G is not 2 is a candidate glue point.

7.7.1 Recognizing molecules

Lemma 7.7.1. If a biconnected component C is the singleton $\{ v \}$, then v is an isolated point of G. ■

Proof. Indeed, if $vu \in E$, then vu is a biconnected subset of G, since every singleton graph is connected. □

Lemma 7.7.2. If C is a biconnected component of a graph $G = (V, E)$ and $v \in C$ is not an articulation point of G, then $vE \subseteq C$. ■

Proof. Let us suppose that $vu \in E$ for some $u \in V \setminus C$ and prove that v is an articulation point of G. It is not possible that $C = \{ v \}$ since v is not

isolated. Therefore there exists $x \in C$, $x \neq v$. If in G there exists a path from u to x which does not visit v, then we can extend C to a greater biconnected component to which u belongs. Therefore such a path does not exists, and v is an articulation point separating u from x. □

Lemma 7.7.3. Let $G = (V, E)$ be a biconnected graph and $D \subseteq V$ be such that $\{ v \in V \mid \deg(v) \neq 2 \} \subseteq D$. Then G is isomorphic to a molecule, with every element of D sent by the isomorphism to a glue point of the molecule, if and only if either

$$|D| = 2 \text{ and } V \setminus D \subseteq dE\,,\ d \in D \tag{i}$$

or

$$|D| < 2 \text{ and } |V| \in \{ 3, 4 \}\,. \tag{ii}$$

■

Proof. Let us consider a biconnected molecule $\theta^{\epsilon,n}_{a,b}$ and observe first that: (a) $\epsilon + n > 0$ since it is connected and (b) $(\epsilon, n) \neq (0, 1)$ since it is biconnected. Let $D \subseteq \{ a, b \}$ be such that $\{ v \in V \mid \deg(v) \neq 2 \} \subseteq D$: clearly $|D| \leq 2$.

If $|D| = 2$, then $D = \{ a, b \}$ and (i) holds. If $|D| < 2$ then $\theta^{\epsilon,n}_{a,b}$ contains at most one vertex of degree distinct from 2. Since $\deg(a) = \deg(b) = \epsilon + n$, this implies that $\theta^{\epsilon,n}_{a,b}$ contains no vertex of degree distinct from 2. We have therefore $(\epsilon, n) = (0, 2)$ and $|V| = 4$ or $(\epsilon, n) = (1, 1)$ and $|V| = 3$ so that (ii) holds.

Conversely, let us consider a biconnected graph $G = (V, E)$ and let $D \subseteq V$ be such that $\{ v \in V \mid \deg(v) \neq 2 \} \subseteq D$.

Let us suppose that (i) holds and let $D = \{ a, b \}$. Therefore if $x \notin D$, then $\deg(x) = 2$ and since $xa, xb \in E$, then $\{ a, b \} = xE$. We have therefore that G is isomorphic to $\theta^{\epsilon,n}_{a,b}$ where $n = |V| - 2$ and $\epsilon = 1$ if $ab \in E$ and otherwise $\epsilon = 0$.

Let us suppose that (ii) holds, i.e. that $|D| < 2$ and $|V| \in \{ 3, 4 \}$. Let us assume first that $|V| = 3$. By looking at the list of graphs with 3 vertices, we

observe that only the total graph K_3 is biconnected, hence G is isomorphic to $\theta^{1,1}_{a,b}$. If $|V| = 4$, then we claim that every vertex of G has degree 2, so that G is isomorphic to $\theta^{0,2}_{a,b}$. To prove the claim, observe that G may have at most one vertex whose degree is not 2. Suppose that such a vertex exists and call it v_0, so that $\deg(v_0) \in \{1, 3\}$ since G is connected. Then the usual formula

$$2|E| = \deg(v_0) + \sum_{v \neq v_0} \deg(v) = \deg(v_0) + 6$$

leads to a contradiction. □

Lemma 7.7.4. Let $G = (V, E)$ be a biconnected graph and $D \subseteq V$ be such that $\{ v \in V \mid \deg(v) \neq 2 \} \subseteq D$. Then G is isomorphic to a molecule $\theta^{\epsilon,n}_{a,b}$, with D isomorphically sent to a subset of $\{a, b\}$, if and only if either (i) $|D| = 2$ and $xd \in E$ for each $x \in V \setminus D$ and $d \in D$ or (ii) $|D| < 2$ and $|V| \in \{3, 4\}$. ■

Therefore the recognition algorithm for a molecule is as follows.

```
IS–MOLECULE(C, A)
if |A| > 2 then return false
let D = { x ∈ C | deg(x) ≠ 2 } ∪ A
if |D| > 2 then return false
if |D| < 2 then
  if |C| ∈ {3,4} then return true
  else return false
foreach x ∈ C \ D
  if D ⊈ xE then return false
return true
```

Let us now argue about time resources of this algorithm.

Fact 7.7.5. Algorithm ENTANGLEMENT-TWO(G) runs in time $O(|V_G|)$. ■

It is clear that the function IS-MOLECULE runs in time $O(|C|)$, so that the loop (lines 7-8) of ENTANGLEMENT-TWO runs in time $O(\sum_{C \in \mathcal{C}(G)} |C|) = O(|V| + |E|)$. Therefore the algorithm requires time $O(|V| + |E|)$.

The following Proposition, whose proof depends on considering a tree with back edges arising from a Depth-First-Search on the graph, elucidates the role of the 3rd line of the algorithm ENTANGLEMENT-TWO.

Proposition 7.7.6. *If a graph (V, E) does not contain a simple cycle C_n with $n \geq k$, then it has at most $(k-2)|V| - 1$ undirected edges.*

Proof. The proof relies on some definitions and lemmas.
Let us recall that a pointed digraph $\langle V, E, v_0 \rangle$ is a tree if for each $v \in V$ there exists a unique path from v_0 to v.

Definition 7.7.7. A tree with back-edges is a tuple $\langle V, T, v_0, B \rangle$ such $\langle V, T, v_0 \rangle$ is a tree, and $B \subseteq V \times V$ is such that if xBy then y is an ancestor of x in the tree $\langle V, T, v_0 \rangle$. ■

For a tree with back edge as above let $\delta(v)$ be the length of the unique path from v_0 to v on the tree. If $(d, a) \in B$ is a back edge, then we define

$$\ell(d, a) = \delta(d) - \delta(a) + 1 \,.$$

We observe that $\ell(d, a)$ is the number of vertices on the path from a to d.

Definition 7.7.8. A TwBE representation of an undirected connected graph $G = \langle V, E \rangle$ is a tree with back edges of the form $\langle V, T, v_0, B \rangle$, satisfying the following conditions:

1. if $ab \in E$, then $(a, b) \in T \cup B$ or $(b, a) \in T \cup B$,
2. $(a, b) \in T \cup B$ implies $(b, a) \notin T \cup B$,
3. if $(a, b) \in T \cup B$, then $ab \in E$.

■

Representations of this kind are obtained by running a DFS on an undirected graph, according to the following standard Lemma (see [CLR90, Theorem 23.9]).

Lemma 7.7.9. Let $(V, \bar{E}_{\mathcal{T}})$ be a DFS tree of an undirected graph (V, E). If an edge ab is in $E \setminus \bar{E}_{\mathcal{T}}$ then either a is an ancestor of b, or a is a descendant of b in T. ■

Lemma 7.7.10. A $\langle V, T, v_0, B\rangle$ tree with back edges such that $0 < \ell(d, a) \leq k$ for each back edge (d, a) has at most $(k + 1)|V| - 1$ egdes. If $\langle V, T, v_0, B\rangle$ is a TwBE representation of an undirected connected graph, then it has at most $(k - 1)|V| - 1$. ■

Proof. The edges from T are as usual $|V| - 1$. Every vertex d can be the source of at most k back edges. Hence we obtain

$$|T \cup B| = |T| + |B| \leq (|V| - 1) + k|V| = (k + 1)|V| - 1\,.$$

If moreover $\langle V, T, v_0, B\rangle$ is the representation of an undirected graph, we cannot have back edges of the form (d, d), not back edges of the form (d, a) if $\ell(d, a) = 1$. Therefore Every vertex d can be the source of at most $k-2$ back edges. The formula follows as before. This ends the proof of lemma 7.7.10 □

Lemma 7.7.11. Let $\langle V, T, v_0, B\rangle$ be a TwBE representation of an undirected connected graph $G = \langle V, E\rangle$ whose simple cycles have length less than k. Then for each backedge (d, a) we have $0 < \ell(d, a) \leq k$. ■

Proof. Suppose that $k < \ell(d, a)$, then the path from a to d together with the edge from d to a form on G an undirected cycle of length $\ell(d, a) > k$. This ends the proof of lemma 7.7.11 □

Summing up the previous Lemmas, the proof of Proposition 7.7.6 follows. □

Line 3 of the algorithm ENTANGLEMENT-TWO ensures $|E_G| = O(|V_G|)$ and that the algorithm runs in time $O(|V_G|)$.

7.8 Conclusion and perspectives

Entanglement is an intrinsically dynamic concept, due to its game theoretic definition. As such it is not an easy object of study, while the two characterizations prepare it for future investigations with standard mathematical tools. They also suggest that entanglement is a quite robust notion, henceforth worth being studied independently of its fix-point theoretic background.

Clearly, a work that still need to be carried out is to look for some useful characterization of *directed* graphs of entanglement at most k. At present, characterizations are known only for $k \leq 1$ [BG05, Proposition 3]. We believe that the results presented here suggest useful directions to achieve this goal. In particular, a suggestive path is to generalize the algebra of molecules and legal 1-Sum to a directed setting. This path might be a feasible one considering that many scientists have recently developed ideas and methods to lift some algebraic framework from an undirected to a directed setting. W.r.t. the algebra of entanglement, a source of ideas might be the recent development of directed homotopy theory from concurrency [GR02].
In the following Chapter, we shall lift the algebraic approach presented in this Chapter to study the undirected graphs of entanglement 3. Furthermore, there the n-connectivity and cyclicity will be the starting point to study entanglement.

Chapter 8

Undirected Graphs of Entanglement 3

In [Tut66] Tutte gave a canonical decomposition of 2*-connected graphs by means of cycles and* 3*-connected components in a tree like structure, called the* 3*-block tree. In this Chapter we continue the investigation of the structure of graphs of bounded entanglement by examining Tutte's decomposition and applying it to undirected graphs of entanglement at most* 3*. We shall give necessary conditions for* 3*-block trees to be a decomposition of* 2*-connected graphs of entanglement at most* 3*.*

8.1 Introduction

Connectivity and cyclicity of graphs are among their basic properties: a graph may be decomposed into maximal articulation point-free blocks in a tree like fashion. An articulation point-free block itself admits a canonical decomposition in terms of cycles and 3-connected graphs in a tree like structure, this decomposition was given by Tutte [Tut66]. Continuing this process, other decompositions of graphs of higher connectivity – a sort of generalization of Tutte's decomposition – have been given in [Hoh92] but they are no longer canonical. Another decomposition of directed graphs in terms of directed cy-

cles, known as *ear* decomposition [BJG01, §7], has shown its use to study the connectivity of directed multigraphs. Such decompositions are important tools in graph theory and they are often used in inductive proofs and constructions. For instance see [RSŠ94] where Tutte's decomposition is used.

In the previous Chapter we have seen that the decomposition of graphs into the canonical 2-connected blocks is well adapted in recognizing the structure of graphs of entanglement 2. With a similar approach, we are asking whether Tutte's decomposition would be of use to recognize the structure of undirected graphs of entanglement 3. Roughly speaking Tutte's decomposition reduces the structure of a given graph into that of its blocks. So what can we say about the blocks themselves if the given graph has entanglement 3 ? To answer this question, we shall examine the notion of entanglement with respect to the two notions of connectivity and cyclicity. This will allow us to characterize the structure of 3-connected blocks, and to impose necessary conditions on the manner by which the blocks are glued together, if the starting graph has entanglement 3. A natural path for generalizing the algebraic approach used to construct the graphs of entanglement 2 to the construction of the graphs of entanglement 3. We mean by the algebraic approach the manner by which a class of graphs is build up out of small pieces of graphs using an appropriate set of gluing operations. Unfortunately, the approach consisting in the construction of the graphs of entanglement 2 – by means of the molecules and the legal 1-Sum operator, see previous Chapter – will only be partially adapted to the construction of the graphs of entanglement 3.
Summarizing, Tutte's decomposition is the main tool used to provide a tree decomposition of entanglement 3. This research path seems to be the most natural one that tries to adapt the results on entanglement 2. This adaptation – even partial and still rises in an experimental stage – interacts with Tutte's decomposition in a deep and not obvious way.

8.2 Preliminaries

All the graphs in this Chapter are finite and undirected until we say otherwise. They are also simple i.e. without multi-edges, but during the decomposition process[1] some mutli-edges may appear. In other words the starting graphs are always simple whereas multi-graphs may appear during the decomposition process.

8.2.1 Cyclicity

A *feedback vertex set* [FPR99] of a digraph G is a subset $X \subseteq V_G$ that meets all directed cycles of G, i.e. the digraph $G \setminus X$ is acyclic. The *cyclicity* of G, denoted $\mathscr{C}(G)$, is the cardinality of the *minimum* feedback vertex set.
When we deal with the cyclicity of an *undirected* graph G we consider the *directed* cycles of the *symmetric directed* graph G. In other words, every (undirected) edge v_1v_2 of G may be viewed as the set of (directed) edges $\{ (v_1, v_2), (v_2, v_1) \}$, and therefore $v_1v_2v_1$ is a directed cycle of length 2. To make the notion of the cyclicity of an undirected graph easier and more intuitive, it is convenient to see the feedback vertex set of an undirected graph as an *edge cover set*, i.e. a set $X \subseteq V_G$ is an edge cover of G if for each edge $v_1v_2 \in E_G$ we have that $v_1 \in X$ or $v_2 \in X$.

Lemma 8.2.1. Let G be an undirected graph, and let $X \subseteq V_G$. Then, X is a feedback vertex set of G if and only if X is an edge cover set of G. ■

Proof.
$\Longrightarrow$
Let $X \subseteq V_G$ be a feedback vertex set of G. Therefore, for every cycle of the form $v_1v_2v_1$ we have that $v_1 \in X$ or $v_2 \in X$. Since the cycle $v_1v_2v_1$ is just the edge v_1v_2 of the undirected graph G, then X is an edge cover set of G.
$\Longleftarrow$
Let $X \subseteq V_G$ be an edge cover set of G and let $C_n = v_1v_2 \dots v_nv_1$ be a cycle

[1] Indeed we mean Tutte's decomposition.

of G. Then, for each edge $v_i v_{i+1}$ of C_n we have $v_i \in X$ or $v_{i+1} \in X$, because X is an edge cover. Therefore X meets C_n, hence X is a feedback vertex set of G. □

It follows by the previous Lemma that the cyclicity of an undirected graph G is the cardinality of the minimum edge cover set of G.

8.2.2 Connectivity

Given two sets A, B of vertices, we call $\pi = v_1 \dots v_n$ an A-B *path* if $V_\pi \cap A = \{ v_1 \}$ and $V_\pi \cap B = \{ v_n \}$. Two or more paths are *independent* if none of them contains an internal vertex of the other. If a, b are two vertices then we shall write "a-b path" instead of "$\{ a \}$-$\{ b \}$ path". In this case, two a-b paths are independent if and only if a and b are their only common vertices.

Definition 8.2.2. A graph is k-connected [Die05, §3] if one needs to remove at least k-vertices to disconnect it. The *connectivity* of a graph G is the maximum k such that G is k-connected. ■

The following Theorem, due to Menger [Die05, §3.3], provides a useful equivalent definition of k-connectivity.

Theorem 8.2.3. A graph is k-connected if and only if it contains k independent paths between any two vertices. ■

A vertex whose removal disconnects the graph is called *articulation point*. A maximal connected subgraph without articulation points is called a *biconnected component*. By their maximality, the biconnected components of G overlap in at most one vertex, which is an articulation point of G. Hence, every edge of G belongs to a unique biconnected component. An isolated vertex is considered trivially as a biconnected component. Let $\mathcal{A}$ be the set of articulation points of G, and $\mathcal{B}$ the set of its 2-connected components. Then we obtain a bipartite graph $T_1(G)$ whose vertices are $\mathcal{A} \cup \mathcal{B}$ and whose edges are aB whenever $a \in B$. It is one of the elementary results to show that if G is connected then $T_1(G)$ is a tree, see Proposition 3.1.2 of [Die05].

8.2.3 Separations, hinges

In this subsection we introduce the definition of separations and hinges. The material presented in this subsection can be found in [Tut66], whereas we shall adopt the notation and the terminology of [DSS95] and [Ric04].

A *bond* is a graph consisting of just two vertices and at least one edge. A *k-bond* is a bond of k edges. observe that, if $k \geq 2$, then a k-bond is a multigraph.

If $A, B \subseteq V_G$ and $S \subset V_G$ are such that every A-B path in G contains a vertex of S, then we say that S separates the sets A and B in G. Observe that this implies that $A \cap B \subseteq S$. We shall say that S separates G if $G \setminus S$ is disconnected, that is, if S separates G into some vertices which are not in S. A separating set of vertices is called a *separator*. A pair (A, B) is a *separation* of G if $A \cup B = V_G$ and G has no edge between $A \setminus B$ and $B \setminus A$. Clearly, this is equivalent to saying that $A \cap B$ separates A from B. We say that a separation (H, K) is a *k-separation* if $|V_{H \cap K}| = k$. A subgraph K of G is an *H-bridge* if K is obtained from a component $\mathcal{C}$ of $G \setminus V_H$ by adding to $\mathcal{C}$ all the edges of G which have at least one end in $\mathcal{C}$. From the above definition of connectivity follows a standard result of graph theory [Tut66]:

Lemma 8.2.4. A graph is k-connected if there is no m-separation, for all $m = 0, \ldots, k-1$. ■

From the definition of connectivity, it follows that the connectivity of the following graphs is infinite: the graph of just one vertex, the k-bonds, for $k \geq 1$, and the 3-clique, see Figure 8.2.1.

Figure 8.2.1: Graphs with infinite connectivity.

Convention 8.2.5. For technical reasons, we shall consider the connectivity of the k-cliques, for $k \geq 3$, equals $k-1$. ■

A *hinge* of G is a 2-separation (A, B) such that at least one of A and B is 2-connected.

8.2.4 Tree decomposition

We reproduce here the definition of the notion of *tree decomposition* as given by Robertson and Seymour [RS86].

Definition 8.2.6. Let G be a graph, T be a tree, and let $X = (V_t)_{t \in T}$ be a family of subsets of V_G. We say that the pair (T, X) is a *tree decomposition* of G if the following conditions hold:

(T-1) $V_G = \bigcup_{t \in T} V_t$,

(T-2) for every edge $v_1 v_2 \in E_G$ there exists V_t such that $v_1, v_2 \in V_t$,

(T-3) if there is a path $t_1 \ldots t_2 \ldots t_3$ in T then $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$.

■

Conditions (T-1) and (T-2) say the the graph G is the union of subgraphs induced by the set of vertices V_t; the sets $V_t, t \in V_T$ as well as the subgraphs induced by $V_t, t \in V_T$ are called the *bags* of the tree decomposition. Condition (T-3) states that the bags of the tree decomposition are organized into a tree like fashion.

One of the most important feature of the tree decomposition concept is that it shows a natural correspondence between the properties of the separations of the graph and its tree decomposition:

Lemma 8.2.7. Let G be a graph and let $(T, (V_t)_{t \in T})$ be a tree decomposition of G. Given an edge $t_1 t_2$ of T and let T_1, T_2 be the components of $T \setminus t_1 t_2$ such that $t_1 \in T_1$ and $t_2 \in T_2$.
Then $V_{t_1} \cap V_{t_2}$ separates $U_1 := \cup_{t \in T_1} V_t$ from $U_2 := \cup_{t \in T_2} V_t$ in G. ■

Proof. Since T is a tree, then every t-t' path in T with $t \in T_1$ and $t' \in T_2$ contains both t_1 and t_2. Therefore, it follows by (T-3) that $U_1 \cap U_2 \subseteq V_{t_1} \cap V_{t_2}$. To accomplish the proof it remains to show that G does not contain an edge u_1u_2 with $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. If such an edge exists then it follows from (T-2) that there exists $t \in V_T$ such that $u_1, u_2 \in V_t$. By assumption, we have chosen u_1 in $U_1 \setminus U_2$ and hence $t \in T_1$. Also, we have chosen u_2 in $U_2 \setminus U_1$ and hence $t \in T_2$. This is a contradiction because by the hypothesis we have $T_1 \cap T_2 = \emptyset$. □

A *bag*, denoted β_t, $t \in V_T$, is the subgraph of G induced by vertices V_t. A *torso*[2], denoted τ_t, is the bag β_t where we add an edge vv' to the multiset of edges of β_t for each $v, v' \in V_t \cap V_{t'}$ such that $tt' \in E_T$. Observe that after adding such edges the torso may become a multigraph even if the starting graph is simple. However, we can split the edges of a torso into two sets, the set of the original edges which belongs to the bag, and the set of edges which we have added, the latter are called the *virtual* edges.

8.3 3-Block decomposition, Tutte's Theorem

The following Theorem, known as *Tutte decomposition Theorem* – which provides a tree decomposition of 2-connected graphs into cycles, bonds and 3-connected components – will be our main working tool.

Theorem 8.3.1. [Tut66] Every 2-connected graph has a tree decomposition $(T, (V_t)_{t \in T})$ such that $|V_t \cap V_{t'}| = 2$ for every $tt' \in E_T$, and moreover every torsos is either a k-bond ($k \geq 3$), or 3-connected or a cycle. Conversely, every graph with such a tree decomposition is 2-connected. Furthermore such a decomposition is unique. ■

The key idea behind Tutte's Theorem consists in considering a particular set of 2-separations of G, these are the 2-separations which are *compatible*

[2] a torso and not a torsor. The plural of torso is torsos. We adopt the terminology of [Die05].

with all the other 2-separations. Two separations (U_1, U_2) and (W_1, W_2) are compatible if we can find $i, j \in \{1, 2\}$ such that $U_i \subseteq W_j$ and $U_{3-j} \supseteq W_{3-j}$. It follows from the proofs of Tutte's Theorem presented in the literature [3] that (U_1, U_2) is a separation which is compatible with all the other separations if and only if $U_1 \cap U_2$ is a hinge. Finally Tutte's decomposition arises from considering a tree decomposition $(T, (V_t)_{t \in T})$ such that $V_t \cap V_{t'}$ is a hinge for all $tt' \in E_T$.

The original proof of this Theorem can be found in [Tut66]. An extension of this Theorem to locally infinite[4] graphs is referenced in [DSS95]. A generalization of this Theorem to arbitrary infinite graphs can be found in [Ric04]. In the sequel we shall refer to such a decomposition as *Tutte's decomposition*. Tutte proved Theorem 8.3.1 for multi-graphs [Tut66], the following Proposition puts an emphasize on that Theorem if the given graph is simple.

Proposition 8.3.2 ([Ric04])**.** *Let G be a graph and let $(T, (V_t)$ be its Tutte's decomposition tree. Then, if G is simple (i.e. without multi-edges) then every 3-connected torso of T is simple too.*

An example of a graph and its decomposition tree is depicted in Figure 8.3.1. Let $U_1 := V_G \setminus \{v_9\}$ and $U_2 := \{v_6, v_8, v_9\}$. Observe that (U_1, U_2) is a 2-separation of G. The set $U_1 \cap U_2 = \{v_6, v_8\}$ is a 2-separator of G but it is not a hinge because neither the subgraph $G[U_1]$ nor $G[U_2]$ is 2-connected. By inspecting the set of 2-separators, we can deduce that the set of hinges is $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_8, v_9\}\}$. The virtual edges arising from this decomposition are represented by dashed lines.

The following Lemma provides some useful Properties of Tutte's Tree.

Lemma 8.3.3. Let G be a 2-connected graph and T its Tutte's tree. Then,

(a) T is a bipartite graph (T_1, T_2) where $t \in T_1$ if and only if V_t is a hinge.

[3] For instance that of [Ric04], which uses the notion of hinges, and of [Die05], that uses the notion of compatible separations.

[4] An infinite graph is locally finite if the degree of each vertex is finite.

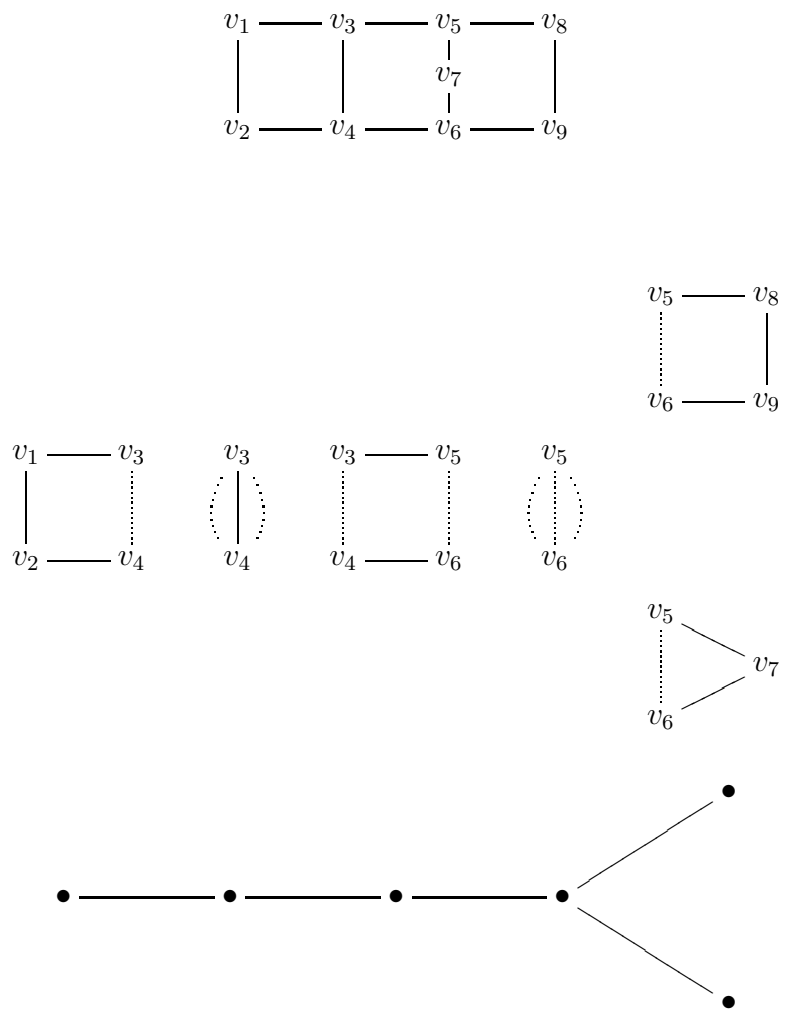


Figure 8.3.1: A graph with its Tutte's decomposition tree

(b) Let $t_1 \dots t_n$ be a path in T then, if h_i denotes the hinge V_{t_i} then

(b.1) for all $i = 1, \dots, n-1$, we have either $h_i \cap h_{i+1} = \emptyset$ or $|h_i \cap h_{i+1}| = 1$,

(b.2) if $h_p \cap h_q = \emptyset$ where $1 \leq p < q \leq n$, then for all $i \leq p$ we have that $h_i \cap h_q = \emptyset$, and

(b.3) if $h_p \cap h_q = \emptyset$ where $1 \leq p < q \leq n$, then for all $i \geq q$ we have that $h_p \cap h_i = \emptyset$.

■

Proof. The statement (a) follows from Tutte's Theorem 8.3.1. The statements (b.1), (b.2), and (b.3) are a direct consequence of Lemma 8.2.7. □

Lemma 8.3.4. [Ric04]. Let G be 2-connected graph. A 2-separator $\{x, y\}$ is a hinge in G if and only if *(i)* either there are at least three $[x, y]$-bridges, or *(ii)* there are two $[x, y]$-bridges at least one of them is 2-connected. ■

In the previous Chapter we have seen that the tree structure of the 2-connected components of a graph gives rise to an algebraic expression of the starting graph in terms of its 2-connected components and the 1-Sum operator. In a similar way, Tutte's decomposition tree of a given 2-connected graph gives rise to an algebraic expression in terms of k-bonds, cycles and 3-connected components and the 2-Sum operator. The latter operation consists in taking two graphs, choosing a 2-clique from each, identifying the vertices in the cliques and deleting the edges of the cliques, see Figure 8.3.2.

The k-sum operators on graphs, for $k = 1, 2, 3$,[5] have been introduced in [Wag37] in the aim to prove a theorem which states that a graph which does not contain a K_5 as a minor may be expressed by means of these operations starting with the class of planar graphs and a particular graph on 8 vertices[6].

[5]The k-sum, for $k = 3$ is defined in a similar way of the 2-Sum operator apart that we take a 3-click of each graph.

[6]This Theorem has been extended to matroids in [Sey81] by extending these operations to matroids as well.

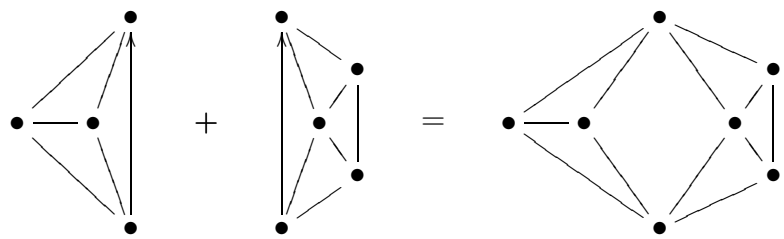


Figure 8.3.2: The 2-Sum operator

Definition 8.3.5. Let G_1 and G_2 be graphs and let $e_1 = (a_1, b_1) \in E_{G_1}$ and $e_2 = (a_2, b_2) \in E_{G_2}$. Define the 2-Sum of G_1 and G_2 on e_1 and e_2 respectively, denoted $G_1 +_{e_1 e_2} G_2$, to be the graph obtained from the union of G_1 and G_2 by identifying the vertex a_1 with a_2 and the vertex b_1 with b_2 and deleting the edges e_1 and e_2, see Figure 8.3.2. ■

It is clear that (see [Tut66]):

Lemma 8.3.6. G and H are 2-connected if and only if $G +_{e_1 e_2} H$ is 2-connected, for each $e_1 \in E_G, e_2 \in E_H$. ■

Note that the 2-bond is the neutral element of the 2-Sum. From the previous Lemma it follows that:

Proposition 8.3.7. *Let $\mathcal{B}_2$ be the least class of graphs containing the k-bonds ($k \geq 3$), the cycles, the 3-connected graphs and closed under the 2-Sum operator. Then, $G \in \mathcal{B}_2$ if and only if G is 2-connected.*

Lemma 8.3.8. Let G be 2-connected and let $(T, (V_t)_{t \in T})$ be its Tutte decomposition, then every torsos τ of T is a minor of G. ■

Proof. Let $t \in T$ and let $t_1, \ldots, t_n$ be the neighbours of t in T. Let $T_i, i = 1, \ldots, n$ be the component of $T \setminus tt_i$ that contains t_i, and let $\overline{T}_i$ be the complement of T_i w.r.t T. Let also $U_i := \cup_{t \in T_i} V_t$. Recall that if $U \subseteq V_G$ then the subgraph of G induced by U is denoted by $G[U]$.
We shall prove that the torso β_t is a minor of G. On the one hand, since

the subgraph $G[U_i]$ is connected, see Theorem IV.20 of [Tut01], then there exists an u_i-w_i path π_i in $G[U_i]$ such that $\{ u_i, w_i \} = V_t \cap V_{t_i}$. On the other hand, since $\{ u_i, w_i \}$ is a 2-separator in G of U_i from $\overline{U}_i$ by Lemma 8.2.7, then $\{ u_i, w_i \}$ is also a 2-separator in G of U_i from U_j where $i \neq j$ because $U_j \subseteq \overline{U}_i$. Therefore, the u_i-w_i paths, for $i = 1, \ldots, n$, are independent. Observe that the graph β_t^+ consisting of the bag β_t where we add the u_i-w_i paths is a subgraph of G. Since the u_i-w_i paths are independent we can contract each path until we get a single edge and moreover the resulting graph is just the torsos τ_t. Hence τ_t is a minor of β_t^+, and β_t^+ is a subgraph of G, hence τ_t is a minor of G. □

From the previous Lemma we get the following particular case:

Lemma 8.3.9. Let G be a 2-connected graph, such that $G = G_1 +_{e_1 e_2} G_2$. Then, $G_i + e_i$ is a minor of G, $i = 1, 2$. ■

8.4 Entanglement, connectivity, and edge covering

On the one hand, atoms of Tutte's decomposition are cycles and 3-connected components, on the other hand cycles have entanglement at most 3. The aim now is to investigate the structure of 3-connected components whenever the starting graph has entanglement 3. To this goal, we first establish the relation between the three notions of entanglement, connectivity and edge covering.

Lemma 8.4.1. Let G be a k-connected graph with $|V_G| \geq k + 1$, then

$$k \leq \mathscr{E}(G) \leq \mathscr{C}(G)$$

■

Proof. Let us prove first the inequality $\mathscr{E}(G) \leq \mathscr{C}(G)$. Let X be an edge cover set of G, we shall show that Cops have a winning strategy in $\mathscr{E}(G, |X|)$.

While Thief is moving on a path π, Cops strategy consists in placing a cop on a vertex v of π if and only if v belongs to X, showing that the number of cops placed on the graph is at most $|X|$. If there is an infinite play then this implies that there is a cycle whose vertices are not covered by X, meaning that X is not an edge cover set of G, this is a contradiction. Therefore, Cops's strategy is winning and hence $\mathscr{E}(G) \leq \mathscr{C}(G)$.
To prove the inequality $k \leq \mathscr{E}(G)$, we use again Menger's Theorem 8.2.3, stating that *a graph is k-connected if and only if it contains k independent paths between any two vertices.*
We shall prove that Thief has a winning strategy in $\mathscr{E}(G, k-1)$. To this goal it is sufficient to show the following conditions to hold: whenever $k-1$ cops are placed on the graph, then

(i) there is at least an edge whose both ends are not occupied by a cop. This is a consequence of the inequalities $\mathscr{C}(G) \geq k > k-1$. The inequality $\mathscr{C}(G) \geq k$ is justified by the fact that every edge cover set of the graph G would disconnect it.

(ii) when it is Thief's turn to move from some vertex v, then by Menger's Theorem, it follows that there is a free path (i.e. its vertices are not occupied by a cop) from v to some vertex w such that $ww' \in E_G$ and both w and w' are not occupied by a cop.

It follows that Thief's strategy consists in looking for an edge which is not occupied by a cop, choosing a free path to it, use this path to reach this edge, and iterating moves on it until Cops place a cop on one of its end points. This strategy can be iterated infinitely often, hence its a winning strategy for Thief in $\mathscr{E}(G, k-1)$. □

Lemma 8.4.2. Let G be k-connected with $|V_G| \geq k+1$. If $\mathscr{E}(G) = k$, then

$$\mathscr{C}(G) = \mathscr{E}(G) = k.$$

■

Proof. We have already mentioned that $\mathscr{C}(G) \geq \mathscr{E}(G) = k$. To prove $\mathscr{C}(G) = k$ we assume that $\mathscr{C}(G) > k$ and we shall deduce a contradiction: we shall prove that Thief has a winning strategy in the game $\mathscr{E}(G, k)$.
We distinguish two cases in the game $\mathscr{E}(G, k)$ according to the number of cops placed on the graph:
Case (i). If the number of cops placed on the graph is at most $k - 1$ then the same conditions *(i)* and *(ii)* provided in the proof of Lemma 8.4.1 still hold, hence in this case Thief will never be caught.
Case (ii). If k cops are placed on the graph then consider the first position of the game for which the number of cops placed on the graph increases from $k - 1$ to k, that is we consider the first Cops' add move $(v, C', Cops) \to (v, C' \cup \{ v \}, Thief)$ where $|C' \cup \{ v \}| = k$.
One the one hand, since $\mathscr{C}(G) > k$ by assumption then there exists an edge ww' which is not covered. On the other hand, since a cop is posted on v and $k - 1$ cops are posted on the remaining vertices, then by Menger's Theorem there are k independent paths linking v to w, therefore there exists a free path π from v to w. Thief's strategy consists in going from v to w through π and iterating moves on the edge ww' until Cops place a cop on either w or w', say w, giving rise to a position of the form $(w, C, Thief)$ where $|C| = k$ and $w \in C$, returning back to the initial configuration. From the latter position Thief uses the same strategy described so far. Such a strategy can be iterated infinitely often, hence Thief has a winning strategy in $\mathscr{E}(G, k)$. This contradicts the hypothesis $\mathscr{E}(G) = k$.

□

8.5 The k-molecules

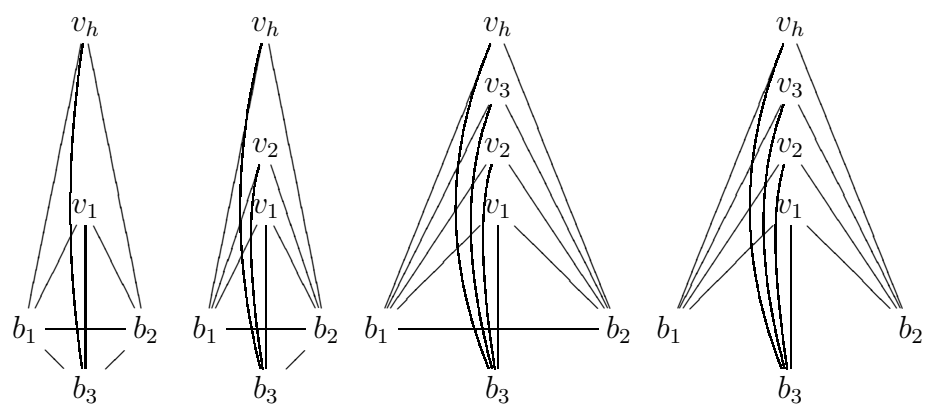
Tutte's Theorem 8.3.1 reduces the structure of a given 2-connected graph into that of its blocks, so what can we say about the structure of the blocks themselves, and in particular the 3-connected ones? On the one hand, atoms of Tutte's decomposition are cycles and 3-connected graphs, on the other

hand cycles have entanglement at most 3. So it is worth to investigate the structure of 3-connected graphs for which the entanglement equals 3. We shall do something more: based on Lemma 8.4.2 we shall investigate the structure of the k-connected graphs of entanglement k, for arbitrary $k \geq 1$.

Definition 8.5.1. Let $k, h \geq 1$, $B_k = \{ b_1, \ldots, b_k \}$ and $\mathcal{B} \subseteq B_k \times B_k$. A k-molecule $\vartheta_{B_k}^{\mathcal{B},h}$ is the graph $G = (V, E)$ such that

1. $V = B_k \cup \{ v_1, \ldots, v_h \}$,
2. $E = \mathcal{B} \cup \{ v_i b_j,\ 1 \leq i \leq h,\ 1 \leq j \leq k \}$.
3. $h \geq k - k'$, where k' is the connectivity of the subgraph of G induced by B_k.

The set B_k is called the base of the k-molecule. ■



$\vartheta_{B_3}^{3,h}, h \geq 1$ $\vartheta_{B_3}^{2,h}, h \geq 2$ $\vartheta_{B_3}^{1,h}, h \geq 3$ $\vartheta_{B_3}^{0,h}, h \geq 3$

Figure 8.5.1: The structure of 3-molecules $\vartheta_{B_3}^{|\mathcal{B}|,h}$.

The 1-molecules are the *stars*, the 2-molecules have been discussed in section 7.3, and the 3-molecules $\vartheta_{B_3}^{\mathcal{B},h}$ are pictured (up to graph isomorphism)

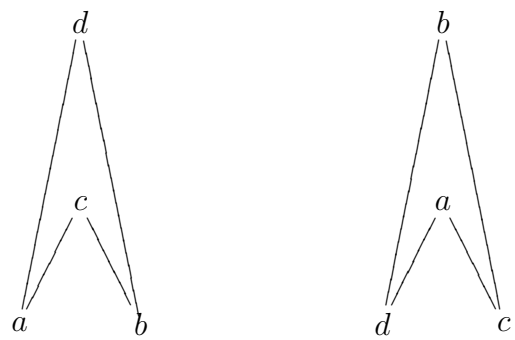


Figure 8.5.2: Two possible ways to view a 2-molecule.

in Figure 8.5.1; since no confusion will arise we have substituted $\mathcal{B}$ by its cardinality $|\mathcal{B}|$[7].

A natural question arises: in how many manners a k-molecule may be written? To illustrate this question consider the graph G depicted in Figure 8.5.2. If we take $A_2 = \{ a, b \}$, then the graph G may be viewed as the two molecule $\vartheta^{\emptyset,2}_{A_2}$. If we take $B_2 = \{ c, d \}$, then G may be considered as the 2-molecule $\vartheta^{\emptyset,2}_{B_2}$. The following Definition formalizes the fact that a graph is a k-molecule.

Definition 8.5.2. Let G be a graph and $B \subset V_G$ with $|B| = k$. We say that the pair (G, A) is a k-premolecule if there exist a k-molecule $\vartheta^{\mathcal{B},h}_{B_k}$ and a graph isomorphism $\psi : G \longrightarrow \vartheta^{\mathcal{B},h}_{B_k}$ sending B to B_k.
We say that a G is an abstract k-molecule if there exists $B \subset V_G$ with $|B| = k$ such that the pair (G, B) is a k-premolecule. ■

Lemma 8.5.3. Let G be an abstract k-molecule. If there exist $A_k, B_k \subset V_G$ such that each pair (G, A_k) and (G, B_k) is a k-premolecule. Then,

4.i the three subgraphs of G induced by $A_k \setminus B_k$, $B_k \setminus A_k$ and $V_G \setminus (A_k \cup B_k)$ are all discrete,

[7]This is possible without confusion when $k = 3$, because a graph on 3 vertices is completely determined (up to isomorphism) by the number of its edges.

4.ii the subgraphs of G induced by A_k and B_k are isomorphic.

■

Proof.

4.i Assume that $A_k \setminus B_k$ is not discrete i.e. there exists $a_1, a_1 \in A_k$ with $a_1a_2 \in E_{G_1}$. If we consider the k-premolecule (G, B_k) then we observe that $a_1a_2 \in V_G \setminus B_k$, this is a contradiction since the graph induced by $V_G \setminus B_k$ must be discrete by the definition of the k-molecules. We get that the sugraph induced by $B_k \setminus A_k$ is discrete by just dualinzing the above proof. To argue that $V_G \setminus (A_k \cup B_k)$ is discrete observe that $V_G \setminus (A_k \cup B_k) \subseteq V_G \setminus A_k$, and since $V_G \setminus A_k$ is discrete by the definition of the k-molecules, then it follows that $V_G \setminus (A_k \cup B_k)$ is also discrete.

4.ii Consider the mapping $\psi : G \longrightarrow G$ such that ψ is a bijection from $A_k \setminus B_k$ to $B_k \setminus A_k$, and it is the identity on both $(A_k \cap B_k)$ and $V_G \setminus (A_k \cup B_k)$. Since the subgraphs induced by $A_k \setminus B_k, B_k \setminus A_K$, and $V_G \setminus (A_K \cup B_k)$ are discrete by 4.i, then for all $v_1, v_2 \in A_k$ and for all $w_1, w_2 \in B_k$, $\psi(v_i) = \psi(w_i), i = 1, 2$, if and only if $\psi(v_1v_2) = \psi(w_1w_2)$. Hence the subgraphs of G induced by A_k and B_k are isomorphic.

□

Definition 8.5.4. Let G be an abstract k-molecule. We say that G is non ambiguous if there exists just one set $B \subset V_G$ such that the pair (G, B) is a k-premolecule. Similarly, a k-molecule ϑ is non ambiguous if ϑ viewed as a graph is non ambiguous. ■

In order to compute the connectivity of the k-molecules, the following Lemma provides a construction of them and a lower bound of their connectivity.

Lemma 8.5.5. Let G be the graph constructed as follows: out of a graph B and a set of vertices $\{ v_1, \ldots, v_h \}$ add an edge between each $v_i, i = 1, \ldots, h$

and each b in B. If B is k'-connected, then the connectivity of G is at least

$$min(|V_B|, k' + h).$$

■

Proof. Let $m = min(|V_B|, k' + h)$, we shall prove that G is m-connected, by proving that every two vertices x, y are linked by at least m disjoint paths. We split the proof in three cases.
Case (i). If $x, y \in V_B$, then there are k' disjoint path in the subgraph induced by V_B from x to y because the latter is k'-connected. Moreover there are h disjoint paths of the form $xv_1y, x_2y, \ldots xv_hy$ where $v_i \in V_G \setminus V_B$.
Case (ii). If $x, y \in V_G \setminus V_B$. In this case there are $|V_B|$ disjoint paths of the form $xb_1y, xb_2y, \ldots, xb_ky$, where $b_i \in V_B$.
Case (iii). If $x \in V_B$ and $y \in V_G \setminus V_B$, then the $k' + h$ disjoint paths are $\Pi_1 \cup \Pi_2 \cup \Pi_3$ where :

- $\Pi_1 = \{ xy \}$, recall that $xy \in E_G$ by definition.
- To exhibit Π_2 recall first that since the graph B is k' connected then x has at least k' neighbors in B, let $b_1, \ldots, b_{k'}$ be such neighbors. Therefore we let $\Pi_2 = \{ xb_1y, \ldots, xb_{k'}y \}$.
- Finally,
$$\Pi_3 = \{ xv_1b'_1y, \ldots, xv_ib'_iy, \ldots xv_{h-1}b'_{h-1}y \}$$
where $\{ b_1, \ldots, b_{k'} \} \cap \{ b'_1, \ldots, b'_{h-1} \} = \emptyset$.
It is straightforward to check that paths in $\Pi_1 \cup \Pi_2 \cup \Pi_3$ are disjoint, they share just their two end points. Moreover $|\Pi_1 \cup \Pi_2 \cup \Pi_3| = 1 + k' + (h - 1) = k' + h$.

□

Now we state the main properties of the k-molecules.

Proposition 8.5.6. *Let $G = \vartheta^{\mathcal{B},h}_{B_k}$ be a k-molecule. Then*

1. *the connectivity of G is k,*

2. *G has B_k as a minimal edge cover and hence $\mathscr{C}(G) = k$, and*

3. *the entanglement of G equals k.*

Proof.

1. Observe that if G is a clique then it should be a $(k+1)$-clique. Moreover, this holds if and only if the subgraph of G induced by $\mathcal{B}_k$ is a k-clique and $h = 1$. In this case the connectivity of G is k by Convention 8.2.5. Assume that G is not a clique. On the one hand, by lemma 8.5.5, it follows that the connectivity of G is at least $min(|B_k|, k' + h) = min(k, k' + h)$ and since $h \geq k - k'$ by the definition of the k-molecules, then $min(k, k' + h) = k$, showing that the connectivity of G is at least k. On the other hand, we need the Claim:

   Claim 8.5.7*:* Let $G := \vartheta_{B_k}^{\mathcal{B},h}$ be a k-molecule. If G is not a clique then $h \geq 2$. ■

   Proof. We distinguish two cases according to the nature of $G[B_k]$.
   If $G[B_k]$ is a k-clique, then we need $h \geq 2$, because if $h = 1$ then G would be a $(k+1)$-clique contradicting the hypothesis.
   If $G[B_k]$ is not a clique, then there exist at least two vertices $b, b' \in B_k$ such that $bb' \notin E_G$, therefore $B_k \setminus \{ b, b' \}$ is a $(k-2)$-separator of b from b' in $G[B_k]$. This implies that the connectivity of $G[B_k]$ is at most $k-2$. From the definition of the k-molecules, we have $h \geq k - k'$ where k' is the connectivity of $G[B_k]$, hence $h \geq k - (k-2) = 2$. This ends the proof of the Claim. □

   Since G is not a clique then, according to the Claim, we have $h \geq 2$ and hence B_k is a k-separator in G of any two vertices in $V_G \setminus B_k$ and in this case G can not be $(k+1)$-connected. We conclude that the connectivity of G is k.

2. Since G is k-connected and $|V_G| \geq k+1$ then by Lemma 8.4.1 we get $\mathscr{C}(G) \geq k$. It is easy to check that B_k is an edge cover of G, hence $\mathscr{C}(G) \leq |B_k| = k$. Therefore $\mathscr{C}(G) = k$.

3. On the one hand, since G is k-connected then from Lemma 8.4.1 we obtain $\mathscr{E}(G) \geq k$. On the other hand, from the same Lemma 8.4.1, we have $\mathscr{E}(G) \leq \mathscr{C}(G)$ and from the previous item we got $\mathscr{C}(G) = k$, thus $\mathscr{E}(G) \leq k$. We conclude that $\mathscr{E}(G) = k$.

□

So far we have stated and proved the main properties of the k-molecules, basically their connectivity, cyclicity, and entanglement. Conversely, the following Proposition states that if the three latter properties coincide, then the graph in question is a k-molecule.

Proposition 8.5.8. *If G is k-connected with $\mathscr{E}(G) = k$, then G is an abstract k-molecule.*

Proof. Let G be a graph as stated. Since $\mathscr{E}(G) = k$ and G is k-connected then by Lemma 8.4.2 it follows that $\mathscr{E}(G) = \mathscr{C}(G) = k$, thus let $B_k = \{ b_1, \ldots, b_k \}$ be a minimal edge cover set of G and let k' be the connectivity of $G[B_k]$.
We claim that vertices in $V_G \setminus B_k$ are at distance one from B_k because B_k is an edge cover set of G. For the same reason the subgraph of G induced by $V_G \setminus B_k$ is discrete. Thus, for each $v \in V_G \setminus B_k$ there exists $b \in B_k$ such that $vb \in E_G$. Let $v \in V_G \setminus B_k$ and $\mathcal{N}_v$ be the set of its neighbors, clearly $\mathcal{N}_v \subseteq B_k$ because $G[V_G \setminus B_k]$ is discrete. If $|\mathcal{N}_v| < k$ meaning that $\mathcal{N}_v \subsetneq B_k$ then clearly $\mathcal{N}_v$ separates v from $B_k \setminus \mathcal{N}_v$. This contradicts the assumption that G is k-connected. We conclude that $\mathcal{N}_v = B_k$ for each $v \in V_G \setminus B_k$.
Finally, to accomplish the proof that G, coming with the desired data, is a k-molecule, it remains just to show that $|V_G \setminus B_k| \geq k - k'$ where k' is the connectivity of $G[B_k]$. Towards a contradiction, assume that $|V_G \setminus B_k| < k - k'$. Since the connectivity of $G[B_k]$ is k' then there exists a k'-separator in $G[B_k]$, let $S_{k'}$ be such a separator and assume that it separates b_1 from b_2; $b_1, b_2 \in B_k$.

Therefore $S_{k'} \cup (V_G \setminus B_k)$ separates also b_1 from b_2. A simple computation shows that $|S_{k'} \cup (V_G \setminus B_k)| < k$, contradicting the fact that G is k-connected. We conclude that the pair (G, B_k) is a k-premolecule, moreover G may be written as $\vartheta_{B_k}^{\mathcal{B},h}$ where $\mathcal{B}$ are the edges of the subgraph of G induced by B_k, and $h = |V_G \setminus B_k|$. □

8.6 A simple proof of the structure of entanglement 2

In this section we provide an alternative proof of the structure of undirected graphs of Entanglement at most 2 already provided in Chapter 7.

Lemma 8.6.1. Let G be a k-molecule, then the only kind of winning strategies for Cops in $\mathscr{E}(G, k)$ is to occupy a minimal edge cover set of G, that is if $(v, C, Thief)$ is a final position then C is a minimal edge cover set of G. ■

Proof. Towards a contradiction: assume that Cops use a winning strategy in $\mathscr{E}(G, k)$ which does not occupy a minimal edge cover set of G. Meaning that, for every position of the form $(v, C, Thief)$ where $|C| = k$, C is not an edge cover set of G. We can assume without loss of generality that $v \in C$. On the one hand, there exists an edge ww' whose ends are not occupied by a cop by assumption that C is not an edge cover. On the other hand, since a cop is placed on v and $k-1$ cops are placed on the remaining vertices of G, then by Menger's Theorem, there are k v-w independent paths, therefore there exists a free path[8] π from v to w. Thief's strategy consists in going from v to w through π and iterating moves on the edge ww' until Cops place a cop on either w or w'. By assumption, the position of the cops on G is not an edge cover set, returning back to the initial configuration. This strategy can be iterated infinitely often and hence it is a winning strategy for Thief in $\mathscr{E}(G, k)$. This is a contradiction because we have assumed that Cops use a winning strategy in $\mathscr{E}(G, k)$. □

[8] This path would be just an edge.

The following Lemma generalizes the notion of *legal* 1-Sum considered in Section 7.3.

Lemma 8.6.2. Let G be a k-molecule and $B = \{ b_1, \dots b_n \} \subset V_G$ such that B is not included in any edge cover set of size k and let $\overline{B} = \{ \bar{b}_1, \dots, \bar{b}_n \}$ be a set of vertices such that $\overline{B} \cap V_G = \emptyset$. Consider the graph $G^B = (V_G \cup \overline{B}, E_G \cup \{ b_i\bar{b}_i, i = 1, \dots, n \})$. Then

$$\mathscr{E}(G^B) \geq k + 1$$

■

Proof. Thief has the following winning strategy in $\mathscr{E}(G^B, k)$. He restricts his moves on the subgraph G of G^B forcing Cops to occupy a minimal edge cover set of G, say C, Lemma 8.6.1. This gives rise to a position of the form $(v, C, Thief)$ where $|C| = k$ and $v \in C$. One the one hand, since $C \cap B \neq \emptyset$ by assumption, then there exists an edge free of cops of the form $b_i\bar{b}_i$ where $b_i \in B \setminus C$. On the other hand, from the above position $(v, C, Thief)$ where $v \in C$, there are $k - 1$ cops placed on the subgraph $G \setminus v$, and since G is k-connected then there are at least k disjoint path linking v to b_i by Menger's Theorem, and then there exists a free path π from v to b_i. Thief's strategy consists in going from v to b_i through π and iterating moves on the edge $b_i\bar{b}_i$ until Cops put a cop on either b_i or $\bar{b}_i$. This implies that the set of vertices of G currently occupied by Cops is no longer a minimal edge cover set. This would allow Thief to restrict his moves on G forcing Cops to occupy a minimal edge cover set of G, Lemma 8.6.1. Such a strategy for Thief can be iterated infinitely often in $\mathscr{E}(G^B, k)$, that is, it is a winning strategy. Therefore $\mathscr{E}(G^B) \geq k + 1$. □

Corollary 8.6.3. Let G be a k-molecule and $B \subset V_G$. Let G^B be defined as in the previous Lemma and let us assume that $\mathscr{E}(G^B) = k$. Then there exists B' such that $B \subseteq B'$ and B' is a minimal edge cover of G ■

Theorem 8.6.4. Let G be a graph of entanglement at most 2. Then

1. each biconnected component B of G comes with a subset $B_2 \subseteq V_B$ such that (B, B_2) is a 2-premolecule,

2. if B is a biconnected component of G and $\mathcal{A}_B$ is the set of the articulation points of G that belong to B, then there exists B_2 such that $\mathcal{A}_B \subseteq B_2$ and B_2 is a minimal edge cover of B.

■

Proof.

1. Let G be such that $\mathscr{E}(G) \leq 2$ and let B be a connected component of G. Then B can not be 3-connected by Lemma 8.4.1, since otherwise we get $\mathscr{E}(G) \geq 3$ contradicting the assumption. By Lemma 8.5.8 there exists $B_2 \subset B$ such that (B, B_2) is a 2-premolecule

2. Let B be a 2-connected component of G, $\mathcal{A}_B$ the set of the articulation points of G belonging to B, then the graph[9] $G[B]^{\mathcal{A}_B}$, defined in Lemma 8.6.2, is a subgraph of G. Hence $\mathscr{E}(G[B]^{\mathcal{A}_B}) \leq 2$, then it follows by Corollary 8.6.3 that there exists $B_2 \subseteq B$ such that $\mathcal{A}_\mathcal{B} \subseteq B_2$ and B_2 is an edge cover set of B of size 2.

□

8.7 Tree decomposition of graphs of entanglement at most 3

The main result of the previous Chapter states that a graph of entanglement 2 has a tree decomposition into 2-connected components such that the latter are the 2-molecules that come with a prescribed set of articulation points. Conversely, starting with the 2-molecules and the 1-sum operator we have

[9]The graph $G[B]^{\mathcal{A}_B}$ is obtained from $G[B]$ and attaching an edge on each vertex belonging to $\mathcal{A}_B$.

been able to generate the class of graphs of entanglement 2. Now we shall follow this approach to deal with the class of graphs of entanglement 3.
We find some *necessary* conditions on the structure of the Tutte's tree to be a tree decomposition of a 2-connected graph of entanglement 3. The necessary conditions deal with three features of the tree: *(i)* conditions on the structure of the 3-connected components: they are the 3-molecules. This is direct consequence of the results proved in the previous sections, *(ii)* conditions on the hinges are given in a similar way of those given on the articulation points when the starting graph has entanglement 2, and *(iii)* conditions on the diameter of the tree.

8.7.1 Classification of the 3-molecules

We have seen in the previous section that a k-molecule may admit many bases giving rise to what we have called *ambiguous* molecules. The following Proposition gives an explicit characterization the class of ambiguous 3-molecules.

Proposition 8.7.1. *Let ϑ be a 3 molecules $\vartheta^{\mathcal{B},h}_{B_3}$, then ϑ is ambiguous if and only if one of the following cases holds:*

I. $|\mathcal{B}| = 3$ *and* $h = 1$*, that is ϑ is the 4-clique,*

II. $|\mathcal{B}| = 2$ *and* $h = 2$*,*

III. $|\mathcal{B}| = 0$ *and* $h = 3$*, that is ϑ is the complete bipartite graph $K_{3,3}$.*

These graphs are depicted in figure 8.7.1

Proof. First, we give a useful property of ambiguous k-molecules:

Claim 8.7.2*:* If G is an ambiguous k-molecule, and A_k, B_k are two different bases of G, then $A_k \cup B_k = V_G$. ■

Proof. Assume that $A_k \cup B_k \subsetneq V_G$ and then let $w \in V_G \setminus (A_k \cup B_k)$. Hence, for each $b \in B_k$ we have by the definition of the k-molecules that $wb \in E_G$.

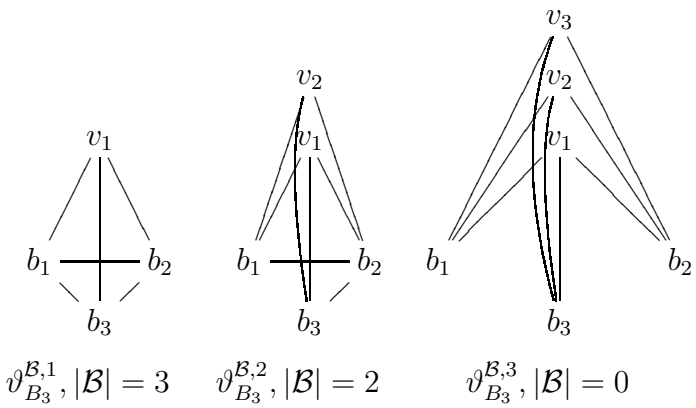


Figure 8.7.1: The set of ambiguous 3-molecules.

This implies that the subgraph of G induced by $V_G \setminus A_k$ is not discrete, contradicting the definition of the k-molecules. We conclude that $A_k \cup B_k = V_G$. This ends the proof of Claim. $\square$

Let $G := \vartheta^{\mathcal{B},h}_{B_3}$ be an ambiguous 3-molecule and A_3, B_3 be two distinct bases of G, thus by the previous Claim we get $A_3 \cup B_3 = V_G$. We recall first that Lemma 8.5.3 states that both the graphs induced by $A_3 \setminus B_3$ and $B_3 \setminus A_3$ are discrete and moreover the graphs induced by A_3 and B_3 are isomorphic. We distinguish three cases according to $|A_3 \cap B_3|$.

1. $|A_3 \cap B_3| = 0$. Since both $G[A_3]$ and $G[B_3]$ are discrete, then G is the complete bipartite graph $K_{3,3}$, i.e. G is the 3-molecule $\vartheta^{\mathcal{B},3}_{B_3}$ where $\mathcal{B} = 0$.

2. $|A_3 \cap B_3| = 1$. Let $A_3 = \{ w, a_1, a_2 \}$ and $B_3 = \{ w, b_1, b_2 \}$. First, $a_1a_2 \notin E_G$ because $G[A_3 \setminus B_3]$ is discrete, and also $b_1b_2 \notin E_G$ because $G[B_3 \setminus A_3]$ is discrete. Second, we shall argue that $\{ wa_i, wb_i,\ i = 1, 2 \} \subset E_G$. Assume that $wa_1 \notin E_G$, then $\{ b_1, b_2 \}$ is a 2-separator in G of a_1 from a_2. This is a contradiction because the 3-molecules are 3-connected. We deduce that $wa_1 \in E_G$. By symmetry, we obtain

also that $wa_2, wb_1, wb_2 \in E_G$. We conclude that, in this case, G is the 3-molecule $\vartheta^{\mathcal{B},2}_{B_3}$ where $\mathcal{B} = 2$.

3. $|A_3 \cap B_3| = 2$. Observe that in this case $h = 1$ in $\vartheta^{\mathcal{B},h}_{B_3}$. We shall argue next that both $G[A_3]$ and $G[B_3]$ are the 3-clique. Recall that in the k-molecule $\vartheta^{\mathcal{B},h}_{B_k}$ we have $h \geq k - k'$ where k' is the connectivity of $G[B_k]$. If $|E_{G[A_3]}| \leq 2$, then the connectivity of $G[A_3]$ is at most 1 and hence in the 3-molecule $\vartheta^{\mathcal{B},h}_{A_3}$ we should have $h \geq k - k' \geq 3 - 1 = 2$, this is a contradiction because we have already mentioned that $h = 1$. We conclude that $|E_{G[A_3]}| = 3$ meaning that G is the 4-clique, i.e. the 3-molecule $\vartheta^{\mathcal{B},1}_{B_3}$ where $|\mathcal{B}| = 3$.

□

8.7.2 Necessary conditions on Tutte's tree

Since a 2-connected graph may be written by means of the 2-Sum operator, we begin by inspecting the main cases for which the 2-Sum operator increases the entanglement [10]. In the next, the symbol + denotes the 2-Sum operator on 2-connected graphs given in Definition 8.3.5.
From now on we shall deal particularly with the 3-molecules and write $\vartheta^{\mathcal{B},h}_{a,b,c}$ instead of $\vartheta^{\mathcal{B},h}_{\{a,b,c\}}$

Lemma 8.7.3. Let $\vartheta^{\mathcal{B},h}_{a,b,c}$ be a 3-molecule and C_3 be the 3-cycle on the vertices $\{a, v_1, z\}$. Define the graph G as follows:

$$G = C_3 +_{av_1} \vartheta^{\mathcal{B},h}_{a,b,c}.$$

If $\{a, v_1\}$ does not belong to any minimal edge cover set of $\vartheta^{\mathcal{B},h}_{a,b,c}$, then $\mathscr{E}(G) \geq 4$ ■

[10] This idea has been already considered in Chapter 7 where we have looked for the cases for which the 1-sum operator increases the entanglement. There, the 1-sum operator which does not increase the entanglement is called the legal 1-Sum.

Proof. The graph G may be viewed as the graph that results from inserting the new vertex z in the edge av_1 of $\vartheta^{\mathcal{B},h}_{a,b,c}$. Let us abbreviate $\vartheta^{\mathcal{B},h}_{a,b,c}$ by ϑ.
Define $f : G \longrightarrow \vartheta$ as follows: $f(v) = v$ if $v \neq z$, and $f(z) = a$. Thief's strategy in $\widetilde{\mathscr{E}}(\vartheta, 3)$[11] (even if it is losing) that forces Cops to occupy a minimal edge cover set of ϑ, this strategy exists by Lemma 8.6.1, will allow us to construct a winning strategy for Thief in $\mathscr{E}(G, 3)$. Every position (v, C_ϑ, P) of $\widetilde{\mathscr{E}}(\vartheta, 3)$ is matched with a position (g, C_G, P) of $\mathscr{E}(G, 3)$ such that the following conditions hold:

- $f(g) = w$ and $f(C_G) = C_\vartheta$. (COPS)
- if Thief moves from $(a, C_\vartheta, Thief)$ to $(v_1, C_\vartheta, Cops)$, then $a \in C_\vartheta$. (COP-ON-a)

The condition (COPS) states essentially that the cops in $\widetilde{\mathscr{E}}(\vartheta, 3)$ are placed on the image of cops in $\mathscr{E}(G, 3)$ by the function f defined above. The condition (COP-ON-a) states that whenever Thief leaves vertex a to v_1 then a cop must already be placed on a.
A Thief's move $M_\vartheta = (u, C_\vartheta, Thief) \to (w, C_\vartheta, Cops)$ in $\widetilde{\mathscr{E}}(\vartheta, 3)$ is simulated either by a move or a sequence of moves in $\mathscr{E}(G, k)$ according to the edge uw.

1. If $uw \neq v_1a$, then the move M_G is simulated in $\mathscr{E}(G, 3)$ by the same Thief's move. Observe that the latter move is possible in $\mathscr{E}(G, 3)$ because $w \in C_G$ then already $f(w) = w \in f(C_G) = C_\vartheta$, and this is impossible.

2. If $(u, w) = (v_1, a)$, then the move M_G is simulated in $\mathscr{E}(G, 3)$ by the following sequence of moves:

$$(v_1, C_G, Thief) \to (z, C_G, Cops) \to (z, C'_G, Thief) \to (a, C'_G, Cops)$$

[11] The game $\widetilde{\mathscr{E}}(G, k)$ is defined as the game $\mathscr{E}(G, k)$ apart that Cops can retire a number of cops, the two versions are equivalent, Proposition 3.4.2.

This sequence is possible. If Thief can not perform such moves then either $z \in C_G$ or $a \in C'_G$. If $z \in C_G$ then $f(z) = a \in f(C_G) = C_\vartheta$, which is impossible. If $a \in C'_G$, then already $a \in C_G$ and hence $f(a) = a \in f(C_G) = C_\vartheta$, which is also impossible. Observe that in this case, Thief's move in $\widetilde{\mathscr{E}}(\vartheta, 3)$ is simulated by a sequence of Thief's moves in $\mathscr{E}(G, 3)$, and the latter are interleaved with Cops moves, and then the position of Cops in $\widetilde{\mathscr{E}}(\vartheta, 3)$ such be updated using the function f. So, it remains to show that Cops related moves in $\widetilde{\mathscr{E}}(\vartheta, 3)$ respect the rule of the game. From the latter position in $\mathscr{E}(G, 3)$, Cops next move is a from $(a, C'_G, Cops) \to (a, C''_G, Thief)$. Hence in $\widetilde{\mathscr{E}}(\vartheta, 3)$ Cops related move should be of the form $(a, C_\vartheta, Cops) \to (a, f(C''_G), Thief)$. Let us compute $f(C''_G)$ in term of C_ϑ. Observe first that $C''_G = (C_G \setminus A) \cup B$ where $\emptyset \subseteq B \subseteq \{ z, a \}$ and $A \subset C_G$ with $|A| \leq 2$.

$$\begin{aligned} f(C''_G) &= f[(C_G \setminus A) \cup B \\ &= f(C_G \setminus A) \cup f(B) \\ &= [f(C_G) \setminus f(A)) \cup Z] \cup f(B) \end{aligned}$$

where $\emptyset \subseteq Z \subseteq \{ z \}$ and $f(B) \subseteq \{ z, a \}$. Therefore,

$$\begin{aligned} f(C''_G) &= (f(C_G) \setminus f(A)) \cup (Z \cup f(B)) \\ &= (C_\vartheta \setminus f(A)) \cup Z' \end{aligned}$$

On the one hand $A \subset C_G$ and hence $f(A) \subset f(C_G) = C_\vartheta$. On the other hand $Z' = Z \cup f(B) \subseteq \{ z \} \cup \{ z, a \} = \{ z, a \}$. We conclude that Cops' move under discussion respects the rules of the game.

3. If $(u, w) = (a, v_1)$, then the move M_ϑ is simulated in $\mathscr{E}(G, 3)$ by Thuief's iteration on az until a cop is placed either on a or z and then Thief

goes to v_1. That is, it is the following sequence:

$$\begin{aligned} M_G^\star = (a, C_G, Thief) \to (z, C_G, Cops) &\to (z, C_G, Thief) \to (a, C_G, Cops) \\ &\to (a, C_G, Thief) \to (z, C_G, Cops) \\ &\to \dots \\ &\to (a, C_G, Cops) \to (a, C'_G, Thief) \\ &\to (z, C'_G, Cops) \to (z, C''_G, Thief) \\ &\to (v_1, C''_G, Cops) \end{aligned}$$

Such that $C'_G \neq C_G$ or $C''_G \neq C_G$. Let us show that Thief can perform such moves, i.e. $z \notin C_G$ and $v_1 \notin C''_G$.
If $v_1 \in C''_G$ then $v_1 \in C_G$, and hence $f(v_1) = v_1 \in f(C_G) = C_\vartheta$, this is impossible.
If $z \in C_G$ the $f(z) = a \in f(C_G) = C_\vartheta$, then let us come back to the previous round of simulation. We mean if we consider Thief's previous moves in $\widetilde{\mathscr{E}}(\vartheta, 3)$, then they are of the form

$$(a^{-1}, C_\vartheta^{-1}, Thief) \to (a, C_\vartheta^{-1}, Cops) \to (a, C_\vartheta, Thief)$$

and since we have supposed that $z \in C_G$, then $f(z) = a \in f(C_G) = C_\vartheta$ and moreover $a \in C_\vartheta^{-1}$, which is impossible.
Let us argue now that Cops' next move in $\widetilde{\mathscr{E}}(\vartheta, 3)$ respects the rules of the game. From the latter position in $\mathscr{E}(G, 3)$, Cops' next move would be of the form $(v_1, C''_G, Cops) \to (v_1, C^\star_G, Thief)$, and hence Cops' in $\widetilde{\mathscr{E}}(\vartheta, 3)$ would reply, according to condition (COPS) by the move $(v_1, C_\vartheta, Cops) \to (v_1, f(C^\star_G), Thief)$. Observe first that $a \in C_\vartheta$ by the condition (COP-ON-a). In order to show that the latter move respect the rules of the game, we compute $f(C^\star_G)$ in term of C_ϑ. Note that $C^\star_G = (C_G \setminus A) \cup B$ where $\emptyset \subseteq B \subseteq \{ a, z, v_1 \}$ and $A \subseteq C_G$. As in the previous case, a simple computation shows that
$f(C^\star_G) = (C_\vartheta \setminus f(A)) \cup f(B')$ where $B' \subseteq \{ a, z, v_1 \}$. On the one hand $A \subseteq C_G$ implying $f(A) \subseteq f(C_G) = C_\vartheta$. On the other hand

$f(B') \subseteq \{ a, v_1 \}$. However, we have mentioned that $a \in C_\vartheta$, and hence $f(C_G^\star) = (C_\vartheta \setminus f(A)) \cup B''$ where $B'' \subseteq \{ v_1 \}$. This shows that Cops' move in question respects the rules of the game.

So far we have described the simulation between the games $\mathscr{E}(G, 3)$ and $\widetilde{\mathscr{E}}(\vartheta, 3)$ and shown that it is consistent. Now we shall show that the hypothesis of the Lemma under proof imply implicitly some restrictions on the 3-molecule $\vartheta_{a,b,c}^{\mathcal{B},h}$ provided in this Lemma.

Claim 8.7.4*:* the 3-molecule $\vartheta_{a,b,c}^{\mathcal{B},h}$ described in Lemma 8.7.3 is not the 4-clique, and hence $h \geq 2$. ■

Proof. If $\vartheta := \vartheta_{a,b,c}^{\mathcal{B},h}$ is the 4-clique, then any set of vertices of size 3 forms a minimal edge cover of ϑ, implying that a, v_1 belongs to some minimal edge cover of ϑ, contradicting the hypothesis. This ends the proof of the Claim. □

Assume that ϑ is ambiguous and let $\{ a', b', c' \}$ be an other minimal edge cover of ϑ. Since the edge $v_1 a$, where the 2-Sum is performed, does not belong to any minimal edge cover of ϑ, then it is invariant w.r.t. changing the bases of ϑ. This shows that we can deal with ϑ like wise $\{ a, b, c \}$ is the unique base.

If Thief is trapped in $\widetilde{\mathscr{E}}(\vartheta, 3)$ then we have a position of the form $(v, \{ a, b, c \}, Thief)$ where $v \in \{ a, b, c \}$. The latter position is matched with a position $(v, C_G, Thief)$ of $\mathscr{E}(G, 3)$ where $f(C_G) = \{ a, b, c \}$. Therefore, either $C_G = \{ a, b, c \}$ or $C_G = \{ z, b, c \}$.

Case (i). If $C_G = \{ a, b, c \}$, then Thief can go to v_1 an iterates moves on $v_1 z$ forcing Cops to put a cop on v_1 or z. If Cops put a cop on v_1 then the image of Cops on ϑ by f is no longer a minimal edge cover, and hence Thief plays in $\widetilde{\mathscr{E}}(\vartheta, 3)$ with the strategy that consists in forcing Cops to occupy a minimal edge cover set of ϑ, i.e. $\{ a, b, c \}$. If Cops put a cop on z, then this cop comes from a, b, or c.

If this cop comes from b or c, then the image of cops on ϑ by f is either $\{ a, c \}$ or $\{ a, b \}$ which is not a minimal edge cover set of ϑ, and hence Thief

forces Cops to occupy again a minimal edge cover of ϑ.
If this cop comes from a then we go to *Case(ii)*.
Case (ii). If $C_G = \{ z, b, c \}$, then assume that the current vertex occupied by Thief is denoted by $x \in \{ z, b, c \}$, and hence Thief's can choose the path xav_2 and[12] iterates moves on v_2a forcing Cops to put a cop on a, coming back to *Case(i)*.

Such a strategy for Thief in $\mathscr{E}(G, 3)$ can be iterated infinitely often, that is, it is a winning strategy, and hence $\mathscr{E}(G) \geq 4$. □

Corollary 8.7.5. Let G be 2-connected such that $\mathscr{E}(G) = 3$ and T be its Tutte decomposition. Let $t_1t_2 \in E_T$ such that the torso τ_{t_1} is 3-connected, then there exists $B_3 \subset V_{\tau_{t_1}}$ such that *(i)* the pair (τ_{t_1}, B_3) is a 3-premolecule, and *(ii)* $V_{t_1} \cap V_{t_2} \subset B_3$. ■

Proof.
(i). Since τ_{t_1} is 3-connected, then by Lemma 8.4.1, we get $\mathscr{E}(\tau_{t_1}) \geq 3$. By Lemma 8.3.8, the torsos τ_{t_1} is a minor of G, hence $\mathscr{E}(\tau_{t_1}) \leq 3$. Hence, $\mathscr{E}(\tau_{t_1}) = 3$. Since τ_{t_1} is 3-connected and and $\mathscr{E}(\tau_{t_1}) = 3$, then it follows from Lemma 8.5.8, that there exists $B_3 \subset V_{\tau_{t_1}}$ such the pair (τ_{t_1}, B_3) is a 3-premolecule.
(ii). We need the Claim:

Claim 8.7.6*:* Let $\{ v_1v_2 \} = V_{t_1} \cap V_{t_2}$, then the graph $\tau_{t_1} +_{v_1v_2} C_3$ is a minor of G. ■

The Claim implies that $\mathscr{E}(\tau_{t_1} +_{v_1v_2} C_3) \leq 3$, then from Lemma 8.7.3 we get that v_1, v_2 should belong to a minimal edge cover of τ_{t_1}. Let B_3' be such a minimal edge cover, then it follows that the pair (τ_{t1}, B_3') is again a 3-premolecule. □

[12]The vertex v_2 exists because $h \geq 2$ in $\vartheta = \vartheta_{\substack{\mathcal{B},h \\ a,b,c}}$

On the diameter of the tree decomposition. Now we shall give conditions on the diameter of the tree decomposition if the graph has entanglement 3. To give a lower bound for the diameter of the tree, we have noticed that a domino of a prescribed length will be a typical excluded minor.

The *domino* D_n is the graph of vertices $V_{D_n} = \{ v_i, w_i \mid i = 0, \ldots, n \}$ and edges $E_{D_n} = \{ v_i v_{i+1}, w_i w_{i+1} \mid i = 0, \ldots, n-1 \} \cup \{ v_i w_i \mid i = 0, \ldots, n \}$. However, when n is even, then we prefer that the set of vertices would be of the form $\{ v_i, w_i \mid i = -\frac{n}{2}, \ldots, 0, \ldots, \frac{n}{2} \}$. For instance, the domino D_8 is depicted in Figure 8.7.2.

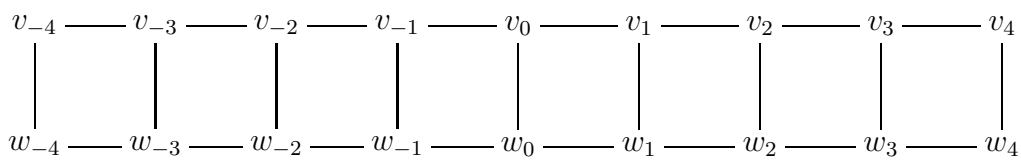


Figure 8.7.2: The 8-Domino D_8

Lemma 8.7.7. The entanglement of the domino D_{14} is at least 4. ■

Proof. We shall describe a winning strategy for Thief in $\mathscr{E}(D_{14}, 3)$. First, let us fix some terminology and notations. Let $V = \{ v_i \mid i = -7, \ldots, 7 \}$ and $W = \{ w_i \mid i = -7, \ldots, 7 \}$. Let $\mathscr{L}_4$, *the left* 4-*domino*, be the subgraph of D_{14} induced by the vertices $\{ v_i, w_i \mid i = -7, \ldots, -3 \}$. Similarly, we let $\mathscr{R}_4$, *the right* 4-*domino*, be the subgraph of D_{14} induced by the vertices $\{ v_i, w_i \mid i = 3, \ldots, 7 \}$. We let also $\mathscr{C}_2$, *the central* 2-*domino*, be the subgraph induced by the vertices $\{ v_i, w_i \mid i = -1, 0, 1 \}$. If S is a sub-domino of D_{14} then the $S \cap V$-path (resp. $S \cap W$-path) is the path induced by vertices $V_S \cap V$ (resp. $V_S \cap W$).

Before giving Thief's winning strategy, we describe it informally:

Step 1. Thief plays on $\mathscr{C}_2$ forcing Cops to place 3 cops on it. According to the last position of Thief in $\mathscr{C}_2$, Thief chooses the left or the right

4-domino, and moreover chooses an extremal vertex of it among $\{ v_i, w_i \mid i = -3, 3 \}$. Let us assume that Thief chooses $\mathscr{L}_4$ and the vertex v_{-3},

Step 2. from the choice $(v_{-3}, \mathscr{L}_4)$, Thief iterates moves on the 4-path $\mathscr{L}_4 \cap V$ starting from v_{-3} until 2 cops are placed on this path, and at this moment

2.1. if there is no cop on $\mathscr{C}_2$, then Thief goes to $\mathscr{C}_2$, and then iterates the strategy from Step 1,

2.2. if there is a cop on the $\mathscr{C}_2$, then there is no cop on $\mathscr{R}_4$, and hence Thief chooses a path to $\mathscr{R}_4$ and an extremal vertex of it among $\{ v_3, w_3 \}$, and then iterates the strategy, up to symmetry, from Step 2.

The formal proof is split into two parts. The first part, called the *intra-steps*, consists in proving that the strategy for Thief described beside each step is realizable. The second part, called the *inter-steps*, consists in proving that the passage between steps is possible. Technically speaking, the inter-steps proofs, are devoted to prove that the path - that leads to the desired sub-domino- is free of Cops.

Let us begin by Step 1. To argue that Thief can play on $\mathscr{C}_2$ in such a way he forces 3 cops to be placed on it, it is sufficient to observe that $\mathscr{E}(\mathscr{C}_2) > 2$ because $\mathscr{C}_2$ does not belong to the class ζ_2 of the graphs of entanglement 2, see the previous Chapter. The following Claim ensures that the passage from Step 1 to Step 2 is possible.

Claim 8.7.8*:* Thief has a strategy to play in $\mathscr{C}_2$ in such a way, once 3 cops are placed on $\mathscr{C}_2$, he can go in a horizontal way either to $\mathscr{L}_4$ or $\mathscr{R}_4$. ■

Proof. Assume that Thief is trapped in $\mathscr{C}_2$ without being able to find a path neither to $\mathscr{L}_4$ nor to $\mathscr{R}_4$. This implies that Thief is on v_0 or on w_0 – otherwise he is able go to $\mathscr{L}_4$ or to $\mathscr{R}_4$– and moreover he is surrounded by 3 Cops in such a way he can not move down, left and right. That gives rise to the final

position $(v_0, \{ v_{-1}, w_0, v_1 \}, Thief)$ [13]. Coming back two moves before, we get a position of the form $(x, \{ v_{-1}, w_0, v_1 \}, Thief)$ where $x \in \{ v_{-1}, w_0, v_1 \}$. From the latter position Thief is clearly able to go to either $\mathscr{L}_4$ or $\mathscr{R}_4$. This ends the proof of the Claim. □

Assume now that 3 cops are placed on $\mathscr{C}_2$ and Thief is on vertex v_{-1}. The other cases are handled by symmetry. It follows from the Claim that Thief is able to choose the pair $(v_{-3}, \mathscr{L}_4)$ by going through the path $v_{-1}v_{-2}v_{-3}$.

Let us prove that the strategy described in Step 2 is possible. Once the pair $(v_{-1}, \mathscr{L}_4)$ is chosen and Thief is on v_{-1}, then Thief restricts his moves on the $\mathscr{L}_4 \cap V$-path, the latter is of length 4 and therefore it has entanglement 2. Hence Thief has a strategy to force 2 cops to be placed on this path. At this moment either there is a cop on $\mathscr{C}_2$ or not.

- If there is a cop on $\mathscr{C}_2$, and since 2 cops are placed on the $\mathscr{L}_4 \cap V$ path, then there is no cop on $\mathscr{R}_4$, and moreover there is a free path leading to one of its left extremal points from the current vertex i.e.

  (a) if there is a cop on $\mathscr{C}_2 \cap W$, then Thief goes from $v_p \in \mathscr{L}_4 \cap V$ to v_3 through the free path $v_p w_p w_{p+1} \ldots w_{-2} v_{-2} v_{-1} \ldots v_3$. Therefore iterates the strategy from Step 2 out of the pair $(v_3, \mathscr{R}_4)$,

  (b) if there is a cop on $\mathscr{C}_2 \cap V$, then Thief goes from $v_p \in \mathscr{L}_4 \cap V$ to w_3 through the free path $v_p w_p w_{p+1} \ldots w_3$. Therefore, iterates the strategy from Step 2 out of the pair $(w_3, \mathscr{R}_4)$

  Remark 8.7.9. We emphasize that in case *(a)*, the path chosen by Thief does not pass through $(\mathscr{R}_4 \cap W) \cup \{ w_2 \}$, the latter vertices are indeed free of cops and hence they might be used later by Thief. Also, in the case *(b)*, the path chosen by Thief does no pass through $(\mathscr{R}_4 \cap V) \cup \{ v_2 \}$, the latter vertices are free of cops, and they would be used latter by Thief. ■

[13] Since we reason up to symmetry, we put the second position $(w_0, \{ w_{-1}, v_0, w_1 \}, Thief)$ in the back ground.

- if there is no cop on $\mathscr{C}_2$ then, from the current vertex $v_p \in \mathscr{L}_4 \cap V$ Thief goes to $\mathscr{C}_2$ through the path $v_p w_p w_{p+1} \dots w_{-1}$. The freennes of this path is ensured by Remark 8.7.9. From the latter position, iterates the strategy from Step 1

The strategy for Thief in $\mathscr{E}(D_{14}, 3)$ described so far can be iterated infinitely often, that is, it is a winning strategy for Thief. Therefore $\mathscr{E}(D_{14}) \geq 4$. □

Given a graph G and its Tutte's decomposition tree T, we call the *spread* of a vertex $v \in V_G$, denoted by δ_v, the number of hinges which v belongs to; i.e.

$$\delta_v = |\{ \{ v, x \} \text{ s.t. } \{ v, x \} \text{ is a hinge } \}|.$$

The spread of the 2-connected graph G, denoted by δ_G, is defined by

$$\delta_G = max\{ \delta_v \text{ s.t. } v \in V_G \}.$$

Definition 8.7.10. A sequence of n hinges $\{ x_1, y_1 \}, \dots, \{ x_n, y_n \}$ is *parallel* if *(i)* each two hinges are disjoint, and *(ii)* there is a path $t_1 \dots t_m$ in T such that for each $i \in \{ 1, \dots, n \}$ there exists $j \in \{ 1, \dots, m \}$ where $\{ x_i, y_i \} \in V_{t_j}$. ■

For instance, an n-domino where $n \geq 2$ contains $n - 1$ parallel hinges. The following Lemma emphasizes an important aspect of hinges, intuitively it states that hinges *do not cross each other*.

Lemma 8.7.11. [Theorem IV.22 of [Tut01] or Lemma 8 of [Ric04]]
Let G be a 2-connected graph and let $\{ x_1, y_1 \}$ be a hinge of G and (U, W) be the 2-separation of G such that $U \cap W = \{ x_1, y_1 \}$. If $\{ x_2, y_2 \}$ is an other hinge of G such that $\{ x_1, y_1 \} \cap \{ x_2, y_2 \} = \emptyset$, then either $x_2, y_2 \in U$ or $x_2, y_2 \in W$. ■

Lemma 8.7.12. Let G be a 2-connected graph and T its Tutte's tree. For all $n \geq 2$, if G contains $4n$ parallel hinges then G has the $(n-1)$-domino as a minor. ■

Proof. Let H be a set of $4n$ hinges of G. First, we construct a graph $\overline{G}$ out of G such that $\overline{G}$ is a minor of G and second, we prove that $\overline{G}$ admits the $(n-1)$-domino as a minor.
Lemma 8.3.4 states that a 2-separator $\{ x, y \}$ is a hinge if and only if *(i)* either there are at least three $[x, y]$-bridges, or *(ii)* there are two $[x, y]$-bridges at least one of them is 2-connected. The construction of $\overline{G}$ follows. For each hinge $\{ x, y \} \in H_n$, where there are at least three $[x, y]$-bridges, do the following operations: (i) delete from G all the $[x, y]$-bridges which do not contain a hinge in H other than the edge xy [14], and (ii) add the edge xy to G.

Fact 8.7.13. The graph $\overline{G}$ constructed above has the following properties:

(1) $\overline{G}$ is 2-connected,

(2) $\overline{G}$ is a minor of G,

(3) the set H is again a set of hinges of $\overline{G}$, and

(4) for every $\{ x, y \} \in H$, there are exactly two $[x, y]$-bridges of $\overline{G}$.

■

Proof. We argue next that the operations (i) and (ii) cited above preserve the properties stated in statements (1),(2),(3), and (4). The statement (1) follows from Lemma 8.3.6 which states that if $\{ x, y \}$ is a 2-separator of G into (U_1, U_2) then $G[U_i] + xy$, for $i = 1, 2$, is also 2-connected. Statement (2) follows from Lemma 8.3.9 which states that $G[U_i] + xy$ is a minor of G, for $i = 1, 2$. Let us show statement (3). Let $\{ x, y \}$ be a hinge in H where $xy \in E_{\overline{G}} \setminus E_G$. On the one hand, $\{ x, y \}$ is again a 2-separator in $\overline{G}$. On the other hand, one of the $[x, y]$-bridges is 2-connected, Lemma 8.3.6, because it contains the edge xy. Finally, statement (4) holds obviously by construction. This ends the proof of the Fact. □

[14] Observe that the hinges of H belong to at most two $[x, y]$-bridges of G.

Second, let us prove that $\overline{G}$ admits the $(n-1)$-domino as a minor. Let $H = \{ \{ a_1, b_1 \}, \ldots, \{ a_{4n}, b_{4n} \} \}$, and v (resp. w) be a vertex in the $[a_1, b_1]$-bridge (resp. in the $[a_{4n}, b_{4n}]$- bridge) which does not contain the remaining hinges. Let π_x, π_y be two (simple) disjoint v-w paths in $\overline{G}$. Such paths do exist because $\overline{G}$ is 2-connected, statement (1) of the Fact 8.7.13. We claim that, for each hinge $\{ a_i, b_i \} \in H$, either $a_i \in \pi_x$ and hence $b_i \in \pi_y$ or $a_i \in \pi_y$ and hence $b_i \in \pi_x$. The argument is that each hinge in H is a 2-separator of v from w in $\overline{G}$. Therefore, from now we assume that the set H of hinges is of form $H = \{ \{ x_i, y_i \}, i = 1, \ldots, 4n \}$ such that $x_i \in \pi_x$ and $y_i \in \pi_y$, for $i = 1, \ldots, 4n$. Intuitively, the two paths π_x and π_y would play the role the horizontal lines of the domino [15] in question. The main remaining technical part is to construct the vertical lines.

Definition 8.7.14. Let $i \in \{ 1, \ldots, 4n-1 \}$, and let $G_i \subset \overline{G}$ be the $[x_{i+1}, y_{i+1}]$-bridge which contains the hinge $\{ x_i, y_i \}$ and let $G_{i+1} \subset \overline{G}$ be the $[x_i, y_i]$-bridge which contains the hinge $\{ x_{i+1}, y_{i+1} \}$. We call an $(i, i+1)$-*block* the graph $G_i \cap G_{i+1}$. If no confusion will arise, we will call an $(i, i+1)$-block with simply a block. ■

Fact 8.7.15. If one of G_i and G_{i+1} (given in the previous definition) is 2-connected then the $(i, i+1)$-block is connected. Moreover, $\overline{G}$ contains at least $2n$ connected blocks. ■

Proof. We prove the first statement of the Fact by assuming that G_i is 2-connected. If G_{i+1} is 2-connected then the proof is similar. Let us denote by $\mathcal{B}$ the $(i, i+1)$-block, and let $v_1, v_2 \in V_{\mathcal{B}}$. We shall show the existence of at least one v_1-v_2 path in $\mathcal{B}$. Since G_i is 2-connected, then there are at least two disjoint (simple) v_1-v_2 paths in G_i. Let π_1 and π_2 be such paths. If both π_1 and π_2 are in $\mathcal{B}$ then we have done. Otherwise, assume that π_1 does no belong to $\mathcal{B}$, i.e. π_1 contains a proper subpath which belongs to $G_i \setminus \mathcal{B}$. We claim that π_1 contains both x_i and y_i because $\{ x_i, x_{i+1} \}$ is a 2-separation in G_i and the path π_1 is supposed to be simple. We mean that if π_1 visits x_i,

[15] if the domino is viewed in an horizontal way as depicted in Figure 8.7.2.

and visits some vertices in $G_i \setminus \mathcal{B}$, then it must visit y_i. Therefore, π_1 may be of the form:

$$\pi_1 = v_1 \dots v_p x_i w_1 \dots w_l y_i v_{p+1} \dots v_q v_2,$$

or of the form:

$$\pi_1 = v_1 \dots v_p y_i w_1 \dots w_l x_i v_{p+1} \dots v_q v_2,$$

where $v_j \in V_{\mathcal{B}}$ for $j = 1, \dots q$ and $w_j \in V_{G_i} \setminus V_{\mathcal{B}}$ for $j = 1, \dots, l$. Since π_1 contains both x_i and y_i, then π_2 contains neither x_i nor y_i, because π_1 and π_2 are supposed to be disjoint. Therefore π_2 belongs to $\mathcal{B}$, i.e. the block $\mathcal{B}$ is connected.

To prove the second statement of the Fact, recall that from the statement (4) of the Fact 8.7.13 we have the number of $[x_j, y_j]$-bridges equals 2, for each hinge $\{ x_j, y_j \}$ in H. From the statement (ii) of Lemma 8.3.4 it follows that one the two $[x_j, y_j]$-bridges is 2-connected. This implies that one of the $(j-1, j)$-block and $(j, j+1)$-block is connected. Hence, $\overline{G}$ does not contain two contiguous[16] blocks which are both not connected. Therefore, $\overline{G}$ contains at least $\frac{4n}{2}$ connected blocks. This ends the proof of the Fact. □

According to the previous Fact, $\overline{G}$ contains at least $2n$ connected blocks. Therefore, $\overline{G}$ contains at least n non contiguous and connected blocks $\mathcal{B}_1, \dots, \mathcal{B}_n$, the latter are pairwise disjoint. Each block $\mathcal{B}_i,\ i = 1, \dots, n$, contains a hinge $\{ a_i, b_i \} \in H$, and moreover it contains a a_i-b_i path π_i because it is connected. On the one hand, the graph consisting of the paths $\pi_i, i = 1, \dots, n$, π_x, and π_y is a subgraph of $\overline{G}$. On the other hand, the π_i paths, for $i = 1, \dots, n$, are pairwise disjoint. By contracting each path among π_i, π_x, and π_y in the desired way we get an $(n-1)$-domino, see Figure 8.7.3.

The latter has a vertical edge $a_i b_i$ for each hinge $\{ a_i, b_i \}$. This ends the proof of Lemma 8.7.12. □

The following Lemma establishes the relation between the spread, the diameter, and the number of parallel hinges.

[16] Two blocks $(i, i+1)$ and $(j, j+1)$ are contiguous if $i = j+1$ or $j = i+1$.

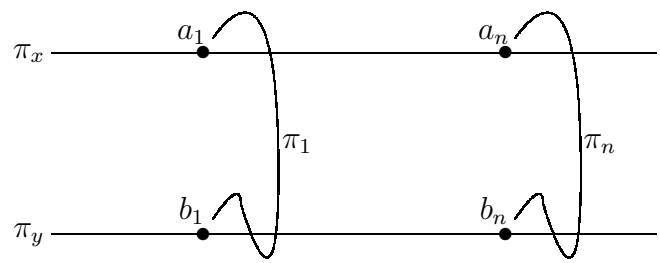


Figure 8.7.3: The construction of the $(n-1)$ domino as a minor

Lemma 8.7.16. Let G be a 2-connected graph and T be its Tutte's tree. Let δ_G be the spread of G. For all $n \geq 2$, if the diameter of T is greater than $4n.\delta_G$ then G contains at least n parallel hinges. ■

Proof. Consider a path in T of length $4n.\delta_G$ of the form

$$t_1 t'_1 \dots t_{2n.\delta_G} t'_{2n.\delta_G} t_{2n.\delta_G+1}$$

where t_1 is a leaf. One of Tutte's tree properties, see Lemma 8.3.3, ensures that T enjoys the following properties:

(a) T is bipartite and moreover every node $t'_i, i = 1, \dots, 2n.\delta_G$ is such $V_{t'_i}$ is a hinge.

(b) if h_i denotes the hinge $V_{t'_i}$ then

  (b.1) for all $i = 1, \dots, 2n.\delta_G - 1$, we have either $h_i \cap h_{i+1} = \emptyset$ or $|h_i \cap h_{i+1}| = 1$,

  (b.2) if $h_p \cap h_q = \emptyset$ where $1 \leq p < q \leq 2n.\delta_G$, then for all $i \leq p$ we have that $h_i \cap h_q = \emptyset$, and

  (b.3) if $h_p \cap h_q = \emptyset$ where $1 \leq p < q \leq 2n.\delta_G$, then for all $i \geq q$ we have that $h_p \cap h_i = \emptyset$.

Let $H = \{ h_1, \ldots, 2n.\delta_G \}$ be a sequence of hinges where $h_i = V_{t'_i}$. Out of H we shall define an undirected graph ∂H in such a way the properties of hinges in H transfer into the properties of ∂H. Define ∂H as follows $V_{\partial H} = \{ h^v \mid h \in H \}$, in other words the vertex h^v is just the hinge h viewed as a vertex; and $h_1^v h_2^v \in E_{\partial H}$ if and only if $h_1 \cap h_1 \neq \emptyset$. We state the main properties of the graph ∂H.

Fact 8.7.17. Let $\mathcal{B}$ be a 2-connected component of ∂H, and let $m = |V_{\mathcal{B}}|$, then $\mathcal{B}$ is an m-clique. ■

Proof. Let $h_1^v, h_2^v \in V_{\mathcal{B}}$. Since $\mathcal{B}$ is 2-connected then it follows by Menger Theorem 8.2.3 that there exist two disjoint h_1^v-h_2^v paths in $\mathcal{B}$ i.e. there is a cycle in $\mathcal{B}$ containing both h_1^v and h_m^v. Let $h_1^v, h_2^v \ldots h_m^v h_1^v$ be such a cycle. Assume that $h_1^v h_3^v \notin E_{\mathcal{B}}$, hence $h_1 \cap h_3 = \emptyset$. It follows from property (b.3) above that for all $i \geq 3$ we have $h_1 \cap h_i = \emptyset$. This is a contradiction since already $h_1 \cap h_n \neq \emptyset$ because $h_1^v h_m^v \in E_{\mathcal{B}}$. We deduce that $h_1^v h_3^v \in E_{\mathcal{B}}$. Using similar arguments, we deduce that for all $i = 3, \ldots, m-1$ we have $h_1^v h_{m-1}^v \in E_{\mathcal{B}}$. We conclude that for every pair h_1^v, h_2^v of vertices in the component $\mathcal{B}$ there is an edge $h_1^v h_1^v$ in $\mathcal{B}$, implying that $\mathcal{B}$ is an m-clique. This ends the proof of the Fact. □

Fact 8.7.18. If ∂H contains an m-clique of vertices $h_1^v, \ldots, h_m^v$ then

$$| \cap_{i=1,\ldots,m} h_i| = 1.$$

■

Proof. The proof is by induction on m.
If $m = 3$, then either $\cap_i h_i \neq \emptyset$ and in this case $| \cap_i h_i| = 1$, or $\cap_i h_i = \emptyset$ and in this case the hinges h_1, h_2, h_3 form a triangle in the following sense: $h_i \cap h_j = \{ v_{ij} \}$ for $i, j = 1, 2, 3$ and $i \neq j$. In the latter case, the hinges h_1, h_2, h_3 do not belong to a unique path in T, contradicting the hypothesis that the hinges in question are parallel, condition *(ii)* of Definition 8.7.10.
Induction step. Consider a $(m+1)$-clique of vertices $h_1^v, \ldots, h_{m+1}^v$. By induction hypothesis $| \cap_{i=2,\ldots,m+1} h_i| = 1$ and $| \cap_{i=1,\ldots,m} h_i| = 1$. This implies that

$\cap_{i=2,\ldots,m+1} h_i = \cap_{i=1,\ldots,m} h_i$, and hence $|\cap_{i=1,\ldots,m+1} h_i| = 1$. This ends the proof of the Fact. □

Summing up the Facts 8.7.17 and 8.7.18 we deduce that the size (i.e. the number of vertices) of the 2-connected components of ∂H is at most δ_G, because, given a 2-connected component $\mathcal{B}$ of vertices $h_1^v, \ldots, h_m^v$ the spread of the vertex $\cap_{i=1,\ldots,m} h_i$ is m. The following Claim allows us to compute a lower bound of the number of 2-connected components of ∂H.

Claim 8.7.19*:* Let H be a graph. If the size (i.e. number of vertices) of its 2-connected components is at most δ then G has got at least $\lfloor \frac{|V_H|}{\delta} \rfloor$ 2-connected components. ■

Proof. Let x be the number of 2-connected components of H. Clearly $|V_H| \leq x.\delta$, hence $\frac{|V_G|}{\delta} \leq x$. Therefore H contains at least $\lfloor \frac{|V_H|}{\delta} \rfloor$ 2-connected components. This ends the proof of the Claim. □

Since $|V_{\partial H}| = 2n.\delta_G$ and the size of the 2-connected components of ∂H is at most δ_G, then, according to the Claim, the number of the 2-connected components of ∂H is at least $2n$.
On the one hand, if the two vertices h_1^v and h_2^v belong to two disjoint 2-connected components then the related hinges h_1 and h_2 are disjoint, because otherwise, there is an edge $h_1^v h_2^v$ in ∂H implying that both h_1^v and h_2^v belong to the same 2-connected component, which is a contradiction. One the other hand ∂H contains at least n pairwise disjoint 2-connected components. We conclude that G contains at least n disjoint hinges, the latter belong to the same path in T, and therefore they are parallel. This ends the proof of Lemma 8.7.16. □

Proposition 8.7.20. *Let G be a 2-connected graph, T be its Tutte's tree, and δ_G be the spread of G. If the entanglement of G is 3 then the diameter of T is at most $2^7.\delta_G$.*

Proof. Let G be as stated in the Proposition with $\mathscr{E}(G) = 3$. If the diameter of G is $c.\delta_G$ then it follows from the Lemma 8.7.16 that G contains at least

$\lfloor c \setminus 4 \rfloor$ parallel hinges. Therefore, from Lemma 8.7.12 we get that G contains a c'-domino $D_{c'}$ as a minor, where $c' = \lfloor \lfloor c \setminus 4 \rfloor \setminus 4 \rfloor - 1$. Hence $\mathscr{E}(D_{c'}) \leq \mathscr{E}(G)$ by Theorem 6.4.2 which states that the class of undirected graphs of entanglement at most k is minor ideal. From Lemma 8.7.7, a 14-domino has entanglement strictly greater than 3. We conclude that $c' \leq 14$ and hence $c \leq 2^7$. □

Now we are ready to state the main result of this Chapter.
We define the *interface* of a torso τ_t as $I_t = \{ v \in V_t \mid \exists tt' \in E_T \text{ s.t } v \in V_{t'} \}$

Theorem 8.7.21. Let G be a 2-connected graph and let $(T, (V_t)_{t \in T})$ be the Tutte decomposition of G. If the entanglement of G is 3 then for every 3-connected torso τ_t of G the following hold:

1. there exits $B_3 \subseteq V_t$ where $|B_3| = 3$ such that (τ_t, B_3) is an abstract 3-molecule,

2. $I_t \subseteq B_3$, and

3. if δ_G is the spread of G, then the diameter of T is at most $2^7.\delta_G$.

■

Proof. Let G be as stated and τ_t a 3-connected torso of G.

1. On the one hand, Lemma 8.3.8 states that τ_t is a minor of G. It follows from Theorem 6.4.2 that $\mathscr{E}(\tau_t) \leq 3$. On the other hand, since τ_t is 3-connected then we get from Lemma 8.4.1 that $3 \leq \mathscr{E}(\tau_t)$, therefore $\mathscr{E}(\tau_t) = 3$. Hence, from Lemma 8.5.8 we deduce that there exists $B_3 \subset V_{\tau_t}$ with $|B_3| = 3$ such that (τ_t, B_3) is an abstract 3-molecule.

2. If the 3-molecule τ_t is not ambiguous, then the property $I_t \subseteq B_3$ follows from statement *(ii)* of Corollary 8.7.5. If τ_t is not ambiguous, then it suffices to generalize Lemma 8.7.3 as follows:

Lemma 8.7.22. Let ϑ be a 3-molecule, and let each C_3, C_3', C_3 and C_3'' be a 3-cycle. Let

$$G = ((\vartheta +_{v_1w_1} C_3) +_{v_2w_2} C_3') +_{v_3w_3} C_3'',$$

where $+$ is the 2-Sum operator and $v_iw_i \in E_\vartheta,\ i = 1, 2, 3$, $v_1w_1 \in E_{C_3}$, $v_2w_2 \in E_{C_3'}$, and $v_3w_3 \in E_{C_3''}$. If the entanglement of G is again 3 then this implies that the vertices $v_i, w_i, i = 1, 2, 3$ belong to a minimal edge cover set of ϑ. ■

3. The condition on the diameter of T follows from Proposition 8.7.20.

□

8.8 Towards sufficient conditions on Tutte's tree

Theorem 8.7.21 provides some necessary conditions on Tutte's tree to be a tree decomposition of a 2-connected graph of entanglement 3. However, these conditions are not sufficient, since the 14-domino is a counter example. A work that needs to be developed consists in computing exact value of diameter of Tutte's tree in order to obtain sufficient conditions. To be more precise, the exact value of the diameter would essentially depend on, besides the spread i.e. statement (3) of Theorem 8.7.21, the length of each cycle which constitutes a torso of Tutte's tree. In the sequel we shall tell something about the entanglement of 2-connected graphs for which Tutte's tree satisfies conditions (1), (2) and (3) of Theorem 8.7.7.

Lemma 8.7.7 states that 4 is a lower bound of the entanglement of the n-domino, where $n \geq 14$. Now we prove that 4 is also an upper bound.

Proposition 8.8.1. *For every $n \in \mathbb{N}$, the n-domino has entanglement at most 4. Therefore, the entanglement of the n-domino, where $n \geq 14$, is exactly 4.*

Proof. Let D_n be an n-domino, we shall describe a winning strategy for Cops in the game $\mathscr{E}(D_n, 4)$. We recall that the set of vertices of D_n is of the form $\{ v_i, w_i \mid i = 0, \ldots, n \}$ as before. Let us call a (p, q)-*domino*, where $0 \leq p < q \leq n$, the sub-domino of D_n induced by the vertices $\{ v_i, w_i \mid i = p, \ldots, q \}$. Cops strategy in $\mathscr{E}(D_n, 4)$ is given by means of the following steps:

Step 1. Starting from the $(0, n)$-domino, and using just 2 cops occupy 2 vertices of the form $\{ v_j, w_j \}$ where $0 < j < n$. At this moment, Thief goes either left or right. Assume that he goes left, then up to symmetry, from the $(0, j)$-domino, iterate the strategy from Step 2.

Step 2. By playing on the $(0, j)$-domino [17] – where 2 cops are placed on $\{ v_j, w_j \}$ – Cops use the other 2 cops to occupy two vertices of form $\{ v_i, w_i \}$ in the $(0, j-1)$-domino. At this moment, either Thief goes left, and in this case iterate the strategy from Step 2 starting from the $(0, i)$-domino, or Thief goes right and iterate the strategy from Step 3 starting from the (i, j)-domino.

Step 3. By playing on the (i, j)-domino – where the 4 cops are placed on $\{ v_i, w_i, v_j, w_j \}$ and Thief is on v_i, (the other positions of Thief are handled by symmetry) – Cops use the two cops on $\{ v_k, w_k \}$ to occupy 2 vertices of the form $\{ v_p, w_p \}$ where $i < p < j$ and k equals j if $j \leq n - i$ and equals i otherwise. Besides, Cops do not remove the 2 cops on $\{ v_j, w_j \}$. At this moment, i.e. from the position v_p (or w_p), Thief either goes left, and in this case iterate the strategy from Step 2 starting from the $(0, p)$-domino, or he goes right, and in this case iterate the strategy from Step 3 starting from the (p, j)-domino.

It remains to argue that (i) Cops strategy described in each step is realizable and (ii) this strategy would not be iterated infinitely often. To prove statement (i), it is enough to prove the following Fact.

[17] We put the case of the (j, n)-domino in the back ground, because we argue up to symmetry.

Fact 8.8.2. Let D_n be an n-domino of vertices $\{ v_i, w_i \mid i = 0, \ldots, n \}$. In the game $\mathscr{E}(D_n, 2)$, Cops have a strategy to occupy 2 vertices of the form $\{ v_p, w_p \}$ where $0 < p < n$ or they win. ■

Proof. If Thief restricts his moves on the path consisting of vertices $\{ v_i, i = 0, n \}$ then Thief uses the standard winning strategy[18] with 2 cops, because paths have entanglement at most 2. Therefore, either Thief decides to go from some v_i to w_i, where $i = 1, \ldots, n-1$, and in this case Cops replace the cop of v_{i-1} to w_i and we have done. Or the game reaches a position where two cops are placed, up to symmetry, on $\{ v_{n-2}, v_{n-1} \}$ and Thief is on v_{n-1}. If Thief goes to from v_{n-1} to v_n then Cops just skip, Thief is forced to go to w_n, and Cops replace the cop from v_{n-2} to w_n. At this moment, Thief returns to v_n – where he is trapped – or he goes to w_{n-1} and here, Cops place the cop from w_n to w_{n-1} and we have done. This ends the proof of the Fact. □

To show statement (ii) above it is enough to observe that whenever a step is revisited a second time, then the size of the (i, j)-domino is strictly lower than that of the first time. This allows to argue that the strategy for Cops described so far is winning in the game $\mathscr{E}(D_n, 4)$, hence $\mathscr{E}(D_n) \leq 4$. If $n \geq 14$, then form Lemma 8.7.7, it follows that $\mathscr{E}(D_n) \leq 4$, and therefore $\mathscr{E}(D_n) = 4$. □

We think that is hopeful to adapt the winning strategy for Cops in the game $\mathscr{E}(D_n, 4)$ – given in the proof of Proposition 8.8.1 – in order to argue that 4 cops are sufficient to win in the 2-connected graphs for which Tutte's tree satisfies the properties 1,2, and 3 of Theorem 8.7.21. This fact is formalized within the following conjecture.

Conjecture 1. Let G be a 2 connected graph for which Tutte's tree satisfies the properties 1,2 and 3 of Theorem 8.7.21. Then G has entanglement 4.

[18] Let us recall it: when Thief walks on the path $v_0 \ldots v_n$ starting from v_0, then Cops place the first cop on v_0, place the second cop on v_1, replace the first cop to v_3, replace the second cop to v_4 ...

Chapter 9

Conclusions

Fixed points is a fundamental concept that has shown its use in computer science. For instance, it allows to express the behaviour of finite and infinite systems. A main part of this thesis is devoted to understand the μ-calculi in their large sense: fixed points over complete lattices.

An interesting path to understand the μ-calculi formalism consists in dealing with the hierarchies induced by the least and the greatest fixed point operators. The alternation hierarchy – the most studied one in the literature – counts the non trivial alternations between different kinds of operators. The star height hierarchy counts the nested depth of those operators, and finally the variable hierarchy counts the number of bound variables. The alternation hierarchy is somehow orthogonal to latter ones. In this thesis we have shown that the variable hierarchy is a sort of refinement of the star height hierarchy, Theorem 3.5.1.

The first main result of this thesis concerns the variable hierarchy for the lattice μ-calculus. The main notion used in our proof is the of the entanglement. Roughly speaking, the entanglement of a μ-term gives us the minimum number of bound variables required in every equivalent μ-term. We have constructed μ-terms (i.e. games) of arbitrary entanglement. And we have shown their hardness w.r.t. the number of bound variables.

The second part of the thesis is devoted to a pure graph theoretic study

of the entanglement.

In Chapter 6 we have shown that the class of undirected graphs of bounded entanglement is closed under taking minors. As a consequence, such class can be characterized by a finite number of excluded minors. Finding the set of excluded minors is challenging. However, this set may be very large and a compact representation of the excluded minors is a major open question.

Chapter 7 focuses on the study of the class of undirected graphs of entanglement 2. We have given two characterizations of that class: one of them by means of excluded subgraphs, and hence by excluded minors, and the other one by means of a construction out of small graphs. The two characterizations are combined to devise a linear times algorithm that decides memberships of that class. Our algorithm is essentially a modified Depth First Search one.

The last Chapter tackles the problem of characterization of the class of undirected graphs of entanglement 3. Our approach consists in studying entanglement w.r.t. the basic graphs theoretic measures: the connectivity and the edge covering. The main technical tools is Tutte's decomposition paradigm of the 2-connected graphs into 3-connected components and cycles. We have provided necessary conditions on Tutte's decomposition tree in order to be a tree decomposition of a 2-connected graph of entanglement 3.

We emphasize that Hohberg's decomposition [Hoh92] of graphs into k-connected components suggests an interesting path to provide a characterization of undirected graphs of entanglement k.
Other decompositions of directed graph [BJG01, §7.2] should be of help in recognizing the structure of directed graphs of bound entanglement.

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