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Linear and 2-frugal choosability of graphs of small maximum average degree

Nathann Cohen * Frédéric Havet *

December 13, 2010

Abstract

A proper vertex colouring of a graph G is *2-frugal* (resp. *linear*) if the graph induced by the vertices of any two colour classes is of maximum degree 2 (resp. is a forest of paths). A graph G is *2-frugally* (resp. *linearly*) *L-colourable* if for a given list assignment $L : V(G) \mapsto 2^{\mathbb{N}}$, there exists a 2-frugal (resp. linear) colouring c of G such that $c(v) \in L(v)$ for all $v \in V(G)$. If G is 2-frugally (resp. linearly) L-list colourable for any list assignment such that $|L(v)| \geq k$ for all $v \in V(G)$, then G is *2-frugally* (resp. *linearly*) *k-choosable*. In this paper, we improve some bounds on the 2-frugal choosability and linear choosability of graphs with small maximum average degree.

1 Introduction

The notion of acyclic colouring was introduced by Grünbaum [3] in 1973 : a vertex colouring is *acyclic* if it is *proper* (no two adjacent vertices have the same colour), and if there is no bicoloured cycle (the subgraph induced by the union of any two colours classes is a forest). A colouring such that for every vertex $v \in V(G)$, no colour appears more than p times in the neighborhood of v, is said to be *p-frugal*, a notion introduced by Hind, Molloy and Reed in [4]. The *p-frugal chromatic number* of a graph G, denoted by $\Phi_p(G)$, is the minimum number of colours in a p-frugal colouring of G and is clearly larger than $\lceil \frac{\Delta}{p} \rceil + 1$. Hind, Molloy, and Reed [4] proved that $\Phi_p(G) \leq \max(p\Delta(G), \frac{e^3}{p}\Delta(G)^{1+1/p})$. In addition, they show that this upper bound is tight up to within a constant factor by showing graphs G such that $\Phi_p(G) \geq \frac{1}{2p}\Delta(G)^{1+1/p}$.

Yuster [5] mixed the notions of 2-frugality and acyclicity, thus introducing the concept of linear colouring. A *linear colouring* of a graph is an acyclic and 2-frugal colouring. It can also be seen as a colouring such that the subgraph induced by the union of any two colour classes is a forest of paths (an acyclic graph with maximum degree at most two). The *linear chromatic number* of a graph G, denoted by $\Lambda(G)$, is the minimum number of colours in a linear colouring of G. As a linear colouring is 2-frugal, $\Lambda(G) \geq \Phi_2(G) \geq \lceil \frac{\Delta}{2} \rceil + 1$. Yuster proved in [5] that $\Lambda(G) = O(\Delta(G)^{3/2})$ in the general case, and he constructed graphs for which $\Lambda(G) = \Omega(\Delta(G)^{3/2})$.

These concepts may be generalized to list colouring. Given a list assignment $L : V(G) \mapsto 2^{\mathbb{N}}$, an *L-colouring* of G is a colouring c such that $c(v) \in L(v)$ for each vertex v. A graph G is *p-frugally* (resp. *linearly*) *L-colourable* if there is an L-colouring of G which is p-frugal (resp. linear). If G is p-frugally (resp. linearly) L-colourable for any assignment L verifying $\forall v \in V(G)$, $|L(v)| \geq k$, then G is said to be *p-frugally k-choosable* (resp. *linearly k-choosable*). The smallest integer k such that the graph G is p-frugally k-choosable is the *p-frugal choosability* or *p-frugal list chromatic number* of G and is denoted by $\Phi_p^l(G)$. The *linear choosability* or *linear list chromatic number* denoted $\Lambda^l(G)$ is defined analogously. The *average degree* of G is $Ad(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G is $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$.

In [2], Esperet et al. proved some upper bounds on the linear choosability of graphs with small maximum average degree.

Theorem 1 (Esperet et al. [2])**.** *Let G be a graph with maximum degree at most* Δ*:*

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1. *If* $\Delta \geq 3$ *and* $Mad(G) < \frac{16}{7}$, *then* $\Lambda^l(G) = \lceil \frac{\Delta}{2} \rceil + 1$.

2. *If* $Mad(G) < \frac{5}{2}$, *then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$.

3. *If* $Mad(G) < \frac{8}{3}$, *then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.

In this article, we show in Theorem. 2 that the above upper bounds on the maximum average degree may be assumed arbitrarily close to 3 when Δ is large enough. When Δ is small, we also improve some of the existing bounds (see Theorem. 3). After the submission of this paper, some results were again improved by Cranston and Yu [1]. Since a linear colouring is 2-frugal, the results are also valid for 2-frugal choosability. However, since being 2-frugal is less restrictive than being linear, we improve some of them in this case (see Theorem. 4). All these results, added to those proved by Esperet et al. [2] which have not been improved, are summarized in the following table :

$Mad(G)$	Δ	$\Lambda^l(G)$	
$< \frac{16}{7} \approx 2.2857$	≥ 3	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	Esperet et al. [2]
$< \frac{39}{16} = 2.4375$	≥ 5	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	Theorem 3-1
$< \frac{48}{19} \approx 2.5263$	≥ 7	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	Theorem 3-2
$< 3 - \frac{3}{\Delta+1}$	≥ 8	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	Theorem 2-1
$< \frac{12}{5}$	≥ 3	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	Cranston and Yu [1]
$< \frac{5}{2}$		$\leq \lceil \frac{\Delta}{2} \rceil + 2$	Esperet et al. [2]
$< \frac{60}{23} \approx 2.6086$	≥ 5	$\leq \lceil \frac{\Delta}{2} \rceil + 2$	Theorem 3-3
$< 3 - \frac{9}{4\Delta+3}$	≥ 7	$\leq \lceil \frac{\Delta}{2} \rceil + 2$	Theorem 2-2
< 3	≥ 9	$\leq \lceil \frac{\Delta}{2} \rceil + 2$	Cranston and Yu [1]
$< \frac{14}{5} = 2.8$		$\leq \lceil \frac{\Delta}{2} \rceil + 3$	Theorem 3-4
< 3	≥ 12	$\leq \lceil \frac{\Delta}{2} \rceil + 3$	Theorem 2-3
< 3		$\leq \lceil \frac{\Delta}{2} \rceil + 4$	Theorem 3-5
$Mad(G)$	$\Phi_2^l(G)$	Δ	
$< \frac{5}{2}$	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	≥ 7	Theorem 4-1
< 3	$\leq \lceil \frac{\Delta}{2} \rceil + 3$		Theorem 4-2

The *girth* $g(G)$ of a graph G is the length of its smallest cycle or $+\infty$ if G has no cycle. Euler's formula implies that a planar graph G has bounded maximum average degree in terms of its girth:

$$Mad(G) < 2 + \frac{4}{g(G) - 2}. \tag{1}$$

This immediately gives to any result on graphs with bounded maximum average degree have an equivalent formulation for planar graphs with large girth. These are summarized in the following table together with those coming from the papers of Esperet et al. [2] and Raspaud and Wang [6, 7] which are not improved here.

girth	$\Lambda^l(G)$	Δ	
≥ 16	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	≥ 3	Esperet et al. [2]
≥ 7	$\leq \lceil \frac{\Delta}{2} \rceil + 1$	≥ 13	Raspaud and Wang [6]
≥ 8	$\leq \lceil \frac{\Delta}{2} \rceil + 1$		
≥ 10	$\leq \lceil \frac{\Delta}{2} \rceil + 2$		Esperet et al
≥ 9	$\leq \lceil \frac{\Delta}{2} \rceil + 2$	≥ 5	
≥ 7	$\leq \lceil \frac{\Delta}{2} \rceil + 3$		
≥ 5	$\leq \lceil \frac{\Delta}{2} \rceil + 4$		Cranston and Yu [1]
	$\leq \lceil \frac{9\Delta}{10} \rceil + 5$	≥ 85	Raspaud and Wang [7]
girth	$\Phi_2^l(G)$	Δ	
≥ 6	$\leq \lceil \frac{\Delta}{2} \rceil + 3$		

The proofs of our results are based on the same general idea. We study graphs which we call *k-frugal-minimal* (resp. *k-linear-minimal*) – i.e. graphs that are not k-frugally colourable (resp. k-linear-colourable), while any of their proper subgraphs is. We first show in Section 2 that some configurations (i.e. subgraphs) may not appear in such a graph. We then use in Section 3 the discharging method to show that a graph containing none of these forbidden configurations must have larger average degree than assumed, giving a contradiction .

2 Forbidden configurations

Before establishing some lemmas, let us give some definitions. Let k be a non-negative integer. A *k-vertex* (resp. *$(\geq k)$-vertex*, *$(\leq k)$-vertex*) is a vertex of degree exactly k (resp. at least k, at most k). A *k-neighbour* of v is a k-vertex adjacent to v. $(\geq k)$ and $(\leq k)$-neighbours are defined similarly.
A *k-thread* in a graph G is an induced path (or cycle) of G with $k+1$ edges, and so at least k internal vertices of degree 2.

Note that a k-frugal-minimal or k-linear-minimal graph is connected and in particular has no 0-vertex. We will sometimes use this easy fact without referring explicitly to it.

2.1 Linear colouring

Lemma 1. *Let H be a k-linear-minimal or k-frugal-minimal graph.*

1. *If $k \geq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 1$, then H has no 1-vertex.*
2. *For every 2-vertex v with $N(v) = \{a,b\}$ and $deg(a) \leq deg(b)$, we have $deg(b) \geq 2(k - deg(a)) + 1$.*
3. *If $k \geq 3$, then H contains no 3-thread.*
4. *If $k \geq 4$, then no 3-vertex is incident to a 2-thread.*
5. *Assume $k \geq 4$. If a 3-vertex has three 2-neighbours, then the second neighbour of each of those is a (≥ 4)-vertex.*
6. *If $k \geq 4$, then no 4-vertex is adjacent to four 2-threads.*
7. *If $k \geq 5$, then no 4-vertex is incident to a 2-thread.*
8. *If $k \geq 5$, then no 5-vertex is incident to five 2-threads.*
9. *If $k \geq 5$, then a 2-vertex has at most one 3-neighbour.*
10. *If $k \geq 5$, then a 4-vertex is not adjacent to two 2-vertices having each a 3-neighbour.*
11. *If $k \geq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 2$, then two 2-vertices are not adjacent.*

12. *If* $k \geq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 3$, *there are no adjacent* 3*-vertex and* 2*-vertex.*

13. *If* $k \geq \left\lceil \frac{\Delta(H)}{2} \right\rceil + 3$, *then every 4-vertex has at most three* 2*-neighbours.*

14. *If* $k \geq \lceil \frac{\Delta(H)}{2} \rceil + 4$, *then every* 4*-vertex has at most two* 2*-neighbours.*

15. *If* $k \geq \max\left(6, \lceil \frac{\Delta(H)}{2} \rceil + 4\right)$, *then every* 5*-vertex has at most four* 2*-neighbours.*

Proof. In the following, we only prove the assertions for linear colouring : 2-frugal colouring being less restrictive, all the proofs translate naturally. Suppose that one of the assertions of Lemma 1 does not hold. Let H be a k-linear-minimal graph for which it fails and L a k-list assignment such that H is not linearly L-colourable.

1. H contains a 1-vertex u. Let v be the neighbour of u. Let c be a linear L-colouring of $H-u$. We now extend c to u. The colour $c(v)$ is forbidden. Moreover to preserve the 2-frugality at v, the colours appearing twice in the neighbourhood of v are also forbidden. There are at most $\lfloor \frac{\Delta(H)-1}{2} \rfloor$ such colours. Hence, at most $\lceil \frac{\Delta(H)}{2} \rceil$ colours in total are forbidden at u. Thus u can be coloured with a non-forbidden colour in its list $L(u)$, and the colouring obtained is a linear L-colouring of H, which is a contradiction.

2 Let v be a 2-vertex of H with $N(v) = \{a,b\}$ and $deg(a) \leq deg(b)$, such that $deg(b) < 2(k - deg(a)) + 1$. Let c be a linear L-colouring of $H - v$.

   - If $c(a) = c(b)$, let us assign to v a colour $c(v) \in L(v)$ different from the ones of the other neighbours of a (i.e. there are at most $deg(a) - 1$ of them), $c(a)$, and every colour which is repeated at least twice in the neighbourhood of b (i.e. there at most $\left\lfloor \frac{deg(b)-1}{2} \right\rfloor < k - deg(a)$ of them). Doing so, we obtain a k-linear L-colouring of H, a contradiction.
   - If $c(a) \neq c(b)$, let us assign to v a colour $c(v) \in L(v)$ different from $c(a)$, $c(b)$, and every colour which is repeated at least twice in the neighbourhood of a or in the neighbourhood of b. The number of forbidden colours is at most $2 + \left\lfloor \frac{deg(a)-1}{2} \right\rfloor + \left\lfloor \frac{deg(b)-1}{2} \right\rfloor < 2 + deg(a) - 2 + k - deg(a) = k$ because $deg(a) \geq 2$ and so $deg(a) \geq 2 + \left\lfloor \frac{deg(a)-1}{2} \right\rfloor$. Hence such an assignment is possible, and yields a k-linear L-colouring of H, a contradiction.

3. It follows directly from 2 which implies that if a 2-vertex has a 2-neighbour then its other neighbour has degree at least 3.

4. There is a configuration as depicted in Figure 1. with possibly $x = u_1$, or $x = u$ and $w = u_1$.

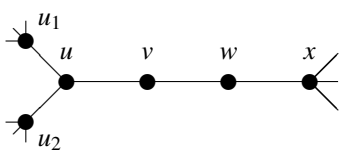

Figure 1: Configuration of Case 4

   By the minimality of H, there exists a linear L-colouring c of $H - v$. We now extend it to v :

   - If $c(u_1) = c(u_2)$, we colour v with $c(v) \in L(v) \backslash \{c(u), c(w), c(u_1)\}$. There can be no bicoloured cycle, as $c(v)$ is different from both $c(u_1)$ and $c(u_2)$, and the 2-frugality at u is preserved.
   - If $c(u_1) \neq c(u_2)$, we colour v with $c(v) \in L(v) \backslash \{c(u), c(w), c(x)\}$. There can be no bicoloured cycle, as $c(v)$ is different from $c(x)$, and c is 2-frugal.

5. There is a configuration as depicted in Figure 2. Possibly $t_i = u_i$ and $w_i = u_i$ for $i = 1, 2$, or some vertices in $\{w_1, w_2, t_1, t_2\}$ are identified.

   By the minimality of H there exists a linear L-colouring c of $H - \{u, v\}$. We now extend it to u and v :

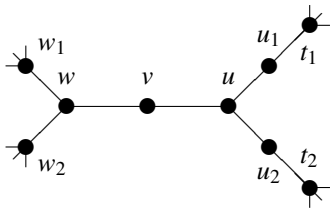

Figure 2: Configuration of Case 5

- If $c(u_1) = c(u_2)$, we colour v with $c(v) \in L(v) \backslash \{c(w), c(w_1), c(u_1)\}$ and u with $c(u) \in L(v) \backslash \{c(v), c(u_1), c(t_1)\}$.
- If $c(u_1) \neq c(u_2)$, we colour u with $c(u) \in L(u) \backslash \{c(u_1), c(u_2), c(w)\}$ and v with $c(v) \in L(v) \backslash \{c(u), c(w), c(w_1)\}$.

6. There is a configuration as depicted in Figure 3. Possibly $t_3 = u = u_3$ and $u_1 = t_2$ and $t_1 = u_2$. In this case, a linear colouring of $H - \{u_1, u_2\}$ can be extended into a linear colouring of H by assigning to u_1 a colour distinct from those of u and w_1 and to u_2 a colour distinct from those of u, u_1 and v_1. This is a contradiction.

   So we may assume that it is not the case. Then possibly some vertices of $\{t_3, u_3, v_3, w_3\}$ may be identified.

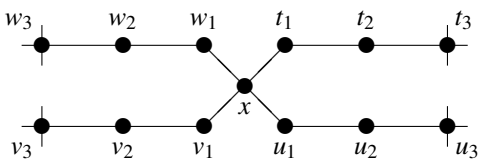

Figure 3: Configuration of Case 6

   By the minimality of H there exists a linear L-colouring c of $H - \{t_1, u_1, v_1, w_1, x\}$. We now extend it to t_1, u_1, v_1, w_1 and x by sequentially colouring:

   - t_1 with $c(t_1) \in L(t_1) \backslash \{c(t_2), c(t_3)\}$.
   - u_1 with $c(u_1) \in L(u_1) \backslash \{c(u_2), c(u_3), c(t_1)\}$.
   - x with $c(x) \in L(x) \backslash \{c(t_1), c(u_1), c(w_2)\}$.
   - w_1 with $c(w_1) \in L(w_1) \backslash \{c(t_2), c(x)\}$.
   - v_1 with $c(v_1) \in L(v_1) \backslash \{c(v_2), c(x), c(w_1)\}$.

   There can be no bicoloured cycle through t_1 (resp. u_1) as its colour is different from $c(t_3)$ (resp. $c(u_3)$), and none going through w_2 as its colour is different from $c(x)$.

7. There is a configuration as depicted in Figure 4. Possibly w_3 is one of $\{t, u, v\}$ or $w_3 = x$ and $w_2 = t$. In the later case, we get a contradiction as in Case 6, so we may assume that it does not occur.

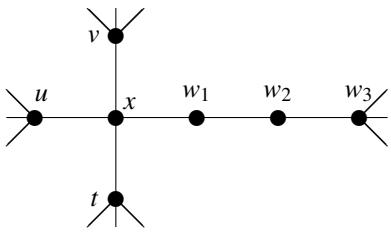

Figure 4: Configuration of Case 7

   By the minimality of H there exists a linear L-colouring c of $H - w_1$. We extend it to w_1 :

   - If $c(w_2) = c(x)$, then we colour w_1 with $c(w_1) \in L(w_1) \backslash \{c(x), c(w_3), c(v), c(t)\}$. There can be no bicoloured cycle as $c(w_1) \neq c(w_3)$, and the 2-frugality at x is preserved.

- If $c(w_2) \neq c(x)$, then we colour w_1 with $c(w_1) \in L(w_1) \backslash \{c(x), c(w_2), c(v), c(t)\}$. There can be no bicoloured cycle as $c(w_2) \neq c(x)$, and the 2-frugality at x is preserved.

8. There is a configuration as depicted in Figure 5. Possibly, some of the vertices in $\{s_3, t_3, u_3, v_3, w_3\}$ are the same or identified with x (by pairs). In the later case, without loss of generality, $s_3 = x = t_3$, $s_1 = t_2$ and $t_1 = s_2$. Then, by minimality of H, there exists a linear colouring c of $H - \{s_1, s_2\}$. It can be extended by colouring s_1 with a colour $c(s_1)$ distinct from $c(x)$ and the possible colour appearing twice on $\{u_1, v_1, w_1\}$, and colouring s_2 with a colour $c(s_2)$ distinct from $c(x)$, $c(s_1)$ and the possible colour appearing twice on $\{u_1, v_1, w_1\}$. Hence we may assume that this case does not appear.

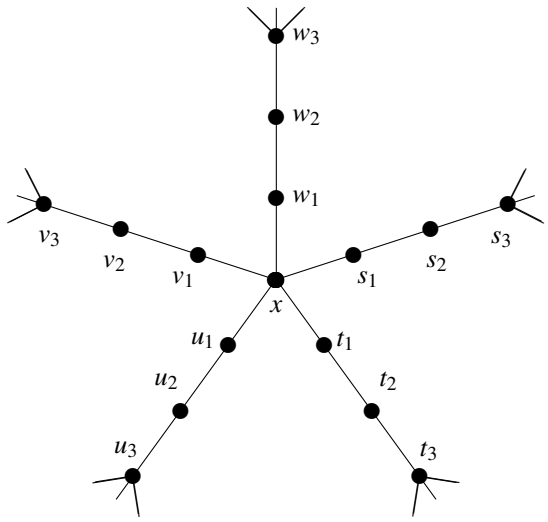

Figure 5: Configuration of Case 8

By the minimality of H there exists a linear L-colouring c of $H - \{x, s_1, t_1, u_1, v_1, w_1\}$. We extend it to x, s_1, t_1, u_1, v_1, and w_1 by sequentially colouring:

- s_1 with $c(s_1) \in L(s_1) \backslash \{c(s_2), c(s_3)\}$.
- t_1 with $c(t_1) \in L(t_1) \backslash \{c(t_2), c(t_3), c(s_1)\}$.
- u_1 with $c(u_1) \in L(u_1) \backslash \{c(u_2), c(u_3), c(s_1), c(t_1)\}$.
- v_1 with $c(v_1) \in L(v_1) \backslash \{c(v_2), c(u_1), c(t_1), c(s_1)\}$.
- x with $c(x) \in L(x) \backslash \{c(s_1), c(t_1), c(u_1), c(v_1)\}$.
- w_1 with $c(w_1) \in L(w) \backslash \{c(w_2), c(x), c(v_1)\}$.

The 2-frugality at x is preserved as 4 different colours are assigned to the vertices s_1, t_1, u_1 and v_1. Furthermore, there can be no bicoloured cycles going through s_1 and s_3, t_1 and t_3, u_1 and u_3 or w_1 and v_1. Thus the obtained L-colouring is linear, a contradiction.

9. There is a configuration as depicted in Figure 6. Possibly, some of the vertices of $\{v_1, v_2, w_1, w_2\}$ are identified or $v_1 = w$ and $w_1 = v$.

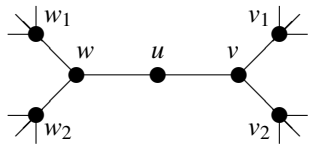

Figure 6: Configuration of Case 9

By the minimality of H, there exists a linear L-colouring c of $H - u$. We extend it to u :

- If $c(v) = c(w)$, we colour u with $c(u) \in L(u) \backslash \{c(v), c(v_1), c(v_2), c(w_1)\}$ to prevent the formation of a bicoloured cycle and preserve the 2-frugality at v and w.
- If $c(v) \neq c(w)$, we colour u with $c(u) \in L(u) \backslash \{c(v), c(v_1), c(w_1), c(w)\}$ to preserve the 2-frugality at v and w. There can be no bicoloured cycles because $c(v) \neq c(w)$.

Hence H is linearly L-colourable, a contradiction.

10. There is a configuration as depicted in Figure 7. Possibly some vertices of $\{v_1'', v_1''', v_2'', v_2''', v_3, v_4\}$ are the same, or, for $i \in \{1,2\}$, $v_i'' = u$ and $v_i' = v_{i+2}$.

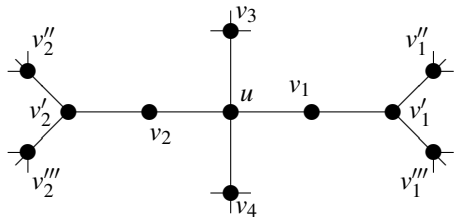

Figure 7: Configuration of Case 10

By the minimality of H there exists a linear L-colouring c of $H - \{v_1, v_2\}$. We extend it to v_1 and v_2:

- If $c(v_1') = c(u)$, we colour v_1 with $c(v_1) \in L(v_1) \backslash \{c(u), c(v_1''), c(v_1'''), c(v_4)\}$. There can be no bicoloured cycle through v_1 as it is different from both v_1'' and v_1''', and the 2-frugality at v_1' is preserved.
- If $c(v_1') \neq c(u)$, we colour v_1 with $c(v_1) \in L(v_1) \backslash \{c(u), c(v_1'), c(v_1''), c(v_4)\}$. There can be no bicoloured cycle through v_1 as $c(v_1') \neq c(u)$, and the 2-frugality at v_1' is preserved.

We colour v_2 with symmetrical rules, replacing v_4 by v_3.

As $c(v_4) \neq c(v_1)$ and $c(v_3) \neq c(v_2)$, the 2-frugality of c is preserved.

11. It follows from 2. Indeed if a 2-vertex would have a 2-neighbour then its second neighbour has degree at least $\Delta(H) + 1$, a contradiction.

12. There is a configuration as depicted in Figure 8 with possibly $v_1' = v_2$.

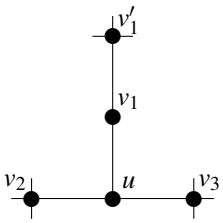

Figure 8: Configuration of Case 12

By the minimality of H, there exists a linear L-colouring c of $H - v_1$.

- If $c(u) = c(v_1')$, we colour v_1 with $c(v_1) \in L(v_1)$ that is different from $c(u)$, the colours appearing twice in the neighbourhood of v_1' (in order to preserve the 2-frugality at this vertex) and different from $c(v_2)$ and $c(v_3)$ to prevent the apparition of bicoloured cycles.
- If $c(u) \neq c(v_1')$, we colour v_1 with $c(v_1) \in L(v_1)$ that is different from $c(u)$ and $c(v_1')$, the colours appearing twice in the neighbourhood of v_1' (in order to preserve 2-frugality at this vertex) and different from $c(v_2)$ to preserve the 2-frugality at u. There can be no bicoloured cycles as $c(u) \neq c(v_1')$.

In both cases, H is linearly L-colourable, a contradiction.

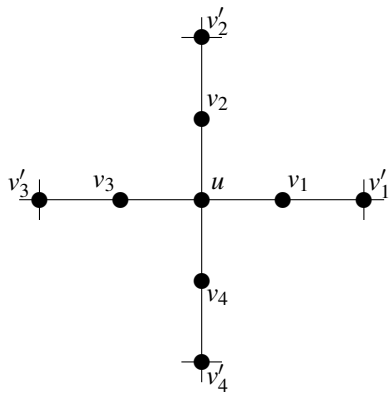

Figure 9: Configuration of Case 13

13. There is a configuration as depicted in Figure 9 with possibly some of the vertices in $\{v'_1, v'_2, v'_3, v'_4\}$ being identified.

    By the minimality of H there exists a linear L-colouring c of $H - \{u, v_1, v_2, v_3, v_4\}$. We extend it to u, v_1, v_2, v_3 and v_4 by sequentially colouring:

    - u with $c(u) \in L(u) \setminus \{c(v'_1), c(v'_2), c(v'_3), c(v'_4)\}$ to prevent the apparition of bicoloured cycles.
    - v_1 with $c(v_1) \in L(v_1)$ different from $c(u)$, $c(v'_1)$ and the colours appearing twice in the neighbourhood of v'_1.
    - v_2 with $c(v_2) \in L(v_2)$ different from $c(u)$, $c(v'_2)$ and the colours appearing twice in the neighbourhood of v'_2.
    - v_3 with $c(v_3) \in L(v_3)$ different from $c(u)$, $c(v'_3)$, $c(v_2)$ and the colours appearing twice in the neighbourhood of v'_3.
    - v_4 with $c(v_4) \in L(v_4)$ different from $c(u)$, $c(v'_4)$, $c(v_1)$ and the colours appearing twice in the neighbourhood of v'_4.

    There can be no bicoloured cycle with this colouring of H and c is 2-frugal because 3 vertices among $\{v_1, v_2, v_3, v_4\}$ cannot share the same colour. Then H is linearly L-colourable, a contradiction.

14. There is a configuration as depicted in Figure 10 with possibly some of the vertices in $\{v'_1, v'_2, v'_3, v_4\}$ being identified.

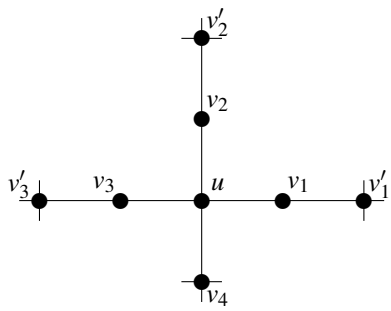

Figure 10: Configuration of Case 14

    By the minimality of H there exists a linear L-colouring c of $H - \{u, v_1, v_2, v_3\}$. We extend it to u, v_1, v_2 and v_3 by sequentially colouring:

    - u with $c(u) \in L(u)$ such that $c(u)$ is different $c(v_4)$, $c(v'_1)$, $c(v'_2)$, $c(v'_3)$ (to avoid any bicoloured cycle) and the colours appearing twice in the neighbourhood of v_4.
    - v_1 with $c(v_1) \in L(v_1)$ different from $c(u)$, $c(v'_1)$, $c(v_4)$, and the colours appearing twice in the neighbourhood of v'_1.

- v_3 with $c(v_3) \in L(v_3)$ different from $c(u)$, $c(v'_3)$, $c(v_4)$, and the colours appearing twice in the neighbourhood of v'_3.
- v_2 with $c(v_2) \in L(v_2)$ different from $c(u)$, $c(v'_2)$, $c(v_3)$, $c(v_1)$, and the colours appearing twice in the neighbourhood of v'_2.

There is no bicoloured cycle containing u because $c(u)$ is different from $c(v'_1)$, $c(v'_2)$ and $c(v'_3)$. Moreover, the 2-frugality at u is assured, as 3 vertices among $\{v_1, v_2, v_3, v_4\}$ cannot share the same colour. Hence H is linearly L-colourable, a contradiction.

15. There is a configuration as depicted in Figure 11 with possibly some of the vertices in $\{v'_1, v'_2, v'_3, v'_4, v'_5\}$ being identified.

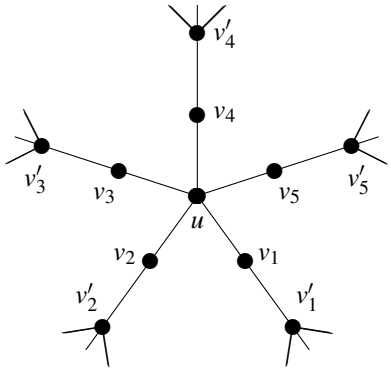

Figure 11: Configuration of Case 15

By the minimality of H there exists a linear L-colouring c of $H - \{u, v_1, v_2, v_3, v_4, v_5\}$. We extend it to u, v_1, v_2, v_3, v_4 and v_5 by sequentially colouring :

- v_1 with $c(v_1) \in L(v_1)$ different from $c(v'_1)$, and the colours appearing twice in the neighbourhood of v'_1.
- v_2 with $c(v_2) \in L(v_2)$ different from $c(v'_2)$, $c(v_1)$, and the colours appearing twice in the neighbourhood of v'_2.
- v_3 with $c(v_3) \in L(v_3)$ different from $c(v'_3)$, $c(v_1)$, $c(v_2)$, and the colours appearing twice in the neighbourhood of v'_3.
- v_5 with $c(v_5) \in L(v_5)$ different from $c(v'_5)$, $c(v_1)$, $c(v_2)$, $c(v_3)$ and the colours appearing twice in the neighbourhood of v'_5.
- u with $c(u) \in L(u)$ different from $c(v_1)$, $c(v_2)$, $c(v_3)$, $c(v_5)$, and $c(v'_4)$.
- v_4 with $c(v_4) \in L(v_4)$ different from $c(v'_4)$, $c(u)$, and the colours appearing twice in the neighbourhood of v'_4.

There is no bicoloured cycle using v'_4 because $c(v'_4) \neq c(u)$. Moreover there is no bicoloured cycle using both v_i and v_j, for $i < j$ and $i, j \in \{1, 2, 3, 5\}$ as $c(v_1)$, $c(v_2)$, $c(v_3)$ and $c(v_5)$ are all distionct. For the same reason, the 2-frugality at u is assured. Thus H is linearly L-colourable, a contradiction.

□

2.2 2-frugal colouring

Lemma 2. *Let H be a k-frugal-minimal graph.*

(i) If $k \geq 4$, then no 4-vertex is incident to two 2-threads.

(ii) If $k \geq \lceil \frac{\Delta}{2} \rceil + 3$, then a 4-vertex has at most one 2-neighbour.

(iii) If $k \geq \max\left(6, \lceil \frac{\Delta}{2} \rceil + 3\right)$, then a 5-vertex has at most four 2-neighbour.

Proof. Suppose that one of the assertions of Lemma 2 does not hold. Let H be a k-frugal-minimal graph for which it fails and L a k-list assignment such that H is not 2-frugally L-colourable.

(i) There is a configuration as depicted in Figure 12.

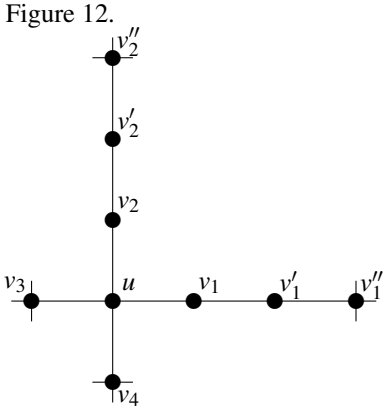

Figure 12: Configuration of Case (ii)

By the minimality of H there exists a 2-frugal L-colouring c of $H - \{v_1, v_2\}$. We extend it to v_1 and v_2 by sequentially colouring:

- v_1 with $c(v_1) \in L(v_1)$ different from $c(u), c(v_4), c(v_1')$.
- v_2 with $c(v_2) \in L(v_2)$ different from $c(u), c(v_3), c(v_2')$.

c is 2-frugal, given that $c(v_1) \neq c(v_4)$ and $c(v_2) \neq c(v_3)$. Hence, H is 2-frugally L-colourable, a contradiction.

(ii) There is a configuration as depicted in Figure 13 with possibly some of the vertices in $\{v_1', v_2', v_3, v_4\}$ being identified.

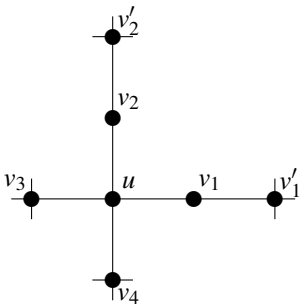

Figure 13: Configuration of Case (iii)

By the minimality of H there exists a 2-frugal L-colouring c of $H - \{v_1, v_2\}$. We extend it to v_1 and v_2 by sequentially colouring:

- v_1 with $c(v_1) \in L(v_1)$ different from $c(u)$, $c(v_4)$, $c(v_1')$ and the colours appearing twice in the neighbourhood of v_1'.
- v_2 with $c(v_2) \in L(v_2)$ different from $c(u)$, $c(v_3)$, $c(v_2')$ and the colours appearing twice in the neighbourhood of v_2'.

c is 2-frugal, given that $c(v_1) \neq c(v_4)$ and $c(v_2) \neq c(v_3)$. Hence, H is 2-frugally L-colourable, a contradiction.

(iii) There is a configuration as depicted in Figure 14 with possibly some vertices in $\{v_1', v_2', v_3', v_4', v_5'\}$ being identified.

By the minimality of H there exists a 2-frugal L-colouring c of $H - \{u, v_1, v_2, v_3, v_4, v_5\}$. We extend it to u, v_1, v_2, v_3, v_4 and v_5 by sequentially colouring:

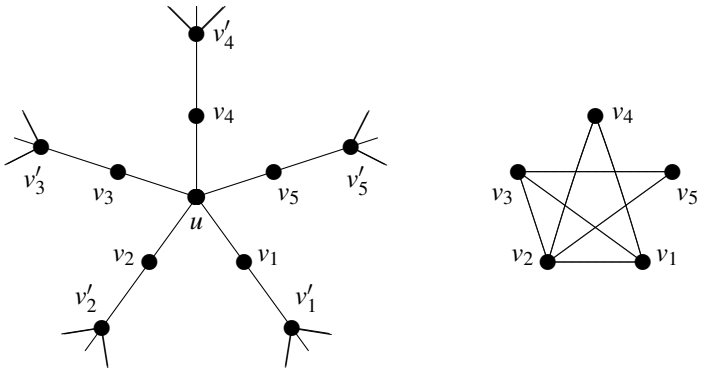

Figure 14: Configuration of Case (iv) and its auxiliary graph

- v_1 with $c(v_1) \in L(v_1)$ different from $c(v'_1)$, and the colours appearing twice in the neighbourhood of v'_1.
- v_2 with $c(v_2) \in L(v_2)$ different from $c(v'_2)$, $c(v_1)$ and the colours appearing twice in the neighbourhood of v'_2.
- v_3 with $c(v_3) \in L(v_3)$ different from $c(v'_3)$, $c(v_1)$, $c(v_2)$ and the colours appearing twice in the neighbourhood of v'_3.
- v_4 with $c(v_4) \in L(v_4)$ different from $c(v'_4)$, $c(v_1)$,$c(v_2)$ and the colours appearing twice in the neighbourhood of v'_4.
- v_5 with $c(v_5) \in L(v_5)$ different from $c(v'_5)$, $c(v_2)$,$c(v_3)$ and the colours appearing twice in the neighbourhood of v'_5.
- u with $c(u) \in L(u)$ different from $c(v_1)$, $c(v_2)$, $c(v_3)$, $c(v_4)$ and $c(v_5)$.

The colouring is 2-frugal at u because no three vertices of $\{v_1, v_2, v_3, v_4, v_5\}$ can share the same colour as there is no stable set of size 3 in the auxiliary graph depicted on the right of Figure 14. Thus H is 2-frugally L-colourable, a contradiction.

□

3 Main results

3.1 Linear colouring – asymptotic result

In this subsection, we prove the following theorem:

Theorem 2. *Let G be a graph of maximum degree at most Δ.*

1. *If $Mad(G) < 3 - \frac{3}{\Delta+1}$ and $\Delta \geq 8$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$.*
2. *If $Mad(G) < 3 - \frac{9}{4\Delta+3}$ and $\Delta \geq 7$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$.*
3. *If $Mad(G) < 3$ and $\Delta \geq 12$ then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.*

Proof. The proof of the three statements are similar.

We assume the existence of a counter-example G with maximum degree at most Δ such that $Mad(G) < 3 - \varepsilon$ (we first consider ε as a variable). G then contains a subgraph H which is $(\lceil \frac{\Delta}{2} \rceil + q)$-linear-minimal with $q = 1$, 2 and 3 depending on the statement. We give to each vertex v of H an initial charge $w(v)$ equal to its degree $deg_H(v)$ in H. The average charge is then equal to the average degree of H which is at most $Mad(G)$.

We then use the following discharging rule :

- Every d-vertex, $d \geq 3$, gives $\alpha_d = \frac{d-(3-\varepsilon)}{d}$ to its 2-neighbours.

We shall prove that after the discharging phase every vertex v has final charge $w^*(v)$ at least $3-\varepsilon$ for some $\varepsilon \geq 0$ to be determined. This implies that

$$Ad(H) = \frac{\sum_{v\in V(H)} w(v)}{|V(H)|} = \frac{\sum_{v\in V(H)} w^*(v)}{|V(H)|} \geq 3-\varepsilon$$

which contradicts $Mad(G) < 3-\varepsilon$.

By Lemma 1-1, there is no (≤ 1)-vertex. For any $d \geq 3$, every d-vertex send at least d times α_d, so its final charge is at least $d - d\cdot\alpha_d = 3-\varepsilon$.

Let us now examine the final charge of 2-vertices. We set $d'_q = 2(\lceil \frac{\Delta}{2} \rceil + q - d) + 1$. By Lemma 1-2, every 2-vertex having a d-neighbour has also a $(\geq d'_q)$-neighbour. Observe that $d'_q > \Delta$ if $d \leq q$, thus no 2-vertex has a $(\leq q)$-neighbour.

If $q \geq 2$, then a 2-vertex v has no 2-neighbour. Hence by Lemma 1-2, it has a d_1-neighbour and a d_2-neighbour, with $2 < d = d_1 \leq d'_q \leq d_2$. Since $\alpha_3 \leq \alpha_4 \leq \cdots \leq \alpha_\Delta$, the final charge of v is at least $w^*(v) = 2+\alpha_{d_1}+\alpha_{d_2} \geq 2+\alpha_d+\alpha_{d'_q}$.

If $q = 1$ a 2-vertex has either no 2-neighbour and as above its final charge is at least $2+\alpha_d+\alpha_{d'_q}$ or it has a 2-neighbour and, by Lemma 1-2, its other neighbour is a Δ-vertex, so it final charge is at least $2+\alpha_\Delta$.

Hence, to prove Theorem 2, it is sufficient to show that $2+\alpha_d+\alpha_{d'_q} \geq 3-\varepsilon$, for all $d \leq d'_q$ and also $2+\alpha_\Delta \geq 3-\varepsilon$ when $q = 1$.

$2+\alpha_d+\alpha_{d'_q} \geq 3-\varepsilon$ is equivalent to $P(d) \geq 0$ with $P(d) = (1+\varepsilon)dd'_q + (\varepsilon-3)(d+d'_q)$. Since $d'_q = 2(\lceil \frac{\Delta}{2} \rceil + q - d) + 1$, $P(d)$ is a polynomial of degree 2 in d of the form $-2(1-\varepsilon)d^2 + A\cdot d + B$ for some constant A and B (note that the coefficient of d^2 is negative). Hence to verify that $P(d) \geq 0$ for all possible values of d it suffices to prove it for the smallest and largest d such that $d \leq d'_q$, namely $\max(3, q+1)$ and $\frac{\Delta+2q+2}{3}$ (it actually is $\frac{\Delta+2q+2}{3}$ if Δ is odd and $\frac{\Delta+2q+1}{3}$ if Δ is even).

If $q \geq 2$, we obtain the following two conditions:

$$\begin{aligned} \varepsilon &\geq \frac{3(q+1)-(q-2)\Delta}{q+1+(q+2)\Delta} \\ \varepsilon &\geq \frac{16-2q-\Delta}{8+2q+\Delta} \end{aligned}$$

For $q = 3$ and $\Delta \geq 12$ the right hand sides of these two inequalities are negative. So they are satisfied for $\varepsilon = 0$, which proves Theorem 2-3. For $q = 2$ and $\Delta \geq 7$ then $\frac{9}{3+4\Delta} \geq \frac{12-\Delta}{12+\Delta}$ and so the above inequalities are satisfied for $\varepsilon = \frac{9}{3+4\Delta}$. This proves Theorem 2-2.

If $q = 1$, we have the three following conditions, the first two given by $P(d)$ and the third one by $2+\alpha_\Delta \geq 3-\varepsilon$.

$$\begin{aligned} \varepsilon &\geq \frac{9}{4\Delta-5} \\ \varepsilon &\geq \frac{14-\Delta}{10+\Delta} \\ \varepsilon &\geq \frac{3}{\Delta+1} \end{aligned}$$

But $\frac{3}{\Delta+1} \geq \frac{9}{4\Delta-5}$ and $\frac{3}{\Delta+1} \geq \frac{14-\Delta}{10+\Delta}$ when $\Delta \geq 8$. So the above inequalities are satisfied for $\varepsilon = \frac{3}{\Delta+1}$. This proves Theorem 2-1.

□

3.2 Graphs of small maximum degree

In this subsection we prove the following theorem.

Theorem 3. *Let G be a graph with maximum degree at most Δ:*

1. *If $\Delta \geq 5$ and $Mad(G) < \frac{39}{16}$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$.*

2. *If* $\Delta \geq 7$ *and* $Mad(G) < \frac{48}{19}$, *then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$.

3. *If* $\Delta \geq 5$ *and* $Mad(G) < \frac{60}{23}$, *then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$.

4. *If* $Mad(G) < \frac{14}{5}$, *then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.

5. *If* $Mad(G) < 3$. *Then* $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 4$.

Proof. The proofs of all statements are similar: we assume the existence of a counter-example G such that $Mad(G) < M$ (we first consider M as a variable), from which we deduce the existence of a subgraph H which is k-linear-minimal. We then use the discharging method to reach a contradiction.

We give to each vertex v of H an initial charge $w(v)$ equal to its degree $deg_H(v)$ in H. The average charge is then equal to the average degree of H which is at most $Mad(G)$. Then, we define discharging rules by which vertices will exchange some of their charge, keeping the average constant. We then want to prove with the help of the lemmas of the previous section that each vertex v has a final charge $w^*(v)$ at least M and so strictly greater than the average charge, which is a contradiction.

Of course, we want to find rules such that M is as large as possible in each case : for this reason, the following proofs actually define the constraints of a Linear Programme in which M is the objective value, and whose variables are the charges exchanged by the vertices during the discharging phase. At the end of each proof, we give an optimal solution of the given Linear Programme which proves the results.

1. Let G be a graph with maximum degree $\Delta \geq 5$ such that $Mad(G) < M$. Set $k_1 = \lceil \frac{\Delta}{2} \rceil + 1$. Suppose by way of contradiction that $\Lambda^l(G) > k_1$. Then G has a subgraph H which is k_1-linear-minimal.

Let us assign to every vertex of H an initial charge $w(v) = deg_H(v)$. Then $\sum_{v \in V(H)} w(v) = \sum_{v \in V(H)} d(v) = Ad(H) \cdot |V(H)|$. We now apply the following discharging rules.

Rule 1. A 2-vertex having two 3-neighbours receives α_3 from each of them.

Rule 2. A 2-vertex having only one 3-neighbour receives α'_3 from it.

Rule 3. A 2-vertex having a 2-neighbour and a (≥ 4)-neighbour receives α_4 from it.

Rule 4. A 2-vertex having a (≥ 3)-neighbour and a (≥ 4)-neighbour receives α'_4 from its (≥ 4)-neighbour.

At the end we want that the final charge of every vertex is at least M. This implies

$$Ad(H) = \frac{\sum_{v \in V(H)} w(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w^*(v)}{|V(H)|} \geq M$$

which contradicts $Mad(G) < M$.

We now define constraints on M and the α_i and α'_i guaranteeing the the final charge $w^*(v)$ of every vertex v of H is at least M.

As there are no (≤ 1)-vertices by Lemma 1-1, let us examine the final charge $w^*(v)$ of a (≥ 2)-vertex v.

- If v is a 2-vertex then by Lemma 1-3 and 1-4 it has either two 3-neighbours, or one 3-neighbour and one (≥ 4)-neighbour, or one 2-neighbour and one (≥ 4)-neighbour. In the first case $w^*(v) = 2 + 2\alpha_3$, on the second $w^*(v) \leq 2 + \alpha'_3 + \alpha'_4$ and in the last one $w^*(v) = 2 + \alpha_4$. So the required constraints are

$$M \leq 2 + 2\alpha_3 \text{ and } M \leq 2 + \alpha'_3 + \alpha'_4 \text{ and } M \leq 2 + \alpha_4.$$

- If v is a 3-vertex adjacent to three 2-neighbours, none of them can be adjacent to another 3-vertex by Lemma 1-5, and so will give at most $3\alpha'_3$. If it is adjacent to two 2-neighbours then it gives at most $\max\{2\alpha_3, 2\alpha'_3\}$. Hence, we obtain the constraints

$$M \leq 3 - 3\alpha'_3 \text{ and } M \leq 3 - 2\alpha_3.$$

- If v is a 4-vertex, by Lemma 1-6 it can not be adjacent to four 2-threads. Then it gives at most $3\alpha_4 + \alpha'_4$. (Here we assume implicitly that $\alpha'_4 \leq \alpha_4$ which is intuitively true but a priori not proved. However this inequality is satisfied by the solution giving the optimal value of M and so our assumption is a posteriori correct). Thus we obtain the constraint $M \leq 4 - 3\alpha_4 - \alpha'_4$.

- If v is a k-vertex with $k \geq 5$, it will give at most give $k\alpha_4$ yielding the constraints $M \leq k - k\alpha_4$.

The optimal value $M = \frac{39}{16}$ is obtained for $\alpha_3 = \frac{7}{32}$, $\alpha'_3 = \frac{3}{16}$, $\alpha_4 = \frac{7}{16}$ and $\alpha'_4 = \frac{1}{4}$.

2. Let G be a graph with maximum degree $\Delta \geq 7$ such that $Mad(G) < M$. Set $k_1 = \lceil \frac{\Delta}{2} \rceil + 1$. Suppose by way of contradiction that $\Lambda^l(G) > k_1$. Then G has a subgraph H which is k_1-linear-minimal.

Let us assign to every vertex of H an initial charge $w(v) = deg_H(v)$ and apply the following discharging rules.

Rule 1. A 3-vertex sends α_3 to each of its 2-neighbours.

Rule 2. A (≥ 4)-vertex sends α_4 to its 2-neighbours which are in a 2-thread and α'_4 to its other 2-neighbours.

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, a contradiction. Again there are no (≤ 1)-vertices by Lemma 1-1.

- Suppose that v is a 2-vertex. Then by Lemma 1-3 it has at most one 2-neighbour. If v has one 2-neighbour then, by Lemma 1-4, it has a (≥ 4)-neighbour from which it receives α_4. We obtain the constraint $M \leq 2 + \alpha_4$. If v has no 2-neighbour then, by Lemma 1-9, it has a (≥ 4)-neighbour from which it receives α'_4. Its other neighbour is a (≥ 3)-neighbour from which it receives α_3. Hence we get $M \leq 2 + \alpha'_4 + \alpha_3$.
- If v is a 3-vertex it sends at most α_3 to each neighbour. This yields the constraints $M \leq 3 - 3\alpha_3$.
- If v is a 4-vertex then by Lemma 1-7, it is incident to no 2-thread, yielding $M \leq 4 - 4\alpha'_4$.
- If v is a 5-vertex then by Lemma 1-8, it is incident to at most four 2-threads, yielding $M \leq 5 - 4\alpha_4 - \alpha'_4$ (again we implicitly assume $\alpha_4 \geq \alpha'_4$ which is satisfied by the solution giving the optimal value).
- If v is a k-vertex with $k \geq 6$, it sends at most α_4 to each neighbour. Thus $M \leq k - k\alpha_4$.

The optimal value $M = \frac{48}{19}$ is obtained for $\alpha_3 = \frac{3}{19}$, $\alpha_4 = \frac{10}{19}$ and $\alpha'_4 = \frac{7}{19}$.

3. Let G be a graph with maximum degree $\Delta \geq 5$ such that $Mad(G) < M$. Set $k_2 = \lceil \frac{\Delta}{2} \rceil + 2$. Suppose by way of contradiction that $\Lambda^l(G) > k_2$. Then G has a subgraph H which is k_2-linear-minimal.

We assign to every vertex of H an initial charge $w(v) = deg_H(v)$ and apply the following discharging rules.

Rule 1. A 3-vertex sends α_3 to each of its 2-neighbours

Rule 2. A (≥ 4)-vertex sends α_4 to each of its 2-neighbours having no (≤ 3)-neighbour.

Rule 3. A (≥ 4)-vertex sends α'_4 to each of its 2-neighbours having a 3-neighbour.

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, which is a contradiction.

- If v is a 2-vertex then it has no 2-neighbour by Lemma 1-11. If it has a 3-neighbour its other neighbour is a (≥ 4)-neighbour according to Lemma 1-9. yielding $M \leq 2 + \alpha_3 + \alpha'_4$. If v has no 3-neighbour, then we get $M \leq 2 + 2\alpha_4$.
- If v is a 3-vertex it sends at most $3\alpha_3$, yielding $M \leq 3 - 3\alpha_3$.
- If v is a 4-vertex then by Lemma 1-10, it has at most one 2-neighbour that has a 3-neighbour, yielding $M \leq 4 - \alpha'_4 - 3\alpha_4$ (with the assumption $\alpha'_4 \geq \alpha_4$).
- If v is a k-vertex with $k \geq 5$, it sends at most $k\alpha'_4$, yielding $M \leq k - k\alpha'_4$.

The optimal value $M = \frac{60}{23}$ is obtained for $\alpha_3 = \frac{3}{23}$, $\alpha_4 = \frac{7}{23}$ and $\alpha'_4 = \frac{11}{23}$.

4. Let G be a graph such that $Mad(G) < M$. Set $k_3 = \lceil \frac{\Delta}{2} \rceil + 3$. Suppose by way of contradiction that $\Lambda^l(G) > k_3$. Then $\Delta \geq 3$, as every graph with maximum degree at most 2 is linearly 3-choosable. Moreover G has a subgraph H which is k_3-linear-minimal.

Let us assign to every vertex of H an initial charge $w(v) = d_H(v)$ and apply the following discharging rule.

Rule 1. A (≥ 4)-vertex sends α_4 to each of its 2-neighbours

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, which is a contradiction.

- If v is a 2-vertex then it has no (≤ 3)-neighbour by Lemmas 1-11 and 1-12. This gives $M \leq 2 + 2\alpha_4$.
- If v is a 3-vertex then its charge does not change, yielding $M \leq 3$.
- If v is a 4-vertex then by Lemma 1-13, it has at most three 2-neighbours. This yields $M \leq 4 - 3\alpha_4$.
- If v is a k-vertex with $k \geq 5$, it sends at most α_4 to each neighbour, yielding $M \leq k - k\alpha_4$.

The optimal value $M = \frac{14}{5}$ is obtained for $\alpha_4 = \frac{2}{5}$.

5. Let G be a graph such that $Mad(G) < M$. Set $k_4 = \lceil \frac{\Delta}{2} \rceil + 4$. Suppose by way of contradiction that $\Lambda^l(G) > k_4$. Then $\Delta \geq 3$, as every graph with maximum degree at most 2 is linearly 3-choosable. Moreover G has a subgraph H which is k_4-linear-minimal.

Let us assign to every vertex of H an initial charge $w(v) = deg_H(v)$ and apply the following discharging rule.

Rule 1. A (≥ 4)-vertex sends α_4 to each of its 2-neighbours.

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, which is a contradiction.

- If v is a 2-vertex then it has no (≤ 3)-neighbour by Lemmas 1-11 and 1-12. This gives $M \leq 2 + 2\alpha_4$.
- If v is a 3-vertex, its charge is unchanged. This gives $M \leq 3$.
- If v is a 4-vertex then by Lemma 1-14, it has at most two 2-neighbours, yielding $M \leq 4 - 2\alpha_4$.
- If v is a 5-vertex then by Lemma 1-15, it has at most four 2-neighbours. This gives $M \leq 5 - 4\alpha_4$.
- If v is a k-vertex with $k \geq 6$, it sends at most $k\alpha_4$. Thus $M \leq k - k\alpha_4$.

The optimal value $M = 3$ is obtained for $\alpha_4 = \frac{1}{2}$.

□

3.3 2-frugal colouring

In this subsection we prove the following theorem.

Theorem 4. *Let G be a graph with maximum degree (at most) Δ*

1. *If $\Delta \geq 7$ and $Mad(G) < \frac{5}{2}$, then $\Phi_2^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$.*
2. *If $Mad(G) < 3$, then $\Phi_2^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.*

Proof. 1. Let G be a graph with maximum degree $\Delta \geq 7$ such that $Mad(G) < M$. Set $k_1 = \lceil \frac{\Delta}{2} \rceil + 1 \geq 5$. Suppose by way of contradiction that $\Phi_2^l(G) > k_1$. Then G has a subgraph H which is k_1-frugal-minimal.

Let us assign to every vertex of H an initial charge $w(v) = d_H(v)$. Then $\sum_{v \in V(H)} w(v) = \sum_{v \in V(H)} d(v) = Ad(H) \cdot |V(H)|$. Let us call a 2-vertex *bad* if it has a 2-neighbour, and *good* otherwise. We now apply the following discharging rules.

Rule 1. 3-vertices give α_3 to each of their 2-neighbours.

Rule 2. (≥ 4)-vertices give α_4^g to each of their good 2-neighbours.

Rule 3. (≥ 4)-vertices give α_4^b to each of their bad 2-neighbours.

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, which is a contradiction

- There are no 1-vertices by Lemma 1-1.
- If v is a 2-vertex then by Lemma 1-3 it does not have two 2-neighbours. In addition, by Lemma 1-4, no bad vertex can have a 3-neighbour, and by Lemma 1-9 a 2-vertex has at most one 3-neighbour. This gives the constraints $M \leq 2+\alpha_3+\alpha_4^g$ and $M \leq 2+\alpha_4^b$.
- If v is a 3-vertex it sends at most α_3 to each of its neighbours, yielding $M \leq 3-3\alpha_3$.
- If v is a 4-vertex, then by Lemma 2-i it has at most one bad neighbour. Hence $M \leq 4-\alpha_4^b-3\alpha_4^g$ (with the assumption $\alpha_4^b \leq \alpha_4^g$).
- If v is a k-vertex with $k \geq 5$ it sends at most α_4^b to each of its neighbours (under the same assumption), yielding $M \leq k-4\alpha_4^b$.

The optimal value $M = \frac{5}{2}$ is obtained for $\alpha_3 = \frac{1}{6}$, $\alpha_4^g = \frac{1}{3}$ and $\alpha_4^b = \frac{1}{2}$.

2. Let G be a graph such that $Mad(G) < M$. Set $k_3 = \lceil \frac{\Delta}{2} \rceil + 3$. Suppose by way of contradiction that $\Phi_2^l(G) > k_3$. Then G has a subgraph H which is k_1-frugal-minimal.

We assign to every vertex of H an initial charge $w(v) = deg_H(v)$ and apply the following discharging rule.

Rule 1. ≥ 4-vertices give α_4 to each of their 2-neighbours.

Let us now define constraints to ensure that the final charge $w^*(v)$ of every vertex v of H is at least M, which is a contradiction.

- There are no 1-vertices by Lemma 1-1.
- If v is a 2-vertex, by Lemma 1-11 it does not have 2-neighbours, and by Lemma 1-12 it can not have any 3-neighbour either. Hence $M \leq 2+2\alpha_4$.
- If v is a 3-vertex, $M \leq w^*(v) = 3$.
- If v is a 4-vertex, by Lemma 2-ii it has at most one 2-neighbour, yielding $M \leq 4-\alpha_4$.
- If v is a 5-vertex, by Lemma 2-iii it has at most four 2-neighbours, giving $M \leq 5-4\alpha_4$.
- If v is a k-vertex with $k \geq 6$, then it sends at most α_4 to each of its neighbours. This yields $M \leq 6-6\alpha_4$.

The optimal value $M = 3$ is obtained for $\alpha_4 = \frac{1}{2}$. □

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