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Facial parity edge colouring

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Abstract

The *facial parity edge colouring* of a connected bridgeless plane graph is such an edge colouring in which no two face-adjacent edges (consecutive edges of a facial walk of some face) receive the same colour, in addition, for each face α and each colour c , either no edge or an odd number of edges incident with α is coloured with c . From the Vizing's theorem it follows that every 3-connected plane graph has a such colouring with at most $\Delta^* + 1$ colours, where Δ^* is the size of the largest face. In this paper we prove that any connected bridgeless plane graph has a facial parity edge colouring with at most 92 colours.

Keywords: plane graph, facial walk, edge colouring

Math. Subj. Class.: 05C10, 05C15

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1 Introduction

One of the motivations for this paper has come from recent papers of Bunde et al. [1, 2] who introduced parity edge colourings of simple graphs. Studying the parity of the usage of colours along walks suggested two edge colouring parameters that have interesting properties and applications. A *parity walk* in an edge colouring of a simple graph is a walk along which each colour is used an even number of times. Bunde et al. [2] introduced two parameters. Let $p(G)$ be the minimum number of colours in an edge colouring of G having no parity path (a *parity edge colouring*). Let $\widehat{p}(G)$ be the minimum number of colours in an edge colouring of G in which every parity walk is closed (a *strong parity edge colouring*). Since incident edges of the same colour would form a parity path of length 2, every parity edge colouring is a proper edge colouring, and hence $p(G) \geq \chi'(G)$, where $\chi'(G)$ is the chromatic index of G . Since a path is an open walk, no strong parity edge colouring has a parity path. Hence, every strong parity edge colouring is a parity edge colouring and $\widehat{p}(G) \geq p(G)$ for every graph G . Although there are graphs G with $\widehat{p}(G) > p(G)$ [2], it remains unknown how large $\widehat{p}(G)$ can be when $p(G) = k$. Elementary results on these parameters appear in [2]. In [1] it is proved that $\widehat{p}(K_n) = 2^{\lceil \log(n) \rceil} - 1$ for all n . Moreover, the optimal strong parity edge colouring of the complete n -vertex graph K_n is unique when n is a power of 2. The authors of [2] mentioned that computing $p(G)$ or $\widehat{p}(G)$ is NP-hard even when G is a tree. Clearly, the parity edge colouring is such a colouring that each path uses at least one colour an odd number of times.

The vertex version of this problem (*strong parity vertex colouring*) with some restrictions was introduced in [3]. The authors of [3] conjectured that there is a constant K such that the vertices of any 2-connected plane graph can be coloured with at most K colours in such a way that for each face α and each colour c , either no vertex or an odd number of vertices incident with α is coloured with c .

Another motivation for this work has come from the papers of Pyber [5] and Mátrai [4]. A graph is called *odd* if the degree of its vertices is odd or zero. Pyber raises the problem of edge covering with odd subgraphs in [5] as the counterpart of even subgraph covering problems. He proved that the edges of every finite simple graph can be covered by at most 4 edge-disjoint odd subgraphs; moreover, if the number of vertices is even then 3 odd subgraphs are sufficient. For not necessarily edge-disjoint coverings we have the following question: Is it true that every graph can be covered by at most 3 odd subgraphs? Mátrai in [4] showed that every finite simple graph can be covered by 3 odd subgraphs and he found an infinite sequence of finite simple connected graphs not coverable by 3 edge-disjoint odd subgraphs.

Pyber's result implies the following: The edges of any 3-connected plane graph G can be coloured by at most 4 colours in such a way that for each face α and each colour c , either no edge or an odd number of edges incident with α is coloured with c . It is sufficient to consider the dual G^* of G and its edge cover with at most 4 edge-disjoint odd subgraphs. This cover induces the required colouring of G .

If we add a requirement that such a colouring must be proper then it is not clear whether there exists such a colouring with K colours, where K is an absolute constant. From Vizing's theorem [6] it follows that every 3-connected plane graph G has such a colouring which uses at most $\Delta^* + 1$ colours, where Δ^* is the size of the largest face. Consider a proper edge colouring of the dual graph G^* . This colouring induces a colouring of G in a natural way. It is such a colouring in which, for each face α of G , all the edges in the boundary of α have distinct colours.

In this paper we show that each connected bridgeless plane graph has a facial parity edge colouring using at most 92 colours.

2 Notation

Let us introduce the notation used in this paper. A graph which can be embedded in the plane is called *planar graph*; a fixed embedding of a planar graph is called *plane graph*.

A *bridge* is an edge whose removal increases the number of components. A graph which contains no bridge is said to be *bridgeless*. In this paper we consider connected bridgeless plane graphs, multiple edges and loops are allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set V , the edge set E , and the face set F . The *degree* of a vertex v , denoted by $\deg(v)$, is the number of edges incident with v , each loop counting as two edges. A k -vertex is a vertex of degree k . For a face α , the *size* of α , $\deg(\alpha)$, is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of α . We often say k -face for a face of size k .

Given a graph G and one of its edges $e = uv$ (the vertices u and v do not have to be different), the *contraction* of e denoted by G/e consists of replacing u and v by a new vertex adjacent to all the former neighbours of u and v , and removing the loop corresponding to the edge e . (We keep multiple edges if they arise.) Analogously we define the contraction of the set of edges $H = \{e_1, \dots, e_k\}$ and we denote it by $G/\{e_1, \dots, e_k\}$ or G/H .

Two faces are *adjacent* if they share an edge. Two (distinct) edges are *face-adjacent* if they are consecutive edges of a facial walk of some face α .

A k -edge colouring of a graph G is a mapping $\varphi : E(G) \rightarrow \{1, \dots, k\}$. The *facial parity edge (FPE) colouring* of a connected bridgeless plane graph is such an edge colouring that no two face-adjacent edges receive the same colour, in addition, for each face α and each colour c , either no edge or an odd number of edges incident with α is coloured with c .

Question 2.1. What is the minimum number of colours $\chi'_{fp}(G)$ that a connected bridgeless plane graph G has a facial parity edge colouring with at most $\chi'_{fp}(G)$ colours?

The number $\chi'_{fp}(G)$ is called the *facial parity chromatic index* of G .

3 Results

Theorem 3.1. *Let G be a connected bridgeless plane graph. Then*

$$\chi'_{fp}(G) \leq 92.$$

The proof uses the method of discharging. Let G be a counterexample with minimal number of edges, then minimal number of 1-faces, and then minimal number of 2-faces. If G is a single cycle of length d , $d \leq 5$, we use exactly d colours. We consider this to be the first step of induction and call this case trivial.

First, we prove several structural properties of G .

We say that a face α is *small* if $1 \leq \deg(\alpha) \leq 44$ and a face β is *big* if $\deg(\beta) \geq 45$.

3.1 Reducible configurations

We find such (forbidden) subgraphs H of G that the facial parity edge colouring of G/H using at most 92 colours can be extended to a required colouring of G using at most 92

colours, which is a contradiction to G being a counterexample. In the sequel, whenever we speak about an FPE colouring, we always mean an FPE colouring using at most 92 colours.

3.1.1 1-faces

Claim 3.2. *Each vertex of G is incident with at most one 1-face.*

Proof. Let v be a vertex incident with at least two 1-faces α_1 and α_2 . If we split v into two vertices v_1 and v_2 in such a way that α_1 and α_2 become a 2-face α and face-adjacency of all edges incident with v is preserved, we obtain a graph G' , see Figure 1. It has the same number of edges as G , but fewer 1-faces. Thus, it is not a counterexample and we can find an FPE colouring φ' of G' . It induces an edge colouring φ of G in a natural way. It is easy to see that φ is an FPE colouring. \square



Figure 1: A vertex incident with at least two 1-faces can be split into two vertices, reducing the number of 1-faces. Similarly, one can reduce a 1-face and a d -face ($2 \leq d \leq 4$) incident with the same vertex.

Claim 3.3. *Each vertex of G incident with a 1-face is not incident with any other d -face for $2 \leq d \leq 4$.*

Proof. We use the same reduction as in the proof of Claim 3.2. We split the vertex v incident with a 1-face γ and a d -face α ($2 \leq d \leq 4$) in such a way that the faces α and γ become a $(d+1)$ -face α' and face-adjacency of all edges incident with v is preserved, see Figure 1 for illustration. Let the reduced graph be G' . It has fewer 1-faces than G , therefore, it has an FPE colouring φ' . The face α' has size at most five, therefore, its edges are coloured using $d+1$ different colours. Thus, the colouring φ of G induced by the colouring φ' of G' is an FPE colouring, too. \square

Claim 3.4. *Each vertex of G incident with a 2-face is not incident with any d -face for $2 \leq d \leq 3$.*

Proof. We use the same reduction as in the proofs of Claims 3.2 and 3.3. We omit the details. \square

3.1.2 Small faces

Claim 3.5. *There are no two adjacent small faces in G .*

Proof. Let α_1 and α_2 be two adjacent small faces in G .

If both α_1 and α_2 are 1-faces, the graph consists of a single vertex and a loop; it has an FPE colouring using 1 colour.

Let α_1 be a 1-face and α_2 be a d -face, $d \geq 2$; let e be the loop they share, see Figure 2. Then the graph $G' = G/e$ has fewer edges than G , therefore, it has an FPE colouring φ' . Let α' be a face in G' corresponding to α_1 and α_2 in G . Since α_2 is a small face, at most 43 colours occur on the edges incident with α' . To extend the colouring φ' of G' to an FPE colouring of G , it suffices to colour the edge e with any colour that does not occur on α' .

Let α_1 and α_2 be two small faces of size at least 2 and let e be the edge they share, see Figure 2. The graph $G' = G/e$ has fewer edges than G , therefore, it has an FPE colouring φ' . Let α'_1 and α'_2 be the faces of G' corresponding to the faces α_1 and α_2 in G . (Since α_1 and α_2 are faces of size at least 2, the size of α'_1 and α'_2 is at least 1.)

Consider the set of colours different from the colours occurring on the edges of α'_1 and α'_2 (the colours admissible for the edge e). Since α_1 and α_2 are small, at most $2 \cdot (44 - 1) = 86$ colours occur on the edges incident with α'_1 and α'_2 . Hence, there is an admissible colour c . We can extend φ' to an FPE colouring φ of G by setting $\varphi(e) = c$. \square

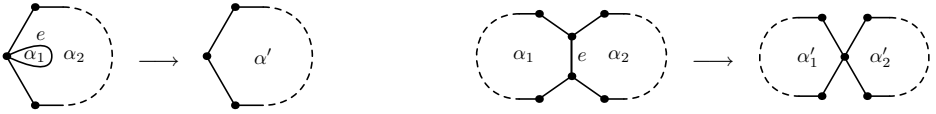


Figure 2: Two adjacent small faces form a reducible configuration: one can contract the edge they share.

Claim 3.6. Let β be a big face adjacent to two small faces α_1 and α_2 . Let e_i be an edge incident with β and α_i , $i = 1, 2$. Then e_1 and e_2 are face-adjacent.

Proof. Let e_1 and e_2 not be face-adjacent. See Figure 3 for illustration. The graph $G' = G/\{e_1, e_2\}$ has fewer edges than G , therefore, it has an FPE colouring φ' . Let $\alpha'_1, \alpha'_2, \beta'$ be the faces of G' corresponding to the faces $\alpha_1, \alpha_2, \beta$ in G , respectively.

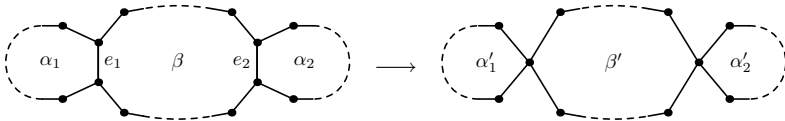


Figure 3: A big face β adjacent to two different small faces α_1 and α_2 forms a reducible configuration unless the edges e_1 and e_2 are face-adjacent.

We extend the colouring φ' of G' to an FPE colouring of G in the following way: Consider the set of colours different from the colours occurring on the edges of α'_1 and α'_2 ; also different from the colours occurring on the edges of G' corresponding to the edges of G face-adjacent to e_1 and e_2 . There are at least $92 - 2 \cdot (44 - 1) - 4 = 2$ such colours, say c_1 and c_2 . If at least one of them, say c_i , already occurs on β' , we set $\varphi(e_1) = \varphi(e_2) = c_i$. If none of them occurs on β' , we set $\varphi(e_i) = c_i$, $i = 1, 2$. \square

Claim 3.7. Each big face is adjacent to at most one 1-face.

Proof. It follows from Claims 3.2 and 3.6. \square

Claim 3.8. *Each big face is adjacent to at most two small faces.*

Proof. Let a big face β be adjacent to small faces α_1 , α_2 , and α_3 . Consider the edges that β shares with α_i , $i = 1, 2, 3$. It is easy to see that there must be a pair of edges e_i and e_j , incident to α_i and α_j , respectively ($i \neq j$), which are not face-adjacent. It is a contradiction with Claim 3.6. \square

3.1.3 Chains of 2-vertices

Claim 3.9. *There is no chain consisting of at least 5 consecutive 2-vertices in G .*

Proof. Let $v_0 e_0 v_1 e_1 \dots v_p e_p v_{p+1}$ be a chain consisting of p vertices v_1, \dots, v_p of degree 2, where $p \geq 5$. The graph $G' = G / \{e_1, e_2, e_3, e_4\}$ has an FPE colouring φ' . Let e'_0 and e'_5 be the edges in G' corresponding to the edges e_0 and e_5 in G . Let $\varphi'(e'_0) = c_1$ and $\varphi'(e'_5) = c_2$. The FPE colouring φ' of G' can be extended to an FPE colouring φ of G by setting $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$, see Figure 4. \square

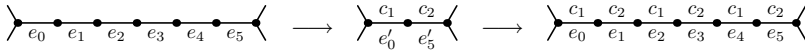


Figure 4: A chain of (at least) five 2-vertices is a reducible configuration.

A d -face α is *hanging* on a vertex v , if all vertices incident with α are 2-vertices except for the vertex v . By Claim 3.9 we have $d \leq 5$. (If $\deg(v) = 2$, the graph G consists of a single cycle of length at most 5, which is the trivial case).

We colour the vertices of G with black, blue and white colour in the following way:

Let all 2-vertices be black, all 3-vertices be blue and all k -vertices for $k \geq 6$ be white.

A 4-vertex v is black if there is a face hanging on it, else it is white. See Figure 5 for illustration of all types of black vertices.

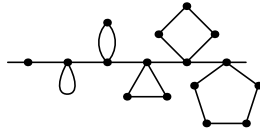


Figure 5: A vertex is black, if it is a 2-vertex, or a 4-vertex with a hanging face.

A 5-vertex v is blue, if there is a face hanging on it, else it is white.

Observe that any 2-vertex v is incident with two faces. We say v is *bad* for both faces it is incident with. Any black 4-vertex v is incident with a small face α of size at most 5 and two other faces. The face β adjacent to α must be big (see Claim 3.5); the vertex v occurs twice on the facial walk of β . The other face γ can be big or small. We say v is *bad* for the face γ .

Claim 3.10. *For any face α , there is no chain of at least 5 consecutive vertices bad for α . Each chain of bad vertices contains at most one bad 4-vertex.*

Proof. Let $v_0e_0v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6$ be a subpath of the facial walk of α containing 5 vertices v_1, \dots, v_5 bad for α . It is easy to see that all the edges e_0, \dots, e_5 are incident with the same face β ($\beta \neq \alpha$). If at least two of v_1, \dots, v_5 are bad 4-vertices, we come to a contradiction with Claim 3.5 or with Claim 3.6. If none of them is a 4-vertex, we are in the case of Claim 3.9. If precisely one of them is a 4-vertex, say v_i , it is incident with a small face γ of size $d \leq 5$.

The graph $G' = G/\{e_1, e_2, e_3, e_4\}$ has an FPE colouring φ' . Let e'_0 and e'_5 be the edges in G' corresponding to the edges e_0 and e_5 in G ; let α', β', γ' be the faces in G' corresponding to the faces α, β, γ in G ; let v' be the vertex in G' corresponding to v_1, \dots, v_5 in G . Let $\varphi'(e'_0) = c_1$ and $\varphi'(e'_5) = c_2$. Since e'_0 and e'_5 are face-adjacent in G' , $c_1 \neq c_2$. To extend φ' to an FPE colouring of G , we proceed in the following way:

If $i = 1, 3$, or 5 , we simply set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$.

If $i = 2$ or $i = 4$, we set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$ and switch the order of colours of edges incident with γ' , see Figure 6 for illustration (here the small face γ is a bigon).

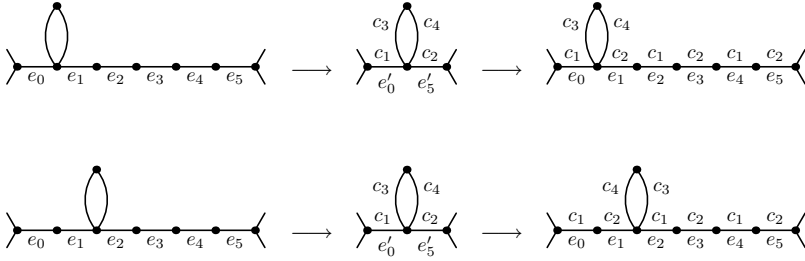


Figure 6: A chain of (at least) five bad black vertices is a reducible configuration as well.

□

Claim 3.11. *Let v be a black vertex bad for a small face α . If v is a 4-vertex, then the face hanging on it is a 1-face.*

Proof. Let v be a black 4-vertex, let γ be the face hanging on v of size at least 2 and let e_1 and e_2 be the edges of α incident with v . It is easy to see that the edges e_1 and e_2 are incident with the same big face, say β . See Figure 7 for illustration.

There is an edge e_γ incident with γ and β , which is not face-adjacent to $e_\alpha \in \{e_1, e_2\}$, which is a contradiction with Claim 3.6. □



Figure 7: Reducible pairs of edges (e_γ and e_α). For details see Claims 3.11 and 3.12.

Claim 3.12. *Let α be a small face sharing at least two bad black vertices with a big face β . Then all the bad black vertices incident with α and β are 2-vertices.*

Proof. Let v_1 and v_2 be black vertices incident with α and β . If v_1 is a 4-vertex, it is bad for α , therefore, there is a small face γ hanging on v_1 , adjacent to β . By Claim 3.11, the face γ is a 1-face. Thus, we can find an edge (a loop) e_γ incident with γ and β and an edge e_α incident with α and β , which are not face-adjacent. This is a contradiction with Claim 3.6. See Figure 7 for illustration. \square

Claim 3.13. *Let v be a black vertex bad for a small d -face α , $d \in \{2, 3, 4\}$. Then v is a 2-vertex.*

Proof. Let v be a black 4-vertex, let γ be the face hanging on v . By Claim 3.11, the face γ is a 1-face. On the other hand, by Claim 3.3, the face γ cannot be a 1-face, which is a contradiction. \square

Claim 3.14. *If a big face β shares a 2-vertex v with a small face α , then β is not adjacent to any other small face.*

Proof. It follows immediately from Claim 3.6. \square

Claim 3.15. *Let γ be a d -face, $d \in \{2, 3, 4, 5\}$, hanging on a vertex v , adjacent to a big face β . Then β is not adjacent to any other small face.*

Proof. It follows immediately from Claim 3.14. \square

3.2 Discharging rules

If G is a minimal counterexample then it contains no reducible configuration. Let the initial charge of each vertex be $\psi(v) = \deg(v) - 6$ and the initial charge of each face be $\psi(\alpha) = 2\deg(\alpha) - 6$. From Euler's formula we can easily derive that

$$\sum_{\alpha \in F} (2\deg(\alpha) - 6) + \sum_{v \in V} (\deg(v) - 6) = -12.$$

It is obvious that all the negative charge is in the vertices of degree 2, 3, 4, and 5 and in the faces of size 1 and 2.

Rule 1: Let β be a big face.

- If β is adjacent to a single small face α , it sends 3 units of charge to α .
- If β is adjacent to two small faces α_1 and α_2 , such that $\deg(\alpha_1) \leq \deg(\alpha_2)$, it sends 2 units of charge to α_1 and 1 unit of charge to α_2 . (If $\deg(\alpha_1) = \deg(\alpha_2)$, it is decided arbitrarily.)

Rule 2: Let β be a big face.

- It sends 2 units of charge to any black vertex bad for β .
- It sends 1 unit of charge to any other black, blue, or white vertex incident with β . (Multiply incident vertices are considered as different.)

Rule 3: Let α be a small face.

- It sends 2 units of charge to any black vertex bad for α .
- It sends 1 unit of charge to any other black or blue vertex incident with α .

Rule 4: Let v be a black 4-vertex.

- It sends 2 units of charge to the incident small hanging face γ .

Rule 5: Let v be a blue 5-vertex.

- It sends 2 units of charge to the incident small hanging face γ .

Rule 6: Let v be a k -vertex, $k \geq 6$.

- It sends 2 units of charge to any incident small hanging face γ .

3.3 Analysis of the discharging process

3.3.1 Vertices

Every 2-vertex is black and bad for both faces incident with it, hence it receives 2 units of charge from both incident faces (Rules 2 and 3). Its new charge is $-4 + 2 + 2 = 0$.

Every 3-vertex is blue, hence it receives 1 unit of charge from all the three incident faces (Rules 2 and 3). Its new charge is $-3 + 3 \cdot 1 = 0$.

Every black 4-vertex v receives 2 units of charge from the face it is bad for (Rules 2 and 3) and $2 \cdot 1$ units of charge from the doubly-incident big face β (Rule 2). It sends 2 units of charge to the hanging face γ (Rule 4). The new charge of v is $-2 + 2 + 2 \cdot 1 - 2 = 0$.

Every white 4-vertex is incident with at least 2 big faces (see Claim 3.5), therefore, its new charge is at least $-2 + 2 \cdot 1 = 0$.

Every blue 5-vertex v is incident with a hanging face γ , doubly-incident with a big face β and incident with two more faces α_1 and α_2 . Therefore, v receives $4 \cdot 1$ units of charge from the incident faces (Rules 2 and 3). It sends 2 units of charge to γ (Rule 5). Therefore, the new charge of v is $-1 + 4 \cdot 1 - 2 = 1$.

Every white 5-vertex is incident with at least 3 big faces (see Claim 3.5), therefore, its new charge is at least $-1 + 3 \cdot 1 = 2$.

Every (white) k -vertex v , $k \geq 6$, has non-negative initial charge. It receives charge from big faces (Rule 2) and sends charge to the hanging faces $\gamma_1, \dots, \gamma_r$ (Rule 6). For each hanging face γ_i , the adjacent big face β_i is doubly-incident to v . The faces β_i and β_j are different for different γ_i and γ_j (see Claims 3.6 and 3.7). Therefore, the new charge of v is at least $r \cdot (2 \cdot 1 - 2) = 0$.

3.3.2 1-faces

Let γ be a 1-face. It is adjacent to a big face β and it is hanging on a vertex v . The face β is adjacent to at most one 1-face (Claim 3.7). Therefore, it sends at least 2 units of charge to the face γ (Rule 1).

If the vertex v is a 2-vertex, we get the trivial graph, which is not a counterexample. If the vertex v is a 3-vertex, the third edge incident with v is a bridge in G , which is not allowed. Therefore, $\deg(v) \geq 4$ and the vertex v sends 2 units of charge to the face γ (Rules 4 – 6). The new charge of γ is at least $-4 + 2 + 2 = 0$.

3.3.3 2-faces

Let α be a 2-face. Its initial charge is -2 . Let v_1 and v_2 be the vertices incident with α . The face α is adjacent to at most two faces, which must be big (see Claim 3.5). Consider the number of black vertices bad for α . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 8 for illustration.

1. Let both v_1 and v_2 be black and bad. Then the graph G consists of a single cycle on two vertices, which is not a counterexample.
2. Let v_1 be a black 2-vertex. Then α is adjacent to a single big face β . The big face β is not adjacent to any other small face (see Claim 3.14). Therefore, β sends 3 units of charge to α (Rule 1). The face α sends 2 units of charge to v_1 and at most 1 unit of charge to v_2 (Rule 3). On the other hand, α is hanging on v_2 , therefore, it receives 2 units of charge from v_2 (Rules 4 – 6). Note that v_2 cannot be a 3-vertex, otherwise there would be a bridge in G . The new charge of α is at least $-2 - 2 - 1 + 2 + 3 = 0$.
3. Let none of v_1 and v_2 be black and bad. Consider the number of faces adjacent to α . If α is adjacent to a single big face β , then β is not adjacent to any other small face. Therefore, β sends 3 units of charge to α . Moreover, in this case none of v_1 and v_2 can be neither black nor blue. The new charge of α is $-2 + 3 = 1$.

If α is adjacent to two big faces β_1 and β_2 , the face α sends at most $2 \cdot 1$ units of charge to v_1 and v_2 . The big face β_i , $i = 1, 2$, is not adjacent to any other small face of size at most 2 (see Claims 3.3, 3.4, and 3.6). Therefore, by Rule 1, the face β_i , $i = 1, 2$, sends at least 2 units of charge to α . The new charge of α is at least $-2 - 2 \cdot 1 + 2 \cdot 2 = 0$.

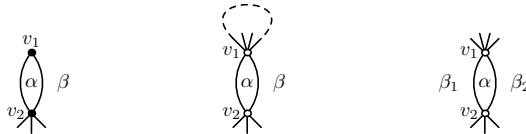


Figure 8: Different possible neighbourhoods of a 2-face α .

3.3.4 3-faces

Let α be a 3-face. Its initial charge is 0. Let v_1, v_2, v_3 be the vertices incident with α . (They do not have to be pairwise different.) Consider the number of black vertices bad for α . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 9 for illustration.

1. Let all the three vertices v_1, v_2, v_3 be black and bad. Then the graph G consists of a single cycle on three vertices, which is not a counterexample.
2. Let v_2 and v_3 be black and bad. Then all the three edges of α are incident with the same big face β and α is hanging on v_1 . Then by Claim 3.15 the face β sends 3 units of charge to α (Rule 1). The face α then sends $2 \cdot 2$ units of charge to the 2-vertices v_2, v_3 and at most 1 unit of charge to the vertex v_1 (Rule 3). If the vertex v_1 is a 3-vertex, the third edge incident with it is a bridge in G . Therefore

$\deg(v_1) \geq 4$ and v_1 sends 2 units of charge to α (Rules 4 – 6). The new charge of α is $3 - 2 \cdot 2 - 1 + 2 = 0$.

3. Let v_3 be black and bad. Let β_1 be the big face incident with the edge v_1v_2 and β_2 be the big face incident with the edges v_2v_3 and v_3v_1 .

The face β_2 sends 3 units of charge to α , β_1 sends at least 1 unit of charge to α . The face α then sends 2 units of charge to v_3 and at most 1 unit of charge to v_1 and v_2 . The new charge of α is at least $3 + 1 - 2 \cdot 1 = 0$.

4. Let none of the vertices v_1, v_2, v_3 be black and bad. Consider the number of faces adjacent to α .

If there are three different faces adjacent to α (they must be big, see Claim 3.5) then the face α receives at least 1 unit of charge from each of them and sends at most 1 unit of charge to each incident vertex. Hence, the new charge of α is at least 0.

If there is a big face β sharing at least two edges with α , these edges are not face-adjacent in β . Therefore, β is not adjacent to any other small face, thus, it sends 3 units of charge to α . The new charge of α is non-negative again.

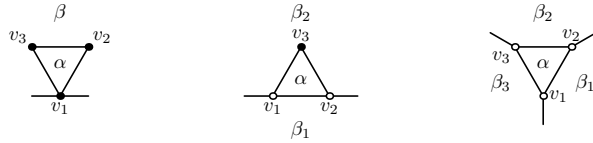


Figure 9: Different possible neighbourhoods of a 3-face α .

3.3.5 4-faces

Let α be a 4-face. Its initial charge is 2. Let v_1, v_2, v_3, v_4 be the vertices incident with α . (They do not have to be pairwise different.) Consider the number of black vertices bad for α . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 10 for illustration.

1. Let all the four vertices v_1, v_2, v_3, v_4 be black and bad. Then the graph G consists of a single cycle on four vertices, which is not a counterexample.
2. Let v_1, v_2 , and v_3 be black and bad. Then all the four edges of α are incident with the same big face β and α is hanging on v_4 . Hence, β sends 3 units of charge to α (Rule 1). The face α then sends $3 \cdot 2$ units of charge to the 2-vertices v_1, v_2, v_3 and at most 1 unit of charge to v_4 (Rule 3). If the vertex v_4 is a 3-vertex, the third edge incident with it is a bridge in G . Therefore, $\deg(v_4) \geq 4$ and v_4 sends 2 units of charge to α (Rules 4 – 6). The new charge of α is $2 + 3 - 3 \cdot 2 - 1 + 2 = 0$.
3. Let v_1 and v_3 be black and bad. Let β_1 be the big face incident with v_1 , let β_2 be the big face incident with v_3 .

If $\beta_1 \neq \beta_2$, both β_1 and β_2 send 3 units of charge to α . The face α then sends 2 units of charge to the vertices v_1 and v_3 and at most 1 unit of charge to the vertices v_2 and v_4 . The new charge of α is at least $2 + 2 \cdot 3 - 2 \cdot 2 - 2 \cdot 1 = 2$. If $\beta_1 = \beta_2$, then v_2 and v_4 are not blue, and the new charge of α is at least $2 + 3 - 2 \cdot 2 = 1$.

4. Let v_1 and v_2 be black and bad. Let β_1 be the big face incident with the edge v_3v_4 and β_2 be the big face incident with the vertices v_1 and v_2 . If $\beta_1 \neq \beta_2$, β_2 sends 3 units of charge to α and β_1 sends at least 1 unit of charge to α . The face α then sends 2 units of charge both to v_1 and v_2 and at most 1 unit of charge to v_3 and v_4 . The new charge of α is at least $2 + 3 + 1 - 2 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then v_3 and v_4 are not blue, and the new charge of α is at least $2 + 3 - 2 \cdot 2 = 1$.
5. Let v_1 be black and bad. The big face β_1 incident with v_1 sends 3 units of charge to α . The face α then sends 2 units of charge to v_1 and at most 1 unit of charge to v_2, v_3 , and v_4 . The new charge of α is at least $2 + 3 - 2 - 3 \cdot 1 = 0$.
6. Let no black and bad vertex be incident with α . Then the big faces adjacent to α send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge), and α sends at most 1 unit of charge to each incident vertex. The new charge of α is at least $2 + 2 - 4 = 0$.

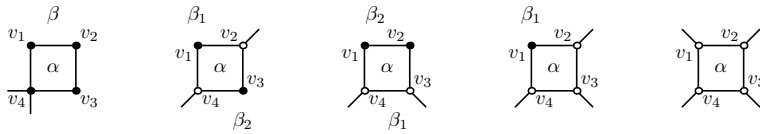


Figure 10: Different possible neighbourhoods of a 4-face α .

3.3.6 5-faces

Let α be a 5-face. Its initial charge is 4. Let v_1, v_2, v_3, v_4, v_5 be the vertices incident with α . (They do not have to be pairwise different.) Consider the number of black vertices bad for α . Note that by Claim 3.11 each such vertex is either a 2-vertex or a 4-vertex with a hanging 1-face. See Figure 11 for illustration.

1. Let all the five vertices v_1, v_2, v_3, v_4, v_5 be black and bad. Then the graph G contains only 5 vertices, hence, it is not a counterexample.
2. Let v_1, v_2, v_3 , and v_4 be black and bad. From Claim 3.6 it follows that none of them is incident with a 1-face. Then all the five edges of α are incident with the same big face β and α is hanging on v_5 . Hence, β sends 3 units of charge to α (Rule 1). The face α then sends $4 \cdot 2$ units of charge to the 2-vertices v_1, v_2, v_3 , and v_4 (Rule 3) and at most 1 unit of charge to the vertex v_5 (Rule 3). If the vertex v_5 is a 3-vertex, the third edge incident with it is a bridge in G . Therefore $\deg(v_5) \geq 4$ and v_5 sends 2 units of charge to α (Rules 4 – 6). The new charge of α is $4 + 3 - 4 \cdot 2 - 1 + 2 = 0$.
3. Let v_1, v_2 , and v_3 be black and bad. Let β_1 be the big face incident with the vertices v_1, v_2 , and v_3 and β_2 be the big face incident with the edge v_4v_5 . By Claim 3.12 v_1, v_2 , and v_3 are 2-vertices. If $\beta_1 \neq \beta_2$, β_1 sends 3 units of charge to α and β_2 sends at least 1 unit of charge to α . The face α then sends 2 units of charge to v_1, v_2 , and v_3 and at most 1 unit of charge to v_4 and v_5 . The new charge of α is at least $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then v_4 and v_5 are not blue, and the new charge of α is at least $4 + 3 - 3 \cdot 2 = 1$.

4. Let v_1, v_2 , and v_4 be black and bad. Let β_1 be the big face incident with v_1 and v_2 , let β_2 be the big face incident with v_4 . By Claim 3.12 v_1 and v_2 are 2-vertices, thus β_1 sends 3 units of charge to α .
If $\beta_1 \neq \beta_2$, then β_2 sends at least 1 unit of charge to α . The face α then sends 2 units of charge to the vertices v_1, v_2 , and v_4 and at most 1 unit of charge to the vertices v_3 and v_5 . The new charge of α is at least $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then v_3 and v_5 are not blue, and the new charge of α is $4 + 3 - 3 \cdot 2 = 1$.
5. Let v_1 and v_2 be black and bad. Let β_1 be the big face incident with the vertices v_1 and v_2 . The face β_1 sends 3 units of charge to α . The face α then sends 2 units of charge both to v_1 and v_2 and at most 1 unit of charge to v_3, v_4 , and v_5 . The new charge of α is at least $4 + 3 - 2 \cdot 2 - 3 \cdot 1 = 0$.
6. Let v_1 and v_3 be black and bad. Let β_1 be the big face incident with v_1 , β_2 be the big face incident with v_3 , and β_3 be the big face incident with the edge v_4v_5 . If β_1, β_2 , and β_3 are three different faces, the new charge of α is at least $4 + 3 - 2 \cdot 2 - 3 \cdot 1 = 0$. If two of them coincide, then at least one of the vertices v_2, v_4 , and v_5 is not blue and the new charge of α is at least $4 + 2 - 2 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2 = \beta_3$, then v_2, v_4 , and v_5 are not blue and the new charge of α is at least $4 + 1 - 2 \cdot 2 = 1$.
7. Let v_1 be black and bad. Then the big faces adjacent to α send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge). The new charge of α is at least $4 + 2 - 2 - 4 \cdot 1 = 0$.
8. Let no black and bad vertex be incident with α . Then the big faces adjacent to α send together at least 1 unit of charge. The new charge of α is at least $4 + 1 - 5 = 0$.

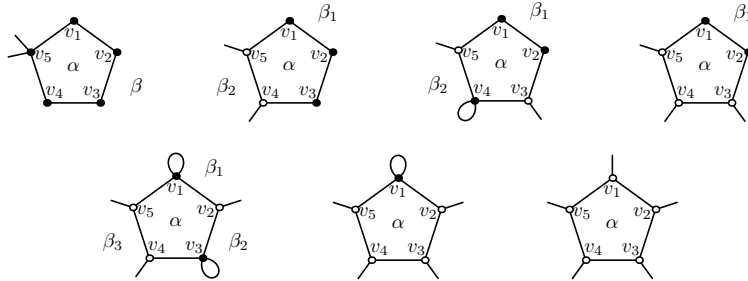


Figure 11: Different possible neighbourhoods of a 5-face α .

3.3.7 Small faces of size at least 6

Let α be a d -face, $6 \leq d \leq 44$. Its initial charge is $2d - 6$. Let v_1, \dots, v_d be the vertices incident with α . (They do not have to be pairwise different.) Consider the black vertices incident with α . Let v_i be a black 4-vertex. It cannot be good for α , since no two small faces are adjacent (see Claim 3.5). Therefore, each black 4-vertex is bad for α . By Claim 3.10 at most $d - 2$ vertices incident with α are bad. Let $k \leq d - 2$ be the number of black vertices incident with α . We can divide the facial walk of α into $d - k \geq 2$ parts, each beginning and ending in a blue or white vertex, each incident with α and a big face β_i ,

$i \in \{1, \dots, d - k\}$. Each of these big faces sends at least 1 unit of charge to α . (If $\beta_i = \beta_j$ for some $1 \leq i < j \leq d - k$, the face β_i cannot be adjacent to another small face but α , therefore, it sends 3 units to α , which is even more than what two different big faces would send.)

The face α then sends 2 units of charge to each of the k incident black vertices, and at most 1 unit of charge to each of the other incident vertices. Together, the new charge of α is at least

$$2d - 6 + (d - k) \cdot 1 - k \cdot 2 - (d - k) \cdot 1 = 2(d - k) - 6.$$

If $d - k \geq 3$, the new charge of α is non-negative.

Let $d - k = 2$. It means there are only two vertices which are not black. Since $d \geq 6$, at least one big face β shares at least 2 black vertices with α , say v_1 and v_2 . By Claim 3.10 at least one from v_1 and v_2 is a 2-vertex, hence, by Claim 3.14 the face β sends 3 units of charge to α . The new charge of α is therefore at least

$$2d - 6 + 3 + 1 - (d - 2) \cdot 2 - 2 \cdot 1 = 0.$$

3.3.8 Big faces

Let β be a d -face, $d \geq 45$. Its initial charge is $2d - 6$. It sends 3 units of charge to the small faces it is adjacent to (Rule 1). It sends 2 units of charge to all bad black vertices; 1 unit of charge to all other vertices. Let k be the number of black vertices bad for β . By Claim 3.10, $k \leq \frac{4}{5} \cdot d$. The new charge of β is therefore at least

$$2d - 6 - 3 - k \cdot 2 - (d - k) \cdot 1 = d - k - 9 \geq d - \frac{4d}{5} - 9 = \frac{d}{5} - 9 = \frac{d - 45}{5} \geq 0.$$

The new charge of all elements of the graph is non-negative, but the sum of all the charges is -12 , which is a contradiction. This contradiction implies that the minimum counterexample does not exist.

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