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Automated verification of equivalence properties of cryptographic protocols

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Abstract. Indistinguishability properties are essential in formal verification of cryptographic protocols. They are needed to model anonymity properties, strong versions of confidentiality and resistance against offline guessing attacks, which can be conveniently modeled using process equivalences. We present a procedure to verify equivalence properties for bounded number of sessions of cryptographic protocols. As in the applied pi-calculus, our protocol specification language is parametrized by a first-order sorted term signature and an equational theory which allows formalization of algebraic properties of cryptographic primitives. Our procedure is able to verify observational equivalence for determinate cryptographic protocols. When protocols are not determinate our procedure can be used for both under- and over-approximations, which proved to be successive on examples. Our procedure can handle a large set of cryptographic primitives, namely those which can be modeled by an optimally reducing convergent rewrite system. The procedure is based on a fully abstract modelling of the traces of a bounded number of sessions of the protocols in first-order Horn clauses on which a dedicated resolution procedure is used to decide equivalence properties. Although, we were unable to prove termination of the resolution procedure, the procedure has been implemented in a prototype tool A-KiSs (Active Knowledge in Security Protocols) and has been effectively tested on examples some of which were outside the scope of existing tools, including checking anonymity of an electronic voting protocol.

1 Introduction

Cryptographic protocols are distributed programs which rely on the use of cryptography to secure electronic transactions such as those that arise in electronic commerce and wireless communication. They are also being applied in new domains such as in Internet voting—legally binding political elections in Estonia, Norway and Switzerland offer the possibility for Internet voting in 2011. This has led to increasing demands on the complexity of desired security properties, leading to more complex cryptographic protocols. Given the socio-economic-political consequences and the history of incorrect design of cryptographic protocols, the need for formal proofs of correctness of protocols is of great importance and has been widely recognized. Formal reasoning about cryptographic protocols is challenging as one has to reason against all potentially malicious behavior—all communication between protocol participants is assumed to be under the control of an adversary.

In order to make the task of formal analysis amenable to automation, usually the assumption of back-box cryptography and unbounded computational power on the part of the adversary is made. This adversarial model is often called the Dolev-Yao model as it is derived from the positions that Dolev and Yao took in their seminal paper [34]. It has proved extremely successful, and there are several automated tools [13, 6, 27, 36] that can automatically check trace-properties such as (weak forms of) confidentiality and authentication. While these trace-based properties are certainly important, many crucial security properties can only be expressed in terms of *indistinguishability* (or equivalence). They include strong flavors of confidentiality [14]; resistance to guessing attacks in password based protocols [10]; and anonymity properties in private authentication [3], electronic voting [31, 9], vehicular networks [28, 29] and RFID protocols [5, 18]. More generally, indistinguishability allows to model security by the means of ideal systems, which are correct by construction [4, 30]. Indistinguishability properties of cryptographic protocols are naturally modeled by the means of *observational* and *testing equivalences* in cryptographic extensions of process calculi, e.g., the spi [4] and the applied-pi calculus [2]. While we have good tools for automated verification of trace properties, the situation is different for indistinguishability properties. This paper is an attempt to address this concern.

State-of-the-art. Many results have been obtained in the restricted case of a pure eavesdropper, i.e., a passive adversary: for *static equivalence* many decidability results have been shown [1, 25, 7] and exact [11, 23] and approximate [15] tools exist for a variety of cryptographic primitives. In the case we consider indistinguishability in the presence of an active adversary, who can interact in an arbitrary way with honest participants less results are known. Hüttel [39] showed undecidability of observational equivalence in the spi calculus, even for the finite control fragment, as well as decidability for the finite, i.e., replication-free, fragment of the spi calculus. The decidability result however only holds for a fixed set of cryptographic primitives and does not yield a practical algorithm. Current results [15] allow to approximate observational equivalence for an unbounded number of sessions. However, this approximation does not suffice to conclude for many applications, such as electronic voting protocols. Symbolic bisimulations have also been devised for the spi [17, 16, 45] and applied pi calculus [32, 41] to avoid unbounded branching due to adversary inputs. However, only [32, 45] and [17] yield a decision procedure, but again only approximating observational equivalence. The results of [32] have been further refined to show a decision procedure on a restricted class of *simple* processes [26]. In particular they rely on a procedure deciding the equivalence of constraint systems, introduced by Baudet [10], for the special case of verifying the existence of guessing attacks. Baudet’s procedure allows arbitrary cryptographic primitives that can be modeled as a subterm convergent rewrite systems [1]. An alternate procedure achieving the same goal was proposed by Chevalier and Rusinowitch [21]. However, both procedures are highly non-deterministic and do not yield a reasonable algorithm which could be implemented. Therefore, Cheval *et al.* [19] have designed a new procedure and a prototype tool to decide the equivalence of constraint systems, but only for a fixed set of primitives. Tools have also been implemented for checking testing equivalence [35], open bisimulation [45] and trace equivalence [20] for a bounded number of sessions but only a limited set of primitives. One may note that [20] is the only decision procedure to consider negative tests, i.e., else branches, which are crucial in several case studies [5, 3].

Our contribution. In this paper we introduce a new procedure for verifying equivalence properties for processes specified in a cryptographic process calculus (without replication). The messages in the calculus are modeled as terms equipped with an equational theory, similar to the applied pi calculus. Our main contributions are as follows.

- Our procedure automatically checks for two equivalences \approx_{ct} and \approx_{ft} which over- and under-approximates the standard notion of trace equivalence \approx_t for cryptographic protocols. The relation \approx_{ft} can thus be used to prove protocols correct while the relation \approx_{ct} can thus be used to rule out incorrect protocols.
- Cortier and Delaune have shown in [26] that observational equivalence coincides with \approx_t for the class of *determinate* processes. They also give a decision procedure for a strict sub-class of determinate processes, namely, *simple* processes. We show that the coarser relation \approx_{ct} coincides with \approx_t , and thus our procedure can be used to verify observational equivalence for the whole class of determinate processes.
- A novelty of our procedure is that it is based on a *fully abstract* modeling of symbolic traces for a *bounded* number of sessions in *first-order Horn clauses*. This is in contrast to the constraint-solving techniques employed by Tiu *et al.* [45], Cheval *et al.* [19, 20], Baudet [10] and Chevalier *et al.* [21] for verifying under-approximations of observational equivalence. Techniques based on Horn clauses have been extensively used, e.g., by Blanchet [13], Weidenbach [46] and Goubault [38], in the case of an unbounded number of sessions. Of these tools, only Blanchet [13] can verify an equivalence property, which happens to be an under-approximation of observational equivalence. Horn clause modeling of an unbounded number of sessions of security protocols may allow false attacks. On the other hand, we have proven our modeling of a bounded number of sessions to be precise.
- Our modelling is fully abstract for arbitrary cryptographic primitives that can be modeled as a convergent rewrite system which has the *finite variant property*. Not only this strictly includes the class of primitives that can be modeled as subterm convergent rewrite systems, this allows us to handle a larger class of cryptographic primitives than [45, 19, 20, 10, 21, 13]. For example, this allows us to handle trapdoor commitment as used by Okamoto for electronic voting in [44]. Although we were unable to prove termination of our procedure, we conjecture it to terminate for the class of cryptographic primitives that can be modeled as subterm convergent rewrite systems. Our conjecture is supported by experimental evidence.

- Our procedure is implemented in the AKiSS (Active Knowledge in Security protocols) prototype tool and used it among others to successfully prove anonymity in an electronic voting protocol [37]. For this electronic voting protocol, this is the first automated proof.

2 Preliminaries

2.1 Terms

Let \mathcal{F} be a signature, i.e., a finite set of function symbols and let ar be a function which assigns to each function symbol a natural number. Given a function symbol $f \in \mathcal{F}$, we say $ar(f) \in \mathbb{N}$ is the arity of f . A function symbol of arity 0 is called a *constant*. Given a set of *atoms* \mathcal{A} and a signature \mathcal{F} , we denote by $\mathcal{T}_{\mathcal{F},\mathcal{A}}$ the set of terms build inductively from \mathcal{A} by applying functions symbols in \mathcal{F} . Given sets of atoms $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$, we denote the set $\mathcal{T}_{\mathcal{F}, \cup_{1 \leq i \leq n} \mathcal{A}_i}$ by $\mathcal{T}_{\mathcal{F}, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}$. We assume that we have the following countably infinite pairwise disjoint sets: a set \mathcal{N} of *private names*, \mathcal{M} of *public names*, a set \mathcal{C} of *public channel names*, a set \mathcal{W} of *parameters*, and a set \mathcal{X} of message variables. Intuitively, elements of the set \mathcal{N} represent nonces generated by honest principals of a protocol, elements of \mathcal{M} represent nonces available both to the adversary and to the honest participants and elements of \mathcal{C} represent names of public channel (e.g. the name of a public network). Elements of \mathcal{W} are pointers used by the adversary to refer to messages output by the honest participants in a protocol. We fix an enumeration w_1, w_2, \dots of the elements of \mathcal{W} . We let x, y, z range over \mathcal{X} . We also define the following set of terms:

- **Terms** denotes the set of all terms $\mathcal{T}_{\mathcal{F}, \mathcal{N}, \mathcal{M}, \mathcal{W}, \mathcal{X}}$
- **Messages** denotes the set of *messages* $\mathcal{T}_{\mathcal{F}, \mathcal{N}, \mathcal{M}}$
- **SMessages** denotes the set of *symbolic messages* $\mathcal{T}_{\mathcal{F}, \mathcal{N}, \mathcal{M}, \mathcal{X}}$.

If t is a term, we denote by $vars(t)$ the set of variables appearing in t , by $names(t)$ the set of names (public or private) appearing in t and $st(t)$ the set of all subterms of t . The functions $vars$, $names$ and st are extended as expected to sequences and sets of terms. A *position* is a string of positive natural numbers and ϵ denotes the empty string. The set $pos(t)$ of positions of a term t is defined as usual. If $p \in pos(t)$ then $t|_p$ is the subterm of t at position p .

Example 1. Consider the signature $\mathcal{F} = \{\text{enc}, \text{dec}, \text{pair}, \text{fst}, \text{snd}\}$ where $ar(\text{enc}) = 3$, $ar(\text{dec}) = ar(\text{pair}) = 2$ and $ar(\text{fst}) = ar(\text{snd}) = 1$. The term $t = \text{pair}(\text{enc}(a, k_1, r_1), \text{enc}(b, k_2, r_2))$ models the pair of the encryptions of public names a and b with keys k_1 , resp. k_2 and randomness r_1 , resp. r_2 . The set of positions $pos(t) = \{\epsilon, 1, 11, 12, 13, 2, 21, 22, 23\}$ and $t_\epsilon = t$, $t_1 = \text{enc}(a, k_1, r_1)$ and $t_{23} = r_2$.

Substitutions. A substitution is a partial function $\sigma : \mathcal{W} \cup \mathcal{X} \rightarrow \text{Terms}$. We only consider substitutions which map elements of \mathcal{W} to elements in **Messages**, elements of \mathcal{X} to elements of **SMessages**. The domain of σ shall be denoted by $dom(\sigma)$. For our purposes, we only consider substitutions with finite domains. We let $range(\sigma) = \{\sigma(u) \in \mathcal{T} \mid u \in dom(\sigma)\}$. If $dom(\sigma) = \{u_1, u_2, \dots, u_n\}$ and $t_i = \sigma(u_i)$ for each $1 \leq i \leq n$ then we shall write σ as $\{u_1 \mapsto t_1, \dots, u_n \mapsto t_n\}$. σ is said to be *ground* if $range(\sigma) \subseteq \text{Messages}$. The notation $names(\sigma)$ will denote the set $names(range(\sigma))$. As usual, a substitution extends homomorphically to a function $\text{apply}_\sigma : \text{Terms} \rightarrow \text{Terms}$ obtained by “applying” σ . Given $t \in \text{Terms}$, we denote $\text{apply}_\sigma(t)$ by $t\sigma$. If σ is a substitution and $X \subseteq dom(\sigma)$, we denoted by $\sigma[X]$ the substitution whose domain is restricted to X .

2.2 Rewriting and unification

Two terms s and t are (syntactically) *unifiable* if there exists a substitution σ such that $s\sigma = t\sigma$. We denote by mgu a function which associates to any two unifiable terms s and t a most general unifier σ of s and t such that $\sigma = \sigma[vars(s, t)]$. It is well known [8] that for any two unifiable terms s and t , there is a most general unifier, unique up to variable renaming.

A rewrite system R is a set of rewrite rules of the form $\ell \rightarrow r$ where $\ell, r \in \text{Terms}$, $\text{names}(\ell, r) = \emptyset$ and $\text{vars}(r) \subseteq \text{vars}(\ell)$. A term t can be rewritten in one step to u , denoted $t \rightarrow_R u$, if there exist a position $p \in \text{pos}(t)$, a rule $\ell \rightarrow r$ in R and a substitution σ such that $t|_p = \ell\sigma$ and u is as t where the subterm $t|_p$ is replaced by $r\sigma$. \rightarrow_R^* denotes the transitive and reflexive closure of \rightarrow_R . A rewrite system is said to be confluent if for any t, t_1, t_2 such that $t \rightarrow_R^* t_1$ and $t \rightarrow_R^* t_2$ there exists u such that $t_1 \rightarrow_R^* u$ and $t_2 \rightarrow_R^* u$. A rewrite system is said to be terminating if it does not admit any infinite sequence $t_0 \rightarrow_R t_1 \rightarrow_R t_2 \rightarrow_R \dots$. It is convergent if it is both confluent and terminating. We denote by $t \downarrow_R$ the normal form of a term t . When R is clear from the context or unimportant we simply write $t \downarrow$. Two terms s and t are said to be equal modulo R , written $s =_R t$, if $s \downarrow_R = t \downarrow_R$. Given a substitution σ we denote by $\sigma \downarrow$ a substitution such that $\text{dom}(\sigma \downarrow) = \text{dom}(\sigma)$ and for all $u \in \text{dom}(\sigma)$ $\sigma \downarrow(u) = \sigma(u) \downarrow$.

Example 2. Continuing Example 1, consider the rewrite system $R = \{\text{dec}(\text{enc}(x, y, z), y) \rightarrow x, \text{fst}(\text{pair}(x, y)) \rightarrow x, \text{snd}(\text{pair}(x, y)) \rightarrow y\}$. The first rewrite rule models that a message can be decrypted, provided decryption uses the same key (represented by variable y) as encryption. The two last rules model projection of the first and second component of a pair. Then we have that $t = \text{fst}(\text{pair}(\text{dec}(\text{enc}(a, k, r), k), b)) \rightarrow_R \text{fst}(\text{pair}(a, b)) \rightarrow_R a = t \downarrow_R$.

Given a convergent rewrite system, we now define the notion of complete set of variants, which was introduced by Common-Lundh and Delaune [24].

Definition 1. A set of substitutions $\text{variants}(t_1, \dots, t_k) = \{\sigma_1, \dots, \sigma_n\}$ is called a complete set of variants of t_1, \dots, t_k if for any substitution ω there exist $1 \leq i \leq n$ and a substitution τ such that for all $1 \leq j \leq k$ we have that $\omega[\text{vars}(t_j)] \downarrow = (\sigma_i \downarrow \tau)[\text{vars}(t_j)]$ and $(t_j \omega) \downarrow = (t_j \sigma_i) \downarrow \tau$.

Intuitively the set of variants of t represents a pre-computation such that any instance of t in normal form is *syntactically* equal to an instance of $t \sigma_i \downarrow$ for some i , without the need to apply further rewrite steps. A rewrite system has the *finite variant property* if for any sequence of terms a finite, complete set of variants exist. In [22], Ciobăcă presents an algorithm for computing such complete sets of variants which is correct whenever the rewrite system is *optimally reducing* [43]. Optimally reducing rewrite systems include subterm convergent systems [1] (and hence the classical Dolev Yao theories for encryption, signatures and hash functions), as well as a theory for modeling blind signatures [40]. Moreover, complete sets of variants can be used to perform unification modulo R [22].

Definition 2. A set of substitutions $\text{mgu}_R(\{s_i \stackrel{?}{=} t_i\}_{i \in I}) = \{\sigma_1, \dots, \sigma_n\}$ is called a complete set of unifiers modulo R of the system of equations $\{s_i \stackrel{?}{=} t_i\}_{i \in I}$ if:

1. $\text{dom}(\sigma_j) \subseteq \text{vars}(X)$ (for all $1 \leq j \leq n$),
2. $s_i \sigma_j =_R t_i \sigma_j$ (for all $i \in I$ and for all $1 \leq j \leq n$) and
3. any σ such that $s_i \sigma =_R t_i \sigma$ for all $i \in I$ is such that $\exists j \exists \tau : \sigma[X] =_R (\sigma_j \tau)[X]$.

where $X = \text{vars}(\{s_i, t_i\}_{i \in I})$.

For singleton systems, we also write $\text{mgu}_R(s, t)$ instead of $\text{mgu}_R(\{s, t\})$.

From now on we assume that the rewrite system is convergent and has the finite variant property.

2.3 Frames, deducibility and static equivalence

Recall that we have fixed an enumeration w_1, w_2, \dots of the elements of the set \mathcal{W} . We will use the notion of a *frame* to represent messages which have been recorded by an attacker.

Definition 3. A frame φ is a substitution $\{w_1 \mapsto t_1, \dots, w_n \mapsto t_n\}$ where $t_i \in \text{Messages}$ ($1 \leq i \leq n$).

Note that in our definition, every frame with $|\text{dom}(\varphi)| = n$ has $\text{dom}(\varphi) = \{w_1, \dots, w_n\}$. We denote the set of all frames as **Frames**. The adversary can use the messages learnt from the run of a protocol to construct new messages. This is modeled as the deducibility relation.

Definition 4. Any term in $\mathcal{T}_{\mathcal{F},\mathcal{M},\mathcal{W}}$ is said to be a recipe. We say that a message t is deducible from φ with a recipe r (written as $\varphi \vdash^r t$) if $t \in \mathbf{Messages}$ and $r\varphi =_{\mathbf{R}} t$. We write $\mathbf{Recipes}$ for the set $\mathcal{T}_{\mathcal{F},\mathcal{M},\mathcal{W}}$.

A frame $\varphi' = \{w_1 \mapsto t'_1, \dots, w_m \mapsto t'_m\}$ extends a frame $\varphi = \{w_1 \mapsto t_1, \dots, w_n \mapsto t_n\}$ if $m \geq n$ and if $t'_i = t_i$ for all $1 \leq i \leq n$. It is easy to see that if φ' extends φ and if $\varphi \vdash^r t$ then $\varphi' \vdash^r t$.

Example 3. Assuming the signature and rewrite system of previous examples we let $\varphi\{w_1 \mapsto \mathbf{enc}(s, k, r), w_2 \mapsto k\}$ where $s, k, r \in \mathcal{N}$ are private names. Then we have that $\varphi \vdash^{\mathbf{dec}(w_1, w_2)} s$. Note that $\mathbf{dec}(w_1, k) \notin \mathbf{Recipes}$ as $k \in \mathcal{N}$. If we had that $s \in \mathcal{M}$ we would also have that $\varphi \vdash^s s$ reflecting that public names are always deducible.

We now define *static equivalence* of frames to capture indistinguishability of sequences of messages.

Definition 5. Let $r_1, r_2 \in \mathbf{Recipes}$. A test $r_1 \stackrel{?}{=} r_2$ holds in a frame φ (written $(r_1 = r_2)\varphi$) if $\varphi \vdash^{r_1} t$ and $\varphi \vdash^{r_2} t$ for some t , i.e., r_1 and r_2 are recipes for the same term in φ .

A frame φ_1 is statically included in φ_2 (written $\varphi_1 \sqsubseteq_s \varphi_2$) iff for all $r_1, r_2 \in \mathbf{Recipes}$ we have that $(r_1 = r_2)\varphi_1$ implies $(r_1 = r_2)\varphi_2$. Two frames φ_1 and φ_2 are statically equivalent (written $\varphi_1 \approx_s \varphi_2$) iff $\varphi_1 \sqsubseteq_s \varphi_2$ and $\varphi_2 \sqsubseteq_s \varphi_1$.

Example 4. Let $a, b \in \mathcal{M}$ and $r, k, k' \in \mathcal{N}$. We have that $\{w_1 \mapsto \mathbf{enc}(a, k, r), w_2 \mapsto k\} \not\approx_s \{w_1 \mapsto \mathbf{enc}(b, k, r), w_2 \mapsto k\}$ because the test $(\mathbf{dec}(w_1, w_2) = a)$ distinguishes both frames. However, $\{w_1 \mapsto \mathbf{enc}(a, k, r), w_2 \mapsto k'\} \approx_s \{w_1 \mapsto \mathbf{enc}(b, k, r), w_2 \mapsto k'\}$. Moreover, we have that $\{w_1 \mapsto a, w_2 \mapsto b\} \sqsubseteq_s \{w_1 \mapsto a, w_2 \mapsto a\}$ while $\{w_1 \mapsto a, w_2 \mapsto a\} \not\sqsubseteq_s \{w_1 \mapsto a, w_2 \mapsto b\}$.

3 A cryptographic process calculus

We shall assume that cryptographic protocols are modeled using a simple process calculus which has similarities with the applied pi-calculus [2]. Applied pi-calculus has proved to be useful for specifying and verifying cryptographic protocols; and there are tools that automate verification of protocols in this model [13]. We shall further restrict our attention to the finite, i.e., replication-free fragment of applied pi-calculus. This restriction is important because observational equivalence becomes undecidable with replication [39]. However, even with the restriction, one can model a bounded number of protocol instances.

We define our process calculus in this section. We begin by defining its syntax.

Syntax. Recall that we have fixed a first-order signature \mathcal{F} , a set \mathcal{N} of *private names*, \mathcal{M} of *public names*, a set \mathcal{C} of public channel names, a set \mathcal{W} of *parameters*, and a set \mathcal{X} of message variables (see Section 2). The terms of the set $\mathcal{T}_{\mathcal{F},\mathcal{N},\mathcal{M},\mathcal{W},\mathcal{X}}$ are also identified modulo a fixed subterm convergent rewrite system \mathbf{R} (see Section 2).

We model a bounded number of instances of a cryptographic protocol as a *finite* set of traces. Traces are defined using sequences of *actions* generated by the following grammar (note that here **in** and **out** are fresh symbols not occurring in \mathcal{F}):

$$\begin{array}{ll} a ::= \mathbf{in}(c, x) & \text{receive action} \\ & \mathbf{out}(c, t) \quad \text{send action} \\ & [s \stackrel{?}{=} t] \quad \text{test action} \end{array}$$

where $x \in \mathcal{X}$, $s, t \in \mathbf{SMessages}$, $c \in \mathcal{C}$. A *trace* T is a sequence of actions $T = a_1.a_2.\dots.a_n$. As usual, a receive action $\mathbf{in}(c, x)$ acts as a binding construct for the variable x . We assume the usual definitions of free and bound variables for traces. We also assume that each variable is bound at most once. A trace is *ground* if it does not contain any free variables. The set of ground traces shall be represented as $\mathbf{GndTraces}$. We also assume the usual definition of a name occurring in a trace.

A *process* P is defined to be a set of traces $P = \{T_1, \dots, T_n\}$. We say that a process is ground if all of its traces are ground. We identify traces with singleton processes.

Remark 1. To ease the presentation of our procedure we have not explicitly included the parallel operator, $|$, in the calculus. One could easily include this in the process algebra and generate the corresponding set of traces. In general, the generated set of traces is *trace equivalent* and not *observational equivalent* to the process. However, for *determinate processes*, a class of process that we shall consider in this paper and shall define shortly, trace equivalence coincides with observational equivalence [26]. Similarly, one can also include the non-deterministic choice operator $+$ in the calculus.

Semantics. The semantics of a process is defined using the semantics of its traces. The semantics of a trace is given in terms of a labeled transition system \mathbb{T} . We assume that all interactions between protocol participants are mediated by the adversary. The labeled transition system records the interaction of the protocol participants with the adversary. The set of labels of \mathbb{T} is defined using the set Recipes . Recall that the set Recipes is the set $\mathcal{T}_{\mathcal{F}, \mathcal{M}, \mathcal{W}}$ (see Section 2). The set of labels, Labels , is

$$\text{Labels} = \{ \mathbf{in}(c, r), \mathbf{out}(c), \mathbf{test} \mid r \in \text{Recipes}, c \in \mathcal{C} \}.$$

The labeled transition system \mathbb{T} is a subset of $(\text{GndTraces} \times \text{Frames}) \times \text{Labels} \times (\text{GndTraces} \times \text{Frames})$ and we shall write $(T, \varphi) \xrightarrow{\ell} (T', \varphi')$ whenever $((T, \varphi), \ell, (T', \varphi')) \in \mathbb{T}$. The frame in the transition system is used to record the messages that the protocol participants have sent in the past. The relation $\xrightarrow{\ell}$ is defined as follows:

$$\begin{array}{c} \text{RECEIVE} \frac{\varphi \vdash^r t}{(\mathbf{in}(c, x).T, \varphi) \xrightarrow{\mathbf{in}(c, r)} (T\{x \mapsto t\}, \varphi)} \\ \\ \text{SEND} \frac{}{(\mathbf{out}(c, t).T, \varphi) \xrightarrow{\mathbf{out}(c)} (T, \varphi \cup \{w_{|dom(\varphi)|+1} \mapsto t\})} \\ \\ \text{TEST} \frac{s =_{\mathbf{R}} t}{([s \stackrel{?}{=} t].T, \varphi) \xrightarrow{\mathbf{test}} (T, \varphi)} \end{array}$$

The label $\mathbf{in}(c, r)$ indicates a message sent by the adversary over the channel c and r is the recipe that adversary uses to create this message. The label $\mathbf{out}(c)$ indicates a message sent over the public channel c and transition rule SEND records the message sent in the frame. Finally, the rule TEST is an internal action.

As usual, we shall write $(T_0, \varphi_0) \xrightarrow{\ell_1, \dots, \ell_n} (T_n, \varphi_n)$ when $(T_0, \varphi_0) \xrightarrow{\ell_1} (T_1, \varphi_1) \dots \xrightarrow{\ell_n} (T_n, \varphi_n)$ and we say that $\ell_1 \dots \ell_n$ is a *run* of (T_0, φ_0) . We shall write $(T, \varphi) \xrightarrow{\ell} (T', \varphi')$ when either $(T, \varphi) \xrightarrow{\mathbf{test}^*, \ell, \mathbf{test}^*} (T', \varphi')$ and $\ell \neq \mathbf{test}$ or $(T, \varphi) \xrightarrow{\mathbf{test}^*} (T', \varphi')$ and $\ell = \mathbf{test}$, where \mathbf{test}^* denotes an arbitrary number of \mathbf{test} actions. We write $(T, \varphi) \xrightarrow{\ell_1, \dots, \ell_n} (T_n, \varphi_n)$ when $(T, \varphi) \xrightarrow{\ell_1} (T_1, \varphi_1) \xrightarrow{\ell_2} \dots \xrightarrow{\ell_n} (T_n, \varphi_n)$. If $P = \{T_1, \dots, T_m\}$ is a process, we write $(P, \varphi) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ (resp. $\xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$) if there exists a trace $T \in P$ such that $(T, \varphi) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ (resp. $(T, \varphi) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$).

Process equivalences. In this section we will define different flavors of trace equivalence which will be useful in this paper. We first recall the standard definition of trace equivalence in cryptographic process algebras.

Definition 6. (Trace equivalence) A ground process P is said to be trace-included in a ground process Q (written $P \sqsubseteq_t Q$) if whenever $(P, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T, \varphi)$ then there exist T', φ' such that $(Q, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ and $\varphi \approx_s \varphi'$. Two processes P and Q are trace-equivalent (written $P \approx_t Q$) if $P \sqsubseteq_t Q$ and $Q \sqsubseteq_t P$.

We will also define two other notions of trace equivalence, one coarser and one more fine-grained. The coarser trace equivalence, which we denote by \approx_{ct} is the trace equivalence that can actually be verified by our procedure.

Definition 7. Given ground processes P and Q , we say that $P \sqsubseteq_{ct} Q$ if whenever $(P, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T, \varphi)$ then there exist T', φ' such that $(Q, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ and $\varphi \sqsubseteq_s \varphi'$. We say that $P \approx_{ct} Q$ if $P \sqsubseteq_{ct} Q$ and $Q \sqsubseteq_{ct} P$.

The following example illustrates the difference between \approx_t and \approx_{ct} .

Example 5. Let P and Q be the ground processes defined as follows: $P = \{\mathbf{out}(c, a).\mathbf{out}(c, a)\}$ and $Q = \{\mathbf{out}(c, a).\mathbf{out}(c, a), \mathbf{out}(c, a).\mathbf{out}(c, b)\}$. Clearly $P \sqsubseteq_{ct} Q$. Observe also that $Q \sqsubseteq_{ct} P$. This is because $\{w_1 \mapsto a, w_2 \mapsto b\} \sqsubseteq_s \{w_1 \mapsto a, w_2 \mapsto a\}$. Thus, $P \approx_{ct} Q$. But $P \not\approx_t Q$.

We will however show that these two notions coincide for a class of *determinate processes*. In the context of the applied pi calculus determinate processes were previously studied by Cortier and Delaune in [26].

Definition 8. (Determinate process) We say that a ground process P is determinate if whenever $(P, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T, \varphi)$ and $(P, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ then $\varphi \approx_s \varphi'$.

Intuitively, determinate processes are processes in which the adversary's static knowledge at any instance is completely determined by its past interaction with the protocol participants. The following is immediate from the definition.

Proposition 1. A ground trace, i.e., a ground process consisting of single trace, is determinate.

As already mentioned above, it was demonstrated in [26] that trace equivalence coincides with observational equivalence for determinate processes. We show that \approx_t and \approx_{ct} also coincide for this class of processes.

Theorem 1. If P and Q are ground processes then $P \approx_t Q$ implies $P \approx_{ct} Q$. Furthermore if P and Q are determinate, then $P \approx_{ct} Q$ implies $P \approx_t Q$.

Proof. Let P and Q be determinate processes.

(\Rightarrow) Follows immediately from definition of \approx_t and \approx_{ct} .

(\Leftarrow) We need to show that $P \approx_{ct} Q$ implies $P \approx_t Q$. We proceed by contradiction. Suppose that $P \approx_{ct} Q$ and $P \not\approx_t Q$. We suppose $P \not\sqsubseteq_t Q$ (the case of $Q \not\sqsubseteq_t P$ being symmetric). As $P \not\sqsubseteq_t Q$ we have that there exist $\ell_1, \dots, \ell_n, T, \varphi$, such that $(P, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T, \varphi)$ and

1. either there exist no φ', T' such that $(Q, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$,
2. or for all φ', T' such that $(Q, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T', \varphi')$ we have that $\varphi \not\approx_s \varphi'$.

In the first case, $P \not\approx_{ct} Q$, contradicting our hypothesis. In the second case, as $\varphi \not\approx_s \varphi'$, there exist r, r' such that $(r = r')\varphi$ and $(r \neq r')\varphi'$ (or vice-versa, the other case is symmetric). As $P \sqsubseteq_w Q$, we have that there exist T'', φ'' such that $(Q, \emptyset) \xrightarrow{\ell_1, \dots, \ell_n} (T'', \varphi'')$ and $\varphi \sqsubseteq_s \varphi''$. Hence, we have $(r = r')\varphi''$. As Q is determinate, we have that $\varphi' \approx_s \varphi''$. This yields a contradiction, as $(r \neq r')\varphi'$ and $(r = r')\varphi''$ would imply $\varphi' \not\approx_s \varphi''$. \square

Additionally, we introduce a more fine-grained notion of trace equivalence, denoted \approx_{ft} .

Definition 9. Given ground processes P and Q , we say that $P \sqsubseteq_{ft} Q$ whenever for all trace $T \in P$ there exists a trace $T' \in Q$ such that $T \approx_t T'$. We say that $P \approx_{ft} Q$ if $P \sqsubseteq_{ft} Q$ and $Q \sqsubseteq_{ft} P$.

It follows directly from the definition that $\approx_{ft} \subset \approx_t$. The difference between these two relations is illustrated by the following example.

Example 6. Let P and Q be ground processes defined as follows:

$$\begin{aligned} P &= \{ \mathbf{out}(c, \mathit{enc}(a, k)).\mathbf{out}(c, \mathit{enc}(b, k)).\mathbf{in}(c, x).[x = \mathit{enc}(a, k)]\mathbf{out}(c, k), \\ &\quad \nu k.\mathbf{out}(c, \mathit{enc}(a, k)).\mathbf{out}(c, \mathit{enc}(b, k)).\mathbf{in}(c, x).[x = \mathit{enc}(b, k)]\mathbf{out}(c, k) \} \\ Q &= \{ \mathbf{out}(c, \mathit{enc}(a, k)).\mathbf{out}(c, \mathit{enc}(b, k)).\mathbf{in}(c, x).[x = \mathit{enc}(\mathit{dec}(x, k), k)]\mathbf{out}(c, k) \} \end{aligned}$$

where $k \in \mathcal{N}$ is a private name and a, b are constants. The test $x = \mathit{enc}(\mathit{dec}(x, k), k)$ simply checks whether x is an encryption with key k . It is not difficult to see that $P \approx_t Q$ but $P \not\approx_{ft} Q$.

As already mentioned our procedure is able check \approx_{ct} which coincides with \approx_t when processes are determinate. In the case where processes are not determinate we can use our procedure to check \approx_{ct} and \approx_{ft} in order to over- and under-approximate \approx_t . Indeed, as traces are determinate processes a procedure for checking \approx_{ct} can be used to verify \approx_{ft} .

4 Modeling traces as Horn clauses

Our decision procedure is based on a fully abstract modelling of a trace in first-order Horn clauses. We give the details of this modelling; we start by giving some definitions that we need for defining the predicates used in the logic.

Symbolic labels and symbolic runs. We define the set of *symbolic labels* as

$$\text{SLabels} = \{\mathbf{in}(c, t), \mathbf{out}(c), \mathbf{test} \mid t \in \text{SMessages}, c \in \mathcal{C}\}$$

and the set of *symbolic runs* as the set of finite sequences of symbolic labels (see Figure 1). The empty sequence is denoted by ϵ . We will often be lazy and write (empty space) for ϵ . Intuitively, a symbolic label stands for a set of possible labels, and a symbolic run stands for a set of possible runs of the protocol.

Symbolic Recipes. We assume a set \mathcal{Y} of *recipe variables* disjoint from \mathcal{X} . The set of terms $\mathcal{T}_{\mathcal{F}, \mathcal{M}, \mathcal{W}, \mathcal{Y}}$ shall be called *symbolic recipes* and denoted by SRecipes . We use capital letters X, Y, Z to range over \mathcal{Y} . Intuitively, a symbolic recipe stands for a set of recipes.

We can extend the definition of substitutions to include variables from \mathcal{Y} in its domain. However, we only consider substitutions that map variables in \mathcal{Y} to SRecipes . A ground substitution must map variables in \mathcal{Y} to Recipes . The notion of most general unifiers is extended to symbolic recipes as expected.

Predicates. The predicates used in our modelling and the semantics of the predicates are given in Figure 1. The predicates are interpreted over a triple— a trace T , a frame φ and a substitution σ . We have four kinds of predicates of the logic, all of whom have a symbolic recipe as an argument. Intuitively, the *reachability predicate* r_w says that each run represented by w is possible. The intruder knowledge predicate $k_w(R, t)$ says that whenever a run represented by w happens, the (symbolic) message t can be constructed by the intruder using the (symbolic) recipe R . The identity predicate $i_w(R, R')$ says that whenever the (symbolic) run SR happens, the (symbolic) recipes R and R' are recipes for the same (symbolic) term. The reachable identity predicate $ri_w(R, R')$ is a short form for the conjunction of the predicates r_w and $i_w(R, R')$.

Formulas and statements. We consider first-order formulas built using the above predicates and the usual connectives (conjunction, disjunction, negation, implication, existential and universal quantification). As in the case of predicates, a formula is interpreted over a triple consisting of a trace T , a frame φ and a substitution σ ; and the semantics is defined as expected. For ground formulas we do not need the substitution σ and when a formula f is ground we simply write $(T, \varphi) \models f$ to denote that this formula holds for (T, φ) . If moreover, $\text{dom}(\varphi) = \emptyset$, we simply write $T \models f$ for $(T, \emptyset) \models f$.

We now identify a subset of the formulas, which we shall call statements. Statements will take the form of Horn clauses, and we shall be mainly concerned with them.

Definition 10. A statement is a Horn clause of the form $H \Leftarrow B_1, \dots, B_n$ where:

1. $H \in \{r_{l_1, \dots, l_k}, k_{l_1, \dots, l_k}(R, t), i_{l_1, \dots, l_k}(R, R'), ri_{l_1, \dots, l_k}(R, R')\}$
2. For each $1 \leq i \leq n$, $B_i = k_{l_1, \dots, l_{j_i}}(X_i, t_i)$

for some $l_1, \dots, l_k \in \text{SLabels}$, $t \in \text{SMessages}$, $R, R' \in \text{SRecipes}$, $j_i \leq k$, $t_1, \dots, t_n \in \text{SMessages}$ and $X_1, \dots, X_n \in \mathcal{Y}$. Furthermore X_1, \dots, X_n are distinct variables and if $H = k_{l_1, \dots, l_k}(R, t)$ then $\text{vars}(t) \subseteq \text{vars}(t_1, \dots, t_n)$.

We implicitly assume that in a Horn clause all variables are universally quantified. Hence, all statements are closed formulas.

Symbolic Runs ($\ell \in \text{SLabels}$):

$u, v, w := \epsilon \mid \ell, w$

Predicates ($w \in \text{SRuns}, R \in \text{SRecipes}, t \in \text{SMessages}$):

r_w (Reachability predicate)

$k_w(R, t)$ (Intruder knowledge predicate)

$i_w(R, R')$ (Identity predicate)

$ri_w(R, R')$ (Reachable identity predicate)

Semantics ($\ell_i \in \text{SLabels}, R \in \text{SRecipes}, t \in \text{SMessages}, T \in \text{GndTraces}, \varphi \in \text{Frames}, \sigma$ a ground substitution):

$(T, \varphi_0, \sigma) \models r_{\ell_1, \dots, \ell_i}$ if $(T, \varphi_0) \xrightarrow{L_1} (T_1, \varphi_1) \xrightarrow{L_2} \dots \xrightarrow{L_n} (T_n, \varphi_n)$
such that $\ell_i \sigma =_R L_i \varphi_{i-1}$ for all $1 \leq i \leq n$

$(T, \varphi_0, \sigma) \models k_{\ell_1, \dots, \ell_i}(R, t)$ if when $(T, \varphi_0) \xrightarrow{L_1} (T_1, \varphi_1) \xrightarrow{L_2} \dots \xrightarrow{L_n} (T_n, \varphi_n)$
such that $\ell_i \sigma =_R L_i \varphi_{i-1}$ for all $1 \leq i \leq n$
then $\varphi'_n \vdash^{R\sigma} t\sigma$

$(T, \varphi_0, \sigma) \models i_{\ell_1, \dots, \ell_i}(R, R')$ if there exists t s.t.
 $(T, \varphi_0, \sigma) \models k_{\ell_1, \dots, \ell_i}(R, t)$ and
 $(T, \varphi_0, \sigma) \models k_{\ell_1, \dots, \ell_i}(R', t)$

$(T, \varphi_0, \sigma) \models ri_{\ell_1, \dots, \ell_i}(R, R')$ if $(T, \varphi_0, \sigma) \models r_{\ell_1, \dots, \ell_i}$ and $(T, \varphi_0, \sigma) \models i_{\ell_1, \dots, \ell_i}(R, R')$

Fig. 1: Predicates

4.1 The set of seed statements

As mentioned above, our decision procedure is based on a fully abstract modelling of a trace in first-order Horn clauses. In this section, given a trace T we will give a set of statements $\text{seed}(T)$ which will serve as a starting point for the modelling. We shall also establish that the set of statements $\text{seed}(T)$ is a sound and (partially) complete abstraction of the trace T . In order to formally define $\text{seed}(T)$, we start by fixing some conventions.

Let $T = a_1.a_2.\dots.a_n$ be a ground trace. We assume w.l.o.g. the following naming conventions:

1. if a_i is a receive action then $a_i = \mathbf{in}(c_i, x_i)$.
2. $x_i \neq x_j$ for any $i \neq j$.
3. if a_i is a send action then $a_i = \mathbf{out}(c_i, t_i)$.
4. if a_i is a test actions then $a_i = [s_i \stackrel{?}{=} t_i]$.

Moreover, for each $1 \leq i \leq n$ let $\ell_i \in \text{SLabels}$ be as follows:

$$\ell_i = \begin{cases} \mathbf{in}(c_i, x_i) & \text{if } a_i = \mathbf{in}(c_i, x_i) \\ \mathbf{out}(c_i) & \text{if } a_i = \mathbf{out}(c_i, t_i) \\ \mathbf{test} & \text{if } a_i = [s_i \stackrel{?}{=} t_i] \end{cases} .$$

For each $0 \leq m \leq n$, let the sets $R(m)$, $S(m)$ and $T(m)$ respectively denote the indices of the receive actions, send actions and test actions amongst a_1, \dots, a_m . Formally,

$$\begin{cases} R(m) = \{i \mid 1 \leq i \leq m, a_i = \mathbf{in}(c_i, x_i)\} \\ S(m) = \{i \mid 1 \leq i \leq m, a_i = \mathbf{out}(c_i, t_i)\} \\ T(m) = \{i \mid 1 \leq i \leq m, a_i = [s_i \stackrel{?}{=} t_i]\} \end{cases} .$$

Given a set of public names $\mathcal{M}_0 \subseteq \mathcal{M}$, *set of seed statements* associated to T and \mathcal{M}_0 , denoted $\text{seed}(T, \mathcal{M}_0)$, is defined to be the set of statements given in Figure 2. If $\mathcal{M}_0 = \mathcal{M}$, then $\text{seed}(T, \mathcal{M})$ is said to be the set of seed statements associated to T and in this case we write $\text{seed}(T)$ as a shortcut for $\text{seed}(T, \mathcal{M})$.

$$\begin{aligned}
& \mathbf{r}_{\ell_1 \sigma \tau \downarrow, \dots, \ell_m \sigma \tau \downarrow} \Leftarrow \{ \mathbf{k}_{\ell_1 \sigma \tau \downarrow, \dots, \ell_{j-1} \sigma \tau \downarrow} (X_j, x_j \sigma \tau \downarrow) \}_{j \in R(m)} \\
& \quad \text{for all } 0 \leq m \leq n \\
& \quad \text{for all } \sigma \in \text{mgu}_{\mathbf{R}}(\{s_k = t_k\}_{k \in T(m)}) \\
& \quad \text{for all } \tau \in \text{variants}(\ell_1 \sigma, \dots, \ell_m \sigma) \\
\\
& \mathbf{k}_{\ell_1 \tau \downarrow, \dots, \ell_m \tau \downarrow} (w_{|S(m)|}, t_m \tau \downarrow) \Leftarrow \{ \mathbf{k}_{\ell_1 \tau \downarrow, \dots, \ell_{j-1} \tau \downarrow} (X_j, x_j \tau \downarrow) \}_{j \in R(m)} \\
& \quad \text{for all } m \in S(n) \\
& \quad \text{for all } \tau \in \text{variants}(\ell_1, \dots, \ell_m, t_m) \\
\\
& \mathbf{k}(c, c) \Leftarrow \\
& \quad \text{for all public names } c \in \mathcal{M}_0 \\
\\
& \mathbf{k}_{\ell_1, \dots, \ell_m} (f(Y_1, \dots, Y_k), f(y_1, \dots, y_k) \tau \downarrow) \Leftarrow \{ \mathbf{k}_{\ell_1, \dots, \ell_m} (Y_j, y_j \tau \downarrow) \}_{j \in \{1, \dots, k\}} \\
& \quad \text{for all } 0 \leq m \leq n \\
& \quad \text{for all function symbols } f \text{ of arity } k \\
& \quad \text{for all } \tau \in \text{variants}(f(y_1, \dots, y_k)).
\end{aligned}$$

Fig. 2: Seed statements

Remark 2. Please note that while constructing the set of seed statements, we apply the most general unifier modulo \mathbf{R} to all tests. In addition, we also apply finite variants. This allows us to *get rid* of rewriting in our procedure.

We shortly show that the set of seed statements is a sound and (partially) complete modelling of a trace. However, we need one more definition to state this fact.

Definition 11. Let K be a set of statements. We define $\mathcal{H}(K)$ to be the smallest set of ground terms such that:

$$\begin{aligned}
& \text{SIMPLE CONSEQUENCE} \frac{\sigma \text{ grounding for } f \quad B_1 \sigma \in \mathcal{H}(K) \quad \dots \quad B_n \sigma \in \mathcal{H}(K)}{H \sigma \in \mathcal{H}(K)} \quad f = \left(H \Leftarrow B_1, \dots, B_n \right) \in K \\
& \text{EXTENDK} \frac{\mathbf{k}_u(R, t) \in \mathcal{H}(K)}{\mathbf{k}_{uv}(R, t) \in \mathcal{H}(K)} \quad \text{EXTENDI} \frac{\mathbf{i}_u(R, R') \in \mathcal{H}(K)}{\mathbf{i}_{uv}(R, R') \in \mathcal{H}(K)}
\end{aligned}$$

(Equivalently, $\mathcal{H}(K)$ is the least Herbrand model of $K \cup \{ \mathbf{k}_{\ell_1, \dots, \ell_{n+1}}(X, x) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_n}(X, x) \}_{n \in \mathbb{N}} \cup \{ \mathbf{i}_{\ell_1, \dots, \ell_{n+1}}(X_1, X_2) \Leftarrow \mathbf{i}_{\ell_1, \dots, \ell_n}(X_1, X_2) \}_{n \in \mathbb{N}}.$)

We show that as far as reachability predicates and intruder knowledge predicates are concerned, the set $\text{seed}(T)$ is a complete abstraction of a trace.

Theorem 2. Let T be a ground trace.

- (Soundness.) For any statement $f \in \text{seed}(T) \cup \mathcal{H}(\text{seed}(T))$, $T \models f$.
- (Completeness.) If $(T, \emptyset) \xrightarrow{L_1, \dots, L_m} (S, \varphi)$ then:
 1. $\mathbf{r}_{L_1 \varphi \downarrow, \dots, L_m \varphi \downarrow} \in \mathcal{H}(\text{seed}(T))$.
 2. if $\varphi \vdash^R t$ then $\mathbf{k}_{L_1 \varphi \downarrow, \dots, L_m \varphi \downarrow}(R, t \downarrow) \in \mathcal{H}(\text{seed}(T))$.

Remark 3. Please note that the set $\text{seed}(T)$ is only partially complete in that we have not shown that in Theorem 2 that if $\varphi \vdash^R t$ and $\varphi \vdash^{R'} t$ then $\mathbf{i}_{L_1 \varphi \downarrow, \dots, L_m \varphi \downarrow} \in \mathcal{H}(\text{seed}(T))$.

We will shortly show how the completeness of $\text{seed}(T)$ can be built upon to achieve a) a full abstraction of the trace T and a b) procedure for checking equivalences \approx_{ct} and \sqsubseteq_{ft} .

$$\begin{array}{c}
\text{RESOLUTION} \frac{f \in K, g \in K_{\text{solved}}, \quad f = (H \Leftarrow \text{k}_{uv}(X, t), B_1, \dots, B_n)}{g = (\text{k}_w(R, t') \Leftarrow B_{n+1}, \dots, B_m) \quad \sigma = \text{mgu}(\text{k}_u(X, t), \text{k}_w(R, t')) \quad t \notin \mathcal{X}} \\
K = K \oplus h \text{ where } h = ((H \Leftarrow B_1, \dots, B_m)\sigma) \\
\\
\text{EQUATION} \frac{f, g \in K_{\text{solved}}, \quad f = (\text{k}_u(R, t) \Leftarrow B_1, \dots, B_n) \quad g = (\text{k}_{u'v'}(R', t') \Leftarrow B_{n+1}, \dots, B_m) \quad \sigma = \text{mgu}(\text{k}_u(-, t), \text{k}_{u'}(-, t'))}{K = K \oplus h \text{ where } h = ((i_{u'v'}(R, R') \Leftarrow B_1, \dots, B_m)\sigma)} \\
\\
\text{TEST} \frac{f, g \in K_{\text{solved}}, \quad f = (i_u(R, R') \Leftarrow B_1, \dots, B_n) \quad g = (r_{u'v'} \Leftarrow B_{n+1}, \dots, B_m) \quad \sigma = \text{mgu}(u, u')}{K = K \oplus h \text{ where } h = ((ri_{u'v'}(R, R') \Leftarrow B_1, \dots, B_m)\sigma)}
\end{array}$$

Fig. 3: Saturation rules

5 Procedure for deciding trace equivalence

We shall now show give a procedure for deciding trace equivalence. At a high level, this consists of two steps.

1. A saturation procedure which constructs a set of *simple* statements from the set $\text{seed}(T)$ which we will call *solved* statements. The saturation procedure ensures that the set of solved statements is a complete abstraction of T .
2. Given two processes P and Q , we saturate the set of seed statements for traces of P and Q and then use the solved statements to decide whether P and Q are trace equivalent.

We shall now give the details of the procedure. We start by the saturation procedure.

5.1 Knowledge bases and saturation

The saturation procedure manipulates a set of statements called a knowledge base:

Definition 12. Given a statement $f = H \Leftarrow B_1, \dots, B_n$,

- f is said to be solved if for all $1 \leq i \leq n$, $B_i = \text{k}_{\ell_1, \dots, \ell_{j_i}}(X_i, x_i)$ for some variables $x_i \in \mathcal{X}$, $X_i \in \mathcal{Y}$.
- f is said to be well-formed if whenever it is solved and $H = \text{k}_{\ell_1, \dots, \ell_k}(R, t)$, we have that $t \notin \mathcal{X}$.

A set of well-formed statements is called a knowledge base. If K is a knowledge base, we define $K_{\text{solved}} = \{f \in K \mid f \text{ is solved}\}$ to be the knowledge base restricted to the solved statements.

Given an initial knowledge base K , the saturation procedure produces another knowledge base $\text{sat}(K)$. The saturation procedure proceeds as follows. First new statements are *generated* and then the knowledge base is *updated* with the new statements. This two-step process continues until a fixed-point is achieved. We describe the two steps in the procedure.

Generating new statements. Given a knowledge base K , new statements f are generated by applying the rules in Figure 3.

Update. The first step while updating the knowledge base by f is to convert f into a canonical form:

Definition 13. Given a solved deduction statement f , we define the canonical form of f to be the statement $f\Downarrow$ obtained by first applying Rule **RENAME** below as many times as possible and then applying Rule **REMOVE** below as many times as possible:

$$\text{RENAME} \frac{H \Leftarrow k_u(X, x), k_{uv}(Y, x), B_1, \dots, B_n}{(H \Leftarrow k_u(X, x), B_1, \dots, B_n)\{Y \mapsto X\}}$$

$$\text{REMOVE} \frac{H \Leftarrow k_u(X, x), B_1, \dots, B_n \quad x \notin \text{vars}(H)}{H \Leftarrow B_1, \dots, B_n}$$

For any other type of statement f , the canonical form $f\Downarrow$ is defined to be equal to f .

It is easy to see that any fact f can be converted into a canonical form. After a canonical form has been obtained, we perform another check before $f\Downarrow$ can be added to the knowledge base. Intuitively, this check ensures that we add enough identity predicates in the the knowledge base.

Definition 14. The set of consequences of a knowledge base K , denoted $\mathbf{conseq}(K)$, is the smallest set such that:

$$\text{AXIOM} \frac{}{k_{uv}(R, t) \Leftarrow k_u(R, t), B_1, \dots, B_m \in \mathbf{conseq}(K)}$$

$$\text{RES} \frac{\begin{array}{c} H \Leftarrow B_1, \dots, B_n \in K \quad \sigma \text{ a substitution} \\ B_1\sigma \Leftarrow C_1, \dots, C_m \in \mathbf{conseq}(K) \quad \dots \quad B_n\sigma \Leftarrow C_1, \dots, C_m \in \mathbf{conseq}(K) \end{array}}{H\sigma \Leftarrow C_1, \dots, C_m \in \mathbf{conseq}(K)}$$

Given a knowledge base K and a statement f , the *update of K by f* , denoted $K \oplus f$, is defined to be $K \cup \{f\Downarrow\}$ if the head of f is not of the form $k_{\ell_1, \dots, \ell_k}(R, t)$. Otherwise, let

$$f\Downarrow = k_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, t_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, t_n)$$

and

$$K \oplus f = \begin{cases} K \cup \{f\Downarrow\} & \text{if } f \text{ is solved and for any } R' \text{ and } 1 \leq i_1, i_2, \dots, i_n \leq k \\ & k_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, t_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, t_n) \notin K' \\ K \cup \{i_{\ell_1, \dots, \ell_k}(R, R') \\ \quad \Leftarrow \{k_{\ell_1, \dots, \ell_{i_j}}(X_j, t_j)\}_{j \in \{1, \dots, n\}}\} & \text{if } f \text{ is solved and } R' \text{ and } 1 \leq i_1, i_2, \dots, i_n \leq k \text{ are such that} \\ & k_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, t_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, t_n) \in K' \\ K \cup \{f\Downarrow\} & \text{if } f \text{ is not solved} \end{cases}$$

where $K' = \mathbf{conseq}(K_{\text{solved}})$.

Please note that update is not a function, namely that there may be several R', i_1, \dots, i_n such that $k_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, t_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, t_n) \in \mathbf{conseq}(K_{\text{solved}})$. However, we need to compute only one such R', i_1, \dots, i_n .

Initial knowledge base. One question that naturally arises is what is the initial knowledge base for the saturation procedure. Given a trace T , the initial knowledge base for the saturation procedure is defined as follows.

Definition 15. Given a set of statements S , the initial knowledge base associated to S , denoted $K_i(S)$, is defined to be the empty knowledge base updated by the set S , i.e., $K_i(T) = \emptyset \oplus_{f \in S} f$. If T is a ground trace, we write $K_i(T)$ for $K_i(\text{seed}(T))$.

Please observe that $K_i(T)$ depends on the order in which statements in $\text{seed}(T)$ are updated. The exact order, however, is not important and our results shall hold regardless of the order chosen. The saturation procedure takes $K_i(T)$ as an input and produces a knowledge base $\text{sat}(K_i(T))$. The reason for choosing $K_i(T)$ instead of $\text{seed}(T)$ as the starting point of the saturation procedure is that $\text{seed}(T)$ may not be a knowledge base (recall that a knowledge base is a set of well-formed statements). The set $K_i(T)$ is, however, a knowledge base.

Proposition 2. *Given a trace T , the set $K_i(T)$ is a knowledge base.*

Soundness and completeness of the saturation procedure. We shall now show that the set of solved statements in $\text{sat}(K_i(T))$ is a sound and complete abstraction of a trace T . We need one more definition which extends $\mathcal{H}(K)$ and allows us to establish that $\text{sat}(K_i(T))$ is a complete abstraction of T .

Definition 16. *Let K be a set of statements. We define $\mathcal{H}_e(K)$ to be the smallest set of ground terms such that $\mathcal{H}(K) \subseteq \mathcal{H}_e(K)$ and:*

$$\begin{array}{c}
\text{REFL} \frac{}{i_w(R, R) \in \mathcal{H}_e(K)} \quad \text{SYM} \frac{i_w(R_1, R_2) \in \mathcal{H}_e(K)}{i_w(R_2, R_1) \in \mathcal{H}_e(K)} \quad \text{TRAN} \frac{i_w(R_1, R_2) \in \mathcal{H}_e(K) \quad i_w(R_1, R_3) \in \mathcal{H}_e(K)}{i_w(R_2, R_3) \in \mathcal{H}_e(K)} \\
\text{CONG} \frac{i_w(R_1, R'_1) \in \mathcal{H}_e(K), \dots, i_w(R_n, R'_n) \in \mathcal{H}_e(K) \quad f \in \mathcal{F}, ar(f) = n}{i_w(f(R_1, \dots, R_n), f(R'_1, \dots, R'_n)) \in \mathcal{H}_e(K)} \quad \text{EXTEND} \frac{i_w(R, R') \in \mathcal{H}_e(K)}{i_{uv}(R, R') \in \mathcal{H}_e(K)} \\
\text{EQUATIONAL CONSEQUENCE} \frac{k_w(R, t) \in \mathcal{H}(K) \quad i_w(R, R') \in \mathcal{H}_e(K)}{k_w(R', t) \in \mathcal{H}_e(K)}
\end{array}$$

We have that the set of solved statements produced by the saturation procedure is a sound and complete abstraction of the trace T .

Theorem 3. *Let T be a ground trace and let $K = \text{sat}(K_i(T))$.*

- (Soundness.) *For any statement $f \in K \cup \mathcal{H}_e(K)$, $T \models f$.*
- (Completeness.) *If $(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (S, \varphi)$ then*
 1. $\uparrow_{L_1 \varphi \downarrow, \dots, L_n \varphi \downarrow} \in \mathcal{H}_e(K_{\text{solved}})$.
 2. *if $\varphi \vdash^R t$ then $k_{L_1 \varphi \downarrow, \dots, L_n \varphi \downarrow}(R, t \downarrow) \in \mathcal{H}_e(K_{\text{solved}})$.*
 3. *if $\varphi \vdash^R t$ and $\varphi \vdash^{R'} t$, then $i_{L_1 \varphi \downarrow, \dots, L_n \varphi \downarrow}(R, R') \in \mathcal{H}_e(K_{\text{solved}})$.*

Effectiveness of the saturation procedure. We have shown that the set of solved statements in $\text{sat}(K_i(T))$ form a sound and complete abstraction for the trace T . However this set is infinite and may not be effectively computable. This may be because of following reasons.

- The set $\text{seed}(T)$ for a ground trace T is infinite. Hence the saturation procedure may continue forever. We will, however, shortly show that for the saturation procedure we only need to consider the saturation of the set $K_i(\text{seed}(T, \mathcal{M}_0))$ where \mathcal{M}_0 is the set of public names occurring in T (see Lemma 1). The set $\text{sat}(K_i(T))$ can then be computed from this set. Since the set $K_i(\text{seed}(T, \mathcal{M}_0))$ is finite, this means that all intermediate knowledge bases in the saturation procedure are finite.
- For the update rule, we have to check that given a knowledge base K , term t , labels ℓ_1, \dots, ℓ_k , indices $1 \leq i_1, \dots, i_n, \leq k$, variables $x_1, \dots, x_n \in \mathcal{X}$ and recipe variables $X_1, \dots, X_n \in \mathcal{Y}$, whether

$$\exists R. k_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K_{\text{solved}}).$$

Furthermore, if the check succeeds then we have to compute one such R . We will show that can be achieved if K is finite (see Lemma 2).

- The saturation procedure may itself not terminate even if the initial knowledge base is finite. As pointed out in the Introduction, we conjecture that the saturation procedure terminates, but were unable to show the termination.

The following lemma allows us to compute the $\text{sat}(K_i(T))$ from the set $\text{sat}(K_i(\text{seed}(M_0, T)))$ where M_0 is the set of public names occurring in T .

Lemma 1. *Let T be a trace and $M_T \subseteq \mathcal{M}$ be the public names occurring in T . Let*

$$K_{\mathcal{M}} = \{\{\mathbf{k}(m, m) \Leftarrow\}_{m \in \mathcal{M}} \cup \{\mathbf{i}(m, m) \Leftarrow\}_{m \in \mathcal{M}} \cup \{\mathbf{ri}(m, m) \Leftarrow\}_{m \in \mathcal{M}}\}.$$

Then $\text{sat}(K_i(T)) = \text{sat}(K_i(\text{seed}(M_T, T))) \cup K_{\mathcal{M}}$.

The following lemma implies that the update step terminates if we only have a finite number of statements in the knowledge base.

Lemma 2. *Given a finite set of solved statements K , term t , labels ℓ_1, \dots, ℓ_k , indices $1 \leq i_1, \dots, i_n \leq k$, variables $x_1, \dots, x_n \in \mathcal{X}$ and recipe variables $X_1, \dots, X_n \in \mathcal{Y}$, it is decidable if there is an R such that $\mathbf{k}_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K_{\text{solved}})$. If the answer to the decision procedure is "Yes," then we can compute one such R .*

5.2 Algorithm

Theorem 4 directly yields an algorithm to decide trace inclusion for determinate processes.

Theorem 4. *Let T be a trace and let P be a determinate process. Let K be the set of solved statements from a saturated knowledge base associated to T . Then $T \sqsubseteq_w P$ iff the following tests hold:*

$$\begin{array}{c} \left(\mathbf{r}_{l_1, \dots, l_n} \Leftarrow \{\mathbf{k}_{w_i}(X_i, x_i)\}_{i \in \{1, \dots, m\}} \right) \in K \\ c_1, \dots, c_k \text{ fresh constants} \quad \sigma : \text{vars}(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\} \text{ is a bijection} \\ \mathbf{k}_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K) \text{ for all } i \text{ such that } l_i = \mathbf{in}(t_i) \\ M_i = l_i \text{ if } l_i \in \{\mathbf{test}, \mathbf{out}(-)\} \quad M_i = \mathbf{in}(d_i, R_i) \text{ if } l_i = \mathbf{in}(d_i, t_i) \\ \text{REACHABILITY} \frac{}{(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (T', \varphi)} \end{array}$$

$$\begin{array}{c} \left(\mathbf{ri}_{l_1, \dots, l_n}(R, R') \Leftarrow \{\mathbf{k}_{w_i}(X_i, x_i)\}_{i \in \{1, \dots, m\}} \right) \in K \\ c_1, \dots, c_k \text{ fresh constants} \quad \sigma : \text{vars}(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\} \text{ is a bijection} \\ \mathbf{k}_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K) \text{ for all } i \text{ such that } l_i = \mathbf{in}(t_i) \\ M_i = l_i \text{ if } l_i \in \{\mathbf{test}, \mathbf{out}(-)\} \quad M_i = \mathbf{in}(d_i, R_i) \text{ if } l_i = \mathbf{in}(d_i, t_i) \\ \text{IDENTITY} \frac{}{(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (T', \varphi) \text{ such that } (R\omega = R'\omega)\varphi \text{ where } \omega = \{X_i \mapsto x_i \sigma\}} \end{array}$$

Note that performing the tests requires deciding if, given t , and w , $\mathbf{k}_w(R, t) \in \mathcal{H}(K)$ for some recipe R for a knowledge base K containing only solved statements.

It is easy to see that this is equivalent to checking if $(\mathbf{k}_w(R, t) \Leftarrow) \in \mathbf{conseq}(K)$ and we have already shown that there is an effective procedure for this (which finds an R if such an R exists).

6 Prototype and case studies

We have implemented the previously described procedure in a tool prototype called AKiSS (Active Knowledge in Security protocols). The tool AKiSS is written in OCaml and has a little more than 2000 lines of source code, including code for computing strongly complete sets of finite variants and complete sets of equational unifiers for subterm convergent equational theories.

We used our tool to verify the equivalences of Examples 5 and 6. We also verified strong and real-or-random secrecy on several protocols. Finally, we have been able to automatically prove anonymity of the FOO electronic voting protocol. We briefly discuss these examples below. The tool including the modeling for all examples of this paper are available on the tool's webpage <http://www.lsv.ens-cachan.fr/~ciobaca/akiss/>.

Strong flavors of confidentiality. The *strong secrecy* property has been introduced by Blanchet in [14] and we rephrase it here in our setting. Let $P = \{t_1, \dots, t_n\}$ be a protocol such that x_1, \dots, x_m are the free variables of P . Then, the variables x_1, \dots, x_m are strongly secret if $\{t_1^1, \dots, t_n^1\} \approx_t \{t_1^2, \dots, t_n^2\}$ where $t_i^j = \mathbf{in}(c, x_1^1). \mathbf{in}(x_1^2) \dots \mathbf{in}(c, x_n^1). \mathbf{in}(c, x_m^2). t_i \{x_k \mapsto x_k^j\}_{1 \leq k \leq m}$ for $1 \leq j \leq 2$, $1 \leq i \leq n$ and $x_1^1, x_1^2, \dots, x_n^1, x_n^2$ are fresh variables and c is a channel name. Intuitively, the attacker cannot distinguish the processes using variables x_k^1 and x_k^2 even though the attacker is allowed to choose arbitrary (public) values for these variables. We illustrate this property on a simple variant of the Denning-Sacco protocol introduced by Blanchet in [14]. Informally, the protocol can be described as follows.

$$\begin{aligned} A \rightarrow B &: \text{aenc}(\text{sign}(\text{pair}(\text{pk}(ska), \text{pair}(\text{pk}(skb, k))), ska), \text{pk}(skb)) \\ B \rightarrow A &: \text{enc}(x, b) \end{aligned}$$

A sends to B a fresh symmetric session key k together with A's and B's public keys. He signs this message with his secret key and encrypts it using B's public key. Upon receiving this message, B decrypts it, checks the signature and uses the fresh session key to symmetrically encrypt a secret x . We have used our tool to show that this protocol respects strong secrecy of x (for one session of A and B). A variant of the protocol [14] consists in letting A also send out a secret y encrypted with k changing the first message to

$$A \rightarrow B : \text{pair}(\text{aenc}(\text{sign}(\text{pair}(\text{pk}(ska), \text{pair}(\text{pk}(skb, k))), ska), \text{pk}(skb)), \text{enc}(y, k))$$

In this case the protocol does not respect strong secrecy of x, y because when choosing $x^1 = y^1$ and $x^2 \neq y^2$ the attacker can distinguish both situations by testing the equality of the encryptions of x and y . This attack is again found by our tool. We have also shown that the Needham-Schroeder-Lowe (NSL) [42] protocol guarantees strong secrecy of the nonce generated by the responder. One may also note that we need to explicitly model randomness for asymmetric encryption for this property to hold. Obviously, Lowe's man-in-the-middle attack on the original Needham-Schroeder protocol makes strong secrecy fail, which can again be found by our tool.

Another variant of secrecy is *real-or-random* secrecy for private names. To check real-or-random secrecy of a name n we postfix all traces with either $\mathbf{out}(c, n)$ or $\mathbf{out}(r)$ where r is some fresh private name and check for indistinguishability. We have been able to show that both variants of Blanchet's protocol described above, as well as the NSL protocol guarantee real-or-random secrecy. Real-or-random secrecy can also be used to model resistance to offline guessing attacks in password protocols [10]. We have shown the EKE protocol [12] to respect this property.

Anonymity for electronic voting protocol. We have been able to automatically verify vote privacy in the FOO electronic voting protocol [37]. This is the most interesting example because, as to the best of our knowledge, no tool can handle this automatically. One may note that Delaune *et al.* [33] were able to apply transformations on the processes such that this verification could be automated by ProVerif. However, the soundness of these transformations has never been proven. If no tool could handle this example this is primarily because of the use of a complex cryptographic primitive, namely *blind signatures*. Our modeling of the protocol follows closely the one in [40]. The property we verify is vote privacy as introduced in [40] and subsequently used in [31, 9]. Vote privacy states that it is not possible to distinguish the situation where two honest voters A and B vote 'yes' and 'no', respectively 'no' and 'yes'.

Efficiency. Currently, on a standard laptop the tool requires a few minutes to carry out such verifications. The use of a multi-core server already reduces these timings by about 40%. We expect however that some optimizations of the saturation procedure could diminish these times significantly. Most of the computational

effort goes into the saturation of the traces. Interleaving individual roles of a protocol introduces an exponential blowup on the number of traces and saturations to perform. However, it would be straightforward to parallelize the saturation of these traces which, e.g. on clusters of machines and scale up to larger protocols and more sessions.

7 Conclusion

In this paper we present a novel procedure for verifying equivalence properties for a bounded number of sessions of cryptographic protocols. The procedure has been implemented in a tool which is able to handle examples which are out of the scope of existing tools.

There are several directions for future work. The implementation of the tool should be optimized and we plan to analyze more examples coming from electronic voting, RFID protocols and auction protocols which all have requirements stated in terms of equivalences.

We would also like to extend the procedure to be able to take disequalities into account. On the one hand, disequalities will allow to verify processes with else branches which are important in a number of practical examples. On the other hand, characterizing disequalities in our decision procedure would allow to directly decide trace equivalence based on static equivalence (rather than static inclusion). Another direction would be to extend the procedure to allow AC operators in order to treat protocols based on exclusive or or Diffie-Hellman exponentiations.

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A Proof of Theorem 2: Soundness and completeness of the set of seed Statements

We prove soundness (see Lemma 3 and Proposition 3) and completeness (see Lemma 4) for the set of seed statements.

Lemma 3 (Soundness of the set of seed statements). *Let T be a ground trace. For any statement f in the set of seed statements $\text{seed}(T)$ we have that $T \models f$.*

Proof. We suppose the same naming conventions for T as in the definition of the set of seed statements (see Section 4.1). We prove that for each statement $f \in \text{seed}(T)$ we have that $T \models f$. There are four kinds of seed statements (see Figure 2). We handle them one-by-one.

1. Let m be such that $0 \leq m \leq n$, let σ and τ be substitutions such that $\sigma \in \text{mgu}_R(\{s_k = t_k\}_{k \in T(m)})$ and $\tau \in \text{variants}(l_1\sigma, \dots, l_m\sigma)$. We show that

$$f = \left((r_{\ell_1\sigma\tau\downarrow, \dots, \ell_m\sigma\tau\downarrow} \Leftarrow \{k_{\ell_1\sigma\tau\downarrow, \dots, \ell_{j-1}\sigma\tau\downarrow}(X_j, x_j\sigma\tau\downarrow)\}_{j \in R(m)}) \right)$$

is a statement that is true in T .

Let ω be an arbitrary substitution grounding for f . Assume furthermore that $T \models (k_{\ell_1\sigma\tau\downarrow, \dots, \ell_{j-1}\sigma\tau\downarrow}(X_j, x_j\sigma\tau\downarrow))\omega$ for all $j \in R(m)$. We show that $T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_m\sigma\tau\downarrow})\omega$. In fact we will show a stronger statement. In particular, we to show that

$$T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_p\sigma\tau\downarrow})\omega$$

for all $0 \leq p \leq m$. We proceed by induction on p .

Base case: $p = 0$. We have $(r_{\ell_1\sigma\tau\downarrow, \dots, \ell_p\sigma\tau\downarrow})\omega = r$. and $T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_p\sigma\tau\downarrow})\omega$ trivially.

Inductive case: $p > 0$. We assume that $T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_{p-1}\sigma\tau\downarrow})\omega$ and we show that $T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_p\sigma\tau\downarrow})\omega$ by case analysis on a_p . Before, we do the case analysis, let us first fix some notations.

Let $T_1 = T$ and $\varphi_1 = \varphi$. As $T \models (r_{\ell_1\sigma\tau\downarrow, \dots, \ell_{p-1}\sigma\tau\downarrow})\omega$, we have that there exist L_1, \dots, L_{p-1} such that

$$(T_i, \varphi_i) \xrightarrow{L_i} (T_{i+1}, \varphi_{i+1})$$

and $L_i\varphi_i =_R \ell_i\sigma\tau\downarrow\omega$ for all $1 \leq i < p$, where $T_i = (a_{i+1} \dots a_n)\{x_j \mapsto x_j\sigma\tau\downarrow\}_{j \in R(i-1)}$ and where φ_i extends φ_{i-1} (for all $1 < i \leq p$). We can now do the case analysis.

- (a) if $a_p = \mathbf{out}(c_p, t_p)$, then $\ell_p = \mathbf{out}(c_p)$ by definition. Let $T_{p+1} = (a_{p+1} \dots a_n)\{x_j \mapsto x_j\sigma\tau\downarrow\}_{j \in R(p)}$ and let $\varphi_{p+1} = \varphi_p \cup \{w_{\text{dom}(\varphi_p)+1} \mapsto t_p\sigma\tau\downarrow\omega\}$. Let $L_p = \mathbf{out}(c_p)$. By the definition, we have that

$$(T_p, \varphi_p) \xrightarrow{L_p} (T_{p+1}, \varphi_{p+1}),$$

which is what we wanted to prove.

- (b) if $a_p = [s_p \stackrel{?}{=} t_p]$, then $\ell_p = \mathbf{test}$. Let $T_{p+1} = (a_{p+1} \dots a_n)\{x_j \mapsto x_j\sigma\tau\downarrow\}_{j \in R(p)}$ and let $\varphi_{p+1} = \varphi_p$. As $\sigma \in \text{mgu}_R(\{s_k = t_k\}_{k \in T(m)})$, we have that $s_p\sigma =_R t_p\sigma$ and therefore $s_p\sigma\tau\downarrow\omega =_R t_p\sigma\tau\downarrow\omega$. Hence,

$$(T_p, \varphi_p) \xrightarrow{\mathbf{test}} (T_{p+1}, \varphi_{p+1}),$$

as we wanted to prove.

- (c) if $a_p = \mathbf{in}(c_p, x_p)$, we know that $p \in R(p)$. Let $T_{p+1} = (a_{p+1} \dots a_n)\{x_j \mapsto x_j\sigma\tau\downarrow\}_{j \in R(p)}$ and let $\varphi_{p+1} = \varphi_p$. As $p \in R(p)$, we have that $T \models (k_{\ell_1\sigma\tau\downarrow, \dots, \ell_{p-1}\sigma\tau\downarrow}(X_p, x_p\sigma\tau\downarrow))\omega$ (this is an antecedent of f). Therefore $\varphi_p \vdash^{X_p\omega} x_p\sigma\tau\downarrow\omega$ and, by letting $L_p = \mathbf{in}(c_p, x_p\sigma\tau\downarrow\omega)$, we obtain by the definition of \rightarrow that

$$(T_p, \varphi_p) \xrightarrow{L_p} (T_{p+1}, \varphi_{p+1}),$$

which is what we wanted to prove.

We have shown that $T \models (r_{\ell_1 \sigma \downarrow, \dots, \ell_p \sigma \downarrow})\omega$.

2. Let $m \in S(n)$ and let $\sigma \in \text{variants}(t_m)$. We show that the statement

$$f = \left((k_{\ell_1 \sigma \downarrow, \dots, \ell_m \sigma \downarrow}(w_{|S(m)|}, (t_m \sigma) \downarrow) \Leftarrow \{k_{\ell_1 \sigma \downarrow, \dots, \ell_{j-1} \sigma \downarrow}(X_j, x_j \sigma \downarrow)\}_{j \in R(m)}) \right)$$

is true in T .

Let ω be a substitution grounding for f . We assume that

$$T \models (k_{\ell_1 \sigma \downarrow, \dots, \ell_{j-1} \sigma \downarrow}(X_j, x_j \sigma \downarrow))\omega$$

for all $j \in R(m)$ and we show that $T \models (k_{\ell_1 \sigma \downarrow, \dots, \ell_m \sigma \downarrow}(w_{|S(m)|}, (t_m \sigma) \downarrow))\omega$.

Let $T_i = (a_i \dots a_n)\{x_j \mapsto x_j \sigma\}_{j \in R(i-1)}$ and $\varphi_i = \cup_{1 \leq j \leq |S(i-1)|} \{w_j \mapsto t_{o(j)} \sigma\}$, where $o(j) = \min\{x \mid |S(x)| = j\}$, i.e. $o(j)$ denotes the index of the j th send action.

We distinguish two cases:

(a) if there exist L_1, \dots, L_m such that $(T_1, \varphi_1) \xrightarrow{L_1} (T_2, \varphi_2) \xrightarrow{L_2} \dots \xrightarrow{L_m} (T_{m+1}, \varphi_{m+1})$ such that $L_i \varphi_i =_{\text{R}} l_i \sigma \downarrow \omega$ for all $1 \leq i \leq m$, we have that

$$\varphi_m(w_{|S(m)|}) = t_{o(S(m))} \sigma \omega = t_m \sigma \omega$$

and we have that $\varphi \vdash^{w_{|S(m)|}} t_m \sigma \omega$ and therefore $\varphi \vdash^{w_{|S(m)|}} (t_m \sigma) \downarrow \omega$ which implies that $T \models (k_{\ell_1 \sigma \downarrow, \dots, \ell_m \sigma \downarrow}(w_{|S(m)|}, t_m \sigma \downarrow))\omega$.

(b) otherwise, we trivially have that $T \models k_{\ell_1 \sigma \downarrow, \dots, \ell_m \sigma \downarrow}(w_{|S(m)|}, (t_m \sigma) \downarrow)\omega$.

We have shown that $T \models f$.

3. Let c be a public name. We show that

$$f = \left(k(c, c) \Leftarrow \right)$$

is true in T because $\emptyset \vdash^c c$.

4. Let g be a function symbol of arity k and let $\sigma \in \text{variants}(g(x_1, \dots, x_k))$. We show that the statement

$$f = \left(k(g(X_1, \dots, X_k), g(x_1, \dots, x_k) \sigma \downarrow) \Leftarrow \{k(X_j, x_j \sigma \downarrow)\}_{j \in \{1, \dots, k\}} \right)$$

is true in T .

Let ω be an arbitrary substitution grounding for f . We assume that $T \models k(X_j, x_j \sigma \downarrow)\omega$ for all $1 \leq j \leq k$ and we show that

$$T \models (k(g(X_1, \dots, X_k), g(x_1, \dots, x_k) \sigma \downarrow))\omega.$$

We have that

$$\emptyset \vdash^{X_j \omega} x_j \sigma \downarrow \omega$$

for all $1 \leq j \leq k$ by our hypothesis. But this implies

$$\emptyset \vdash^{g(X_1 \omega, \dots, X_k \omega)} g(x_1 \sigma \downarrow \omega, \dots, x_k \sigma \downarrow \omega) =_{\text{R}} g(x_1, \dots, x_k) \sigma \downarrow \omega$$

which immediately implies that $T \models (k(g(X_1, \dots, X_k), g(x_1, \dots, x_k) \sigma \downarrow))\omega$.

We have shown that $T \models f$.

We have shown for every statement $f \in \text{seed}(T)$ that $T \models f$. □

Proposition 3 (Soundness of $\mathcal{H}()$). *Let T be a ground trace and K be a set of statements such that for all $f \in K$ we have that $T \models f$. Then for all $f \in \mathcal{H}(K)$ we also have that $T \models f$.*

Proof. The proof of this proposition is a straightforward induction on the size of the smallest proof of $f \in \mathcal{H}(K)$.

Base case. The proof of $f \in \mathcal{H}(K)$ is obtained by applying the rule SIMPLE CONSEQUENCE when $n = 0$. We have that $f' = (H \Leftarrow) \in K$ and $f = f'\sigma$ where σ is a substitution grounding for f' . As $f' \in K$, by hypothesis, $T \models f'$. Hence, as all variables in f' are universally quantified, $T \models f'\sigma$.

Inductive case. We proceed by case distinction on the last rule which has been applied.

- SIMPLE CONSEQUENCE: We have that $f' = (H \Leftarrow B_1 \dots B_n) \in K$, σ is a substitution grounding for f' such that $f = H\sigma$ and $B_i\sigma \in \mathcal{H}(K)$ for $1 \leq i \leq n$. As $H \Leftarrow B_1 \dots B_n \in K$ we have by hypothesis that $T \models H \Leftarrow B_1 \dots B_n$ and hence $T \models (H \Leftarrow B_1 \dots B_n)\sigma$. By induction hypothesis we also have that $T \models B_i\sigma$. Hence, we conclude that $T \models H\sigma$.
- EXTENDK: We have that $k_u(R, t) \in \mathcal{H}(K)$. By induction hypothesis $T \models k_u(R, t)$. It follows from the semantics of k that $T \models k_{uv}(R, t)$.
- EXTENDI: Similar to the EXTENDK.

Lemma 4. *Let T and S be traces and let φ be a frame. If $(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (S, \varphi)$ then:*

- (A) $r_{L_1\varphi\downarrow, \dots, L_n\varphi\downarrow} \in \mathcal{H}(\text{seed}(T))$
- (B) if $\varphi \vdash^R t$ then $k_{L_1\varphi\downarrow, \dots, L_n\varphi\downarrow}(R, t\downarrow) \in \mathcal{H}(\text{seed}(T))$

Proof. We prove the two statements by induction on n . We assume that the two statements hold for any index less than n and we prove them for n .

As $(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (S, \varphi)$, we have that:

- there exists ω such that $(L_1\varphi\downarrow, \dots, L_n\varphi\downarrow) = (\ell_1, \dots, \ell_n)\omega$.
- $s_k\omega =_R t_k\omega$ for all $k \in T(n)$.

We prove each of statements in turn:

- (A) As $s_k\omega =_R t_k\omega$ for all $k \in T(n)$, it follows by the definition of mgu_R that there exists $\sigma \in \text{mgu}_R(\{s_k \stackrel{?}{=} t_k\}_{k \in T(n)})$ such that:

- (a) $\text{dom}(\sigma) \subseteq X$,
- (b) $s_k\sigma =_R t_k\sigma$ for all $k \in T(n)$ and
- (c) $\omega[X] =_R (\sigma\pi)[X]$ for some substitution π

where $X = \text{vars}(\{s_k, t_k\}_{k \in T(n)})$.

It follows that $(\ell_1, \dots, \ell_n)\omega\downarrow = (\ell_1, \dots, \ell_n)\sigma\pi\downarrow$ for some substitution π .

By the definition of variants($(\ell_1, \dots, \ell_n)\sigma$), we have that there exists $\tau \in \text{variants}((\ell_1, \dots, \ell_n)\sigma)$ such that $(\ell_1, \dots, \ell_n)\sigma\pi\downarrow = (\ell_1, \dots, \ell_n)\sigma\tau\downarrow\tau'$ for some substitution τ' .

By the definition of the seed knowledge base $\text{seed}(T)$, we have that the statement

$$f = \left(r_{\ell_1\sigma\tau\downarrow, \dots, \ell_n\sigma\tau\downarrow} \Leftarrow k_{\ell_1\sigma\tau\downarrow, \dots, \ell_{j-1}\sigma\tau\downarrow}(X_j, x_j\sigma\tau\downarrow)_{j \in R(n)} \right) \in \text{seed}(T)$$

is in the seed knowledge base $\text{seed}(T)$.

Let τ'' be the substitution that extends τ' by sending X_j to R_j for all $j \in R(n)$ (where R_j are recipes for $x_j\omega$).

We have by the induction hypothesis that each component of the right hand side of $f\tau''$ is in $\mathcal{H}(\text{seed}(T))$. Therefore the head of $f\tau''$ is in $\mathcal{H}(\text{seed}(T))$:

$$r_{\ell_1\sigma\tau\downarrow\tau'', \dots, \ell_n\sigma\tau\downarrow\tau''} = r_{\ell_1\sigma\tau\downarrow\tau', \dots, \ell_n\sigma\tau\downarrow\tau'} \in \mathcal{H}(\text{seed}(T)).$$

- (B) By induction on R , we show that:

$$k_{L_1\varphi\downarrow, \dots, L_n\varphi\downarrow}(R, R\varphi\downarrow) \in \mathcal{H}(\text{seed}(T))$$

- (a) If $R = c$ is a public name, and as the statement $f = \left(\mathbf{k}(c, c) \Leftarrow \right)$ is in the set of seed statements by definition, we have that $\mathbf{k}(R, R\varphi\downarrow) = \mathbf{k}(c, c) \in \mathcal{H}(\text{seed}(T))$ by definition and therefore $\mathbf{k}_{L_1\varphi\downarrow, \dots, L_n\varphi\downarrow}(R, R\varphi\downarrow) \in \mathcal{H}(\text{seed}(T))$ by the EXTENDK rule.
- (b) If $R = f(R_1, \dots, R_k)$, let γ be the substitution of domain $\text{dom}(\gamma) \subseteq \{y_1, \dots, y_k\}$ that maps y_j to $R_j\varphi\downarrow$ for all $1 \leq j \leq k$.

Let $\tau \in \text{variants}(f(y_1, \dots, y_k))$ and τ' be such that $R\varphi\downarrow = (f(y_1, \dots, y_k)\tau)\downarrow\tau'$.
By the definition of the seed knowledge base, we have that the statement

$$g = \left(\mathbf{k}_{\ell_1, \dots, \ell_n}(f(Y_1, \dots, Y_k), f(y_1, \dots, y_k)\tau\downarrow) \Leftarrow \{ \mathbf{k}_{\ell_1, \dots, \ell_n}(Y_j, y_j\tau\downarrow) \}_{j \in \{1, \dots, k\}} \right) \in \text{seed}(T).$$

Let $\tau'' = \omega\downarrow \cup \tau' \cup \{Y_j \mapsto R_j\}_{j \in \{1, \dots, k\}}$. We have that all antecedents of $g\tau''$ are in $\mathcal{H}(\text{seed}(T))$ by the induction hypothesis. Therefore, the head of $g\tau''$ is also in $\mathcal{H}(\text{seed}(T))$.

- (c) If $R = w_j$, let m be the smallest index such that $|S(m)| = j$ (i.e. m is the index of the action a_m that output the content of w_j) and let t_m be the term such that $a_m = \mathbf{out}(c, t_m)$ for some channel c . Let $\tau \in \text{variants}(\ell_1, \dots, \ell_m, t_m)$ and τ' be substitutions such that $(\ell_1, \dots, \ell_m, t_m)\omega\downarrow = (\ell_1, \dots, \ell_m, t_m)\tau\downarrow\tau'$. We have by the definition of the seed knowledge base that the statement

$$h = \left(\mathbf{k}_{\ell_1\tau\downarrow, \dots, \ell_m\tau\downarrow}(w_j, t_m\tau\downarrow) \Leftarrow \{ \mathbf{k}_{\ell_1\tau\downarrow, \dots, \ell_{k-1}\tau\downarrow}(X_k, x_k\tau\downarrow) \}_{k \in R(m)} \right) \in \text{seed}(T).$$

Let R_k be recipes of $x_k\tau\downarrow\tau' =_{\mathbf{R}} x_k\omega$ in the smallest possible prefix of φ . Let $\tau'' = \tau' \cup \{X_k \mapsto R_k\}_{k \in R(m)}$. We have that the antecedents of $h\tau''$ are in $\mathcal{H}(\text{seed}(T))$ by the induction hypothesis. Therefore the head of $h\tau''$:

$$\mathbf{k}_{\ell_1\tau\downarrow\tau'', \dots, \ell_m\tau\downarrow\tau''}(w_j, t_m\tau\downarrow\tau'') = \mathbf{k}_{\ell_1\tau\downarrow\tau', \dots, \ell_m\tau\downarrow\tau'}(w_j, t_m\tau\downarrow\tau') = \mathbf{k}_{\ell_1\omega\downarrow, \dots, \ell_m\omega\downarrow}(w_j, t_m\omega\downarrow) \in \mathcal{H}(\text{seed}(T)).$$

But $\ell_1\omega\downarrow, \dots, \ell_m\omega\downarrow$ is a prefix of w and therefore by the EXTENDK rule $\mathbf{k}_w(w_j, t_m\omega\downarrow) = \mathbf{k}_w(R_j, R_j\varphi\downarrow) \in \mathcal{H}(\text{seed}(T))$, which is what we had to prove.

B Soundness and completeness of saturation: Proof of Theorem 3

We start by showing soundness.

B.1 Soundness of saturation

Soundness is an immediate consequence of Lemma 5, Lemma 6, Lemma 8, Lemma 9 and Lemma 10 proved below.

Lemma 5 (Soundness of canonicalization). *If $T \models f$ then $T \models f\downarrow$.*

Proof. We will show that each canonicalization rule is sound:

1. For the RENAME rule, consider a statement

$$f = \left(H \Leftarrow \mathbf{k}_{t_1, \dots, t_k}(X, x), \mathbf{k}_{t_1, \dots, t_l}(Y, x), B_1, \dots, B_n \right)$$

where $k \leq l$ and we show that if $T \models f$ then $T \models g$ where

$$g = \left((H \Leftarrow \mathbf{k}_{t_1, \dots, t_k}(X, x), B_1, \dots, B_n) \{Y \mapsto X\} \right)$$

Let τ be a grounding substitution for g such that $T \models \mathbf{k}_{t_1, \dots, t_k}(X, x)\{Y \mapsto X\}\tau, B_1\{Y \mapsto X\}\tau, \dots, B_n\{Y \mapsto X\}\tau$. We show that if $T \models f$ then $T \models H\{Y \mapsto X\}\tau$.

Let τ' be a substitution identical to τ , except for $\tau'(Y) = \tau(X)$. We will show that all the antecedents in $f\tau'$ are true in T .

Indeed, $k_{t_1, \dots, t_k}(X, x)\tau' = k_{t_1, \dots, t_k}(X, x)\{Y \mapsto X\}\tau$ holds by hypothesis. As $k \leq l$ and $T \models k_{t_1, \dots, t_k}(X, x)\tau'$, we also have that $T \models k_{t_1, \dots, t_l}(X, x)\tau' = k_{t_1, \dots, t_l}(Y, x)\tau'$. Furthermore $T \models B_1\tau' = B_1\{Y \mapsto X\}\tau, \dots, B_n\tau' = B_n\{Y \mapsto X\}\tau$ by hypothesis. As $T \models f$, and all antecedents of $f\tau'$ are true in T , we obtain that $T \models H\tau'$. But $H\tau' = H\{Y \mapsto X\}\tau$ and therefore we have that $T \models H\{Y \mapsto X\}\tau$. As we have chosen τ arbitrarily, it follows that $T \models g$.

2. For the REMOVE rule, consider a solved statement

$$f = \left(H \Leftarrow k_{t_1, \dots, t_k}(X, x), B_1, \dots, B_n \right)$$

such that the rule RENAME does not apply to f and such that $x \notin \text{vars}(H)$. We show that if $T \models f$ then $T \models g$ where

$$g = \left(H \Leftarrow B_1, \dots, B_n \right)$$

Let τ be an arbitrary substitution such that $T \models B_1\tau, \dots, B_n\tau$. We will show that $T \models H\tau$ and hence $T \models g$.

Let $(T_1, \varphi_1) = (T, \emptyset)$. We distinguish between two cases:

(a) If

$$(T_1, \varphi_1) \xrightarrow{L_1} (T_2, \varphi_2) \xrightarrow{L_2} \dots \xrightarrow{L_k} (T_{k+1}, \varphi_{k+1}) \text{ such that } L_i\varphi_i = t_i\tau \text{ for all } 1 \leq i \leq k$$

, we consider the substitution τ' to be identical to τ except for $\tau'(x) = (X\tau)\varphi_{k+1}$.

As $x \notin \text{vars}(H)$ and because f is solved and the rule RENAME does not apply, we have that $x \notin \text{vars}(B_1, \dots, B_n)$ and therefore $T \models B_1\tau' = B_1\tau, \dots, B_n\tau' = B_n\tau$.

Furthermore, we have that $T \models k_{t_1, \dots, t_k}(X, x)\tau'$ by the definition of k .

As all antecedents of $f\tau'$ are true in T and $T \models f$, it follows that $T \models H\tau'$. But $H\tau = H\tau'$ since $x \notin \text{vars}(H)$ and therefore $T \models H\tau$.

(b) Otherwise, we trivially have that $T \models k_{t_1, \dots, t_k}(X, x)\tau$. We have that all antecedents of $f\tau$ are true in T and therefore, as $T \models f$, it follows that $T \models H\tau$.

We have shown that $T \models g$, therefore the rule REMOVE is sound.

We have shown that both rules for computing the canonical form are sound and therefore $T \models f \Downarrow$ whenever $T \models f$. \square

Lemma 6 (Soundness of the consequence). *If for all $f \in K$ we have that $T \models f$, then for all $f \in \text{conseq}(K)$ we have that $T \models f$.*

Proof. We show that both inference rules are sound.

For the AXIOM rule, soundness follows immediately from the semantics of k .

For the RES rule, let

$$f = \left(H \Leftarrow B_1, \dots, B_n \right)$$

and

$$g_i = \left(B_i\sigma \Leftarrow C_1, \dots, C_m \right)$$

for $1 \leq i \leq n$ be statements such that $T \models f$ and $T \models g_i$ ($1 \leq i \leq n$).

We will show that

$$T \models \left(H\sigma \Leftarrow C_1, \dots, C_m \right)$$

by letting τ be a substitution such that $T \models C_1\tau, \dots, C_m\tau$ and proving that $T \models H\sigma\tau$. Indeed, as $T \models C_1\tau, \dots, C_m\tau$ and as $T \models g_i$ ($1 \leq i \leq n$), we have that $T \models B_i\sigma\tau$ ($1 \leq i \leq n$). But $T \models f$ and therefore $T \models H\sigma\tau$ as well, which is what we had to show.

Lemma 7 (Monotonicity of k). *If $T \models k_u(R, t)$ then $T \models k_{uv}(R, t)$.*

Proof. Immediate by the semantics of k .

Lemma 8 (Soundness of the Resolution saturation rule). *Let f , g and h be defined as in the RESOLUTION rule. If $T \models f$ and $T \models g$ then $T \models h$.*

Proof. We consider the following statements:

$$\begin{aligned} f &= \left(H \Leftarrow k_{\ell_1, \dots, \ell_i}(X, t), B_1, \dots, B_n \right) \\ g &= \left(k_{\ell'_1, \dots, \ell'_j}(R, t') \Leftarrow B_{n+1}, \dots, B_m \right) \\ h &= \left((H \Leftarrow B_1, \dots, B_m) \sigma \right) \end{aligned}$$

with $j \leq i$ and where $\sigma = \text{mgu}(k_{\ell'_1, \dots, \ell'_j}(R, t'), k_{\ell_1, \dots, \ell_i}(X, t))$. We will show that if $T \models f$ and $T \models g$ then $T \models h$.

Indeed, let τ be an arbitrary substitution grounding for h and assume that $T \models B_1 \sigma \tau, \dots, B_m \sigma \tau$. We will show that $T \models H \sigma \tau$.

As $T \models B_{n+1} \sigma \tau, \dots, B_m \sigma \tau$ and because $T \models g$, we have that $T \models \text{mgu}(k_{\ell'_1, \dots, \ell'_j}(R, t') \sigma \tau, B_{n+1}, \dots, B_m) \sigma \tau = k_{\ell_1, \dots, \ell_i}(X, t) \sigma \tau$ by choice of $\sigma = \text{mgu}(k_{\ell'_1, \dots, \ell'_j}(R, t'), k_{\ell_1, \dots, \ell_i}(X, t))$.

As $j \leq i$, it follows by Lemma 7 that $T \models k_{\ell_1, \dots, \ell_i}(X, t) \sigma \tau$ as well. As all antecedents of $f \sigma \tau$ are true in T and because $T \models f$, we have that $T \models H \sigma \tau$.

As τ was chosen arbitrarily, it follows that $T \models h$. \square

Lemma 9 (Soundness of the Equation saturation rule). *Let f , g and h be defined as in the EQUATION rule. If $T \models f$ and $T \models g$ then $T \models h$.*

Proof. We consider the following statements:

$$\begin{aligned} f &= \left(k_u(R, t) \Leftarrow B_1, \dots, B_n \right) \\ g &= \left(k_{u'v'}(R', t') \Leftarrow B_{n+1}, \dots, B_m \right) \\ h &= \left((i_{u'v'}(R, R') \Leftarrow B_1, \dots, B_m) \sigma \right) \end{aligned}$$

where $\sigma = \text{mgu}(k_u(R, t), k_{u'}(R, t'))$.

We will show that if $T \models f$ and $T \models g$ then $T \models h$. Let τ be an arbitrary substitution grounding for h . We assume that $T \models B_1 \sigma \tau, \dots, B_m \sigma \tau$ and we show that $T \models i_{u'v'}(R, R') \sigma \tau$.

As $T \models B_1 \sigma \tau, \dots, B_n \sigma \tau$ and because $T \models f$ we have that $T \models k_u(R, t) \sigma \tau$. But $k_u(R, t) \sigma \tau = k_{u'}(R, t) \sigma \tau$ by choice of $\sigma = \text{mgu}(k_u(R, t), k_{u'}(R, t))$ and therefore $T \models k_{u'}(R, t) \sigma \tau$. By monotonicity of k (Lemma 7) we also have that $T \models k_{u'v'}(R, t) \sigma \tau$.

As $T \models B_{n+1} \sigma \tau, \dots, B_m \sigma \tau$ and because $T \models g$ we also obtain that $T \models k_{u'v'}(R', t') \sigma \tau$. As $T \models k_{u'v'}(R, t) \sigma \tau$ and $T \models k_{u'v'}(R', t') \sigma \tau$, we have by definition that $T \models i_{u'v'}(R, R') \sigma \tau$.

We have shown that the head of $h \tau$ is true in T . As τ was chosen arbitrarily, it follows that h holds in T . \square

Lemma 10 (Soundness of the Test saturation rule). *Let f , g , h be statements as in the TEST saturation rule. If $T \models f$ and $T \models g$ then $T \models h$.*

Proof. We consider the following statements:

$$\begin{aligned} f &= \left(i_u(R, R') \Leftarrow B_1, \dots, B_n \right) \\ g &= \left(r_{u'v'} \Leftarrow B_{n+1}, \dots, B_m \right) \\ h &= \left((ri_{u'v'}(R, R') \Leftarrow B_1, \dots, B_m) \sigma \right) \end{aligned}$$

where $\sigma = \text{mgu}(u, u')$.

Let τ be an arbitrary substitution grounding for h . We assume that $T \models B_1\sigma\tau, \dots, B_m\sigma\tau$ and we show that $T \models \text{ri}_u(R, R')\tau$. Indeed, as $T \models B_1\sigma\tau, \dots, B_n\sigma\tau$ and as $T \models f$, we have that

$$T \models \text{i}_u(R, R')\sigma\tau. \quad (1)$$

As $T \models B_{n+1}\sigma\tau, \dots, B_m\sigma\tau$ and as $T \models g$, we have that

$$T \models \text{r}_{u'v'}\sigma\tau. \quad (2)$$

But $\sigma = \text{mgu}(u, u')$ and therefore $u\sigma\tau = u'\sigma\tau$. Therefore, by Equations (1) and (2), we immediately obtain $T \models \text{ri}_{u'v'}(R, R')\sigma\tau$, which is what we wanted. As τ was chosen arbitrarily, it follows that $T \models h$.

Lemma 11 (Soundness of the update). *If for all $f \in K$ we have that $T \models f$ and if $T \models g$, then for any $f \in (K \oplus g)$ we have that $T \models f$.*

Proof. If $K \oplus g = K \cup \{g\downarrow\}$, we immediately conclude by Lemma 6. Otherwise, it must be that

$$g\downarrow = \left(\text{k}_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow \text{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \text{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \right)$$

for some $R, t, \ell_1, \dots, \ell_k, i_1, \dots, i_n, X_1, \dots, X_n, x_1, \dots, x_n$ and $K \oplus g = K \cup \{h\}$, where

$$h = \left(\text{i}_{\ell_1, \dots, \ell_k}(R, R') \Leftarrow \text{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \text{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \right)$$

and where

$$g' = \left(\text{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \text{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \text{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \right) \in \mathbf{conseq}(K_{\text{solved}}).$$

It is sufficient to show that $T \models h$. As $K_{\text{solved}} \subseteq K$, it immediately follows that $g' \in \mathbf{conseq}(K)$ and, by Lemma 6, $T \models g'$.

We now show that $T \models h$. Let τ be an arbitrary substitution grounding for h such that the antecedents of $h\tau$ are true in T . As the antecedents of $h\tau$ are the same as the antecedents of $g\downarrow\tau$ and those of $g'\tau$, and as $T \models g$ and $T \models g'$ we have that $T \models \text{k}_{\ell_1, \dots, \ell_k}(R, t)\tau$ and $T \models \text{k}_{\ell_1, \dots, \ell_k}(R', t)\tau$.

But this immediately implies that $T \models \text{i}_{\ell_1, \dots, \ell_k}(R, R')\tau$ (the head of $h\tau$). As τ was chosen arbitrarily, it follows that $T \models h$. \square

Proof. Immediate by Lemma 3 to Lemma 11.

B.2 Completeness of saturation

The first two items of the completeness are immediate consequences of Lemma 4 and of Lemma 21 proved below.

The third item follows immediately from the second item, Proposition 5 and Lemma 16, also proved below.

Proposition 4. *Let K be a knowledge base, let $f = (H' \Leftarrow C_1, \dots, C_m)$ be a statement such that $f \in \mathbf{conseq}(K)$ and let τ be a substitution grounding for f such that $C_i\tau \in \mathcal{H}(K)$ for all $1 \leq i \leq n$. Then $H'\tau \in \mathcal{H}(K)$.*

Proof. Induction on the proof tree of $f \in \mathbf{conseq}(K)$.

If the AXIOM rule was used, then we have $H\sigma \in \mathcal{H}(K)$ by the EXTENDK rule.

If the RES rule was used, we have that there exists $(H \Leftarrow B_1, \dots, B_n) \in K$ and a substitution σ such that $H' = H\sigma$ and $B_i\sigma \Leftarrow C_1, \dots, C_m \in \mathbf{conseq}(K)$ ($1 \leq i \leq n$).

By the induction hypothesis, we have that $B_i\sigma\tau \in \mathcal{H}(K)$. As $(H \Leftarrow B_1, \dots, B_n) \in K$, it follows that $H\sigma\tau = H'\tau \in \mathcal{H}(K)$, which is what we had to show.

Definition 17. Let $\mathcal{S}(H, K)$ be the size of the smallest proof tree of $H \in \mathcal{H}(K)$ (defined only when $H \in \mathcal{H}(K)$).

Proposition 5. Let K be a knowledge base. If $k_w(R, t) \in \mathcal{H}_e(K)$ and $i_w(R, R') \in \mathcal{H}_e(K)$, then

$$k_w(R', t) \in \mathcal{H}_e(K).$$

Proof. As $k_w(R, t) \in \mathcal{H}_e(K)$, it follows that there exist R'' such that

$$k_w(R'', t) \in \mathcal{H}(K) \tag{3}$$

and such that $i_w(R, R'') \in \mathcal{H}_e(K)$. But $i_w(R, R') \in \mathcal{H}_e(K)$ and therefore, by the symmetry and transitivity of $i_w(-, -)$, we have that

$$i_w(R'', R') \in \mathcal{H}_e(K). \tag{4}$$

Using Equations 3 and 4 we immediately obtain by the definition of \mathcal{H}_e that $k_w(R', T) \in \mathcal{H}_e(K)$.

Definition 18. We write $w \sqsubseteq w'$ whenever w is a prefix of w' : i.e. there exists ℓ_1, \dots, ℓ_n such that $w' = \ell_1, \dots, \ell_n$ and $w = \ell_1, \dots, \ell_m$ for some $0 \leq m \leq n$.

Proposition 6. If $k_w(R, t) \in \mathcal{H}(K)$ (resp. $i_w(R, S) \in \mathcal{H}(K)$) then there exist a statement $f = (k_{w'}(R', t') \Leftarrow B_1, \dots, B_m) \in K$ (resp. $f = (i_{w'}(R', S') \Leftarrow B_1, \dots, B_m) \in K$) and a substitution σ such that $R'\sigma = R$, $t'\sigma = t$ (resp. $S'\sigma = S$), $w'\sigma \sqsubseteq w$, $B_i\sigma \in \mathcal{H}(K)$ for all $1 \leq i \leq m$ and $\sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K) < \mathcal{S}(k_w(R, t), K)$.

Proof. We prove the proposition by induction on the smallest proof tree of $H = k_w(R, t) \in \mathcal{H}(K)$ (resp. $H = i_w(R, S) \in \mathcal{H}(K)$). We proceed by case distinction on the last proof rule that has been applied.

- SIMPLE CONSEQUENCE: In this case we have that there exist a statement $f = (H' \Leftarrow B_1, \dots, B_m) \in K$ and a substitution σ such that $H'\sigma = H$, $B_i\sigma \in \mathcal{H}(K)$ for all $1 \leq i \leq m$ and $\sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K) + 1 = \mathcal{S}(k_{w'}(R', t')\sigma, K)$. Hence we directly conclude.
- EXTENDK: In this case $H = k_w(R, t)$ and we have that $w = uv$ for some u, v and $k_u(R, t) \in \mathcal{H}(K)$. By induction hypothesis, we have that there exists $f = k_{u'}(R', t') \Leftarrow B_1, \dots, B_m \in K$ and σ such that $R'\sigma = R$, $t'\sigma = t$, $u'\sigma \sqsubseteq u$, $B_i\sigma \in \mathcal{H}(K)$ for all $1 \leq i \leq m$ and $\sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K) < \mathcal{S}(k_w(R, t), K)$. As $u \sqsubseteq w$, we also have that $u'\sigma \sqsubseteq w$. Moreover, $\mathcal{S}(k_w(R, t) \in \mathcal{H}(K)) = \mathcal{S}(k_u(R, t) \in \mathcal{H}(K)) + 1 > \sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K)$ which allows us to conclude.
- EXTENDI: In this case $H = i_w(R, S)$ and we have that $w = uv$ for some u, v and $i_u(R, S) \in \mathcal{H}(K)$. By induction hypothesis, we have that there exists $f = i_{u'}(R', S') \Leftarrow B_1, \dots, B_m \in K$ and σ such that $R'\sigma = R$, $S'\sigma = S$, $u'\sigma \sqsubseteq u$, $B_i\sigma \in \mathcal{H}(K)$ for all $1 \leq i \leq m$ and $\sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K) < \mathcal{S}(k_w(R, t), K)$. As $u \sqsubseteq w$, we also have that $u'\sigma \sqsubseteq w$. Moreover, $\mathcal{S}(i_w(R, S) \in \mathcal{H}(K)) = \mathcal{S}(i_u(R, S) \in \mathcal{H}(K)) + 1 > \sum_{1 \leq i \leq m} \mathcal{S}(B_i\sigma, K)$ which allows us to conclude.

Lemma 12. Let K be a saturated knowledge base, let $f \in K$ be a statement

$$f = (i_w(R, R') \Leftarrow B_1, \dots, B_n)$$

and let σ be a substitution grounding for f such that $B_i\sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$. Then we have that

$$(i_w(R, R'))\sigma \in \mathcal{H}(K_{\text{solved}}).$$

Proof. Let $\mathcal{G} = \sum_{i \in \{1, \dots, n\}} \mathcal{S}(B_i\sigma, K_{\text{solved}})$. We prove the lemma by induction on \mathcal{G} . If f is a solved statement, the conclusion is immediate by the definition of \mathcal{H} .

Otherwise, if f is not a solved statement, there exists some B_j ($1 \leq j \leq n$) such that $B_j = k_{w_j}(X_j, t_j)$ and $t_j \notin \mathcal{X}$.

As $B_j\sigma \in \mathcal{H}(K_{\text{solved}})$, it follows by Proposition 6 that $w_j = u_jv_j$ for some u_j, v_j and that there exists

$$g = \left(k_{u'_j}(R'_j, t'_j) \Leftarrow B_{n+1}, \dots, B_m \right) \in K_{\text{solved}}$$

and a substitution σ' grounding for g such that $B_{n+1}\sigma', \dots, B_m\sigma' \in \mathcal{H}(K_{\text{solved}})$, $R'_j\sigma' = X_j\sigma$, $t'_j\sigma' = t_j\sigma$, $u'_j\sigma' = u_j\sigma$ and $\mathcal{S}(B_j\sigma) = 1 + \sum_{i \in \{n+1, \dots, m\}} \mathcal{S}(B_i\sigma')$.

As $\omega = \sigma \cup \sigma'$ is a unifier of $H = k_{u'_j}(R'_j, t'_j)$ and $k_{u_j}(X_j, t_j)$, it follows that the two terms are unifiable. Let $\tau = \text{mgu}(H, k_{u_j}(X_j, t_j))$ denote their most general unifier. As K is saturated, it follows that the RESOLUTION saturation rule was applied to f and g and therefore the resulting equational statement

$$h = \left(i_w(R, R') \Leftarrow B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_m \right) \tau \in K$$

must be in K (by the update function, equational statements are added to the knowledge base).

As ω is a unifier of H and B_j and as $\tau = \text{mgu}(H, k_{u_j}(X_j, t_j))$, it follows that there exists ω' such that $\omega = \tau\omega'$. We have that ω' is a substitution grounding for h , that

$$B_i\tau\omega' \in \mathcal{H}(K_{\text{solved}})$$

for $i \in \{1, \dots, j-1, j+1, \dots, m\}$ and that $\sum_{i \in \{1, \dots, j-1, j+1, \dots, m\}} \mathcal{S}(B_i\tau\omega') = \mathcal{G} - 1$.

Therefore we can apply the induction hypothesis to h and ω' and conclude.

Lemma 13. *Let K be a saturated knowledge base, let $f \in K$ be a statement*

$$f = \left(ri_w(R, R') \Leftarrow B_1, \dots, B_n \right)$$

and let σ be a substitution grounding for f such that $B_i\sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$.

Then we have that

$$(ri_w(R, R'))\sigma \in \mathcal{H}(K_{\text{solved}}).$$

Proof. Identical to Lemma 12.

Lemma 14. *Let K be a saturated knowledge base, let $f \in K$ be a statement*

$$f = \left(r_w \Leftarrow B_1, \dots, B_n \right)$$

and let σ be a substitution grounding for f such that $B_i\sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$.

Then we have that

$$r_w\sigma \in \mathcal{H}(K_{\text{solved}}).$$

Proof. Identical to Lemma 12 and Lemma 13.

Lemma 15. *Let K be a saturated knowledge base. If $r_u \in \mathcal{H}(K_{\text{solved}})$ and $i_u(R, R') \in \mathcal{H}(K_{\text{solved}})$, then $ri_u(R, R') \in \mathcal{H}(K_{\text{solved}})$.*

Proof. As $r_u \in \mathcal{H}(K_{\text{solved}})$, there exists a solved statement $f = \left(r_v \Leftarrow B_1, \dots, B_n \right) \in K_{\text{solved}}$ and a substitution σ grounding for f such that $B_i\sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$ and such that $u = v\sigma$.

As $i_u(R, R') \in \mathcal{H}(K_{\text{solved}})$, there exists by Proposition 6 a solved statement $g = \left(i_w(T, T') \Leftarrow B_{n+1}, \dots, B_m \right)$ and a substitution τ grounding for g such that $B_i\tau \in \mathcal{H}(K_{\text{solved}})$ for all $n+1 \leq i \leq m$ and such that $u \sqsupseteq w\tau$, $R = T\tau$ and $R' = T'\tau$.

As $v\sigma = u \sqsupseteq w\tau$, it follows that $v = v_0v_1$ such that v_0 and w are unifiable ($\sigma \cup \tau$ is such a unifier). Let $\omega = \text{mgu}(v_0, w)$ and let π be such that $\sigma \cup \tau = \omega \circ \pi$.

As the knowledge base is saturated, the TEST saturation rule must have fired for f and g and therefore K must have been updated by h where

$$h = \left((ri_v(T, T') \Leftarrow B_1, \dots, B_m) \omega \right).$$

But as h is not a deduction fact, the update must have simply added h to K and therefore $h \in K$.

We have that $B_i \omega \pi = B_i \sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$ and that $B_i \omega \pi = B_i \tau \in \mathcal{H}(K_{\text{solved}})$ for all $n+1 \leq i \leq m$. By applying Lemma 13 to the statement h and the substitution π , we obtain that $ri_v(T, T') \omega \pi = ri_u(R, R') \in \mathcal{H}(K_{\text{solved}})$. \square

Lemma 16. *Let K be a saturated knowledge base such that $k_u(R, t) \in \mathcal{H}(K_{\text{solved}})$ and $k_{uv}(R', t) \in \mathcal{H}(K_{\text{solved}})$. Then we have that $i_{uv}(R, R') \in \mathcal{H}(K_{\text{solved}})$.*

Proof. Let $u = \ell_1, \dots, \ell_k$ and $v = \ell_{k+1}, \dots, \ell_l$. As $k_u(R, t) \in \mathcal{H}(K_{\text{solved}})$, it follows by Proposition 6 that there exist

$$f = \left(k_w(S, s) \Leftarrow B_1, \dots, B_n \right) \in K_{\text{solved}}$$

and a substitution σ grounding for f such that $B_i \sigma \in \mathcal{H}(K_{\text{solved}})$ ($1 \leq i \leq n$) and $k_w(S, s) \sigma = k_{u'}(R, t)$ for some $u' \sqsubseteq u$ a prefix of u .

Similarly, as $k_{uv}(R', t) \in \mathcal{H}(K_{\text{solved}})$, it follows that there exist

$$f' = \left(k_{w'}(S', s') \Leftarrow B'_1, \dots, B'_m \right) \in K_{\text{solved}}$$

and a substitution σ' grounding for f' such that $B'_i \sigma' \in \mathcal{H}(K_{\text{solved}})$ ($1 \leq i \leq m$) and $k_{w'}(S', s') \sigma' = k_{u''}(R', t)$ for $u'' \sqsubseteq uv$ a prefix of uv .

We have that $w \sigma \sqsubseteq u$, which trivially implies $w \sigma \sqsubseteq uv$. We also have $w' \sigma' \sqsubseteq uv$. Let $w = \ell'_1, \dots, \ell'_p$ and $w' = \ell''_1, \dots, \ell''_q$ and let $r = \max\{p, q\}$. We have that $(\ell'_1, \dots, \ell'_r) \sigma = (\ell''_1, \dots, \ell''_r) \sigma'$.

We have that $\sigma \cup \sigma'$ is a unifier of $k_{\ell'_1, \dots, \ell'_r}(-, s)$ and $k_{\ell''_1, \dots, \ell''_r}(-, s')$, it follows that $\tau = \text{mgu}(k_{\ell'_1, \dots, \ell'_r}(-, s), k_{\ell''_1, \dots, \ell''_r}(-, s'))$ exists. As K is saturated, it follows that the equational fact

$$h = \left(i_{\ell'_1, \dots, \ell'_r}(S, S') \Leftarrow B_1, \dots, B_n, B'_1, \dots, B'_m \right) \tau \in K$$

resulting from applying the EQUATION saturation rule to f and f' is in K .

As $\sigma \cup \sigma'$ is a unifier of $k_{\ell'_1, \dots, \ell'_r}(-, s)$ and $k_{\ell''_1, \dots, \ell''_r}(-, s')$ and as $\tau = \text{mgu}(k_{\ell'_1, \dots, \ell'_r}(-, s), k_{\ell''_1, \dots, \ell''_r}(-, s'))$, it follows that there exists ω such that $\sigma \cup \sigma' = \tau \omega$.

We have that ω is grounding for h and that $B_1 \tau \omega, \dots, B_n \tau \omega, B'_1 \tau \omega, \dots, B'_m \tau \omega \in \mathcal{H}(K_{\text{solved}})$. Therefore, we have by Lemma 12 that

$$i_{\ell'_1, \dots, \ell'_r}(S, S') \tau \omega = i_{\ell'_1 \sigma, \dots, \ell'_r \sigma}(R, R') \in \mathcal{H}(K_{\text{solved}}).$$

But $(\ell'_1, \dots, \ell'_r) \sigma$ is a prefix of uv and therefore

$$i_{uv}(R, R') \in \mathcal{H}(K_{\text{solved}})$$

by the EXTENDI rule, which is what we wanted to prove.

Lemma 17. *Let K be a saturated knowledge base, let*

$$f = \left(k_w(R, t) \Leftarrow B_1, \dots, B_n \right)$$

be a statement such that $f \Downarrow \in K_{\text{solved}}$ and let σ be a substitution grounding for f such that $B_i \sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$. Then we have that

$$(k_w(R, t)) \sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

Proof. We prove this by induction on the number of canonicalization steps.

If f is already in canonical form, then the conclusion is immediately true by definition of \mathcal{H} .

Otherwise, there must be a canonicalization rule which can be applied to f . We distinguish between two cases:

1. If the RENAME canonicalization rule can be applied, then f must be of the form:

$$f = \left(\mathbf{k}_w(R, t) \Leftarrow \mathbf{k}_u(X, x), \mathbf{k}_{uv}(Y, x), B_3, \dots, B_n \right).$$

Let us consider the statement f' obtained by applying RENAME to f :

$$f' = \left(\mathbf{k}_w(R, t) \Leftarrow \mathbf{k}_u(X, x), B_3, \dots, B_n \right) \{Y \mapsto X\}.$$

By the definition of a statement, Y has at most one occurrence in B_1, \dots, B_n and therefore we have that $(B_1, B_3, \dots, B_n) \{Y \mapsto X\} = (B_1, B_3, \dots, B_n)$. Therefore $(B_1, B_3, \dots, B_n) \{Y \mapsto X\} \sigma = (B_1, B_3, \dots, B_n) \sigma$. We can therefore apply the induction hypothesis on f' and σ to obtain that

$$\mathbf{k}_w(R, t) \{Y \mapsto X\} \sigma \in \mathcal{H}_e(K_{\text{solved}}). \quad (5)$$

But $\mathbf{k}_u(X, x) \sigma \in \mathcal{H}(K_{\text{solved}})$ and $\mathbf{k}_{uv}(Y, x) \sigma \in \mathcal{H}(K_{\text{solved}})$. By Lemma 16, we have that

$$\mathbf{i}_{uv}(X, Y) \sigma \in \mathcal{H}(K_{\text{solved}}). \quad (6)$$

From Equation 5 and Equation 6 and as uv is a prefix of w by the definition of a statement, we conclude by Proposition 5 that

$$\mathbf{k}_w(R, t) \sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

2. If the REMOVE canonicalization rule can be applied, then f must be of the form:

$$f = \left(\mathbf{k}_w(R, t) \Leftarrow \mathbf{k}_u(X, x), B_2, \dots, B_n \right).$$

Let f' be the statement obtained from f by applying REMOVE. We have that

$$f' = \left(\mathbf{k}_w(R, t) \Leftarrow B_2, \dots, B_n \right).$$

By applying the induction hypothesis on f' and σ , we immediately obtain our conclusion:

$$\mathbf{k}_w(R, t) \sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

Lemma 18. *Let K be a saturated knowledge base, let*

$$f = \left(\mathbf{k}_w(R, t) \Leftarrow B_1, \dots, B_n \right)$$

be a statement such that $f \Downarrow = \left(\mathbf{k}_w(R', t) \Leftarrow C_1, \dots, C_m \right)$ for some R', C_1, \dots, C_m and let R'' be a recipe such that

$$g = \left(\mathbf{k}_w(R'', t) \Leftarrow C_1, \dots, C_m \right) \in \mathbf{conseq}(K_{\text{solved}})$$

and such that

$$h = \left(\mathbf{i}_w(R'', R') \Leftarrow C_1, \dots, C_m \right) \in K_{\text{solved}}.$$

Let σ be a substitution grounding for f such that $B_i \sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$. Then we have that

$$(\mathbf{k}_w(R, t)) \sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

Proof. We prove the lemma by induction on the number of steps to reach the canonical form.

If f is already in canonical form we have that $B_1, \dots, B_n = C_1, \dots, C_m$ and, by applying Proposition 4 to g and σ , we have that

$$k_w(R', t)\sigma \in \mathcal{H}(K_{\text{solved}}).$$

Furthermore, as $h \in K_{\text{solved}}$ and as all antecedents $B_1\sigma, \dots, B_n\sigma = C_1\sigma, \dots, C_m\sigma$ of $h\sigma$ are in $\mathcal{H}(K_{\text{solved}})$, we have that

$$i_w(R'', R')\sigma \in \mathcal{H}(K_{\text{solved}}).$$

It immediately follows that

$$k_w(R'', t)\sigma \in \mathcal{H}_e(K_{\text{solved}}),$$

which is what we had to prove.

Otherwise, there must be a canonicalization rule which can be applied to f . We distinguish between two cases:

1. If the RENAME canonicalization rule can be applied, then f must be of the form:

$$f = \left(k_w(R, t) \Leftarrow k_u(X, x), k_{uv}(Y, x), B_3, \dots, B_n \right).$$

Let us consider the statement f' obtained by applying RENAME to f :

$$f' = \left(k_w(R, t) \Leftarrow k_u(X, x), B_3, \dots, B_n \right) \{Y \mapsto X\}.$$

By the definition of a statement, Y has at most one occurrence in B_1, \dots, B_n and therefore we have that $(B_1, B_3, \dots, B_n) \{Y \mapsto X\} = (B_1, B_3, \dots, B_n)$. Therefore $(B_1, B_3, \dots, B_n) \{Y \mapsto X\} \sigma = (B_1, B_3, \dots, B_n) \sigma$. We can therefore apply the induction hypothesis on f' and σ to obtain that

$$k_w(R, t) \{Y \mapsto X\} \sigma \in \mathcal{H}_e(K_{\text{solved}}). \quad (7)$$

But $k_u(X, x)\sigma \in \mathcal{H}(K_{\text{solved}})$ and $k_{uv}(Y, x)\sigma \in \mathcal{H}(K_{\text{solved}})$. By Lemma 16, we have that

$$i_{uv}(X, Y)\sigma \in \mathcal{H}(K_{\text{solved}}). \quad (8)$$

From Equation 7 and Equation 8 and as uv is a prefix of w by the definition of a statement, we conclude by Proposition 5 that

$$k_w(R, t)\sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

2. If the REMOVE canonicalization rule can be applied, then f must be of the form:

$$f = \left(k_w(R, t) \Leftarrow k_u(X, x), B_2, \dots, B_n \right).$$

Let f' be the statement obtained from f by applying REMOVE. We have that

$$f' = \left(k_w(R, t) \Leftarrow B_2, \dots, B_n \right).$$

By applying the induction hypothesis on f' and σ , we immediately obtain our conclusion:

$$k_w(R, t)\sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

Lemma 19. *Let K be a saturated knowledge base, let $f \in K$ be a statement*

$$f = \left(k_w(R, t) \Leftarrow B_1, \dots, B_n \right)$$

and let σ be a substitution grounding for f such that $B_i\sigma \in \mathcal{H}(K_{\text{solved}})$ for all $1 \leq i \leq n$. Then we have that

$$(k_w(R, t))\sigma \in \mathcal{H}_e(K_{\text{solved}}).$$

Proof. Let $\mathcal{G} = \sum_{i \in \{1, \dots, n\}} \mathcal{S}(B_i \sigma, K_{\text{solved}})$. We prove the lemma by induction on \mathcal{G} .

If f is a solved statement, the conclusion is trivial by the definitions of \mathcal{H} , \mathcal{H}_e .

Otherwise, there exists some $B_j = k_{w_j}(X_j, t_j)$ (with $1 \leq j \leq n$) such that $t_j \notin \mathcal{X}$.

As $B_j \sigma \in \mathcal{H}(K_{\text{solved}})$, we have by Proposition 6 that there exist

$$g = \left(k_{u'}(R', t') \Leftarrow B'_1, \dots, B'_m \right) \in K_{\text{solved}},$$

a substitution σ' grounding for g such that $B'_1 \sigma', \dots, B'_m \sigma' \in \mathcal{H}(K_{\text{solved}})$, $k_{u'}(R', t') \sigma' = k_u(X_j, t_j) \sigma$ for some prefix $u \sqsubseteq w_j$ of w_j and $\mathcal{S}(B_j \sigma, K_{\text{solved}}) > \sum_{i \in \{1, \dots, m\}} \mathcal{S}(B'_i \sigma', K_{\text{solved}})$.

As $\sigma \cup \sigma'$ is a unifier of $k_u(X_j, t_j)$ and $k_{u'}(R', t')$, it follows that $\tau = \text{mgu}(k_u(X_j, t_j), k_{u'}(R', t'))$ exists. Let $\sigma \cup \sigma'$ must be an instance of the most general unifier, let ω be a substitution such that $\sigma \cup \sigma' = \tau \omega$.

As K is saturated, it follows that the RESOLUTION saturation rule was applied to f and g . Let h be the resulting statement:

$$h = \left(k_w(R, t) \Leftarrow B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_n, B'_1, \dots, B'_m \right) \tau.$$

We distinguish two cases:

1. if h is not solved we have that $h \in K$ by the update function (as K is saturated). We can therefore apply the induction hypothesis on h and on the substitution ω to immediately conclude.
2. if h is solved, we distinguish two cases:
 - (a) either $h \Downarrow \in K$, in which case we conclude by applying Lemma 17 to h and ω .
 - (b) or $h \Downarrow = \left(k_w(R'', t) \Leftarrow C_1, \dots, C_k \right)$ and

$$h' = \left(k_w(R''', t) \Leftarrow C_1, \dots, C_k \right) \in \mathbf{conseq}(K_{\text{solved}})$$

and

$$h'' = \left(i_w(R''', R'') \Leftarrow C_1, \dots, C_k \right) \in K_{\text{solved}}$$

for some R''' , in which case we conclude by applying Lemma 18.

Proposition 7. *If $k_u(R, t) \in \mathcal{H}_e(K)$, then $k_{uv} \in \mathcal{H}_e(K)$.*

Proof. As $k_u(R, t) \in \mathcal{H}_e(K)$, it follows that $k_u(R', t) \in \mathcal{H}(K)$ and $i_u(R', R) \in \mathcal{H}_e(K)$ for some R' . By the EXTENDK rule, we have that $k_{uv}(R', t) \in \mathcal{H}(K)$ and by the EXTEND rule, we have that $i_{uv}(R', R) \in \mathcal{H}_e(K)$. We conclude by rule EQUATIONAL CONSEQUENCE that $k_{uv}(R, t) \in \mathcal{H}_e(K)$, which is what we had to show.

Lemma 20. *Let K be a saturated knowledge base such that $k_w(R, t) \in \mathcal{H}(K)$. Then*

$$k_w(R, t) \in \mathcal{H}_e(K_{\text{solved}}).$$

Proof. By induction on $\mathcal{S}(k_w(R, t), K)$.

We distinguish two cases:

1. either $k_u(R, t) \in \mathcal{H}(K)$ for some u prefix of w , in which case we obtain by the induction hypothesis that $k_u(R, t) \in \mathcal{H}_e(K_{\text{solved}})$. We conclude by Proposition 7.
2. or there exist

$$f = \left(k_u(S, s) \Leftarrow B_1, \dots, B_n \right) \in K$$

(with $B_i = k_{w_i}(X_i, t_i)$) and a substitution σ grounding for f such that $B_i \sigma \in \mathcal{H}(K)$ for all $1 \leq i \leq n$, $s\sigma = t$, $S\sigma = R$, $u\sigma = w$ and

$$\mathcal{S}(k_w(R, t), K) = 1 + \sum_{i \in \{1, \dots, n\}} \mathcal{S}(B_i \sigma, K).$$

We apply the induction hypothesis on $B_i\sigma$ for all $1 \leq i \leq n$ and we have that $B_i\sigma \in \mathcal{H}_e(K_{\text{solved}})$. By the definition of \mathcal{H}_e it follows that there exist $\{R'_i\}_{1 \leq i \leq n}$ such that

$$k_{w_i\sigma}(R'_i, t_i\sigma) \in \mathcal{H}(K_{\text{solved}})$$

and such that

$$i_{w_i\sigma}(R'_i, X_i\sigma) \in \mathcal{H}_e(K_{\text{solved}}) \text{ for all } 1 \leq i \leq n.$$

As $w_i\sigma$ is a prefix of $u\sigma = w$, we have by rule EXTEND that

$$i_w(R'_i, X_i\sigma) \in \mathcal{H}_e(K_{\text{solved}}) \text{ for all } 1 \leq i \leq n. \quad (9)$$

Let σ' be a substitution defined to be σ except on $\{X_1, \dots, X_n\}$, where σ' maps X_i to R'_i for all $1 \leq i \leq n$. We apply Lemma 19 on f and σ' . We obtain that

$$k_u(S, s)\sigma' = k_w(S\sigma', t) = k_w(S\{X_i \mapsto R'_i\}\sigma, t) \in \mathcal{H}_e(K_{\text{solved}}). \quad (10)$$

Therefore, combining Equation 10 with Equation 9 (i.e. replacing R'_i by $X_i\sigma$), we immediately obtain our conclusion:

$$k_w(S\{X_i \mapsto X_i\sigma\}\sigma, t) = k_w(S\sigma, t) = k_w(R, t) \in \mathcal{H}_e(K_{\text{solved}}).$$

Lemma 21. *Let K be a set of statements and let K' be the saturation of the knowledge base obtained by updating the empty knowledge base by each statement in K .*

Then $\mathcal{H}(K) \subseteq \mathcal{H}_e(K'_{\text{solved}})$

Proof. Let $H \in \mathcal{H}(K)$. We will prove by induction on the proof tree of $H \in \mathcal{H}(K)$ that each node of the tree is in $\mathcal{H}_e(K'_{\text{solved}})$. We distinguish two cases:

1. if $H = k_w(R, t)$ and $k_u(R, t) \in \mathcal{H}(K)$ for some prefix u of w , in which case by the induction hypothesis we have that $k_u(R, t) \in \mathcal{H}_e(K'_{\text{solved}})$ and we conclude by Proposition 7.
2. if $H = i_w(R, R')$ and $i_u(R, R') \in \mathcal{H}(K)$ for some prefix u of w , we have that $i_u(R, R') \in \mathcal{H}_e(K'_{\text{solved}})$ by the induction hypothesis and therefore $i_w(R, R') \in \mathcal{H}_e(K'_{\text{solved}})$ by rule EXTEND.
3. otherwise there is a statement

$$f = (H' \leftarrow B'_1, \dots, B'_n) \in K$$

and a substitution σ grounding for f such that $H = H'\sigma$ and $B'_i\sigma \in \mathcal{H}(K)$.

By the induction hypothesis, we have that $B'_i\sigma \in \mathcal{H}_e(K'_{\text{solved}})$. W.l.o.g. assume that $B'_i = k_{w'_i}(X_i, t'_i)$. As $B'_i\sigma \in \mathcal{H}_e(K'_{\text{solved}})$, we have by definition of \mathcal{H}_e that there exist R'_i such that

$$k_{w'_i\sigma}(R'_i, t'_i\sigma) \in \mathcal{H}(K'_{\text{solved}}), \quad (11)$$

$$i_{w'_i\sigma}(R'_i, X_i\sigma) \in \mathcal{H}_e(K'_{\text{solved}}) \quad (12)$$

for all $1 \leq i \leq n$.

But $w'_i\sigma$ is a prefix of w , where w is such that $H = \text{predicate}_w(\dots)$ with $\text{predicate} \in \{r, i, ri, k\}$. Therefore, by applying the EXTEND rule to Equation (12), we obtain

$$i_w(R'_i, X_i\sigma) \in \mathcal{H}_e(K_{\text{solved}}). \quad (13)$$

Let σ' be the substitution defined to be σ except that it maps X_i to R'_i for all $1 \leq i \leq n$.

We will show that $H'\sigma' \in \mathcal{H}_e(K'_{\text{solved}})$. As K' was updated by f , there are three cases:

- (a) if $f \in K'$, we conclude by Lemma 19, Lemma 12, Lemma 13 or Lemma 14.
- (b) or $f \downarrow \in K'$ and $f \notin K'$, in which case f must be a solved deduction statement. In this case, by Lemma 17, we obtain that $H'\sigma' \in \mathcal{H}_e(K'_{\text{solved}})$.

(c) or $f \Downarrow = (\mathbf{k}_w(R, t) \Leftarrow C_1, \dots, C_m)$ and there exists R' such that

$$(\mathbf{k}_w(R', t) \Leftarrow C_1, \dots, C_m) \in \mathbf{conseq}(K'_{\text{solved}})$$

and such that

$$(i_w(R, R') \Leftarrow C_1, \dots, C_m) \in K'_{\text{solved}}.$$

In this case, we have that $H'\sigma' \in \mathcal{H}_e(K'_{\text{solved}})$ by Lemma 18.

We have shown that $H'\sigma' \in \mathcal{H}_e(K'_{\text{solved}})$. But $H'\sigma' = H'\{X_i \mapsto R'_i\}\sigma$ and therefore $H'\{X_i \mapsto R_i\}\sigma \in \mathcal{H}_e(K'_{\text{solved}})$. By Equation (13), we obtain that $H'\{X_i \mapsto X_i\sigma\}\sigma = H'\sigma \in \mathcal{H}_e(K'_{\text{solved}})$, which is what we had to show.

C Proof of Lemma 1

We let \Rightarrow denote the saturation relation. We let $\Rightarrow^=$ denote the reflexive closure of \Rightarrow .

Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be public names and let

$$K_{\text{names}} = \{\{\mathbf{k}(m, m) \Leftarrow\}_{m \in \mathcal{M}_0} \cup \{i(m, m) \Leftarrow\}_{m \in \mathcal{M}_0} \cup \{\text{ri}(m, m) \Leftarrow\}_{m \in \mathcal{M}_0}\}$$

be a set of statements involving names in \mathcal{M}_0 .

Lemma 22. *Let K be a knowledge base such that $\text{names}(K) \cap \mathcal{M}_0 = \emptyset$. Let $K_1 \subseteq K_{\text{names}}$. If h is a statement such that $\text{names}(h) \cap \mathcal{M}_0 = \emptyset$, then*

$$(K \uplus K_1) \oplus h = (K \oplus h) \uplus K_1.$$

Proof. If h is not solved or if it is not a deduction statement, we have that $(K \uplus K_1) \oplus h = (K \uplus K_1) \cup \{h\} = (K \cup \{h\}) \uplus K_1 = (K \oplus h) \uplus K_1$. If h is a solved deduction statement, let

$$h \Downarrow = \mathbf{k}_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n).$$

We distinguish two cases:

1. either $\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \notin \mathbf{conseq}((K \uplus K_1)_{\text{solved}})$ for any R' , in which case

$$(K \uplus K_1) \oplus h = (K \uplus K_1) \cup \{h \Downarrow\} = (K \cup \{h \Downarrow\}) \uplus K_1.$$

It follows that $\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \notin \mathbf{conseq}(K_{\text{solved}})$ for any R' either (since $K \subseteq K \uplus K_1$). Therefore $K \oplus h = K \cup \{h \Downarrow\}$ and we immediately conclude by replacing $K \cup \{h \Downarrow\}$ by $K \oplus h$ in the equation above.

2. or $\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}((K \uplus K_1)_{\text{solved}})$ for some R' . In this case, $(K \uplus K_1) \oplus h = (K \uplus K_1) \cup \{f\}$ where

$$f = \left(i_{\ell_1, \dots, \ell_k}(R, R') \Leftarrow \{\mathbf{k}_{\ell_1, \dots, \ell_{i_j}}(X_j, x_j)\}_{j \in \{1, \dots, n\}} \right).$$

To conclude we show the following claim.

If $\text{names}(t) \cap \mathcal{M}_0 = \emptyset$ and

$$\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}((K \uplus K_1)_{\text{solved}})$$

then

$$\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K_{\text{solved}})$$

To proof this claim we proceed by induction on the size of the proof tree of

$$\mathbf{k}_{\ell_1, \dots, \ell_k}(R', t) \Leftarrow \mathbf{k}_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, \mathbf{k}_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}((K \uplus K_1)_{\text{solved}}).$$

Base case: we need to consider two cases according to which rule has been applied.

- AXIOM: the rule does not depend on the knowledge base and we trivially conclude.
- RES: we have that $n = 0$, i.e., $H \Leftarrow \in (K \cup K_1)_{\text{solved}}$ and $H\sigma = k_{\ell_1, \dots, \ell_k}(R', t)$. As $\text{names}(t) \cap \mathcal{M}_0 = \emptyset$ we have that $H \Leftarrow \in K_{\text{solved}}$. Hence, $k_{\ell_1, \dots, \ell_k}(R', t) \in \mathbf{conseq}(K_{\text{solved}})$.

Inductive case: We suppose that the proof ends with an application of the RES rule. We have that $H \Leftarrow B_1, \dots, B_m \in (K \cup K_1)_{\text{solved}}$, $B_i\sigma \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}((K \uplus K_1)_{\text{solved}})$ and $H\sigma = k_{\ell_1, \dots, \ell_k}(R', t)$. Let $H = k_u(S, t')$ and $B_i = k_{u_i}(Y_i, y_i)$. As $H\sigma = k_{\ell_1, \dots, \ell_k}(R', t)$ and $\text{names}(t) \cap \mathcal{M}_0 = \emptyset$, by inspection of the statements in K_1 , it must be that $H \Leftarrow B_1, \dots, B_m \in K_{\text{solved}}$. Moreover, as $t'\sigma = t$ we have by hypothesis that $t'\sigma \cap \mathcal{M}_0 = \emptyset$ and hence $t' \cap \mathcal{M}_0 = \emptyset$. As $y_i \in \text{vars}(t')$ we have that $y_i\sigma \cap \mathcal{M}_0 = \emptyset$ and we can apply our induction hypothesis to conclude that $B_i\sigma \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K_{\text{solved}})$ for $1 \leq i \leq n$. Hence, as

$$H \Leftarrow B_1, \dots, B_m \in K_{\text{solved}}$$

and

$$B_i\sigma \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K_{\text{solved}})$$

for $1 \leq i \leq n$ we conclude that $k_{\ell_1, \dots, \ell_k}(R', t) \in \mathbf{conseq}(K)$.

Lemma 23. *If K is a knowledge base such that $\text{names}(K) \cap \mathcal{M}_0 = \emptyset$, $K_1 \subseteq K_{\text{names}}$ and*

$$K \uplus K_1 \Rightarrow K''$$

then $K'' = K' \uplus K_2$ with $K \Rightarrow^= K'$, $K_2 \subseteq K_{\text{names}}$ and $\text{names}(K') \cap \mathcal{M}_0 = \emptyset$.

Proof. We perform a case distinction depending on which saturation rule triggered:

1. if rule RESOLUTION triggered, we will show that $f, g \in K$.
Indeed, no statement $(k(m, m) \Leftarrow) \in K_1$ can play the role of g in the RESOLUTION saturation rule since $t' = m$ must unify with $t \notin X$. Therefore t must be m , but $m \notin \text{names}(K)$ by hypothesis and therefore t cannot be m .
No statement in K_1 can play the role of f in the RESOLUTION saturation rule since they have no antecedents.
Therefore $f, g \in K$ and $\text{names}(h) \notin \mathcal{M}_0$. We choose $K' = K \oplus h$, $K_2 = K_1$ and we conclude by Lemma 22.
2. if rule EQUATION triggered, we distinguish three cases:
 - (a) if a statement $(k(m, m) \Leftarrow) \in K_1$ plays the role of f in the EQUATION saturation rule, we have that $t = m$. As t' unifies with m , we have that either $t' = m$ or that t' is a variable. The second case is not possible since g must be well-formed. Therefore $t' = m$. As $m \notin \text{names}(K)$ by hypothesis it follows that $g \in K_1$ and therefore $g = k(m, m)$. Therefore the resulting statement is $i(m, m)$. We choose $K_2 = K_1 \cup \{i(m, m)\}$, $K' = K$ to conclude.
 - (b) if a statement $(k(m, m) \Leftarrow) \in K_1$ plays the role of g , the reasoning is analogous to the case above
 - (c) otherwise $f, g \in K$. Therefore $\text{names}(h) \cap \mathcal{M}_0 = \emptyset$. We choose $K' = K \oplus h$ and $K_2 = K_1$ to conclude.
3. if rule TEST triggered, we distinguish two cases:
 - (a) if $(i(m, m) \Leftarrow) \in K_1$ plays the role of f , we choose $K' = K$ and $K_2 = K_1 \cup \{ri(m, m)\}$ to conclude.
 - (b) otherwise $f \in K$. The statement g must also be in K since g is a reachability statement and K_1 does not contain reachability statements. We choose $K' = K \oplus h$ and $K_2 = K_1$ to conclude.

From the above lemma we can immediately conclude that if

$$K \cup \{k(m, m)\}_{m \in \mathcal{M} \setminus \text{names}(K)} \Rightarrow^* K'$$

and K' is saturated, then

$$K \Rightarrow^* K''$$

with K'' saturated and $K' = K'' \cup K_{\text{names}}$. This means that there is no need to keep track of all (an infinite number of) names during the saturation process.

D Proof of Lemma 2

The definition of $\mathbf{conseq}(K)$ yields a direct recursive algorithm which moreover computes R :

- (Axiom) Check whether $t = x_j$ for $1 \leq j \leq n$. If this is the case return (yes, X_j).
- (Res) Otherwise, guess a (solved) statement $k_u(R', t') \Leftarrow k_{u_1}(Y_1, y_1), \dots, k_{u_k}(Y_k, y_k) \in K$ and compute substitution σ such that $k_{\ell_1, \dots, \ell_k}(R', t) = k_u(R', t')\sigma$. Check recursively whether

$$\exists R_i. k_{u_i}(R_i, y_i)\sigma \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K)$$

for $1 \leq i \leq k$. In that case return (yes, $R'[Y_i \mapsto R_i]_{1 \leq i \leq n}$). Otherwise return no.

Termination is ensured because the size of t when checking whether

$$\exists R. k_{\ell_1, \dots, \ell_k}(R, t) \Leftarrow k_{\ell_1, \dots, \ell_{i_1}}(X_1, x_1), \dots, k_{\ell_1, \dots, \ell_{i_n}}(X_n, x_n) \in \mathbf{conseq}(K)$$

strictly decreases in each recursive call. Indeed, when $k_u(R', t') \Leftarrow k_{u_1}(Y_1, y_1), \dots, k_{u_k}(Y_k, y_k) \in K$ we have that $t' \notin X$ because it is well-formed and $y_i \in \mathit{vars}(t')$ by definition of a statement. Hence, $|y_i\sigma| < |t'\sigma| = |t|$.

E Proof of the algorithm

In order to prove Theorem 4 we need the following technical lemmas.

Lemma 24. *Let T be a trace and let K be a saturated knowledge base associated to T . Then for any statement $f \in K$, we have that:*

1. *if $f = \left(r_{l_1, \dots, l_n} \leftarrow \{k_{w_i}(X_i, t_i)\}_{i \in \{1, \dots, m\}} \right)$ and $x \in \text{vars}(l_k)$ then there exists $w_j = l_1, \dots, l_{k'}$ with $k' < k$ such that $x \in \text{vars}(t_j)$.*
2. *if $f = \left(k_{l_1, \dots, l_n}(R, t) \leftarrow \{k_{w_i}(X_i, t_i)\}_{i \in \{1, \dots, m\}} \right)$ and $x \in \text{vars}(t)$ then $x \in \text{vars}(t_1, \dots, t_m)$.*

Proof. The seed knowledge base satisfies the above properties and they are preserved by canonicalization, update and saturation.

Lemma 25. *If $\text{dom}(\varphi) \subseteq \text{dom}(\varphi')$ and $\varphi \vdash^r t$, then $\varphi' \vdash^r t$.*

Proof. The same rewrite steps to obtain t from $r\varphi$ are used to obtain t from $r\varphi'$.

Lemma 26. *Let T_0 be a trace, $\varphi_0 = \emptyset$ the empty frame, and $\{c_1, \dots, c_k\}$ names such that $c_i \notin \text{names}(T_0)$ for all $1 \leq i \leq k$.*

If

$$(T_0, \varphi_0) \xrightarrow{L_1} (T_1, \varphi_1) \xrightarrow{L_2} \dots \xrightarrow{L_n} (T_n, \varphi_n)$$

and $\forall 1 \leq i \leq k$

- *either $c_i \notin \text{names}(L_1, \dots, L_n)$*
- *or $\varphi_{\text{idx}(c_i)-1} \vdash^{R_i} t_i$ for some t_i where $\text{idx}(c_i) = \min\{j \mid c_i \in \text{names}(L_j)\}$*

then

$$(T_0, \varphi_0) \xrightarrow{L_1 \pi'} (T_1 \pi, \varphi_1 \pi) \xrightarrow{L_2 \pi'} \dots \xrightarrow{L_n \pi'} (T_n \pi, \varphi_n \pi),$$

where $\pi' = \{c_i \mapsto R_i\}_{i \in \{1, \dots, k\}}$ and $\pi = \{c_i \mapsto t_i\}_{i \in \{1, \dots, k\}}$.

Proof. By induction on n , the same operational steps will take place with the new labels.

Lemma 27. *Let T be a trace, let $\{c_1, \dots, c_k\}$ be public names not appearing in T and let $\pi : \{c_1, \dots, c_k\} \rightarrow \text{Messages}$ and $\pi' : \{c_1, \dots, c_k\} \rightarrow \text{Recipes}$ be mappings from names to terms. If $T \models k_{l_1, \dots, l_i}(R_{i+1}, t_{i+1})$ and $T \models k_{l_1 \pi, \dots, l_i \pi}(c_i \pi', c_i \pi)$ then $T \models k_{l_1 \pi, \dots, l_i \pi}(R_{i+1} \pi', t_{i+1} \pi)$.*

Proof. By induction on R .

Lemma 28. *Let T be a trace and φ a frame such that $(T, \varphi) \xrightarrow{L} (T', \varphi')$ and such that*

1. *either $M = L$,*
2. *or $L = \mathbf{in}(d, R)$ and $M = \mathbf{in}(d, R')$ such that $(R = R')\varphi$.*

Then we have that $(T, \varphi) \xrightarrow{M} (T', \varphi')$.

Proof. If $M = L$ then the result is obvious. Otherwise, R and R' are recipes for the same term in φ and therefore the transition still holds. \square

Theorem 4. Let T be a trace and let P be a determinate process. Let K be the set of solved statements from a saturated knowledge base associated to T . Then $T \sqsubseteq_w P$ iff the following tests hold:

$$\text{REACHABILITY} \frac{\begin{array}{l} (r_{l_1, \dots, l_n} \Leftarrow \{k_{w_i}(X_i, x_i)\}_{i \in \{1, \dots, m\}}) \in K \\ c_1, \dots, c_k \text{ fresh constants } \sigma : \text{vars}(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\} \text{ is a bijection} \\ k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K) \text{ for all } i \text{ such that } l_i = \mathbf{in}(t_i) \\ M_i = l_i \text{ if } l_i \in \{\mathbf{test}, \mathbf{out}(-)\} \quad M_i = \mathbf{in}(d_i, R_i) \text{ if } l_i = \mathbf{in}(d_i, t_i) \end{array}}{(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (T', \varphi)}$$

$$\text{IDENTITY} \frac{\begin{array}{l} (r_{l_1, \dots, l_n}(R, R') \Leftarrow \{k_{w_i}(X_i, x_i)\}_{i \in \{1, \dots, m\}}) \in K \\ c_1, \dots, c_k \text{ fresh constants } \sigma : \text{vars}(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\} \text{ is a bijection} \\ k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K) \text{ for all } i \text{ such that } l_i = \mathbf{in}(t_i) \\ M_i = l_i \text{ if } l_i \in \{\mathbf{test}, \mathbf{out}(-)\} \quad M_i = \mathbf{in}(d_i, R_i) \text{ if } l_i = \mathbf{in}(d_i, t_i) \end{array}}{(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (T', \varphi) \text{ such that } (R\omega = R'\omega)\varphi \text{ where } \omega = \{X_i \mapsto x_i \sigma\}}$$

Proof. We first prove that if any of the tests fail, $T \not\sqsubseteq P$. Indeed, if the REACHABILITY test fails, we have that $P \not\xrightarrow{M_1, \dots, M_n} (-, -)$, but, by the soundness of K , we have that $(T, \emptyset) \xrightarrow{M_1, \dots, M_n} (-, -)$ and therefore $T \not\sqsubseteq P$.

If the IDENTITY test fails, we have that:

1. either $(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (-, -)$, in which case, by the soundness of K , we have that $(T, \emptyset) \xrightarrow{M_1, \dots, M_n} (-, -)$ and therefore $T \not\sqsubseteq P$.
2. or for any φ' such that $(P, \emptyset) \xrightarrow{M_1, \dots, M_n} (-, \varphi')$ we have $(R\pi \neq R'\pi)\varphi'$. By the soundness of K , we have however that $(T, \emptyset) \xrightarrow{M_1, \dots, M_n} (-, \varphi)$ and $(R\pi = R'\pi)\varphi$. Therefore $T \not\sqsubseteq P$.

Next, we prove that if $T \not\sqsubseteq P$, then at least one test fails. We assume by contradiction that $T \sqsubseteq P$, that all tests pass and we derive a contradiction.

As $T \not\sqsubseteq P$, it follows that there exist L_1, \dots, L_n, φ such that either:

$$(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (T', \varphi) \quad \text{and} \\ \forall S \in P : (S, \emptyset) \not\xrightarrow{L_1, \dots, L_n} (S', \psi)$$

or:

$$(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (T', \varphi) \text{ and } (R = R')\varphi \quad \text{and} \\ \forall S \in P, (S, \emptyset) \xrightarrow{L_1, \dots, L_n} (S', \psi) \text{ implies } (R \neq R')\psi$$

Let n be the smallest index such that one of the above holds. We then have that:

$$(T, \emptyset) \xrightarrow{L_1} (T_1, \varphi_1) \xrightarrow{L_2} \dots \xrightarrow{L_{n-1}} (T_{n-1}, \varphi_{n-1}) \xrightarrow{L_n} (T_n, \varphi_n)$$

and there exists $S \in P$ such that:

$$(S, \emptyset) \xrightarrow{L_1} (S_1, \psi_1) \xrightarrow{L_2} \dots \xrightarrow{L_{n-1}} (S_{n-1}, \psi_{n-1}),$$

such that for all R, R' we have $(R = R')\varphi_i \implies (R = R')\psi_i$ ($1 \leq i \leq n-1$) and:

1. either for all $U \in P$ we have $(U, \emptyset) \not\xrightarrow{L_1, \dots, L_n} (-, \psi)$
2. or there exist recipes R, R' such that for any $U' \in P$ such that $(U', \emptyset) \xrightarrow{L_1, \dots, L_n} (-, \psi')$ we have $(R \neq R')\psi'$.

We consider each of the cases separately:

1. If we are in the case of Item 1, as $(T, \emptyset) \xrightarrow{L_1, \dots, L_n} (T_n, \varphi_n)$, we have by Theorem ?? that $r_{L_1 \varphi_n \downarrow, \dots, L_n \varphi_n \downarrow} \in \mathcal{H}_e(K)$. By the definition of \mathcal{H}_e , we have that it contains no reachability statements in addition to those in \mathcal{H} : therefore $r_{L_1 \varphi_n \downarrow, \dots, L_n \varphi_n \downarrow} \in \mathcal{H}(K)$.

Therefore there exist a statement $f = (r_{l_1, \dots, l_n} \Leftarrow k_{w_i}(X_i, x_i)_{i \in \{1, \dots, m\}}) \in K$ and a substitution τ grounding for f such that $l_i \tau = L_i \varphi_n \downarrow$ (for all $1 \leq i \leq n$) and such that $k_{w_i \tau}(X_i \tau, x_i \tau) \in \mathcal{H}(K)$.

Let c_1, \dots, c_k be fresh public names and let $\sigma : \text{vars}(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\}$ be a bijection. As $(k_{\ell_1, \dots, \ell_{|w_i|}}(c_j, c_j) \Leftarrow) \in K(T)$ for all $1 \leq i \leq m$ and all $1 \leq j \leq k$ and $w_i \sigma$ is an instance of $\ell_1, \dots, \ell_{|w_i|}$, it follows by Theorem ?? that there exists a substitution σ' such that $k_{w_i \sigma}(X_i \sigma', x_i \sigma) \in \mathcal{H}_e(K)$ for all $1 \leq i \leq m$, which implies by the definition of \mathcal{H}_e that there exists a substitution σ'' such that $k_{w_i \sigma}(X_i \sigma'', x_i \sigma) \in \mathcal{H}(K)$ for all $1 \leq i \leq m$.

By instantiating f with $\sigma \cup (\sigma''[\{X_1, \dots, X_m\}])$, we obtain that $r_{l_1 \sigma, \dots, l_n \sigma} \in \mathcal{H}(K)$. By the soundness of K and of \mathcal{H} , it follows that $T \models r_{l_1 \sigma, \dots, l_n \sigma}$. Therefore, there exist recipes R'_i (for all $1 \leq i \leq n$ such that $l_i = \mathbf{in}(-, t_i)$) such that $T \models k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R'_i, t_i \sigma)$. By the completeness of K , it follows that there exist recipes R_i such that $k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K)$.

Let $M_i = l_i$ if $l_i \in \{\mathbf{test}, \mathbf{out}(-)\}$ and let $M_i = \mathbf{in}(d_i, R_i)$ if $l_i = \mathbf{in}(d_i, t_i)$ for all $1 \leq i \leq n$. Because the REACHABILITY test worked, it follows that there exists $S'_0 \in P$ such that, if we let $\psi'_0 = \emptyset$, we have

$$(S'_0, \psi'_0) \xrightarrow{M_1} (S'_1, \psi'_1) \xrightarrow{M_2} \dots \xrightarrow{M_n} (S'_n, \psi'_n).$$

We already have that $k_{w_j \tau}(X_j \tau, x_j \tau) \in \mathcal{H}(K)$ by choice of f and of τ . By the soundness of \mathcal{H} and K , we obtain $T \models k_{w_j \tau}(X_j \tau, x_j \tau)$. By Lemma 24, we have that $|w_j| < i$, and as w_j is a prefix of l_1, \dots, l_{i-1} , we have that $T \models k_{l_1 \tau, \dots, l_{i-1} \tau}(X_j \tau, x_j \tau)$.

Let $\pi : \{c_1, \dots, c_k\} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{N}, \mathcal{M})$ and $\pi' : \{c_1, \dots, c_k\} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{M}, \mathcal{W})$ be defined such that $\pi(c_l) = x_j \tau$ and $\pi'(c_l) = X_j \tau$ when $\sigma(x_j) = c_l$. As $X_j \tau = c_l \pi'$, $x_j \tau = c_l \pi$ and $l_1 \tau, \dots, l_{i-1} \tau = l_1 \sigma \pi, \dots, l_{i-1} \sigma \pi$ therefore we have that $T \models k_{l_1 \sigma \pi, \dots, l_{i-1} \sigma \pi}(c_l \pi', c_l \pi)$.

We also have that $k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma) \in \mathcal{H}(K)$. By the soundness of \mathcal{H} and K , we have that $T \models k_{l_1 \sigma, \dots, l_{i-1} \sigma}(R_i, t_i \sigma)$. We apply Lemma 27 to obtain that $T \models k_{l_1 \sigma \pi, \dots, l_{i-1} \sigma \pi}(R_i \pi', t_i \sigma \pi)$. But $t_i \sigma \pi = t_i \tau$ and $l_1 \sigma \pi, \dots, l_{i-1} \sigma \pi = l_1 \tau, \dots, l_{i-1} \tau$ and therefore we have that $T \models k_{l_1 \tau, \dots, l_{i-1} \tau}(R_i \pi', t_i \tau)$.

Let R''_i be such that $L_i = \mathbf{in}(d_i, R''_i)$ for some channel d_i for all i such that the i th action of T is a receive. By the definition of \models , we have therefore that $T \models k_{l_1 \tau, \dots, l_{i-1} \tau}(R''_i, t_i \tau)$.

We conclude that $T \models i_{l_1 \tau, \dots, l_{i-1} \tau}(R_i \pi', R''_i)$, or, equivalently, $(R_i \pi' = R''_i) \varphi_{i-1}$. By the hypothesis, we have that $(R_i \pi' = R''_i) \psi_{i-1}$ as well.

From Lemma 26, we obtain that

$$(S'_0, \psi'_0) = (S'_0 \pi, \psi_0 \pi) \xrightarrow{M_1 \pi'} (S'_1 \pi, \psi'_1 \pi) \xrightarrow{M_2 \pi'} \dots \xrightarrow{M_n \pi'} (S'_n \pi, \psi'_n \pi).$$

We will show by induction on n that

$$(S'_0 \pi, \psi_0 \pi) \xrightarrow{L_1} (S'_1 \pi, \psi'_1 \pi) \xrightarrow{L_2} \dots \xrightarrow{L_n} (S'_n \pi, \psi'_n \pi).$$

We assume by the induction hypothesis that

$$(S'_0 \pi, \psi_0 \pi) \xrightarrow{L_1} (S'_1 \pi, \psi'_1 \pi) \xrightarrow{L_2} \dots \xrightarrow{L_i} (S'_i \pi, \psi'_i \pi)$$

and we show that

$$(S'_i \pi, \psi'_i \pi) \xrightarrow{L_{i+1}} (S'_{i+1} \pi, \psi'_{i+1} \pi).$$

We will show that L_{i+1} and $M_{i+1} \pi'$ satisfy the conditions of Lemma 28 and then we can easily conclude by it. Indeed, either $L_{i+1} = M_{i+1} \pi'$ (in the case of a **test** or **out** action), or $L_{i+1} = \mathbf{in}(d_{i+1}, R''_{i+1})$ and $M_{i+1} \pi' = \mathbf{in}(d_{i+1}, R_{i+1} \pi')$ (in the case of a **in** action). In the second case, as we have shown

$(R_{i+1}\pi' = R''_{i+1})\psi_i$, by determinacy of P it follows that $\psi_i\pi \approx_s \psi'_i\pi$ and therefore $(R_{i+1}\pi' = R''_{i+1})\psi'_i$ as well. As the hypothesis of Lemma 28 are satisfied, we can conclude.

Let $U = S'_0$. We have shown that $(U, \emptyset) \xrightarrow{L_1, \dots, L_n} (-, _)$, therefore obtaining a contradiction. Therefore Item 1 cannot hold.

2. In the case of Item 2, we assume that for all $U \in P$ such that $(U, \emptyset) \xrightarrow{L_1, \dots, L_n} (-, \psi)$ we have $(R \neq R')\psi$ to obtain a contradiction.

As $(R = R')\varphi_n$, it follows by Theorem ?? that $i_{L_1\varphi_n\downarrow, \dots, L_n\varphi_n\downarrow}(R, R') \in \mathcal{H}_e(K)$.

As $(R \neq R')\psi$ it follows that there exist recipes Q, Q' such that $i_{L_1\varphi_n\downarrow, \dots, L_n\varphi_n\downarrow}(Q, Q') \in \mathcal{H}(K)$ but $(Q \neq Q')\psi$.

As $r_{L_1\varphi_n\downarrow, \dots, L_n\varphi_n\downarrow} \in \mathcal{H}(K)$, we have by Lemma 15 that $ri_{L_1\varphi_n\downarrow, \dots, L_n\varphi_n\downarrow}(Q, Q') \in \mathcal{H}(K)$. Therefore there exists a statement

$$f = \left(ri_{l_1, \dots, l_n}(P, P') \Leftarrow \{k_{w_i}(X_i, x_i)\}_{i \in \{1, \dots, m\}} \right) \in K$$

and a substitution τ grounding for f such that $k_{w_i\tau}(X_i\tau, x_i\tau) \in \mathcal{H}(K)$ (for all $1 \leq i \leq m$), $l_1\tau, \dots, l_n\tau = L_1\varphi_n\downarrow, \dots, L_n\varphi_n\downarrow$, $P\tau = Q$ and $P'\tau = Q'$.

Let c_1, \dots, c_k be fresh public names and let $\sigma : vars(l_1, \dots, l_n) \rightarrow \{c_1, \dots, c_k\}$ be a bijection. As

$\left(k_{\ell_1, \dots, \ell_{|w_i|}}(c_j, c_j) \Leftarrow \right) \in K(T)$ for all $1 \leq i \leq m$ and all $1 \leq j \leq k$ and $w_i\sigma$ is an instance of $\ell_1, \dots, \ell_{|w_i|}$, it follows by Theorem ?? that there exists a substitution σ' such that $k_{w_i\sigma'}(X_i\sigma', x_i\sigma') \in \mathcal{H}_e(K)$ for all $1 \leq i \leq m$, which implies by the definition of \mathcal{H}_e that there exists a substitution σ'' such that $k_{w_i\sigma''}(X_i\sigma'', x_i\sigma'') \in \mathcal{H}(K)$ for all $1 \leq i \leq m$.

By instantiating f with $\sigma \cup (\sigma''[\{X_1, \dots, X_m\}])$, we obtain that $r_{l_1\sigma, \dots, l_n\sigma} \in \mathcal{H}(K)$. By the soundness of K and of \mathcal{H} , it follows that $T \models r_{l_1\sigma, \dots, l_n\sigma}$. Therefore, there exist recipes R'_i (for all $1 \leq i \leq n$ such that $l_i = \mathbf{in}(-, t_i)$) such that $T \models k_{l_1\sigma, \dots, l_{i-1}\sigma}(R'_i, t_i\sigma)$. By the completeness of K , it follows that there exist recipes R_i such that $k_{l_1\sigma, \dots, l_{i-1}\sigma}(R_i, t_i\sigma) \in \mathcal{H}(K)$.

Let $M_i = l_i$ if $l_i \in \{\mathbf{test}, \mathbf{out}(-)\}$ and let $M_i = \mathbf{in}(d_i, R_i)$ if $l_i = \mathbf{in}(d_i, t_i)$ for all $1 \leq i \leq n$. Because the REACHABILITY test worked, it follows that there exists $S'_0 \in P$ such that, if we let $\psi'_0 = \emptyset$, we have

$$(S'_0, \psi'_0) \xrightarrow{M_1} (S'_1, \psi'_1) \xrightarrow{M_2} \dots \xrightarrow{M_n} (S'_n, \psi'_n).$$

Similarly to Item 1, we can show that there exists $S'_0 \in P$ such that

$$(S'_0, \psi'_0) = (S'_0\pi, \psi_0\pi) \xrightarrow{L_1} (S'_1\pi, \psi'_1\pi) \xrightarrow{L_2} \dots \xrightarrow{L_n} (S'_n\pi, \psi'_n\pi)$$

where π and π' are defined as in Item 1.

Furthermore, as the IDENTITY test worked, it follows that $(P\omega = P'\omega)\psi'_n$, where $\omega = \{X_i \mapsto x_i\sigma\}$. As the equational theory is stable by substitution of terms for names, we have that $(P\omega\pi' = P'\omega\pi')\psi'_n\pi$. But $P\omega\pi' = P\tau = Q$ and $P'\omega\pi' = P'\tau = Q'$, which means that $(Q = Q')\psi'_n\pi$.

We have shown that there exists $S'_0 \in P$ such that $(S'_0, \emptyset) \xrightarrow{L_1, \dots, L_n} (-, \psi'_n\pi)$ and $(Q = Q')\psi'_n\pi$. By determinacy of P , it follows that $(Q = Q')\psi$ for any ψ and any $U \in P$ such that $(U, \emptyset) \xrightarrow{L_1, \dots, L_n} (-, \psi)$, therefore we obtain a contradiction.

As both cases yield a contradiction, it follows that if $T \not\sqsubseteq P$, there exists at least one test that fails.