On the cactus rank of cubic forms

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THE CACTUS RANK OF CUBIC FORMS

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Abstract. We prove that the smallest degree of an apolar 0-dimensional scheme of a general cubic form in $n+1$ variables is at most $2n+2$, when $n \geq 8$, and therefore smaller than the rank of the form. For the general reducible cubic form the smallest degree of an apolar subscheme is $n+2$, while the rank is at least $2n$.

Introduction

The rank of a homogeneous form F of degree d is the minimal number of linear forms $L_1, \dots, L_r$ needed to write F as a sum of pure d-powers:

$$F = L_1^d + \cdots + L_r^d.$$

Various other notions of rank, such as cactus rank and border rank, appear in the study of higher secant varieties and are closely related to the rank. The cactus rank is the minimal length of an apolar subscheme to F, while the border rank is the minimal r such that F is a limit of forms of rank r. The notion of cactus rank is inspired by the cactus varieties studied in [Buczynska, Buczynski 2010]. For irreducible cubic forms that do not define a cone, the cactus rank is minimal for cubics of Fermat type, e.g. $F = x_0^3 + \cdots + x_n^3$. In this case all three ranks coincide. There are however other cubic forms with minimal cactus rank whose border rank is strictly higher. We show that the cactus rank is smaller than the rank for a general form, as soon as the degree is at least 3 and the number of variables is at least 9. This result follows from the computation of a natural upper bound for the cactus rank that we conjecture is sharp for general cubic forms.

The rank of forms has seen growing interest in recent years. This work is close in line to [Iarrobino 1994], [Iarrobino, Kanev 1999] and [Elias, Rossi 2011], in their study of apolarity and the local Gorenstein algebra associated to a polynomial. Applications to higher secant varieties can be found in [Chiantini, Ciliberto, 2002], [Buczynska, Buczynski 2010] and [Landsberg, Ottaviani 2011], while the papers [Landsberg, Teitler 2010], [Brachat et al. 2010], [Bernardi et al. 2011] and [Carlini et al. 2011] concentrate on effective methods to compute the rank and to compute an explicit decomposition of a form. In a different direction, the rank of cubic forms associated to canonical curves has been computed in [De Poi, Zucconi 2011a] and [De Poi, Zucconi 2011b].

1. Apolar Gorenstein subschemes

We consider homogeneous polynomials $F \in S = \mathbb{C}[x_0, \dots, x_n]$, and consider the dual ring $T = \mathbb{C}[y_0, \dots, y_n]$ acting on S by differention:

$$y_j(x_i) = \frac{d}{dx_j}(x_i) = \delta_{ij}.$$

With respect to this action S_1 and T_1 are natural dual spaces and $< x_0, \dots, x_n >$ and $< y_0, \dots, y_n >$ are dual bases. In particular T is naturally the coordinate ring of $\mathbb{P}(S_1)$ the projective space of 1-dimensional subspaces of S_1, and vice versa. The annihilator of F is an ideal $F^\perp \subset T$, and the quotient $T_F = T/F^\perp$ is graded Artinian and Gorenstein.

Definition 1. A subscheme $X \subset \mathbb{P}(S_1)$ is apolar to F if the homogeneous ideal $I_X \subset F^\perp \subset T$.

F admits some natural finite local apolar Gorenstein subschemes. Any local Gorenstein subscheme is defined by an (in)homogeneous polynomial: For any $f \in S_{x_0} = \mathbb{C}[x_1, \dots, x_n]$, the annihilator $f^\perp \subset T_{y_0} = \mathbb{C}[y_1, \dots, y_n]$ defines an Artinian Gorenstein quotient $T_f = T_{y_0}/f^\perp$. Now, $\mathrm{Spec}(T_f)$ is naturally a subscheme in $\mathbb{P}(S_1)$. Taking $f = F(1, x_1, \dots, x_n)$ we show that $\mathrm{Spec}(T_f)$ is apolar to F. In fact, F admits a natural apolar Gorenstein subscheme for any linear form in S.

Any nonzero linear form $l \in S$ belongs to a basis $(l, l_1, \dots, l_n)$ of S_1, with dual basis $(l', l'_1, \dots, l'_n)$ of T_1. In particular the homogeneous ideal in T of the point $[l] \in \mathbb{P}(S_1)$ is generated by $\{l'_1, \dots, l'_n\}$, while $\{l_1, \dots, l_n\}$ generate the ideal of the point $\phi([l]) \in \mathbb{P}(T_1)$, where $\phi : \mathbb{P}(T_1) \to \mathbb{P}(S_1),\ y_i \mapsto x_i, i = 0, \dots, n$.

The form $F \in S$ defines a hypersurface $\{F = 0\} \subset \mathbb{P}(T_1)$. The Taylor expansion of F with respect to the point $\phi([l])$ may naturally be expressed in the coordinates functions $(l, l_1, \dots, l_n)$. Thus

$$F = a_0 l^d + a_1 l^{d-1} f_1(l_1, \dots, l_n) + \dots + a_d f_d(l_1, \dots, l_n).$$

We denote the corresponding dehomogenization of F with respect to l by F_l, i.e.

$$F_l = a_0 + a_1 f_1(l_1, \dots, l_n) + \dots + a_d f_d(l_1, \dots, l_n).$$

Also, we denote the subring of T generated by $\{l'_1, \dots, l'_n\}$ by $T_{l'}$. It is the natural coordinate ring of the affine subspace $\{l' \neq 0\} \subset \mathbb{P}(S_1)$.

Lemma 1. *The Artinian Gorenstein scheme $\Gamma(F_l)$ defined by $F_l^\perp \subset T_{l'}$ is apolar to F, i.e. the homogenization $(F_l^\perp)^h \subset F^\perp \subset T$.*

Proof. If $g \in F_l^\perp \subset \mathbb{C}[l'_1, \dots, l'_n]$, then $g = g_1 + \dots + g_r$ where g_i is homogeneous in degree i. Similarly $F_l = f = f_0 + \dots + f_d$. The annihilation $g(f) = 0$ means that for each $e \geq 0$, $\sum_j g_j f_{e+j} = 0$. Homogenizing we get

$$g^h = G = (l')^{r-1} g_1 + \dots + g_r, \quad f^h = F = l^d f_0 + \dots + f_d$$

and

$$G(F) = \sum_e \sum_j l^{d-r-e} g_j f_{e+j} = \sum_e l^{d-r-e} \sum_j g_j f_{e+j} = 0.$$

□

Remark 1. (Suggested by Mats Boij) The ideal $(F_l^\perp)^h$ may be obtained without dehomogenizing F. Write $F = l^e F_{d-e}$, such that l does not divide F_{d-e}. Consider the form $F_{2(d-e)} = l^{d-e} F_{d-e}$. Unless $d - e = 0$, i.e. $F = l^d$, the degree $d - e$ part of the annihilator $(F_{2(d-e)})^\perp_{d-e}$ generates an ideal in $(l)^\perp$ and the saturation $\mathrm{sat}(F_{2(d-e)})^\perp_{d-e})$ coincides with $(F_l^\perp)^h$. In fact if $G \in T_{d-e}$ then

$$G(F_{2(d-e)}) = G(l^{d-e} F_{d-e}) = G(l^{d-e}) F_{d-e} + l G(l^{d-e-1}) F_{d-e}$$

so $G(F_{2(d-e)}) = 0$ only if $G(l^{d-e}) = 0$.

Apolarity has attracted interest since it characterizes powersum decompositions of F, cf. [Iarrobino, Kanev 1999], [Ranestad, Schreyer 2000]. The annihilator of a power of a linear form $l^d \in S$ is the ideal of the corresponding point $p_l \in \mathbb{P}_T$ in degrees at most d. Therefore $F = \sum_{i=1}^r l_i^d$ only if $I_\Gamma \subset F^\perp$ where $\Gamma = \{p_{l_1}, \dots, p_{l_r}\} \subset \mathbb{P}_T$. On the other hand, if

$I_{\Gamma,d} \subset F_d^{\perp} \subset T_d$, then any differential form that annihilates each l_i^d also annihilates F, so, by duality, $[F]$ must lie in the linear span of the $[l_i^d]$ in $\mathbb{P}(S_d)$. Thus $F = \sum_{i=1}^r l_i^d$ if and only if $I_\Gamma \subset F^{\perp}$.

Various notions of rank for F are therefore naturally defined by apolarity : The cactus rank $cr(F)$ is defined as

$$cr(F) = \min\{\mathrm{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1), \dim\Gamma = 0, I_\Gamma \subset F^{\perp}\},$$

the smoothable rank $sr(F)$ is defined as

$$sr(F) = \min\{\mathrm{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1) \ \text{ smoothable}, \dim\Gamma = 0, I_\Gamma \subset F^{\perp}\}$$

and the rank $r(F)$ is defined as

$$r(F) = \min\{\mathrm{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1) \ \text{ smooth}, \dim\Gamma = 0, I_\Gamma \subset F^{\perp}\}.$$

Clearly $cr(F) \leq sr(F) \leq r(F)$. A separate notion of border rank, $br(F)$, often considered, is not defined by apolarity. The border rank is rather the minimal r, such that F is the limit of polynomials of rank r. Thus $br(F) \leq sr(F)$. These notions of rank coincide with the notions of length of annihilating schemes in Iarrobino and Kanev book [Iarrobino, Kanev 1999, Definition 5.66]: Thus cactus rank coincides with the scheme length, $cr(F) = l\mathrm{sch}(F)$, and smoothable rank coincides with the smoothable scheme length, $sr(F) = l\mathrm{schsm}(F)$, while border rank coincides with length $br(F) = l(F)$. In addition they consider the differential length $l\mathrm{diff}(F)$, the maximum of the dimensions of the space of k-th order partials of F as k varies between 0 and $\deg F$. This length is the maximal rank of a catalecticant or Hankel matrix at F, and is always a lower bound for the cactus rank: $l\mathrm{diff}(F) \leq cr(F)$.

For a general form F in S of degree d the rank, the smoothable rank and the border rank coincide and equals, by the Alexander Hischowitz theorem,

$$br(F) = sr(F) = r(F) = \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil,$$

when $d > 2$, $(n,d) \neq (2,4), (3,4), (4,3), (4,4)$. The local Gorenstein subschemes considered above show that the cactus rank for a general polynomial may be smaller. Let

$$N_d = \begin{cases} 2\binom{n+k}{k} & \text{when } d = 2k+1 \\ \binom{n+k}{k} + \binom{n+k+1}{k+1} & \text{when } d = 2k+2 \end{cases} \tag{1}$$

and denote by $\mathrm{Diff}(F)$ the subspace of S generated by the partials of F of all orders, i.e. of order $0, \ldots, d = \deg F$.

Theorem 1. *Let $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous form of degree d, and let $l \in S_1 =< x_0, \ldots, x_n >$ be any linear form. Let F_l be a dehomogenization of F with respect to l. Then*

$$cr(F) \leq \dim_K \mathrm{Diff}(F_l).$$

In particular

$$cr(F) \leq N_d.$$

Proof. According to Lemma 1 the subscheme $\Gamma(F_l) \subset \mathbb{P}(T_1)$ is apolar to F. The subscheme $\Gamma(F_l)$ is affine and has length equal to

$$\dim_k T_{l'}/F_l^{\perp} = \dim_K \mathrm{Diff}(\mathrm{F}_l).$$

If all the partial derivatives of F_l of order at most $\lfloor \frac{d}{2} \rfloor$ are linearly independent, and the partial derivatives of higher order span the space of polynomials of degree at most $\lfloor \frac{d}{2} \rfloor$, then

$$\dim_K \mathrm{Diff(F_l)} = 1 + n + \binom{n+1}{n-1} + \cdots + \binom{n+\lfloor \frac{d}{2} \rfloor}{n-1} + \cdots + n + 1 = N_d.$$

Clearly this is an upper bound so the theorem follows. □

Question 1. What is the cactus rank $cr(n,d)$ for a general form $F \in \mathbb{C}[x_0, \ldots, x_n]_d$?

If Γ computes the rank (resp. smoothable rank or cactus rank) of F, then Γ is locally Gorenstein [Buczynska, Buczynski 2010, proof of Proposition 2.2]. Every local Gorenstein scheme Γ in $\mathbb{P}_T$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_r]/g^{\perp})$ for some polynomial $g \in \mathbb{C}[x_1, \ldots, x_r]$ (cf. [Iarrobino 1994, Lemma 1.2]). If Γ is a local Gorenstein component of an apolar subscheme to F, then the number r of variables in the polynomial g defining Γ is at most n but the degree of g may be larger than the degree of F. In particular, even local apolar subschemes of minimal length may not be of the kind $\Gamma(F_l)$, described above.

2. Cubic forms

If $F \in S$ is a general cubic form, then the cactus rank according to Theorem 1 is at most $2n + 2$.

If F is a general reducible cubic form in S and l is a linear factor, then $f = F_l$ is a quadratic polynomial and $\Gamma(f)$ is smoothable of length at most $n + 2$: The partials of a nonsingular quadratic polynomial in n variables form a vector space of dimension $n + 2$, so this is the length of $\Gamma(f)$. On the other hand let E be an elliptic normal curve of degree $n + 2$ in $\mathbb{P}^{n+1}$. Let $T(E)$ be the homogeneous coordinate ring of E. A quotient of $T(E)$ by two general linear forms is artinian Gorenstein with Hilbert function $(1, n, 1)$ isomorphic to T_q for a quadric q of rank n. Thus T_f is isomorphic to T_q and $\Gamma(f)$ is smoothable.

Theorem 2. *For a general cubic form $F \in \mathbb{C}[x_0, \ldots, x_n]$, the cactus rank is*

$$cr(F) \leq 2n + 2.$$

For a general reducible cubic form $F \in \mathbb{C}[x_0, \ldots, x_n]$ with $n > 1$, the cactus rank and the smoothable rank are

$$cr(F) = sr(F) = n + 2.$$

Proof. It remains to show that for a general reducible cubic form $cr(F) \geq n + 2$. If $\Gamma \subset \mathbb{P}_T$ has length less than $n + 1$ it is contained in a hyperplane, so $I_\Gamma \subset F^{\perp}$ only if the latter contains a linear form. If $\{F = 0\}$ is not a cone, this is not the case. If $\Gamma \subset \mathbb{P}_T$ has length $n + 1$, then, for the same reason, this subscheme must span $\mathbb{P}_T$. Its ideal in that case is generated by $\binom{n+1}{2}$ quadratic forms. If F is general, $F_2^{\perp}$ is also generated by $\binom{n+1}{2}$, so they would have to coincide. On the other hand this equality is a closed condition on cubic forms. If $F = x_0(x_0^2 + \cdots + x_n^2)$, then

$$F_2^{\perp} =< y_1 y_2, \ldots, y_{n-1} y_n, y_0^2 - y_1^2, \ldots, y_0^2 - y_n^2 > .$$

In particular $\dim F_2^{\perp} = \binom{n+1}{2}$, But the quadrics $F_2^{\perp}$ do not have any common zeros, so $cr(F) \geq n + 2$. The general reducible cubic must therefore also have cactus rank at least $n + 2$ and the theorem follows. □

Remark 2. By [Landsberg, Teitler 2010, Theorem 1.3] the lower bound for the rank of a reducible cubic form that depends on $n + 1$ variables and not less, is $2n$.

If $F = x_0F_1(x_1, \dots, x_n)$ where F_1 is a quadratic form of rank n, then $cr(F) = sr(F) = n+1$, the same as for a Fermat cubic, while the rank is at least $2n$.

We give another example with $cr(F) = n+1 < sr(F)$. Let $G \in \mathbb{C}[x_1, \dots, x_m]$ be a cubic form such that the scheme $\Gamma(G) = \mathrm{Spec}(\mathbb{C}[y_1, \dots, y_m]/G^{\perp})$ has length $2m+2$ and is not smoothable. By [Iarrobino 1984, Section 4A] $m \geq 6$ suffice. Denote by $G_1 = y_1(G), \dots, G_m = y_m(G)$ the first order partials of G. Let

$$F = G + x_0x_1x_{m+1} + \cdots + x_0x_mx_{2m} + x_0^2x_{2m+1} \in \mathbb{C}[x_0, \dots, x_{2m+1}].$$

Then

$$F_{x_0} = G + x_1x_{m+1} + \cdots + x_mx_{2m} + x_{2m+1}$$

and

$$\mathrm{Diff}(F_{x_0}) =< F_{x_0}, G_1 + x_{m+1}, \dots, G_m + x_{2m}, x_1, \dots, x_m, 1 >$$

so $\dim\mathrm{Diff}(F_{x_0}) = 2m+2$. Therefore $\Gamma(F_{x_0})$ is apolar to F and computes the cactus rank of F. Since $\{F = 0\}$ is not a cone, $\Gamma(F_{x_0})$ is nondegenerate, so its homogeneous ideal is generated by the quadrics in the ideal of $F^{\perp}$. In particular $\Gamma(F_{x_0})$ is the unique apolar subscheme of length $2m+2$. Since this is not smoothable, the smoothable rank is strictly bigger.

By Theorem 2 the cactus rank of a generic cubic form $F \in \mathbb{C}[x_0, \dots, x_n]$ is at most $2n+2$. The first n for which $2n+2$ is smaller than the rank $r(F) = \lceil \frac{1}{n+1}\binom{n+3}{3} \rceil$ of the generic cubic form in $n+1$ variables is $n = 8$, where $r(F) = 19$ and $cr(F) \leq 18$.

Conjecture 1. The cactus rank $cr(F)$ of a generic homogeneous cubic $F \in k[x_0, \dots, x_n]$ equals the rank when $n \leq 7$ and equals $2n+2$ when $n \geq 8$.

In the third Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+3}{3}-1}$, the span of each subscheme $\Gamma \subset \mathbb{P}^n$ of length d is a linear space L_Γ of dimension at most $d-1$. The cactus rank of a general cubic form is then the minimal d such that the linear spaces L_Γ spanned by length d subschemes Γ fill $\mathbb{P}^{\binom{n+3}{3}-1}$.

For this minimum it suffices to consider the subscheme of the Hilbert scheme parameterizing locally Gorenstein schemes, i.e. all of whose componenets are local Gorenstein schemes, cf. [Buczynska, Buczynski 2010]. Furthermore, any local Gorenstein scheme of length at most 10 is smoothable (cf. [Casnati, Notari 2011a]), so the conjecture holds for $n \leq 5$. Casnati and Notari has recently extended their result to length at most 11, (cf. [Casnati, Notari 2011b]), which means that the conjecture holds also when $n = 6$. There are nonsmoothable local Gorenstein algebras of length 14 (cf. [Iarrobino 1984]), so for $n \geq 7$ a different argument is needed to confirm or disprove the conjecture.

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