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Tutorial on Differential Games

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► **To cite this version:**

Marc Quincampoix. Tutorial on Differential Games. SADCO Summer School 2011 - Optimal Control, Sep 2011, London, United Kingdom. inria-00630050

HAL Id: inria-00630050

<https://inria.hal.science/inria-00630050>

Submitted on 7 Oct 2011

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Differential Games III

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Joint Work P. Cardaliaguet , M.Q

The differential Game

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), & t \in [0, T] \\ u(t) \in U, v(t) \in V \end{cases} \quad (1)$$

where $f : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}^N$, U and V being the control sets of the players. To any initial condition $x(t_0) = x_0$ we associate $t \rightarrow X_t^{t_0, x_0, u, v}$ the solution to (1).

The first player—choosing u —wants to minimize a final cost of the form

$$g(x(T))$$

while the second player,—playing with v —wants to maximize it.

the state-space x_0 is only imperfectly known by the players : they only know that the initial position is randomly distributed under some fixed probability measure μ_0 . Both players are assumed to know this probability μ_0 , and have a perfect knowledge of the control of the other player.

So the "lack" of information is very specific :

- it is symmetric for both player
- it is only concerned with the current position of the game.

We denote by \mathcal{M} the Borel probability measures μ s.t.

$$\int_{\mathbb{R}^N} |x|^2 d\mu(x) < +\infty .$$

Assumptions

$$\left\{ \begin{array}{l} (i) \quad U \text{ and } V \text{ are compact subsets of some finite} \\ \quad \text{dimensional spaces} \\ (ii) \quad f : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}^N \text{ is continuous and} \\ \quad \text{Lipschitz continuous with respect to} \\ (iii) \quad \forall (x, u, v), |f(x, u, v)| \leq M \\ (iv) \quad g : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is Lipschitz continuous and bounded} \end{array} \right. \quad (2)$$

$$\mathcal{U}(t_0) = \{u : [t_0, T] \rightarrow U, \text{ Lebesgue measurable}\}$$

$$\mathcal{V}(t_0) = \{v : [t_0, T] \rightarrow V, \text{ Lebesgue measurable}\}$$

Strategies

Definition 1 A *NAD strategy* is a map $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ such that there is some $\tau > 0$ such that $\forall t$,

$\forall v_1, v_2 \in \mathcal{V}(t_0)$, $v_1 = v_2$ on $[t_0, t] \Rightarrow \alpha(v_1) = \alpha(v_2)$ on $[t_0, t + \tau]$.

$\mathcal{A}(t_0)$ is the set of such α . Symmetrically, $\mathcal{B}(t_0)$ is the set of *NAD strategies* $\beta : \mathcal{U}(t_0) \rightarrow \mathcal{V}(t_0)$ for the second player.

Lemma 2 $\forall (\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, $\exists!(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$,

$$\alpha(v_0) = u_0 \text{ and } \beta(u_0) = v_0 .$$

$$X_t^{t_0, x, \alpha, \beta} := X_t^{t_0, x, u_0, v_0} \quad \forall t \in [t_0, T] .$$

Payoffs and Values

$g : \mathbb{R}^N \mapsto \mathbb{R}$ which is Lipschitz and bounded. For any $(t_0, \mu_0) \in [0, T) \times \mathcal{M}$ and for any $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ we set

$$J(t_0, \mu_0, u, v) = \int_{\mathbb{R}^N} g \left(X_T^{t_0, x, u, v} \right) d\mu(x) .$$

For any pair of strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, we define

$$J(t_0, \mu_0, \alpha, \beta) = J(t_0, \mu_0, u_0, v_0)$$

where $(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ is associated to (α, β) by the Lemma

Definition of the value functions:

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha, \beta)$$

and

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} J(t_0, \mu_0, \alpha, \beta) .$$

Obviously we have

$$V^-(t_0, \mu_0) \leq V^+(t_0, \mu_0) \quad \forall (t_0, \mu_0) \in [0, T] \times \mathcal{M} .$$

Remark

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} J(t_0, \mu_0, \alpha(v), v)$$

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) .$$

Preliminaries on Probability Measures

For $\mu \in \mathcal{M}$, we denote by $L^2_\mu(\mathbb{R}^N, \mathbb{R})$ (resp. $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$) the set of μ -measurable maps $p : \mathbb{R}^N \rightarrow \mathbb{R}$ (resp. $p : \mathbb{R}^N \rightarrow \mathbb{R}^N$) such that $\|p\|_{L^2_\mu} := \int_{\mathbb{R}^N} |p|^2 d\mu < +\infty$

For $\mu \in \mathcal{M}$ and $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ a Borel measurable with linear growth, $\varphi\#\mu$ is the push-forward of μ by φ ,

$$\varphi\#\mu(A) = \mu\left(\varphi^{-1}(A)\right) \quad \forall A \subset \mathbb{R}^N, \text{ Borel measurable}$$

or, equivalently, such that, $\forall f : \mathbb{R}^N \rightarrow \mathbb{R}$, Borel measurable and bounded,

$$\int_{\mathbb{R}^N} f d(\varphi\#\mu) = \int_{\mathbb{R}^N} f(\varphi(x)) d\mu(x) .$$

Wasserstein Distance **cf** book of Villani

$$\mathbf{d}(\mu, \nu) = \inf \left\{ \left(\int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma \right)^{\frac{1}{2}} \right\} \quad (3)$$

where the infimum is taken over all the probability measures γ in \mathbb{R}^{2N} such that

$$\pi_1\#\gamma = \mu \quad \text{and} \quad \pi_2\#\gamma = \nu, \quad (4)$$

π_1 and π_2 being respectively the projections on the first and the second coordinates: $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. A measure γ satisfying (4) is an admissible transport plan from μ to ν . The optimal γ are called optimal plans.

Lemma 3 *If $h : \mathbb{R}^N \rightarrow \mathbb{R}$ is k -Lipschitz continuous, then*

$$\left| \int_{\mathbb{R}^N} h(x) d\mu(x) - \int_{\mathbb{R}^N} h(x) d\nu(x) \right| \leq k \mathbf{d}(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{M}.$$

Lemma 4 *Let $\mu, \nu \in \mathcal{M}$ and γ be optimal for $\mathbf{d}(\mu, \nu)$. Then there exist $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ and $q \in L^2_\nu(\mathbb{R}^N, \mathbb{R}^N)$ s. t.*

$$\int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x) \quad (5)$$

$$\int_{\mathbb{R}^N} \langle \varphi(x), q(x) \rangle d\nu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(y), x - y \rangle d\gamma(x) \quad (6)$$

for any Borel measurable map $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ with at most a linear growth.

proof Let γ be an optimal plan from μ to ν . Then

$$\begin{aligned}\int_{\mathbb{R}^N} h(x) d\mu(x) &= \int_{\mathbb{R}^{2N}} h(x) d\gamma(x, y) \\ &\leq \int_{\mathbb{R}^{2N}} h(y) d\gamma(x, y) + k \int_{\mathbb{R}^{2N}} |x - y| d\gamma(x, y) \\ &\leq \int_{\mathbb{R}^{2N}} h(y) d\nu(y) + k \mathbf{d}(\mu, \nu)\end{aligned}$$

□

proof We just show the existence of p , since the proof for q can be obtained in the same way. Let us consider the linear map Φ on $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ defined by

$$\Phi(\varphi) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

Then

$$|\Phi(\varphi)| \leq \left(\int_{\mathbb{R}^{2N}} |\varphi(x)|^2 d\gamma(x) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x) \right)^{\frac{1}{2}} \leq \mathbf{d}(\mu, \nu) \|\varphi\|_{L^2_\mu}$$

for any $\varphi \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$. Therefore Φ is bounded on $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ whence the existence of $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ from Riesz Representation Theorem. \square

Regularity of the Values

Proposition 5 *The value functions V^+ and V^- are Lipschitz continuous.*

proof for V^+

We shall first prove that the values are Lipschitz continuous with respect to the second variable. Fix $t_0 \in [0, T]$, $\mu_0 \in \mathcal{M}$, $\nu_0 \in \mathcal{M}$ and $\varepsilon > 0$. There exists a nonanticipative strategy $\alpha_\varepsilon \in \mathcal{A}(t_0)$ such that

$$V^+(t_0, \nu_0) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \nu_0, \alpha_\varepsilon, \beta) \leq V^+(t_0, \nu_0) + \varepsilon.$$

Hence

$$\begin{aligned} & V^+(t_0, \mu_0) - V^+(t_0, \nu_0) \leq \\ & \varepsilon + \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) - \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \nu_0, \alpha_\varepsilon, \beta) \\ & \leq 2\varepsilon + J(t_0, \mu_0, \alpha_\varepsilon, \beta_\varepsilon) - J(t_0, \nu_0, \alpha_\varepsilon, \beta_\varepsilon) \end{aligned}$$

where $\beta_\varepsilon \in \mathcal{B}(t_0)$ is such that

$$\sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) - \varepsilon \leq J(t_0, \mu_0, \alpha_\varepsilon, \beta_\varepsilon) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta).$$

Thus

$$\begin{aligned} V^+(t_0, \mu_0) - V^+(t_0, \nu_0) &\leq \\ 2\varepsilon + \int_{\mathbb{R}^N} g \left(X_T^{t_0, x, \alpha_\varepsilon, \beta_\varepsilon} \right) d\mu_0(x) - \int_{\mathbb{R}^N} g \left(X_T^{t_0, x, \alpha_\varepsilon, \beta_\varepsilon} \right) d\nu_0(x) \\ &\leq k^2 e^{kT} d(\mu_0, \nu_0) + 2\varepsilon \end{aligned}$$

thanks to Lemma 3 and because $x \mapsto X_T^{t_0, x, u, v}$ is $k^2 e^{kT}$ Lipschitz.

Consider now $0 < t_0 < s_0 < T$ and the strategy α_ε . Let $u_0 \in \mathcal{U}(t_0)$ and $v_0 \in \mathcal{V}(t_0)$ two given control. We define the

nonanticipative strategy $\alpha_1 \in A(s_0)$ as follows

$$\forall v \in \mathcal{V}(s_0), \alpha_1(v) := \alpha_\varepsilon(v_1)$$

where $v_1(t) = v_0(t)$ if $t \in [t_0, s_0)$ and $v_1(t) = v(t)$ if $t \in [s_0, T]$.

$$\begin{aligned} & V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \\ & \leq \varepsilon + \sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta) - \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) \\ & \leq 2\varepsilon + J(s_0, \mu_0, \alpha_1, \beta_\varepsilon) - J(t_0, \mu_0, \alpha_\varepsilon, \beta_1) \end{aligned}$$

where $\beta_\varepsilon \in \mathcal{B}(s_0)$ is such that

$$\sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta) - \varepsilon \leq J(s_0, \mu_0, \alpha_1, \beta_\varepsilon) \leq \sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta)$$

and $\beta_1 \in \mathcal{B}(t_0)$ is defined as follows:

$\forall u \in \mathcal{U}(t_0), \beta_1(u)(t) = v_0(t)$ if $t \in [t_0, s_0)$ and $\beta_1(u)|_{[s_0, T]} = \beta_\varepsilon(u|_{[s_0, T]})$. **Hence**

$$\begin{aligned}
& V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \\
& \leq 2\varepsilon + \int_{\mathbb{R}^N} g \left(X_T^{s_0, x, \alpha_1, \beta_\varepsilon} \right) d\mu_0(x) - \int_{\mathbb{R}^N} g \left(X_T^{t_0, x, \alpha_\varepsilon, \beta_1} \right) d\mu_0(x) \\
& = 2\varepsilon + \int_{\mathbb{R}^N} g \left(X_T^{s_0, x, \alpha_1, \beta_\varepsilon} \right) - g \left(X_T^{s_0, X_{s_0}^{t_0, x, \alpha_1, v_0}, \alpha_1, \beta_\varepsilon} \right) d\mu_0(x)
\end{aligned}$$

by noticing that

$$X_T^{t_0, x, \alpha_\varepsilon, \beta_1} = X_T^{s_0, X_{s_0}^{t_0, x, \alpha_1, v_0}, \alpha_1, \beta_\varepsilon}.$$

Consequently

$$V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \leq 2\varepsilon + Mk^2 e^{kT} (t_0 - s_0),$$

using the fact that $|x - X_{s_0}^{t_0, x, \alpha_1, v_0}| \leq M(t_0 - s_0)$ □

Dynamic Programming

Proposition 6 [Dynamic programming] *Let $(t_0, t_1, \mu_0) \in [0, T) \times [0, T] \times \mathcal{M}$ be fixed with $t_0 < t_1$. Then*

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} V^+ \left(t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \# \mu_0 \right)$$

and

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} V^- \left(t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \# \mu_0 \right)$$

proof We only prove the dynamic programming for

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) .$$

$V^-(t_0, \mu_0) = W(t_0, t_1, \mu_0)$ where we set

$$W(t_0, t_1, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^- \left(t_1, X_{t_1}^{t_0, \cdot, u, \beta} \# \mu_0 \right)$$

Let us prove first that $V^-(t_0, \mu_0) \leq W(t_0, t_1, \mu_0)$. Fix $\beta_0 \in \mathcal{B}(t_0)$ and $u_0 \in \mathcal{U}(t_0)$. Define $\beta_1 \in \mathcal{B}(t_1)$ as follows

$$\forall u \in \mathcal{U}(t_1), \beta_1(u) := \beta_0(u_1)$$

where $u_1(t) = u_0(t)$ if $t \in [t_0, t_1)$ and $u_1(t) = u(t)$ if $t \in [s_0, T]$.

Clearly β_1 is nonanticipative such that

$$\forall t \in [t_1, T], X_t^{t_0, x, u, \beta_0(u)} = X_t^{t_1, X_{t_1}^{t_0, x, u_0, \beta_0(u_0)}, u, \beta_1(u)}.$$

Hence for any $u \in \mathcal{U}(t_1)$,

$$\begin{aligned} & J(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0, u, \beta_1) = \\ &= \int_{\mathbb{R}^N} g \left(X_T^{t_1, x, u, \beta_1} \right) d(X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0)(x) = \\ & \int_{\mathbb{R}^N} g \left(X_T^{t_0, x, u_1, \beta_0(u_1)} \right) d\mu_0(x) \end{aligned}$$

So

$$\inf_{u \in \mathcal{U}(t_1)} J(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0, u, \beta_1) =$$

v

$$\inf_{u \in \mathcal{U}(t_0) \text{ with } u|_{[t_0, t_1]} = u_0|_{[t_0, t_1]}} J(t_0, \mu_0, u, \beta_0).$$

Hence

$$V^-(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)}) \geq \inf_{u \in \mathcal{U}(t_0) \text{ with } u|_{[t_0, t_1]} = u_0|_{[t_0, t_1]}} J(t_0, \mu_0, u, \beta_0).$$

Consequently, u_0 and β_0 being arbitrary, we have $V^-(t_0, \mu_0) \leq W(t_0, t_1, \mu_0)$.

Let us prove the reverse inequality

$$V^-(t_0, \mu_0) \geq W(t_0, t_1, \mu_0) := \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^-\left(t_1, X_{t_1}^{t_0, \cdot, u, \beta} \# \mu_0\right)$$

Fix $\varepsilon > 0$. For any $\mu \in \mathcal{M}$ there exists some $\beta_\mu \in \mathcal{B}(t_1)$ such that

$$\inf_{u \in \mathcal{U}(t_1)} J(t_1, \mu, u, \beta_\mu) \geq V^-(t_1, \mu) - \varepsilon.$$

Fix $\beta_0 \in \mathcal{U}(t_0)$. Define $\beta_0 \in \mathcal{B}(t_0)$ as follows: for any $u \in \mathcal{U}(t_0)$ we have

$$\beta(u)|_{[t_0, t_1]} = \beta_0(u)|_{[t_0, t_1]}, \quad \beta(u)|_{[t_1, T]} = \beta_{\mu_1}(u|_{[t_1, T]}),$$

where $\mu_1 = X_{t_1}^{t_0, \cdot, u, \beta_0} \# \mu_0$. Hence for any $u \in \mathcal{U}(t_0)$ we obtain

$$J(t_0, \mu_0, u, \beta(u)) = J(t_1, \mu_1, u|_{[t_1, T]}, \beta_{\mu_1}(u|_{[t_1, T]})) \geq V^-(t_1, \mu_1) - \varepsilon.$$

Hence

$$V^-(t_0, \mu_0) \geq \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) \geq \inf_{u \in \mathcal{U}(t_0)} V^-(t_1, X_{t_1}^{t_0, \cdot, u, \beta_0} \# \mu_0) - \varepsilon.$$

We obtained the wished conclusion passing to the supremum in β_0 because ε is arbitrary. \square

Hamilton Jacobi Isaacs Equation

$$w_t + \mathcal{H}(\mu, Dw) = 0 \tag{7}$$

where $\mathcal{H} = \mathcal{H}(\mu, p)$ is an Hamiltonian defined for any $\mu \in \mathcal{M}$ and $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$.

Definition 7 (Sub- and super-differential) *Let $w : [0, T] \times \mathcal{M} \rightarrow \mathbb{R}$ be a function, $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$ and let $\delta > 0$. $(p_t, p_\mu) \in \mathbb{R} \times L_\mu^2(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the δ -super-differential $D_\delta^+ w(t_0, \mu_0)$ to w at (t_0, μ_0) if, $\forall \varphi \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$,*

$$\limsup_{\|\varphi\|_{L_\mu^2} \rightarrow 0, t \rightarrow t_0} [w(t, (\text{id}_{\mathbb{R}^N} + \varphi) \# \mu_0) - w(t_0, \mu_0) - p_t(t - t_0)$$

$$- \int_{\mathbb{R}^N} \langle \varphi(x), p_\mu(x) \rangle d\mu_0(x)] \frac{1}{\|\varphi\|_{L_\mu^2} + |t - t_0|} \leq \delta$$

A pair $(p_t, p_\mu) \in \mathbb{R} \times L_\mu^2(\mathbb{R}^N, \mathbb{R}^N)$ belongs to the δ -sub-differential $D_\delta^- w(t_0, \mu_0)$ to w at (t_0, μ_0) if $(-p_t, -p_\mu)$ belongs to the δ -super-differential to $-w$ at (t_0, μ_0) .

Solutions of Hamilton-Jacobi equation

Definition 8 *We say that a map $w : [0, T] \times \mathcal{M} \rightarrow \mathbb{R}$ is a sub-solution of the HJ equation (7) if w is upper semi-continuous and if, for any $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$, for any $(p_t, p_\mu) \in D_\delta^+ w(t_0, \mu_0)$, we have for any $\delta > 0$,*

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C\delta \quad (8)$$

where $C > 0$ is a constant which depends only of \mathcal{H} .

In a similar way, w is a super-solution of the HJ equation (7) if w is lower semicontinuous and if, for any $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$, for any $(p_t, p_\mu) \in D_\delta^- w(t_0, \mu_0)$, we have

$$p_t + \mathcal{H}(\mu_0, p_\mu) \leq C\delta . \quad (9)$$

Values and HJI Equations

$$\mathcal{H}^+(\mu, p) = \inf_{u \in U} \sup_{v \in V} \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x)$$

$$\mathcal{H}^-(\mu, p) = \sup_{v \in V} \inf_{u \in U} \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x) .$$

Lemma 9 *Let $\mu, \nu \in \mathcal{M}$, γ be an optimal plan from μ to ν , and $p \in L^2_\mu$ and $q \in L^2_\nu$ be defined by (5) and (6) respectively. Then, for $\mathcal{H} = \mathcal{H}^+$ or $\mathcal{H} = \mathcal{H}^-$*

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2 ,$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

proof for \mathcal{H}^-

$$\begin{aligned} \mathcal{H}^-(\mu, p) &= \sup_v \inf_u \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x) \\ &= \sup_v \inf_u \int_{\mathbb{R}^{2N}} \langle f(x, u, v), x - y \rangle d\gamma(x, y) \\ &\leq \sup_v \inf_u \int_{\mathbb{R}^{2N}} \langle f(y, u, v), x - y \rangle d\gamma(x, y) \\ &\quad + k \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x, y) \\ &\leq \sup_v \inf_u \int_{\mathbb{R}^N} \langle f(y, u, v), q(y) \rangle d\nu(y) \\ &\quad + k \mathbf{d}^2(\mu, \nu) \\ &\leq \mathcal{H}^-(\nu, q) + k \mathbf{d}^2(\mu, \nu) \end{aligned}$$

□

Proposition 10 *The upper value function V^+ is a solution to HJI with $\mathcal{H} := \mathcal{H}^+$ while the lower value function V^- is a solution to HJI with $\mathcal{H} := \mathcal{H}^-$.*

proof of V^+ is a subsolution

Fix $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$, $\delta > 0$ and $(p_t, p_\mu) \in D_\delta^+ V^+(t_0, \mu_0)$.

We will prove that

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -\delta \quad (10)$$

Consider $t \in (t_0, T)$. **For any** $\alpha \in \mathcal{A}(t_0)$ and $\beta \in \mathcal{B}(t_0)$ **define** $\varphi_{\alpha, \beta} \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$ **such that**

$$(\text{id}_{\mathbb{R}^N} + \varphi_{\alpha, \beta})(x) = X_t^{t_0, x, \alpha, \beta} = x + \int_{t_0}^t f(x(s), u(s), v(s)) ds,$$

where (u, v) **is associated with** (α, β) **and** $x(s) = X_s^{t_0, x, \alpha, \beta}$.

$$\begin{aligned}
& V^+(t, X_t^{t_0, \cdot, \alpha, v} \# \mu_0) - V^+(t_0, \mu_0) - p_t(t - t_0) \\
& - \int_{\mathbb{R}^N} < \int_{t_0}^t f(X_s^{t_0, x, \alpha, \beta}, u(s), v(s)) ds, p_\mu(x) > d\mu(x) \quad (11) \\
& \leq (\|\varphi_{\alpha, \beta}\|_{L_\mu^2} + |t - t_0|)(\varepsilon(t, \varphi_{\alpha, \beta}) + \delta)
\end{aligned}$$

where $\varepsilon(t, \varphi_{\alpha, \beta}) \rightarrow 0$ as $t \rightarrow t_0$ and $\varphi_{\alpha, \beta} \rightarrow 0$ in L_μ^2 . Passing to the sup on v and inf on α , we obtain by DDP

$$\begin{aligned}
0 \leq & \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} \left[\int_{\mathbb{R}^N} < \int_{t_0}^t f(X_s^{t_0, x, \alpha(v), v}, \alpha(v)(s), v(s)) ds, p_\mu(x) > d\mu \right. \\
& \left. + p_t(t - t_0) + (\|\varphi_{\alpha(v), v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{\alpha, \beta})) \right]
\end{aligned}$$

for $\|\varphi_{\alpha(v), v}\|_{L_\mu^2} + |t - t_0|$ small enough.

For t close enough to t_0 we obtain

$$0 \leq \inf_{u \in U} \sup_{v \in \mathcal{V}(t_0)} \left[\int_{\mathbb{R}^N} < \int_{t_0}^t f(x, u, v(s)) ds, p_\mu(x) > d\mu(x) \right. \\ \left. + p_t(t - t_0) + (\|\varphi_{u,v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{u,v})) \right]$$

when we restrict the infimum to nonanticipative strategies α which has constant control values. Hence

$$0 \leq \inf_{u \in U} \sup_{v \in V} \left[(t - t_0) \int_{\mathbb{R}^N} < f(x, u, v) ds, p_\mu(x) > d\mu(x) \right. \\ \left. + p_t(t - t_0) + (\|\varphi_{u,v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{u,v})) \right]$$

Dividing this inequality by $t - t_0$ and letting $t \rightarrow t_0^+$ gives, since $\|\varphi_{u,v}\|_{L_\mu^2} = O(t - t_0)$,

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -(1 + M)\delta .$$

Comparison Principle for HJI

$$w_t + \mathcal{H}(\mu, Dw) = 0$$

Assumptions on \mathcal{H}

- $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N) \mapsto \mathcal{H}(\mu, \cdot)$ is positively homogeneous.
- for any $\mu, \nu \in \mathcal{M}$, if γ is the optimal plan from μ to ν , and $p \in L^2_\mu$ and $q \in L^2_\nu$ are defined by (5) and (6) respectively,

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2. \quad (12)$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), q(x) \rangle d\nu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(y), x - y \rangle d\gamma(x)$$

Comparison principle for HJI

Theorem 11 *Let w_1 be a bounded and Lipschitz continuous subsolution and w_2 be a bounded and Lipschitz continuous supersolution to (7). Then*

$$\inf_{[0,T] \times \mathcal{M}} (w_2 - w_1) = \inf_{\mathcal{M}} w_2(T, \cdot) - w_1(T, \cdot) .$$

Proof of Comparison Principle

$$A = \inf_{\mu \in \mathcal{M}} w_2(T, \mu) - w_1(T, \mu) .$$

Since \mathcal{H} is independant of w , $w_1 - A$ is still a subsolution.
So we suppose without loss of generality that $A = 0$.

By Contradiction

$$-\xi := \inf_{\mu \in \mathcal{M}, t \in [0, T]} w_2(t, \mu) - w_1(t, \mu) < 0 .$$

And choose $(t_0, \mu_0) \in [0, T] \times \mathcal{M}$ such that

$$(w_2 - w_1)(t_0, \mu_0) < -\xi/2. \tag{13}$$

Let $C > 0$ such that $\forall \delta > 0$

$$\forall (p_t, p_\mu) \in D_\delta^+ w_1(t_0, \mu_0), p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C\delta$$

$$\forall (p_t, p_\mu) \in D_\delta^- w_2(t_0, \mu_0) p_t + \mathcal{H}(\mu_0, p_\mu) \leq C\delta .$$

Fix $\varepsilon > 0$, $\eta > 0$ and $\delta > 0$ sufficiently small such that

$$\xi > 2\eta T + \frac{k^2\varepsilon}{2} \quad \text{and} \quad 2C\delta + 2k(\delta + k)^2\varepsilon < \eta . \quad (14)$$

We consider the following continuous function defined on $([0, T] \times \mathcal{M})^2$:

$$\Phi(s, \mu, t, \nu) = -w_1(s, \mu) + w_2(t, \nu) + \frac{1}{\varepsilon} (\mathbf{d}^2(\mu, \nu) + (t - s)^2) - \eta s .$$

Define

$$(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in \text{Arg} \min_{[0, T] \times \mathcal{M}} \Phi$$

From Ekeland Variational Principle that $\exists(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in \mathcal{M}^2 \times [0, T]^2$ such that for any $(s, \mu, t, \nu) \in ([0, T] \times \mathcal{M})^2$

$$\left\{ \begin{array}{l} i) \quad \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \\ ii) \quad \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(s, \mu, t, \nu) \\ \quad \quad + \delta([\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + [\mathbf{d}^2(\nu, \bar{\nu}) + |t - \bar{t}|^2]^{\frac{1}{2}}) \end{array} \right. \quad (15)$$

CLAIM 1 $\rho^2 := \mathbf{d}^2(\bar{\mu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2 \leq (k + \delta)^2 \varepsilon^2$ **Indeed,**

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(\bar{s}, \bar{\mu}, \bar{s}, \bar{\mu}) + \delta[\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$$

Since w_2 is k -Lipschitz continuous,

$$\begin{aligned} \delta\rho + w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - \eta\bar{s} &\geq w_2(\bar{t}, \bar{\nu}) - w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon}\rho^2 - \eta\bar{s} \\ &\geq w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - k\rho + \frac{1}{\varepsilon}\rho^2 - \eta\bar{s} \end{aligned}$$

Hence $\rho \leq (k + \delta)\varepsilon$,

Assume $\bar{s}, \bar{t} \in (0, T)$. Let γ be the optimal transport plan between $\bar{\mu}$ and $\bar{\nu}$ and $p \in L^2(\bar{\mu})$ and $q \in L^2(\bar{\nu})$ associated.

CLAIM 2

$$\left(\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta, \frac{2}{\varepsilon}p \right) \in D_{\delta}^+ w_1(\bar{s}, \bar{\mu}), \quad \left(\frac{2}{\varepsilon}(\bar{s} - \bar{t}), \frac{2}{\varepsilon}q \right) \in D_{\delta}^- w_2(\bar{t}, \bar{\nu}),$$

From (15)-ii), we have for any (s, μ) ,

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(s, \mu, \bar{t}, \bar{\nu}) + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}}.$$

$$\begin{aligned} w_1(s, \mu) &\leq w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon}(\mathbf{d}^2(\mu, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \bar{\nu}) + (s - \bar{t})^2 - (\bar{s} - \bar{t})^2) \\ &\quad + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s} - s) \end{aligned} \tag{16}$$

We will replace μ by $(\text{id}_{\mathbb{R}^N} + \varphi)\# \bar{\mu}$ with $\varphi \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$

Estimation

Observe $\gamma' = (\text{id}_{\mathbb{R}^N} + \varphi, \text{id}_{\mathbb{R}^N})\#\gamma$ is an admissible transport plan from $(\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}$ to $\bar{\nu}$. Hence

$$\begin{aligned} \mathbf{d}^2((\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \bar{\nu}) &\leq \\ \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma'(x, y) - \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x, y) & \\ = \int_{\mathbb{R}^{2N}} (|x + \varphi(x) - y|^2 - |x - y|^2) d\gamma(x, y) & \\ = \int_{\mathbb{R}^{2N}} (2 \langle x - y, \varphi(x) \rangle + |\varphi(x)|^2) d\gamma(x, y) & \\ = \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \quad (\text{from the def. of } p) & \end{aligned}$$

$$\begin{aligned} w_1(s, (\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}) &\leq w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \\ &+ \frac{1}{\varepsilon} (s - \bar{s}) [(s - \bar{s}) + 2(\bar{s} - \bar{t})] + \delta [\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s} - s) \end{aligned}$$

So

$$\begin{aligned} & w_1(s, (\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - (s - \bar{s})\left[\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta\right] \\ & - \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \leq \\ & + \frac{1}{\varepsilon}(s - \bar{s})^2 + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} \end{aligned}$$

Hence

$$\left(\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta, \frac{2}{\varepsilon}p \right) \in D_{\delta}^+ w_1(\bar{s}, \bar{\mu})$$

which the claim 2.

By Claim 2, because w_1 sub solution, w_2 supersolution

$$\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta + \mathcal{H}\left(\bar{\mu}, \frac{2}{\varepsilon}p\right) \geq -C\delta .$$

$$\frac{2}{\varepsilon}(\bar{s} - \bar{t}) + \mathcal{H}\left(\bar{\mu}, \frac{2}{\varepsilon}q\right) \leq C\delta .$$

Subtracting we obtain $\mathcal{H}(\bar{\nu}, q) - \mathcal{H}(\bar{\mu}, p) \leq 2C\delta - \eta$.

But from the assumption on \mathcal{H} we have $\mathcal{H}(\bar{\nu}, q) - \mathcal{H}(\bar{\mu}, p) \geq -k\mathbf{d}^2(\bar{\mu}, \bar{\nu})$. So

$$-k(k + \delta)^2\varepsilon \leq 2C\delta - \eta ,$$

a contradiction with (14).

We have to check that \bar{s} and \bar{t} cannot be equal to 0 or T . Let us assume for instance that $\bar{s} = T$. We first note that

$$\Phi(t_0, \mu_0, t_0, \mu_0) = w_2(t_0, \mu_0) - w_1(t_0, \mu_0) - \eta t_0 \leq -\xi/2$$

From (Ekeland), i), we have

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \leq -\xi/2 .$$

Since $\bar{s} = T$ and w_2 is k -Lipschitz continuous, we get (setting as before $\rho := [\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$)

$$\begin{aligned} -\xi/2 &\geq w_2(\bar{t}, \bar{\nu}) - w_1(T, \bar{\mu}) + \frac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq w_2(T, \bar{\mu}) - w_1(T, \bar{\mu}) - k\rho + \frac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq -k\rho + \frac{1}{\varepsilon}\rho^2 - \eta T , \end{aligned}$$

A contradiction with the choices of $\eta, \delta, \varepsilon$ and the estimation of ρ .

To show that $\bar{s} \neq 0$ and $\bar{t} \neq 0$, it is enough to use the standard fact that w_1 and w_2 are respectively sub- and supersolutions up to $t = 0$,

Lemma 12 *If w is a subsolution (resp. a supersolution) of (7) on the time interval $(0, T)$, then w is also a subsolution (resp. a supersolution) on $[0, T)$.*

□

Uniqueness Result for HJI

Corollary 13 *There exists at most one lipschitz continuous solution of (7) satisfying*

$$w(T, \mu) = \int_{\mathbb{R}^N} g(x) d\mu(x) , \forall \mu \in \mathcal{M}.$$

Existence of A value

Theorem 14 *We suppose the following Isaacs condition:*

$$\mathcal{H}^+ = \mathcal{H}^-.$$

Then the game has a value. Namely:

$$V^+(t, \mu) = V^-(t, \mu) \quad \forall (t, \mu) \in [0, T] \times \mathcal{M}.$$

Furthermore $V^+ = V^-$ is the unique solution of the Hamilton-Jacobi equation (7) with $\mathcal{H} = \mathcal{H}^+ = \mathcal{H}^-$.

Thank You for your Attention
