

Differential Games II

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Recall

$x' = f(x, u, v)$ where $u(t) \in U, v(t) \in V$

$$C(t_0, x_0, u(\cdot), v(\cdot)) = g(x[t_0, x_0, u(\cdot), v(\cdot)](T))$$

A map $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ is a NA Strategy iff,

$\forall v_1, v_2 \in \mathcal{V}(t_0), \forall t, v_1 = v_2$ **on** $[t_0, t] \Rightarrow \alpha(v_1) = \alpha(v_2)$ **on** $[t_0, t]$.

Value Function

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v)$$

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} C(t_0, x_0, u, \beta(u))$$

Dynamic Programming Principle

Theorem 1 *Let* $t_0 < t_0 + h < T$

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^-(t_0, x[t_0, x_0, u, \beta(u)](t_0 + h))$$

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} V^+(t_0 + h, x[t_0, x_0, \alpha(v), v](t_0 + h))$$

Proof We will prove that $V^-(t_0, x_0) = W(t_0, t_0 + h, x_0)$ with

$$W(t_0, t_0 + h, x_0) := \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^-(t_0 + h, x[t_0, x_0, u, \beta(u)](t_0 + h))$$

Step 1 Show that $V^-(t_0, x_0) \leq W(t_0, t_0 + h, x_0)$.

Fix $\beta_0 \in \mathcal{B}(t_0)$ and $u_0 \in \mathcal{U}(t_0)$. **Denote** $x(t) =: x[t_0, x_0, u, \beta_0(u)](t)$.

We define $\beta \in \mathcal{B}(t_0 + h)$ by

$$\forall u \in \mathcal{U}(t_0 + h), \beta(u) := \beta_0(\tilde{u}) \text{ with } \tilde{u} := \begin{cases} u_0 \text{ on } [t_0, t_0 + h] \\ u \text{ on } [t_0 + h, T] \end{cases}$$

Observe that

$$x[t_0, x_0, u, \beta(u)](t) = x[t_0, x_0, \tilde{u}, \beta_0(\tilde{u})](t), \quad \forall t \in [t_0 + h, T]$$

So

$$\inf_{u \in \mathcal{U}(t_0+h)} C(t_0 + h, x(t_0 + h), u, \beta(u)) =$$

$$\inf_{\begin{cases} u \in \mathcal{U}(t_0 + h) \\ u = u_0 \text{ on } [t_0, t_0 + h] \end{cases}} C(t_0, x_0, u, \beta_0(u))$$

Then

$$V^-(t_0 + h, x(t_0 + h)) \geq \inf_{u \in \mathcal{U}(t_0+h)} C(t_0 + h, x(t_0 + h), u, \beta(u)) =$$

$$\inf_{\begin{cases} u \in \mathcal{U}(t_0) \\ u = u_0 \text{ on } [t_0, t_0 + h] \end{cases}} C(t_0, x_0, u, \beta_0(u))$$

Passing to the inf on u_0 and the sup on β_0

$$\sup_{\beta_0 \in \mathcal{B}(t_0)} \inf_{u_0 \in \mathcal{U}(t_0)} V^-(t_0 + h, x(t_0 + h)) \geq$$

$$\sup_{\beta_0 \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} C(t_0, x_0, u, \beta_0(u)) = V^-(t_0, x_0)$$

We have $W(t_0, t_0 + h, x_0) \geq V^-(t_0, x_0)$.

Step 2 Show that $V^-(t_0, x_0) \geq W(t_0, t_0 + h, x_0)$.

Fix $\varepsilon > 0$. **For every** x , **there exists** $\bar{\beta}_x \in \mathcal{B}(t_0 + h)$ **s.t.**

$$\inf_{u \in \mathcal{U}(t_0 + h)} C(t_0 + h, x, u, \bar{\beta}_x(u)) \geq V^-(t_0 + h, x) - \varepsilon$$

Fix β_0 **and define**

$$\forall u \in \mathcal{U}(t_0), \beta(u) := \begin{cases} \beta_0(u) & \text{on } [t_0, t_0 + h] \\ \bar{\beta}_{x(t_0+h)}(u|[t_0 + h, T]) & \end{cases}$$

where $x(t) = x[t_0, x_0, u, \beta_0(u)](t)$.

$$\begin{aligned} C(t_0, x_0, u, \beta(u)) &= C(t_0+h, x(t_0+h), u|[t_0+h, T], \bar{\beta}_{x(t_0+h)}(u|[t_0+h, T])) \\ &\geq V^-(t_0 + h, x(t_0 + h)) - \varepsilon \end{aligned}$$

So passing to the inf on u

$$\inf_{u \in \mathcal{U}(t_0)} C(t_0, x_0, u, \beta(u)) \geq \inf_{u \in \mathcal{U}(t_0)} V^-(t_0 + h, x(t_0 + h)) - \varepsilon$$

So

$$V^-(t_0, x_0) \geq \inf_{u \in \mathcal{U}(t_0)} V^-(t_0 + h, x(t_0 + h)) - \varepsilon$$

Taking the sup on β_0 we have $V^-(t_0, x_0) \geq W(t_0, t_0 + h, x_0) - \varepsilon$.

Since ε is arbitrary, the proof is complete.

q.e.d.

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Viscosity Solution

As we have seen before value functions of differential game problems have very much to do with partial differential equations. These equations can be of the evolutionary type, as for Bolza problem:

$$V_t(t, x) + H(t, x, V_x(t, x)) = 0 \quad \text{for all } (t, x) \in [0, T) \times \mathbb{R}^N$$

or of stationary type, as for the minimal time problem or for the infinite horizon problem:

$$H(x, V(x), \nabla V(x)) = 0 \quad \text{for all } x \in \Omega .$$

These first order equations are usually called Hamilton-Jacobi-Isaacs equations. In order to give a suitable definition of solutions for these equations, without having to consider the different forms of these equations, we write them in a generic way:

$$H(t, x, V(t, x), V_t(t, x), V_x(t, x)) = 0 \quad \text{for all } (t, x) \in \Omega. \quad (1)$$

where now $H = H(t, x, s, p_t, p_x)$ is the so-called *hamiltonian* of the problem and Ω is some open subset of $[0, +\infty) \times \mathbb{R}^N$.

Definition 2 (Viscosity solutions) *Let $V : \Omega \rightarrow \mathbb{R}$ be a continuous function. We say that:*

- *V is a supersolution of equation (1) if V is lower semi-continuous and if, for any $(t, x) \in \Omega$, for any test function $\varphi \in \mathcal{C}^1(\Omega)$ such that $V - \varphi$ has a local minimum at*

(t, x) , we have

$$H(t, x, V(t, x), \varphi_t(t, x), \varphi_x(t, x)) \leq 0 .$$

- V is a subsolution of equation (1) if V is upper semicontinuous and if, for any $(t, x) \in \Omega$, for any test function $\varphi \in C^1(\Omega)$ where $V - \varphi$ has a local maximum at (t, x) , we have

$$H(t, x, V(t, x), \varphi_t(t, x), \varphi_x(t, x)) \geq 0 .$$

- V is a solution of equation (1) if V is a super- and a subsolution of equation (1).

Let us first point out that a classical solution—when it exists—is always a viscosity solution:

Proposition 3 *Assume that V is a C^1 solution of (1). Then V is a viscosity solution of (1).*

Proof of Proposition 3 : We only prove that V is a supersolution, the proof for the subsolution being symmetric. Suppose that a \mathcal{C}^1 test function φ is such that $V - \varphi$ has a local minimum at some point $(t, x) \in \Omega$. The functions V and φ being \mathcal{C}^1 , the necessary condition of optimality implies that

$$V_t(t, x) = \varphi_t(t, x) \text{ and } V_x(t, x) = \varphi_x(t, x) .$$

Since V is a solution of (1), this gives:

$$0 = H(t, x, V(t, x), V_t(t, x), V_x(t, x)) = H(t, x, V(t, x), \varphi_t(t, x), \varphi_x(t, x))$$

which is the desired result, with an equality instead of an inequality.

q.e.d.

From now on, we state several standard properties of viscosity solutions. Let us start with some purely algebraic properties.

Lemma 4 *Let us assume that V is a supersolution of (1). Then $-V$ is a subsolution of equation*

$$\tilde{H}(t, x, w(t, x), w_t(t, x), w_x(t, x)) = 0 \quad \text{in } \Omega \quad (2)$$

where

$$\tilde{H}(t, x, s, p_t, p_x) = -H(t, x, -s, -p_t, -p_x) .$$

Conversely, if V is a subsolution of (1), then $-V$ is a supersolution of (2).

Proof of Lemma 4 : Assume that V is a supersolution of (1). Let φ be a \mathcal{C}^1 test function such that $-V - \varphi$ has a local maximum at some point (t_0, x_0) . This means that

$V - (-\varphi)$ has a local minimum at (t_0, x_0) and thus we have, since V is a supersolution of (1),

$$H(t_0, x_0, V(t_0, x_0), -\varphi_t(t_0, x_0), -\varphi_x(t_0, x_0)) \leq 0 .$$

This inequality can be rewritten as

$$-H(t_0, x_0, -(-V(t_0, x_0)), -\varphi_t(t_0, x_0), -\varphi_x(t_0, x_0)) \geq 0 ,$$

whence

$$\tilde{H}(t, x, w(t, x), w_t(t, x), w_x(t, x)) \geq 0 \quad \text{in } \Omega .$$

This proves that V is a subsolution of (2).

q.e.d.

Existence Result

$$C(t_0, x_0, u(\cdot), v(\cdot)) = \int_{t_0}^T L(t, x(t), u(t), v(t))e^{-\lambda t} dt + g(x(T))$$

Lemma 5 *Let us assume that the following superdynamic programming holds for any $h \in (0, h_0)$, for some $h_0 > 0$, at some point (t_0, x_0) .*

$$V^-(t_0, x_0) \geq \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u(\cdot) \in \mathcal{U}(t_0)} \left\{ \int_{t_0}^{t_0+h} L(s, x(s), u(s), \beta(u)(s)) ds + e^{-\lambda h} V^-(t_0 + h, x(t_0 + h)) \right\} \quad (3)$$

Let φ be a \mathcal{C}^1 such that $V^-\varphi$ has a local minimum at some point (t_0, x_0) . Then

$$\inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \left\{ \varphi_t(t_0, x_0) + \langle f(t_0, x_0, u, v), \varphi_x(t_0, x_0) \rangle - \lambda V^-(t_0, x_0) + L(t_0, x_0, u, v) \right\} \leq 0 \quad (4)$$

If, on the contrary, the subdynamic programming holds

$$V^-(t_0, x_0) \leq \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u(\cdot) \in \mathcal{U}(t_0)} \left\{ \int_{t_0}^{t_0+h} L(s, x(s), u(s), \beta(u)(s)) ds + e^{-\lambda h} V^-(t_0 + h, x(t_0 + h)) \right\} \quad (5)$$

for any $h \in (0, h_0)$ (for some $h_0 > 0$) and if $V - \varphi$ has a local maximum at (t_0, x_0) , then

$$\inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \{ \varphi_t(t_0, x_0) + \langle f(t_0, x_0, u, v), \varphi_x(t_0, x_0) \rangle - \lambda V^-(t_0, x_0) + L(t_0, x_0, u, v) \} \geq 0. \quad (6)$$

Remark In other words, V^- is a viscosity solution of $D_t V(t, x) + H^-(t, x, D_x V(t, x)) - \lambda V(t, x) = 0$

Technical Lemma

this technical remark will explain later on why the expression $\sup_{\beta \in \mathcal{B}(t_0)} \inf_{u(\cdot) \in \mathcal{U}(t_0)}$ in the DPP becomes $\inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}}$ in the HJI equations.

Lemma 6 *Let $\theta : \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R}$ be a continuous function and let $\bar{\mathbf{V}}$ be the set of Borell measurable maps $\bar{v} : \mathbf{U} \rightarrow \mathbf{V}$:*

$$\bar{\mathbf{V}} = \{ \bar{v} : \mathbf{U} \rightarrow \mathbf{V}, \bar{v} \text{ Borell measurable} \} .$$

Then

$$\min_{u \in \mathbf{U}} \max_{v \in \mathbf{V}} \theta(u, v) = \sup_{\bar{v} \in \bar{\mathbf{V}}} \inf_{u \in \mathbf{U}} \theta(u, \bar{v}(u)) .$$

Proof of Lemma 6: Let $\bar{v} \in \bar{V}$ be fixed. Then, for any $u \in \mathbf{U}$, we have

$$\theta(u, \bar{v}(u)) \leq \sup_{v \in \mathbf{V}} \theta(u, v) ,$$

which implies that

$$\inf_{u \in \mathbf{U}} \theta(u, \bar{v}(u)) \leq \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \theta(u, v) .$$

Whence the inequality

$$\sup_{\bar{v} \in \bar{V}} \inf_{u \in \mathbf{U}} \theta(u, \bar{v}(u)) \leq \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \theta(u, v) .$$

In order to prove the reverse inequality, let us fix $\varepsilon > 0$ and, for any $u \in \mathbf{U}$, let $v_u \in \mathbf{V}$ be such that

$$\theta(u, v_u) \geq \sup_{v \in \mathbf{V}} \theta(u, v) - \frac{\varepsilon}{2} .$$

Since θ is continuous and \mathbf{V} is compact, the map $u \rightarrow \sup_{v \in \mathbf{V}} \theta(u, v)$ is continuous. Therefore we can find a covering of \mathbf{V} by a finite family of open sets $\mathcal{O}_i \subset \mathbf{V}$ (for $i = 1, \dots, k$) and a finite family of controls $v_i \in \mathbf{V}$ (for $i = 1, \dots, k$) such that

$$\theta(u, v_i) \geq \sup_{v \in \mathbf{V}} \theta(u, v) - \varepsilon \quad \forall u \in \mathcal{O}_i .$$

Then the Borel map \tilde{v} defined by

$$\tilde{v}(u) = v_i \text{ if } u \in \mathcal{O}_i \setminus \bigcup_{j < i} \mathcal{O}_j$$

satisfies

$$\theta(u, \tilde{v}(u)) \geq \sup_{v \in \mathbf{V}} \theta(u, v) - \varepsilon \quad \forall u \in \mathbf{U} .$$

Therefore

$$\inf_{u \in \mathbf{U}} \theta(u, \tilde{v}(u)) \geq \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \theta(u, v) - \varepsilon ,$$

which finally implies

$$\sup_{\bar{v} \in \bar{\mathbf{V}}} \inf_{u \in \mathbf{U}} \theta(u, \bar{v}(u)) \geq \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \theta(u, v) - \varepsilon .$$

Hence the result is proved since ε is arbitrary.

q.e.d.

We are now ready to start the proof of Lemma 5.

Proof of Lemma 5: We first prove that (3) entails (4). Let $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$ and $\varphi \in \mathcal{C}^1$ be a test function such that $V^+ - \varphi$ has a local minimum at (t_0, x_0) . This means

that there is a neighbourhood \mathcal{O} of (t_0, x_0) such that

$$V(t, x) - V(t_0, x_0) \geq \varphi(t, x) - \varphi(t_0, x_0) \quad \forall (t, x) \in \mathcal{O}. \quad (7)$$

We can rewrite the superdynamic programming as

$$\sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \left[e^{-\lambda h} V(t_0 + h, x(t_0 + h)) - V(t_0, x_0) + \int_{t_0}^{t_0+h} \bar{L}(s) ds \right] \leq 0.$$

where we have set $x(\cdot) = x[t_0, x_0, u, \beta(u)]$ and $\bar{L}(s) = L(s, x(s), u(s), \beta(u))$.
Owing to inequality (7) this implies that

$$\begin{aligned} \sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} [& e^{-\lambda h} (\varphi(t_0 + h, x(t_0 + h)) - \varphi(t_0, x_0)) \\ & + (e^{-\lambda h} - 1)V(t_0, x_0) + \int_{t_0}^{t_0+h} \bar{L}(s) ds] \leq 0 \end{aligned} \quad (8)$$

for any $h > 0$ sufficiently small. Since φ is \mathcal{C}^1 , we have

$$\varphi(t_0 + h, x(t_0 + h)) - \varphi(t_0, x_0) = \int_{t_0}^{t_0+h} [\bar{\varphi}_t(s) + \langle \bar{\varphi}_x(s), \bar{f}(s) \rangle] ds$$

where we have set, for simplicity $\bar{\varphi}_t(s) = \varphi_t(s, x(s))$ and $\bar{\varphi}_x(s) = \varphi_x(s, x(s))$ and $\bar{f}(s) = f(s, x(s), u(s), \beta(u)(s))$. Moreover, the term $(e^{-\lambda h} - 1)$ can be written as

$$e^{-\lambda h} - 1 = - \int_{t_0}^{t_0+h} \lambda e^{-\lambda(s-t_0)} ds .$$

Thus inequality (8) can be put under the form

$$\sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \int_{t_0}^{t_0+h} [e^{-\lambda h} (\bar{\varphi}_t(s) + \langle \bar{\varphi}_x(s), \bar{f}(s) \rangle) - \lambda e^{-\lambda(s-t_0)} V(t_0, x_0) + \bar{L}(s)] ds \leq 0$$

Since f , L , φ_t and φ_x are continuous with respect to (s, x) , for any fixed $\varepsilon > 0$, there is some $h_0 > 0$ (which does not

depend on β and on u), such that, for all $h \in (0, h_0)$, we have

$$\begin{aligned} & \frac{1}{h} \int_{t_0}^{t_0+h} [e^{-\lambda h} (\bar{\varphi}_t(s) + \langle \bar{\varphi}_x(s), \bar{f}(s) \rangle) - \lambda e^{-\lambda(s-t_0)} V(t_0, x_0) + \bar{L}(s)] ds \\ & \geq \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u(s), \beta(u)(s)) \rangle \\ & \quad - \lambda V(t_0, x_0) + L(t_0, x_0, u(s), \beta(u)(s))] ds - \varepsilon . \end{aligned}$$

With this inequality, we have

$$\begin{aligned} & \sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u(s), \beta(u)(s)) \rangle \\ & \quad - \lambda V(t_0, x_0) + L(t_0, x_0, u(s), \beta(u)(s))] ds \leq \varepsilon . \end{aligned}$$

Let us now notice that, for all Borell measurable function $\bar{v} : \mathbf{U} \rightarrow \mathbf{V}$, the map $\beta : \mathcal{U}(t_0) \rightarrow \mathcal{V}(t_0)$ defined by

$$\forall u \in \mathcal{U}(t_0), \beta(u)(s) = \bar{v}(u(s))$$

is a nonanticipative strategy. Therefore, when the second player plays only such strategies we get

$$\sup_{\bar{v} \in \bar{V}} \inf_{u(\cdot)} \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u(s), \bar{v}(u(s))) \rangle - \lambda V(t_0, x_0) + L(t_0, x_0, u(s), \bar{v}(u(t)))] ds \leq \varepsilon ,$$

where

$$\bar{V} = \{ \bar{v} : \mathbf{U} \rightarrow \mathbf{V} , \bar{v} \text{ Borell measurable} \} .$$

But clearly

$$\begin{aligned}
& \sup_{\bar{v} \in \bar{V}} \inf_{u(\cdot)} \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u(t), \bar{v}(u(t))) \rangle \\
& \quad - \lambda V(t_0, x_0) + L(t_0, x_0, u(s), \bar{v}(u(t)))] ds \\
& = \sup_{\bar{v} \in \bar{V}} \inf_{u \in \mathbf{U}} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, \bar{v}(u)) \rangle \\
& \quad - \lambda V(t_0, x_0) + L(t_0, x_0, u, \bar{v}(u))]
\end{aligned}$$

because the integrand does not depend on the time. Then we deduce from Lemma 6 that

$$\begin{aligned}
& \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, v) \rangle - \lambda V(t_0, x_0) + \\
& L(t_0, x_0, u, v)] \leq \varepsilon . \text{ Since } \varepsilon \text{ can be chosen arbitrarily small,}
\end{aligned}$$

we get the desired result:

$$\inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \{ \varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, v) \rangle - \lambda V(t_0, x_0) + L(t_0, x_0, u, v) \} \leq 0 .$$

Let us now prove that the subdynamic programming principle (5) implies (6). Let φ be a \mathcal{C}^1 test function such that $V - \varphi$ has a local maximum at (t_0, x_0) . This means that there is a neighbourhood \mathcal{O} of (t_0, x_0) such that

$$\forall (t, x) \in \mathcal{O}, V(t, x) - V(t_0, x_0) \leq \varphi(t, x) - \varphi(t_0, x_0) .$$

From the subdynamic programming principle (5), we get as previously, for any $h > 0$ sufficiently small:

$$\sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \left[e^{-\lambda h} V(t_0 + h, x(t_0 + h)) - V(t_0, x_0) + \int_{t_0}^{t_0+h} \bar{L}(s) ds \right] \geq 0$$

(where we have set $\bar{L}(s) = L(s, x(s), u(s), \beta(u)(s))$), which now implies that

$$\sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \left[e^{-\lambda h} (\varphi(t_0 + h, x(t_0 + h)) - \varphi(t_0, x_0)) + (e^{-\lambda h} - 1)V(t_0, x_0) \right. \\ \left. + \int_{t_0}^{t_0+h} \bar{L}(s) ds \right] \geq 0$$

for any $h > 0$ sufficiently small. Arguing as previously, this inequality implies that, for any $\varepsilon > 0$, there is some $h_0 > 0$ such that for all $h \in (0, h_0)$,

$$\sup_{\beta} \inf_{u(\cdot)} \frac{1}{h} \int_{t_0}^{t_0+h} \left[\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u(s), \beta(u)(s)) \rangle \right. \\ \left. - \lambda V(t_0, x_0) + L(t_0, x_0, u(s), \beta(u)(s)) \right] ds \geq -\varepsilon .$$

In particular, if the first player only plays constant controls,

this gives:

$$\sup_{\beta} \inf_{u \in \mathbf{U}} \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, \beta(u)(s)) \rangle - \lambda V(t_0, x_0) + L(t_0, x_0, u, \beta(u)(s))] ds \geq -\varepsilon .$$

But now note that the integrand does not depend on the time, and thus the second player can as well play strategies of the form $\bar{v} \in \bar{V}$:

$$\begin{aligned} & \sup_{\beta} \inf_{u \in \mathbf{U}} \frac{1}{h} \int_{t_0}^{t_0+h} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, \beta(u)(s)) \rangle - \lambda V(t_0, x_0) + L(t_0, x_0, u, \beta(u)(s))] ds \\ &= \sup_{\bar{v} \in \bar{V}} \inf_{u \in \mathbf{U}} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, \bar{v}(u)) \rangle - \lambda V(t_0, x_0) + L(t_0, x_0, u, \bar{v}(u))] \end{aligned}$$

We can then deduce from Lemma 6 that

$$\inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} [\varphi_t(t_0, x_0) + \langle \varphi_x(t_0, x_0), f(t_0, x_0, u, v) \rangle - \lambda V(t_0, x_0)$$

+L(t_0, x_0, u, v)] $\geq -\varepsilon$. This completes the proof since ε is arbitrary.

q.e.d.

Continuity of the Values

Infinite Horizon cost

$$C(x_0, u(\cdot), v(\cdot)) = \int_0^{+\infty} e^{-\lambda t} L(x(t), u(t), v(t)) dt$$

- $$\left\{ \begin{array}{l} i) \mathbf{U} \text{ and } \mathbf{V} \text{ are compact subsets} \\ \text{of some finite dimensional space} \\ ii) f : \mathbb{R}^N \times \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R}^N \text{ is continuous and bounded,} \\ \text{and Lipschitz continuous with respect to the } x \text{ variable} \\ iii) L : \mathbb{R}^N \times \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R} \text{ is continuous and bounded,} \\ \text{and Lipschitz continuous with respect to the } x \text{ variable} \end{array} \right. \quad (9)$$

Proposition 7 *Under assumption (9), the value functions V^+ and V^- are Hölder continuous and bounded in \mathbb{R}^N .*

Hamilton Jacobi Isaacs Equations

$$H^+(x, p) = \inf_{u \in \mathbf{U}} \sup_{v \in \mathbf{V}} \{L(x, u, v) + \langle f(x, u, v), p \rangle\} \quad \forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N$$

$$H^-(x, p) = \sup_{v \in \mathbf{V}} \inf_{u \in \mathbf{U}} \{L(x, u, v) + \langle f(x, u, v), p \rangle\} \quad \forall (x, s, p) \in \mathbb{R}^N \times \mathbb{R}^N$$

Proposition 8 *Under assumption (9), the functions V^+ and V^- are respectively viscosity solutions of*

$$-\lambda V(x) + H^+(x, \nabla V(x)) = 0 \quad \text{in } \mathbb{R}^N \quad (10)$$

and of

$$-\lambda V(x) + H^-(x, \nabla V(x)) = 0 \quad \text{in } \mathbb{R}^N \quad (11)$$

Comparison principle

Stationary Hamilton-Jacobi Isaacs equation of the form:

$$-\lambda V(x) + H(x, \nabla V(x)) = 0 \quad \text{in } \mathbb{R}^N \quad (12)$$

where $H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$.

$$|H(y, p) - H(x, p)| \leq C_R \|y - x\| (1 + \|p\|)$$

for some $C_r > 0$, for any $x, y \in \Omega$, $-R \leq s \leq R$, $\forall p \in \mathbb{R}^N$, $\forall R > 0$
(13)

Theorem 9 Assume that (13). If V_1 is a bounded viscosity subsolution of (12) while V_2 is a bounded viscosity supersolution of (12) then

$$V_1(x) \geq V_2(x) \quad \forall x$$

We immediately deduce from this result that equation (12), with a fixed boundary condition has at most one continuous viscosity solution.

Corollary 10 Assume that H satisfies (13). Then, the problem

$$H(x, V(x), \nabla V(x)) = 0$$

has at most one bounded continuous viscosity solution.

Existence of The Value

Theorem 11 *If moreover Isaacs condition holds:*

$$H^+(x, p) = H^-(x, p) \quad \forall (x, p) \quad (14)$$

then the game has a value. Namely:

$$V^+(x) = V^-(x) \quad \forall x$$

Thank You for your Attention
