

Differential Games I

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Introduction : Preliminaries

We consider a two player differential game with dynamics

$$x' = f(x, u, v) \text{ where } u(t) \in U, v(t) \in V$$

Player I plays with u while **player II** plays with v .

$x(\cdot) = x[t_0, x_0, u(\cdot), v(\cdot)]$ is the solution with initial condition $x(t_0) = x_0$

- i) \mathbf{U} and \mathbf{V} are compact subsets of some finite dimensional space
- ii) $f : \mathbb{R}^N \times \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R}^N$ is continuous and bounded, and Lipschitz continuous with respect to the x variable

First player wants to minimize a cost (while the second player wants to maximize it)

Bolza cost : (Mayer cost is the case $L = 0$)

$$C(t_0, x_0, u(\cdot), v(\cdot)) = \int_{t_0}^T L(t, x(t), u(t), v(t))dt + g(x(T))$$

Minimal time cost

$$C(x_0, u(\cdot), v(\cdot)) = \theta_C(x[x_0, u(\cdot), v(\cdot)])$$

where C is a target and

$$\theta_C(x(\cdot)) = \inf\{t \geq 0 \mid x(t) \in C\} ,$$

Infinite Horizon cost

$$C(x_0, u(\cdot), v(\cdot)) = \int_0^{+\infty} e^{-\lambda t} L(x(t), u(t), v(t))dt$$

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Introduction : Lion and Man

Pursuit Game on a closed circular arena $B(0, 1)$:

Man : $y'(t) = u(t) \in MB$

Lion : $z'(t) = v(t) \in LB$

Questions (Man point of View)

- How long at least the man can escape the lion?
- If he succeeds to escape forever, what is the minimal distance he can put between him and the Lion?

Lion and Man

Case 1: $L > M$

$$\bar{v}(y(t), z(t)) := L \frac{y(t) - z(t)}{\|y(t) - z(t)\|}$$

Lion "solves" $z'(t) = \bar{v}(y(t), z(t))$

Remark: This is a way to react to the behavior of the man (Feedback Strategy)

WHATEVER THE MAN DOES the capture occurs before time

$$\frac{\|y(0) - z(0)\|}{L - M}$$

Lion and Man

Case 2: $L < M$

The Man can always escape the Lion

Polar coordinates (ρ_M, θ_M) and (ρ_L, θ_L)

$$\begin{cases} (\rho'_M)^2 + \rho_M^2 (\theta'_M)^2 \leq M^2 \\ (\rho'_L)^2 + \rho_L^2 (\theta'_L)^2 \leq L^2 \end{cases}$$

Man strategy (assume that the initial position $\|y(0)\| = 0$)

$$\begin{cases} \rho'_M = 0 \\ \theta'_M = +M \text{ if } \theta_M - \theta_L \geq 0 \\ \theta'_M = -M \text{ if } \theta_M - \theta_L < 0 \end{cases}$$

Exercise

Lion and Man

Case 2: $L = M$

The Man can always escape the Lion

The Lion can get as close he wants to the Man

Lion strategy (assume that $y(0)$ and $z(0)$ are in the same radius and $\|y(0)\| < 1$)

$$\begin{cases} \rho'_L = \rho'_M \\ \rho'_L = \sqrt{M^2 - \rho_L^2 (\theta'_M)^2} \end{cases}$$

A strategy (not a feedback strategy).

A verification Theorem for Pursuit Evasion Game

$$x' = f(x, u, v) \text{ where } u(t) \in U, v(t) \in V$$

$C \subset \mathbb{R}^N$ is a target that the second player wants to reach in minimum time while the first player wants to avoid it as long as possible

Player 1 "maximize "

$$\theta_C(x(\cdot)) = \inf\{t \geq 0 \mid x(t) \in C\} \in \mathbb{R}_+ \cup \{+\infty\}$$

while player 2 wants to "minimize" it.

Structure of Information : feedback strategies

Definition 1 *A feedback strategy for Player I is a map $\bar{u} : IR \times IR^N \mapsto U$. A feedback strategy for Player II is a map $\bar{v} : IR \times IR^N \mapsto V$.*

PROBLEM: Replacing strategies in the dynamic one obtains

$$x'(t) = f(x(t), \bar{u}(t, x(t)), \bar{v}(t, x(t))), \quad x(0) = x_0 \quad (1)$$

which does not necessarily have a solution. Thus we restrict the notion of strategies

Definition 2 *A subset (\bar{U}, \bar{V}) of the set of feedback strategies is admissible iff*

- 1. it contains all "open-loop strategies" (Lebesgue measurable maps $u: \mathbb{R} \mapsto U$ $v: \mathbb{R} \mapsto V$)*
- 2. for all pair $(\bar{u}, \bar{v}) \in (\bar{U}, \bar{V})$, and any x_0 , (1) as a unique solution.*
- 3. Concatenation property*
- 4. Shift property : $\forall \tau, u(\cdot + \tau) \in \bar{U}$*

To any (\bar{u}, \bar{v}) we associate a cost $C(x_0, \bar{u}, \bar{v}) = \theta_C(x[x_0, \bar{u}, \bar{v}])$

Definition 3 *For a class of strategies (\bar{U}, \bar{V}) we define the values*

$$V^-(x_0) = \sup_{\bar{u} \in \bar{U}} \inf_{\bar{v} \in \bar{V}} C(x_0, \bar{u}, \bar{v}), \quad V^+(x_0) = \inf_{\bar{v} \in \bar{V}} \sup_{\bar{u} \in \bar{U}} C(x_0, \bar{u}, \bar{v})$$

We say that the game has a value if $V^+ = V^-$. A strategy u^ is said optimal for player I, if*

$$V^-(x_0) = \inf_{\bar{v} \in \bar{V}} C(x_0, u^*, \bar{v})$$

A strategy v^ is said optimal for player I, if*

$$V^+(x_0) = \sup_{\bar{u} \in \bar{U}} C(x_0, \bar{u}, v^*)$$

Hamiltonians

$$H^-(x, p) = \sup_{u \in U} \inf_{v \in V} \langle f(x, u, v), p \rangle, \quad H^+(x, p) = \inf_{v \in V} \sup_{u \in U} \langle f(x, u, v), p \rangle$$

Assume **Isaacs' Condition**

$$\forall (x, p) \in \mathbb{R}^N \times \mathbb{R}^N, H^-(x, p) = H^+(x, p)$$

Denote for all x, p

$$\tilde{u}(x, p) \in \text{Arg} \max_{u \in U} \inf_{v \in V} \langle f(x, u, v), p \rangle$$

$$\tilde{v}(x, p) \in \text{Arg} \min_{v \in V} \sup_{u \in U} \langle f(x, u, v), p \rangle$$

Theorem 4 (*Isaacs Verification Theorem*) *Assume that C is closed and that Isaacs' condition holds true. If there exists a continuous $V : \mathbb{R}^N \mapsto \mathbb{R}$ with $V(x) = 0 \forall x \in C$ and of class C^1 on $\mathbb{R}^N \setminus C$ which satisfies*

$$H(x, DV(x)) + 1 = 0 \quad \forall x \in \mathbb{R}^N \setminus C. \quad (2)$$

If furthermore $\bar{u}^(x) := \tilde{u}(x, DV(x))$ and $\bar{v}^*(x) := \tilde{v}(x, DV(x))$ belong to \bar{U} and \bar{V} then V is the value of the game, $\bar{u}^*(x)$ and $\bar{v}^*(x)$ are optimal.*

Proof We prove first $\sup_{\bar{u}} C(x_0, \bar{u}, \bar{v}^*) \leq V(x_0)$. Indeed for $\bar{u} \in \bar{U}$ define $x(\cdot) := x(x_0, \bar{u}, \bar{v}^*)$ then $\forall t < \tau := C(x_0, \bar{u}, \bar{v}^*)$

$$\frac{d}{dt}V(x(t)) = \langle DV(x(t)), f(x(t), \bar{u}(t, x(t)), \bar{v}^*(x(t))) \rangle$$

$$\leq \max_{u \in U} \langle DV(x(t)), f(x(t), u, \bar{v}^*(x(t))) \rangle = H(x(t), DV(x(t))) = -1$$

Thus $V(x(t)) - V(x_0) \leq -t$. **So for** $t \rightarrow \tau^-$ **we obtain**

$$-V(x_0) \leq -\tau = -C(x_0, \bar{u}, \bar{v}^*)$$

Similarly $V(x_0) \leq \inf_{\bar{v} \in \bar{V}} C(x_0, \bar{u}^*, \bar{v})$. **Hence**

$$V^+(x_0) = \sup_{\bar{u}} C(x_0, \bar{u}, \bar{v}^*) \leq V(x_0) \leq \inf_{\bar{v} \in \bar{V}} C(x_0, \bar{u}^*, \bar{v}) = V^-(x_0)$$

So $V^+ = V = V^-$.

q.e.d.

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Introduction : Non anticipative Strategies

Varayia, Roxin, Elliot, Kalton

$$x' = f(x, u, v) \text{ where } u(t) \in U, v(t) \in V$$

$$C(t_0, x_0, u(\cdot), v(\cdot)) = g(x[t_0, x_0, u(\cdot), v(\cdot)](T))$$

For any $t_0 \in [0, T]$, we denote by $\mathcal{U}(t_0)$ (resp. $\mathcal{V}(t_0)$) the set of measurable maps $u : [t_0, T] \rightarrow U$ (resp. $v : [t_0, T] \rightarrow V$).

Definition 5 *A map $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ is a nonanticipative strategy if, for any v_1 and v_2 of $\mathcal{V}(t_0)$, which coincide almost everywhere on some subinterval $[t_0, t]$ of $[0, T]$, the images $\alpha(v_1)$ and $\alpha(v_2)$ coincide almost everywhere on $[t_0, t]$.*

Definition 6 *Value Function*

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v)$$

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} C(t_0, x_0, u, \beta(u))$$

Non anticipative Strategies and feedback Strategies

Proposition 7 *”Any Feedback strategy is a an NA strategy” To any feedback strategy and any intial position is canonically associated a NA strategy*

A feedback strategy for Player I is a map $\bar{u} : IR \times IR^N \mapsto U$. A feedback strategy for Player II is a map $\bar{v} : IR \times IR^N \mapsto V$.

Non anticipative Strategies with Delay

Definition 8 *A map $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ is a nonanticipative strategy with delay if there is a partition $t_0 < t_1 < \dots < t_n = T$ of $[t_0, T]$ such that, for any controls v_1 and v_2 of $\mathcal{V}(t_0)$, which coincide almost everywhere on a subinterval $[t_0, t_i]$ for some $i = 0, \dots, n - 1$, the images $\alpha(v_1)$ and $\alpha(v_2)$ coincide almost everywhere on $[t_0, t_{i+1}]$.*

Remark this definition means that $\alpha(v)$ does not depend on v on the interval $[t_0, t_1]$: $\forall v_1, v_2 \in \mathcal{V}(t_0)$, $\alpha(v_1) = \alpha(v_2)$ on $[t_0, t_1]$.

Nonanticipative strategies and nonanticipative strategies with delay $\beta : \mathcal{U}(t_0) \rightarrow \mathcal{V}(t_0)$ are defined in a symmetric way.

Lemma 9 *Let $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$ be a nonanticipative strategy with delay (resp. a nonanticipative strategy) and $\beta : \mathcal{U}(t_0) \rightarrow \mathcal{V}(t_0)$ be a nonanticipative strategy (resp. a nonanticipative strategy with delay). Then there is a unique pair of controls $(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that*

$$\alpha(v_0) = u_0 \text{ and } \beta(u_0) = v_0 .$$

Proof : We assume that α is a nonanticipative strategy with delay while β is a nonanticipative strategy. Let $t_0 < t_1 < \dots < t_n = T$ be a partition associated with α . We are going to define the pair of controls (u_0, v_0) by induction on the interval $[t_0, t_k)$. For that purpose, let us firstly notice that the restriction of $\alpha(v)$ to the interval $[t_0, t_1)$ does not depend on v . Hence u_0 is equal to this restriction on $[t_0, t_1)$. Moreover, $v_0 = \beta(u_0)$ is uniquely defined on $[t_0, t_1)$ because β is nonanticipative.

Let us assume we have constructed u_0 and v_0 on the interval $[t_0, t_k)$.

The restriction of $\alpha(v)$ to $[t_k, t_{k+1})$ only depends on the restriction of v to $[t_0, t_k)$. Hence u_0 is uniquely defined on $[t_0, t_{k+1})$ by $u_0 = \alpha(v_0)$. From which follows that $v_0 = \beta(u_0)$ on $[t_0, t_{k+1})$ because β is nonanticipative.

By induction we have uniquely defined u_0 and v_0 on $[t_0, T]$.

q.e.d.

Game in Normal Form

To any pair of strategies $(\alpha, \beta) \in \mathcal{A}_d \times \mathcal{B}_d$ one can associated a unique trajectory

$$x[t_0, x_0, \alpha, \beta] := x[t_0, x_0, u, v]$$

where (u, v) are associated to (α, β) by the Lemma.

Definition 10 *Value Function*

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} C(t_0, x_0, \alpha, \beta)$$

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{\alpha \in \mathcal{A}_d(t_0)} C(t_0, x_0, \alpha, \beta)$$

Obviously $V^- \leq V^+$

Proposition 11 *Equivalent Definition of Value*

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v)$$

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{u \in \mathcal{U}(t_0)} C(t_0, x_0, u, \beta(u))$$

Proof : $V^+(t_0, x_0) \geq \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v)$ by considering "constant" maps β .

For every NAD $(\alpha, \beta) \exists!(u, v)$ associated by the Lemma.

$$C(t_0, x_0, \alpha, \beta) = C(t_0, x_0, \alpha(v), v) \leq \sup_{v' \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v'), v')$$

$$\sup_{\beta \in \mathcal{B}_d(t_0)} C(t_0, x_0, \alpha, \beta) \leq \sup_{v' \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v'), v')$$

Taking the inf on α , the proof is complete

Comparison with NA strategies and NA with Delay

A NA with Delay is a NA strategy

there is no hope to approximate a nonanticipative strategy by nonanticipative strategy with delay “trajectory by trajectory”.

An example : Let us consider $U = V = [-1, 1]$ and the dynamics $f : \mathbb{R} \times U \times V \rightarrow \mathbb{R}$ defined by

$$f(x, u, v) = uv .$$

Let us point out that f satisfies Isaacs' condition (??). Let α be the following nonanticipative strategy:

$$\alpha(v)(t) = \text{sign}(v(t))$$

where

$$\text{sign}(s) = \begin{cases} 1 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0 \end{cases}$$

We claim that, for any nonanticipative strategy with delay $\bar{\alpha}$, there is some control \bar{v} such that

$$|x[0, 0, \bar{\alpha}(\bar{v}), \bar{v}](1) - x[0, 0, \alpha(\bar{v}), \bar{v}](1)| \geq 1 .$$

Proof : Let $t_0 = 0 < t_1 < \dots < t_n = 1$ be the partition associated with $\bar{\alpha}$. The restriction u_0 of $\bar{\alpha}(v)$ to $[0, t_1)$ does not depend on v . Let us set $\bar{v}(t) = -\text{sign}(u_0(t))$ on $[0, t_1)$. Let us now assume that we have constructed \bar{v} on some interval $[0, t_k)$. Then, the restriction of $\bar{\alpha}(v)$ on $[t_k, t_{k+1})$ only depends on the restriction of v on $[0, t_k)$. Let u_k be this restriction for $v = \bar{v}$. Let us then define \bar{v} on $[t_k, t_{k+1})$ by setting $\bar{v}(t) = -\text{sign}(u_k(t))$ on this interval.

By induction we have defined \bar{v} on $[0, 1]$, such that $\bar{\alpha}(\bar{v})(t)v(t) \leq 0$ for any $t \in [0, 1)$. However, since $|\bar{v}(t)| = 1$ for any $t \in [0, 1)$, we have $\alpha(\bar{v})(t)\bar{v}(t) = 1$ for any $t \in [0, 1)$. Hence

$$x[0, 0, \bar{\alpha}(v), v](1) \leq 0 \text{ and } x[0, 0, \alpha(v), v](1) = 1 .$$

This proves our claim.

q.e.d.

For any nonanticipative strategy $\alpha \in \mathcal{A}(t_0)$, let us set

$$R(\alpha)(t) = \text{closure}\{y \in \mathbb{R}^N \mid \exists v \in \mathcal{V}(t_0) \text{ with } y = x[t_0, x_0, \alpha(v), v](t), \}$$

We have the following result :

Theorem 12 *Let us assume that f satisfy Isaacs condition. Then, for any nonanticipative strategy $\alpha \in \mathcal{A}(t_0)$ and for any $\varepsilon > 0$, there is some nonanticipative strategy with delay $\bar{\alpha} \in \mathcal{A}_d^d(t_0)$ such that*

$$\forall t \in [t_0, T], \forall v \in \mathcal{V}(t_0), d_{R(\alpha)(t)}(x[t_0, x_0, \bar{\alpha}(v), v](t)) \leq \varepsilon .$$

where $d_{R(\alpha)(t)}(y) = \inf_{z \in R(\alpha)(t)} \|y - z\|$.

This means that the points one can reach at time t with the strategy with delay are very close from the point one reaches at time t with the original strategy.

Proof : It is quite long. Hence we divide it into several steps.

1. Let us first fix the notations. Let α be the nonanticipative strategy we want to approximate. Let us fix n very large (to be chosen later). The partition of the nonanticipative strategy with delay is $t_0 < t_1 < \dots < t_n = T$ with $t_k = kT/n$. We are going to construct the nonanticipative strategy with delay $\bar{\alpha}$ by induction on $k \in [[0, n - 1]]$.
2. Since the dynamics have at most a linear growth, there is some constant R such that, for any control u and v

$$\forall t \in [t_0, T], \|x[t_0, x_0, u, v](t)\| \leq R .$$

Let us set

$$M = \sup_{\|x\| \leq R, u \in U, v \in V} \|f(x, u, v)\| .$$

Since all the solution we are dealing with are bounded by M , we can do as if f is bounded by M . We also denote by L a Lipschitz constant for f with respect to x .

3. For $k = 0$, we choose any $u_0 \in U$ and we set $\bar{\alpha}(\bar{v}) = u_0$ on $[t_0, t_1)$ for any $\bar{v} \in \mathcal{V}(t_0)$.
4. Assume that we have constructed $\bar{\alpha}$ on $[t_0, t_k)$. Let us now fix $\bar{v} \in \mathcal{V}(t_0)$ and let us set $x_k = x[t_0, x_0, \bar{\alpha}(\bar{v}), \bar{v}](t_k)$. If x_k belongs to $R(\alpha)(t_k)$, then we choose some $u_k \in U$ and we set $\bar{\alpha}(\bar{v}) = u_k$ on $[t_k, t_{k+1})$. From now on, we assume that x_k does not belong to $R(\alpha)(t_k)$

Let us choose $y_k \in R(\alpha)(t_k)$ such that there is some control $v_k \in \mathcal{V}(t_0)$ with $x[t_0, x_0, \alpha(\bar{v}), \bar{v}](t_k) = y_k$ and such that

$$d_{R(\alpha)(t_k)}^2(x_k) \geq \|y_k - x_k\|^2 - \frac{1}{n^2}.$$

Let us define the nonanticipative strategy α_k by setting

$$\forall v \in \mathcal{V}(t_k), \alpha_k(v) = \alpha(\tilde{v})$$

where

$$\tilde{v} = \begin{cases} v_k & \text{on } [t_0, t_k) \\ v & \text{on } [t_k, T] \end{cases}$$

We claim that, for any $v \in \mathcal{V}(t_k)$ and for any $t \in [t_k, T]$, the point $x[t_k, y_k, \alpha_k(v), v](t)$ belongs to $R(\alpha)(t)$. Indeed

$$x[t_k, y_k, \alpha_k(v), v](t) = x[t_0, x_0, \alpha(\tilde{v}), \tilde{v}](t) \in \mathcal{R}(\alpha)(t)$$

since α is nonanticipative. Therefore, for any $u \in \mathcal{U}(t_k)$,

$$d_{R(\alpha)}(t)(x[t_k, y_k, u, \bar{v}](t)) \leq \inf_{v \in \mathcal{V}(t_k)} \|x[t_k, y_k, u, \bar{v}](t) - x[t_k, y_k, \alpha_k(v), v](t)\| \quad (3)$$

Let us set $p_k = x_k - y_k$ and let $u_k \in U$ be such that

$$\sup_{v \in V} \langle f(x_k, u_k, v), p_k \rangle = \inf_{u \in U} \sup_{v \in V} \langle f(x_k, u, v), p_k \rangle .$$

Then we set $\bar{\alpha}(\bar{v}) = u_k$ on $[t_k, t_{k+1})$. Let us point out that $\bar{\alpha}(\bar{v})$ on the interval $[t_k, t_{k+1})$ only depends on the restriction of \bar{v} on $[t_0, t_k)$ (because it only depends on x_k). The lengthy part of the proof consists in estimating the right-hand side of inequality (3).

5. Let us set $x(\cdot) = x[t_k, y_k, u_k, \bar{v}]$. We have, for any $v \in \mathcal{V}(t_k)$,

$$\begin{aligned} & \|x(t) - x[t_k, y_k, \alpha_k(v), v](t)\|^2 = \\ & \|x(t) - x_k\|^2 + \|x[t_k, y_k, \alpha_k(v), v](t) - y_k\|^2 + \|p_k\|^2 + \\ & 2 \langle x(t) - x_k, p_k \rangle - 2 \langle x[t_k, y_k, \alpha_k(v), v](t) - y_k, p_k \rangle \\ & + 2 \langle x(t) - x_k, y_k - x[t_k, y_k, \alpha_k(v), v](t) \rangle . \end{aligned}$$

For any $t \in [t_k, t_{k+1})$, we have

$$\|x(t) - x_k\|^2 = \left\| \int_{t_k}^t f(x(s), u_k, \bar{v}(s)) ds \right\|^2 \leq M^2(t - t_k)^2 \leq M^2/n^2$$

because f is bounded by M , and, in the same way,

$$\|x[t_k, y_k, \alpha_k(v), v](t) - y_k\|^2 \leq M^2(t - t_k)^2 \leq M^2/n^2 .$$

6. We now estimate the term $\langle x(t) - x_k, p_k \rangle$. For any

$t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} & \langle x(t) - x_k, p_k \rangle \\ &= \int_{t_k}^t \langle f(x(s), u_k, \bar{v}(s)), p_k \rangle ds \\ &\leq \int_{t_k}^t \langle f(x_k, u_k, \bar{v}(s)), p_k \rangle ds + ML\|p_k\|/n^2 \\ &\leq (t - t_k) \sup_{v \in V} \langle f(x_k, u_k, v), p_k \rangle + ML\|p_k\|/n^2 \\ &\leq (t - t_k) \inf_u \sup_v \langle f(x_k, u, v), p_k \rangle + ML\|p_k\|/n^2 \end{aligned}$$

7. We now deal with the term $\langle x[t_k, y_k, \alpha_k(v), v](t) - y_k, p_k \rangle$ (for simplicity, we set $y(\cdot) = x[t_k, y_k, \alpha_k(v), v]$): For any

$t \in [t_k, t_{k+1})$, we have

$$\begin{aligned}
& \langle y(t) - y_k, p_k \rangle \\
&= \int_{t_k}^t \langle f(y(s), \alpha_k(v)(s), v(s)), p_k \rangle ds \\
&\geq \int_{t_k}^t \langle f(y_k, \alpha_k(v)(s), v(s)), p_k \rangle ds - ML\|p_k\|/n^2 \\
&\geq \int_{t_k}^t \inf_u \langle f(x_k, u, v(s)), p_k \rangle ds - L\|p_k\|^2/n - ML\|p_k\|/n^2
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sup_{v \in \mathcal{V}(t_k)} \langle x[t_k, y_k, \alpha_k(v), v](t) - y_k, p_k \rangle \geq \\
& (t - t_k) \sup_{v \in V} \inf_u \langle f(x_k, u, v(s)), p_k \rangle - L\|p_k\|^2/n - ML\|p_k\|/n^2
\end{aligned}$$

8. Using Isaacs assumption (??), the three previous steps with inequality (3) give, for any $t \in [t_k, t_{k+1})$,

$$d_{R(\alpha)}^2(t)(x[t_k, y_k, u_k, \bar{v}](t)) \leq \|p_k\|^2(1+2L/n) + (4ML\|p_k\| + 2M^2)/n^2$$

where $\|p_k\| \leq 2R$ from the definition of R . Hence

$$d_{R(\alpha)(t)}^2(x[t_k, y_k, u_k, \bar{v}](t)) \leq d_{R(\alpha)(t_k)}^2(x_k)(1 + 2L/n) + C/n^2$$

for some constant C depending only on R , L and M because

$$\|p_k\|^2 \leq d_{R(\alpha)(t_k)}^2(x_k) + \frac{1}{n^2}.$$

9. By induction we have constructed some nonanticipative strategy with delay such that, for any $\bar{v} \in \mathcal{V}(t_0)$, for any $t \in [t_0, T]$,

$$d_{R(\alpha)(t)}^2 \leq \frac{C}{n^2}(1 + 2L/n) \frac{(1 + 2L/n)^n - 1}{2L/n}$$

The right-hand side goes to 0 when $n \rightarrow +\infty$. Thus we can choose n large enough so that the right-hand side is

smaller than ε . With this choice of n , the Theorem is proved.

q.e.d.

Corollary 13 *If g is continuous*

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} C(t_0, x_0, \alpha, \beta) =$$

$$\inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v) =$$

$$\inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} C(t_0, x_0, \alpha(v), v) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} C(t_0, x_0, \alpha, \beta)$$

Similarly for V^-

Random Non anticipative Strategies with Delay

Definition 14 *A Random Non Anticipative strategy with Delay is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha), \alpha)$ where $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ is a probability space and*

- $\alpha : \Omega \times \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ is a measurable map
- $\exists t_0 < t_1 < \dots < t_n = T$ such that, for any controls v_1 and v_2 of $\mathcal{V}(t_0)$, which coincide almost everywhere on a subinterval $[t_0, t_i]$ for some $i = 0, \dots, n - 1$, then

$$\alpha(v_1)(\omega, t) = \alpha(v_2)(\omega, t), \quad \forall t \in [t_0, t_{i+1}], \forall \omega \in \Omega_\alpha.$$

Thank You for your Attention
