Tutorial: Introduction to Interior Point Methods for Predictive Control

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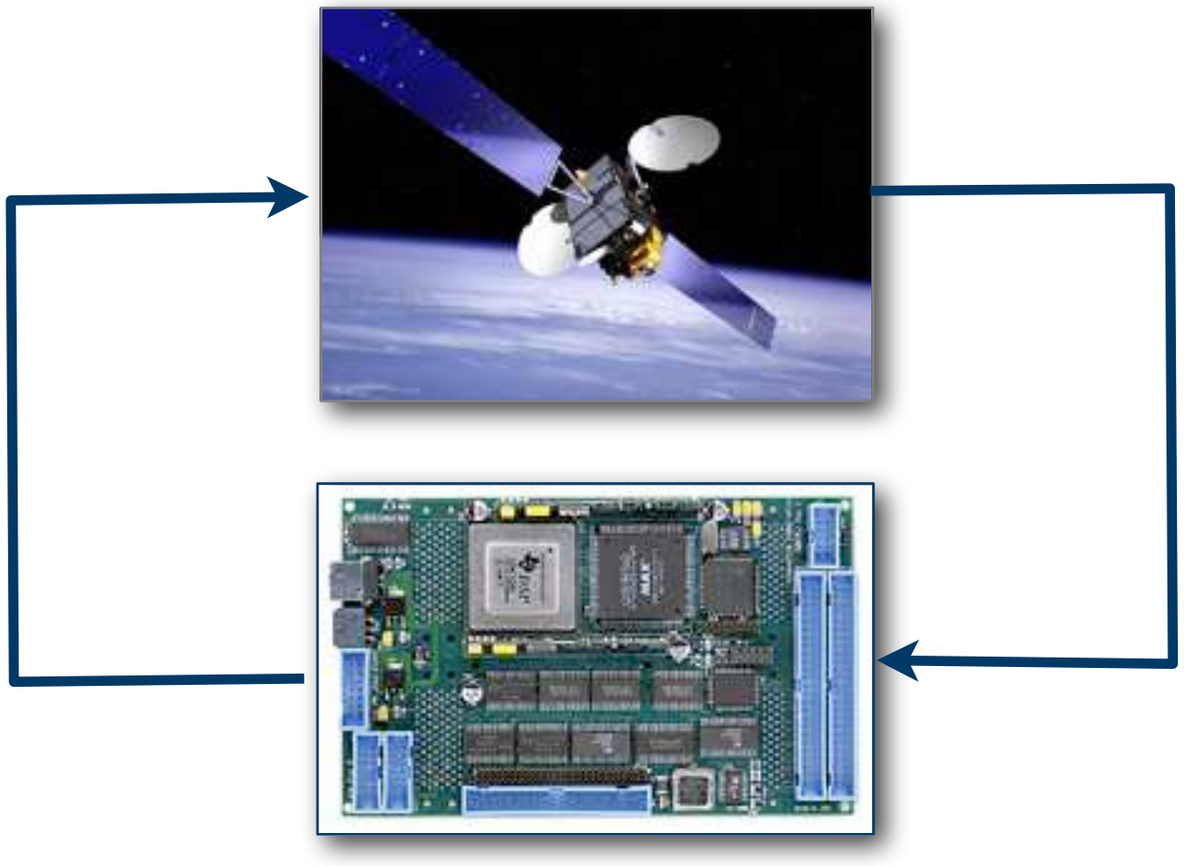
Introduction to Interior Point Methods for Predictive Control



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Overview

Introduction and motivation
- Why control & why feedback?
- Predictive control as a feedback strategy

Constrained LQR (CLQR) to Quadratic Program (QP)
- Sampled-data CLQR and equivalent discrete-time CLQR
- Non-condensed and condensed QP formulations

Algorithms for solving QP
- Interior point methods
- Exploiting structure of QP

Hardware issues
- Interaction between algorithm choice and hardware
- Parallel computation and pipelining

Wrap-up and questions

Motivation for Control

Predict behaviour of a dynamical system in an **uncertain environment**

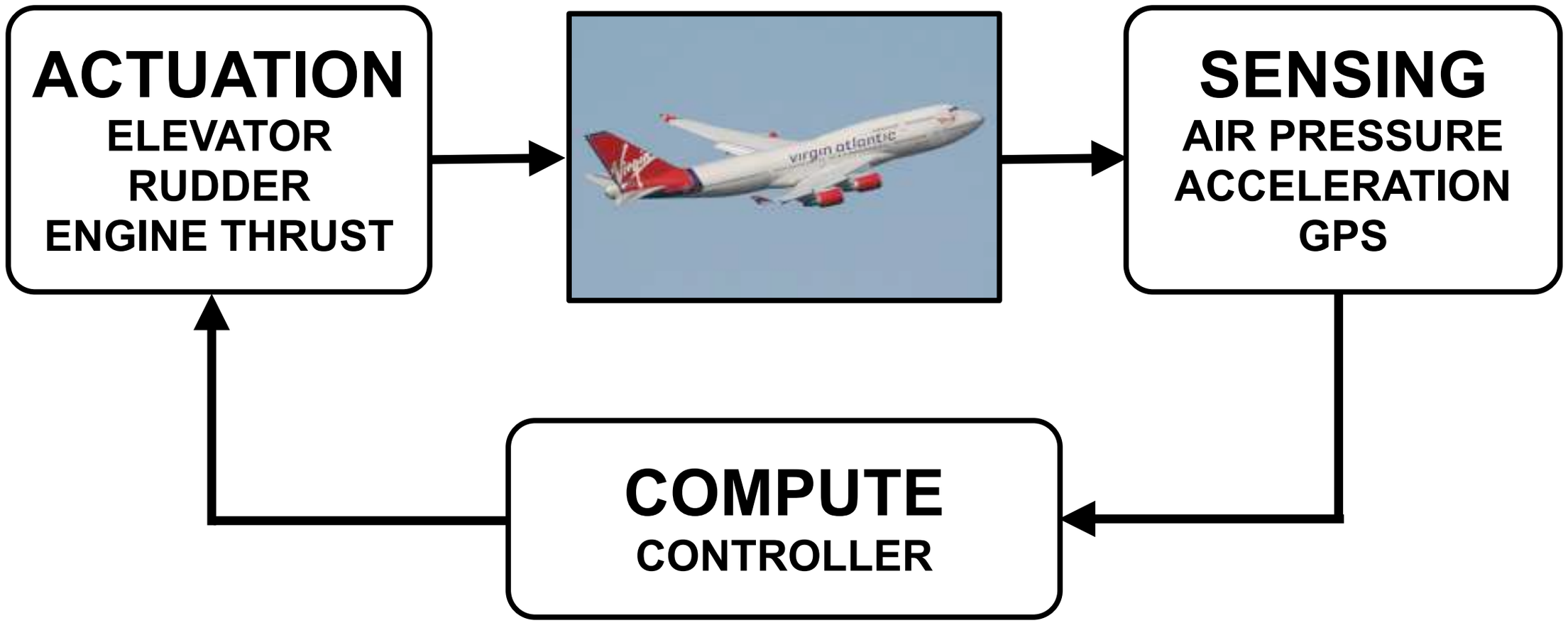
Uncertain Environment = unmodelled dynamics + disturbances + measurement noise + other

Unknowns
????

Behaviour
????

What is Control?

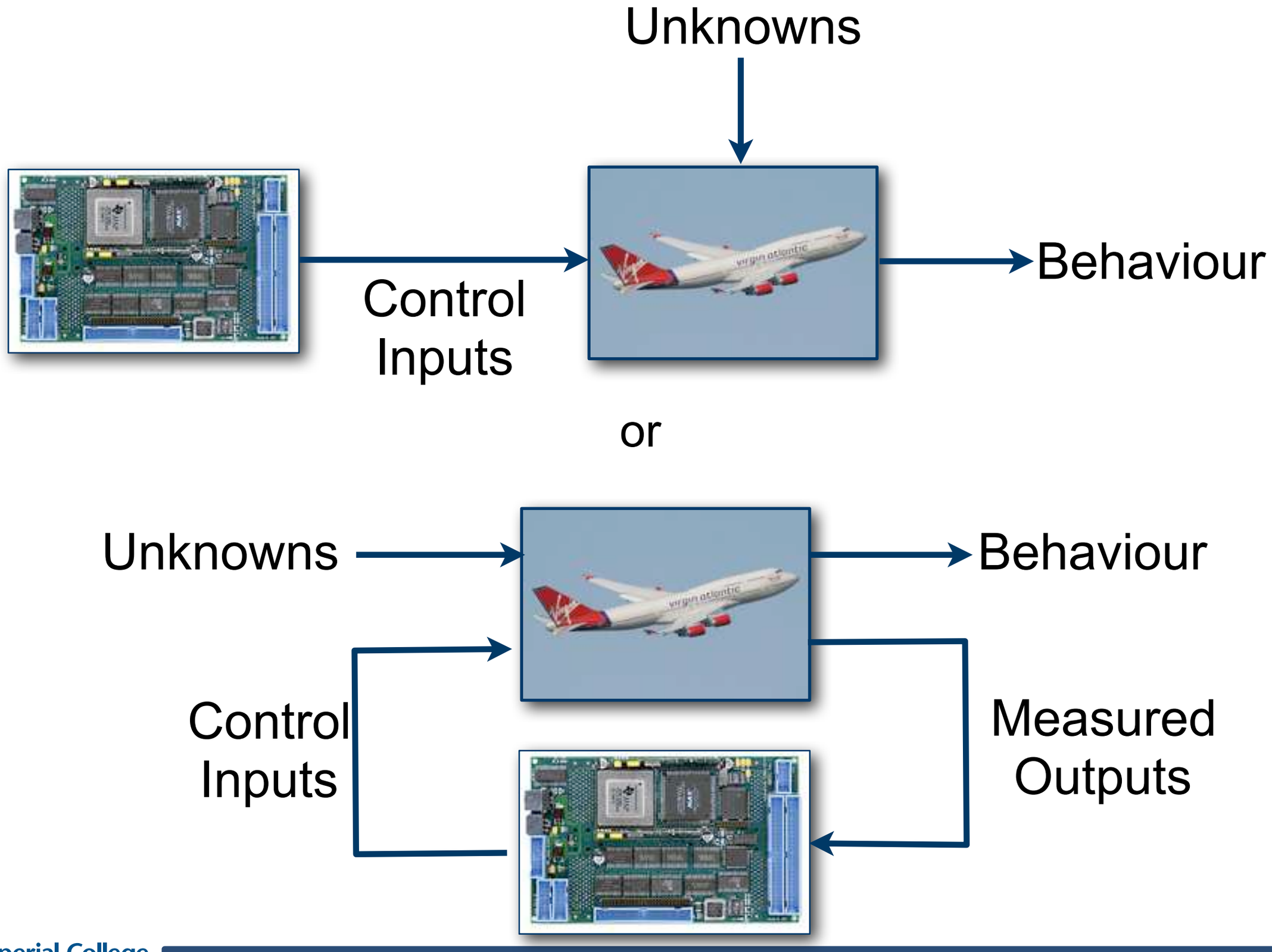
Control = sensing + computation + actuation



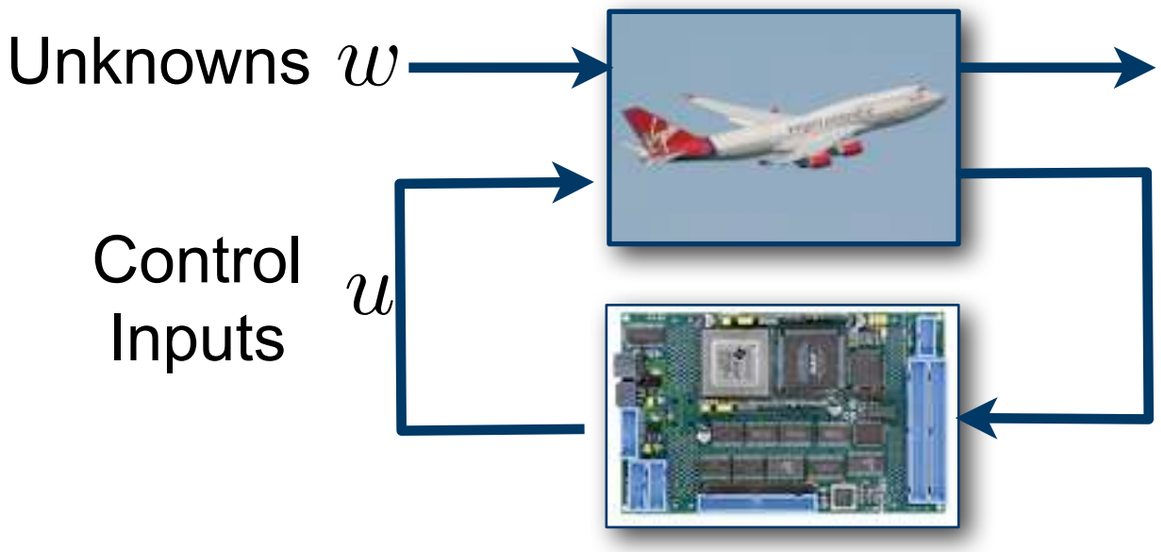
Use feedback **if and only** if there is uncertainty

uncertainty includes noise, disturbances, modelling error

Open-loop or Closed-loop?



Why Closed-loop?



Linear Dynamics

$$z = Fu + Gw$$

$$y = Hu + Jw$$

$$u = Ky$$

No control: $z = Gw \qquad (u = 0)$

Open-loop: $z = Fu + Gw$

Feedback: $z = \left[FK\left(I - HK\right)^{-1} J + G\right] w$

Feedback is the ONLY way to attenuate unknown w

Drawbacks of Feedback

Can destabilise a system

- Example: JAS 39 Gripen crash
- Open-loop unstable
- Input constraints saturated
- Outside controllable region

Couples different parts of the system

- Might inject measurement noise

Complicates manufacture and commissioning

This is why we use feedback ***only if*** there is uncertainty

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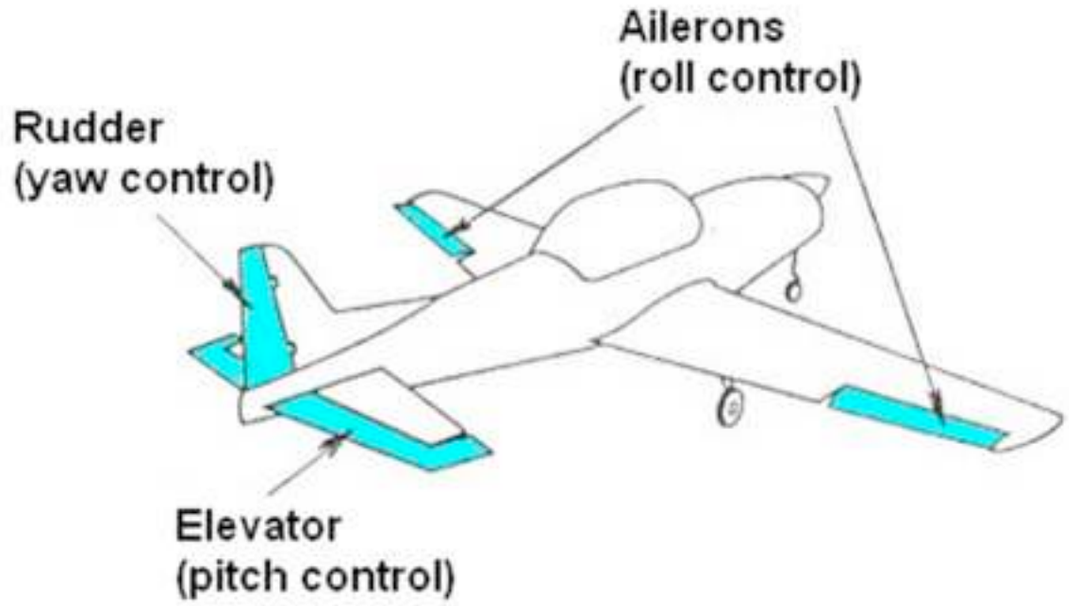
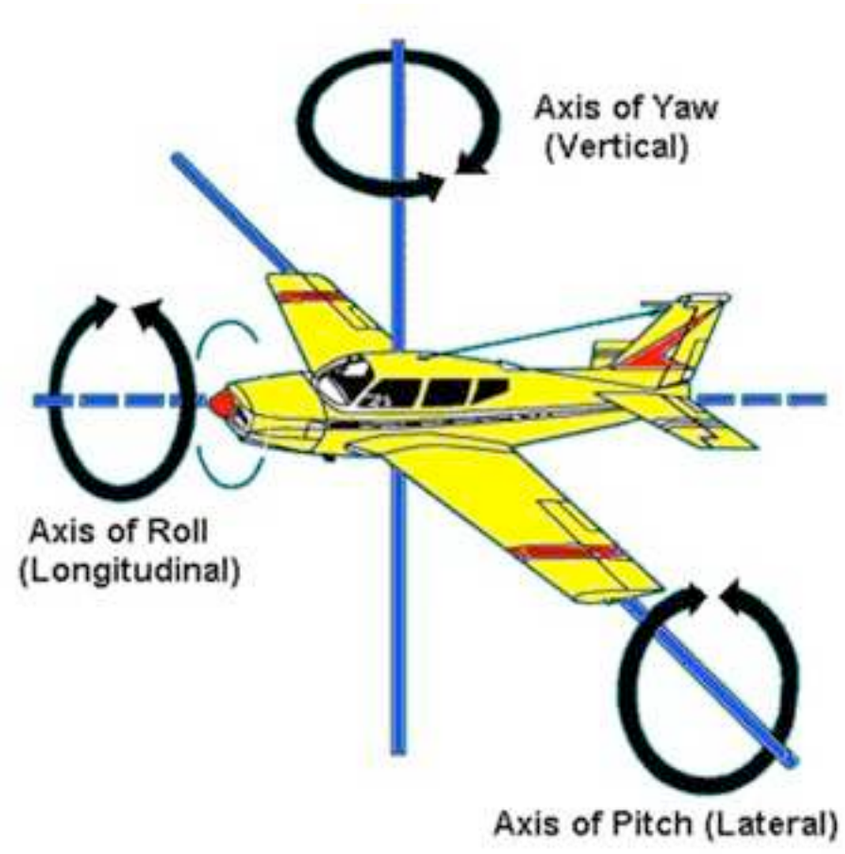
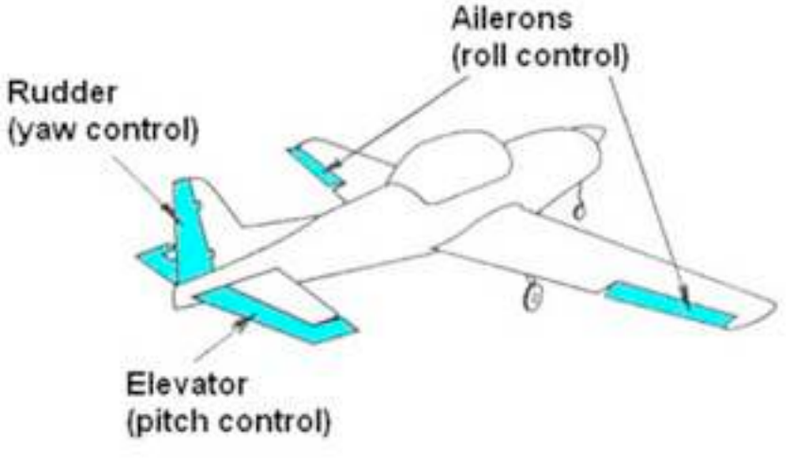
This is why we use feedback ***only if*** there is uncertainty

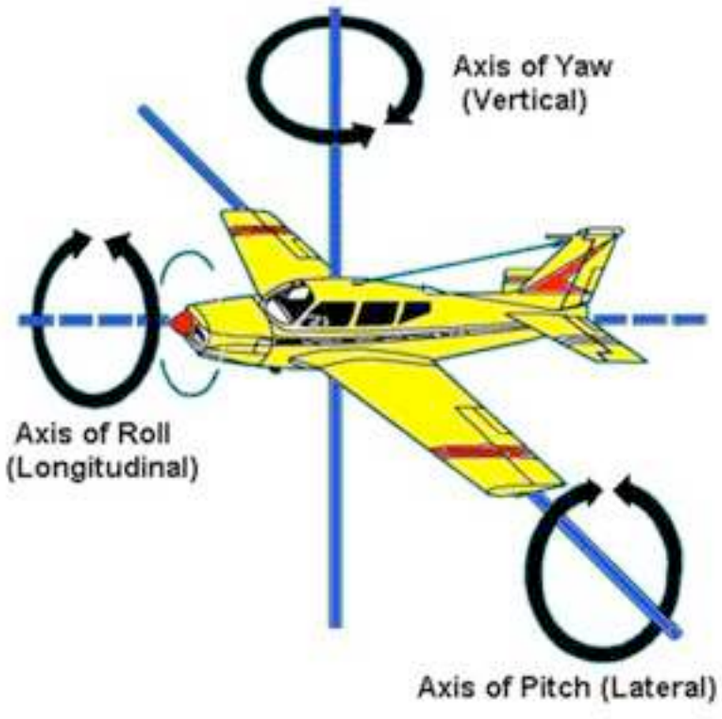
Example: Cessna Citation

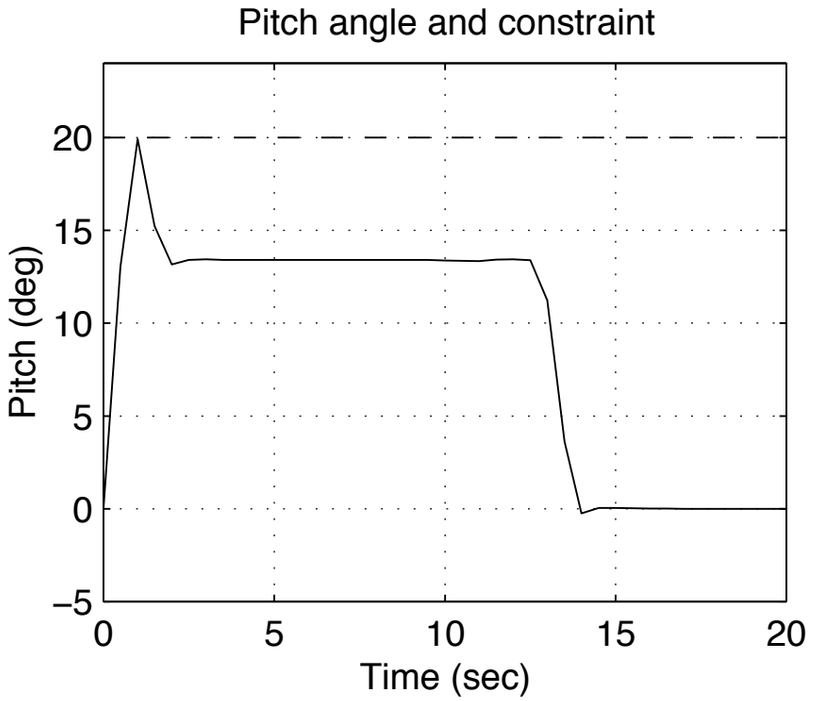


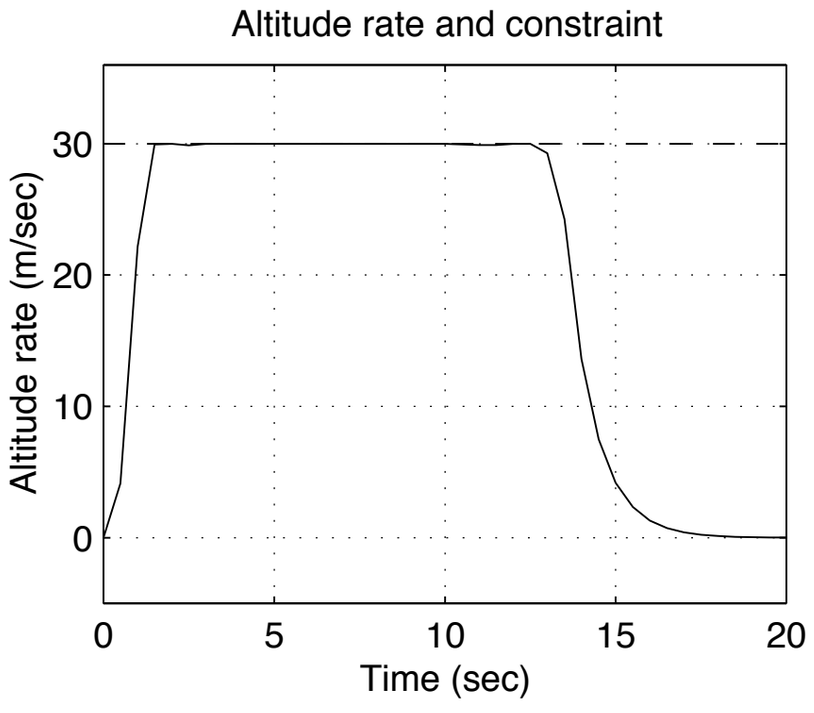


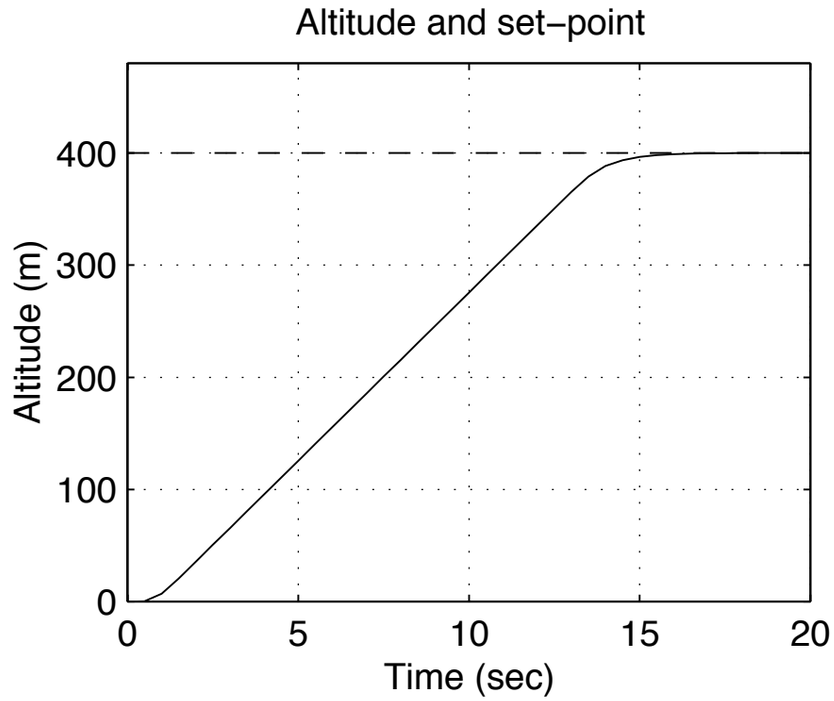
Example: What, not How

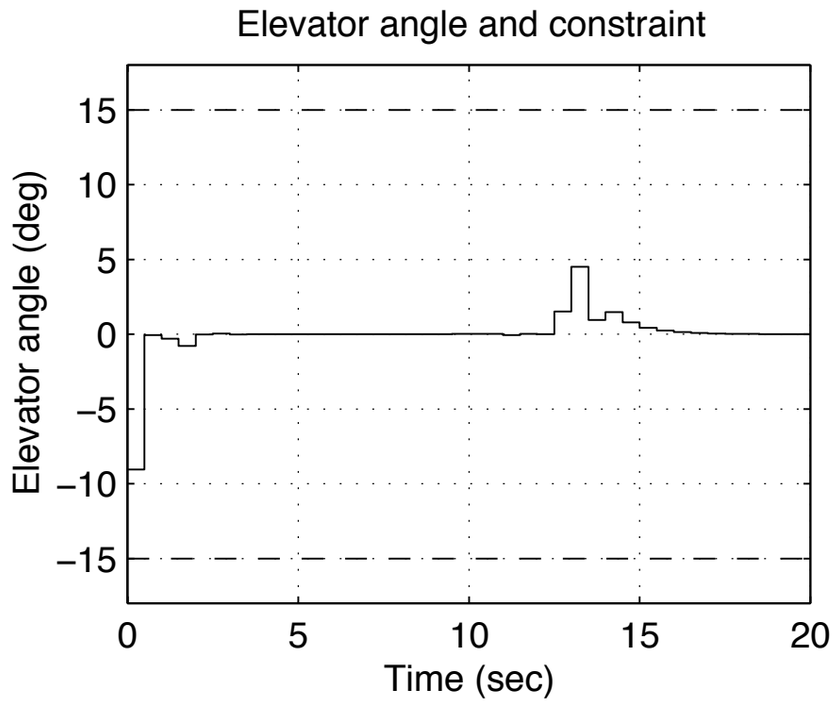












Constraints in Control

All physical systems have constraints:

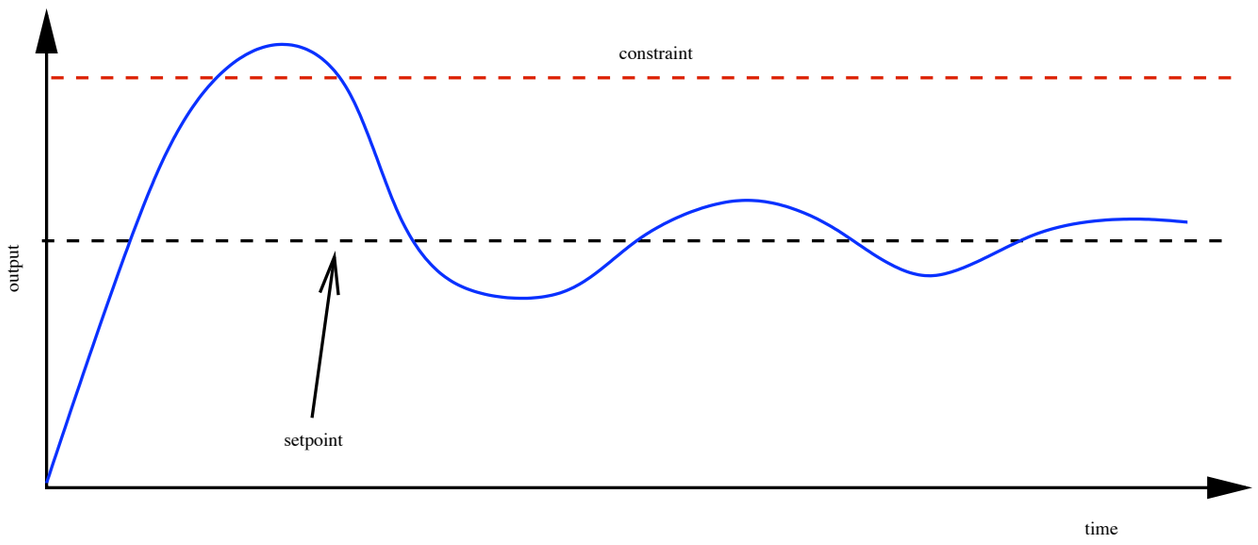
- Physical constraints, e.g. actuator limits
- Safety constraints, e.g. temperature/pressure limits
- Performance constraints, e.g. overshoot

Optimal operating points are often near constraints

Most control methods address constraints a posteriori:

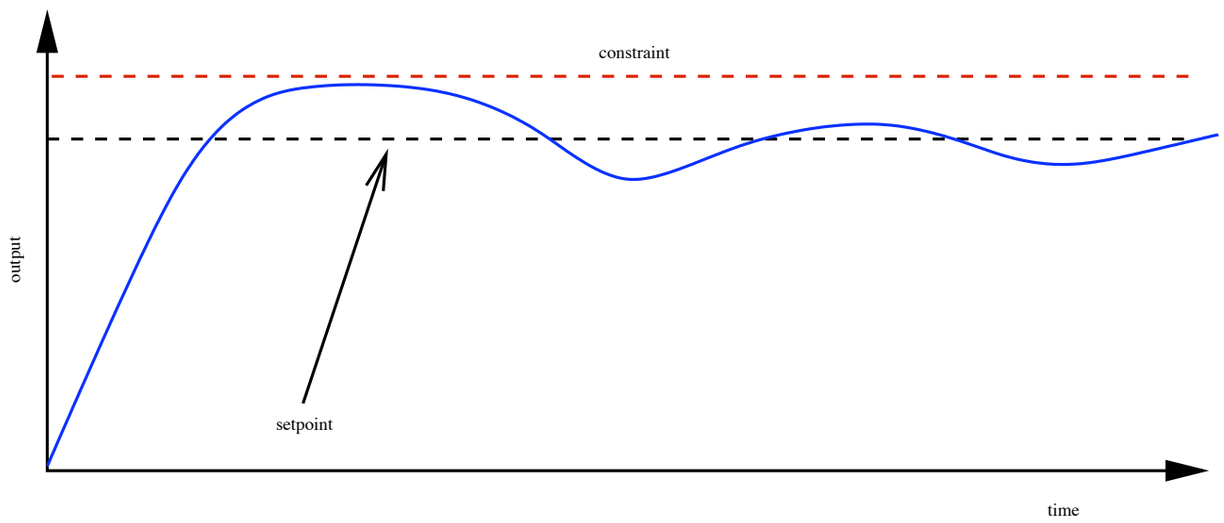
- Anti-windup methods, trial and error

Optimal Operation and Constraints



Classical Control

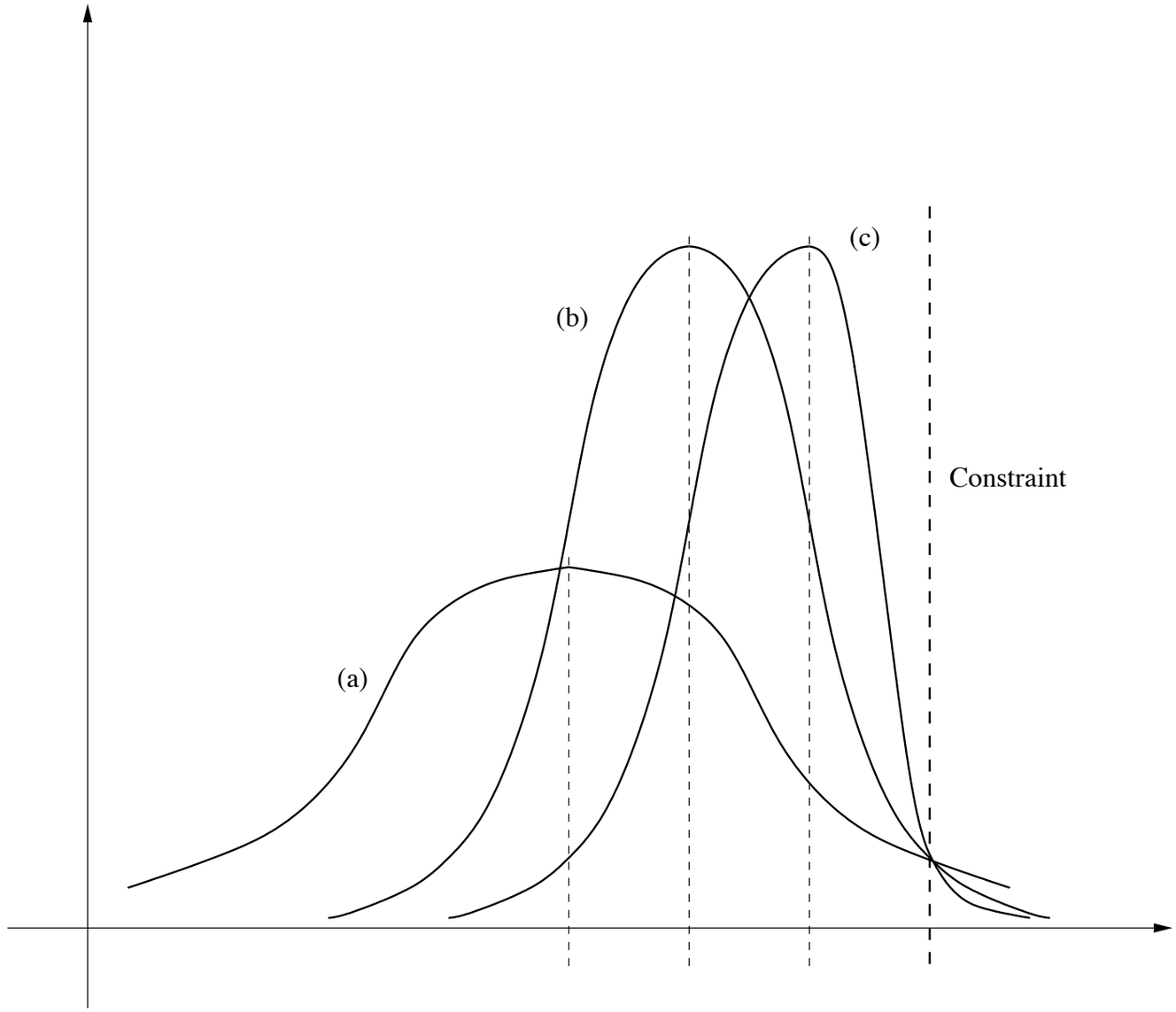
- No knowledge of constraints
- Setpoint far from constraints
- Suboptimal plant operation



Predictive Control

- Constraints included in design
- Setpoint closer to optimal
- Improved plant operation

Getting Closer to Constraints

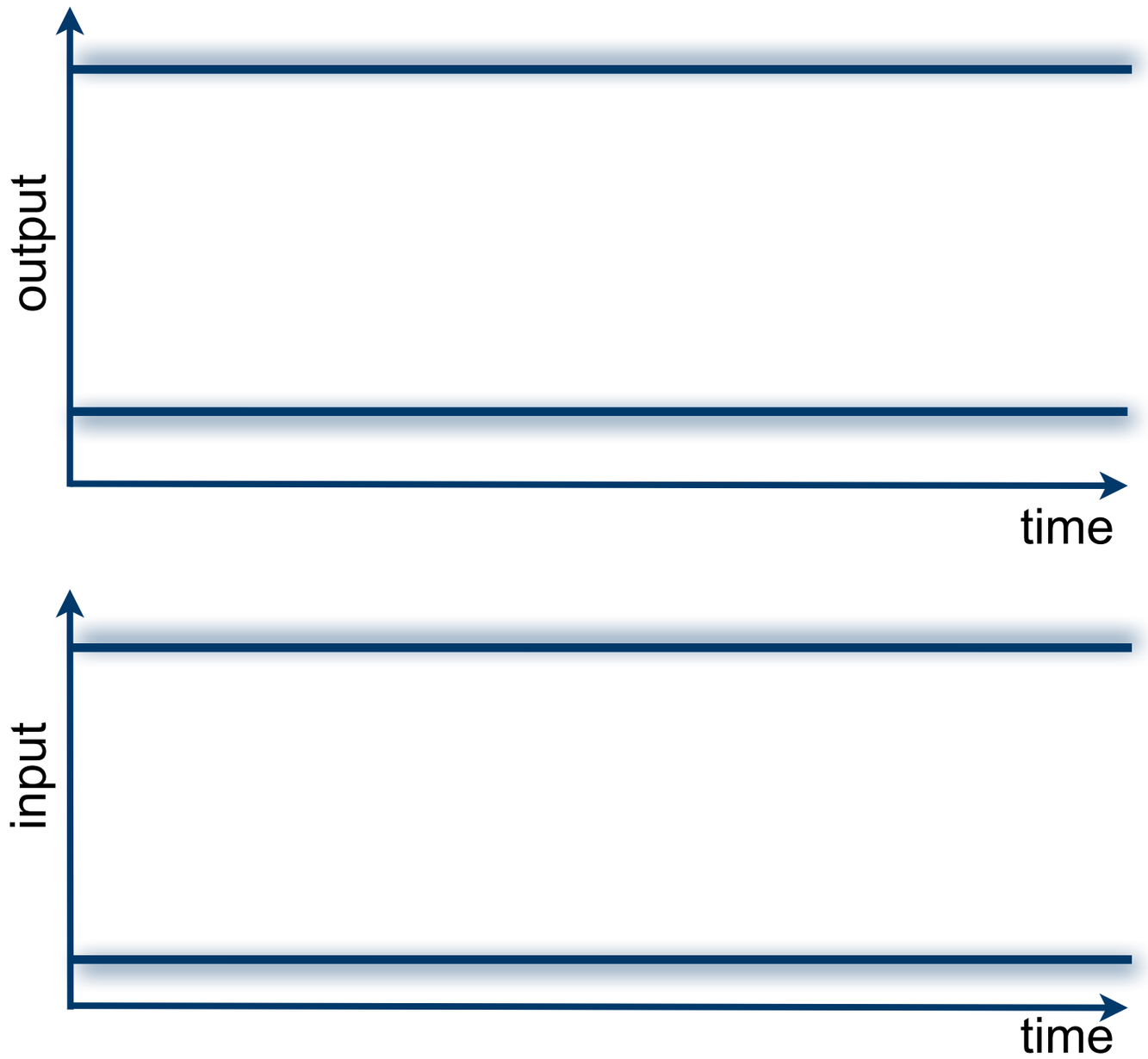


(a) PID control

(b) LQG control

(c) Predictive Control

Receding Horizon Principle



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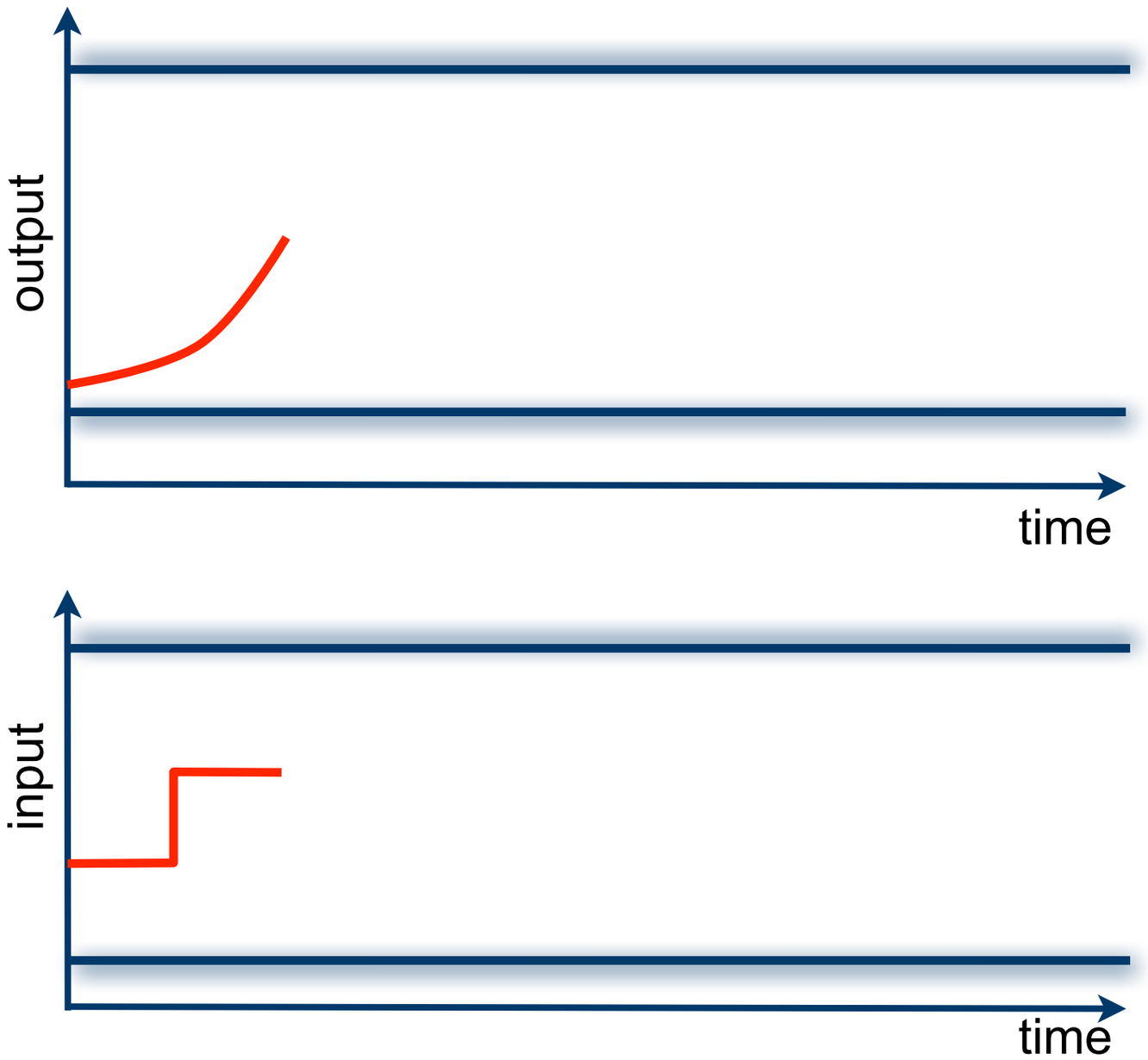

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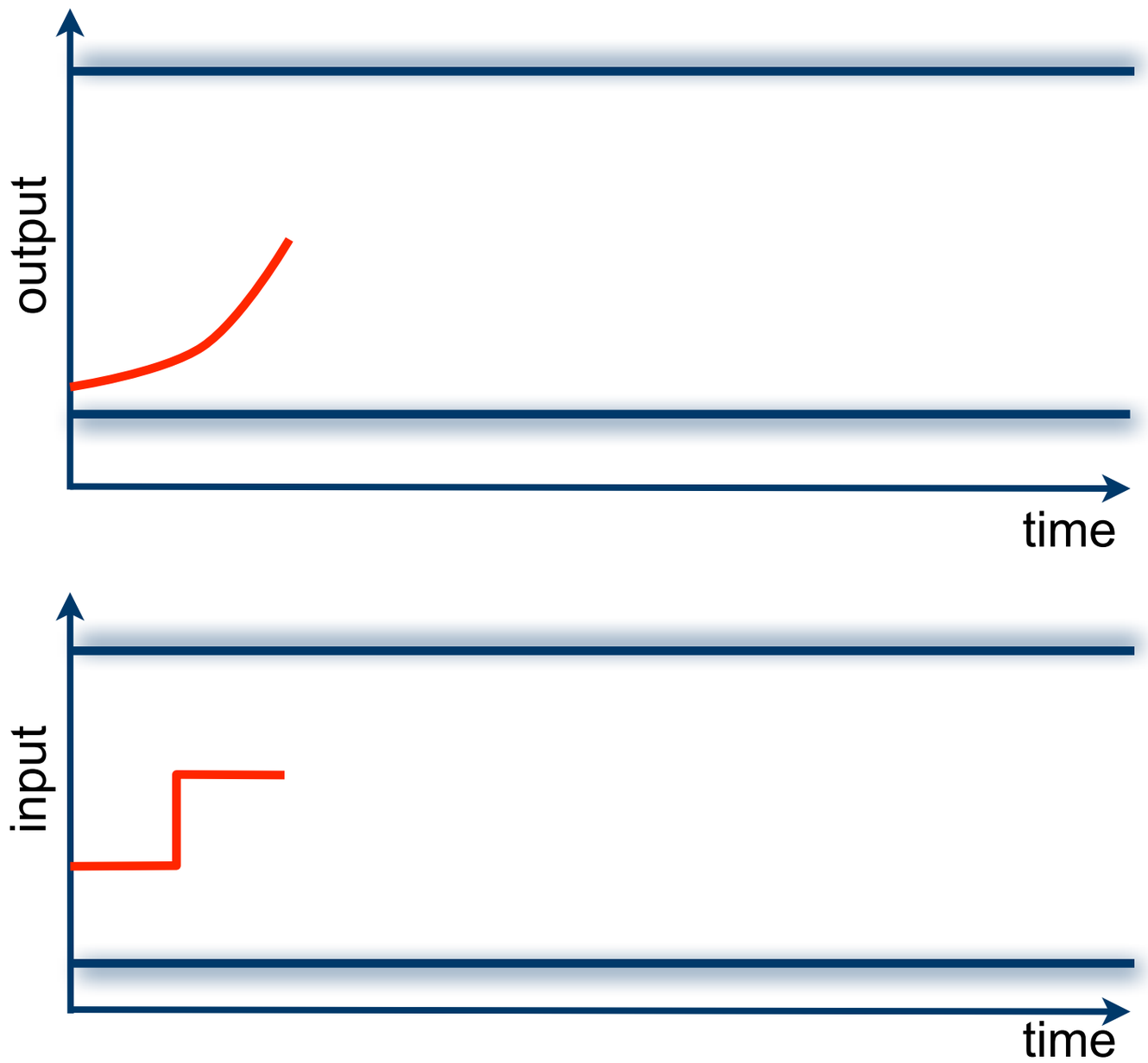
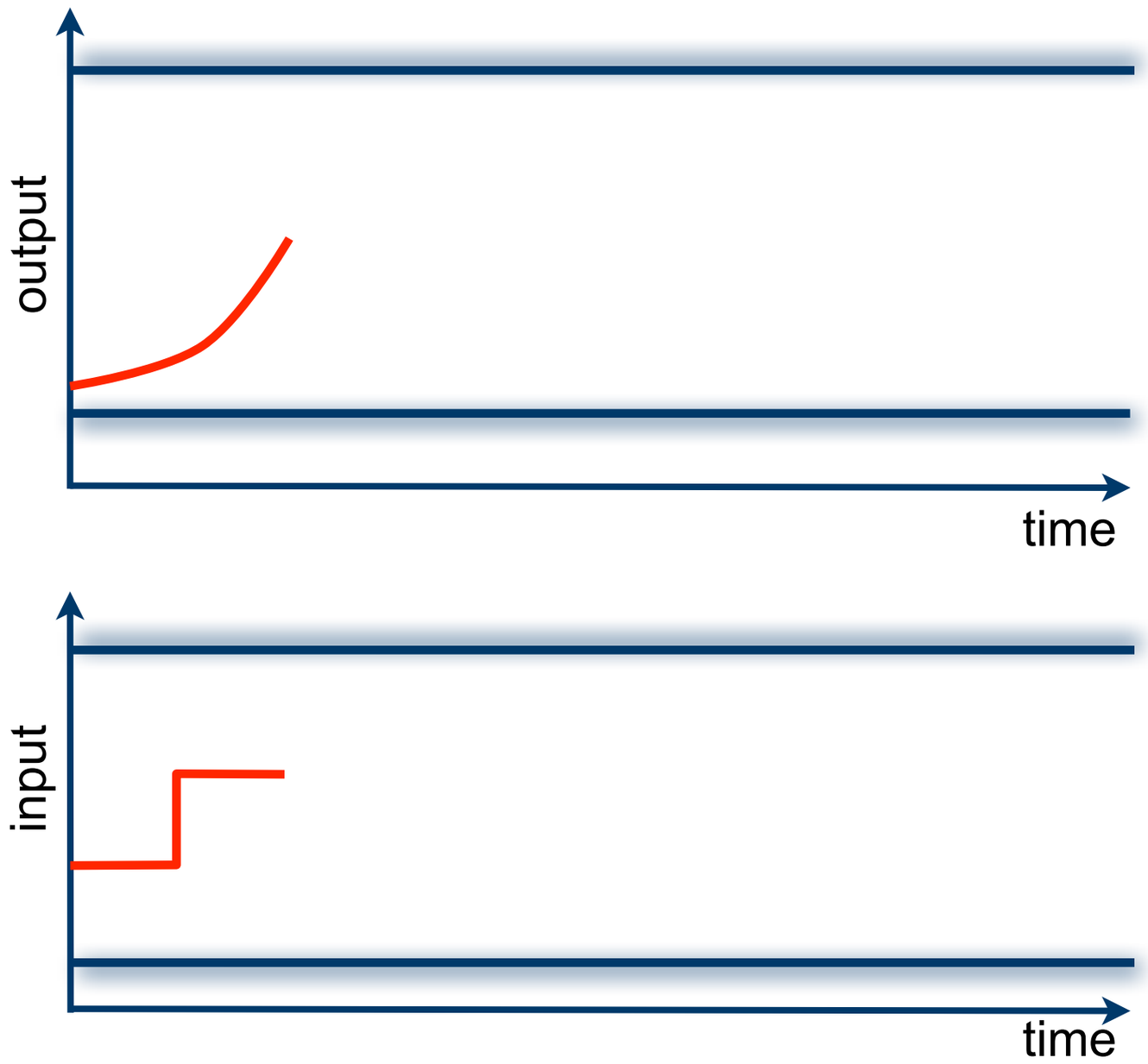


1. Take measurement

Receding Horizon Principle



1. Take measurement
2. Solve optimal control problem

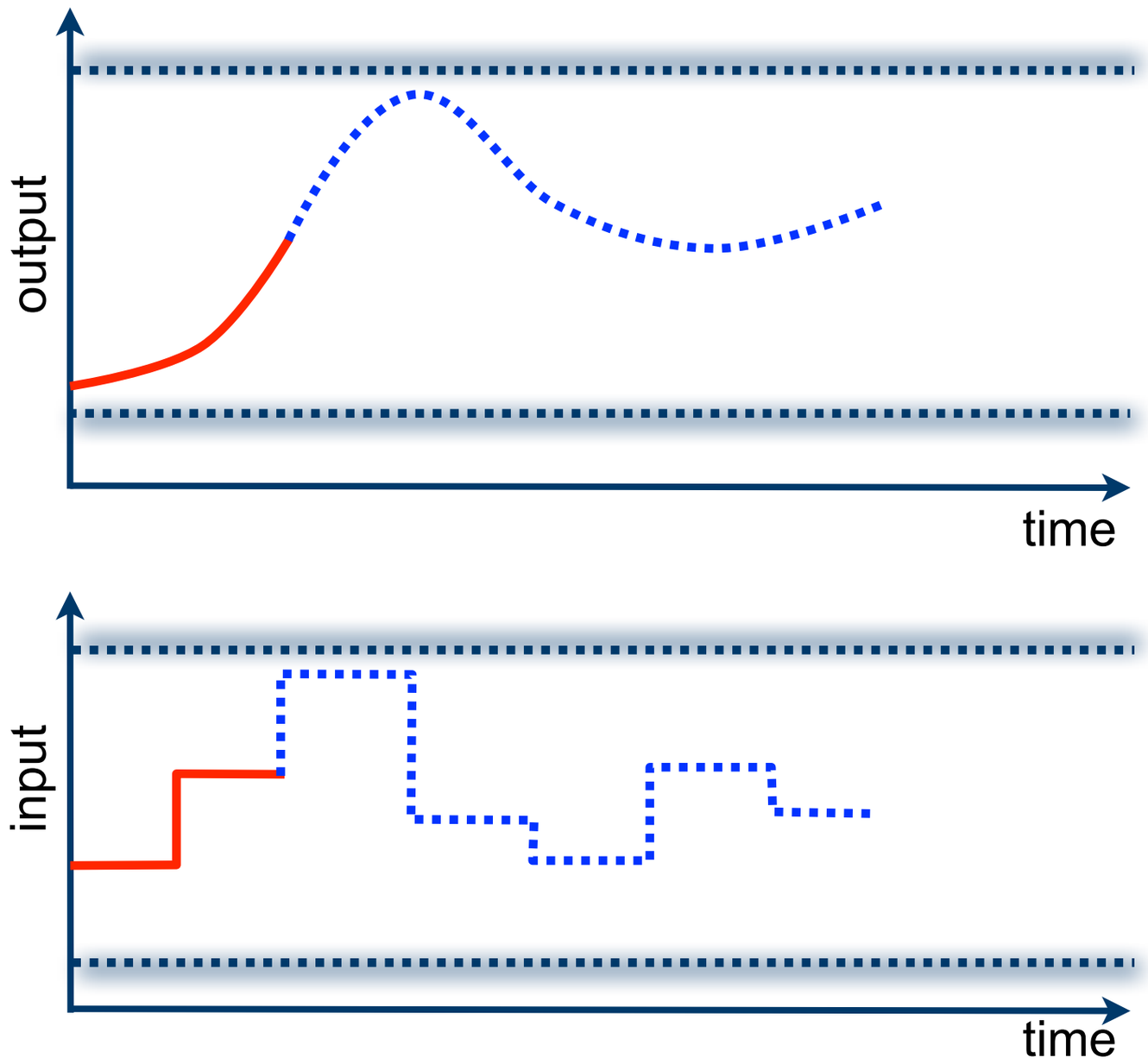
$$\min_{x(\cdot),u(\cdot)} V(x(\cdot),u(\cdot))$$

$$x(0) = \hat{x},$$

$$\dot{x}(t) = f(x(t),u(t))$$

$$c(x(\cdot),u(\cdot)) \leq 0$$

Receding Horizon Principle



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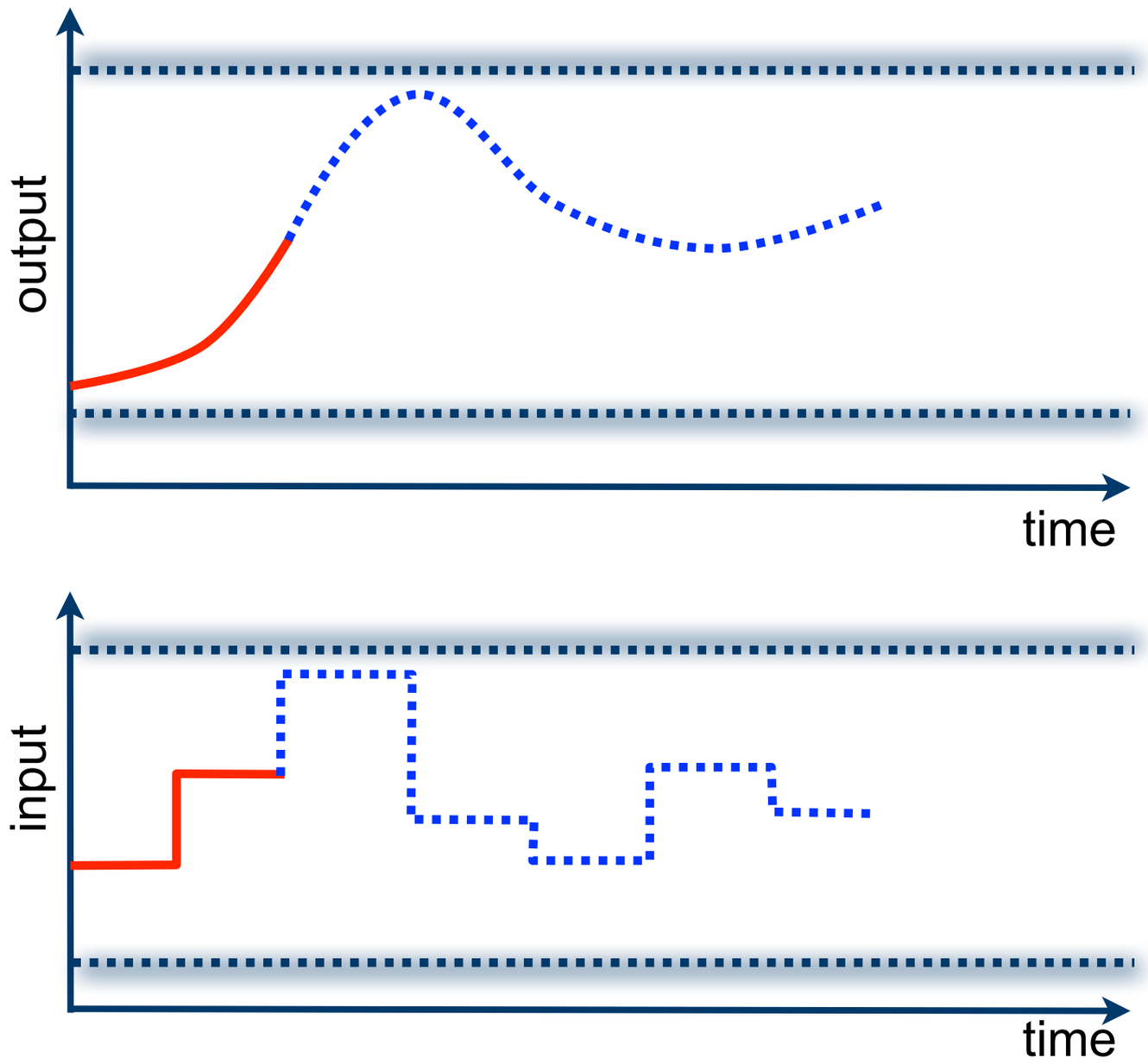

1. Take measurement
2. Solve optimal control problem
3. Implement only first part

$$\min_{x(\cdot),u(\cdot)} V(x(\cdot),u(\cdot))$$

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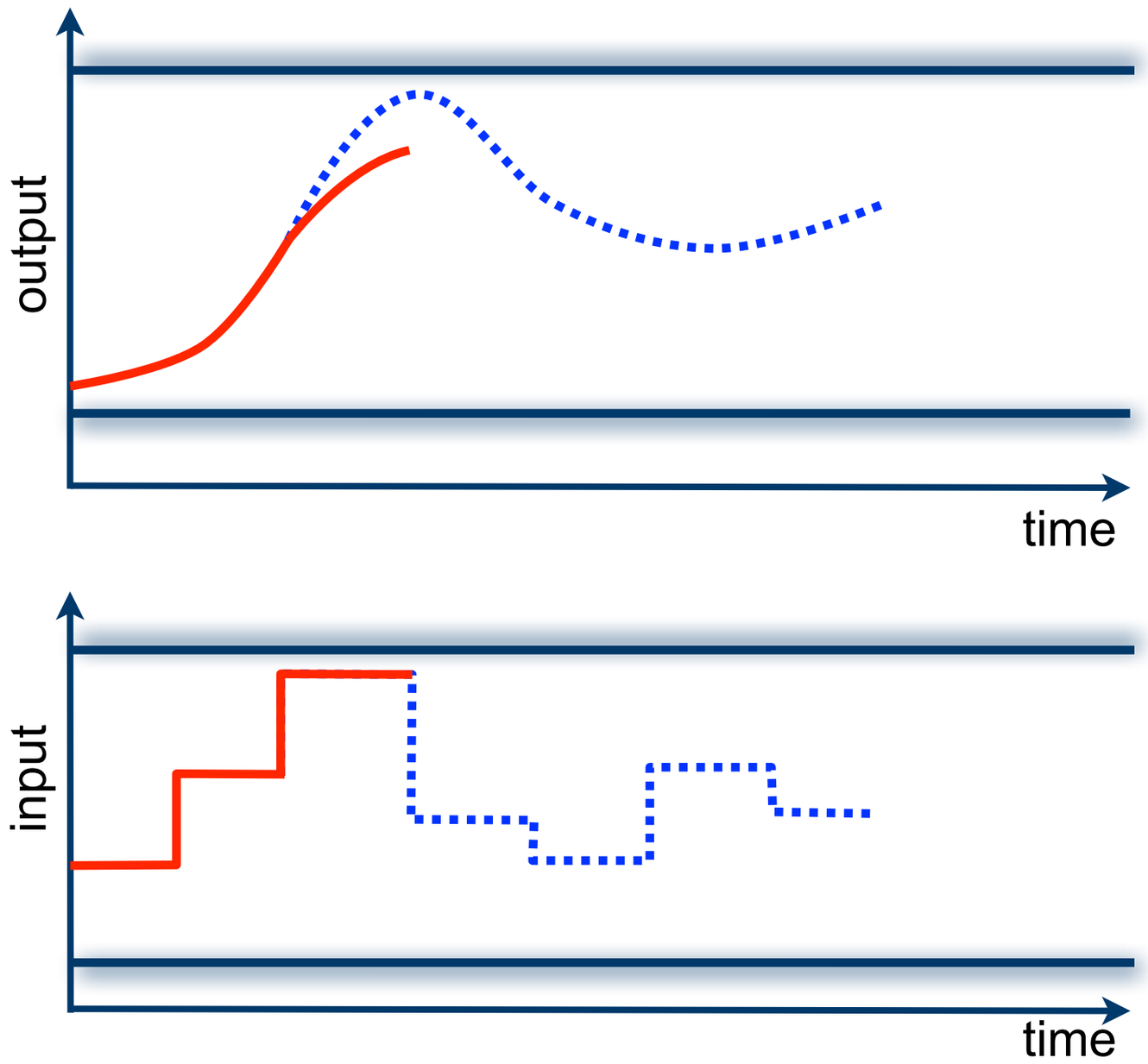
Receding Horizon Principle



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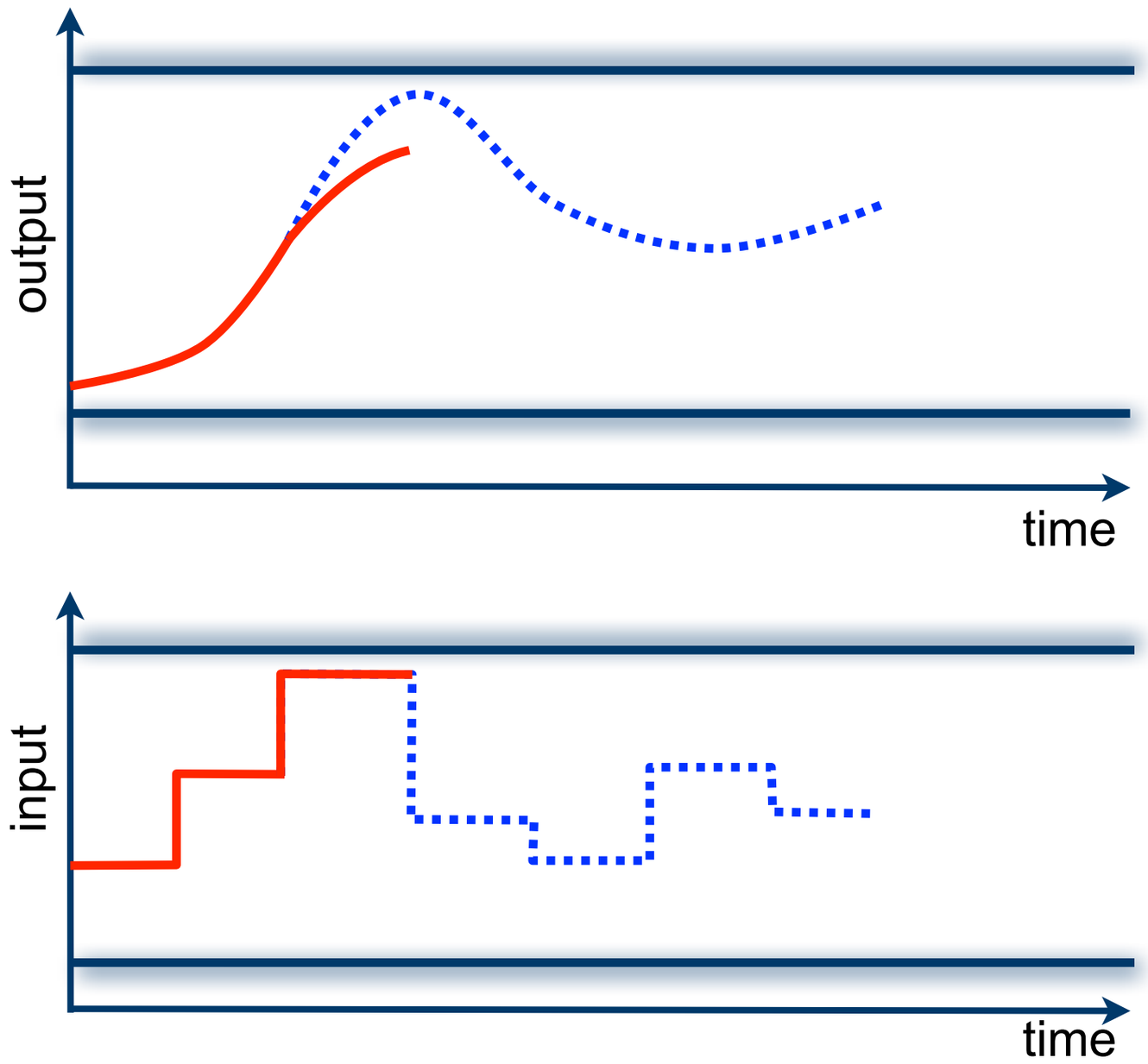

1. Take measurement
2. Solve optimal control problem
3. Implement only first part
4. Go to step 1

$$\min_{x(\cdot),u(\cdot)} V(x(\cdot),u(\cdot))$$

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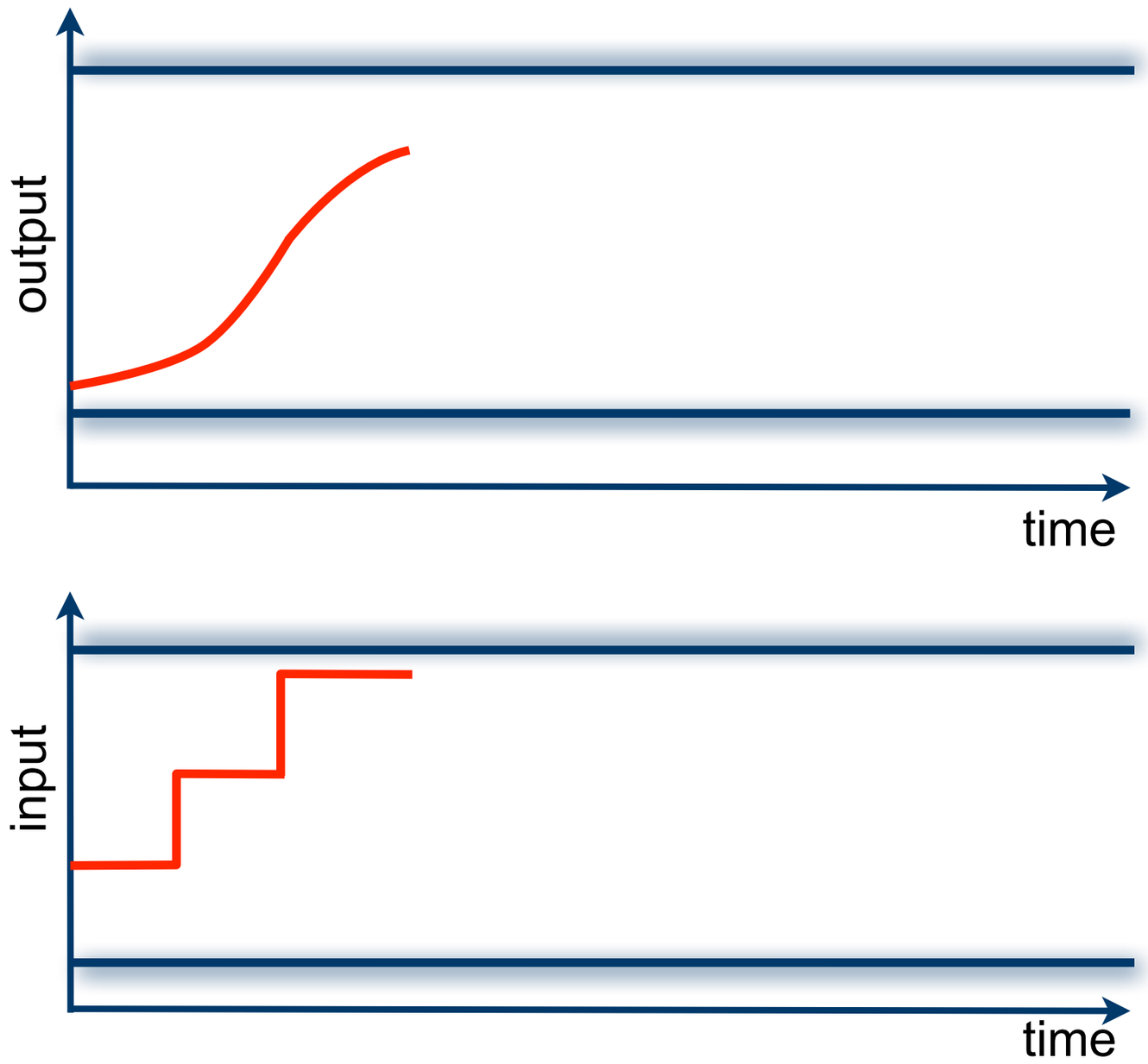

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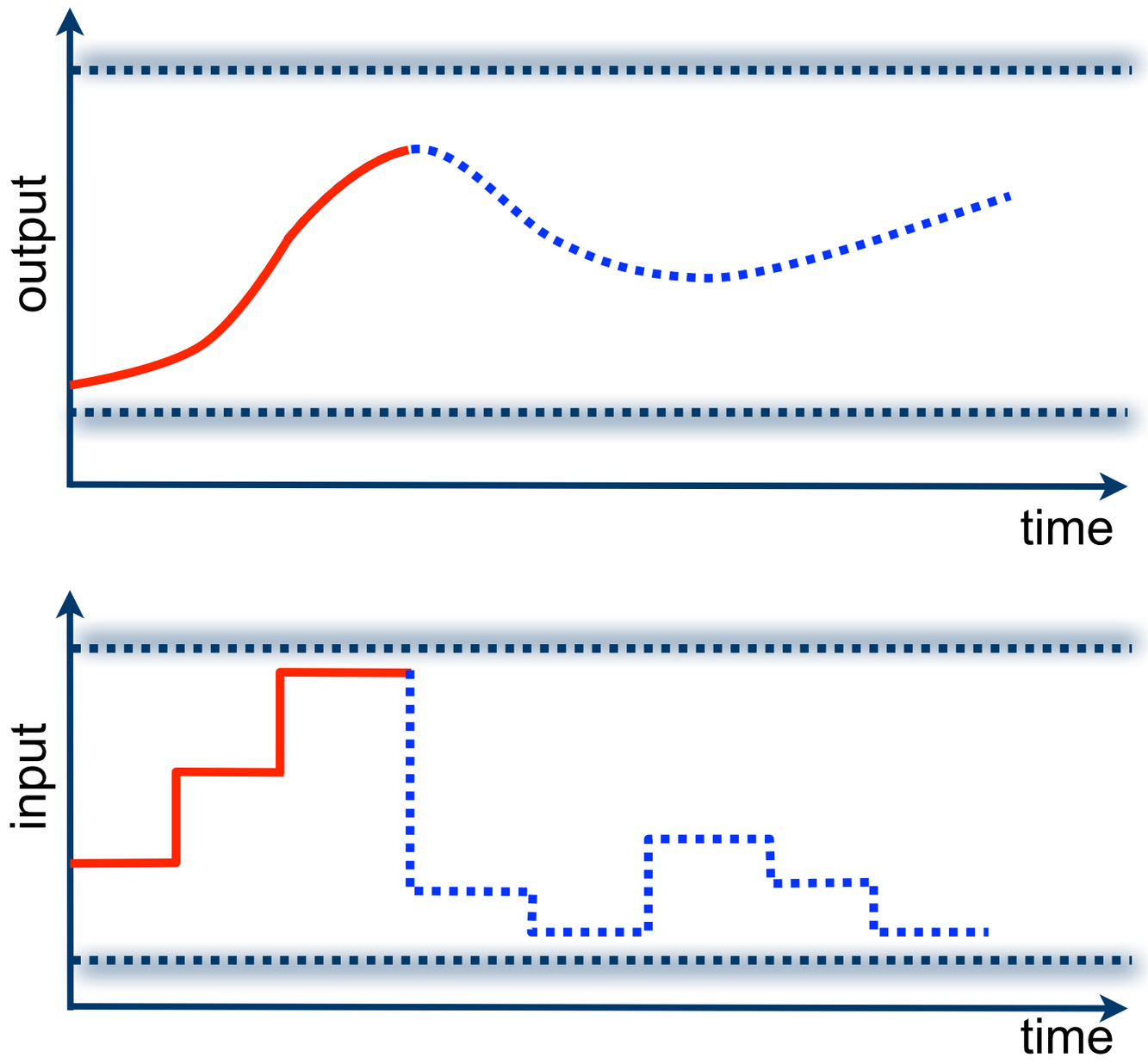

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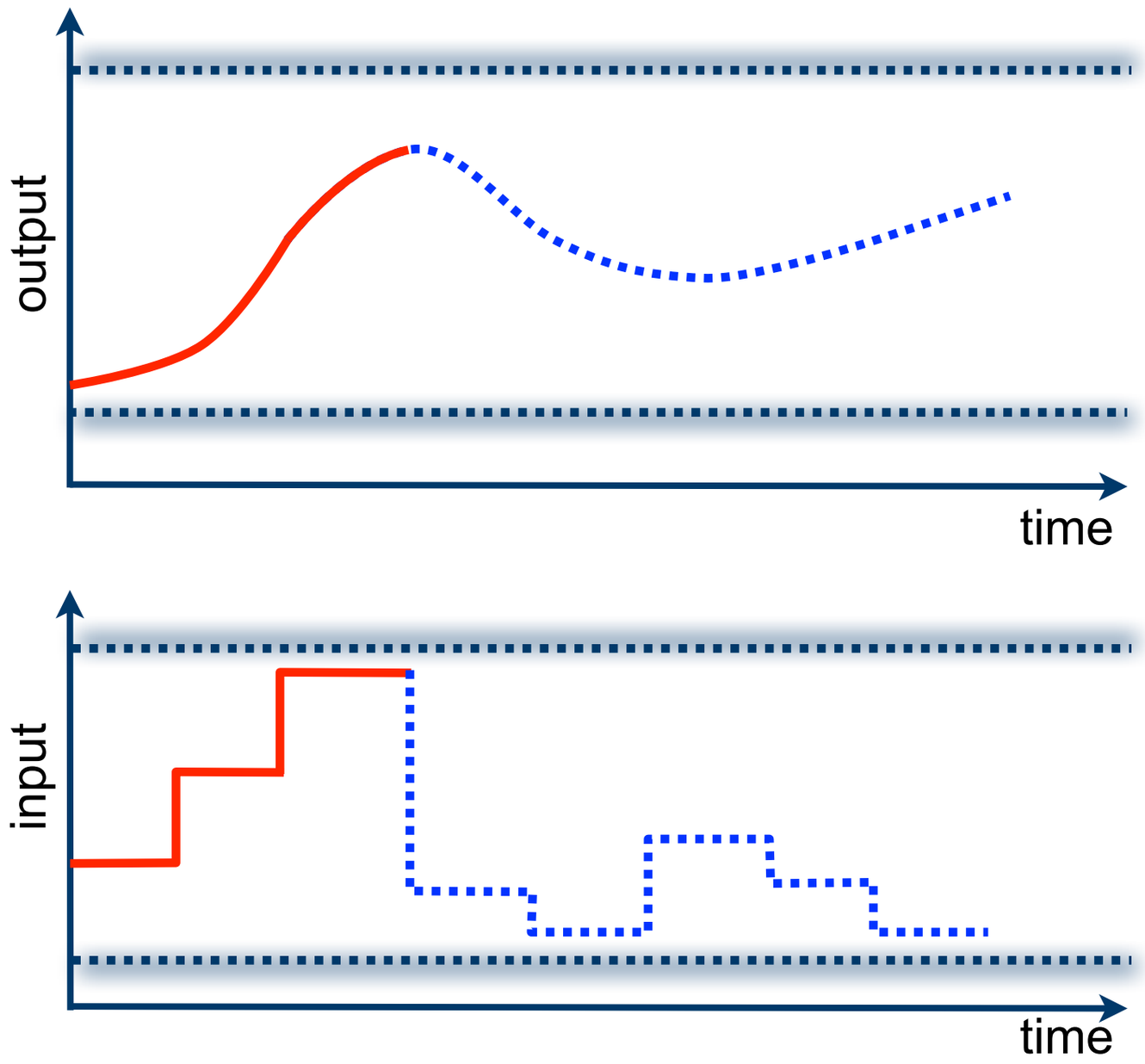

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$$c(x(\cdot),u(\cdot)) \leq 0$$

Solution is a function of current state, hence we have ***feedback***

Properties of Predictive Control

Is this a ‘new’ idea?

- Standard finite horizon optimal control
- Since the 1950s (Coales and Noton, 1956)
- Optimization in the loop, in ‘real time’
- Repeatedly solving an optimal control problem

The main problems:

- Optimization needs to be fast enough
- The resulting control law might not be stable

The main advantages:

- Systematic method for handling constraints
- Flexible performance specifications
- Easy to understand

Also Known As...

Other Names in Industry and Academia:

- Dynamic Matrix Control (DMC)
- Generalised Predictive Control (GPC)
- ...

Generic names:

- Model Predictive Control (MPC)
- Model-Based Predictive Control (MBPC)

Predictive Control ≠ Receding Horizon Control (RHC)

- Decreasing Horizon Control (DHC)
- Variable Horizon Control (VHC)

Computational Speed and Applications

Historically, MPC has been used on ‘slow’ processes:

- Petrochemical and process industries, pulp and paper
- Sample time of seconds to hours

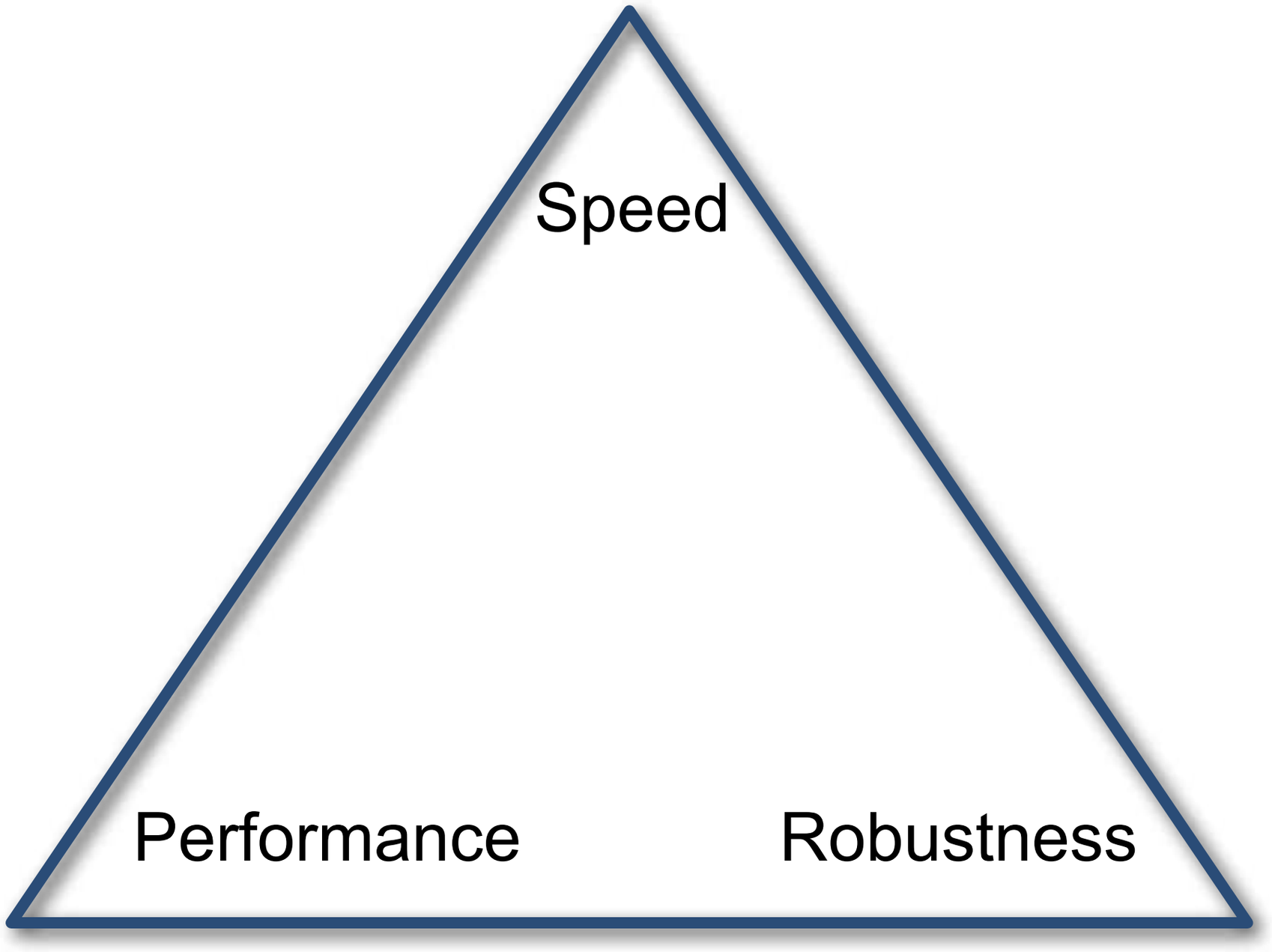
Major advances in hardware and algorithms

- Computation of 1 minute in 1990 ⇒ now less than 1s

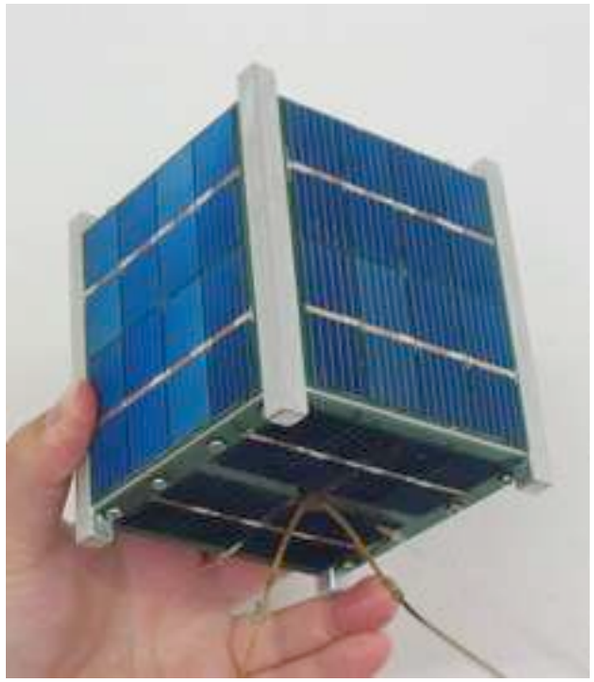
MPC now being proposed for “fast” processes:

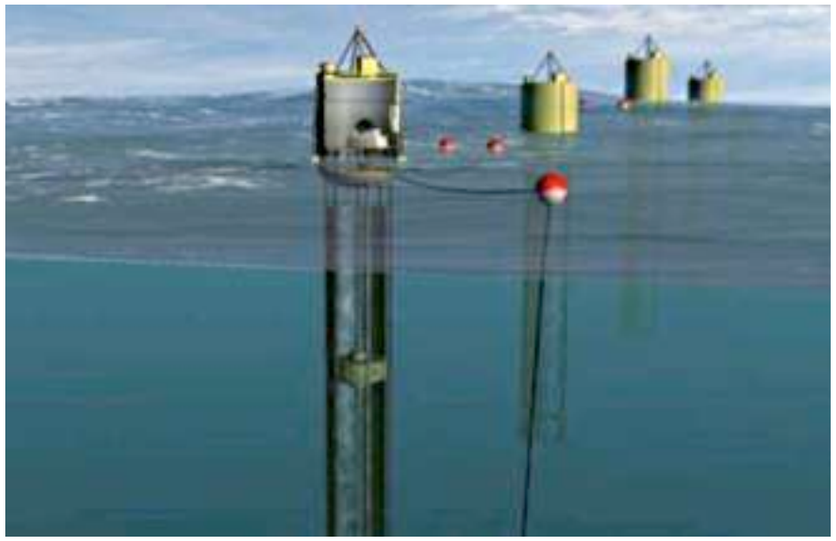
- Automotive traction and engine control
- Aerospace applications
- Autonomous vehicles
- Electricity generation and distribution

Trade-Offs in MPC

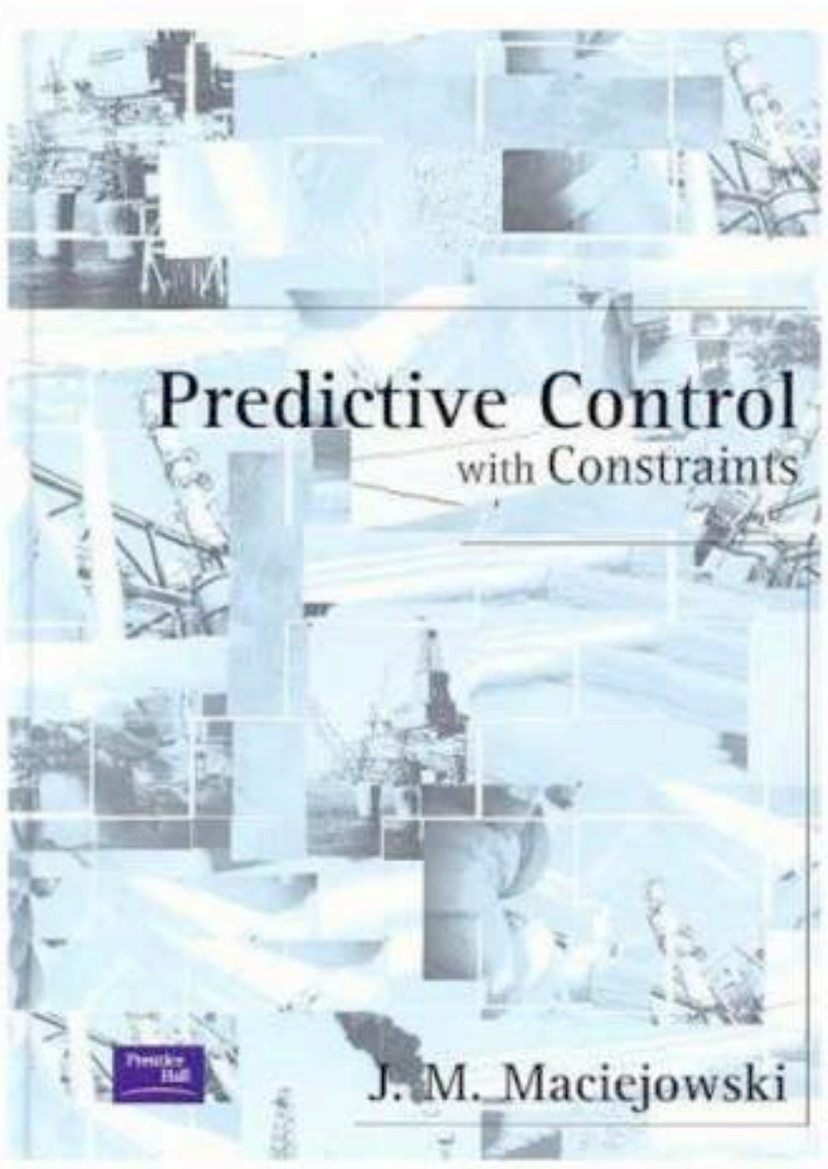


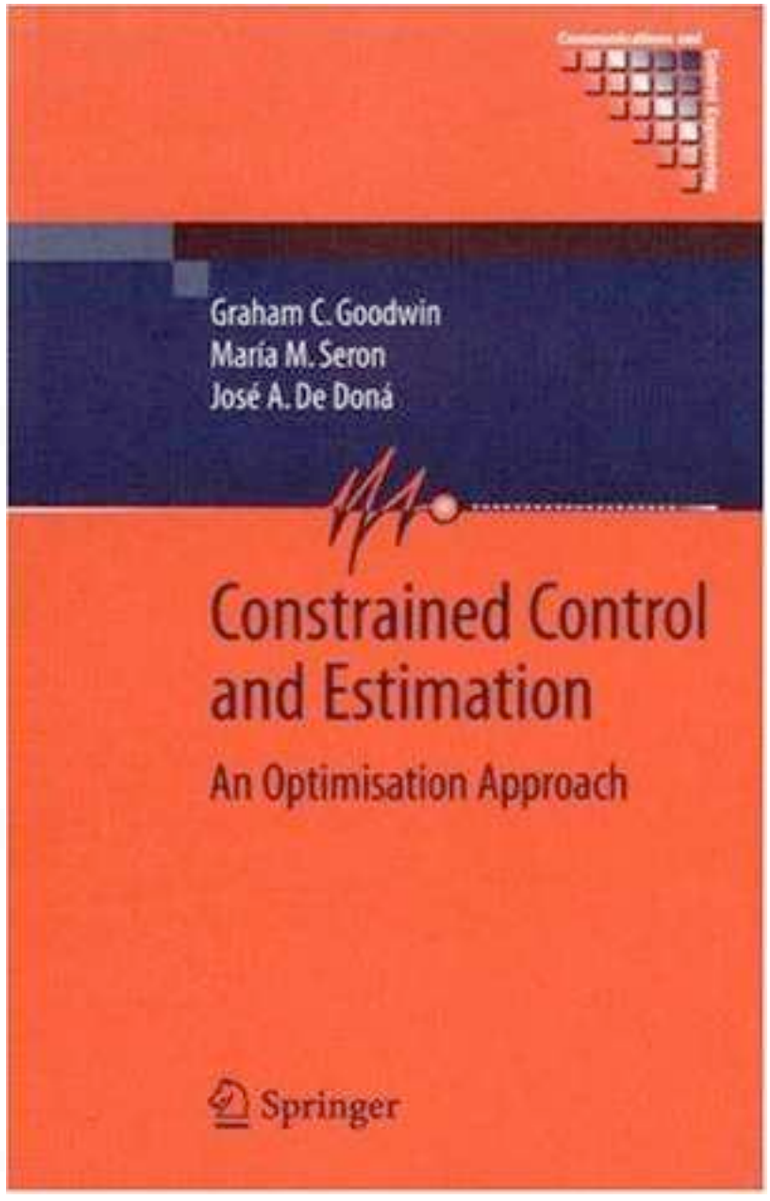
Future Applications of MPC

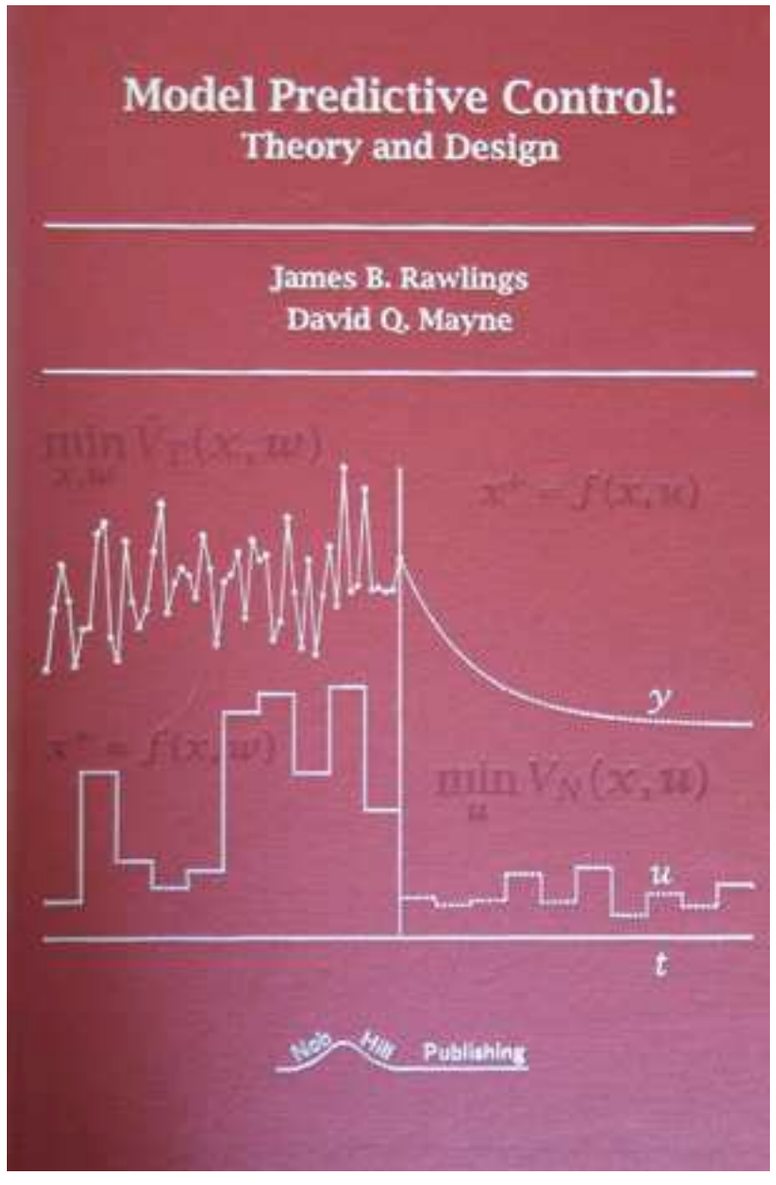




Books on Predictive Control



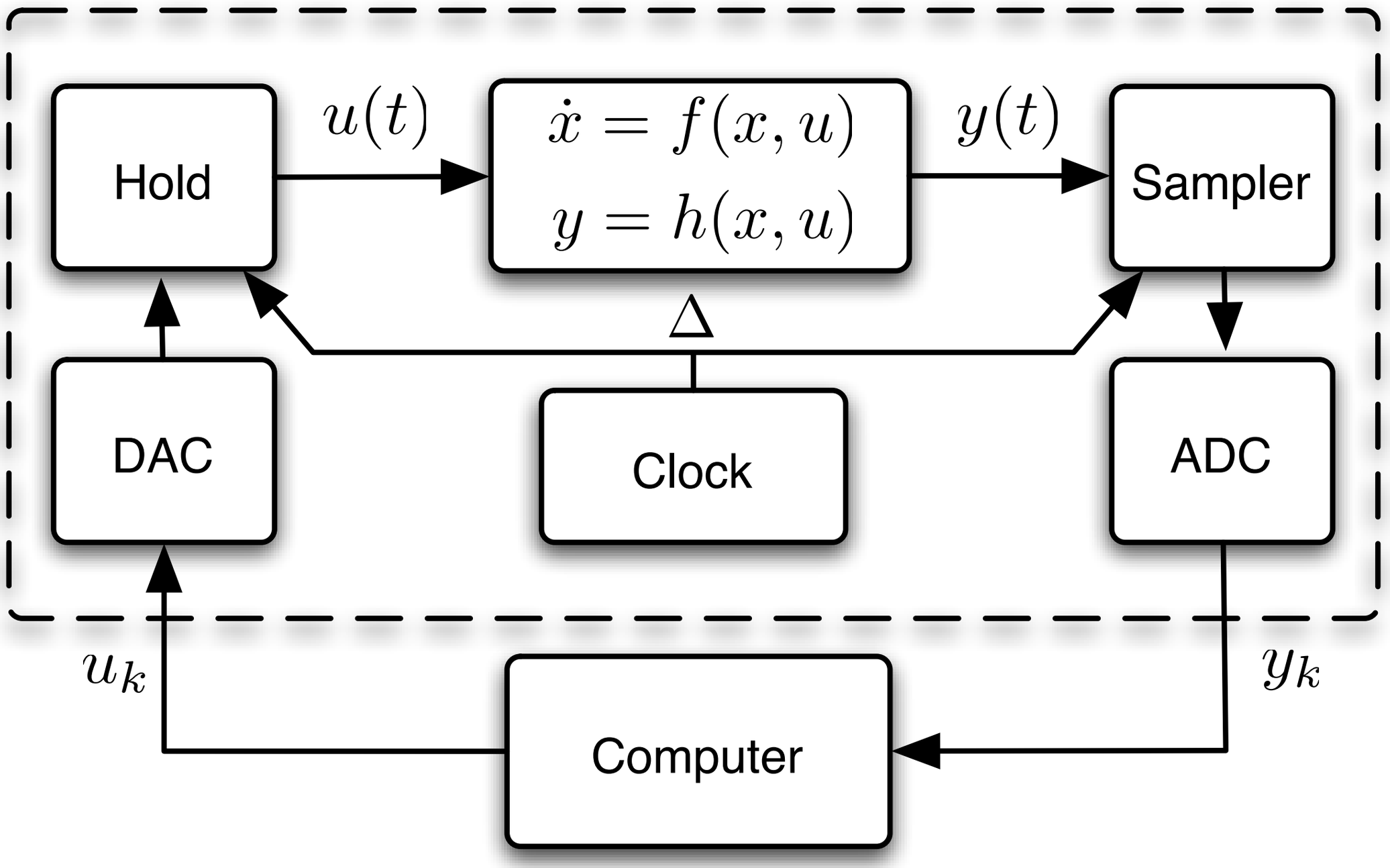




Sampled-data Control

$$x_{k+1} = g(x_k, u_k)$$

$$y_k = h(x_k, u_k)$$



Continuous-time CLQR

$$\min_{x(\cdot),u(\cdot)} x(T)'P_c x(T) + \int_0^T x(t)'Q_c x(t) + u(t)'R_c u(t)dt$$

$$x(0) = \hat{x},\ \dot{x}(t) = A_c x(t) + B_c u(t), \quad \forall t \in (0,T)$$

$$E_c x(t) + J_c u(t) \leq c_c, \quad \forall t \in [0,T)$$

$$x(\cdot), u(\cdot) \in \mathcal{L}_2[0,T]$$

Standing assumptions:

- Solution exists
- Convex terminal and stage cost

Sampled-data CLQR

$$\min_{x(\cdot),u(\cdot)} x(T)'P_c x(T) + \int_0^T x(t)'Q_c x(t) + u(t)'R_c u(t)dt$$

$$x(0) = \hat{x},\ \dot{x}(t) = A_c x(t) + B_c u(t), \quad \forall t \in (0, T)$$

$$E_\Delta x(t) + J_\Delta u(t) \leq c_\Delta, \quad \forall t \in \{0, \Delta, 2\Delta, \ldots (N-1)\Delta\}$$

Zero-order hold (ZOH) constraint commonly employed:

$$u(t) = u(k\Delta), \quad \forall t \in [k\Delta, (k+1)\Delta),\ k = 0, 1, \ldots, N-1$$

Conditions for convergence to continuous-time solution as sample period tends to zero:

J. Yuz, G. Goodwin, A. Feuer, and J. De Doná, “Control of constrained linear systems using fast sampling rates,” Systems & Control Letters, vol. 54, no. 10, pp. 981–990, 2005.

Equivalent Discrete-time CLQR

$$\min_{\substack{u_0,u_1,\ldots,u_{N-1}\\ x_0,x_1,x_2,\ldots,x_N}} \frac{1}{2}x'_N P x_N + \sum_{k=0}^{N-1} \frac{1}{2}\left(x'_k Q x_k + u'_k R u_k + 2x'_k S u_k\right)$$

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1$$

$$Ex_k + Ju_k \leq c, \quad k = 0, 1, \ldots N-1$$

Computing various matrices (all functions of sample period, in general):

- Inside Matlab's c2d and lqrd function (*not* dlqr)
- K. Astrom and B. Wittenmark, Computer-Controlled Systems, 3rd ed., Prentice-Hall, 1997.
- C. Van Loan, "Computing integrals involving the matrix exponential", IEEE Trans. Automatic Control, vol. 23, no. 3, pp. 395–404, 1978.
- J. Yuz, G. Goodwin, A. Feuer, and J. De Doná,"Control of constrained linear systems using fast sampling rates," Systems & Control Letters, vol. 54, no. 10, pp. 981–990, 2005.
- G. Pannocchia, J. B. Rawlings, D. Q. Mayne, and W. Marquardt, "On computing solutions to the continuous time constrained linear quadratic regulator," IEEE Trans. Automatic Control, vol. 55, no. 9, pp. 2192–2198, 2010.

Discrete Minimum Principle = KKT for QP

Pearson & Sridhar, IEEE Trans. Automat. Control, 1966: Stage cost and terminal cost *convex*, dynamics *linear* and inequality constraints *convex,* so **discrete minimum principle** gives *necessary and sufficient* conditions for the solution:

$$A'\nu_{k+1} = Qx_k + Su_k + \nu_k + E'\lambda_k, \qquad k = 1, \ldots, N-1$$

$$\nu_N + Px_N = 0,$$

$$B'\nu_{k+1} = J'\lambda_k + Ru_k + S'x_k, \qquad k = 0, \ldots, N-1$$

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \qquad k = 0, \ldots, N-1$$

$$\lambda_k \geq 0, \quad Ex_k + Ju_k \leq c, \qquad k = 0, \ldots, N-1$$

$$\mathrm{diag}(\lambda_k)(Ex_k + Ju_k - c) = 0, \qquad k = 0, \ldots, N-1$$

Same as **KKT** conditions for a suitably-defined quadratic program (**QP**), where the $\nu_k,\ \lambda_k$ are the Lagrange multipliers.

Quadratic Program (QP)

$$\min_{\theta} \ \frac{1}{2}\theta' H \theta + h'\theta$$
$$F\theta = f$$
$$G\theta \leq g$$

Matlab: QUADPROG

Octave: QP

Scilab: QPSOLVE

Free: OOQP, BPMPD, CLP, QPC, qpOASES, CVX/CVXGEN

Other: CPLEX, MOSEK, NAG, QPOPT, TOMLAB, XPRESS

Free SOCP/SDP solvers (also solves QPs): SDPT3, SEDUMI, SDPA, SDPNAL, CSDP, DSDP

Interfaces/modelling: YALMIP, AMPL, CVX/CVXGEN, TOMLAB

Solving the CLQR Problem as a QP

Optimal input sequence is nonlinear function of current state

Formulate as a multi-parametric quadratic program (QP):

- Off-line: Compute solution as a the piecewise-affine expression for all possible values of state
- On-line: Implement solution as lookup table
- Limited to easy/small problems
- Bemporad et al. (Automatica, 2002)
- Multi-Parametric Toolbox (MPT), http://control.ee.ethz.ch/~mpt/

Philosophy of Predictive Control:

- Off-line: Don't compute explicit expression for solution
- On-line: Solve a QP in 'real-time' at each sample instant
- Gives *exactly* the same input sequence as explicit expression
- Faster for complex/medium/large problems
- More flexible than computing explicit expression

Formulating CLQR Problem as a QP

Non-condensed

$$\theta := \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix}$$

$$\Rightarrow \quad \min_{\theta} \ \frac{1}{2}\theta' H \theta$$
$$F\theta = f(\hat{x})$$
$$G\theta \leq g$$

Condensed

$$\theta := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$\Rightarrow \quad \min_{\theta} \ \frac{1}{2}\theta' H \theta + h(\hat{x})'\theta$$
$$G\theta \leq g(\hat{x})$$

Non-condensed Formulation

$$\theta := \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \min_{\theta} \ & \frac{1}{2}\theta' H \theta \\ & F\theta = f(\hat{x}) \\ & G\theta \leq g \end{aligned}$$

NB (why formulation and choice of QP solver matters):

- Matrices (*F*,*G*,*H*) are *sparse* and *banded.*

Use a QP solver that exploits sparsity and bandedness:

- SJ Wright, *Parallel Computing*, 1990
- SJ Wright, *J. Optimization Theory Applications*, 1993

Tailor-made QP solvers can do better than general-purpose QP solvers:

- CV Rao, SJ Wright and JB Rawlings, *J. Optimization Theory Applications*, 1998

Why Order of Unknowns Matters

YES

$$\theta := \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix}$$

NO

$$\theta := \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

QP solvers rely on linear algebra solvers, which do not always perform well at detecting that re-ordering rows and/or columns results in a banded matrix.

Matlab Example: Why Order Matters

```
>> n = 1e7;
>> b1 = rand(n,1);
>> permrows = randperm(n);
>> b2 = b1(permrows); % randomly permute rows of b1
>> A1 = speye(n); % identity matrix stored as sparse matrix
>> A2 = A1(permrows,:); % randomly permute rows of A1
```

```
>> % solve linear system A1*x1 = b1, with A1 diagonal
>> tic; x1 = A1\b1; toc
Elapsed time is 0.230493 seconds.

>> % solve same linear system, but with rows permuted
>> tic; x2 = A2\b2; toc
Elapsed time is 9.231288 seconds.
```

Equality Constraints (Non-condensed)

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1$$

$$\Updownarrow$$

$$\begin{bmatrix} I & & & & & & \\ -A & -B & I & & & & \\ & & -A & -B & & & \\ & & & & \ddots & & \\ & & & & & I & \\ & & & & & -A & -B & I \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} \hat{x} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Using the Kronecker Product

$$M \in \mathbb{R}^{m \times n}$$

$$I_N \otimes M = \begin{bmatrix} M & & \\ & \ddots & \\ & & M \end{bmatrix} \in \mathbb{R}^{Nm \times Nn}$$

$$\mathbf{1}_N \otimes M = \begin{bmatrix} M \\ \vdots \\ M \end{bmatrix} \in \mathbb{R}^{Nm \times n}$$

$\mathbf{1}_N$ column vector of *N* ones

Inequality Constraints (Non-condensed)

$$Ex_k + Ju_k \leq c, \quad k = 0, 1, \ldots N-1$$

$$\Updownarrow$$

$$\begin{bmatrix} I_N \otimes [E \ J] & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix} \leq \mathbf{1}_N \otimes c$$

$\mathbf{1}_N$ column vector of *N* ones

Cost Function (Non-condensed)

$$\frac{1}{2}x'_N P x_N + \sum_{k=0}^{N-1} \frac{1}{2}\left(x'_k Q x_k + u'_k R u_k + 2x'_k S u_k\right)$$

$$= \frac{1}{2}\begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_N \end{bmatrix}' \begin{bmatrix} I_N \otimes \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_N \end{bmatrix}$$

Homework: Non-condensed Formulation

$$\theta := \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ u_{N-1} \\ x_N \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \min_{\theta} \ & \frac{1}{2}\theta' H \theta \\ & F\theta = f(\hat{x}) \\ & G\theta \leq g \end{aligned}$$

Write Matlab/Octave/Scilab code that uses the previous three slides to compute the QP matrices (F, G, H) and vectors $(f(\hat{x}), g)$ *without* using any for- or while-loops in your code.

Hint: Rewrite the matrix in the equality constraints using Kronecker products.

Condensed Formulation

$$\theta := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \min_{\theta} \ & \frac{1}{2}\theta' H \theta + h(\hat{x})'\theta \\ & G\theta \leq g(\hat{x}) \end{aligned}$$

NB (why smaller is not always better):

- Matrices (*G*,*H*) are *not* sparse and banded, in general, hence not always faster to solve than non-condensed case, even though less unknowns.
- Numerically ill-conditioned QP if dynamic system is unstable.

For a suitable change of variables, which introduces sparsity and bandedness and allows one to work with unstable systems, see:

- JL Jerez, EC Kerrigan and GA Constantinides. A Condensed and Sparse QP Formulation for Predictive Control, *Proc. IEEE CDC-ECC*, Dec. 2011

Inequality Constraints (Condensed)

$$Ex_k + Ju_k \leq c, \quad k = 0, 1, \ldots N-1$$

$$\Updownarrow$$

$$[I_N \otimes E \ 0] \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} + [I_N \otimes J] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \leq \mathbf{1}_N \otimes c$$

$\mathbf{1}_N$ column vector of *N* ones

Cost Function (Condensed)

$$\frac{1}{2}x'_N P x_N + \sum_{k=0}^{N-1} \frac{1}{2}\left(x'_k Q x_k + u'_k R u_k + 2x'_k S u_k\right)$$

$$= \frac{1}{2}\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}' \begin{bmatrix} I_N \otimes Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}' I_N \otimes R \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$+ \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}' \begin{bmatrix} I_N \otimes S \\ 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Dynamic System Equations (Condensed)

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1$$

$$\Updownarrow$$

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ I_N \otimes B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} \hat{x} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \left[I - \begin{bmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ I_N \otimes B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} + \left[I - \begin{bmatrix} 0 & 0 \\ I_N \otimes A & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \hat{x}$$

Homework: Condensed Formulation

$$\theta := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \min_{\theta} \ & \frac{1}{2}\theta' H \theta + h(\hat{x})'\theta \\ & G\theta \leq g(\hat{x}) \end{aligned}$$

Write Matlab/Octave/Scilab code that uses the previous three slides to compute the QP matrices (G, H) and vectors $(g(\hat{x}), h(\hat{x}))$ *without* using any for- or while-loops in your code, or computing inverses of matrices.

Hint: Start by computing the matrices in the expression for the state sequence, which is a linear function of the input sequence and current state $\hat{x}$, and substituting the expression for the state sequence into the expressions for the cost and inequality constraints.

Algorithms for Solving a Convex QP

$$\min_{\theta} \frac{1}{2}\theta' H\theta + h'\theta$$
$$F\theta = f$$
$$G\theta \leq g$$

Assume convex QP, i.e. $H \geq 0$

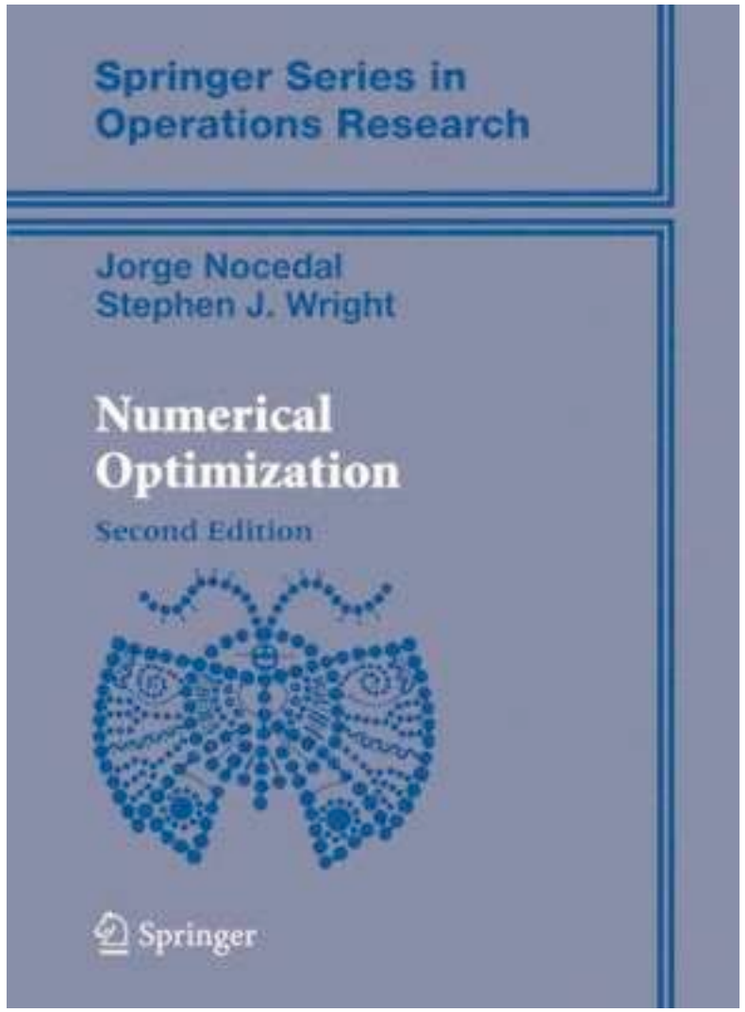
Interior Point Methods:

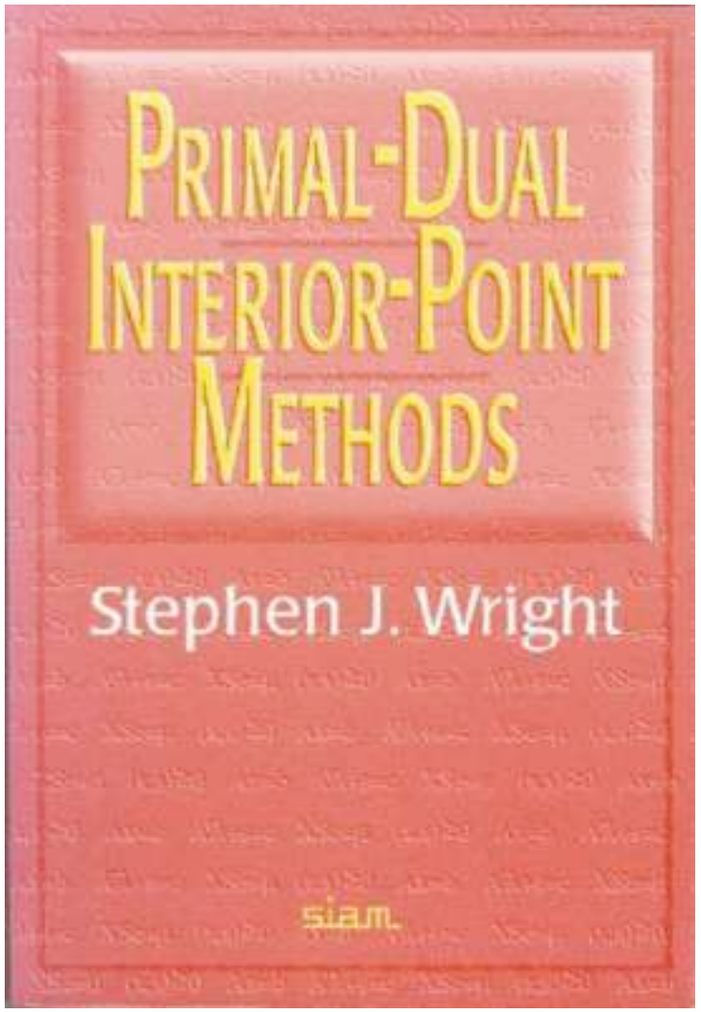
- Easy to code
- Easy to design custom hardware/circuits
- Numerical ill-conditioning an issue for some problems
- (Conservative) polynomial-time bounds
- "Predictable" run-time

Active Set Methods:

- Not so easy to code
- Not so easy to design custom hardware/circuits
- Exponential run-time in worst case
- "Not-so-predictable" run-time
- Very fast for some problems

Books on Optimization





Software for Solving a Convex QP

$$\min_{\theta} \frac{1}{2}\theta' H \theta + h'\theta$$
$$F\theta = f$$
$$G\theta \leq g$$

Matlab: QUADPROG

Octave: QP

Scilab: QPSOLVE

Free: OOQP, BPMPD, CLP, QPC, qpOASES, CVX/CVXGEN

Other: CPLEX, MOSEK, NAG, QPOPT, TOMLAB, XPRESS

Free SOCP/SDP solvers (also solves QPs): SDPT3, SEDUMI, SDPA, SDPNAL, CSDP, DSDP

Interfaces/modelling: YALMIP, AMPL, CVX/CVXGEN, TOMLAB

Optimality Conditions for a Convex QP

Karush-Kuhn-Tucker (KKT) optimality conditions are *necessary and sufficient:* there exist (ν, λ, s) s.t.

$$
\begin{aligned}
H\theta + F'\nu + G'\lambda + h &= 0 \\
F\theta - f &= 0 \\
G\theta + s - g &= 0 \\
\Lambda S\mathbf{1} &= 0 \\
s \geq 0,\ \ \lambda &\geq 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\theta = \arg\min_{\vartheta} \ & \frac{1}{2}\vartheta' H \vartheta + h'\vartheta \\
& F\vartheta = f \\
& G\vartheta \leq g
\end{aligned}
$$

$\Lambda := \mathrm{diag}(\lambda), S := \mathrm{diag}(s)$

s is a slack vector, i.e. $G\theta \leq g \Leftrightarrow \exists s \geq 0 : G\theta + s - g = 0$

Lagrange multipliers:

ν associated with equality constraints

λ associated with inequality constraints

Discrete Minimum Principle = KKT for QP

Pearson & Sridhar, IEEE Trans. Automat. Control, 1966: Stage cost and terminal cost *convex*, dynamics *linear* and inequality constraints *convex,* so **discrete minimum principle** gives *necessary and sufficient* conditions for the solution:

$$A'\nu_{k+1} = Qx_k + Su_k + \nu_k + E'\lambda_k, \qquad k = 1, \ldots, N-1$$

$$\nu_N + Px_N = 0,$$

$$B'\nu_{k+1} = J'\lambda_k + Ru_k + S'x_k, \qquad k = 0, \ldots, N-1$$

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \qquad k = 0, \ldots, N-1$$

$$\lambda_k \geq 0, \quad Ex_k + Ju_k \leq c, \qquad k = 0, \ldots, N-1$$

$$\mathrm{diag}(\lambda_k)(Ex_k + Ju_k - c) = 0, \qquad k = 0, \ldots, N-1$$

Exactly the same as the **KKT** conditions for the non-condensed **QP** formulation, where $\nu_k,\ \lambda_k$ are the Lagrange multipliers.

Modified KKT Equations

Most infeasible (start) interior point methods use Newton's method applied to a sequence of linearizations of the *modified* KKT equations:

$$H\theta + F'\nu + G'\lambda + h = 0$$

$$F\theta - f = 0$$

$$G\theta + s - g = 0$$

$$\Lambda S\mathbf{1} = \mathbf{1}\tau$$

where the scalar $\tau > 0$ is driven to zero, while ensuring the *interior point condition* is satisfied at each iteration:

$$(\lambda, s) > 0$$

NB: Just setting $\tau = 0$ doesn't work in practice!

Logarithmic Barrier Methods

$$\min_{\theta,s} \ \frac{1}{2}\theta' H\theta + h'\theta - \tau \mathbf{1}' \ln(s)$$

$$F\theta = f$$

$$G\theta + s = g$$

Provided that $s > 0$ one can show that the stationary points of the Lagrangian of the above satisfy:

$$H\theta + F'\nu + G'\lambda + h = 0$$

$$F\theta - f = 0$$

$$G\theta + s - g = 0$$

$$\Lambda S \mathbf{1} = \mathbf{1}\tau$$

Use Newton's method to solve above, then reduce $\tau > 0$ and repeat process till KKT conditions are satisfied

Newton Step and Max Step Length

Residual: $\rho := \begin{bmatrix} \rho_{\text{dual}} \\ \rho_{\text{eq}} \\ \rho_{\text{in}} \\ \rho_{\text{cent}} \end{bmatrix} := \begin{bmatrix} H\theta + F'\nu + G'\lambda + h \\ F\theta - f \\ G\theta + s - g \\ \Lambda S\mathbf{1} - \mathbf{1}\tau \end{bmatrix}$

The *Newton step* $(\Delta\theta, \Delta\nu, \Delta\lambda, \Delta s)$ is computed by linearizing the residual about the current iterate and solving the linear system:

$$\begin{bmatrix} H & F' & G' & 0 \\ F & 0 & 0 & 0 \\ G & 0 & 0 & I \\ 0 & 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \\ \Delta\lambda \\ \Delta s \end{bmatrix} = -\rho,$$

Max step length: $\alpha_{\max} := \max_{\alpha \in (0,1]} \{\alpha \mid (\lambda, s) + \alpha(\Delta\lambda, \Delta s) \geq 0\}$

Primal-dual Methods

Do not solve the modified KKT equations exactly.

1.Compute Newton step by solving the linear system:

$$\begin{bmatrix} H & F' & G' & 0 \\ F & 0 & 0 & 0 \\ G & 0 & 0 & I \\ 0 & 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \\ \Delta\lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} H\theta + F'\nu + G'\lambda + h \\ F\theta - f \\ G\theta + s - g \\ \Lambda S\mathbf{1} - \mathbf{1}\tau \end{bmatrix}$$

2.Compute step length α according to some criterion

3.Compute new iterate

$$(\theta, \nu, \lambda, s) \leftarrow (\theta, \nu, \lambda, s) + \alpha(\Delta\theta, \Delta\nu, \Delta\lambda, \Delta s)$$

4.Decrease τ according to some criterion

5.If KKT conditions not satisfied, go to step 1

Faster than logarithmic barrier methods in practice.

Computational Bottleneck

The Newton step is a descent direction for

$$(\theta, \nu, \lambda, s) \mapsto \|\rho\|_2^2$$

Most interior point methods don't just use the residual for determining the step length

$$\alpha \in (0, \alpha_{\max}]$$

The criteria used for computing the step length and reducing τ are important, since they determine the number of iterations.

However: Computing the *Newton step* takes up most of the time and resources per iteration.

Computing a Newton Step

Solve the linear system:

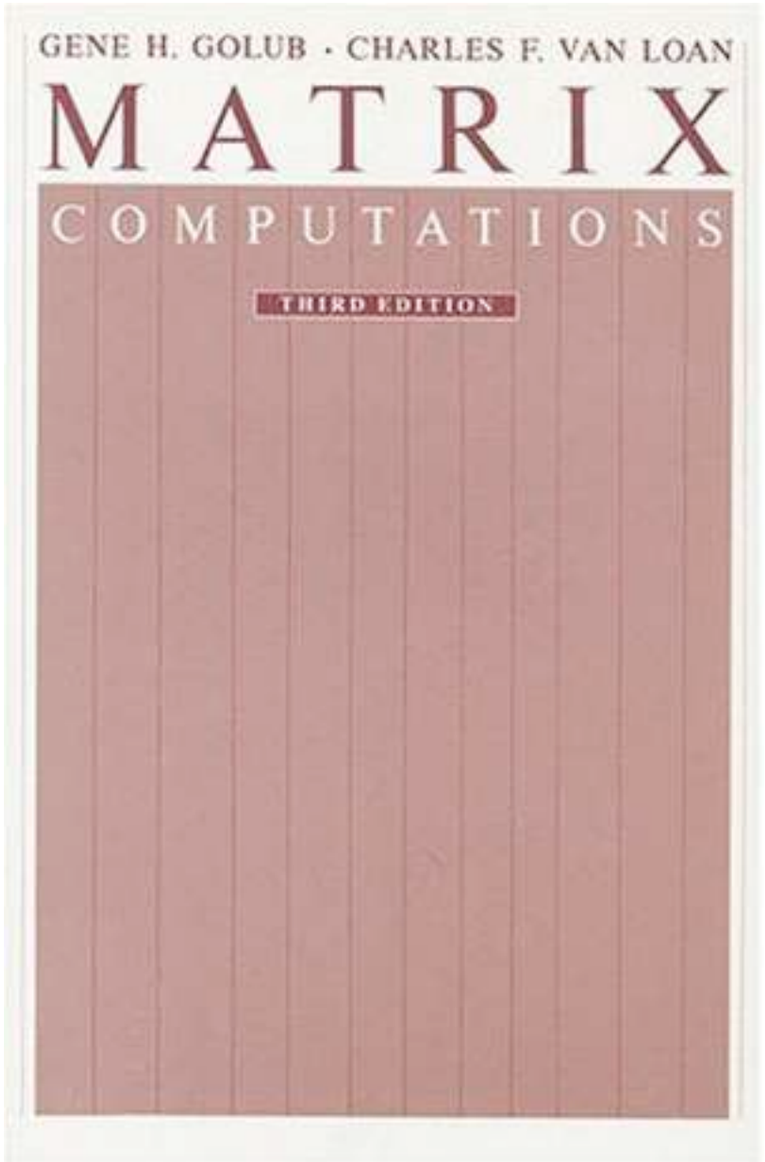
$$\begin{bmatrix} H & F' & G' & 0 \\ F & 0 & 0 & 0 \\ G & 0 & 0 & I \\ 0 & 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \\ \Delta\lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} \rho_{\text{dual}} \\ \rho_{\text{eq}} \\ \rho_{\text{in}} \\ \rho_{\text{cent}} \end{bmatrix}$$

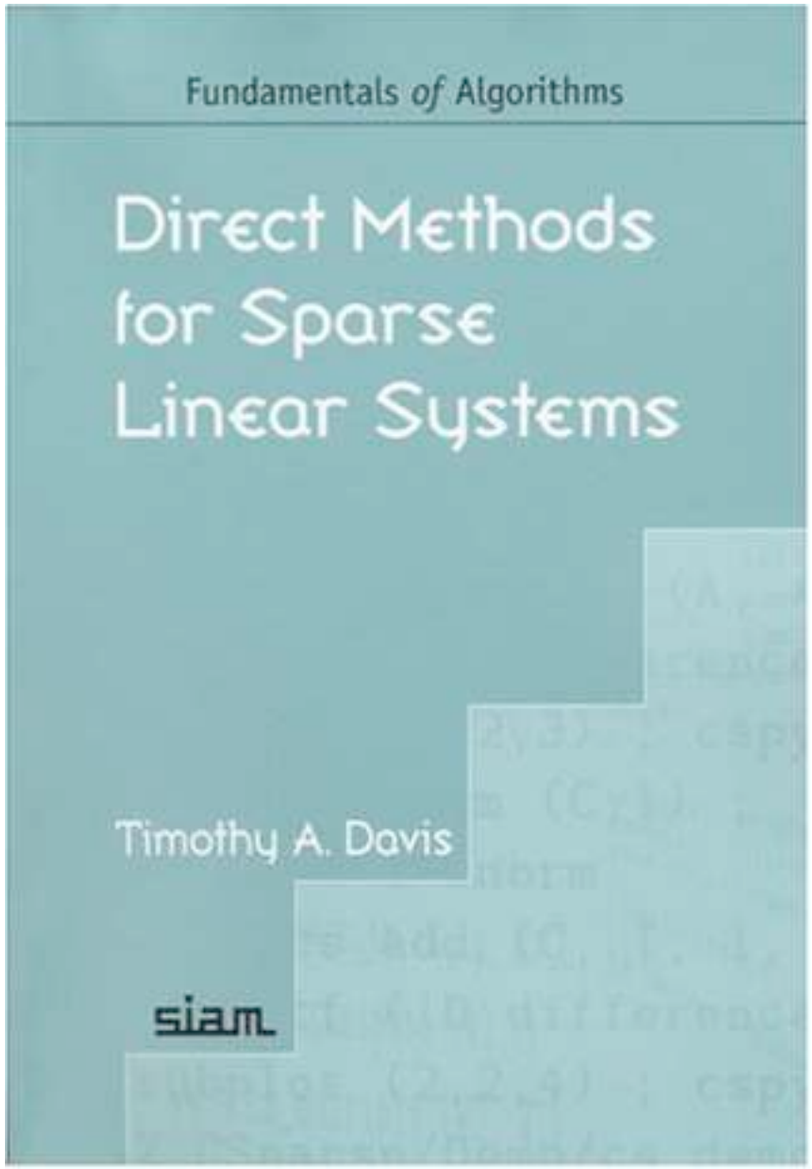
The *KKT matrix* is:

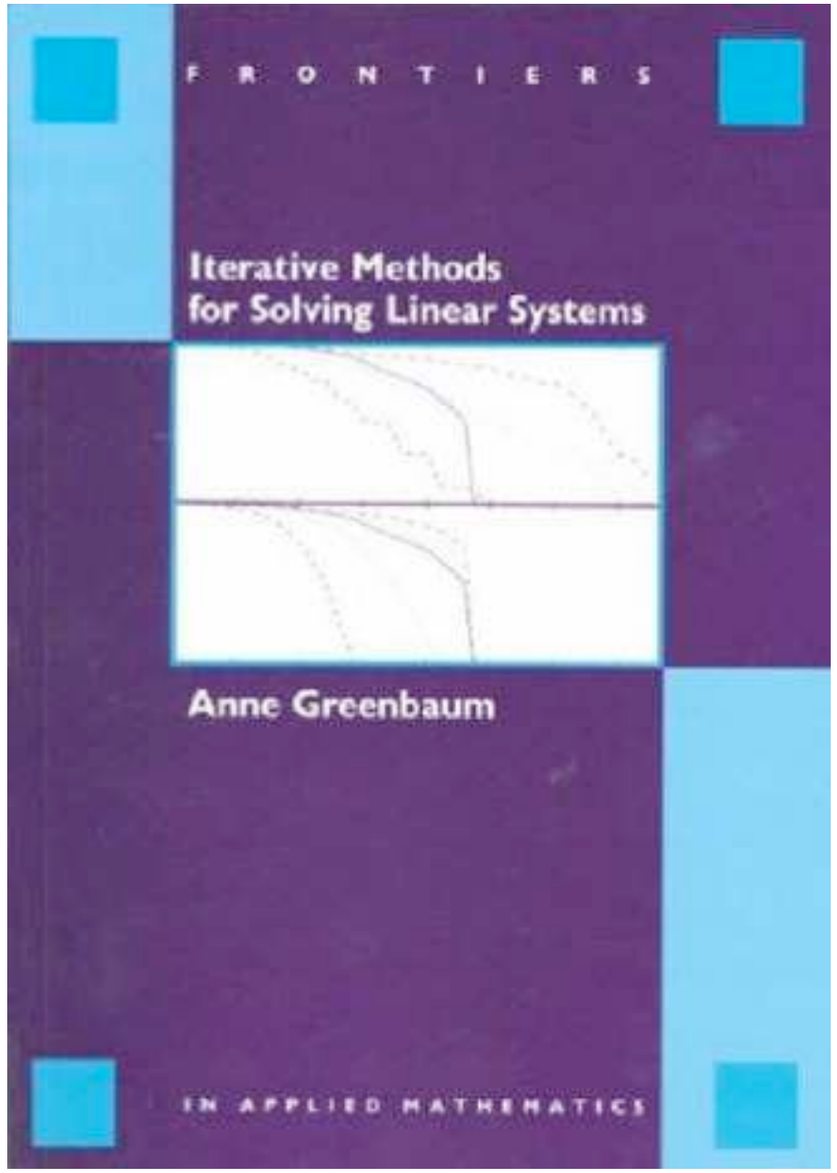
- Non-symmetric
- Indefinite
- Can be very sparse and structured
- Can be ill-conditioned

Solution method extremely important.

Books on Solving Linear Systems







Methods for Solving Linear Systems

	Non-symmetric matrix	**Symmetric and indefinite matrix**	**Symmetric and positive definite matrix**
Direct methods	LU factorization	LDL factorization	Cholesky factorization
Iterative methods	Generalized Minimal Residual (GMRES)	Minimal Residual (MINRES)	Conjugate Gradient (CG)

Important Questions to Ask

Block eliminate some variables before solving?

What is the numerical stability of the method?

Direct or iterative solver?

Sparse or dense solver?

Is there structure that can be exploited, e.g. can re-ordering variables result in a banded KKT matrix?

What is the computational hardware architecture?

Which Block Eliminations?

Solve for some variables first using block elimination.

Many possible choices - order can affect numerical conditioning and efficiency of computation.

A popular choice, which introduces symmetry:

$$\begin{bmatrix} H & F' & G' \\ F & 0 & 0 \\ G & 0 & -\Lambda^{-1}S \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \rho_{\text{dual}} \\ \rho_{\text{eq}} \\ \rho_{\text{in}} - \Lambda^{-1}\rho_{\text{cent}} \end{bmatrix}$$

$$\Delta s = -\Lambda^{-1}\left(\rho_{\text{cent}} + S\Delta\lambda\right)$$

Further Block Eliminations

One can reduce the system further, at potential expense of reducing sparsity:

$$\begin{bmatrix} H + G'\Lambda S^{-1} G & F' \\ F & 0 \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \end{bmatrix} = - \begin{bmatrix} \rho_{\text{dual}} + G'\Lambda S^{-1}\rho_{\text{in}} - G'S\rho_{\text{cent}} \\ \rho_{\text{eq}} \end{bmatrix}$$

$$\Delta\lambda = \Lambda S^{-1} G \Delta\theta + \Lambda S^{-1}\rho_{\text{in}} - S\rho_{\text{cent}}$$

$$\Delta s = -\Lambda^{-1}\left(\rho_{\text{cent}} + S\Delta\lambda\right)$$

If there are no equality constraints (F = 0), then the reduced KKT matrix is positive semi-definite:

$$\left[H + G'\Lambda S^{-1} G\right]\Delta\theta = -\rho_{\text{dual}} - G'\Lambda S^{-1}\rho_{\text{in}} + G'S\rho_{\text{cent}}$$

What About Numerical Stability?

Original, non-symmetric KKT matrix:

- Ill-conditioned in later iterations if QP is degenerate, i.e. when Lagrange multipliers are zero for active constraints

Reduced, symmetric KKT matrices:

- Ill-conditioned in later iterations, since some components of $\Lambda^{-1}S$ tend to zero and some to infinity.

Iterative methods need good pre-conditioners if KKT matrix is ill-conditioned.

Active area of research.

Direct or Iterative Solvers?

Direct, e.g. LU

Advantages:

- Mature theory
- Numerically stable
- Large choice of software

Disadvantages:

- Not necessarily good for sparse or large matrices
- Cannot terminate early
- Not easily parallelizable

Iterative, e.g. GMRES

Advantages:

- Good for sparse or large matrices
- Early termination possible
- Easily parallelizable

Disadvantages:

- Theory not mature
- Numerically sensitive
- Small choice of software

Sparse or Dense Solvers?

Big and sparse often faster and requires less memory than small and dense.

- Non-condensed formulation could be better than condensed formulation if care is taken.

Iterative methods often preferred for sparse matrices.

Some direct methods can minimize amount of “fill-in” during factorization of sparse matrices.

Matlab Example: Why Size Isn't Everything

```
>> n=4e3; % Form a dense matrix
>> A2=randn(n,n);
>> b2=randn(n);
>> tic;x2=A2\b2;toc
Elapsed time is 9.021262 seconds.
>> n=1e6; % Form larger, but sparse (banded) matrix
>> A1=diag(sprandn(n,1,1))+[spalloc(n-1,1,0)
diag(sprandn(n-1,1,1)); spalloc(1,n,0)];
>> b1=randn(n,1);
>> tic;x1=A1\b1;toc
Elapsed time is 1.048625 seconds.
```

Why Structure, and Not Just Size, Matters

Suppose we want to solve a linear system $Ax = b$, where A is an n x n positive definite matrix.

One can use Cholesky factorization to solve for x.

The number of floating point operations needed grows with $O(n^3)$.

If the matrix is banded with width k, then the number of floating point operations grows with $O(k^2 n)$.

Is There Structure to Exploit?

Non-condensed: Interleave input and state variables with Lagrange multipliers to get a **banded,** reduced KKT matrix:

- SJ Wright, *Parallel Computing*, 1990
- SJ Wright, *J. Optimization Theory Applications*, 1993
- CV Rao, SJ Wright and JB Rawlings, *J. Optimization Theory Applications*, 1998

Condensed: Same, but need a change of variables:

- JL Jerez, EC Kerrigan and GA Constantinides. A Condensed and Sparse QP Formulation for Predictive Control, *Proc. IEEE CDC-ECC*, Dec. 2011

Growth of # Floating Point Operations

If upper and lower bounds on all inputs and states

	Horizon Length *N*	# inputs *m*	# states *n*
Condensed Dense Cholesky	**Cubic**	**Cubic**	**Linear**
Non-condensed Sparse *LDL'*	**Linear**	**Cubic**	**Cubic**

Relation to Unconstrained LQR Control

The reduced linear system:

$$\begin{bmatrix} H + G'\Lambda S^{-1}G & F' \\ F & 0 \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\nu \end{bmatrix} = - \begin{bmatrix} \rho_{\text{dual}} + G'\Lambda S^{-1}\rho_{\text{in}} - G'S\rho_{\text{cent}} \\ \rho_{\text{eq}} \end{bmatrix}$$

can be shown to be the necessary and sufficient optimality conditions of a time-varying LQR problem without inequality constraints and suitably-defined cost:

$$\min_{\substack{u_0,u_1,\ldots,u_{N-1} \\ x_0,x_1,x_2,\ldots,x_N}} \frac{1}{2}x'_N\tilde{Q}_N x_N + \sum_{k=0}^{N-1} \frac{1}{2}\left(x'_k\tilde{Q}_k x_k + u'_k\tilde{R}_k u_k + 2x'_k\tilde{S}_k u_k\right)$$

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1$$

Rao, Wright, Rawlings, *J. Optimization Theory Applications*, 1998

Discrete Minimum Principle = KKT for QP

Pearson & Sridhar, IEEE Trans. Automat. Control, 1966: Stage cost and terminal cost *convex*, dynamics *linear* and inequality constraints *convex,* so **discrete minimum principle** gives *necessary and sufficient* conditions for the solution:

$$A'\nu_{k+1} = Qx_k + Su_k + \nu_k + E'\lambda_k, \qquad k = 1, \ldots, N-1$$

$$\nu_N + Px_N = 0,$$

$$B'\nu_{k+1} = J'\lambda_k + Ru_k + S'x_k, \qquad k = 0, \ldots, N-1$$

$$x_0 = \hat{x},\ x_{k+1} = Ax_k + Bu_k, \qquad k = 0, \ldots, N-1$$

$$\lambda_k \geq 0, \quad Ex_k + Ju_k \leq c, \qquad k = 0, \ldots, N-1$$

$$\mathrm{diag}(\lambda_k)(Ex_k + Ju_k - c) = 0, \qquad k = 0, \ldots, N-1$$

Exactly the same as the **KKT** conditions for the non-condensed **QP** formulation, where $\nu_k,\ \lambda_k$ are the Lagrange multipliers.

Which Hardware is Faster?

Concorde

Shinkansen

Which Hardware is Faster?

Concorde

100 passengers
Mach 2 (2160 km/h)

1630 passengers
300 km/h

Shinkansen

Which Hardware is Faster?

Concorde

100 passengers
Mach 2 (2160 km/h)

300km:
100 passengers = 8-9 min
1630 passengers = 6hr 35 min

1630 passengers
300 km/h

Shinkansen

300km:
100 passengers = 1hr
1630 passengers = 1hr

What About the Hardware?

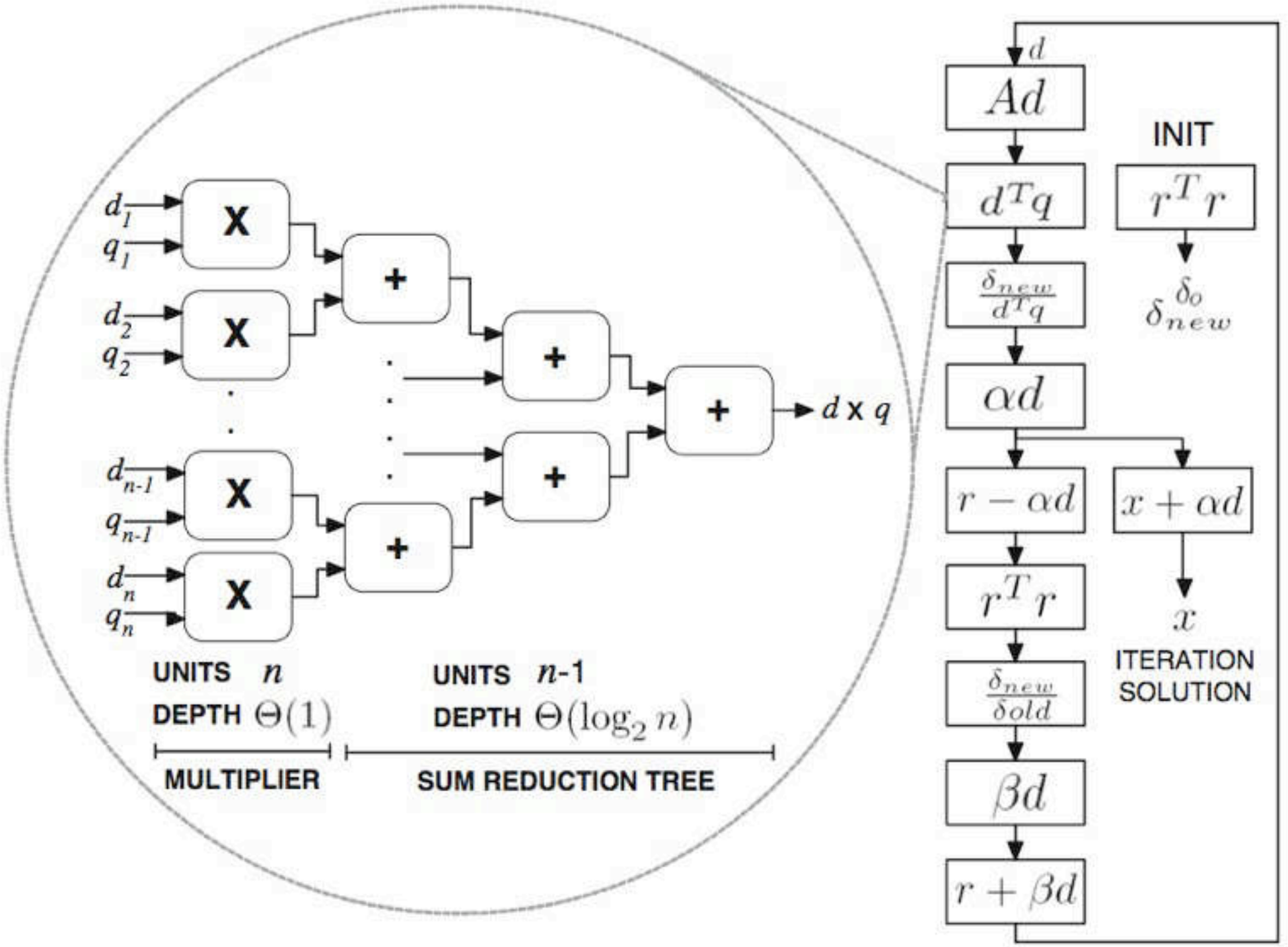
It's not just about:

- the software
- how fast the clock is
- the number of transistors
- how much memory there is
- the energy available
- the computational delay
- the sample rate
- the accuracy of the computation
- the closed-loop performance
- the robustness of the closed-loop

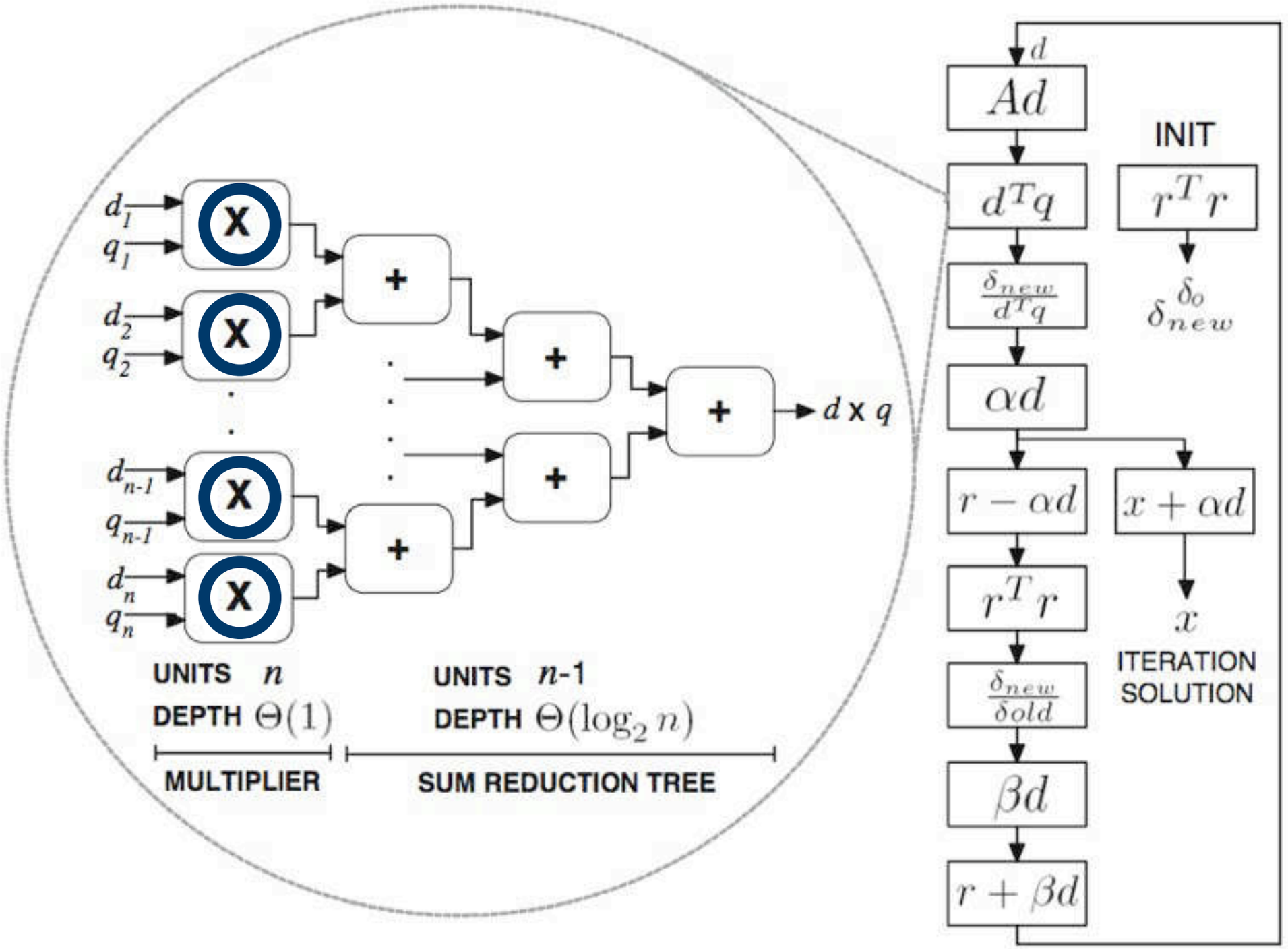
There is no optimal hardware implementation.

There is always a trade-off - "good enough" is possible

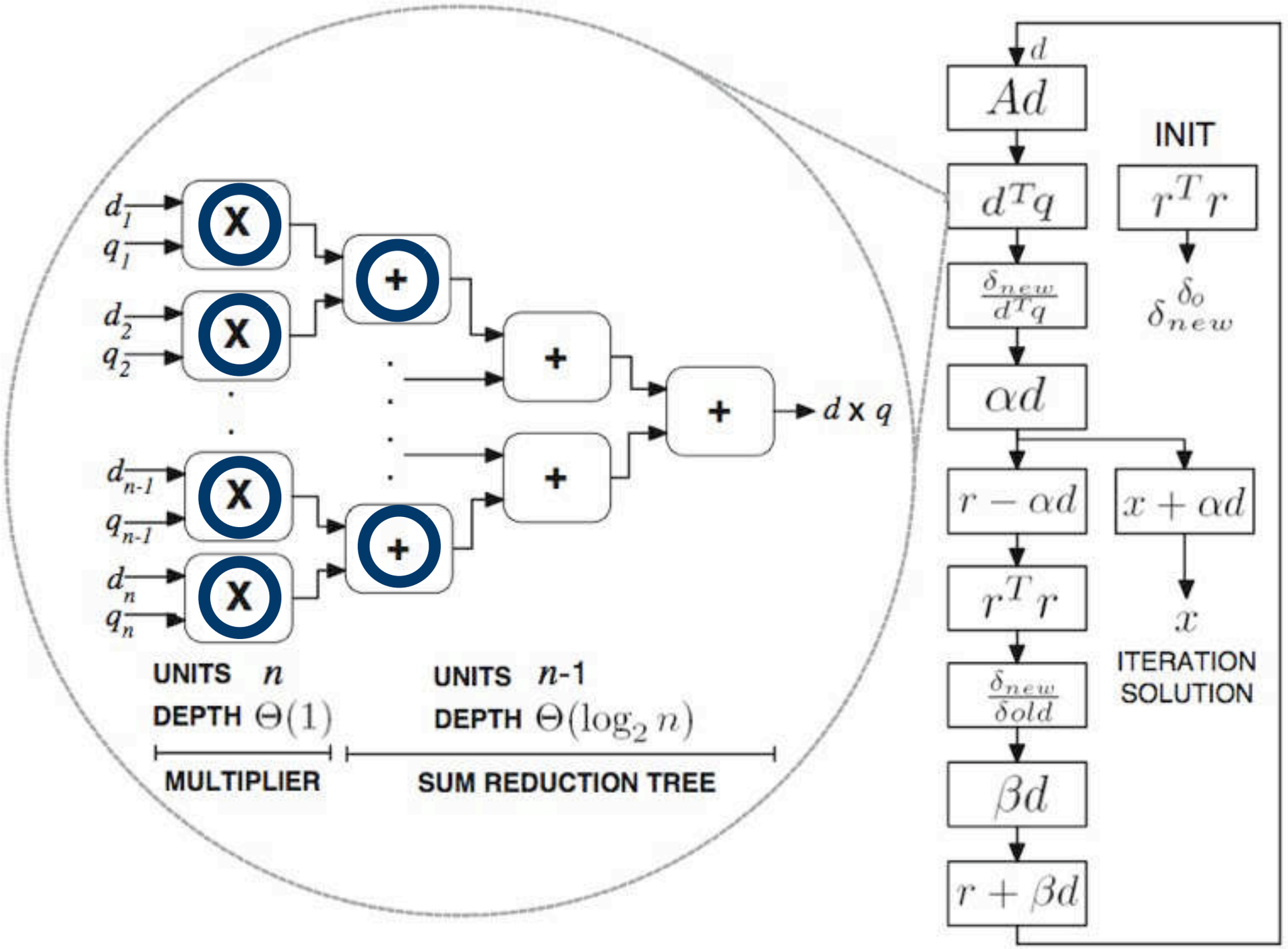
Pipelining in CG Method



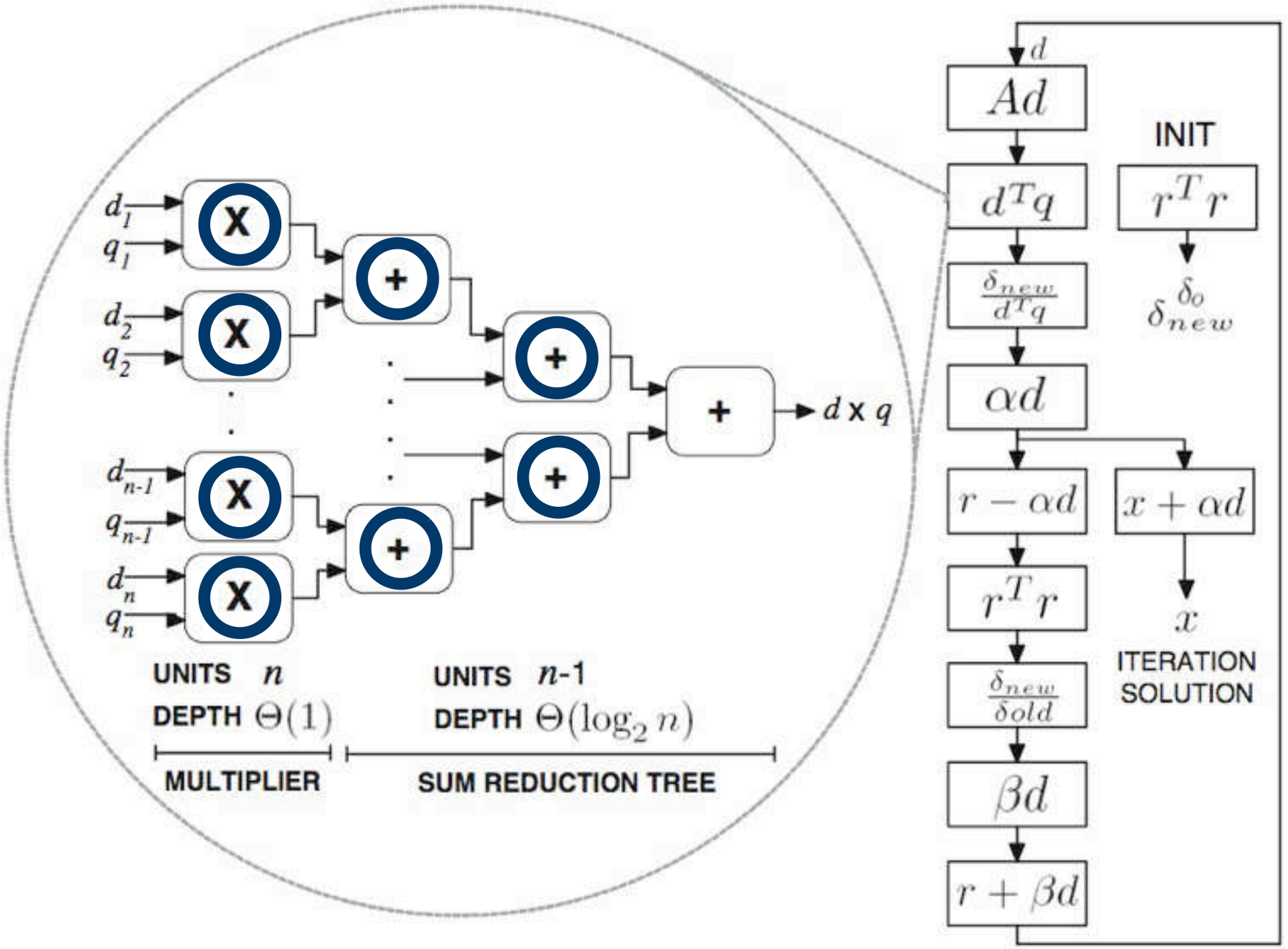
Pipelining in CG Method



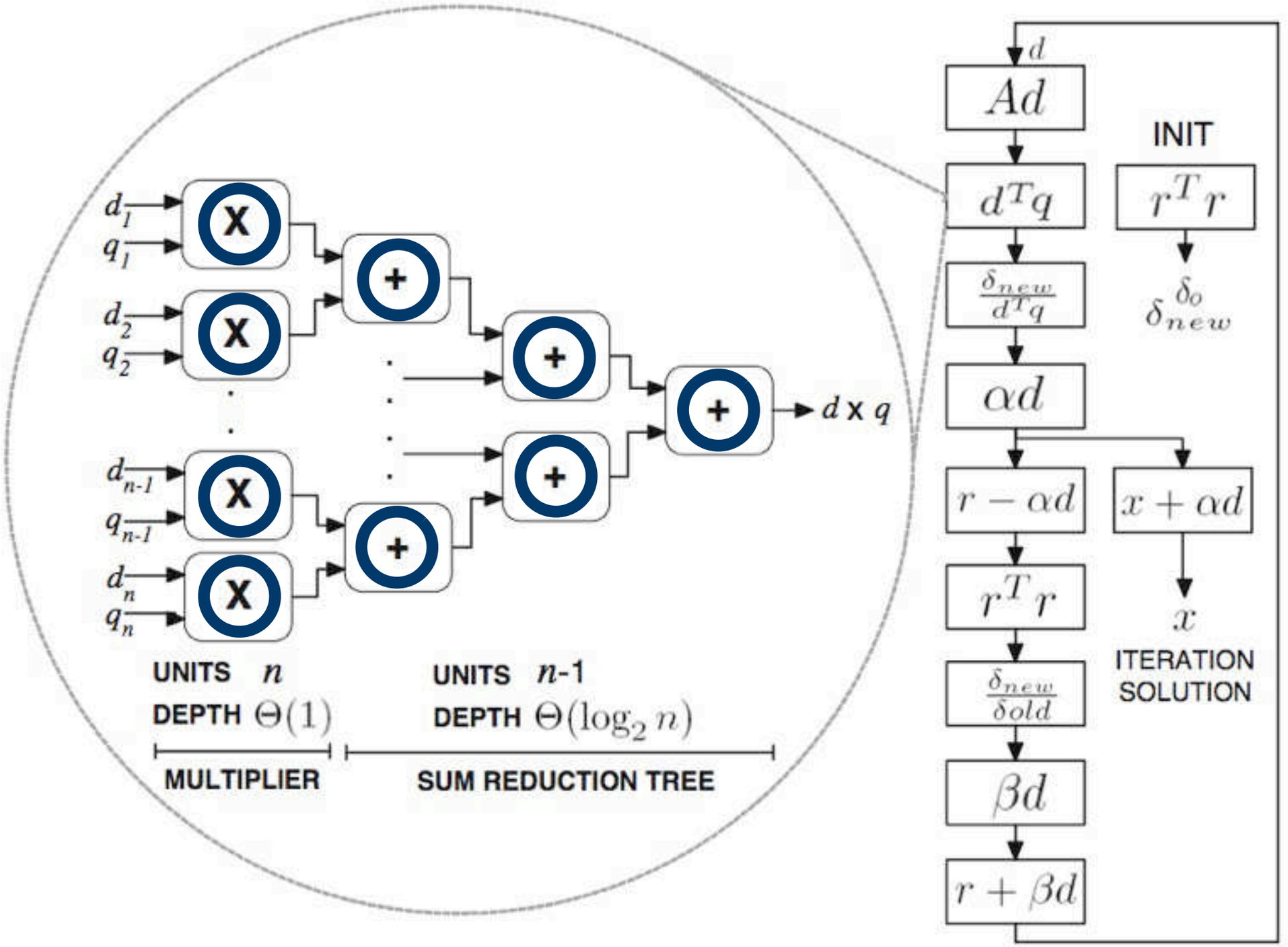
Pipelining in CG Method



Pipelining in CG Method

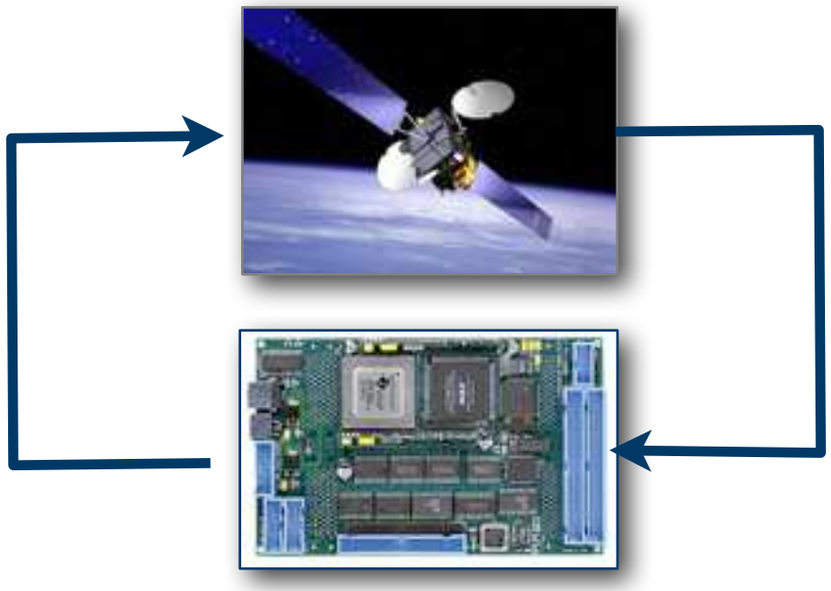


Pipelining in CG Method



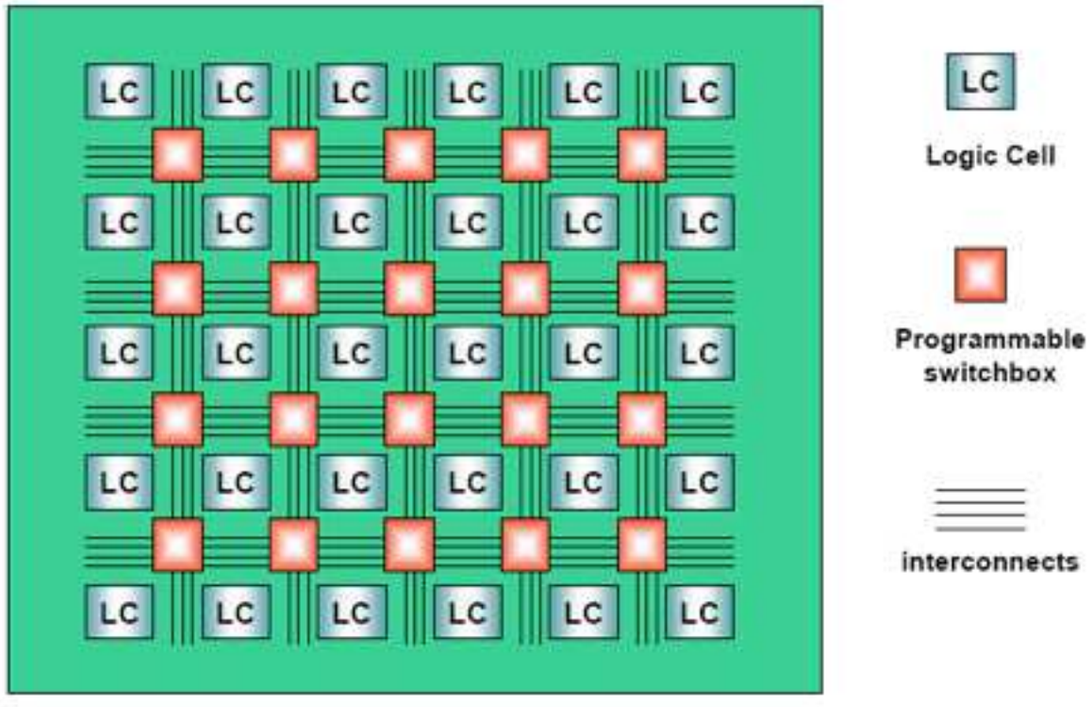
Embedded Optimization and Control

Algorithm Design ⟷ Silicon Implementation



Field-Programmable Gate Array (FPGA)

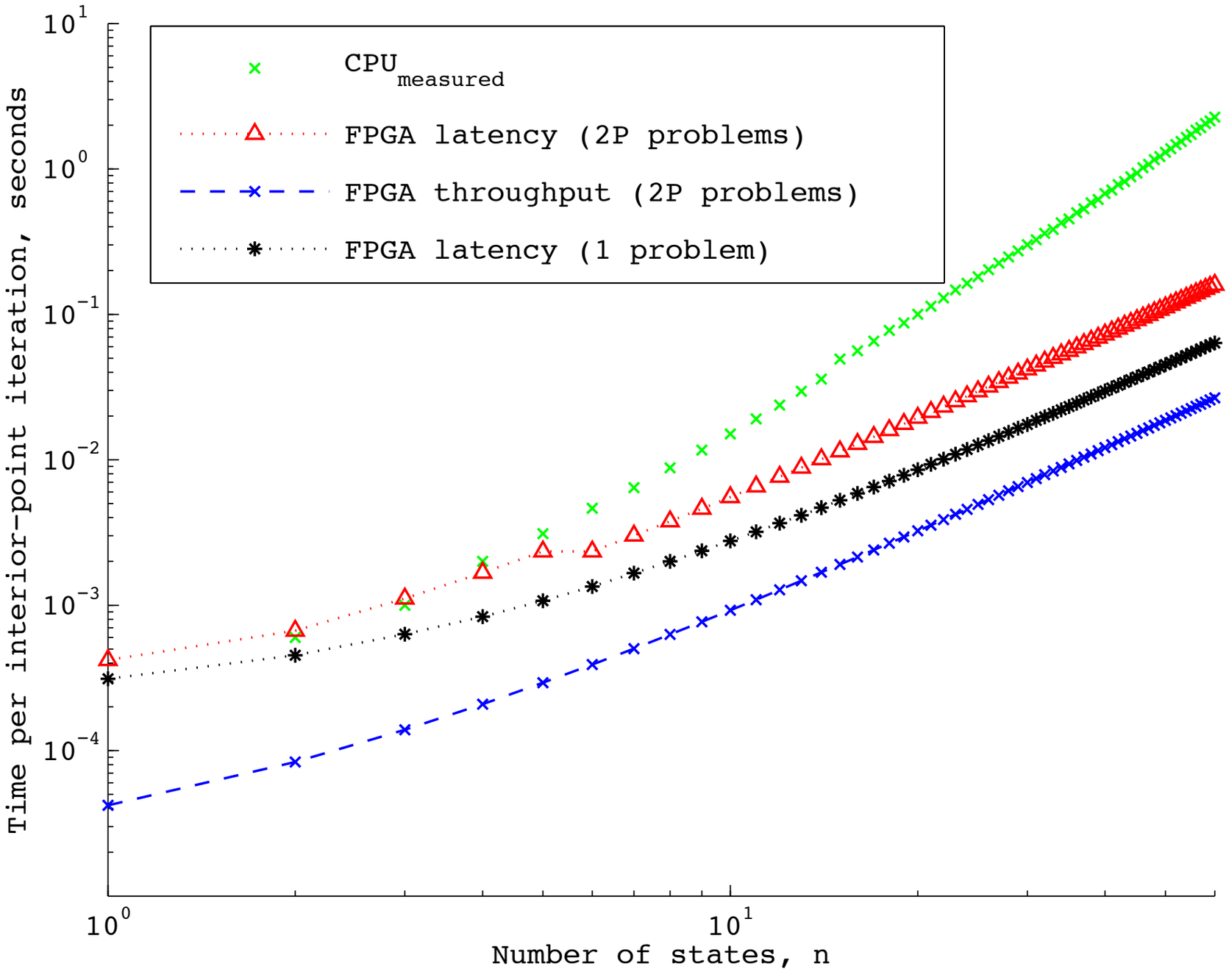
- Parallel and pipelined computations
- Flexible number representation
- Design memory subsystem
- Predictable
- Energy efficient (<10mW-10W)
- Budget to match demand (<$10-$20k)



Time: Non-condensed QP Formulation

Hardware : Xilinx Virtex 6 SX 475T @ 250MHz (40nm)

Software : Intel Core2 Q8300 @ 2.5GHz, 3GB RAM, 4MB L2 Cache (45nm)

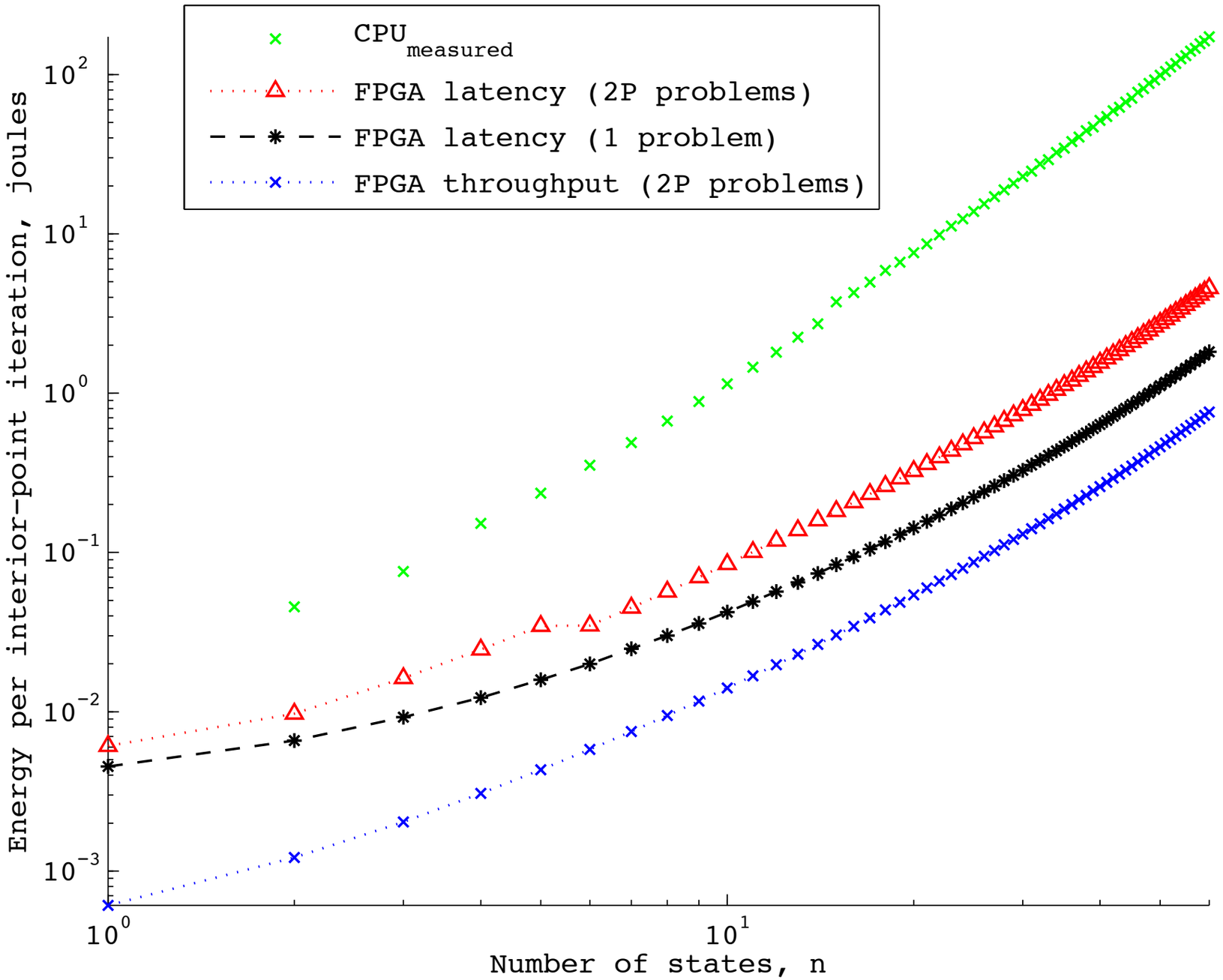


- Red curve: 14x
- Black curve: 36x
- Blue curve: 85x

Energy: Non-condensed QP Formulation

Hardware : Xilinx Virtex 6 SX 475T @ 250MHz (40nm)

Software : Intel Core2 Q8300 @ 2.5GHz, 3GB RAM, 4MB L2 Cache (45nm)



CPU power $\sim$ 76W

FPGA board power $\sim$ 14-28W

For the largest problem, the improvement is:

- Red curve: 38x
- Black curve: 95x
- Blue curve: 227x

Selected Papers

GA Constantinides. Tutorial Paper:
Parallel Architectures for Model Predictive Control. *Proc. European Control Conference* 2009.

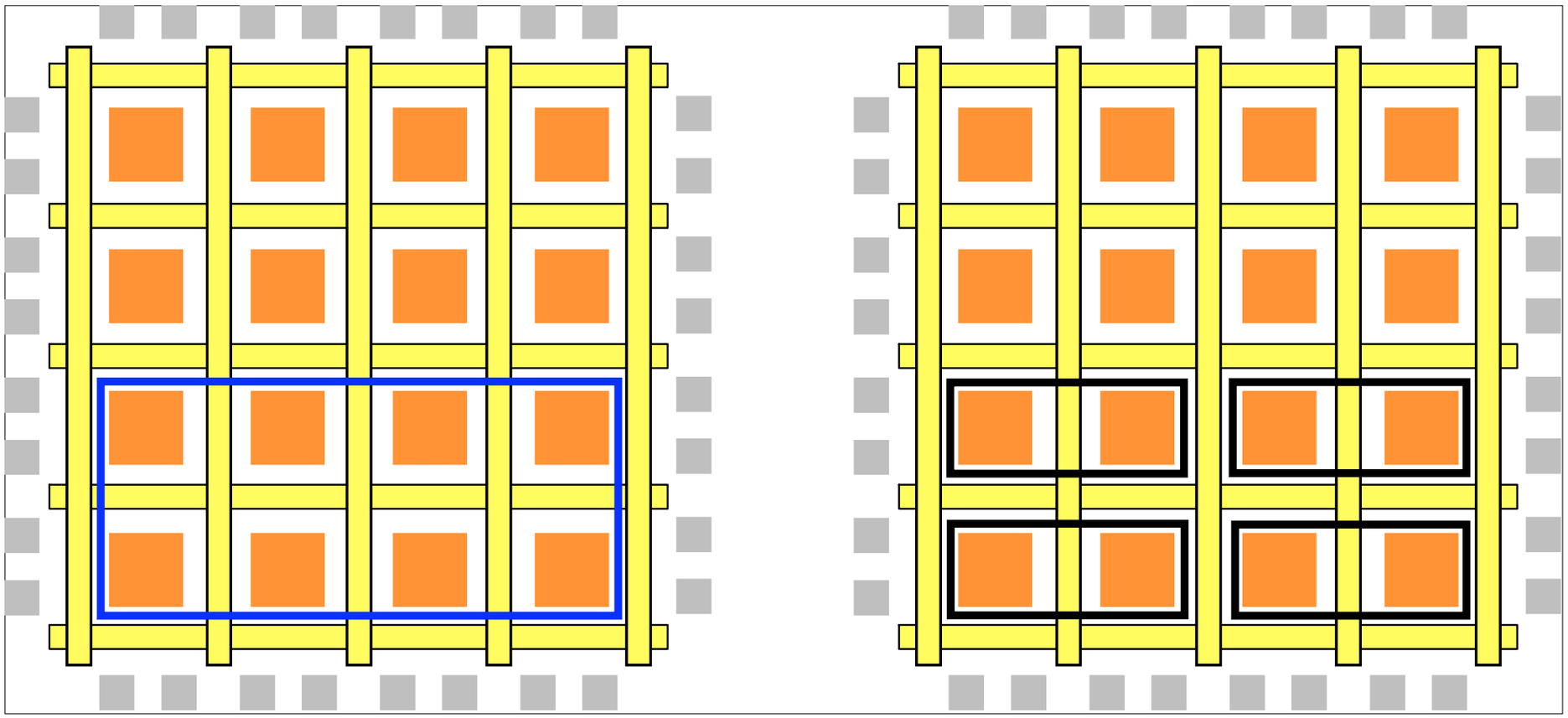
AG Wills, G Knagge, B Ninness. Fast Linear Model Predictive Control Via Custom Integrated Circuit Architecture. *IEEE Trans. Control Systems Technology* 2011.

JL Jerez, GA Constantinides, EC Kerrigan, K-V Ling. Parallel MPC for Real-Time FPGA-based Implementation. *Proc. 18th IFAC World Congress*, 2011.

Precision/speed Trade-offs in Silicon

Silicon area used grows quadratically with precision

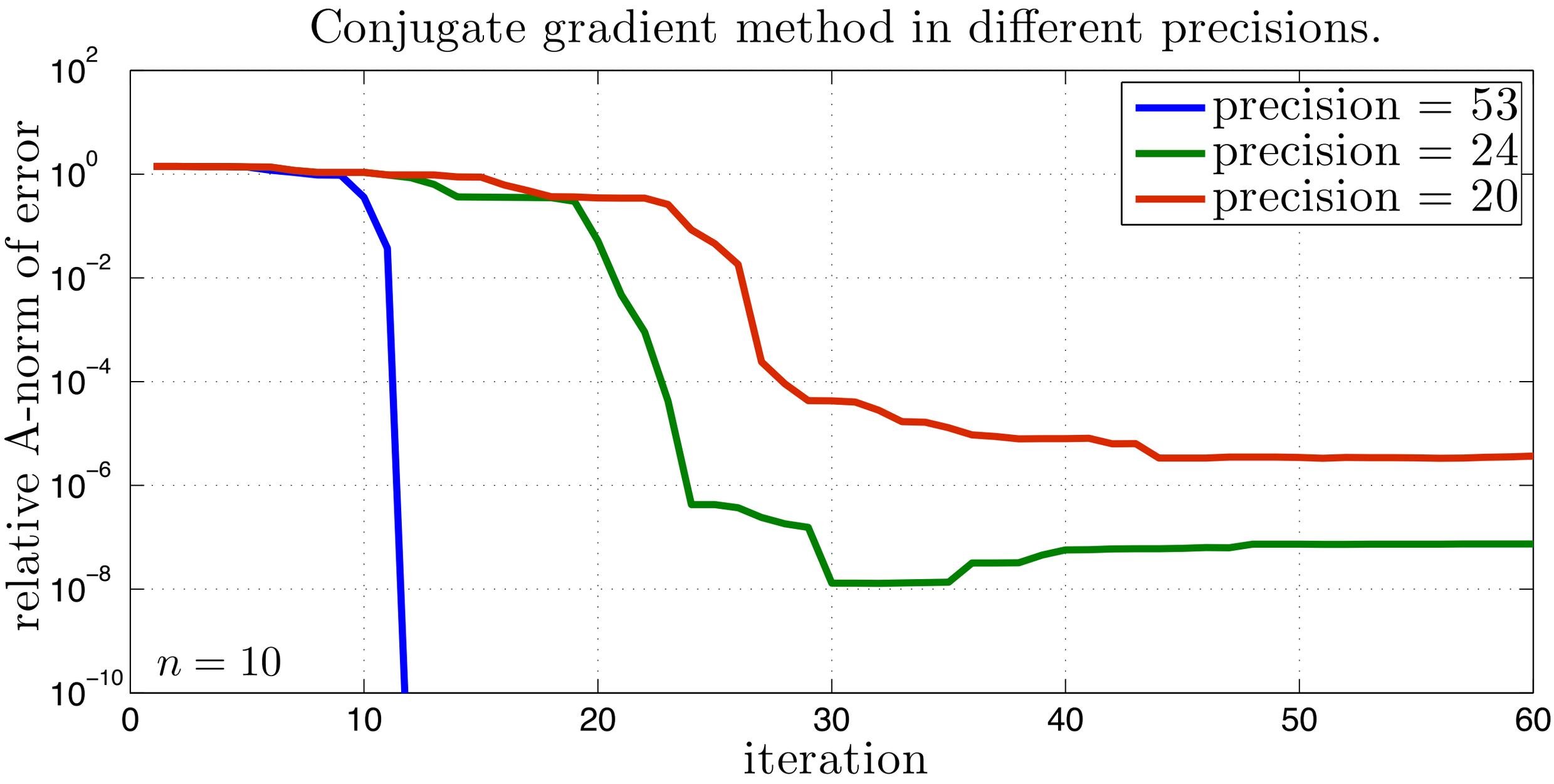
Less precision => more parallelism => potential speed up



1x high precision circuit

4x low (half) precision circuits

Finite Precision Effects



Summary

Motivation

- Use feedback if and only if uncertainty is present
- Predictive control is a constrained feedback strategy

Repeatedly solve an open-loop CLQR problem

- Solve sampled-data CLQR problem by solving a QP
- Non-condensed and condensed QP formulations

Interior point methods

- Use Newton's method to solve modified KKT conditions
- Computing Newton step uses most of the resources

Hardware issues

- Hardware should inform choice of algorithm and vice versa
- Trade-offs always have to be made

Extensions

Nonlinear dynamics

Time-varying dynamics and constraints

Non-quadratic costs, e.g. 1-norm, infinity-norm, time

Robust formulations, e.g. min-max, stochastic

State estimation with constraints and nonlinearities

Further Reading