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Tutorial on Control and State Constrained Optimal Control Problems – Part 2 : Mixed Control-State Constraints

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Imperial College London, September 5, 2011

Outline

- 1 Optimal Control Problems with Control and State Constraints
- 2 Numerical Method: Discretize and Optimize
- 3 Theory of Optimal Control Problems with Mixed Control-State Constraints
- 4 Example: Rayleigh Problem with Different Constraints and Objectives
- 5 Example: Optimal Exploitation of Renewable Resources

Optimal Control Problem

state $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$.

Dynamics and Boundary Conditions

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, t_f], \\ x(t) &= \bar{x}_0, \quad \psi(x(t_f)) = 0 \quad (\psi : \mathbb{R}^n \rightarrow \mathbb{R}^r). \end{aligned}$$

Control and State Constraints

$$\begin{aligned} c(x(t), u(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k) \\ s(x(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (s : \mathbb{R}^n \rightarrow \mathbb{R}^l) \end{aligned}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t)) dt$$

Discretization

For simplicity consider a **MAYER-type problem** with cost functional

$$J(u, x) = g(x(t_f)).$$

This can be achieved by considering the **additional state variable** x_0 with

$$\dot{x}_0 = f_0(x, u), \quad x_0(0) = 0.$$

Then we have

$$x_0(t_f) = \int_0^{t_f} f_0(t, x(t), u(t)) dt.$$

Choose an integer $N \in \mathbb{N}$, a **stepsize** h and **grid points** t_i :

$$h = t_f/N, \quad t_i := ih, \quad (i = 0, 1, \dots, N).$$

Approximation of control and state at grid points:

$$u(t_i) \approx u_i \in \mathbb{R}^m, \quad x(t_i) \approx x_i \in \mathbb{R}^n \quad (i = 0, \dots, N)$$

Large-scale NLP using EULER's method

Minimize

$$J(u, x) = g(x_N)$$

subject to

$$\begin{aligned} x_{i+1} &= x_i + h \cdot f(t_i, x_i, u_i), & i = 0, \dots, N-1, \\ x_0 &= \bar{x}_0, \quad \psi(x_N) = 0, \\ c(x_i, u_i) &\leq 0, & i = 0, \dots, N, \\ s(x_i) &\leq 0, & i = 0, \dots, N, \end{aligned}$$

Optimization variable for full discretization:

$$z := (u_0, x_1, u_1, x_2, \dots, u_{N-1}, x_N, u_N) \in \mathbb{R}^{N(m+n)+m}$$

NLP Solvers

- AMPL : [Programming language](#) (Fourer, Gay, Kernighan)
- IPOPT : [Interior point method](#) (Andreas Wächter)
- LOQO : [Interior point method](#) (Vanderbei et al.)
- Other NLP solvers embedded in AMPL : cf. NEOS server
- NUDOCSS : optimal control package (Christof Büskens)
- WORHP : SQP solver (Christof Büskens, Matthias Gerdts)
- Special feature: solvers provide [LAGRANGE-multipliers as approximations of the adjoint variables](#).

Optimal Control Problem with Control-State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$.

All functions are assumed to be sufficiently smooth

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0, t_f],$$

$$x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,$$

$$(0 = \varphi(x(0), x(t_f)) \quad \text{mixed boundary conditions})$$

Mixed Control-State Constraints

$$\alpha \leq c(x(t), u(t)) \leq \beta, \quad t \in [0, t_f], \quad c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$

Control bounds $\alpha \leq u(t) \leq \beta$ are included by $c(x, u) = u$.

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u) dt$$

Hamiltonian

Hamiltonian

$$H(x, \lambda, u) = \lambda_0 f(x, u) + \lambda f(x, u) \quad \lambda \in \mathbb{R}^n \text{ (row vector)}$$

Augmented Hamiltonian

$$\begin{aligned} \mathcal{H}(x, \lambda, \mu, u) &= H(x, \lambda, u) + \mu c(x, u) \\ &= \lambda_0 f(x, u) + \lambda f(x, u) + \mu c(x, u), \quad \mu \in \mathbb{R}. \end{aligned}$$

Let $(u, x) \in \mathcal{L}^\infty([0, T], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, T], \mathbb{R}^n)$ be a **locally optimal pair of functions**.

Regularity assumption

$$c_u(x(t), u(t)) \neq 0 \quad \forall t \in J_a$$

$$J_a := \{ t \in [0, t_f] \mid c(x(t), u(t)) = \alpha \text{ or } = \beta \}$$

Minimum Principle of Pontryagin et al. and Hestenes

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a **locally optimal pair of functions** that satisfies the regularity assumption. Then there exist

- an adjoint (costate) function $\lambda \in \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$,
- a multiplier function $\mu \in \mathcal{L}^\infty([0, t_f], \mathbb{R})$,
- and a multiplier $\rho \in \mathbb{R}^r$ associated to the boundary condition $\psi(x(t_f)) = 0$

that satisfy the following conditions for a.a. $t \in [0, t_f]$, where the argument (t) denotes evaluation along the trajectory $(x(t), u(t), \lambda(t))$:

Minimum Principle of Pontryagin et al. and Hestenes

(i) **Adjoint ODE and transversality condition:**

$$\begin{aligned}\dot{\lambda}(t) &= -\mathcal{H}_x(t) = -(\lambda_0 f_0 + \lambda f)_x(t) - \mu(t) c_x(t), \\ \lambda(t_f) &= (\lambda_0 g + \rho\psi)_x(x(t_f)),\end{aligned}$$

(iia) **Minimum Condition for Hamiltonian:**

$$H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq c(x(t), u) \leq \beta \}$$

(iib) **Local Minimum Condition for Augmented Hamiltonian:**

$$0 = \mathcal{H}_u(t) = (\lambda_0 f_0 + \lambda f)_u(t) + \mu(t) c_u(t)$$

(iii) **Sign of multiplier μ and complementarity condition:**

$$\begin{aligned}\mu(t) &\leq 0, \text{ if } c(x(t), u(t)) = \alpha; \quad \mu(t) \geq 0, \text{ if } c(x(t), u(t)) = \beta, \\ \mu(t) &= 0, \text{ if } \alpha < c(x(t), u(t)) < \beta.\end{aligned}$$

Evaluation of the Minimum Principle: boundary arc

Boundary arc: Let $[t_1, t_2]$, $0 \leq t_1 < t_2 < t_f$, be an interval with

$$c(x(t), u(t)) = \alpha \quad \text{or} \quad c(x(t), u(t)) = \beta \quad \forall t_1 \leq t \leq t_2.$$

For simplicity assume a **scalar control**, i.e., $m = 1$.

Due to the regularity condition $c_u(x(t), u(t)) \neq 0$ there exists a smooth function $u_b(x)$ satisfying

$$c(x, u_b(x)) \equiv \alpha \quad (\equiv \beta) \quad \forall x \text{ in a neighborhood of the trajectory.}$$

The control $u_b(x)$ is called the **boundary control** and yields the **optimal control** by the relation $u(t) = u_b(x(t))$.

It follows from the local minimum condition $0 = \mathcal{H}_u = H_u + \mu c_u$ that the **multiplier** μ is given by

$$\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)).$$

Case I : Regular Hamiltonian, u is continuous

CASE I : Consider optimal control problems which satisfy the

Assumption: The Hamiltonian $H(x, \lambda, u)$ is **regular**, i.e., it admits a **unique minimum** u . The **strict Legendre condition** holds:

$$H_{uu}(t) > 0 \quad \forall t \in [0, t_f].$$

(a) Then there exists a "**free control**" $u = u_{free}(x, \lambda)$ satisfying

$$H_u(x, \lambda, u_{free}(x, \lambda)) \equiv 0.$$

(b) The **optimal control** $u(t)$ is **continuous** in $[0, t_f]$.

Claim (b) follows from the continuity and regularity of H .

The continuity of the control implies **junction conditions at junction points** t_k ($k = 1, 2$) **with the boundary**:

$$u_{free}(x(t_k), \lambda(t_k)) = u_b(x(t_k)), \quad \mu(t_k) = 0 \quad (k = 1, 2).$$

Rayleigh Problem with Quadratic Control

The Rayleigh problem is a variant of the van der Pol Oscillator, where x_1 denotes the electric current.

Control problem for the Rayleigh Equation

$$\text{Minimize } J(x, u) = \int_0^{t_f} (u^2 + x_1^2) dt \quad (t_f = 4.5)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5. \end{aligned}$$

Three types of constraints:

Case (a) : no control constraints.

Case (b) : control constraint $-1 \leq u(t) \leq 1$.

Case (c) : **mixed control-state constraint**

$$\alpha \leq u(t) + x_1(t)/6 \leq 0, \quad \alpha = -1, -2$$

Case I (a) : Rayleigh problem, no constraint

Normal Hamiltonian:

$$H(x, \lambda, u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2)) + 4u$$

Adjoint Equations:

$$\dot{\lambda}_1 = -H_{x_1} = -2x_1 + \lambda_2 \qquad \lambda_1(t_f) = 0,$$

$$\dot{\lambda}_2 = -H_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) \qquad \lambda_2(t_f) = 0,$$

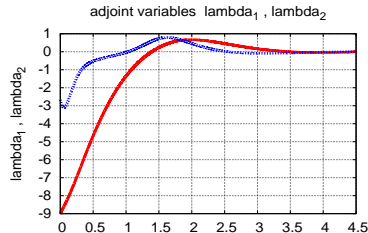
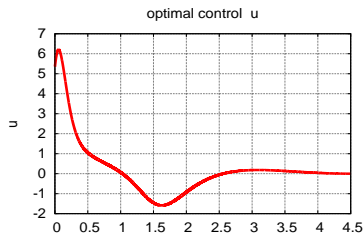
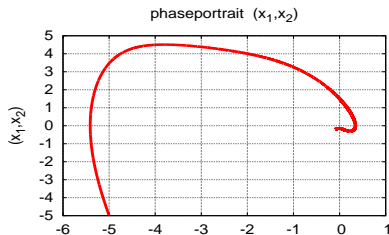
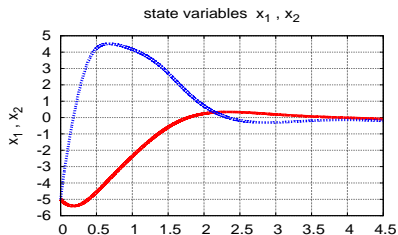
Minimum condition:

$$0 = H_u = 2u + 4\lambda_2 \quad \Rightarrow \quad u = u_{free}(x, \lambda) = -2\lambda_2.$$

Shooting method for solving the boundary value problem for (x, λ) :

Determine unknown shooting vector $s = \lambda(0) \in \mathbb{R}^2$ that satisfies the terminal condition $\lambda(t_f) = 0$: use Newton's method

Case I : Rayleigh problem without constraints



Note: Hamiltonian is regular, control $u(t)$ is continuous (analytic).

Case I (b) : Rayleigh problem, constraint $-1 \leq u(t) \leq 1$

Hamiltonian H and adjoint equations are as in Case (a).

The free control is given by $u_{free}(x, \lambda) = -2\lambda_2$.

Structure of optimal control:

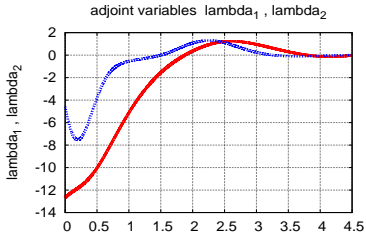
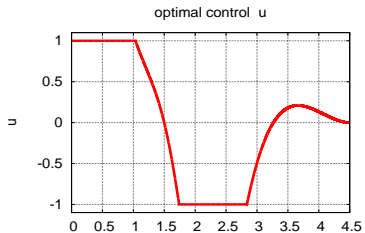
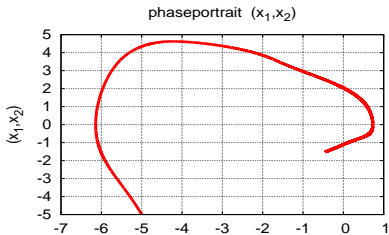
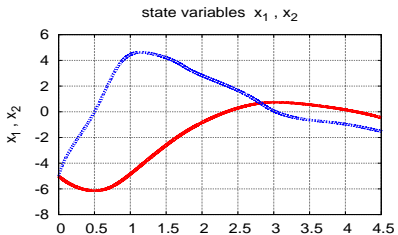
$$u(t) = \left\{ \begin{array}{ll} 1 & \text{for } 0 \leq t \leq t_1 \\ -2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\ -1 & \text{for } t_2 \leq t \leq t_3 \\ -2\lambda_2(t) & \text{for } t_3 \leq t \leq t_f \end{array} \right\}$$

Junction conditions: Continuity of the control implies

$$u(t_k) = -2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3.$$

Shooting method for solving the boundary value problem for (x, λ) :

Determine shooting vector $s = (\lambda(0), t_1, t_2, t_3) \in \mathbb{R}^{2+3}$ that satisfies 2 terminal conditions $\lambda(t_f) = 0$ and 3 junction conditions.

Rayleigh problem with control constraint $|u(t)| \leq 1$ 

Note: Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3$

Case I (c) : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

Augmented (normal) Hamiltonian:

$$\begin{aligned} \mathcal{H}(x, \lambda, \mu, u) = & u^2 + x_1^2 + \lambda_1 x_2 \\ & + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2)) + 4u + \mu(u + x_1/6) \end{aligned}$$

Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 = -\mathcal{H}_{x_1} = & -2x_1 + \lambda_2 - \mu/6, & \lambda_1(t_f) = 0, \\ \dot{\lambda}_2 = -\mathcal{H}_{x_2} = & -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) & \lambda_2(t_f) = 0, \end{aligned}$$

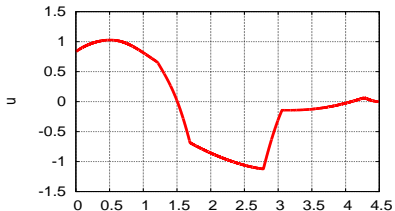
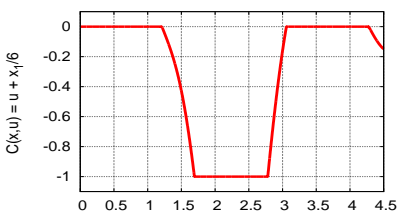
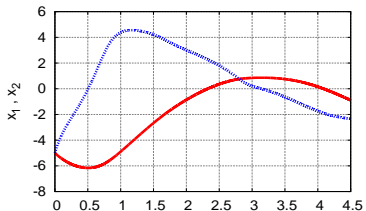
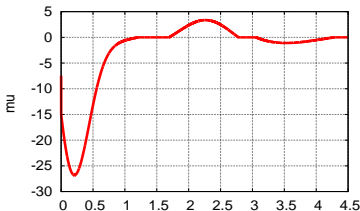
Free control : $u_{free}(x, \lambda) = -2\lambda_2.$

Boundary control : $u_b(x) = \alpha - x_1/6$ for $\alpha \in \{-1, 0\}.$

Multiplier :

$$\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)) = 2u_b(x) + 4\lambda_2.$$

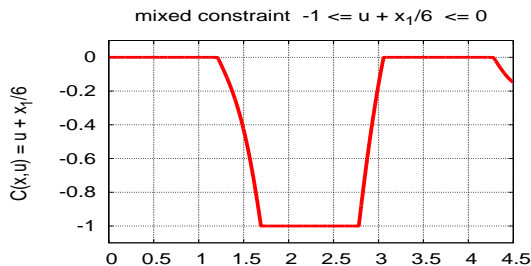
Case I (c) : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

optimal control u mixed constraint $-1 \leq u + x_1/6 \leq 0$ state variables x_1, x_2 multiplier for constraint $-1 \leq u + x_1/6 \leq 0$ 

Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda(t_k) = \alpha - x_1(t_k)/6, \alpha \in \{0, -1\}$.

Case I (c) : Rayleigh problem, structure of optimal control for mixed constraint $-1 \leq u + x_1/6 \leq 0$



$$u(t) = \left\{ \begin{array}{ll} -x_1/6 & \text{for } 0 \leq t \leq t_1 \\ -2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\ -1 - x_1/6 & \text{for } t_2 \leq t \leq t_3 \\ -2\lambda_2(t) & \text{for } t_3 \leq t \leq t_4 \\ -x_1/6 & \text{for } t_4 \leq t \leq t_5 \\ -2\lambda_2(t) & \text{for } t_5 \leq t \leq t_f \end{array} \right\}$$

Case II : control u appears linearly

CASE II : Control **appears linearly** in the cost functional, dynamics and mixed control-state constraint. Let u be scalar.

Dynamics and Boundary Conditions

$$\begin{aligned}\dot{x}(t) &= f_1(x(t)) + f_2(x(t)) \cdot u(t), \quad \text{a.e. } t \in [0, t_f], \\ x(0) &= x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,\end{aligned}$$

Mixed Control-State Constraints

$$\alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u(t) \leq \beta \quad \forall t \in [0, t_f]. \quad c_1, c_2 : \mathbb{R}^n \rightarrow \mathbb{R}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} (f_{01}(x(t)) + f_{02}(x(t)) \cdot u(t)) dt$$

Case II : Hamiltonian and switching function

Normal Hamiltonian

$$H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + [f_{02}(x) + \lambda f_2(x)] \cdot u.$$

Augmented Hamiltonian

$$\mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu (c_1(x) + c_2(x) \cdot u)$$

The optimal control $u(t)$ solves the **minimization problem**

$$\min \{ H(x(t), \lambda(t), u) \mid \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \}$$

Define the **switching function**

$$\sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x), \quad \sigma(t) = \sigma(x(t), \lambda(t)).$$

Case II : Hamiltonian

The **minimum condition** is equivalent to the **minimization problem**

$$\min \{ \sigma(t) \cdot u \mid \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \}$$

We deduce the **control law**

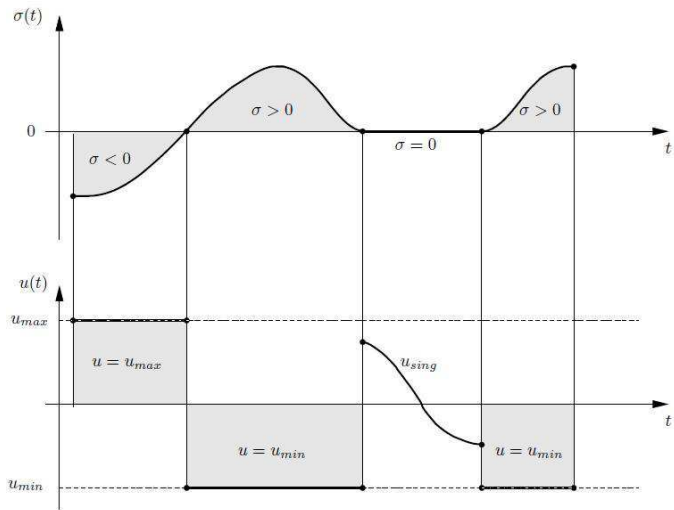
$$c_1(x(t)) + c_2(x(t)) \cdot u(t) = \begin{cases} \alpha, & \text{if } \sigma(t) \cdot c_2(x(t)) > 0 \\ \beta, & \text{if } \sigma(t) \cdot c_2(x(t)) < 0 \\ \text{undetermined} & \text{if } \sigma(t) \equiv 0 \end{cases}$$

The control u is called **bang-bang** in an interval $I \subset [0, t_f]$, if $\sigma(t) \cdot c_2(x(t)) \neq 0$ for all $t \in I$. The control u is called **singular** in an interval $I_{\text{sing}} \subset [0, t_f]$, if $\sigma(t) \cdot c_2(x(t)) \equiv 0$ for all $t \in I_{\text{sing}}$.

For the control constraint $\alpha \leq u(t) \leq \beta$ with $c_1(x) = 0$, $c_2(x) = 1$ we get the **classical control law**

$$u(t) = \begin{cases} \alpha, & \text{if } \sigma(t) > 0 \\ \beta, & \text{if } \sigma(t) < 0 \\ \text{undetermined}, & \text{if } \sigma(t) \equiv 0 \end{cases}$$

Bang-Bang and Singular Controls



Case II : Rayleigh problem with $-1 \leq u(t) \leq 1$

Rayleigh problem with control appearing linearly

Minimize $J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) dt \quad (t_f = 4.5)$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5, \\ -1 &\leq u(t) \leq 1. \end{aligned}$$

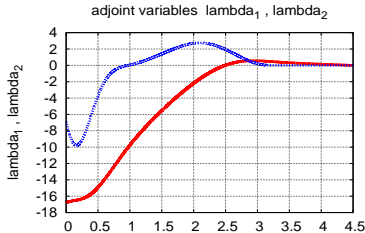
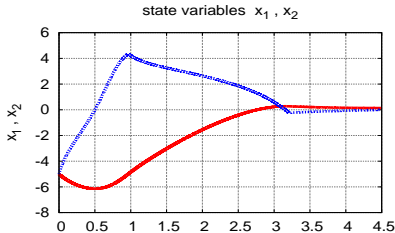
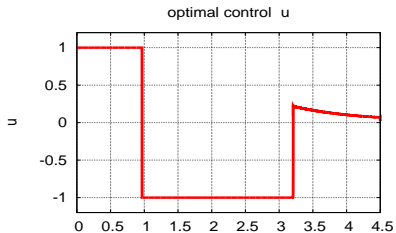
Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2), & \lambda_2(t_f) &= 0, \end{aligned}$$

The switching function $\sigma(t) = H_u(t) = 4\lambda_2(t)$ gives the control law

$$u(t) = -\text{sign}(\lambda_2(t))$$

Case II : Rayleigh problem, $-1 \leq u(t) \leq 1$



Control $u(t)$ is bang-bang-singular.

Switching conditions: $\lambda(t_1) = 0$, $\lambda_2(t) \equiv 0 \forall t \in [t_2, t_f]$.

Case II : Rayleigh problem, $\alpha \leq u + x_1/6 \leq 0$

$$\text{Minimize } J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) dt \quad (t_f = 4.5)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5, \end{aligned}$$

and the **mixed control-state constraint**

$$\alpha \leq u(t) + x_1(t)/6 \leq 0 \quad \forall 0 \leq t \leq t_f.$$

Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2 - \mu/6, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2), & \lambda_2(t_f) &= 0, \end{aligned}$$

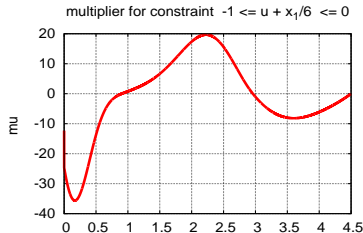
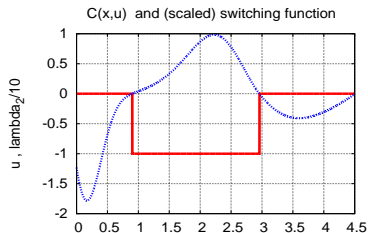
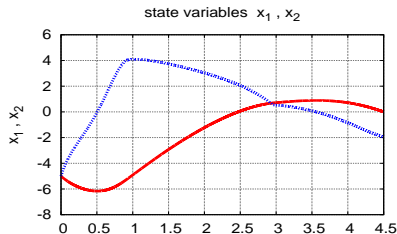
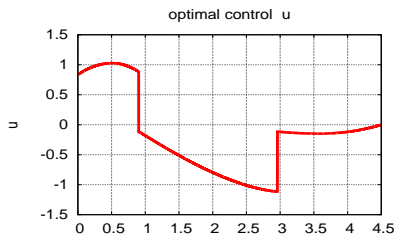
Control law for $\alpha \leq u + x_1/6 \leq 0$

The **switching function** is $\sigma(t) = H_u(t) = 4\lambda_2(t)$.

In view of $c_2(x) \equiv 1$ we have the **control law**

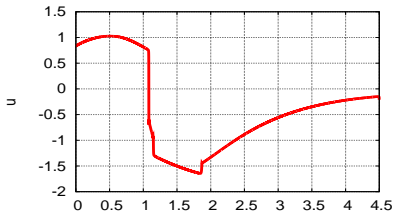
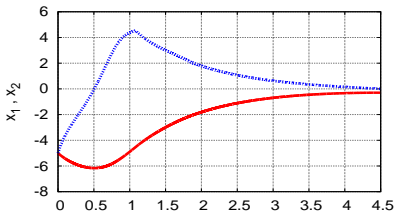
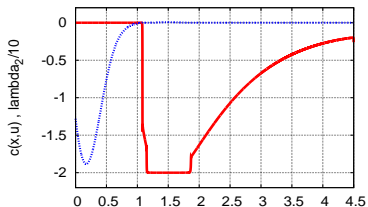
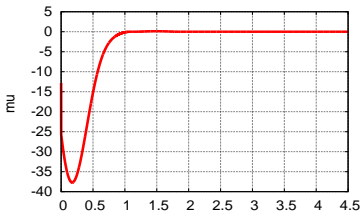
$$u + x_1/6 = \left\{ \begin{array}{ll} \alpha < 0 & , \text{ if } \lambda_2(t) > 0 \\ 0 & , \text{ if } \lambda_2(t) < 0 \\ \text{undetermined} & , \text{ if } \lambda_2(t) \equiv 0 \end{array} \right\}$$

Case II : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$



Note: constraint $u(t) + x_1(t)/6$ is "bang-bang".

Case II : Rayleigh problem, $-2 \leq u + x_1/6 \leq 0$

optimal control u state variables x_1, x_2  $c(x,u)$ and (scaled) switching functionmultiplier for constraint $-1 \leq u + x_1/6 \leq 0$ 

Note: constraint $u(t) + x_1(t)/6$ is "bang-singular-bang-singular".

Optimal Fishing, Clark, Clarke, Munro

COLIN W. CLARK, FRANK H. CLARKE, GORDON R. MUNRO:
The optimal exploitation of renewable resource stock: problem of irreversible investment, *Econometrica* **47**, pp. 25–47 (1979).

State variables and control variables:

$x(t)$: population biomass at time $t \in [0, t_f]$,
 renewable resource, e.g., fish,

$K(t)$: amount of capital invested in the fishery,
 e.g., number of "standardized" fishing vessels available,

$E(t)$: fishing effort (control), $h(t) = E(t)x(t)$ is harvest rate ,

$I(t)$: investment rate (control),

Optimal Fishing: optimal control model

Dynamics in $[0, t_f]$ (here: $a = 1, b = 5, \gamma = 0$)

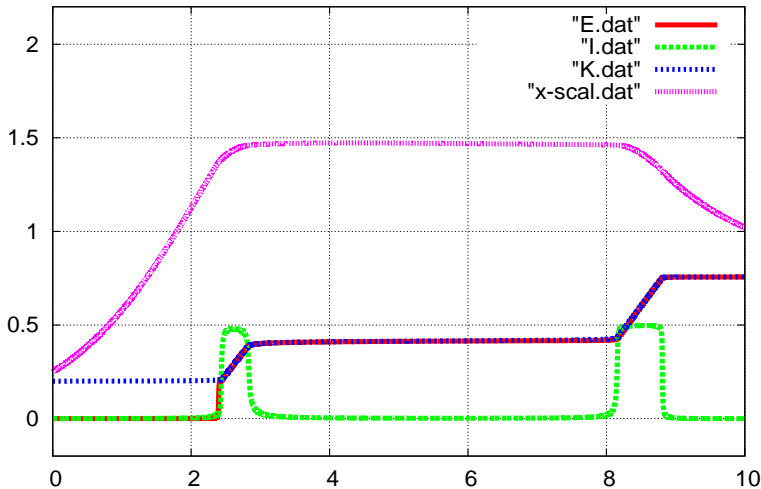
$$\begin{aligned} \dot{x}(t) &= a \cdot x(t) \cdot (1 - x(t)/b) - E(t) \cdot x(t), & x(0) &= x_0, \\ \dot{K}(t) &= I(t) - \gamma \cdot K(t), & K(0) &= K_0. \end{aligned}$$

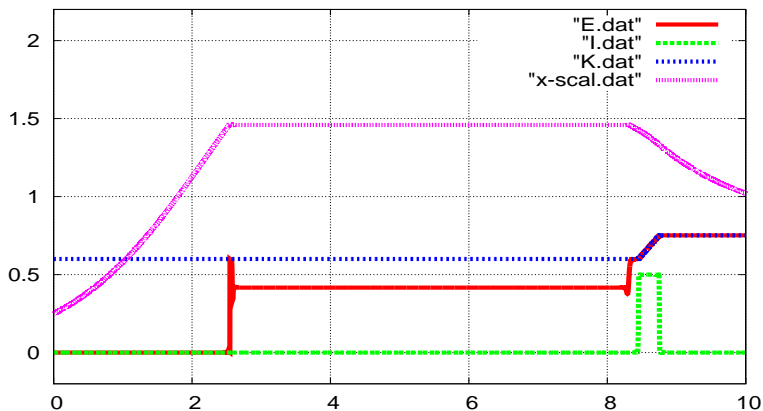
Mixed Control-State Constraint and Control Constraint

$$0 \leq E(t) \leq K(t), \quad 0 \leq I(t) \leq I_{max}, \quad t \in [0, t_f],$$

Maximize benefit (parameters: $r = 0.05, c_E = 2, c_I = 1.1$)

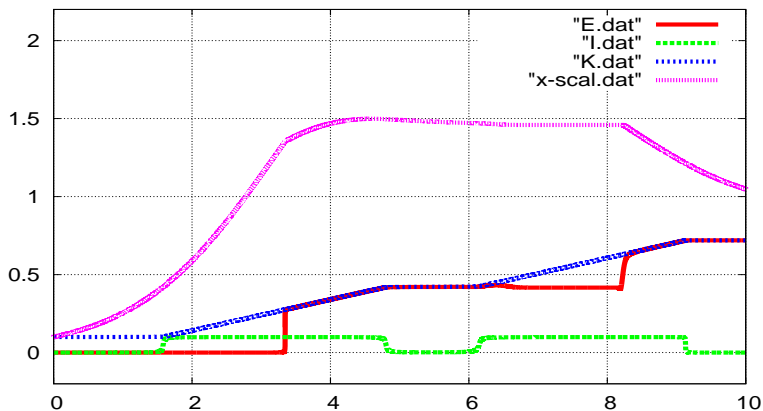
$$J(u, x) = \int_0^{t_f} \exp(-r \cdot t) (p \cdot E(t) \cdot x(t) - c_E \cdot E(t) - c_I \cdot I(t)) dt$$

Optimal Fishing: $x_0 = 0.5$, $K_0 = 0.2$, $I_{max} = 0.5$ controls E, I and state variables $0.5 \cdot x$, K

Optimal Fishing: $x_0 = 0.5, K_0 = 0.6, I_{max} = 0.5$ controls E, I and state variables $0.5*x, K$ 

Fishing rate : $E(t) = 0, E(t) = \text{singular}, E(t) = K(t).$

Investment rate : $I(t) = 0, I(t) = I_{max}, I(t) = 0.$

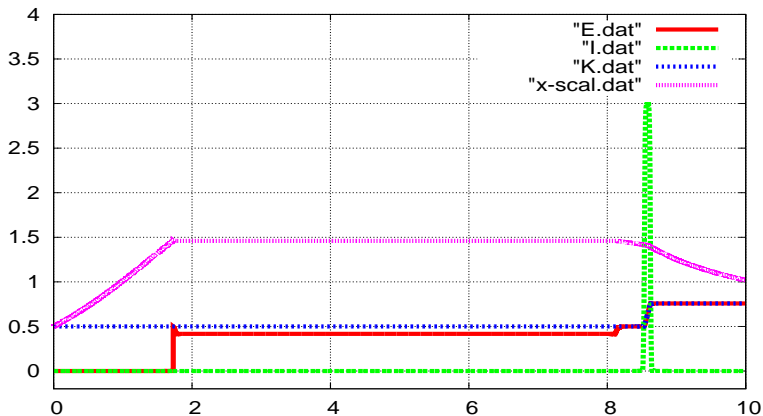
Optimal Fishing: $x_0 = 0.2, K_0 = 0.1, I_{max} = 0.1$ controls E, I and state variables $0.5 \cdot x, K$ 

Fishing rate : $E(t) = 0, E(t) = K(t), E(t)$ singular, $E(t) = K(t)$.

Investment rate : 2 arcs with $I(t) = I_{max}$.

Optimal Fishing: $x_0 = 1.0$, $K_0 = 0.5$, $I_{max} = 3$

controls E, I and state variables $0.5 \cdot x$, K



Fishing rate : $E(t) = 0$, $E(t)$ singular $E(t) = K(t)$,

Investment rate : 1 "impulse" with $I(t) = I_{max}$.