



HAL
open science

Tutorial on Control and State Constrained Optimal Control Problems

Helmut Maurer

► **To cite this version:**

Helmut Maurer. Tutorial on Control and State Constrained Optimal Control Problems. SADCO Summer School 2011 - Optimal Control, Sep 2011, London, United Kingdom. inria-00629518

HAL Id: inria-00629518

<https://inria.hal.science/inria-00629518>

Submitted on 6 Oct 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Tutorial on Control and State Constrained Optimal Control Problems – Part 2 : Mixed Control-State Constraints

Helmut Maurer

University of Münster, Germany
Institute of Computational and Applied Mathematics

SADCO Summer School
Imperial College London, September 5, 2011

Outline

- 1 Optimal Control Problems with Control and State Constraints
- 2 Numrical Method: Discretize and Optimize
- 3 Theory of Optimal Control Problems with Mixed Control-State Constraints
- 4 Example: Rayleigh Problem with Different Constraints and Objectives
- 5 Example: Optimal Exploitation of Renewable Resources

Optimal Control Problem

state $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$.

Dynamics and Boundary Conditions

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, t_f], \\ x(t) &= \bar{x}_0, \quad \psi(x(t_f)) = 0 \quad (\psi : \mathbb{R}^n \rightarrow \mathbb{R}^r).\end{aligned}$$

Control and State Constraints

$$\begin{aligned}c(x(t), u(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k) \\ s(x(t)) &\leq 0, \quad 0 \leq t \leq t_f, \quad (s : \mathbb{R}^n \rightarrow \mathbb{R}^l)\end{aligned}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t)) dt$$

Discretization

For simplicity consider a **MAYER-type problem** with cost functional

$$J(u, x) = g(x(t_f)).$$

This can be achieved by considering the **additional state variable** x_0 with

$$\dot{x}_0 = f_0(x, u), \quad x_0(0) = 0.$$

Then we have

$$x_0(t_f) = \int_0^{t_f} f_0(t, x(t), u(t)) dt.$$

Choose an integer $N \in \mathbb{N}$, a **stepsize** h and **grid points** t_i :

$$h = t_f/N, \quad t_i := ih, \quad (i = 0, 1, \dots, N).$$

Approximation of control and state at grid points:

$$u(t_i) \approx u_i \in \mathbb{R}^m, \quad x(t_i) \approx x_i \in \mathbb{R}^n \quad (i = 0, \dots, N)$$

Large-scale NLP using EULER's method

Minimize

$$J(u, x) = g(x_N)$$

subject to

$$\begin{aligned} x_{i+1} &= x_i + h \cdot f(t_i, x_i, u_i), & i = 0, \dots, N-1, \\ x_0 &= \bar{x}_0, \quad \psi(x_N) = 0, \\ c(x_i, u_i) &\leq 0, & i = 0, \dots, N, \\ s(x_i) &\leq 0, & i = 0, \dots, N, \end{aligned}$$

Optimization variable for full discretization:

$$z := (u_0, x_1, u_1, x_2, \dots, u_{N-1}, x_N, u_N) \in \mathbb{R}^{N(m+n)+m}$$

NLP Solvers

- AMPL : [Programming language](#) (Fourer, Gay, Kernighan)
- IPOPT : [Interior point method](#) (Andreas Wächter)
- LOQO : [Interior point method](#) (Vanderbei et al.)
- Other NLP solvers embedded in AMPL : cf. NEOS server
- NUDOCSS : optimal control package (Christof Büskens)
- WORHP : SQP solver (Christof Büskens, Matthias Gerdts)
- Special feature: solvers provide [LAGRANGE-multipliers as approximations of the adjoint variables](#).

Optimal Control Problem with Control-State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$.

All functions are assumed to be sufficiently smooth

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in [0, t_f],$$

$$x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,$$

$$(0 = \varphi(x(0), x(t_f)) \quad \text{mixed boundary conditions})$$

Mixed Control-State Constraints

$$\alpha \leq c(x(t), u(t)) \leq \beta, \quad t \in [0, t_f], \quad c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$

Control bounds $\alpha \leq u(t) \leq \beta$ are included by $c(x, u) = u$.

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u) dt$$

Hamiltonian

Hamiltonian

$$H(x, \lambda, u) = \lambda_0 f(x, u) + \lambda f(x, u) \quad \lambda \in \mathbb{R}^n \text{ (row vector)}$$

Augmented Hamiltonian

$$\begin{aligned} \mathcal{H}(x, \lambda, \mu, u) &= H(x, \lambda, u) + \mu c(x, u) \\ &= \lambda_0 f(x, u) + \lambda f(x, u) + \mu c(x, u), \quad \mu \in \mathbb{R}. \end{aligned}$$

Let $(u, x) \in \mathcal{L}^\infty([0, T], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, T], \mathbb{R}^n)$ be a locally optimal pair of functions.

Regularity assumption

$$c_u(x(t), u(t)) \neq 0 \quad \forall t \in J_a$$

$$J_a := \{ t \in [0, t_f] \mid c(x(t), u(t)) = \alpha \text{ or } = \beta \}$$

Minimum Principle of Pontryagin et al. and Hestenes

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a **locally optimal pair of functions** that satisfies the regularity assumption. Then there exist

- an adjoint (costate) function $\lambda \in \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$,
- a multiplier function $\mu \in \mathcal{L}^\infty([0, t_f], \mathbb{R})$,
- and a multiplier $\rho \in \mathbb{R}^r$ associated to the boundary condition $\psi(x(t_f)) = 0$

that satisfy the following conditions for a.a. $t \in [0, t_f]$, where the argument (t) denotes evaluation along the trajectory $(x(t), u(t), \lambda(t))$:

Minimum Principle of Pontryagin et al. and Hestenes

(i) **Adjoint ODE and transversality condition:**

$$\begin{aligned}\dot{\lambda}(t) &= -\mathcal{H}_x(t) = -(\lambda_0 f_0 + \lambda f)_x(t) - \mu(t) c_x(t), \\ \lambda(t_f) &= (\lambda_0 g + \rho\psi)_x(x(t_f)),\end{aligned}$$

(iia) **Minimum Condition for Hamiltonian:**

$$H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq c(x(t), u) \leq \beta \}$$

(iib) **Local Minimum Condition for Augmented Hamiltonian:**

$$0 = \mathcal{H}_u(t) = (\lambda_0 f_0 + \lambda f)_u(t) + \mu(t) c_u(t)$$

(iii) **Sign of multiplier μ and complementarity condition:**

$$\begin{aligned}\mu(t) &\leq 0, \text{ if } c(x(t), u(t)) = \alpha; \quad \mu(t) \geq 0, \text{ if } c(x(t), u(t)) = \beta, \\ \mu(t) &= 0, \text{ if } \alpha < c(x(t), u(t)) < \beta.\end{aligned}$$

Evaluation of the Minimum Principle: boundary arc

Boundary arc: Let $[t_1, t_2]$, $0 \leq t_1 < t_2 < t_f$, be an interval with

$$c(x(t), u(t)) = \alpha \quad \text{or} \quad c(x(t), u(t)) = \beta \quad \forall t_1 \leq t \leq t_2.$$

For simplicity assume a **scalar control**, i.e., $m = 1$.

Due to the regularity condition $c_u(x(t), u(t)) \neq 0$ there exists a smooth function $u_b(x)$ satisfying

$$c(x, u_b(x)) \equiv \alpha \quad (\equiv \beta) \quad \forall x \text{ in a neighborhood of the trajectory.}$$

The control $u_b(x)$ is called the **boundary control** and yields the **optimal control** by the relation $u(t) = u_b(x(t))$.

It follows from the local minimum condition $0 = \mathcal{H}_u = H_u + \mu c_u$ that the **multiplier** μ is given by

$$\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)).$$

Case I : Regular Hamiltonian, u is continuous

CASE I : Consider optimal control problems which satisfy the

Assumption: The Hamiltonian $H(x, \lambda, u)$ is **regular**, i.e., it admits a **unique minimum** u . The **strict Legendre condition** holds:

$$H_{uu}(t) > 0 \quad \forall t \in [0, t_f].$$

(a) Then there exists a "**free control**" $u = u_{free}(x, \lambda)$ satisfying

$$H_u(x, \lambda, u_{free}(x, \lambda)) \equiv 0.$$

(b) The **optimal control** $u(t)$ is **continuous** in $[0, t_f]$.

Claim (b) follows from the continuity and regularity of H .

The continuity of the control implies **junction conditions at junction points** t_k ($k = 1, 2$) **with the boundary**:

$$u_{free}(x(t_k), \lambda(t_k)) = u_b(x(t_k)), \quad \mu(t_k) = 0 \quad (k = 1, 2).$$

Rayleigh Problem with Quadratic Control

The Rayleigh problem is a variant of the van der Pol Oscillator, where x_1 denotes the electric current.

Control problem for the Rayleigh Equation

$$\text{Minimize } J(x, u) = \int_0^{t_f} (u^2 + x_1^2) dt \quad (t_f = 4.5)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5. \end{aligned}$$

Three types of constraints:

Case (a) : no control constraints.

Case (b) : control constraint $-1 \leq u(t) \leq 1$.

Case (c) : **mixed control-state constraint**

$$\alpha \leq u(t) + x_1(t)/6 \leq 0, \quad \alpha = -1, -2$$

Case I (a) : Rayleigh problem, no constraint

Normal Hamiltonian:

$$H(x, \lambda, u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2)) + 4u$$

Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2 & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) & \lambda_2(t_f) &= 0, \end{aligned}$$

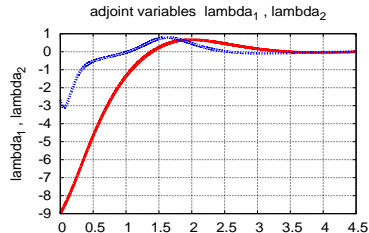
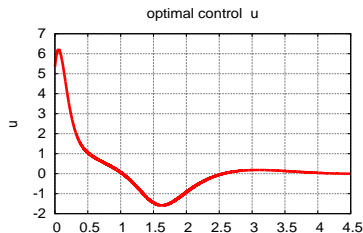
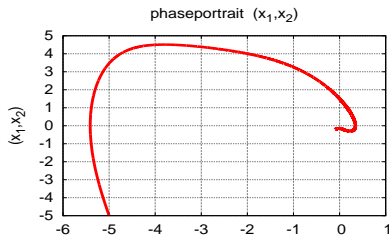
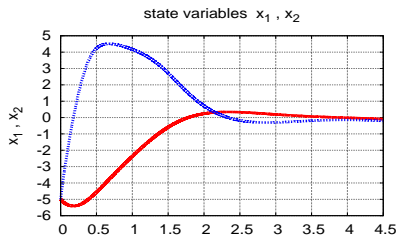
Minimum condition:

$$0 = H_u = 2u + 4\lambda_2 \quad \Rightarrow \quad u = u_{free}(x, \lambda) = -2\lambda_2.$$

Shooting method for solving the boundary value problem for (x, λ) :

Determine unknown shooting vector $s = \lambda(0) \in \mathbb{R}^2$ that satisfies the terminal condition $\lambda(t_f) = 0$: use Newton's method

Case I : Rayleigh problem without constraints



Note: Hamiltonian is regular, control $u(t)$ is continuous (analytic).

Case I (b) : Rayleigh problem, constraint $-1 \leq u(t) \leq 1$

Hamiltonian H and adjoint equations are as in Case (a).

The free control is given by $u_{free}(x, \lambda) = -2\lambda_2$.

Structure of optimal control:

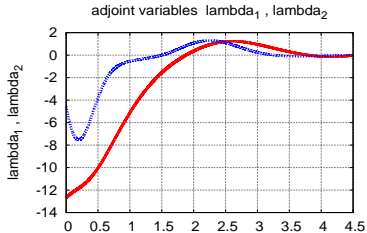
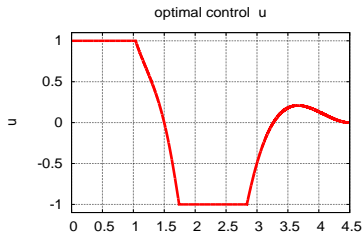
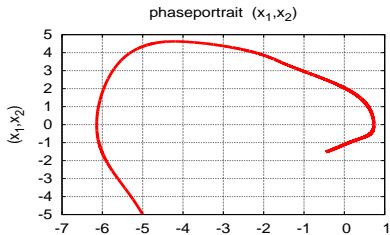
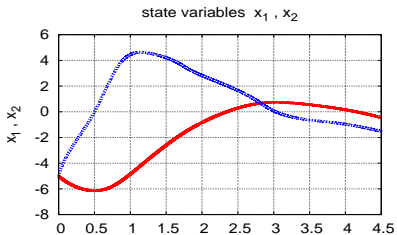
$$u(t) = \left\{ \begin{array}{ll} 1 & \text{for } 0 \leq t \leq t_1 \\ -2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\ -1 & \text{for } t_2 \leq t \leq t_3 \\ -2\lambda_2(t) & \text{for } t_3 \leq t \leq t_f \end{array} \right\}$$

Junction conditions: Continuity of the control implies

$$u(t_k) = -2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3.$$

Shooting method for solving the boundary value problem for (x, λ) :

Determine shooting vector $s = (\lambda(0), t_1, t_2, t_3) \in \mathbb{R}^{2+3}$ that satisfies 2 terminal conditions $\lambda(t_f) = 0$ and 3 junction conditions.

Rayleigh problem with control constraint $|u(t)| \leq 1$ 

Note: Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3$

Case I (c) : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

Augmented (normal) Hamiltonian:

$$\begin{aligned} \mathcal{H}(x, \lambda, \mu, u) = & u^2 + x_1^2 + \lambda_1 x_2 \\ & + \lambda_2 (-x_1 + x_2(1.4 - 0.14x_2^2)) + 4u + \mu(u + x_1/6) \end{aligned}$$

Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 = -\mathcal{H}_{x_1} = & -2x_1 + \lambda_2 - \mu/6, & \lambda_1(t_f) = 0, \\ \dot{\lambda}_2 = -\mathcal{H}_{x_2} = & -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) & \lambda_2(t_f) = 0, \end{aligned}$$

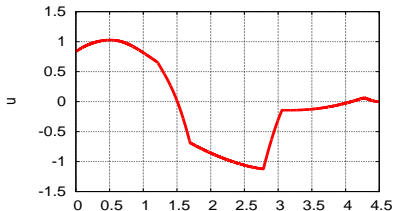
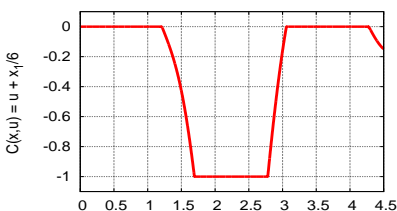
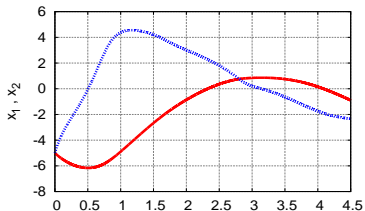
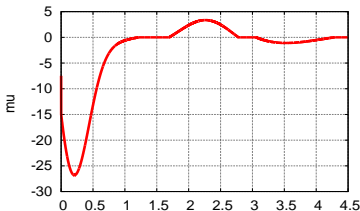
Free control : $u_{free}(x, \lambda) = -2\lambda_2.$

Boundary control : $u_b(x) = \alpha - x_1/6$ for $\alpha \in \{-1, 0\}.$

Multiplier :

$$\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) / c_u(x, u_b(x)) = 2u_b(x) + 4\lambda_2.$$

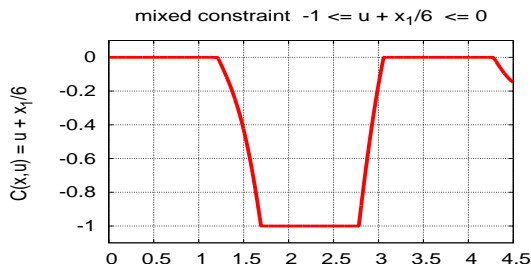
Case I (c) : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$

optimal control u mixed constraint $-1 \leq u + x_1/6 \leq 0$ state variables x_1, x_2 multiplier for constraint $-1 \leq u + x_1/6 \leq 0$ 

Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda(t_k) = \alpha - x_1(t_k)/6, \alpha \in \{0, -1\}$.

Case I (c) : Rayleigh problem, structure of optimal control for mixed constraint $-1 \leq u + x_1/6 \leq 0$



$$u(t) = \left\{ \begin{array}{ll} -x_1/6 & \text{for } 0 \leq t \leq t_1 \\ -2\lambda_2(t) & \text{for } t_1 \leq t \leq t_2 \\ -1 - x_1/6 & \text{for } t_2 \leq t \leq t_3 \\ -2\lambda_2(t) & \text{for } t_3 \leq t \leq t_4 \\ -x_1/6 & \text{for } t_4 \leq t \leq t_5 \\ -2\lambda_2(t) & \text{for } t_5 \leq t \leq t_f \end{array} \right\}$$

Case II : control u appears linearly

CASE II : Control **appears linearly** in the cost functional, dynamics and mixed control-state constraint. Let u be scalar.

Dynamics and Boundary Conditions

$$\begin{aligned}\dot{x}(t) &= f_1(x(t)) + f_2(x(t)) \cdot u(t), \quad \text{a.e. } t \in [0, t_f], \\ x(0) &= x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,\end{aligned}$$

Mixed Control-State Constraints

$$\alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u(t) \leq \beta \quad \forall t \in [0, t_f]. \quad c_1, c_2 : \mathbb{R}^n \rightarrow \mathbb{R}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} (f_{01}(x(t)) + f_{02}(x(t)) \cdot u(t)) dt$$

Case II : Hamiltonian and switching function

Normal Hamiltonian

$$H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + [f_{02}(x) + \lambda f_2(x)] \cdot u.$$

Augmented Hamiltonian

$$\mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu (c_1(x) + c_2(x) \cdot u)$$

The optimal control $u(t)$ solves the **minimization problem**

$$\min \{ H(x(t), \lambda(t), u) \mid \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \}$$

Define the **switching function**

$$\sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x), \quad \sigma(t) = \sigma(x(t), \lambda(t)).$$

Case II : Hamiltonian

The **minimum condition** is equivalent to the **minimization problem**

$$\min \{ \sigma(t) \cdot u \mid \alpha \leq c_1(x(t)) + c_2(x(t)) \cdot u \leq \beta \}$$

We deduce the **control law**

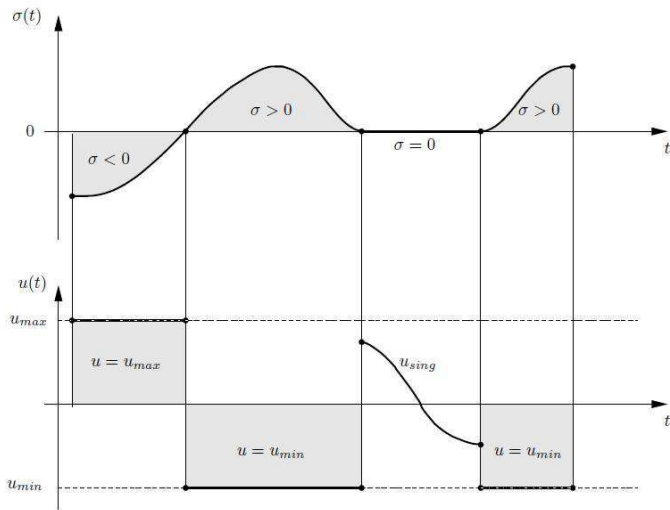
$$c_1(x(t)) + c_2(x(t)) \cdot u(t) = \begin{cases} \alpha, & \text{if } \sigma(t) \cdot c_2(x(t)) > 0 \\ \beta, & \text{if } \sigma(t) \cdot c_2(x(t)) < 0 \\ \text{undetermined} & \text{if } \sigma(t) \equiv 0 \end{cases}$$

The control u is called **bang-bang** in an interval $I \subset [0, t_f]$, if $\sigma(t) \cdot c_2(x(t)) \neq 0$ for all $t \in I$. The control u is called **singular** in an interval $I_{\text{sing}} \subset [0, t_f]$, if $\sigma(t) \cdot c_2(x(t)) \equiv 0$ for all $t \in I_{\text{sing}}$.

For the control constraint $\alpha \leq u(t) \leq \beta$ with $c_1(x) = 0$, $c_2(x) = 1$ we get the **classical control law**

$$u(t) = \begin{cases} \alpha, & \text{if } \sigma(t) > 0 \\ \beta, & \text{if } \sigma(t) < 0 \\ \text{undetermined}, & \text{if } \sigma(t) \equiv 0 \end{cases}$$

Bang-Bang and Singular Controls



Case II : Rayleigh problem with $-1 \leq u(t) \leq 1$

Rayleigh problem with control appearing linearly

$$\text{Minimize } J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) dt \quad (t_f = 4.5)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5, \\ -1 &\leq u(t) \leq 1. \end{aligned}$$

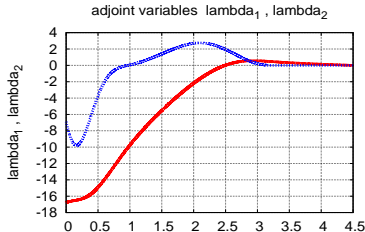
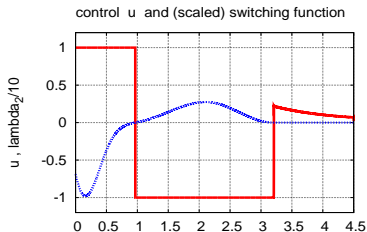
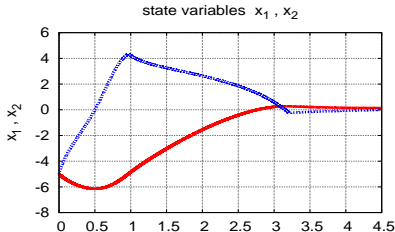
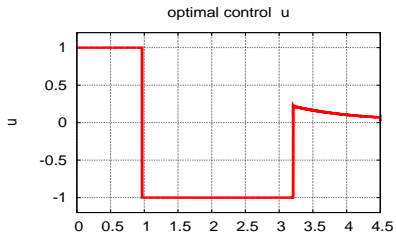
Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2), & \lambda_2(t_f) &= 0, \end{aligned}$$

The switching function $\sigma(t) = H_u(t) = 4\lambda_2(t)$ gives the control law

$$u(t) = -\text{sign}(\lambda_2(t))$$

Case II : Rayleigh problem, $-1 \leq u(t) \leq 1$



Control $u(t)$ is bang-bang-singular.

Switching conditions: $\lambda(t_1) = 0$, $\lambda_2(t) \equiv 0 \forall t \in [t_2, t_f]$.

Case II : Rayleigh problem, $\alpha \leq u + x_1/6 \leq 0$

$$\text{Minimize } J(x, u) = \int_0^{t_f} (x_1^2 + x_2^2) dt \quad (t_f = 4.5)$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, & x_2(0) &= -5, \end{aligned}$$

and the **mixed control-state constraint**

$$\alpha \leq u(t) + x_1(t)/6 \leq 0 \quad \forall 0 \leq t \leq t_f.$$

Adjoint Equations:

$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2 - \mu/6, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2), & \lambda_2(t_f) &= 0, \end{aligned}$$

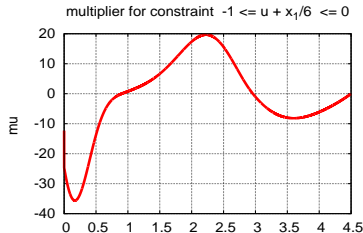
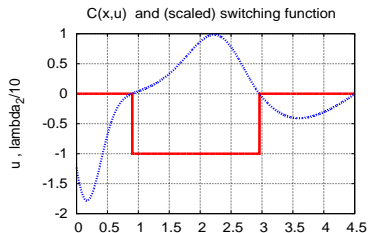
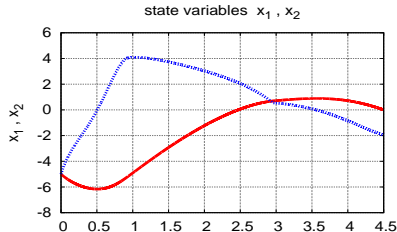
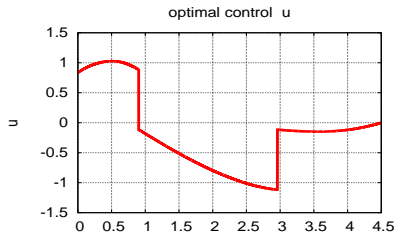
Control law for $\alpha \leq u + x_1/6 \leq 0$

The **switching function** is $\sigma(t) = H_u(t) = 4\lambda_2(t)$.

In view of $c_2(x) \equiv 1$ we have the **control law**

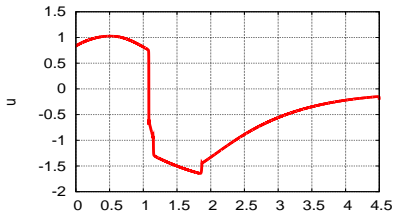
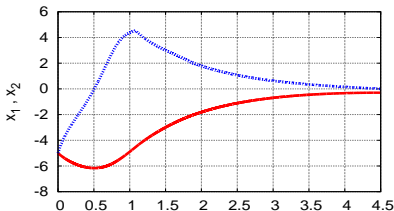
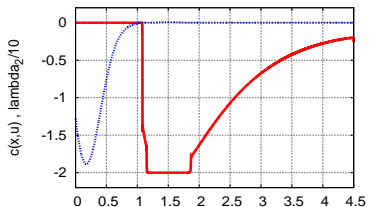
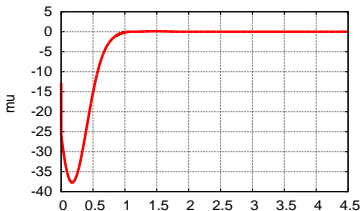
$$u + x_1/6 = \left\{ \begin{array}{ll} \alpha < 0 & , \text{ if } \lambda_2(t) > 0 \\ 0 & , \text{ if } \lambda_2(t) < 0 \\ \text{undetermined} & , \text{ if } \lambda_2(t) \equiv 0 \end{array} \right\}$$

Case II : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$



Note: constraint $u(t) + x_1(t)/6$ is "bang-bang".

Case II : Rayleigh problem, $-2 \leq u + x_1/6 \leq 0$

optimal control u state variables x_1, x_2  $c(x,u)$ and (scaled) switching functionmultiplier for constraint $-1 \leq u + x_1/6 \leq 0$ 

Note: constraint $u(t) + x_1(t)/6$ is "bang-singular-bang-singular".

Optimal Fishing, Clark, Clarke, Munro

COLIN W. CLARK, FRANK H. CLARKE, GORDON R. MUNRO:
The optimal exploitation of renewable resource stock: problem of irreversible investment, *Econometric* **47**, pp. 25–47 (1979).

State variables and control variables:

$x(t)$: population biomass at time $t \in [0, t_f]$,
 renewable resource, e.g., fish,

$K(t)$: amount of capital invested in the fishery,
 e.g., number of "standardized" fishing vessels available,

$E(t)$: fishing effort (control), $h(t) = E(t)x(t)$ is harvest rate ,

$I(t)$: investment rate (control),

Optimal Fishing: optimal control model

Dynamics in $[0, t_f]$ (here: $a = 1, b = 5, \gamma = 0$)

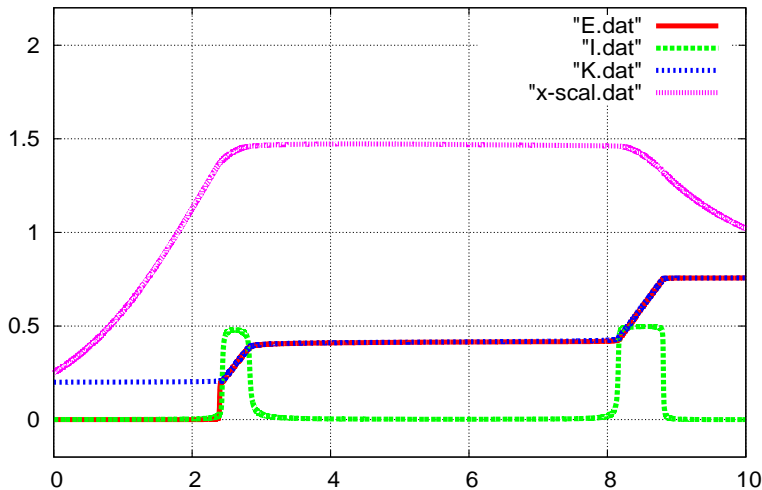
$$\begin{aligned}\dot{x}(t) &= a \cdot x(t) \cdot (1 - x(t)/b) - E(t) \cdot x(t), & x(0) &= x_0, \\ \dot{K}(t) &= I(t) - \gamma \cdot K(t), & K(0) &= K_0.\end{aligned}$$

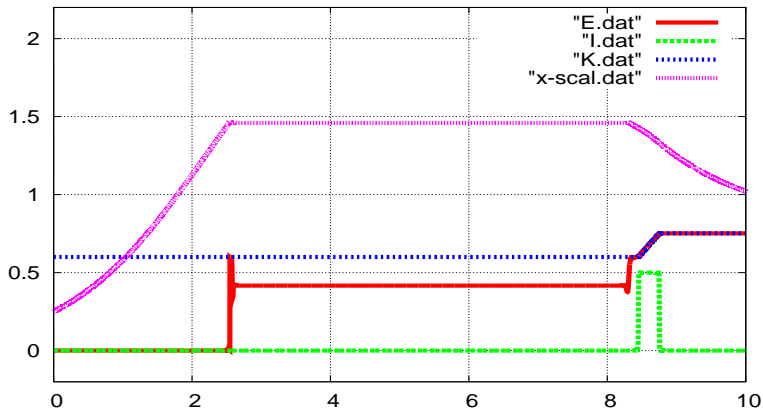
Mixed Control-State Constraint and Control Constraint

$$0 \leq E(t) \leq K(t), \quad 0 \leq I(t) \leq I_{max}, \quad t \in [0, t_f],$$

Maximize benefit (parameters: $r = 0.05, c_E = 2, c_I = 1.1$)

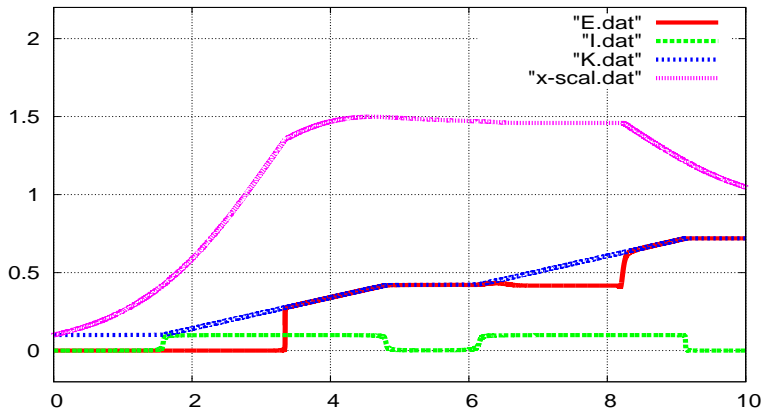
$$J(u, x) = \int_0^{t_f} \exp(-r \cdot t) (p \cdot E(t) \cdot x(t) - c_E \cdot E(t) - c_I \cdot I(t)) dt$$

Optimal Fishing: $x_0 = 0.5$, $K_0 = 0.2$, $I_{max} = 0.5$ controls E, I and state variables $0.5 \cdot x$, K

Optimal Fishing: $x_0 = 0.5, K_0 = 0.6, I_{max} = 0.5$ controls E, I and state variables $0.5*x, K$ 

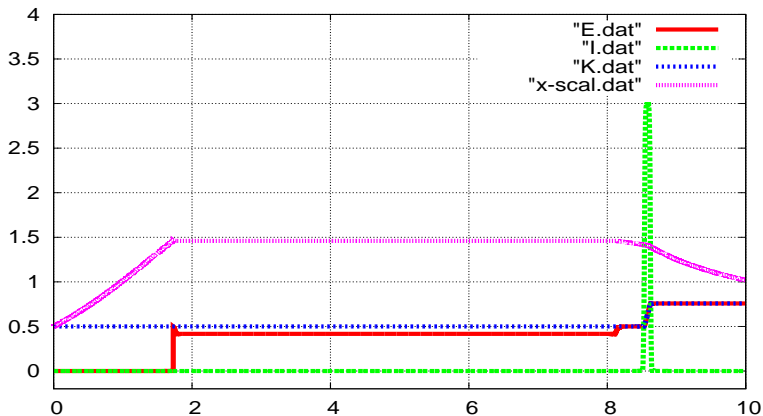
Fishing rate : $E(t) = 0, E(t) = \text{singular}, E(t) = K(t).$

Investment rate : $I(t) = 0, I(t) = I_{max}, I(t) = 0.$

Optimal Fishing: $x_0 = 0.2, K_0 = 0.1, I_{max} = 0.1$ controls E, I and state variables $0.5 \cdot x, K$ 

Fishing rate : $E(t) = 0, E(t) = K(t), E(t)$ singular, $E(t) = K(t)$.

Investment rate : 2 arcs with $I(t) = I_{max}$.

Optimal Fishing: $x_0 = 1.0$, $K_0 = 0.5$, $I_{max} = 3$ controls E, I and state variables $0.5 \cdot x$, K

Fishing rate : $E(t) = 0$, $E(t)$ singular $E(t) = K(t)$,

Investment rate : 1 "impulse" with $I(t) = I_{max}$.