

Tutorial on Control and State Constrained Optimal Control Problems and Applications – Part 3 : Pure State Constraints

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Outline

- 1 Theory of Optimal Control Problems with Pure State Constraints
- 2 Academic Example: order $q = 1$ of the state constraint
- 3 Van der Pol Oscillator : order $q = 1$ of the state constraint
- 4 Example: Immune Response
- 5 Example: Optimal Control of a Model of Climate Change

Optimal Control Problem with Pure State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}$ (for simplicity).

All functions are assumed to be sufficiently smooth.

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, t_f],$$

$$x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,$$

$$(0 = \varphi(x(0), x(t_f)), \quad \text{mixed boundary conditions})$$

Pure State Constraints and Control Bounds

$$s(x(t)) \leq 0, \quad t \in [0, t_f], \quad (s : \mathbb{R}^n \rightarrow \mathbb{R})$$

$$\alpha \leq u(t) \leq \beta, \quad t \in [0, t_f].$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u(t)) dt$$

Hamiltonian and Minimum Principle

Hamiltonian

$$H(x, \lambda, u) = \lambda_0 f(x, u) + \lambda f(x, u) \quad \lambda \in \mathbb{R}^n \text{ (row vector)}$$

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a **locally optimal pair of functions**. Then there exist

- an adjoint (costate) function $\lambda \in \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$,
- a multiplier function of bounded variation $\mu \in \mathcal{BV}^\infty([0, t_f], \mathbb{R})$,
- a multiplier $\rho \in \mathbb{R}$ associated to the boundary condition $\psi(x(t_f)) = 0$,

that satisfy the following conditions for a.a. $t \in [0, t_f]$, where the argument (t) denotes evaluations along the trajectory $(x(t), u(t), \lambda(t))$:

Minimum Principle of Pontryagin et al. (Hestenes)

(i) Adjoint integral equation and transversality condition:

$$\lambda(t) = \int_t^{t_f} H_x(s) ds + \int_t^{t_f} s_x(x(s)) d\mu(s) \\ + (\lambda_0 g + \rho\psi)_x(x(t_f)) \quad (\text{if } s(x(t_f)) < 0),$$

(ii) Minimum Condition for Hamiltonian:

$$H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq u \leq \beta \}$$

(iii) positive measure $d\mu$ and complementarity condition:

$$d\mu(t) \leq 0 \text{ and } \int_0^{t_f} s_x(t) d\mu(t) = 0$$

Order of a state constraint $s(x(t)) \leq 0$

Define recursively functions $s^{(k)}(x, u)$ by

$$\begin{aligned} s^{(0)}(x, u) &= s(x), \\ s^{(k+1)}(x, u) &= \frac{\partial s^{(k)}}{\partial x}(x, u) f(x, u), \quad (k = 0, 1, \dots) \end{aligned}$$

Suppose there exist $q \in \mathbb{N}$ with

$$\begin{aligned} \frac{\partial s^{(k)}}{\partial u}(x, u) &\equiv 0, \text{ i.e., } s^{(k)} = s^{(k)}(x), \quad (k = 0, 1, \dots, q-1), \\ \frac{\partial s^{(q)}}{\partial u}(x, u) &\neq 0. \end{aligned}$$

Then along a solution of $\dot{x}(t) = f(x(t), u(t))$ we have

$$\begin{aligned} s^{(k)}(x(t)) &= \frac{d^k}{dt^k} s(x(t)) \quad (k = 0, 1, \dots, q-1), \\ s^{(q)}(x(t), u(t)) &= \frac{d^q}{dt^q} s(x(t)) \end{aligned}$$

Regularity conditions to ensure $d\mu(t) = \eta(t)dt$

Regularity assumption on a boundary arc $s(x(t)) = 0$, $t_1 \leq t \leq t_2$

$$\frac{\partial s^{(q)}}{\partial u}(x(t), u(t)) \neq 0 \quad \forall t_1 \leq t \leq t_2.$$

Assumption on boundary control

There exists a sufficiently smooth **boundary control** $u = u_b(x)$ with $s(x, u_b(x)) \equiv 0$.

Assumption: $\alpha < u(t) = u_b(x(t)) < \beta \quad \forall t_1 < t < t_2$.

Regularity of multiplier (measure) μ

The regularity and assumption on boundary control imply that there exist a smooth multiplier $\eta(t)$ with

$$d\mu(t) = \eta(t) dt \quad t_1 < t < t_2$$

Minimum Principle under regularity

Augmented Hamiltonian

$$\mathcal{H}(x, \lambda, \eta, u) = H(x, \lambda, u) + \eta \cdot s(x)$$

Adjoint equation and jump conditions

$$\dot{\lambda}(t) = -\mathcal{H}_x(t) = -H_x(t) - \eta(t)s_x(x(t))$$

$$\lambda(t_k+) = \lambda(t_k-) - \nu_k s_x(x(t_k)), \quad \nu_k \geq 0$$

at each contact or junction time t_k , $\nu_k = \mu(t_k+) - \mu(t_k-)$

Minimum condition

$$H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq u \leq \beta \}$$

$$H_u(t) = 0 \quad \text{on boundary arcs} \quad t_1^+ < t < t_2^-$$

Academic Example

Minimize

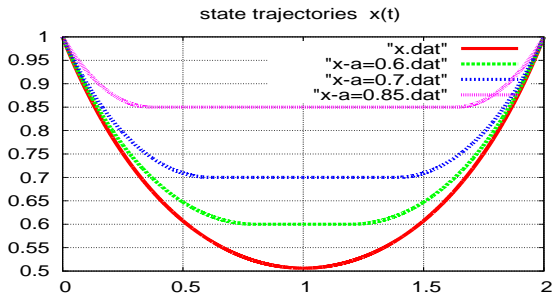
$$J(x, u) = \int_0^2 (u^2 + x^2) dt$$

subject to

$$\dot{x} = x^2 - u, \quad x(0) = 1, \quad x(2) = 1,$$

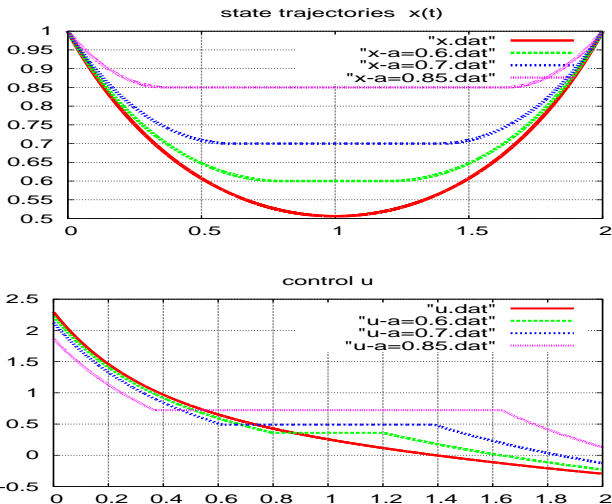
and the **state constraint**

$$x(t) \geq a, \quad \forall 0 \leq t \leq 2.$$



Boundary arc $x(t) \equiv a = 0.7$ for $t_1 = 0.614 \leq t \leq t_2 = 1.386$.

Academic Example: state and control trajectories



Academic Example: Computation of Multiplier η

State constraint: $s(x) = a - x \leq 0$.

Order of the state constraint is $q = 1$, since

$$s^{(1)}(x, u) = -\dot{x} = -x^2 + u, \quad (s^{(1)})_u = 1.$$

Boundary control $u = u_b(x)$ with $s^{(1)}(x, u_b(x)) \equiv 0$ is given by

$$u_b(x) = x^2 = a^2.$$

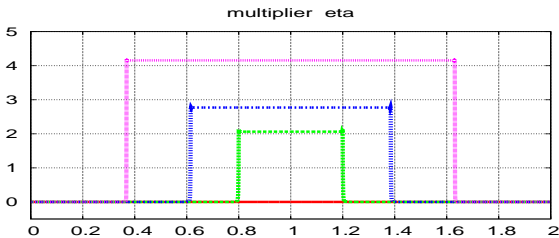
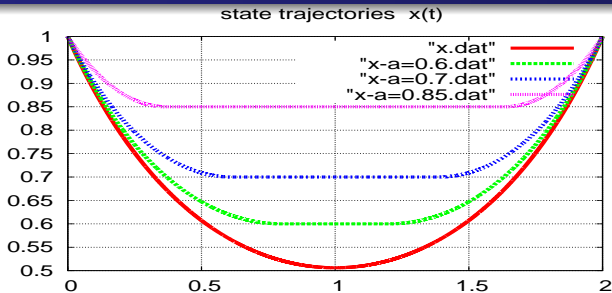
Augmented Hamiltonian and adjoint equation:

$$\begin{aligned} \mathcal{H}(x, \lambda, \eta, u) &= u^2 + x^2 + \lambda(x^2 - u) + \eta(a - x), \\ \dot{\lambda} &= -\mathcal{H}_x = -2x - 2\lambda x + \eta. \end{aligned}$$

The minimum condition $0 = H_u = 2u - \lambda$ gives $u = \lambda/2$. Since $u = a^2$ holds on a boundary arc, we get $\dot{\lambda} = 0$ and hence the multiplier η on the boundary

$$\eta(t) \equiv 2a(1 + 2a^2) > 0.$$

Academic Example: state and control trajectories

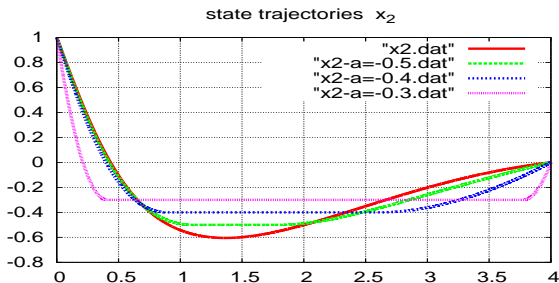


Van der Pol Oscillator

Minimize $J(x, u) = \int_0^{t_f} (u^2 + x_1^2 + x_2^2) dt \quad (t_f = 4),$

subject to $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_2(1 - x_1^2) + u,$
 $x_1(0) = x_2(0) = 1, \quad x_1(t_f) = x_2(t_f) = 0,$

state constraint $x_2(t) \geq a \quad \forall 0 \leq t \leq t_f.$



Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$.

Van der Pol : Computation of Multiplier η

Order of the state constraint is $q = 1$, since $s(x) = a - x$ and

$$s^{(1)}(x, u) = -\dot{x}_2 = x_1 + x_2(x_1^2 - 1) - u, \quad (s^{(1)})_u = -1 \neq 0.$$

Boundary control $u = u_b(x)$ with $s^{(1)}(x, u_b(x)) \equiv 0$ is given by

$$u_b(x) = x_1 + x_2(x_1^2 - 1) = x_1 + a(x_1^2 - 1).$$

Augmented Hamiltonian and adjoint equation:

$$\begin{aligned} \mathcal{H}(x, \lambda, \eta, u) &= u^2 + x_1^2 + x_2^2 + \lambda_1 x_2 + \lambda_2(-x_1 + x_2(1 - x_1^2)) + u \\ &\quad + \eta(a - x), \\ \dot{\lambda}_1 &= -\mathcal{H}_{x_1} = -2x_1 + \lambda_2(1 + 2x_1x_2), \\ \dot{\lambda}_2 &= -\mathcal{H}_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1 - x_1^2) + \eta. \end{aligned}$$

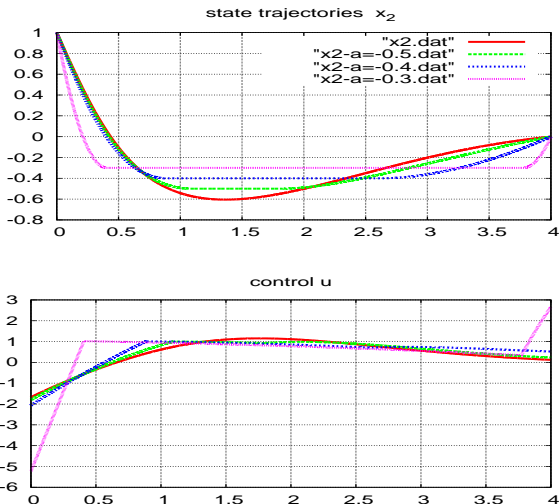
The minimum condition $0 = H_u = 2u + \lambda_2$ gives $u = -\lambda_2/2$.

Hence, $x_1 - a + ax_1^2 = -\lambda_2/2$ holds on the boundary.

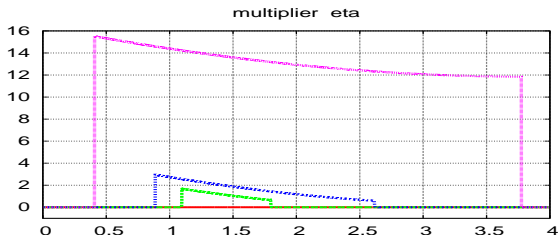
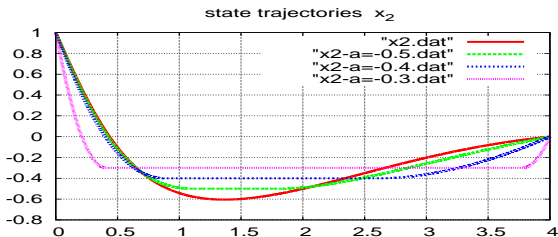
Differentiation yields

$$\eta = \eta(x, \lambda) = -4a^2x_1 + \lambda_1 + \lambda_2(1 - x_1^2).$$

Van der Pol Oscillator : optimal control



Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$.

Van der Pol Oscillator : multiplier η 

Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$.

CASE I : Hamiltonian is regular

CASE I : control u appears "nonlinearly" and $U = \mathbb{R}$

Assume that

- the Hamiltonian is regular, i.e., $H(x, \lambda, u)$ admits a unique minimum with respect to u ,
- the strict Legendre condition $H_{uu}(t) > 0$ holds.

Let $q \geq 1$ be the order of the state constraint $s(x) \leq 0$ and consider a **boundary arc** with $s(x(t)) = 0 \quad \forall t_1 \leq t \leq t_2$.

Junction conditions

- $q = 1$: The **control** $u(t)$ and the **adjoint variable** $\lambda(t)$ are **continuous** at t_k , $k = 1, 2$.
- $q = 2$: The **control** $u(t)$ is **continuous** but the **adjoint variable** may have jumps according to $\lambda(t_k+) = \lambda(t_k-) - \nu_k s_x(t_k)$.
- $q \geq 3$ and q odd: If the **control** $u(t)$ is **piecewise analytic** then there are no **boundary arcs** but only contact points.

Case II : control u appears linearly

Dynamics and Boundary Conditions

$$\begin{aligned}\dot{x}(t) &= f_1(x(t)) + f_2(x(t)) \cdot u(t), \text{ a.e. } t \in [0, t_f], \\ x(0) &= x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,\end{aligned}$$

Control and State Constraints

$$\alpha \leq u(t) \leq \beta \quad s(x(t)) \leq 0 \quad \forall t \in [0, t_f].$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} (f_{01}(x(t)) + f_{02}(x(t)) \cdot u(t)) dt$$

Case II : Hamiltonian and switching function

Normal Hamiltonian

$$H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + [f_{02}(x) + \lambda f_2(x)] \cdot u.$$

Augmented Hamiltonian $\mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu s(x)$.

Switching function

$$\sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x), \quad \sigma(t) = \sigma(x(t), \lambda(t)).$$

On a boundary arc we have

$$\begin{aligned} s(x(t)) &= 0, & t_1 \leq t \leq t_2, \\ \alpha < u(t) < \beta, & t_1 < t < t_2. \end{aligned}$$

The **minimum condition** for the control implies

$$0 = H_u(x(t), \lambda(t), u(t)) = \sigma(t), \quad t_1+ \leq t \leq t_2-.$$

Formally, the **boundary control behaves like a singular control**.

Case II : Boundary Control and Junction Theorem

Let q be the order of the state constraint:

$$s^{(q)}(x, u) = \frac{d^q}{dt^q} s(x) = s_1(x) + s_2(x) \cdot u.$$

The boundary control $u = u_b(x)$ is determined from $s^{(q)}(x, u) = 0$ as

$$u = u_b(x) = -s_1(x)/s_2(x).$$

Junction Theorem for $q = 1$

Let $q = 1$ and let a bang-bang arc be joined with a boundary arc at $t_1 \in (0, T)$.

Claim: If the control is **discontinuous** at t_1 , then the **adjoint variable is continuous** at t_1 .

Model of the immune response

Dynamic model of the immune response:

Asachenko A, Marchuk G, Mohler R, Zuev S,
Disease Dynamics, Birkhäuser, Boston, 1994.

Optimal control:

Stengel RF, Ghigliazza R, Kulkarni N, Laplace O,
Optimal control of innate immune response,
Optimal Control Applications and Methods **23**, 91–104 (2002),

Lisa Poppe, Julia Meskauskas: Diploma theses, Universität
Münster (2006,2008).

Innate Immune Response: state and control variables

State variables:

- $x_1(t)$: concentration of **pathogen**
(=concentration of associated **antigen**)
- $x_2(t)$: concentration of **plasma cells**,
which are carriers and producers of antibodies
- $x_3(t)$: concentration of **antibodies**, which kill the pathogen
(=concentration of **immunoglobulins**)
- $x_4(t)$: relative characteristic of a **damaged organ**
(**0 = healthy, 1 = dead**)

Control variables:

- $u_1(t)$: pathogen killer
- $u_2(t)$: plasma cell enhancer
- $u_3(t)$: antibody enhancer
- $u_4(t)$: organ healing factor

Generic dynamical model of the immune response

$$\dot{x}_1(t) = (1 - x_3(t))x_1(t) - u_1(t),$$

$$\dot{x}_2(t) = 3A(x_4(t))x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t),$$

$$\dot{x}_3(t) = x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t),$$

$$\dot{x}_4(t) = x_1(t) - x_4(t) - u_4(t).$$

Immune deficiency function triggered by target organ damage

$$A(x_4) = \left\{ \begin{array}{ll} \cos(\pi x_4), & 0 \leq x_4 \leq 0.5 \\ 0 & 0.5 \leq x_4 \end{array} \right\}.$$

For $0.5 \leq x_4(t)$ the production of plasma cells stops.

State delay $d \geq 0$ in variables x_1 and x_3

Initial conditions ($d = 0$) : $x_2(0) = 2$, $x_3(0) = 4/3$, $x_4(0) = 0$

Case 1 : $x_1(0) = 1.5$, decay, requires no therapy (control)

Case 2 : $x_1(0) = 2.0$, slower decay, requires no therapy

Case 3 : $x_1(0) = 3.0$, **diverges without control (lethal case)**

Optimal control model: cost functional

State $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, **Control** $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$

L^2 -functional quadratic in control: Stengel et al.

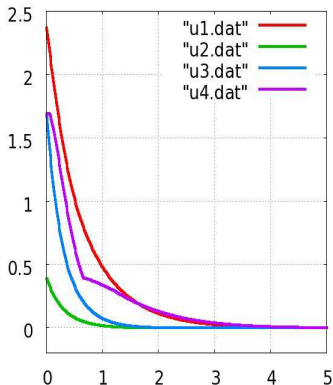
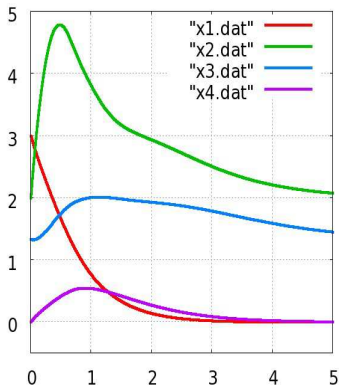
$$\begin{aligned} \text{Minimize } J_2(x, u) = & x_1(t_f)^2 + x_4(t_f)^2 \\ & + \int_0^{t_f} (x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2) dt \end{aligned}$$

L^1 -functional linear in control

$$\begin{aligned} \text{Minimize } J_1(x, u) = & x_1(t_f)^2 + x_4(t_f)^2 \\ & + \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) dt \end{aligned}$$

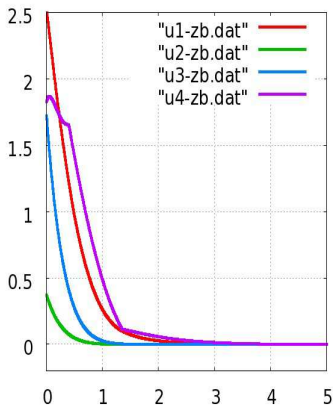
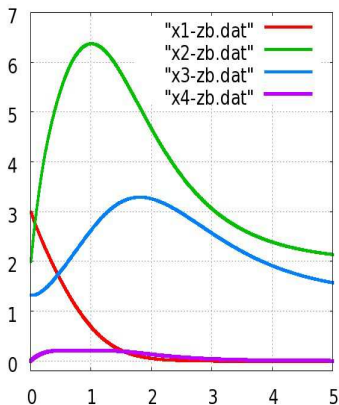
Control constraints: $0 \leq u_i(t) \leq u_{\max}$, $i = 1, \dots, 4$

Final time: $t_f = 10$

L^2 -functional, $d = 0$: optimal state and control variables

State variables x_1, x_2, x_3, x_4 and optimal controls u_1, u_2, u_3, u_4 :
second-order sufficient conditions via matrix Riccati equation

L^2 -functional, $d = 0$: state constraint $x_4(t) \leq 0.2$

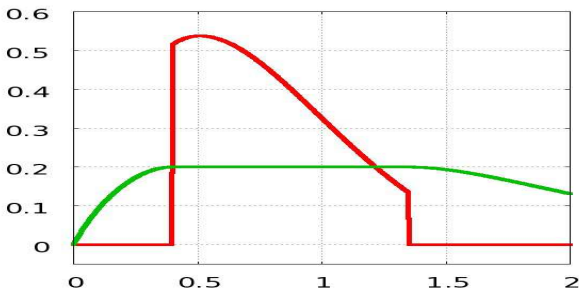


State and control variables for **state constraint** $x_4(t) \leq 0.2$.
Boundary arc $x_4(t) \equiv 0.2$ for $t_1 = 0.398 \leq t \leq t_2 = 1.35$

L^2 -functional, multiplier $\eta(t)$ for constraint $x_4(t) \leq 0.2$

Compute multiplier η as function of (x, λ) :

$$\eta(x, \lambda) = \lambda_2 3\pi \sin(\pi x_4) x_1 x_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$



Scaled multiplier $0.1\eta(t)$ and boundary arc $x_4(t) = 0.2$

L^2 -functional, delay $d > 0$, constraint $x_4(t) \leq \alpha$

Dynamics with state delay $d > 0$

$$\dot{x}_1(t) = (1 - x_3(t))x_1(t) - u_1(t),$$

$$\dot{x}_2(t) = 3 \cos(\pi x_4) x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t),$$

$$\dot{x}_3(t) = x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t),$$

$$\dot{x}_4(t) = x_1(t) - x_4(t) - u_4(t)$$

$$x_4(t) \leq \alpha \leq 0.5$$

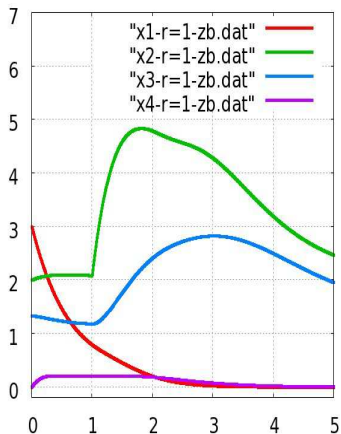
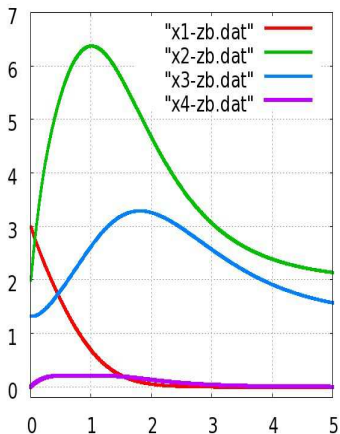
Initial conditions

$$x_1(t) = 0, \quad -d \leq t < 0, \quad x_1(0) = 3,$$

$$x_3(t) = 4/3, \quad -d \leq t \leq 0,$$

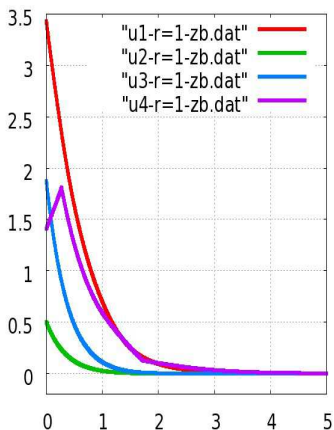
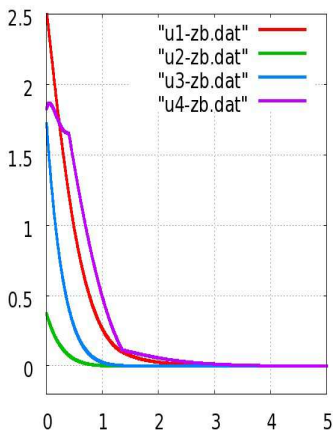
$$x_2(0) = 2, \quad x_4(0) = 0.$$

L^2 -functional : delay $d = 1$ and $x_4(t) \leq 0.2$



State variables for $d = 0$ and $d = 1$

L^2 -functional : delay $d = 1$ and $x_4(t) \leq 0.2$



Optimal controls for $d = 0$ and $d = 1$

L^2 -functional, $d = 1$: multiplier $\eta(t)$ for $x_4(t) \leq 0.2$

Compute multiplier η as function of (x, λ) :

$$\eta(x, y, \lambda) = \lambda_2 3\pi \sin(\pi x_4) y_1 y_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$



Scaled multiplier $0.1 \eta(t)$ and boundary arc $x_4(t) = 0.2$;

$\eta(t)$ is discontinuous at $t = d = 1$

L^1 -functional : no delays

Minimize

$$J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 + \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) dt$$

Dynamics with delay d and control constraints

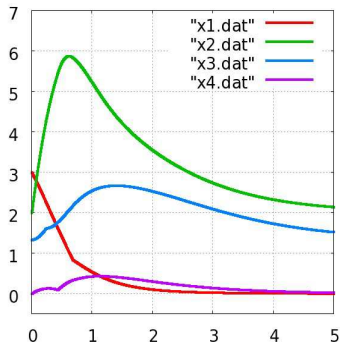
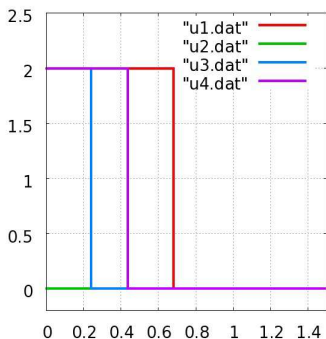
$$\dot{x}_1(t) = (1 - x_3(t))x_1(t) - u_1(t),$$

$$\dot{x}_2(t) = 3A(x_4(t))x_1(t)x_3(t) - (x_2(t) - 2) + u_2(t),$$

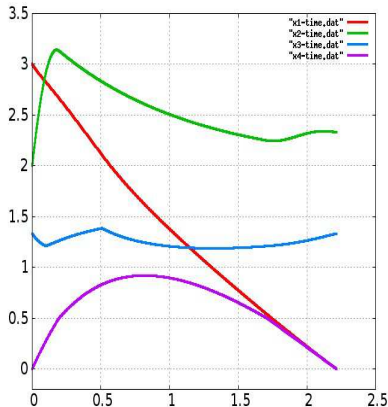
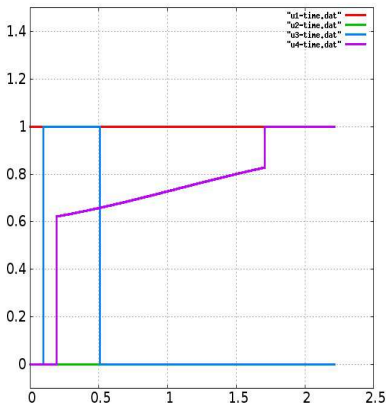
$$\dot{x}_3(t) = x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t),$$

$$\dot{x}_4(t) = x_1(t) - x_4(t) - u_4(t),$$

$$0 \leq u_i(t) \leq u_{\max}, \quad 0 \leq t \leq t_f \quad (i = 1, \dots, 4)$$

L^1 -functional : $u_{\max} = 2$ 

L^1 -functional : non-delayed, time-optimal control for
 $x_1(t_f) = x_4(t_f) = 0, x_3(t_f) = 4/3$



$u_{\max} = 1$: minimal time $t_f = 2.2151$, singular arc for $u_4(t)$

Dynamical Model of Climate Change

A. GREINER, L. GRÜNE, AND W. SEMMLER,
Growth and Climate Change: Threshold and Multiple Equilibria, Working Paper, Schwartz Center for Economic Policy Studies, The New School, 2009, to appear.

State Variables:

$K(t)$: Capital (per capita)

$M(t)$: CO₂ concentration in the atmosphere

$T(t)$: Temperature (Kelvin)

Control Variables

$C(t)$: Consumption

$A(t)$: Abatement per capita

Dynamical Model of Climate Change

Production : $Y = K^{0.18} \cdot D(T - T_o), \quad T_o = 288 (K)$

Damage : $D(T - T_o) = (0.025(T - T_o)^2 + 1)^{-0.025}$

Dynamics of per-capita capital K

$$\dot{K} = Y - C - A - (\delta + n)K, \quad K(0) = K_0.$$
$$(\delta = 0.075, \quad n = 0.03)$$

Emission : $E = 3.5 \cdot 10^{-4} \cdot K/A$

Dynamics of CO_2 concentration M

$$\dot{M} = 0.49 E - 0.1 M, \quad M(0) = M_0.$$

Dynamical Model of Climate Change (continued)

Albedo (non-reflected energy) at temperature T (Kelvin):

$$1 - \alpha_1(T) = k_1 \frac{2}{\pi} \arctan\left(\frac{\pi(T-293)}{2}\right) + k_2,$$

$$k_1 = 5.6 \cdot 10^{-3}, \quad k_2 = 0.1795.$$

Radiative forcing : $5.35 \ln(M)$.

Outgoing radiative flux (Stefan-Boltzman-law) : $\epsilon \sigma_T T^4$.

Parameters : $\epsilon = 0.95$, $\sigma_T = 5.67 \cdot 10^{-8}$,
 $cth = 0.149707$, $Q = 1367$.

Dynamics of temperature T with delay $d \geq 0$

$$\dot{T}(t) = cth \cdot \left[(1 - \alpha_1(T(t))) \cdot \frac{Q}{4} - \frac{19}{116} \cdot \epsilon \cdot \sigma_T \cdot T(t)^4 + 5.35 \cdot \ln(M(t-d)) \right],$$

$$T(0) = T_0.$$

Optimal Control Model of Climate Change

Control constraints for $0 \leq t \leq t_f = 200$:

$$0 < C(t) \leq C_{max} = 1, \quad 7 \cdot 10^{-4} \leq A(t) \leq 3 \cdot 10^{-3}$$

State constraints of order 2 and 3 :

$$M(t) \leq M_{max}, \quad 0 \leq t \leq t_f = 200,$$

$$T(t) \leq T_{max}, \quad t_e = 20 \leq t \leq t_f = 200.$$

Maximize consumption

$$J(K, M, T, C, A) = \int_0^{t_f} e^{-(n-\rho)t} \ln(C(t)) dt$$

Stationary points of the canonical system

Constant Abatement $A = 1.21 \cdot 10^{-3}$: 3 stationary points

$$T_s := 291.607, \quad M_s = 2.05196, \quad K_s = 1.44720,$$

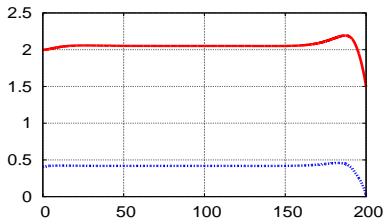
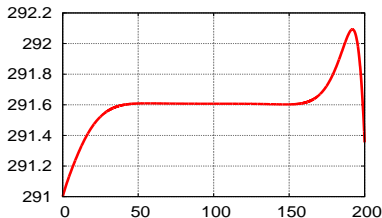
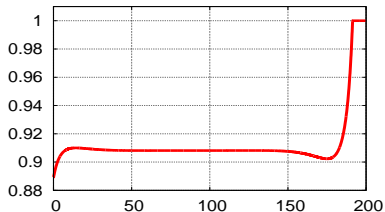
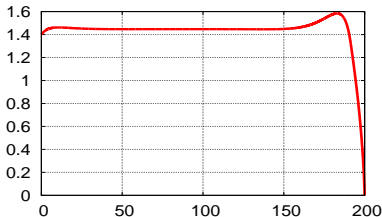
$$T_s := 294.258, \quad M_s = 1.90954, \quad K_s = 1.34726,$$

$$T_s := 294.969, \quad M_s = 2.07792, \quad K_s = 1.46606.$$

Control variable abatement $A(t)$: Social Optimum

$$T_s := 288.286, \quad M_s = 1.28500, \quad K_s = 1.79647.$$

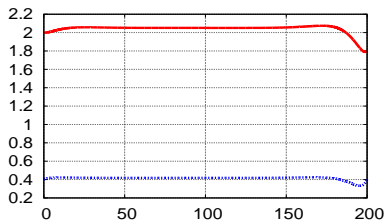
$$T(0) = 291, M(0) = 2.0, K(0) = 1.4$$

Concentration M , Emission E Temperature T (Kelvin)Consumption C Capital K 

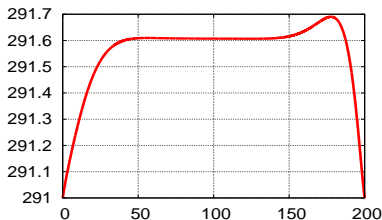
$$T(0) = 291, M(0) = 2.0, K(0) = 1.4$$

$$T(t_f) = 291, M(t_f) = 1.8, K(t_f) = 1.4$$

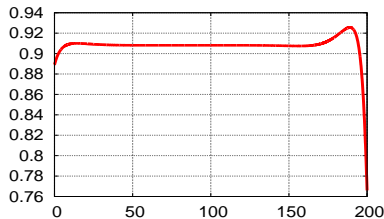
Concentration M, Emission E



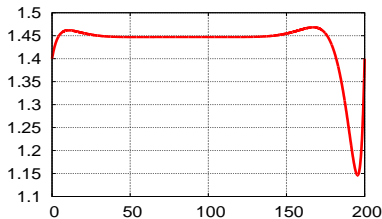
Temperature T (Kelvin)



Consumption C



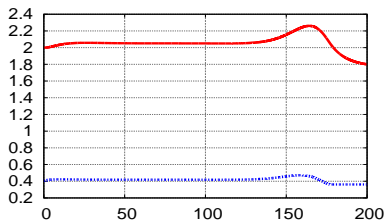
Capital K



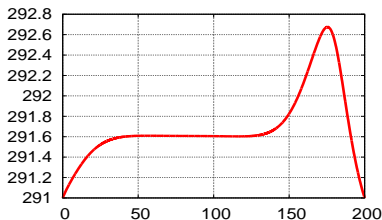
$$T(0) = 291, M(0) = 2.0, K(0) = 1.4$$

$$T(t_f) = 291, M(t_f) = 1.8, K(t) \geq 1.2, 0.85 \leq C(t) \leq 1$$

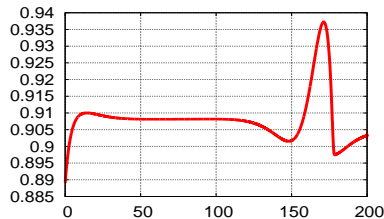
Concentration M, Emission E



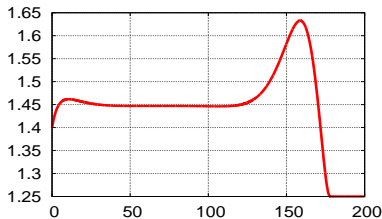
Temperature T (Kelvin)



Consumption C

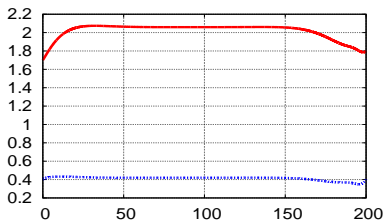


Capital K

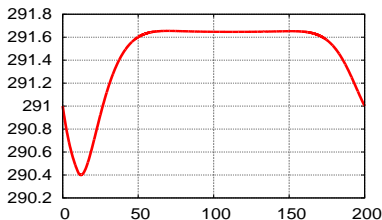


$$d = 10 : T(0) = 291, M(0) = 1.8, K(0) = 1.4$$
$$T(t_f) = 291, M(t_f) = 1.8, K(t_f) = 1.4$$

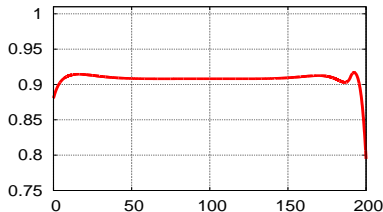
Concentration M, Emission E



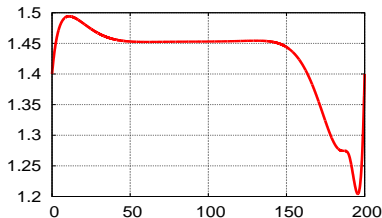
Temperature T (Kelvin)



Consumption C



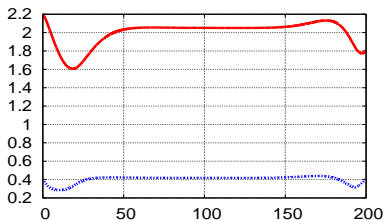
Capital K



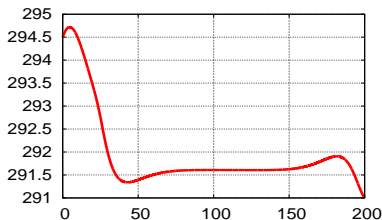
$$T(0) = 294.5, M(0) = 2.2, K(0) = 1.4$$

$$T(t_f) = 291, M(t_f) = 1.8, K(t_f) = 1.4, 0.85 \leq C(t) \leq 1$$

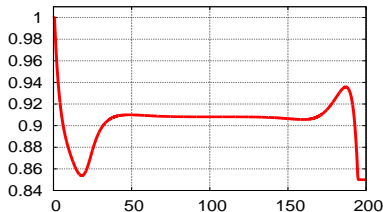
Concentration M, Emission E



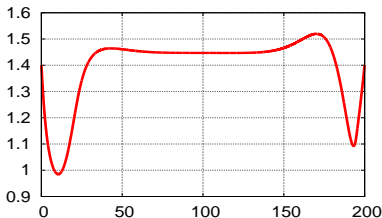
Temperature T (Kelvin)



Consumption C



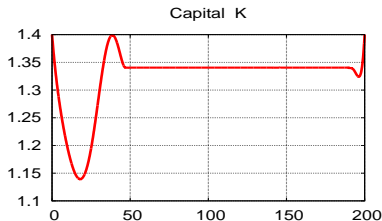
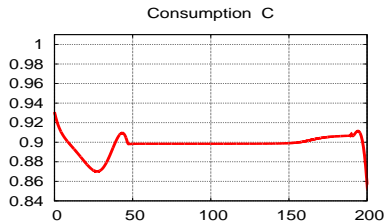
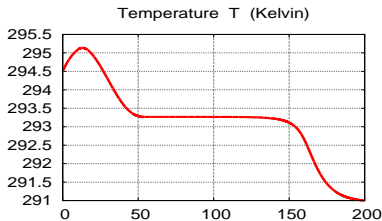
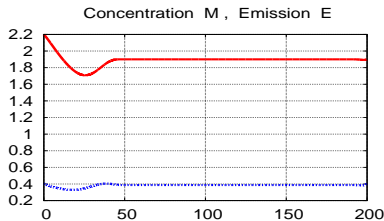
Capital K



$$d = 10 : T(0) = 294.5, M(0) = 2.2, K(0) = 1.4$$

$$T(t_f) = 291, K(t) = 1.4, 0.85 \leq C(t) \leq 1$$

State constraint of order two : $M(t) \leq 1.9$ for $30 \leq t \leq t_f = 200$



$$d = 10 : T(0) = 294.5, M(0) = 2.2, K(0) = 1.4$$

$$T(t_f) = 291, K(t) = 1.4, 0.85 \leq C(t) \leq 1$$

State constraint of order two : $M(t) \leq 1.9$ for $30 \leq t \leq t_f = 200$

