



HAL
open science

Tutorial on Control of Non Holonomic Systems by Active Constraints

Franco Rampazzo

► **To cite this version:**

Franco Rampazzo. Tutorial on Control of Non Holonomic Systems by Active Constraints. SADCO Summer School 2011 - Optimal Control, Sep 2011, London, United Kingdom. inria-00629512

HAL Id: inria-00629512

<https://inria.hal.science/inria-00629512>

Submitted on 6 Oct 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CONTROL OF NON HOLONOMIC SYSTEMS BY ACTIVE CONSTRAINTS

Franco Rampazzo,
University of Padova, Italy

SADCO Summer School
Imperial College , London, September 5-9, 2011,

FIRST MOVEMENT

—*Allegro*—

Introduction

INVESTIGATE **analysis** AND **geometry** RELATED TO
THE FOLLOWING PROGRAM:

INVESTIGATE **analysis** AND **geometry** RELATED TO THE FOLLOWING PROGRAM:

Consider a $(N + M)$ -dimensional mechanical system and

INVESTIGATE **analysis** AND **geometry** RELATED TO THE FOLLOWING PROGRAM:

Consider a $(N + M)$ -dimensional mechanical system and let Q be the $(N + M)$ -dimensional configuration manifold, locally parameterized by

$$q = (q^1, \dots, q^N, q^{N+1}, \dots, q^{N+M})$$

INVESTIGATE **analysis** AND **geometry** RELATED TO THE FOLLOWING PROGRAM:

Consider a $(N + M)$ -dimensional mechanical system and let Q be the $(N + M)$ -dimensional configuration manifold, locally parameterized by

$$q = (q^1, \dots, q^N, q^{N+1}, \dots, q^{N+M})$$

Assign the "control"

$$u(t) = (u^1, \dots, u^M) \equiv (q^{N+1}(t), \dots, q^{N+M}(t))$$

INVESTIGATE **analysis** AND **geometry** RELATED TO THE FOLLOWING PROGRAM:

Consider a $(N + M)$ -dimensional mechanical system and let Q be the $(N + M)$ -dimensional configuration manifold, locally parameterized by

$$q = (q^1, \dots, q^N, q^{N+1}, \dots, q^{N+M})$$

Assign the "control"

$$u(t) = (u^1, \dots, u^M) \equiv (q^{N+1}(t), \dots, q^{N+M}(t))$$

(i.e. give the evolution of the last M coordinates)

INVESTIGATE **analysis** AND **geometry** RELATED TO THE FOLLOWING PROGRAM:

Consider a $(N + M)$ -dimensional mechanical system and let Q be the $(N + M)$ -dimensional configuration manifold, locally parameterized by

$$q = (q^1, \dots, q^N, q^{N+1}, \dots, q^{N+M})$$

Assign the "control"

$$u(t) = (u^1, \dots, u^M) \equiv (q^{N+1}(t), \dots, q^{N+M}(t))$$

(i.e. give the evolution of the last M coordinates)

PROBLEM:

What can we say on the *whole* motion $q = q(t)$?

Standard goals:

Standard goals:

OPTIMIZATION

Standard goals:

OPTIMIZATION

CONTROLLABILITY

Standard goals:

OPTIMIZATION

CONTROLLABILITY

STABILIZABILITY

Standard goals:

Optimization

CONTROLLABILITY

STABILIZABILITY

Standard goals:

Optimization

CONTROLLABILITY

Stabilizability

Standard goals:

Optimization ...

mostly for \mathcal{NC} (=non-centrifugal) systems

CONTROLLABILITY

Stabilizability

Standard goals:

Optimization ...

mostly for \mathcal{NC} (=non-centrifugal) systems

CONTROLLABILITY

Stabilizability

Mostly for \mathcal{C} (=centrifugal) systems

WARNING

WARNING

In this presentation the words "CENTRIFUGAL" and "NON-CENTRIFUGAL" do not refer to precise definitions.

WARNING

In this presentation the words "CENTRIFUGAL" and "NON-CENTRIFUGAL" do not refer to precise definitions. Rather they have a *pedagogic, allusive* function. For this reason their use will be (hopefully) more acceptable once the presentation is concluded.

NOTICE:

NOTICE:

In the conventional applications of Control Theory to Mechanics

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

INSTEAD

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

INSTEAD

here

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

INSTEAD

here

controls are identified with some of the coordinates:

$$u(t) = (q^{N+1}, \dots, q^{N+M})$$

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

INSTEAD

here

controls are identified with some of the coordinates:

$$u(t) = (q^{N+1}, \dots, q^{N+M})$$

Which is a local way of imposing moving constraints as controls.

NOTICE:

In the conventional applications of Control Theory to Mechanics
controls are forces (or powers)

INSTEAD

here

controls are identified with some of the coordinates:

$$u(t) = (q^{N+1}, \dots, q^{N+M})$$

Which is a local way of imposing moving constraints as controls.

We assume that d'Alembert principle is verified, i.e., reaction forces are "orthogonal" to the reduced system

Let us begin with

Let us begin with

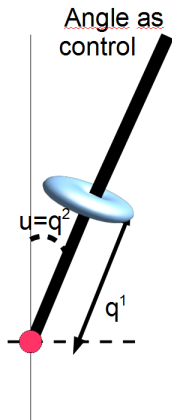
SOME EXAMPLES of MECHANICAL SYSTEMS:

THREE CLASS \mathcal{C} EXAMPLES:

THREE **CLASS C** EXAMPLES:

(*C* stands for *centrifugal*)

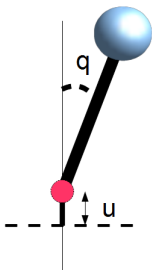
(1C) The angle as control



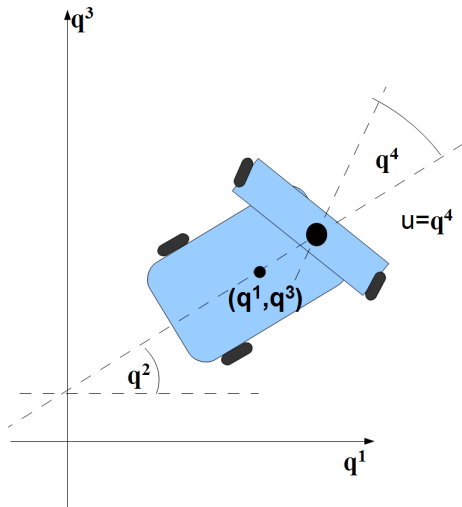
The control is
the angle.

(2C) The pendulum with oscillating pivot

Vertically moving
pivot



(3C) The "Roller Racer"



(The Roller Racer is a well-known toy):



NOW, THREE **CLASS** \mathcal{NC} EXAMPLES:

NOW, THREE **CLASS** \mathcal{NC} EXAMPLES:

(\mathcal{NC} stands for *non centrifugal*)

(1 $\mathcal{N}\mathcal{C}$) The pendulum with length as control

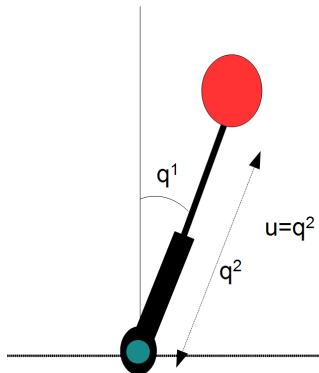
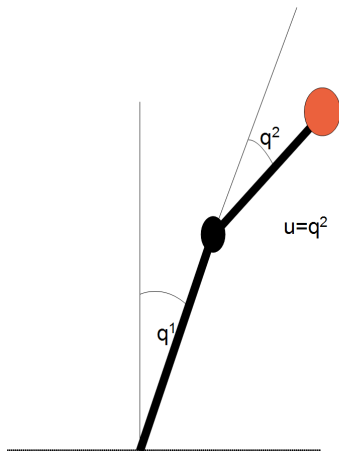
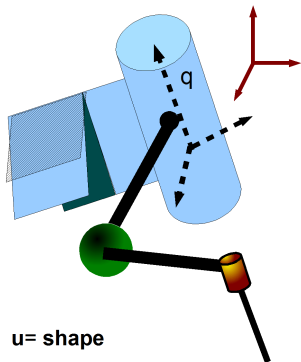


Figure: Length as control

(2 $\mathcal{N}\mathcal{C}$) The pendulum with a second pendulum as control

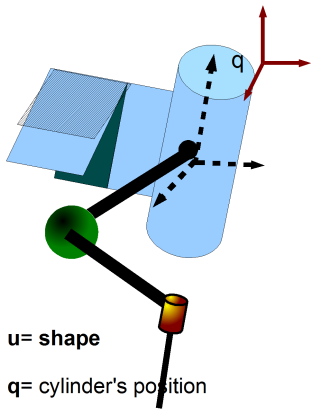


(3NC) "Shape" as control

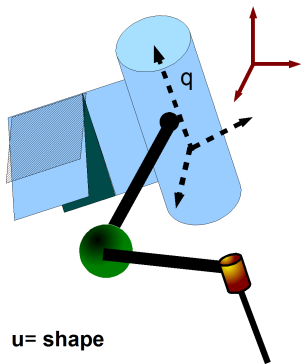


u = shape

q = cylinder's position

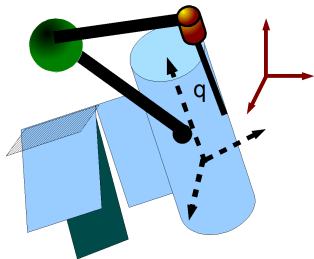


A rigid movement (the shape u is unchanged)



$u = \text{shape}$

$q = \text{cylinder's position}$



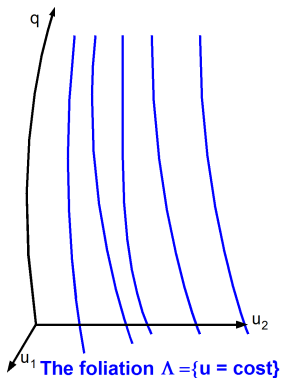
u = shape

q = cylinder's position

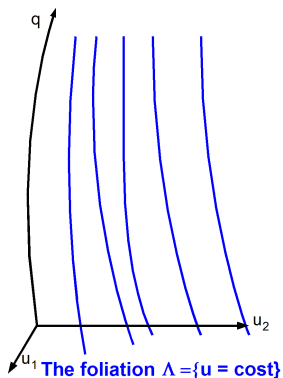
Change of shape (i.e. change of u)

Exposition

The abstract framework:

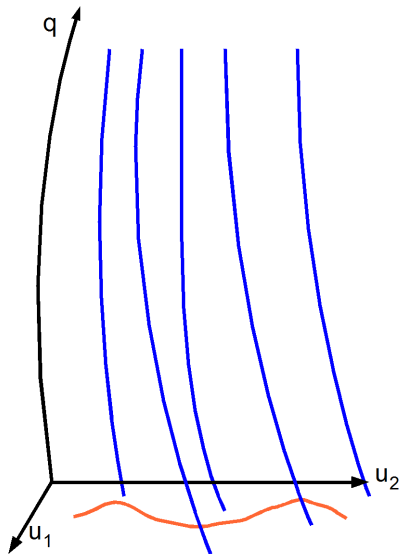


The abstract framework:

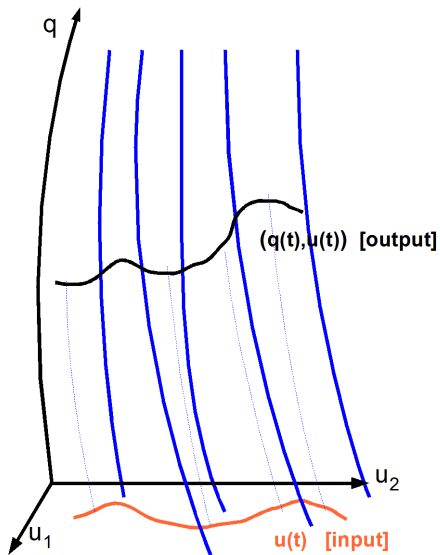


Given $u = u(t)$, we are interested in the corresponding motion $q(t)$ along the leaves of Λ .

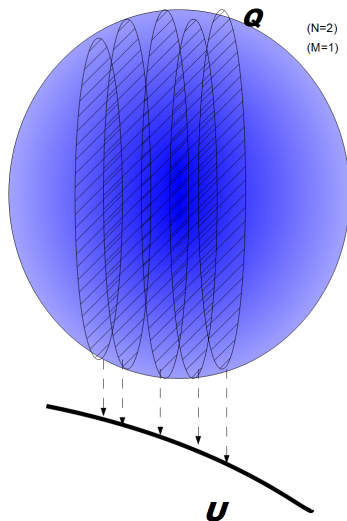
2-dimensional controls, 1-dimensional leaves



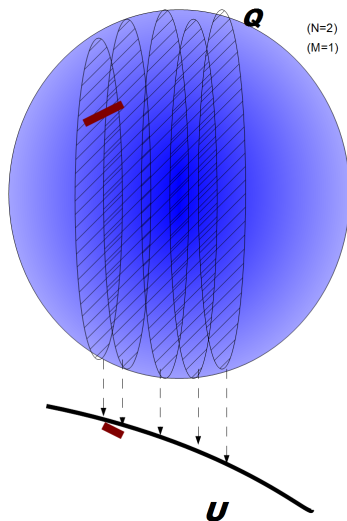
2-dimensional controls, 1-dimensional leaves



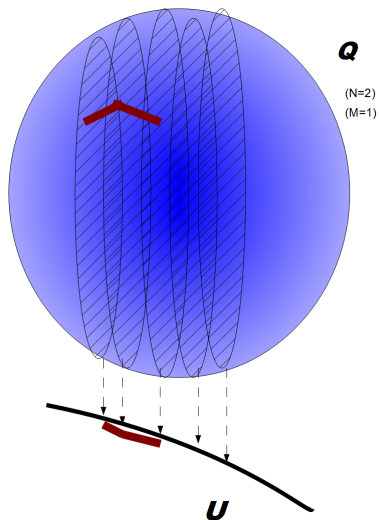
1-dimensional controls, 2-dimensional leaves



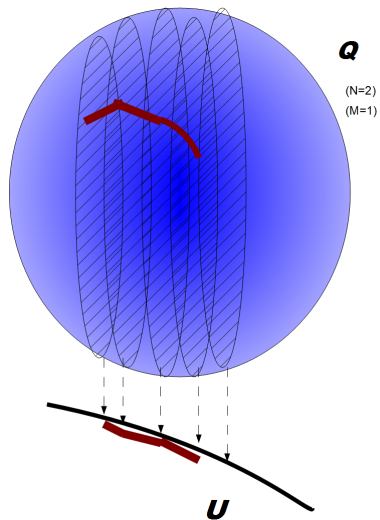
1-dimensional controls, 2-dimensional leaves



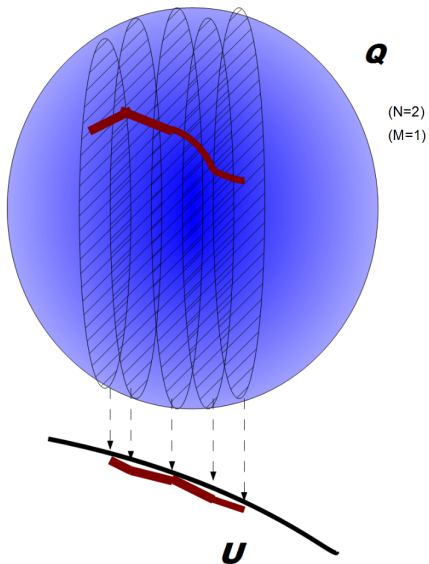
1-dimensional controls, 2-dimensional leaves



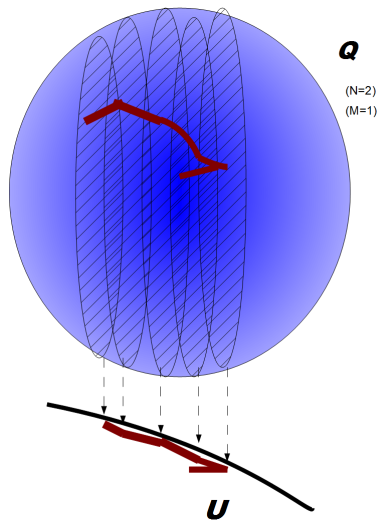
1-dimensional controls, 2-dimensional leaves



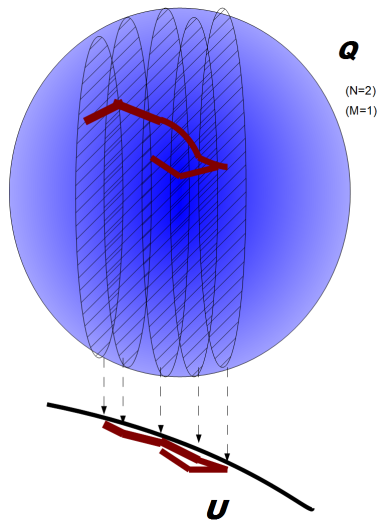
1-dimensional controls, 2-dimensional leaves



1-dimensional controls, 2-dimensional leaves



1-dimensional controls, 2-dimensional leaves



We shall focus on TWO MAIN KINDS OF CONTROLS u :

We shall focus on TWO MAIN KINDS OF CONTROLS u :

(1) RAPIDLY OSCILLATING CONTROLS

We shall focus on TWO MAIN KINDS OF CONTROLS u :

(1) RAPIDLY OSCILLATING CONTROLS

(2) BOUNDED VARIATION (POSSIBLY DISCONTINUOUS)
CONTROLS

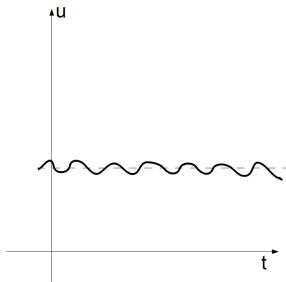


Figure: Rapidly oscillating controls

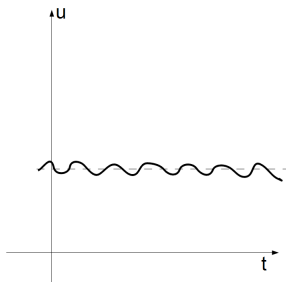


Figure: Rapidly oscillating controls

For instance:

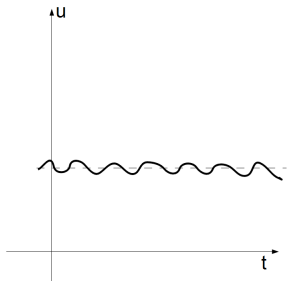


Figure: Rapidly oscillating controls

For instance: $\epsilon \sin(\mathbf{t}/\epsilon)$

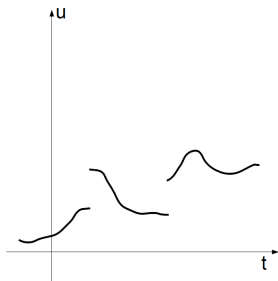


Figure: Control with bounded variation

Rapidly oscillating controls

Let us consider

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

PRODUCE a "FORCE"

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

PRODUCE a "FORCE"

NOW GUESS:

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

PRODUCE a "FORCE"

NOW GUESS:

Which of the previous examples do the job?

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

PRODUCE a "FORCE"

NOW GUESS:

Which of the previous examples do the job? The \mathcal{C} (=centrifugal)

Let us consider

THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

PRODUCE a "FORCE"

NOW GUESS:

Which of the previous examples do the job? The \mathcal{C} (=centrifugal) or the \mathcal{NC} (=non-centrifugal) systems?

(1C) The angle as control

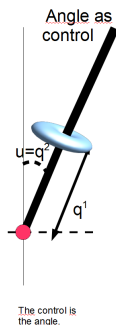


Figure: Oscillations of the angle generate a (centrifugal!) force on the sliding ring

(2 C) The pendulum with oscillating pivot

Vertically moving
pivot

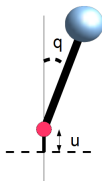


Figure: Oscillations of the pivot stabilize the unstable equilibrium
(*Kapiza pendulum*)

(3C) The "Roller Racer"

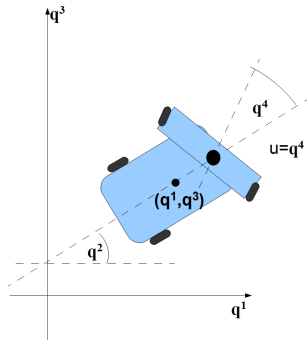


Figure: Oscillations of handlebar generate forward motion

Well,

Well,

YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

Well,

YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

Notice that they all are \mathcal{C} (=centrifugal)!

Well,

YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

Notice that they all are \mathcal{C} (=centrifugal)!

... A SUSPECT ARISES:

Well,

YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

Notice that they all are \mathcal{C} (=centrifugal)!

... A SUSPECT ARISES:

Are \mathcal{NC} (=non-centrifugal) systems such that

”OSCILLATIONS DO NOT GENERATE FORCES”?

Well,

YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

Notice that they all are \mathcal{C} (=centrifugal)!

... A SUSPECT ARISES:

Are \mathcal{NC} (=non-centrifugal) systems such that

”OSCILLATIONS DO NOT GENERATE FORCES”?

To feed our intuition let us give one more look to what *we have called* \mathcal{NC} (=non-centrifugal) systems:

(1 $\mathcal{N}\mathcal{C}$) The pendulum with length as control

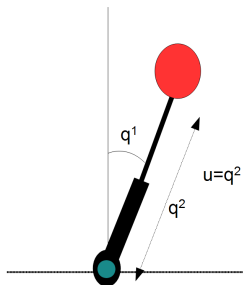
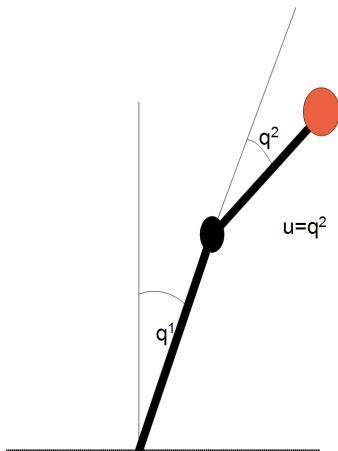
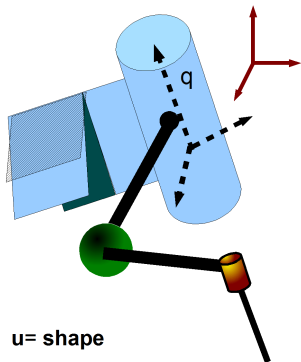


Figure: Almost insensitive to small oscillation of length

$(2\mathcal{N}C)$ The pendulum with a second pendulum as control

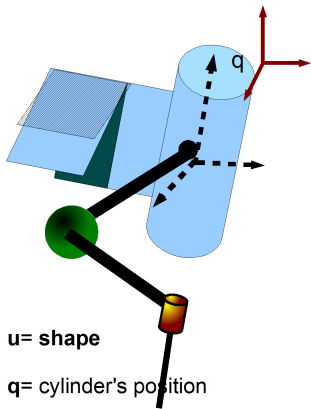


(3NC) "Shape" as control

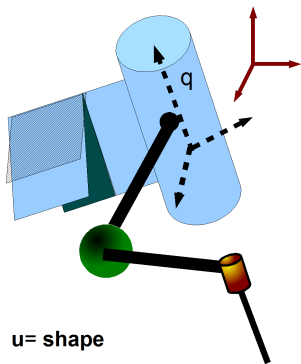


u = shape

q = cylinder's position

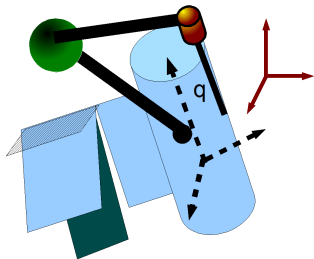


A rigid movement (the shape u is unchanged)



$u = \text{shape}$

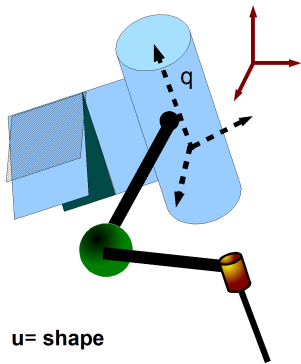
$q = \text{cylinder's position}$



u = shape

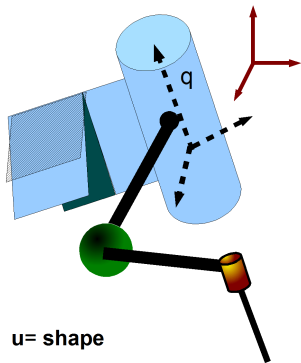
q = cylinder's position

Change of shape (i.e. change of u)



$u = \text{shape}$

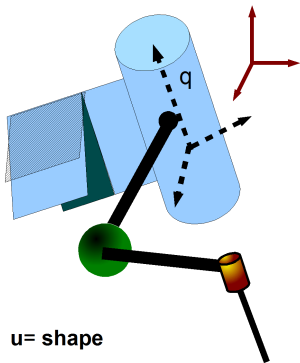
$q = \text{cylinder's position}$



$u = \text{shape}$

$q = \text{cylinder's position}$

Also in this case,



$u = \text{shape}$

$q = \text{cylinder's position}$

Also in this case, the body is almost insensitive to small rapid oscillation of the control (=the shape)

THE SUSPECT is *reasonable!*

THE SUSPECT is reasonable!

In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

THE SUSPECT is reasonable!

In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

"DO NOT PRODUCE FORCES"

THE SUSPECT is reasonable!

In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

"DO NOT PRODUCE FORCES"

At this stage, this statement is just an act of *faith*,

THE SUSPECT is reasonable!

In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

"DO NOT PRODUCE FORCES "

At this stage, this statement is just an act of *faith*,...not to mention that

THE SUSPECT is reasonable!

In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

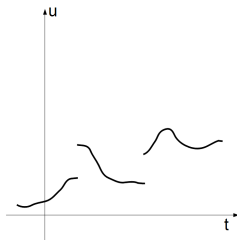
"DO NOT PRODUCE FORCES"

At this stage, this statement is just an act of *faith*,...not to mention that it is quite tautological

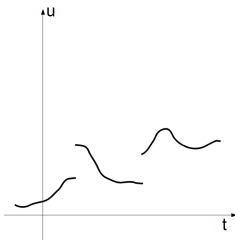


Later we shall mention *BV controls*

Later we shall mention *BV controls*



Later we shall mention *BV controls*



NOTICE: *the derivative \dot{u} is a **measure**....THEY CAN BE IMPLEMENTED ONLY IN "NON CENTRIFUGAL" SYSTEMS (WHERE THEY ARE INTERESTING FOR OPTIMIZATION PURPOSES)*

Development

IT IS TIME FOR SOME MATHEMATICS:

IT IS TIME FOR SOME MATHEMATICS:

Let us forget mechanics for a while

IT IS TIME FOR SOME MATHEMATICS:

Let us forget mechanics for a while

and let us consider a **particular class of control systems**:

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Notice: *the actual "controls" are the derivatives \dot{u}*

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Notice: *the actual "controls" are the derivatives \dot{u}*

As a matter of fact,

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Notice: *the actual "controls" are the derivatives \dot{u}*

As a matter of fact, , we can even neglect the dependence on u just by adding variables

$$\dot{x}^{n+\alpha} = \dot{u}^{\alpha}$$

so obtaining

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Notice: *the actual "controls" are the derivatives \dot{u}*

As a matter of fact, , we can even neglect the dependence on u just by adding variables

$$\dot{x}^{n+\alpha} = \dot{u}^{\alpha}$$

so obtaining

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$
(non-centrifugal!)

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$
(non-centrifugal!)
- **The affine case** : $h_{\alpha\beta} = 0$

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$
(non-centrifugal!)
- **The affine case** : $h_{\alpha\beta} = 0$ (non-centrifugal!)

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$
(non-centrifugal!)
- **The affine case** : $h_{\alpha\beta} = 0$ (non-centrifugal!)
- **The general case** : $h_{\alpha\beta} \neq 0$

Flashforward to Mechanics:

If we set

$$(q^1, \dots, q^N, p^1, \dots, p^n) = x$$

in the mechanical examples above, we obtain control equations of the form

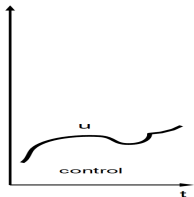
$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

In view of mechanical applications, we distinguish among:

- **The trivial case** : $g_{\alpha} = 0$ $h_{\alpha\beta} = 0$
(non-centrifugal!)
- **The affine case** : $h_{\alpha\beta} = 0$ (non-centrifugal!)
- **The general case** : $h_{\alpha\beta} \neq 0$... (centrifugal!)

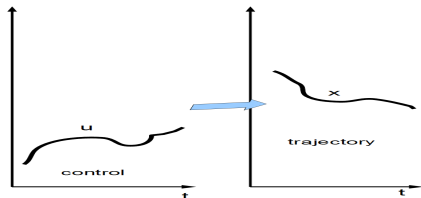
It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



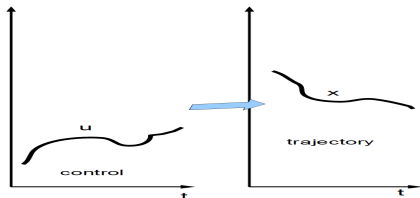
It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



It is interesting to investigate properties of the *input-output map*

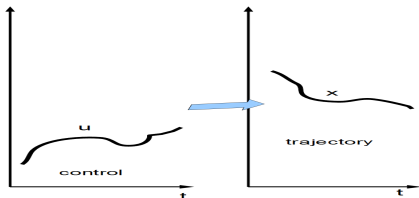
$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :

It is interesting to investigate properties of the *input-output map*

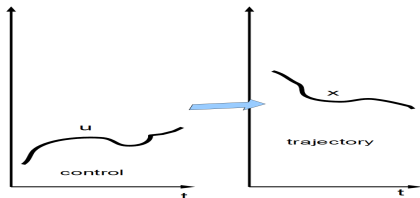
$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :
linearly?...

It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$

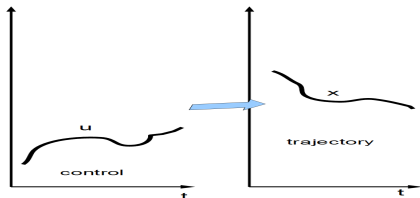


Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :

linearly?... $(\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}^{\alpha})$

It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



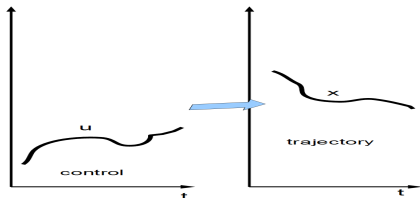
Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :

linearly?... $(\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}^{\alpha})$

or

It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



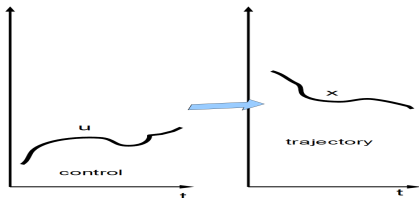
Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :

linearly?... $(\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}^{\alpha})$

or **quadratically?**...

It is interesting to investigate properties of the *input-output map*

$$u(\cdot) \mapsto x[u(\cdot)](\cdot)$$



Of course its *regularity* depends on how the right hand of dynamic equations depend on \dot{u} :

linearly?... $(\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}^{\alpha})$

or **quadratically?**... $(\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}^{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}^{\alpha}\dot{u}^{\beta})$

TWO BASIC QUESTIONS:

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

IN THIS TUTORIAL
WE SHALL BE MAINLY CONCERNED WITH (2)

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

IN THIS TUTORIAL
WE SHALL BE MAINLY CONCERNED WITH (2)

but we are also recalling crucial issues concerning (1),

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

IN THIS TUTORIAL
WE SHALL BE MAINLY CONCERNED WITH (2)

but we are also recalling crucial issues concerning (1), because of:
(i) its geometrical meaning,

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

IN THIS TUTORIAL

WE SHALL BE MAINLY CONCERNED WITH (2)

but we are also recalling crucial issues concerning (1), because of:

- (i) its geometrical meaning,
- (ii) the vast related literature,

TWO BASIC QUESTIONS:

(1) Can discontinuous controls $u(\cdot)$ be implemented?

Comment: Question (1) is important in optimization with no coercivity

(2) What happens with highly oscillatory $u(\cdot)$?

Comment: Question (2) is important for controllability and stabilizability questions

IN THIS TUTORIAL

WE SHALL BE MAINLY CONCERNED WITH (2)

but we are also recalling crucial issues concerning (1), because of:

- (i) its geometrical meaning,
- (ii) the vast related literature,
- (iii) its relation with (2)

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

- (1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case.

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

- (1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case. (It would correspond to integrate "squares of Dirac functions")

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

- (1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case. (It would correspond to integrate "squares of Dirac functions")
(On the contrary, in the linear case discontinuous controls CAN be implemented...)

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

(1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case. (It would correspond to integrate "squares of Dirac functions")

(On the contrary, in the linear case discontinuous controls CAN be implemented...but this is a delicate question, which will be briefly mentioned later.)

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

- (1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case. (It would correspond to integrate "squares of Dirac functions")
(On the contrary, in the linear case discontinuous controls CAN be implemented...but this is a delicate question, which will be briefly mentioned later.)
- (2) Despite (1), limits of solutions can well be discontinuous.

The general, quadratic, case

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}^{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}^{\alpha} \dot{u}^{\beta}$$

FACTS:

- (1) Discontinuous controls $u(\cdot)$ CANNOT be implemented in this case. (It would correspond to integrate "squares of Dirac functions")
(On the contrary, in the linear case discontinuous controls CAN be implemented...but this is a delicate question, which will be briefly mentioned later.)
- (2) Despite (1), limits of solutions can well be discontinuous.

(see *A. Bressan - F.R. (1999) SIAM Control Opt.* for the closure of the set of solutions when $\int_0^T |u|^2 dt \leq K$, in the case of **scalar** u)

Definition of STABILIZABILITY

- The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

Definition of STABILIZABILITY

- The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

is

stabilizable at $\bar{x} \in \mathbb{R}^n$

Definition of STABILIZABILITY

- The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

is

stabilizable at $\bar{x} \in \mathbb{R}^n$

if,

$\forall \varepsilon > 0 \exists \delta > 0$ such that : $\forall \hat{x} \in B(\bar{x}, \delta)$ *there exists a piecewise smooth control function* $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ *such that*

$$x(t, u) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

Definition of STABILIZABILITY

- The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x)\dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}_{\alpha}\dot{u}_{\beta}$$

is

stabilizable at $\bar{x} \in \mathbb{R}^n$

if,

$\forall \varepsilon > 0 \exists \delta > 0$ such that : $\forall \hat{x} \in B(\bar{x}, \delta)$ *there exists a piecewise smooth control function* $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ *such that*

$$x(t, u) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

- If, in addition,

$$\lim_{t \rightarrow \infty} x(t, u) = \bar{x}$$

Definition of STABILIZABILITY

- The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x)\dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}_{\alpha}\dot{u}_{\beta}$$

is

stabilizable at $\bar{x} \in \mathbb{R}^n$

if,

$\forall \varepsilon > 0 \exists \delta > 0$ such that : $\forall \hat{x} \in B(\bar{x}, \delta)$ *there exists a piecewise smooth control function* $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ *such that*

$$x(t, u) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

- If, in addition,

$$\lim_{t \rightarrow \infty} x(t, u) = \bar{x}$$

(Eq) is called

asymptotically stabilizable at \bar{x} .

The differential inclusion

The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

can be associated with the following convexified **DIFFERENTIAL INCLUSION**:

The differential inclusion

The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

can be associated with the following convexified **DIFFERENTIAL INCLUSION**:

$$\frac{dx}{dt} \in \mathcal{F}(x),$$

The differential inclusion

The control system

$$(Eq) \quad \dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

can be associated with the following convexified **DIFFERENTIAL INCLUSION**:

$$\frac{dx}{dt} \in \mathcal{F}(x),$$

where

$$\mathcal{F}(x) \doteq \overline{\text{co}} \left\{ f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) w_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) w_{\alpha} w_{\beta}; \quad (w_1, \dots, w_m) \in \mathbb{R}^m \right\}$$

THE "SYMMETRIZED DIFFERENTIAL INCLUSION"

Another differential inclusion can be considered, having in mind u -oscillation:

THE "SYMMETRIZED DIFFERENTIAL INCLUSION"

Another differential inclusion can be considered, having in mind u -oscillation:

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

THE "SYMMETRIZED DIFFERENTIAL INCLUSION"

Another differential inclusion can be considered, having in mind u -oscillation:

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

where

$$\mathcal{G}(x) \doteq \overline{\text{co}} \left\{ f(x) + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) w_{\alpha} w_{\beta} \quad (w_1, \dots, w_m) \in \mathbb{R}^m \right\}$$

THE "SYMMETRIZED DIFFERENTIAL INCLUSION"

Another differential inclusion can be considered, having in mind u -oscillation:

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

where

$$\mathcal{G}(x) \doteq \overline{\text{co}} \left\{ f(x) + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) w_{\alpha} w_{\beta} \quad (w_1, \dots, w_m) \in \mathbb{R}^m \right\}$$

- Notice that it is the differential inclusion associated with

$$\text{(Eq)} \quad \dot{x} = f(x) + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

where there is no \dot{u} -linear term;

THE "SYMMETRIZED DIFFERENTIAL INCLUSION"

Another differential inclusion can be considered, having in mind u -oscillation:

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

where

$$\mathcal{G}(x) \doteq \overline{\text{co}} \left\{ f(x) + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) w_{\alpha} w_{\beta} \quad (w_1, \dots, w_m) \in \mathbb{R}^m \right\}$$

- Notice that it is the differential inclusion associated with

$$\text{(Eq)} \quad \dot{x} = f(x) + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

where there is no \dot{u} -linear term;

- $\mathcal{G}(x) \subset \mathcal{F}(x)$

Definition of WEAK STABILITY for differential inclusions

Definition of WEAK STABILITY for differential inclusions

The point \bar{x} is **weakly stable** for the differential inclusion

$$\frac{dx}{dt} \in \mathcal{F}(x)$$

Definition of WEAK STABILITY for differential inclusions

The point \bar{x} is **weakly stable** for the differential inclusion

$$\frac{dx}{dt} \in \mathcal{F}(x)$$

if, $\forall \varepsilon > 0 \exists \delta > 0$ such that, $\forall \hat{x} \in B(\bar{x}, \delta)$, *there exists a trajectory* $x(\cdot)$

$$x(t) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

Definition of WEAK STABILITY for differential inclusions

The point \bar{x} is **weakly stable** for the differential inclusion

$$\frac{dx}{dt} \in \mathcal{F}(x)$$

if, $\forall \varepsilon > 0 \exists \delta > 0$ such that, $\forall \hat{x} \in B(\bar{x}, \delta)$, *there exists a trajectory* $x(\cdot)$

$$x(t) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

If, in addition,

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}$$

Definition of WEAK STABILITY for differential inclusions

The point \bar{x} is **weakly stable** for the differential inclusion

$$\frac{dx}{dt} \in \mathcal{F}(x)$$

if, $\forall \varepsilon > 0 \exists \delta > 0$ such that, $\forall \hat{x} \in B(\bar{x}, \delta)$, *there exists a trajectory* $x(\cdot)$

$$x(t) \in B(\bar{x}, \varepsilon) \quad \forall t \geq 0$$

If, in addition,

$$\lim_{t \rightarrow \infty} x(t) = \bar{x}$$

the differential inclusion is called **weakly asymptotically stable** (at \bar{x}).

Weak Lyapunov functions for differential inclusions

Weak Lyapunov functions for differential inclusions

Definition. A scalar function V is a **weak Lyapunov function** if

:

Weak Lyapunov functions for differential inclusions

Definition. A scalar function V is a **weak Lyapunov function** if

:

- V is continuous on \mathcal{N} , and continuously differentiable on $\mathcal{N} \setminus \{\bar{x}\}$.

Weak Lyapunov functions for differential inclusions

Definition. A scalar function V is a **weak Lyapunov function** if

:

- V is continuous on \mathcal{N} , and continuously differentiable on $\mathcal{N} \setminus \{\bar{x}\}$.
- $V(\bar{x}) = 0$ while $V(x) > 0$ for all $x \neq \bar{x}$.

Weak Lyapunov functions for differential inclusions

Definition. A scalar function V is a **weak Lyapunov function** if

:

- V is continuous on \mathcal{N} , and continuously differentiable on $\mathcal{N} \setminus \{\bar{x}\}$.
- $V(\bar{x}) = 0$ while $V(x) > 0$ for all $x \neq \bar{x}$.
- For each $\delta > 0$ sufficiently small, the sublevel set $\{x; V(x) \leq \delta\}$ is compact.
- At each $x \neq \bar{x}$ one has

$$\inf_{y \in F(x)} \nabla V(x) \cdot y \leq 0.$$

Theorem

Theorem Assume that the differential inclusion

$$\frac{dx}{ds} \in \mathcal{F}(x)$$

admits a weak Lyapunov function $V = V(x)$ defined on a neighborhood \mathcal{N} at \bar{x} .

Theorem Assume that the differential inclusion

$$\frac{dx}{ds} \in \mathcal{F}(x)$$

admits a weak Lyapunov function $V = V(x)$ defined on a neighborhood \mathcal{N} at \bar{x} .

Then the control system

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

can be stabilized at \bar{x} .

Corollary.

Corollary. Assume that the
"symmetrized" differential inclusion

$$\frac{dx}{ds} \in \mathcal{G}(x)$$

admits a weak Lyapunov function $V = V(x)$ defined on a neighborhood \mathcal{N} at \bar{x} .

Corollary. Assume that the
"symmetrized" differential inclusion

$$\frac{dx}{ds} \in \mathcal{G}(x)$$

admits a weak Lyapunov function $V = V(x)$ defined on a neighborhood \mathcal{N} at \bar{x} .

Then the control system

$$\dot{x} = f(x) + \sum_{\alpha}^m g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

can be stabilized at \bar{x} .

It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

has *something to do with* ”VIBRATIONAL CONTROLS”.

It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

has *something to do with* ”VIBRATIONAL CONTROLS”.

Indeed $\mathcal{G}(x)$ is the convexification of

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}_{\alpha}\dot{u}_{\beta}$$

without the (= \dot{u} -linear) term $\sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha}$,

It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

has *something to do with* ”VIBRATIONAL CONTROLS”.

Indeed $\mathcal{G}(x)$ is the convexification of

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}_{\alpha}\dot{u}_{\beta}$$

without the (= \dot{u} -linear) term $\sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha}$,

while $\mathcal{F}(x)$ is the convexification

with the (= \dot{u} -linear) term $\sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha}$.

It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

has *something to do with* ”VIBRATIONAL CONTROLS”.

Indeed $\mathcal{G}(x)$ is the convexification of

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\dot{u}_{\alpha}\dot{u}_{\beta}$$

without the (= \dot{u} -linear) term $\sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha}$,

while $\mathcal{F}(x)$ is the convexification

with the (= \dot{u} -linear) term $\sum_{\alpha=1}^m g_{\alpha}(x)\dot{u}_{\alpha}$.

(**Intuitively:** *Rapid oscillations of u cancel out the \dot{u} -linear term.*)

A main technical device: reparameterization

NOTICE:

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x .

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . *To overcome this drawback one exploits L^2 -reparameterizations.*

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . *To overcome this drawback one exploits L^2 -reparameterizations.* For a control $u \in W^{1,2}$ consider a *new time parameter*

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . To overcome this drawback one exploits **L^2 -reparameterizations**. For a control $u \in W^{1,2}$ consider a new time parameter

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

$$\begin{aligned} \phi_0(s) = t(s) &\doteq \sigma^{-1}(s) & v_0^2 &\doteq \frac{dt}{ds} \\ \phi_\alpha(s) &\doteq u_\alpha(t(s)) & \alpha = 1 \dots, m, & v_\alpha \doteq \frac{\phi_\alpha}{ds} \end{aligned}$$

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . To overcome this drawback one exploits **L^2 -reparameterizations**. For a control $u \in W^{1,2}$ consider a new time parameter

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

$$\phi_0(s) = t(s) \doteq \sigma^{-1}(s) \quad v_0^2 \doteq \frac{dt}{ds}$$

$$\phi_\alpha(s) \doteq u_\alpha(t(s)) \quad \alpha = 1 \dots, m, \quad v_\alpha \doteq \frac{d\phi_\alpha}{ds}$$

Setting $y(s) = x(t(s))$,

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . To overcome this drawback one exploits **L^2 -reparameterizations**. For a control $u \in W^{1,2}$ consider a new time parameter

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

$$\phi_0(s) = t(s) \doteq \sigma^{-1}(s) \quad v_0^2 \doteq \frac{dt}{ds}$$

$$\phi_\alpha(s) \doteq u_\alpha(t(s)) \quad \alpha = 1 \dots, m, \quad v_\alpha \doteq \frac{d\phi_\alpha}{ds}$$

Setting $y(s) = x(t(s))$, the original control system

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_\alpha(x) \dot{u}_\alpha + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x) \dot{u}_\alpha \dot{u}_\beta$$

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . To overcome this drawback one exploits **L^2 -reparameterizations**. For a control $u \in W^{1,2}$ consider a new time parameter

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

$$\begin{aligned}\phi_0(s) &= t(s) \doteq \sigma^{-1}(s) & v_0^2 &\doteq \frac{dt}{ds} \\ \phi_\alpha(s) &\doteq u_\alpha(t(s)) & \alpha = 1 \dots, m, & v_\alpha \doteq \frac{\phi_\alpha}{ds}\end{aligned}$$

Setting $y(s) = x(t(s))$, the original control system

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_\alpha(x) \dot{u}_\alpha + \sum_{\alpha, \beta=1}^m h_{\alpha, \beta}(x) \dot{u}_\alpha \dot{u}_\beta$$

is turned into the reparameterized system

$$\frac{dy}{ds} = f(y) v_0^2 + \sum_{\alpha=1}^m g_\alpha(y) v_0 v_\alpha + h_{\alpha, \beta}(y) v_\alpha v_\beta$$

A main technical device: reparameterization

NOTICE: The above differential inclusions have unbounded values, i.e. $\mathcal{F}(x)$ is unbounded at each x . To overcome this drawback one exploits L^2 -reparameterizations. For a control $u \in W^{1,2}$ consider a new time parameter

$$\sigma(t) \doteq \frac{\int_0^t (1 + |\dot{u}|^2) d\tau}{T + \|\dot{u}\|_2^2}$$

and set

$$\phi_0(s) = t(s) \doteq \sigma^{-1}(s) \quad v_0^2 \doteq \frac{dt}{ds}$$

$$\phi_\alpha(s) \doteq u_\alpha(t(s)) \quad \alpha = 1 \dots, m, \quad v_\alpha \doteq \frac{d\phi_\alpha}{ds}$$

Setting $y(s) = x(t(s))$, the original control system

$$\dot{x} = f(x) + \sum_{\alpha=1}^m g_\alpha(x) \dot{u}_\alpha + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x) \dot{u}_\alpha \dot{u}_\beta$$

is turned into the reparameterized system

$$\frac{dy}{ds} = f(y) v_0^2 + \sum_{\alpha=1}^m g_\alpha(y) v_0 v_\alpha + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(y) v_\alpha v_\beta$$

Correspondingly, one has the

REPARAMETERIZED DIFFERENTIAL INCLUSION:

$$\frac{dy}{ds} \in \mathbf{F}(y)$$

$$\mathbf{F}(y) \doteq \overline{\text{co}} \left\{ f(y)v_0^2 + \sum_{\alpha=1}^m g_{\alpha}(y)v_0v_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(y)v_{\alpha}v_{\beta}; \right. \\ \left. v_0 \in [0, 1], v_0^2 + \dots + v_m^2 = 1 \right\}.$$

Trivial model equation

$$\dot{x} = \dot{u} + (\dot{u})^2 \quad x(0) = 0$$

Trivial model equation

$$\dot{x} = \dot{u} + (\dot{u})^2 \quad x(0) = 0$$

Consider the controls

$$u_n(t) \doteq \frac{\sin(nt)}{n} \quad \text{so} \quad \dot{u}_n(t) = \cos(nt)$$

Therefore, for each $\tau > 0$,

$$x_n(\tau) = x_n(\tau) - x_n(0) = \int_0^\tau \cos(nt) dt + \int_0^\tau \cos^2(nt) dt =$$

Trivial model equation

$$\dot{x} = \dot{u} + (\dot{u})^2 \quad x(0) = 0$$

Consider the controls

$$u_n(t) \doteq \frac{\sin(nt)}{n} \quad \text{so} \quad \dot{u}_n(t) = \cos(nt)$$

Therefore, for each $\tau > 0$,

$$\begin{aligned} x_n(\tau) &= x_n(\tau) - x_n(0) = \int_0^\tau \cos(nt) dt + \int_0^\tau \cos^2(nt) dt = \\ &= \frac{1}{n} \int_0^{n\tau} \cos(s) ds + \frac{1}{n} \int_0^{n\tau} \cos^2(s) ds = \end{aligned}$$

Trivial model equation

$$\dot{x} = \dot{u} + (\dot{u})^2 \quad x(0) = 0$$

Consider the controls

$$u_n(t) \doteq \frac{\sin(nt)}{n} \quad \text{so} \quad \dot{u}_n(t) = \cos(nt)$$

Therefore, for each $\tau > 0$,

$$\begin{aligned} x_n(\tau) &= x_n(\tau) - x_n(0) = \int_0^\tau \cos(nt) dt + \int_0^\tau \cos^2(nt) dt = \\ &= \frac{1}{n} \int_0^{n\tau} \cos(s) ds + \frac{1}{n} \int_0^{n\tau} \cos^2(s) ds = \\ &= \frac{1}{n} \frac{n\tau}{2\pi} \int_0^{2\pi} \cos(s) ds + \frac{1}{n} \frac{n\tau}{2\pi} \int_0^{2\pi} \cos^2(s) ds + O(1/n) = \frac{\tau}{2} + O(1/n) \end{aligned}$$

Trivial model equation

$$\dot{x} = \dot{u} + (\dot{u})^2 \quad x(0) = 0$$

Consider the controls

$$u_n(t) \doteq \frac{\sin(nt)}{n} \quad \text{so} \quad \dot{u}_n(t) = \cos(nt)$$

Therefore, for each $\tau > 0$,

$$\begin{aligned} x_n(\tau) - x_n(0) &= \int_0^\tau \cos(nt) dt + \int_0^\tau \cos^2(nt) dt = \\ &= \frac{1}{n} \int_0^{n\tau} \cos(s) ds + \frac{1}{n} \int_0^{n\tau} \cos^2(s) ds = \\ &= \frac{1}{n} \frac{n\tau}{2\pi} \int_0^{2\pi} \cos(s) ds + \frac{1}{n} \frac{n\tau}{2\pi} \int_0^{2\pi} \cos^2(s) ds + O(1/n) = \frac{\tau}{2} + O(1/n) \end{aligned}$$

Hence

$$x_n(\tau) - x_n(0) \rightarrow \frac{\tau}{2}.$$

Notice: on each interval $[0, \tau]$, as n tends to infinity

Notice: on each interval $[0, \tau]$, as n tends to infinity

(1) the integral of the term \dot{u}_n cancels out.

Notice: on each interval $[0, \tau]$, as n tends to infinity

(1) the integral of the term \dot{u}_n cancels out.

(2) the term \dot{u}_n^2 has the same effect as the constant $\frac{1}{2}$, i.e. it produces the output $t \rightarrow \frac{t}{2}$

Recapitulation

BACK TO MECHANICS

- **assigning the control $u = u(t)$** is nothing but **adding the new holonomic constraint $u = u(t)$** ... to the original system.

- **assigning the control $u = u(t)$** is nothing but **adding the new holonomic constraint $u = u(t)$...** to the original system.
- As is well-known **in order to determine equations one needs to specify *how* the control constraint $u = u(t)$ is physically implemented :**

- **assigning the control $u = u(t)$ is nothing but adding the new holonomic constraint $u = u(t)$... to the original system.**
- **As is well-known in order to determine equations one needs to specify how the control constraint $u = u(t)$ is physically implemented :**
Here we assume **D'Alembert Hypothesis**, namely, *the work of constraint reactions is equal to zero.*

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a quadratic form in (\dot{q}, \dot{u}) ,

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

In particular \mathcal{H} is quadratic in the momenta (p, π)

$$\mathcal{H} = (p, \pi) \mathbf{A}^{-1} (p, \pi)^t \quad \mathbf{A}^{-1} = (a^{r,s})_{r,s=1,\dots,N+M}$$

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e., the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Now there is an isomorphism between the momentum (p, π) and the velocity (\dot{q}, \dot{u}) .

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Now there is an isomorphism between the momentum (p, π) and the velocity (\dot{q}, \dot{u}) . By partially inverting this isomorphism we obtain

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Now there is an isomorphism between the momentum (p, π) and the velocity (\dot{q}, \dot{u}) . By partially inverting this isomorphism we obtain

π as a linear combination of (p, \dot{u}) .

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Now there is an isomorphism between the momentum (p, π) and the velocity (\dot{q}, \dot{u}) . By partially inverting this isomorphism we obtain

π as a linear combination of (p, \dot{u}) .

Therefore, we get the control equations for (q, p) :

The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{\mathbf{q}}_i = \frac{\partial \mathcal{H}}{\partial \mathbf{p}_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{\mathbf{p}}_i = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Now there is an isomorphism between the momentum (p, π) and the velocity (\dot{q}, \dot{u}) . By partially inverting this isomorphism we obtain

π as a linear combination of (p, \dot{u}) .

Therefore, we get the control equations for (q, p) :

The equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^m g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

The equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^m g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

Notice:

The equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^m g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

Notice: Setting $x = (q, p)$ the equations take the same form as the control equations we have considered above, namely

The equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^m g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

Notice: Setting $x = (q, p)$ the equations take the same form as the control equations we have considered above, namely

$$\dot{x} = f(x) + \sum_{\alpha=1}^M g_{\alpha}(x) \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha\beta}(x) \dot{u}_{\alpha} \dot{u}_{\beta}$$

The quadratic term:

The quadratic term:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix} + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

The quadratic term:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix} + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \cdot \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix}$$

where

The quadratic term:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix} + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \cdot \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix} \quad \text{where}$$

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} \doteq \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

Remark:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

the matrix

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

Remark:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

the matrix

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

does not depend on q .

Stabilizability of mechanical systems

Theorem 3.

Let us assume that the force Γ is a conservative, and let $U = U(q, u)$ be a potential of Γ .

Let us fix \bar{u} , and let us assume that there exist positive real coefficients $\lambda_1, \dots, \lambda_k$ verifying $\sum_{r=1}^k \lambda_r = 1$, and vectors w_1, \dots, w_s such that *the effective potential*

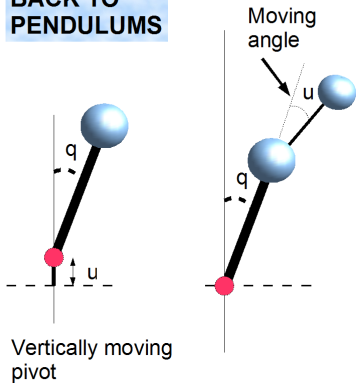
$$\hat{U}(q) \doteq U(\bar{u}, q) - \frac{1}{2} \sum_{r=1}^k \lambda_r \sum_{\alpha, \beta=1}^m e_{\alpha, \beta}(q, \bar{u}) w_r^\alpha w_r^\beta$$

has a strict minimum at $q = \bar{q}$.

Then, the control mechanical system can be stabilized to (\bar{q}, \bar{u}) .

- **Idea of the proof:** the choice of the λ_r, w_r selects from the corresponding differential inclusion $\dot{x} \in \mathcal{G}$ a conservative mechanical system with potential energy equal to \hat{U}
- If \bar{q} is unstable for the frozen control $u = \bar{u}$, a **necessary condition for stabilizability** is that **the matrix $e_{\alpha,\beta}$ be q -dependent.**

BACK TO PENDULUMS



Is the state $q=0$
stabilizable
by a vibrational u ?

Pendulum with moving pivot

$$\begin{cases} \dot{q} = p + (\sin q)\dot{u} \\ \dot{p} = -\frac{\partial U}{\partial q} - p \cos q \dot{u} - (\sin q \cos q)(\dot{u})^2, \end{cases}$$

where $U(q, c) \doteq \cos q$.

The kinetic matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -\sin q \\ -\sin q & 2 \end{pmatrix}$$

...so that the 1×1 matrix E is

$$E = e_{11} = 1 + \cos^2 q$$

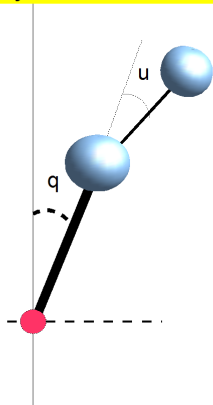
Notice: The necessary condition (i.e. " E depends on q ") is verified; Moreover: the **effective potential**

$$U_{\{1\}\{w\}} = \cos q - \frac{1}{2}(1 + (\cos q)^2)w^2.$$

has a minimum at $q = 0$ as soon as $w^2 \geq 1$. Therefore: The system is vibrationally stabilizable at 0.

Coda

Why NO stabilizability?



In order to give the answer one
need not even know the equation of motion...

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

the 1×1 matrix E is equal to 1...

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

the 1×1 matrix E is equal to 1...

In particular,

E is independent of q !

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

the 1×1 matrix E is equal to 1...

In particular,

E is independent of q !

i.e. the necessary condition for vibrational stabilizability is not satisfied

Indeed the kinetic matrix is

$$\mathbf{A} = \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

the 1×1 matrix E is equal to 1...

In particular,

E is independent of q !

i.e. the necessary condition for vibrational stabilizability is not satisfied

in other words the control equations are affine in \dot{u} .

SECOND MOVEMENT

—*Adagio*—

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

$$(h_{\alpha,\beta} = 0)$$

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

($h_{\alpha,\beta} = 0$) These systems are called "impulsive", because **one might want to implement discontinuous controls u .**

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{\mathbf{u}}^{\alpha}$$

($h_{\alpha,\beta} = 0$) These systems are called "impulsive", because **one might want to implement discontinuous controls \mathbf{u}** . (This is natural in optimal control problems like

minimize $\psi(\mathbf{x}(\mathbf{T}))$,

because minimizing sequences u_n could exist with larger and larger derivatives $\dot{\mathbf{u}}_n$)

FACTS:

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{\mathbf{u}}^{\alpha}$$

($h_{\alpha,\beta} = 0$) These systems are called "impulsive", because **one might want to implement discontinuous controls \mathbf{u}** . (This is natural in optimal control problems like

minimize $\psi(\mathbf{x}(\mathbf{T}))$,

because minimizing sequences u_n could exist with larger and larger derivatives $\dot{\mathbf{u}}_n$)

FACTS:

(i) **A distributional approach does not work.**

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{\mathbf{u}}^{\alpha}$$

($h_{\alpha,\beta} = 0$) These systems are called "impulsive", because **one might want to implement discontinuous controls \mathbf{u}** . (This is natural in optimal control problems like

$$\text{minimize } \psi(\mathbf{x}(\mathbf{T})),$$

because minimizing sequences u_n could exist with larger and larger derivatives $\dot{\mathbf{u}}_n$)

FACTS:

- (i) **A distributional approach does not work.**
- (ii) **A robust definition of solution can be given by a density argument (on the graphs)**

The affine case: IMPULSIVE SYSTEMS

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

($h_{\alpha,\beta} = 0$) These systems are called "impulsive", because **one might want to implement discontinuous controls u** . (This is natural in optimal control problems like

minimize $\psi(\mathbf{x}(T))$,

because minimizing sequences u_n could exist with larger and larger derivatives \dot{u}_n)

FACTS:

- (i) **A distributional approach does not work.**
- (ii) **A robust definition of solution can be given by a density argument (on the graphs)**
- (iii) **Such a definition allows for jumps of u .**

(i) A distributional approach does not work.

WHY?

(i) A distributional approach does not work.

WHY?

Because

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation...

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation... **More intuitively:**

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation... **More intuitively**: consider the simple equation

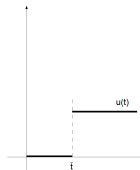
$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{x}(\mathbf{t})\dot{\mathbf{u}}(\mathbf{t})$$

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation... **More intuitively**: consider the simple equation

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{x}(\mathbf{t})\dot{\mathbf{u}}(\mathbf{t})$$



and take \mathbf{u} to be the Heaveside function, measure $\delta_{\bar{\mathbf{t}}}$.

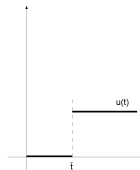
so $\dot{\mathbf{u}}$ is the Dirac

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation... **More intuitively**: consider the simple equation

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{x}(\mathbf{t})\dot{\mathbf{u}}(\mathbf{t})$$



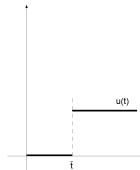
and take \mathbf{u} to be the Heaveside function, so $\dot{\mathbf{u}}$ is the Dirac measure $\delta_{\bar{t}}$. Which value should we assign to x at the jump time \bar{t} ?...

(i) A distributional approach does not work.

WHY?

Because a distributional (=in measure) approach cannot work in a nonlinear equation... **More intuitively**: consider the simple equation

$$\dot{x}(t) = x(t)u(t)$$



and take u to be the Heaviside function, so \dot{u} is the Dirac measure $\delta_{\bar{t}}$. Which value should we assign to x at the jump time \bar{t} ?...**Simply, there is no reasonable a priori choice**

(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

(a) Approximate a L^1 control u by means of smooth u_n ;

(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;

(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;

(ii) A robust definition of solution can be given by a density argument:

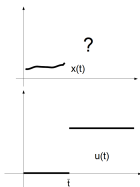
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;
- (d) Define x as the solution corresponding to u ;

(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

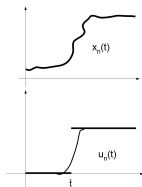
- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;
- (d) Define x as the solution corresponding to u ;



(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

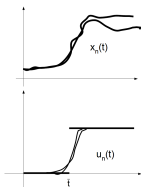
- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;
- (d) Define x as the solution corresponding to u ;



(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

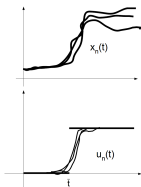
- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;
- (d) Define x as the solution corresponding to u ;



(ii) A robust definition of solution can be given by a density argument:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

- (a) Approximate a L^1 control u by means of smooth u_n ;
- (b) Consider the solutions x_n corresponding to u_n ;
- (c) Let $n \rightarrow \infty$ and verify that x_n tends to a unique function x ;
- (d) Define x as the solution corresponding to u ;



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{\mathbf{u}}^{\alpha}$$

IT WORKS!

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{\mathbf{u}}^{\alpha}$$

**IT WORKS!
BUT**

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

**IT WORKS!
BUT
IT WORKS ONLY FOR SCALAR u**

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

**IT WORKS!
BUT
IT WORKS ONLY FOR SCALAR u**

More precisely:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

THE REASON FOR THIS IS:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

THE REASON FOR THIS IS: **NON COMMUTATIVITY MATTERS:**

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

THE REASON FOR THIS IS: **NON COMMUTATIVITY MATTERS:**

$$[\mathbf{g}_{\alpha}, \mathbf{g}_{\beta}] \doteq \mathbf{D}\mathbf{g}_{\beta} \cdot \mathbf{g}_{\alpha} - \mathbf{D}\mathbf{g}_{\alpha} \cdot \mathbf{g}_{\beta} \neq \mathbf{0} \quad \alpha \neq \beta$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

IT WORKS!

BUT

IT WORKS ONLY FOR SCALAR u

More precisely: for **vector-valued controls** $u \in \mathcal{R}^m$ the limit of the x_n depends on the sequence u_n approximating u

THE REASON FOR THIS IS: **NON COMMUTATIVITY MATTERS:**

$$[\mathbf{g}_{\alpha}, \mathbf{g}_{\beta}] \doteq \mathbf{D}\mathbf{g}_{\beta} \cdot \mathbf{g}_{\alpha} - \mathbf{D}\mathbf{g}_{\alpha} \cdot \mathbf{g}_{\beta} \neq \mathbf{0} \quad \alpha \neq \beta$$

($[\cdot, \cdot]$ is the *LIE BRACKET*).

Vector-valued impulses

If one uses (possibly discontinuous) controls with bounded variation the situation nowadays is quite clear:

Vector-valued impulses

If one uses (possibly discontinuous) controls with bounded variation the situation nowadays is quite clear:

- To have a well-defined solution besides u one has to give a graph completion of u

Vector-valued impulses

If one uses (possibly discontinuous) controls with bounded variation the situation nowadays is quite clear:

- To have a well-defined solution besides u one has to give a **graph completion** of u
- Whenever $[g_\alpha, g_\beta](x) \neq 0$,

Vector-valued impulses

If one uses (possibly discontinuous) controls with bounded variation the situation nowadays is quite clear:

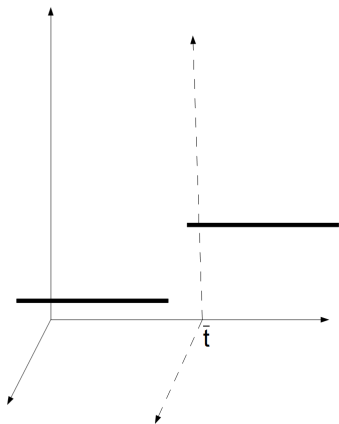
- To have a well-defined solution besides u one has to give a **graph completion** of u
- Whenever $[g_\alpha, g_\beta](x) \neq 0$, it is essential WHICH **graph completion** we choose to approximate.

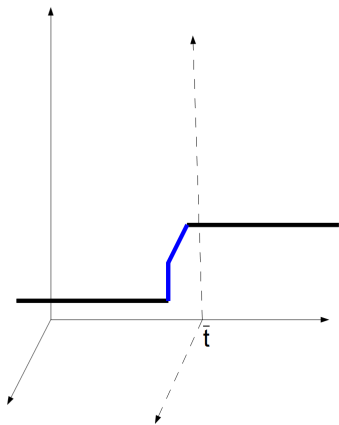
Vector-valued impulses

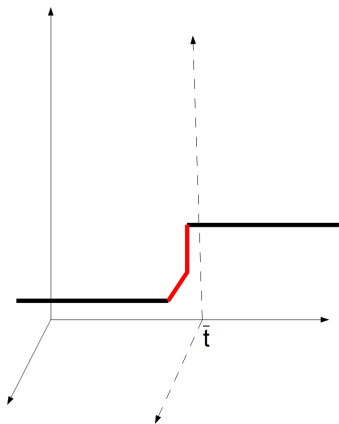
If one uses (possibly discontinuous) controls with bounded variation the situation nowadays is quite clear:

- To have a well-defined solution besides u one has to give a **graph completion** of u
- Whenever $[g_\alpha, g_\beta](x) \neq 0$, it is essential WHICH **graph completion** we choose to approximate.

Roughly speaking: at a jump of u one has to assign the *trajectory of the jump*







NOTICE:

NOTICE:

All graph completions are equivalent if and only if the vector fields g_α commute, namely:

NOTICE:

All graph completions are equivalent if and only if the vector fields g_α commute, namely:

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{0} \quad \forall \alpha, \beta$$

NOTICE:

All graph completions are equivalent if and only if the vector fields g_α commute, namely:

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{0} \quad \forall \alpha, \beta$$

(in particular this is true in the scalar case: $[\mathbf{g}, \mathbf{g}] = \mathbf{0}$)

NOTICE:

All graph completions are equivalent if and only if the vector fields g_α commute, namely:

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{0} \quad \forall \alpha, \beta$$

(in particular this is true in the scalar case: $[\mathbf{g}, \mathbf{g}] = \mathbf{0}$)

The deep geometrical reason for this is the
Simultaneous Flow-box Theorem:

NOTICE:

All graph completions are equivalent if and only if the vector fields g_α commute, namely:

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{0} \quad \forall \alpha, \beta$$

(in particular this is true in the scalar case: $[\mathbf{g}, \mathbf{g}] = \mathbf{0}$)

The deep geometrical reason for this is the

Simultaneous Flow-box Theorem:

All vector fields g_α commute

IF AND ONLY IF

*there exists a system of coordinates where they all are **constant***

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

if a control $\tilde{u}(\cdot)$ oscillates near a constant value \bar{u} ,

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

*if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.*

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

*if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.*

Sounds familiar?

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x})\dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

*if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.*

Sounds familiar?

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

*if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.*

Sounds familiar?

In any case, do not forget it,

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.

Sounds familiar?

In any case, do not forget it, thinking in particular to the non-stabilizable double pendulum

An important property of affine systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=1}^m \mathbf{g}_{\alpha}(\mathbf{x}) \dot{u}^{\alpha}$$

Forget discontinuous u , and take only continuous ones (with bounded variation)

The input-output map

$$\phi : u(\cdot) \rightarrow \Phi(u) = x(\cdot)$$

is continuous e.g. with respect to C^0 topologies!

For instance:

if a control $\tilde{u}(\cdot)$ oscillates **near** a constant value \bar{u} ,
then the outputs $\tilde{x}(\cdot) = \Phi(\tilde{u}(\cdot))$ and $\bar{x}(\cdot) = \Phi(\bar{u})$ are **close**.

Sounds familiar?

In any case, do not forget it, thinking in particular to the
non-stabilizable double pendulum ... wait for the *Third Movement*

An incomplete list of authors who have given contributions on the subject of impulsive system:

An incomplete list of authors who have given contributions on the subject of impulsive system:

H. Sussmann, A. Bressan, A.Bressan-F.R, M.Motta- F.R., B. Miller, Dal Maso-F.R., C. Sartori-F.R., J. Dorroh-G.Ferreyra, A. Sarychev, R. Vinter-G. Silva, F.Lobo Pereira, P. Mason, P. Wolenski-S.Zabic, H. Zidani...

THIRD MOVEMENT (*Scherzo*)

—*Con moto*—

Back to Mechanics

Back to Mechanics

As before, we consider a mechanical systems where the last M coordinates $(x^{N+1}, \dots, x^{N+M}) = (u^1, \dots, u^M)$ are regarded as a control.

Back to Mechanics

As before, we consider a mechanical systems where the last M coordinates $(x^{N+1}, \dots, x^{N+M}) = (u^1, \dots, u^M)$ are regarded as a control.

QUESTION:

Back to Mechanics

As before, we consider a mechanical systems where the last M coordinates $(x^{N+1}, \dots, x^{N+M}) = (u^1, \dots, u^M)$ are regarded as a control.

QUESTION:

*Is the fact that the control system is **affine in \dot{u}** —generally it is **quadratic**— related to some differential geometric property?*

Remember:

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a quadratic form in (\dot{q}, \dot{u}) ,

Remember:

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

Remember:

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

Remember:

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

In particular \mathcal{H} is quadratic in the momenta (p, π)

$$\mathcal{H} = (p, \pi) \mathbf{A}^{-1} (p, \pi)^t \quad \mathbf{A}^{-1} = (a^{r,s})_{r,s=1,\dots,N+M}$$

Remember: The Hamiltonian equations of motion before the imposition of the holonomic constraint (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Remember: the equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix} + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \cdot \\ \cdot \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix}$$

where

$$\begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 \\ F^u(\cdot) \end{pmatrix} + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \vdots \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix} \quad \text{where}$$

$$\begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

(=inverse of the lower right $M \times M$ minor of \mathbf{A}^{-1})

Remember:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

THE MATRIX

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

IS INDEPENDENT OF q .

Remember:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

THE MATRIX

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

IS INDEPENDENT OF q .

The question now is:

Remember:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

THE MATRIX

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

IS INDEPENDENT OF q .

The question now is:

Is this property a "geometric property"?

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

(1) The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

- (1) The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero
IF AND ONLY IF

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

- (1) The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero
IF AND ONLY IF
- (2) Geodesics orthogonal to one leaf $\{u = \text{constant}\}$ are orthogonal to all leaves they meet.

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

(1) The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

(2) Geodesics orthogonal to one leaf $\{u = \text{constant}\}$ are orthogonal to all leaves they meet.

IF AND ONLY IF

SOME GEOMETRY RELATED TO THE QUADRATIC DEPENDENCE ON \dot{u} :

THEOREM

(1) The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

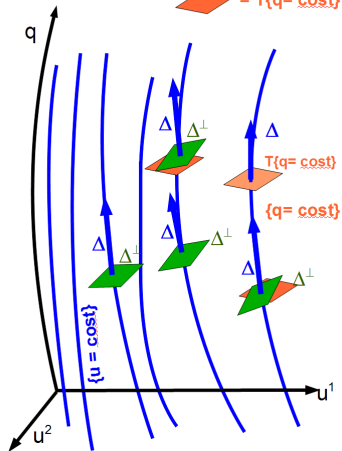
IF AND ONLY IF

(2) Geodesics orthogonal to one leaf $\{u = \text{constant}\}$ are orthogonal to all leaves they meet.

IF AND ONLY IF

(3) Any pair of leaves $\{u = \text{constant}\}$ have keep the same distance at every point

$\uparrow = \Delta = T\{u = \text{cost}\}$ $\text{green plane} = \Delta^\perp = \text{orth. bundle}$
 $\text{orange plane} = T\{q = \text{cost}\}$



The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

MEASURES

The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

MEASURES **how much geodesics which are orthogonal to**
 $\Lambda = \{u = \text{constant}\}$ **at a point**

The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

MEASURES **how much geodesics which are orthogonal to $\Lambda = \{u = \text{constant}\}$ at a point FAIL to remain orthogonal at the other points.**

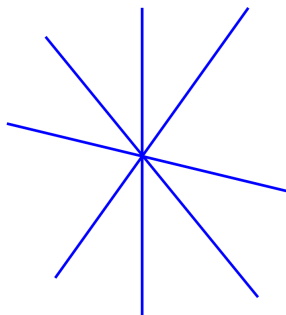
The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

MEASURES **how much geodesics which are orthogonal to $\Lambda = \{u = \text{constant}\}$ at a point FAIL to remain orthogonal at the other points.**

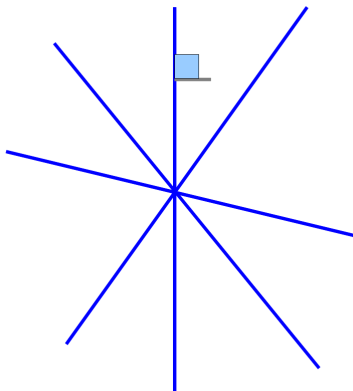
The orthogonal curvature $\frac{\partial e_{\alpha,\beta}}{\partial q^i}$ is a *tensor* with respect to the coordinate transformations that respect the foliation structure.

EXAMPLES



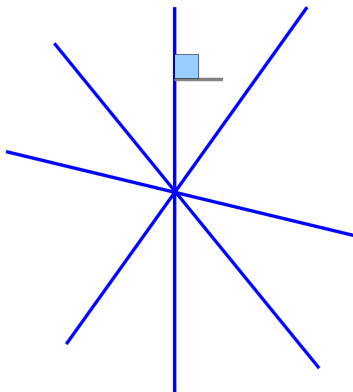
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



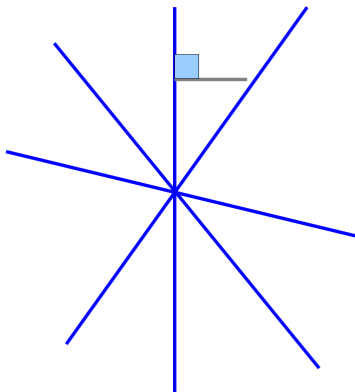
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



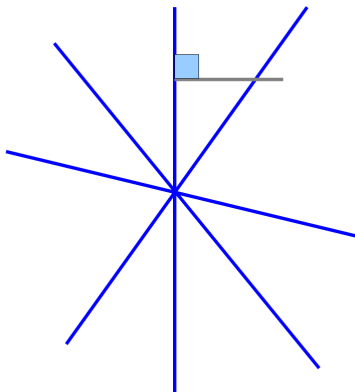
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



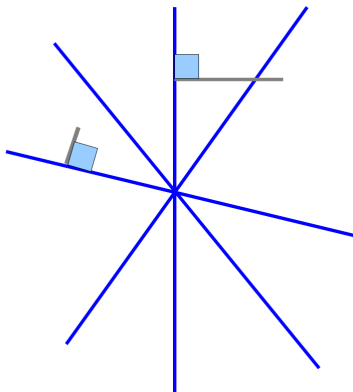
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



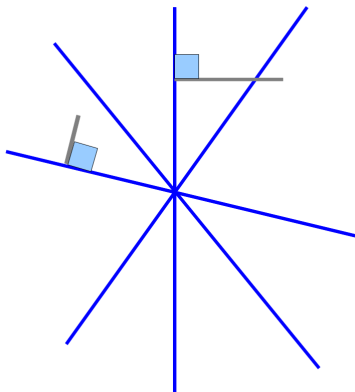
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



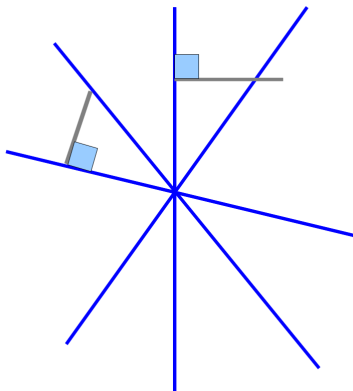
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



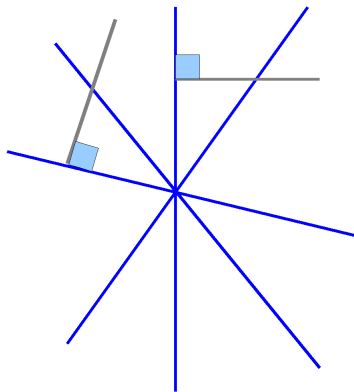
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



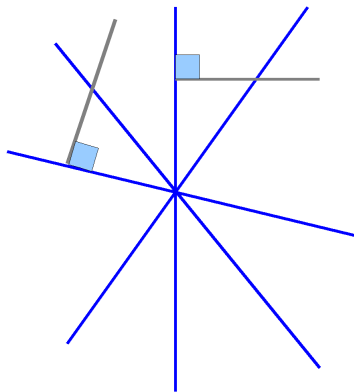
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



Non-Riemannian Foliation: geodesics lose orthogonality

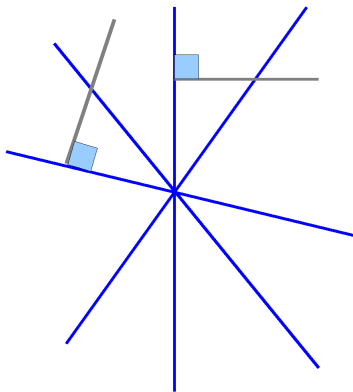
The angle as control:



Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:
quadratic part!

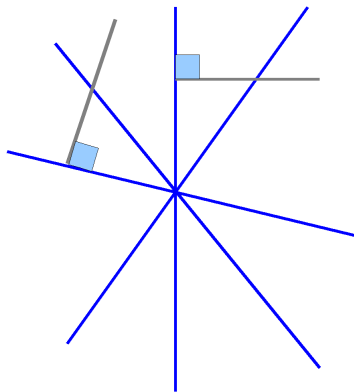
Non-zero



Non-Riemannian Foliation: geodesics lose orthogonality

Non-zero

The angle as control:
quadratic part!
Hence,

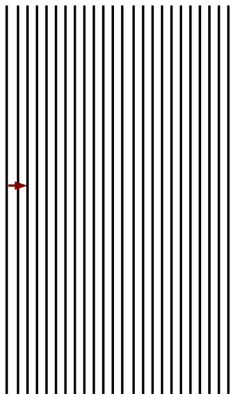


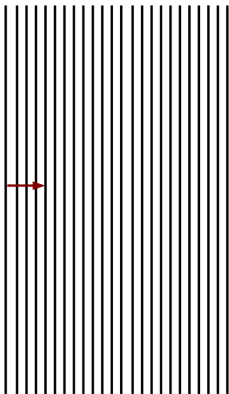
Non-Riemannian Foliation: geodesics lose orthogonality

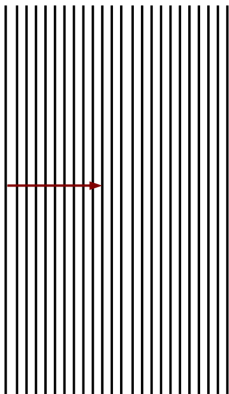
The angle as control:
quadratic part!

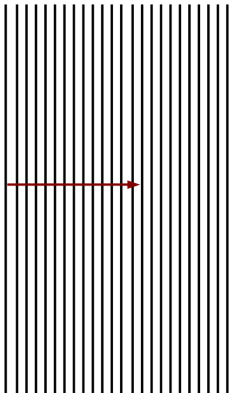
Hence, chance of vibrational stabilization!

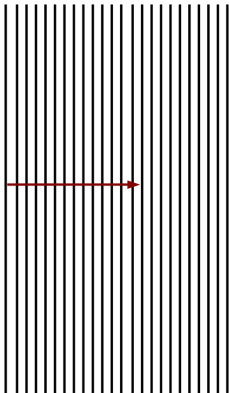
Non-zero



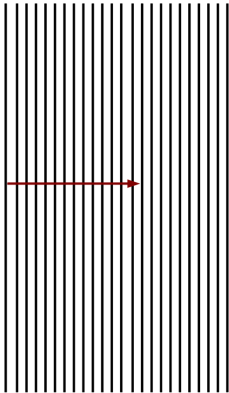






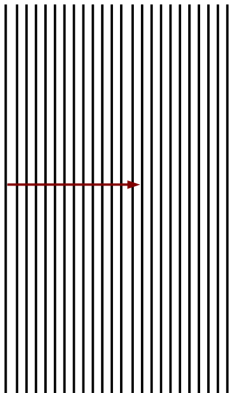


No quadratic part!



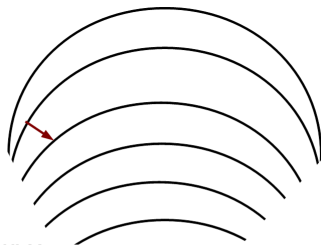
Hence,

No quadratic part!

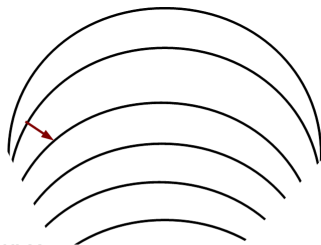


No quadratic part!

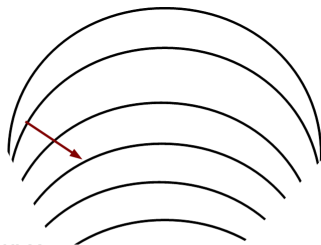
Hence, no vibrational stabilization!



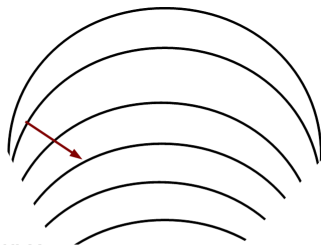
Pendulum with oscillating pivot



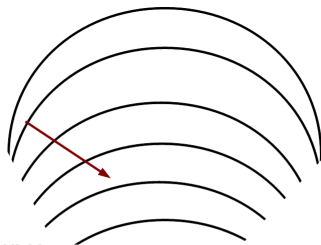
Pendulum with oscillating pivot



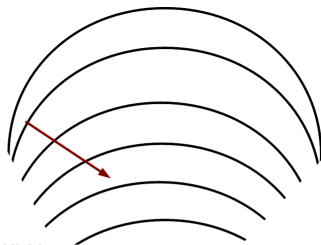
Pendulum with oscillating pivot



Pendulum with oscillating pivot

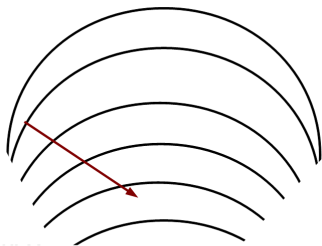


Pendulum with oscillating pivot

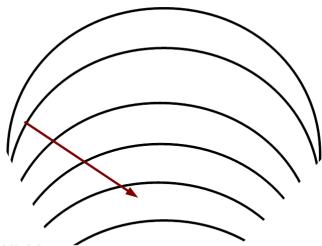


Pendulum with oscillating pivot

Pendulum with oscillating pivot

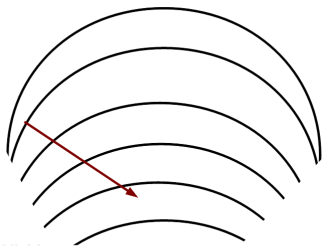


Pendulum with oscillating pivot



Non-zero quadratic part!

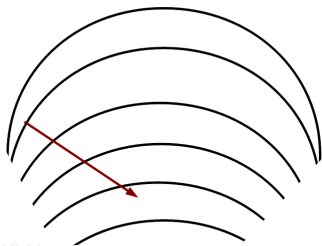
Pendulum with oscillating pivot



Hence,

Non-zero quadratic part!

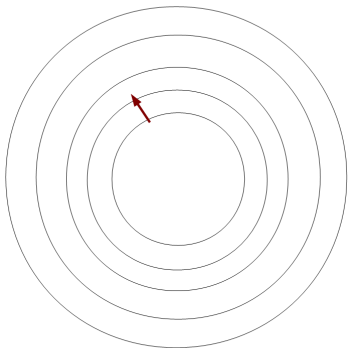
Pendulum with oscillating pivot



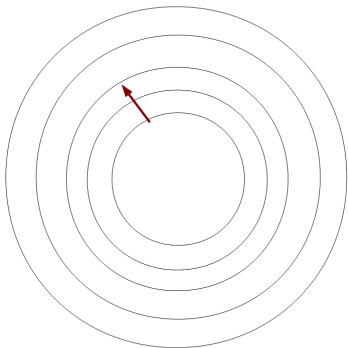
Non-zero quadratic part!

Hence, chance of vibrational stabilization!

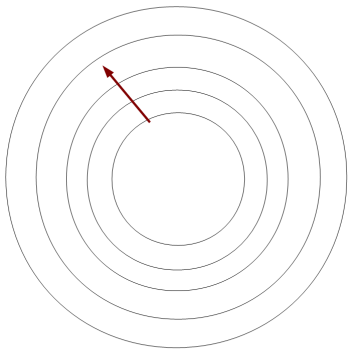
The length of the pendulum as control



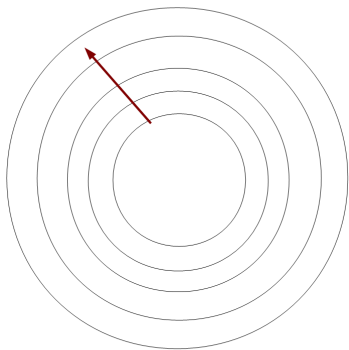
The length of the pendulum as control



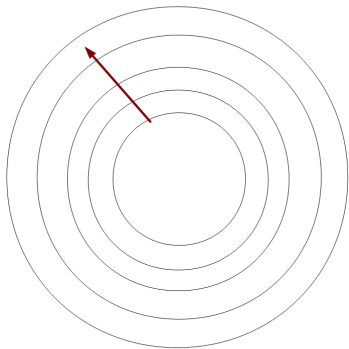
The length of the pendulum as control



The length of the pendulum as control

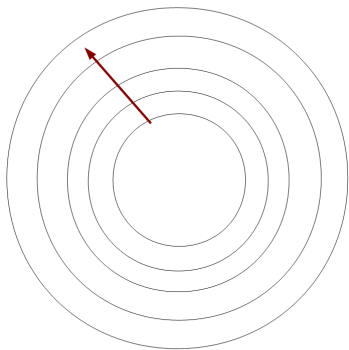


The length of the pendulum as control



No quadratic part!

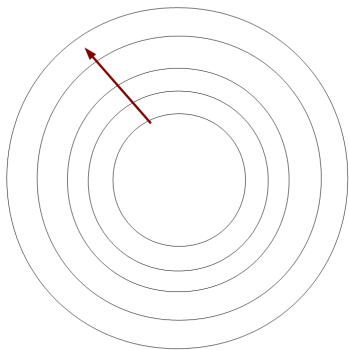
The length of the pendulum as control



Hence,

No quadratic part!

The length of the pendulum as control

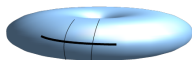


No quadratic part!

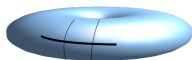
Hence, no vibrational stabilization!

Something similar happens for the double pendulum:

Something similar happens for the double pendulum:

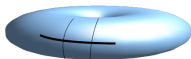


Something similar happens for the double pendulum:



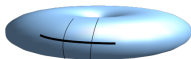
Geodesics keep orthogonality to leaves.

Something similar happens for the double pendulum:



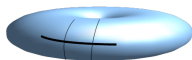
Geodesics keep orthogonality to leaves. Hence no quadratic part!

Something similar happens for the double pendulum:



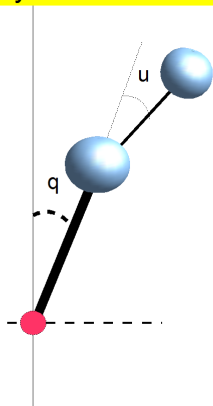
Geodesics keep orthogonality to leaves. Hence no quadratic part! Hence,

Something similar happens for the double pendulum:



Geodesics keep orthogonality to leaves. Hence no quadratic part! Hence, no vibrational stabilization!

Why NO stabilizability?



FOURTH MOVEMENT (*Finale*)

—*Allegro molto*—

The title of this tutorial contain the expression
Non holonomic

The title of this tutorial contain the expression
Non holonomic

BUT

The title of this tutorial contain the expression
Non holonomic

BUT

UP TO NOW EVERYTHING HAS BEEN HOLONOMIC...

The title of this tutorial contain the expression
Non holonomic

BUT

UP TO NOW EVERYTHING HAS BEEN HOLONOMIC...

So, this Final is devoted to "Non holonomic constraints"

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

A *non holonomic constraint* is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation by a constraint on configuration q .

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

A *non holonomic constraint* is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation by a constraint on configuration q .

EQUIVALENTLY:

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

A *non holonomic constraint* is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation by a constraint on configuration q .

EQUIVALENTLY:

At each point a subspace $\Delta(q) \subset T_q\mathcal{Q}$ is given, and

$$\dot{q} \in \Delta(q)$$

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

A *non holonomic constraint* is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation by a constraint on configuration q .

EQUIVALENTLY:

At each point a subspace $\Delta(q) \subset T_q\mathcal{Q}$ is given, and

$$\dot{q} \in \Delta(q)$$

but $\Delta(q)$ is **not integrable**

We assume our mechanical system, locally described by coordinates (q^1, \dots, q^{N+M}) , is subject to a **non holonomic** constraint.

WHAT DOES THIS MEAN?

A *non holonomic constraint* is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation by a constraint on configuration q .

EQUIVALENTLY:

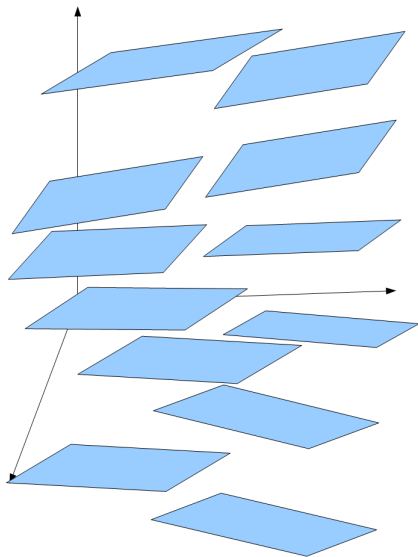
At each point a subspace $\Delta(q) \subset T_q\mathcal{Q}$ is given, and

$$\dot{q} \in \Delta(q)$$

but $\Delta(q)$ is **not integrable**

i.e., there is no *foliation* of \mathcal{Q} whose leaves have $\Delta(q)$ as tangent space at any q .

EXAMPLE in \mathcal{R}^3



ANOTHER EXAMPLE in \mathcal{R}^3

$$\Delta = \text{span} \{f_1, f_2\}$$

where

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}$$

("Brockett example").

ANOTHER EXAMPLE in \mathcal{R}^3

$$\Delta = \text{span} \{f_1, f_2\}$$

where

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}$$

("Brockett example").

Δ is not integrable

ANOTHER EXAMPLE in \mathcal{R}^3

$$\Delta = \text{span} \{f_1, f_2\}$$

where

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}$$

("Brockett example").

Δ is not integrable because it is *not involutive*,

ANOTHER EXAMPLE in \mathcal{R}^3

$$\Delta = \text{span} \{f_1, f_2\}$$

where

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}$$

("Brockett example").

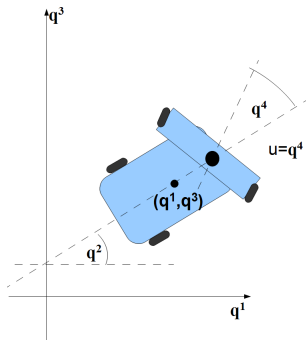
Δ is not integrable because it is *not involutive*,
i.e the Lie Bracket

$$[f_1, f_2]$$

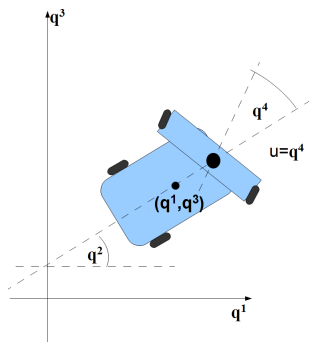
does not belong to Δ .

A mechanical example: The Roller Racer

A mechanical example: The Roller Racer

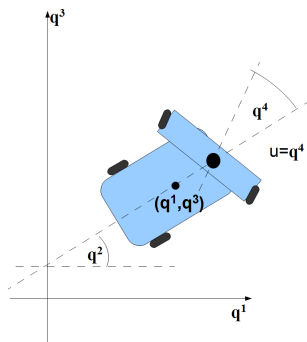


A mechanical example: The Roller Racer



The fact that the velocity of the first body must be directed as the angle q^2 and the analog fact for the velocity of the second body,

A mechanical example: The Roller Racer



The fact that the velocity of the first body must be directed as the angle q^2 and the analog fact for the velocity of the second body, is **non holonomic constraint** on the system.



If the Roller Racer were on a icy surface, it would be not subject to

If the Roller Racer were on a icy surface, it would be not subject to the holonomic constraints,

If the Roller Racer were on a icy surface, it would be not subject to the holonomic constraints, and the control u would be a "shape" control (as in the double pendulum.)

If the Roller Racer were on a icy surface, it would be not subject to the holonomic constraints, and the control u would be a "shape" control (as in the double pendulum.) This would mean

$$\text{orthogonal curvature} = 0$$

If the Roller Racer were on a icy surface, it would be not subject to the holonomic constraints, and the control u would be a "shape" control (as in the double pendulum.) This would mean

$$\text{orthogonal curvature} = 0$$

i.e. the control system would be affine in \dot{u} .

If the Roller Racer were on a icy surface, it would be not subject to the holonomic constraints, and the control u would be a "shape" control (as in the double pendulum.) This would mean

$$\text{orthogonal curvature} = 0$$

i.e. the control system would be affine in \dot{u} .

In particular,

no forces would be produced by rapid small oscillations of u .

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

So one could conjecture that

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

So one could conjecture that

*"the system is affine in \dot{u}
if and only if
this was true before the imposition of the non holonomic
constraint."*

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

So one could conjecture that

*"the system is affine in \dot{u}
if and only if
this was true before the imposition of the non holonomic
constraint."*

IN FACT, **THIS IS WRONG.**

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

So one could conjecture that

*"the system is affine in \dot{u}
if and only if
this was true before the imposition of the non holonomic
constraint."*

IN FACT, **THIS IS WRONG.**

The non holonomic constraint, strikingly, **adds quadratic terms (to \dot{u}) in the equations.**

OBSERVE: One could think that the imposition of the non holonomic constraint $\dot{q} \in \Delta(q)$ is just a fact of **linear projection of the equations on Δ** .

So one could conjecture that

*"the system is affine in \dot{u}
if and only if
this was true before the imposition of the non holonomic
constraint."*

IN FACT, **THIS IS WRONG.**

The non holonomic constraint, strikingly, **adds quadratic terms (to \dot{u}) in the equations.**

Physically one can say the non holonomic constraint produces a *centrifugal effect*

The control equations for the Roller Racer (on a surface with friction):

$$\left\{ \begin{array}{l} \dot{q}^1 = 2\rho \cos u \sin q^2 \cdot \xi - \frac{J\rho \sin q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{q}^2 = 2 \sin u \cdot \xi - \frac{J \sin^2 u}{\Delta_0} \cdot \dot{u} \\ \dot{q}^3 = 2\rho \cos q^2 \cos u \cdot \xi - \frac{J\rho \cos q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{\xi} = -\sin 2u \left(\frac{(I+J-\rho^2)}{\Delta_1} + \frac{1}{2(\rho^2/\Delta_4 + \sin^2 u)} \right) \cdot \xi \dot{u} - \frac{2J\rho^2 \cos u}{\Delta_1^2} \cdot \dot{u}^2. \end{array} \right.$$

The control equations for the Roller Racer (on a surface with friction):

$$\left\{ \begin{array}{l} \dot{q}^1 = 2\rho \cos u \sin q^2 \cdot \xi - \frac{J\rho \sin q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{q}^2 = 2 \sin u \cdot \xi - \frac{J \sin^2 u}{\Delta_0} \cdot \dot{u} \\ \dot{q}^3 = 2\rho \cos q^2 \cos u \cdot \xi - \frac{J\rho \cos q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{\xi} = -\sin 2u \left(\frac{(I+J-\rho^2)}{\Delta_1} + \frac{1}{2(\rho^2/\Delta_4 + \sin^2 u)} \right) \cdot \xi \dot{u} - \frac{2J\rho^2 \cos u}{\Delta_1^2} \cdot \dot{u}^2. \end{array} \right.$$

RollerRacers' race ...



RollerRacers' race ...

GRAZIE

GRAZIE

PER L'ATTENZIONE