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Tutorial on Control of Non Holonomic Systems by Active Constraints

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Franco Rampazzo. Tutorial on Control of Non Holonomic Systems by Active Constraints. SADCO Summer School 2011 - Optimal Control, Sep 2011, London, United Kingdom. inria-00629512

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CONTROL OF NON HOLONOMIC SYSTEMS BY ACTIVE CONSTRAINTS

Franco Rampazzo,
University of Padova, Italy

SADCO Summer School
Imperial College , London, September 5-9, 2011,

FIRST MOVEMENT

—*Allegro*—

Introduction

INVESTIGATE **analysis** AND **geometry** RELATED TO
THE FOLLOWING PROGRAM:

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$$u(t) = (u^1, \dots, u^M) \equiv (q^{N+1}(t), \dots, q^{N+M}(t))$$

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PROBLEM:

What can we say on the *whole* motion $q = q(t)$?

Standard goals:

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OPTIMIZATION

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CONTROLLABILITY

Standard goals:

OPTIMIZATION

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STABILIZABILITY

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mostly for \mathcal{NC} (=non-centrifugal) systems

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Mostly for \mathcal{C} (=centrifugal) systems

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Which is a local way of imposing moving constraints as controls.

We assume that d'Alembert principle is verified, i.e., reaction forces are "orthogonal" to the reduced system

Let us begin with

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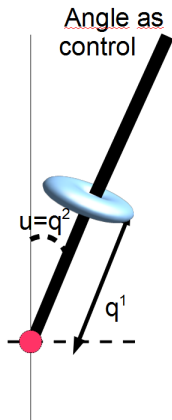
SOME EXAMPLES of MECHANICAL SYSTEMS:

THREE CLASS \mathcal{C} EXAMPLES:

THREE **CLASS C** EXAMPLES:

(*C* stands for *centrifugal*)

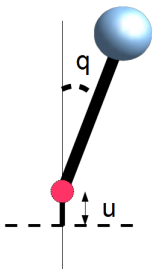
(1C) The angle as control



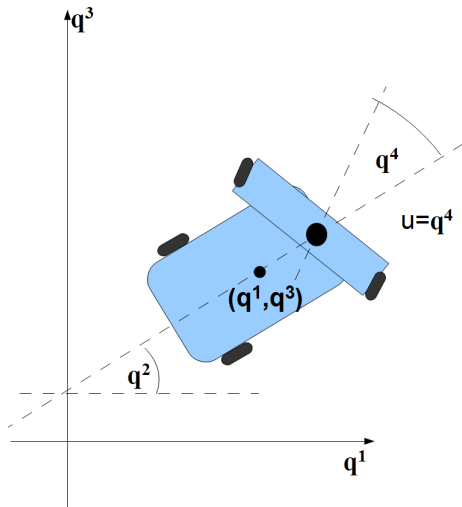
The control is
the angle.

(2C) The pendulum with oscillating pivot

Vertically moving
pivot



(3C) The "Roller Racer"



(The Roller Racer is a well-known toy):



NOW, THREE **CLASS** \mathcal{NC} EXAMPLES:

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(\mathcal{NC} stands for *non centrifugal*)

(1 $\mathcal{N}\mathcal{C}$) The pendulum with length as control

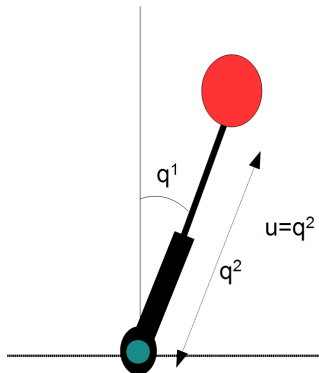
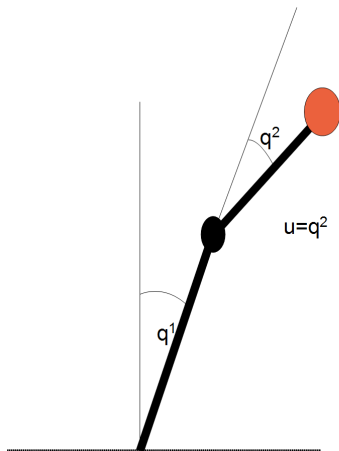
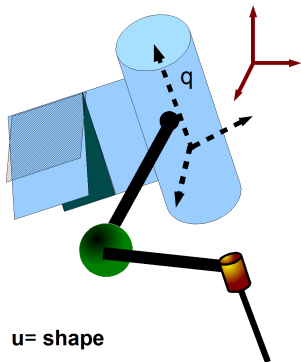


Figure: Length as control

(2 $\mathcal{N}C$) The pendulum with a second pendulum as control

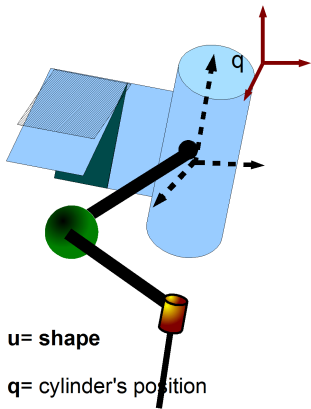


(3NC) "Shape" as control

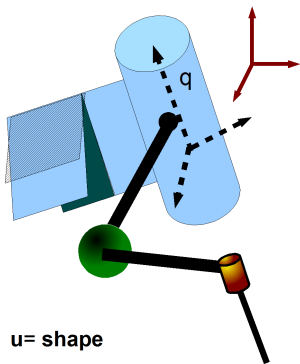


u = shape

q = cylinder's position

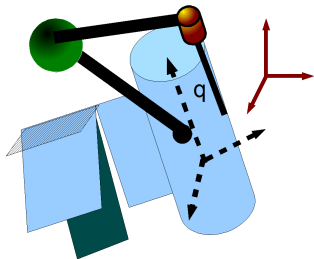


A rigid movement (the shape u is unchanged)



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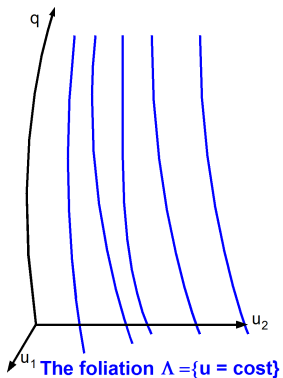
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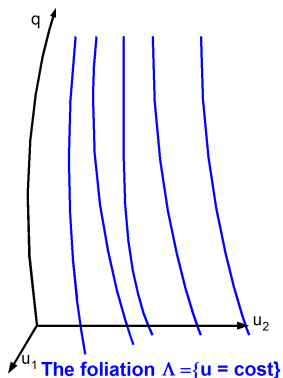
Change of shape (i.e. change of u)

Exposition

The abstract framework:

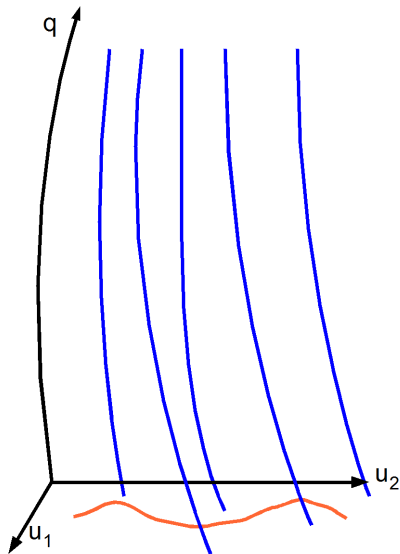


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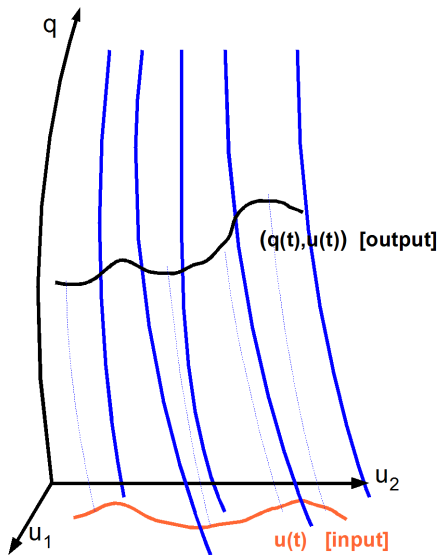


Given $u = u(t)$, we are interested in the corresponding motion $q(t)$ along the leaves of Λ .

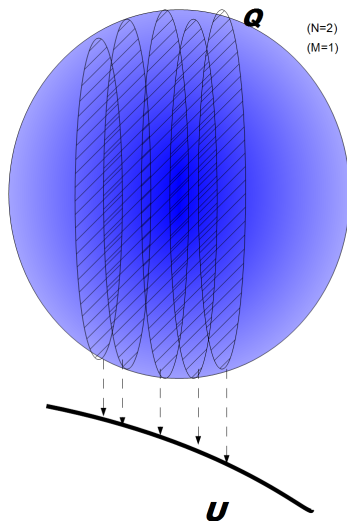
2-dimensional controls, 1-dimensional leaves



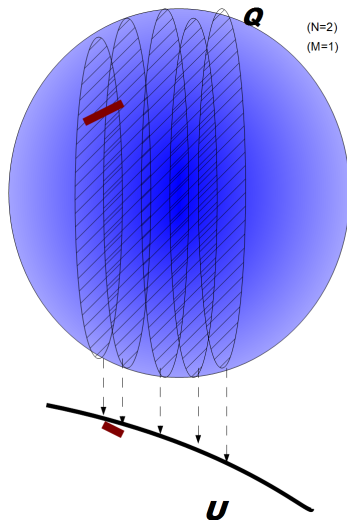
2-dimensional controls, 1-dimensional leaves



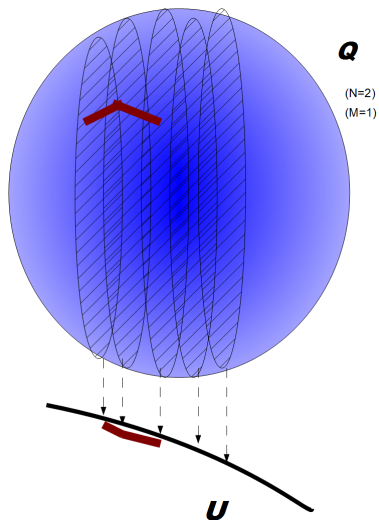
1-dimensional controls, 2-dimensional leaves



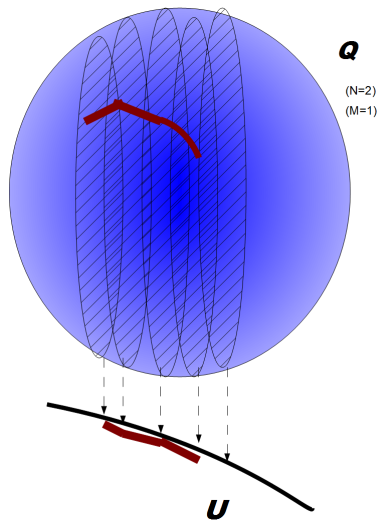
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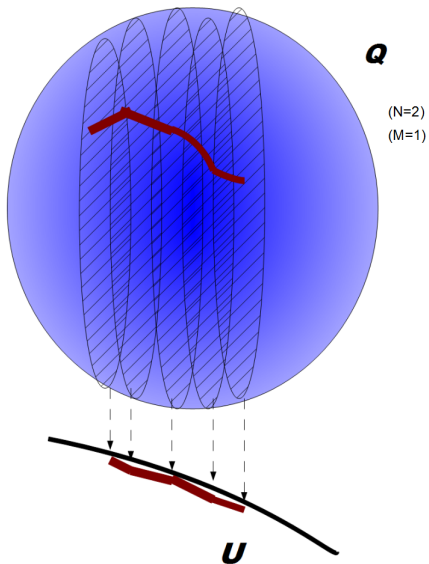
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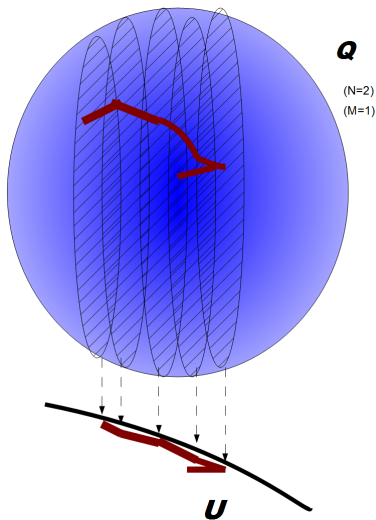
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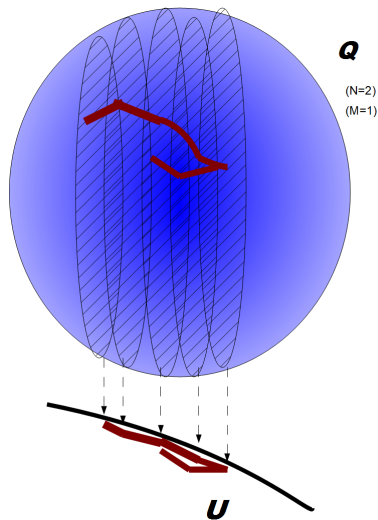
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1-dimensional controls, 2-dimensional leaves



We shall focus on TWO MAIN KINDS OF CONTROLS u :

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(1) RAPIDLY OSCILLATING CONTROLS

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(1) RAPIDLY OSCILLATING CONTROLS

(2) BOUNDED VARIATION (POSSIBLY DISCONTINUOUS)
CONTROLS

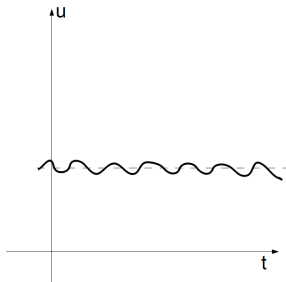


Figure: Rapidly oscillating controls

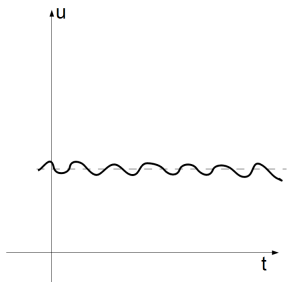


Figure: Rapidly oscillating controls

For instance:

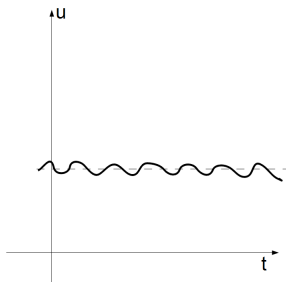


Figure: Rapidly oscillating controls

For instance: $\epsilon \sin(\mathbf{t}/\epsilon)$

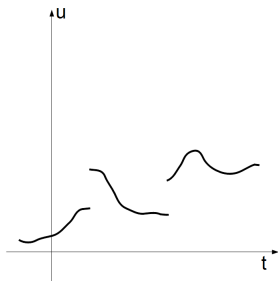


Figure: Control with bounded variation

Rapidly oscillating controls

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THREE CASES where **RAPID OSCILLATIONS OF THE CONTROL COORDINATES**

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PRODUCE a "FORCE"

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NOW GUESS:

Which of the previous examples do the job? The \mathcal{C} (=centrifugal) or the \mathcal{NC} (=non-centrifugal) systems?

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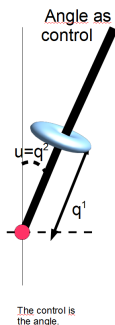


Figure: Oscillations of the angle generate a (centrifugal!) force on the sliding ring

(2 C) The pendulum with oscillating pivot

Vertically moving
pivot

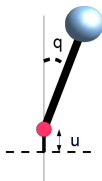


Figure: Oscillations of the pivot stabilize the unstable equilibrium
(*Kapiza pendulum*)

(3C) The "Roller Racer"

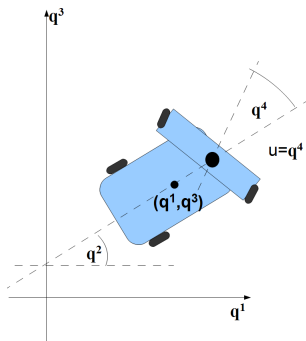


Figure: Oscillations of handlebar generate forward motion

Well,

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YES,

in the above three examples,

OSCILLATIONS ARE FORCE-GENERATING.

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To feed our intuition let us give one more look to what *we have called* \mathcal{NC} (=non-centrifugal) systems:

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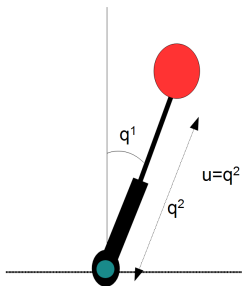
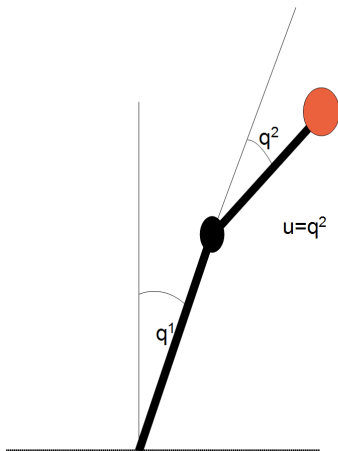
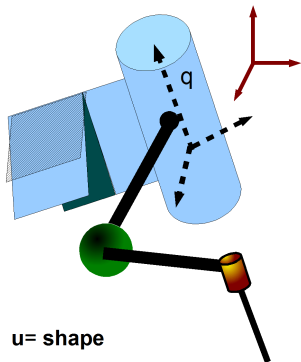


Figure: Almost insensitive to small oscillation of length

(2 $\mathcal{N}\mathcal{C}$) The pendulum with a second pendulum as control

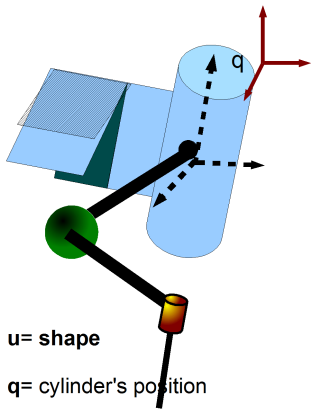


(3NC) "Shape" as control

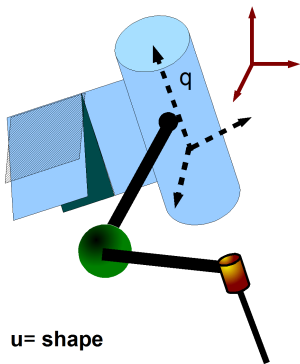


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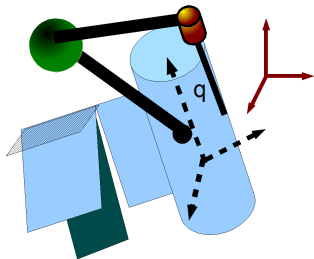


A rigid movement (the shape u is unchanged)



$u = \text{shape}$

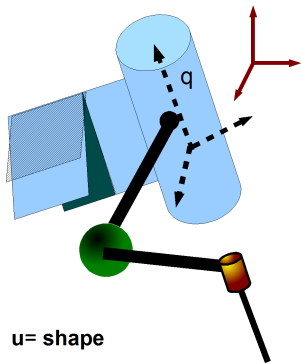
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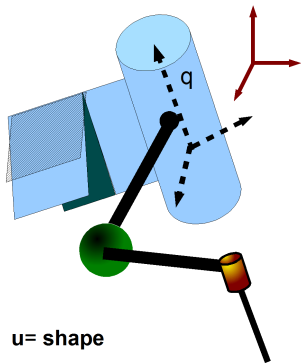
q = cylinder's position

Change of shape (i.e. change of u)



u= shape

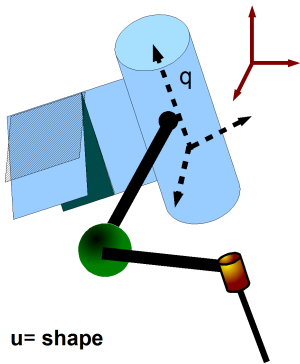
q= cylinder's position



$u = \text{shape}$

$q = \text{cylinder's position}$

Also in this case,



$u = \text{shape}$

$q = \text{cylinder's position}$

Also in this case, the body is almost insensitive to small rapid oscillation of the control (=the shape)

THE SUSPECT is *reasonable!*

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In fact, in the case of \mathcal{NC} (=non centrifugal) systems, small rapid oscillation of the control-coordinate

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At this stage, this statement is just an act of *faith*,

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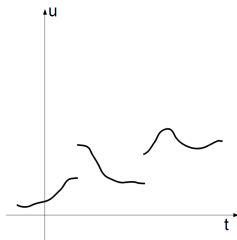
"DO NOT PRODUCE FORCES"

At this stage, this statement is just an act of *faith*,...not to mention that it is quite tautological

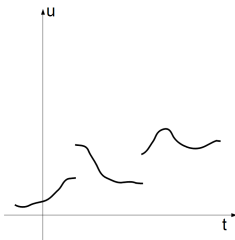


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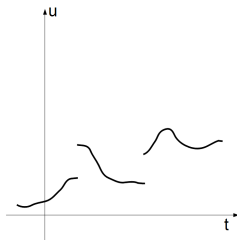


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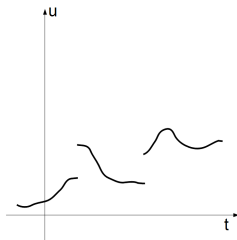
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NOTICE: *the derivative \dot{u} is a **measure**....THEY CAN BE IMPLEMENTED ONLY IN "NON CENTRIFUGAL" SYSTEMS (WHERE THEY ARE INTERESTING FOR OPTIMIZATION PURPOSES)*

Development

IT IS TIME FOR SOME MATHEMATICS:

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and let us consider a **particular class of control systems**:

ODE's depending quadratically on the control derivative \dot{u}

We are interested in control equations having the form

$$\dot{x} = f(x, u) + \sum_{\alpha=1}^m g_{\alpha}(x, u) \dot{u}_{\alpha} + \sum_{\alpha, \beta=1}^m h_{\alpha\beta}(x, u) \dot{u}_{\alpha} \dot{u}_{\beta}$$

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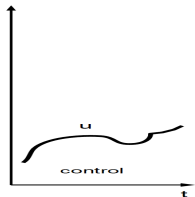
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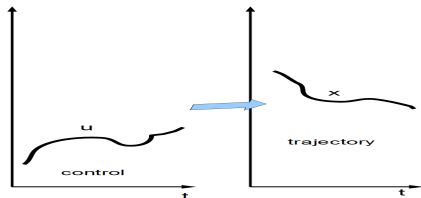
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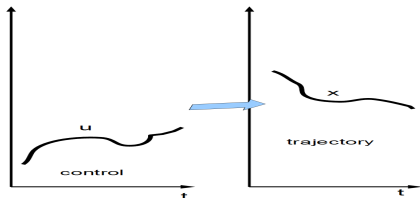
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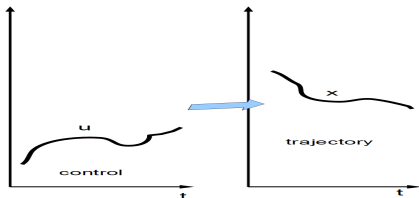
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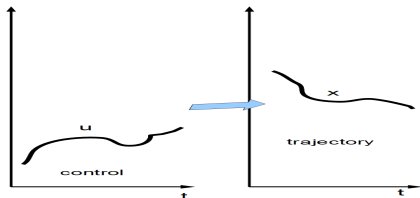


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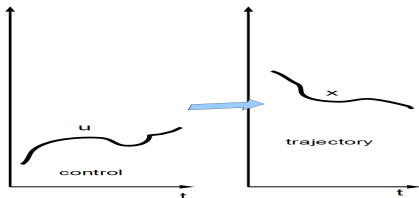


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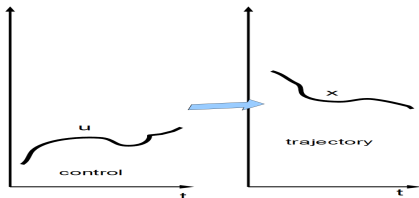
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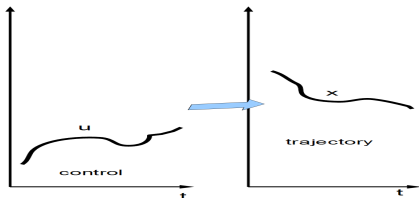
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(see *A. Bressan - F.R. (1999) SIAM Control Opt.* for the closure of the set of solutions when $\int_0^T |u|^2 dt \leq K$, in the case of **scalar** u)

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- At each $x \neq \bar{x}$ one has

$$\inf_{y \in F(x)} \nabla V(x) \cdot y \leq 0.$$

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It is intuitive that the weak stability of the symmetrized diff. inclusion

$$\frac{dx}{dt} \in \mathcal{G}(x)$$

has *something to do with* ”VIBRATIONAL CONTROLS”.

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(**Intuitively:** *Rapid oscillations of u cancel out the \dot{u} -linear term.*)

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Correspondingly, one has the

REPARAMETERIZED DIFFERENTIAL INCLUSION:

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$$\mathbf{F}(y) \doteq \overline{\text{co}} \left\{ f(y)v_0^2 + \sum_{\alpha=1}^m g_{\alpha}(y)v_0v_{\alpha} + \sum_{\alpha,\beta=1}^m h_{\alpha,\beta}(y)v_{\alpha}v_{\beta}; \right. \\ \left. v_0 \in [0, 1], v_0^2 + \dots + v_m^2 = 1 \right\}.$$

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Therefore, for each $\tau > 0$,

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Hence

$$x_n(\tau) - x_n(0) \rightarrow \frac{\tau}{2}.$$

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(2) the term \dot{u}_n^2 has the same effect as the constant $\frac{1}{2}$, i.e. it produces the output $t \rightarrow \frac{t}{2}$

Recapitulation

BACK TO MECHANICS

- **assigning the control $u = u(t)$** is nothing but **adding the new holonomic constraint $u = u(t)$** ... to the original system.

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- **assigning the control $u = u(t)$ is nothing but adding the new holonomic constraint $u = u(t)$... to the original system.**
- **As is well-known in order to determine equations one needs to specify how the control constraint $u = u(t)$ is physically implemented :**
Here we assume **D'Alembert Hypothesis**, namely, *the work of constraint reactions is equal to zero.*

The Kinetic Energy

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In particular \mathcal{H} is quadratic in the momenta (p, π)

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The Hamiltonian equations of motion before the imposition of the constraint-control (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

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$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \cdot \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix}$$

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$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} \doteq \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

Remark:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

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does not depend on q .

Stabilizability of mechanical systems

Theorem 3.

Let us assume that the force Γ is a conservative, and let $U = U(q, u)$ be a potential of Γ .

Let us fix \bar{u} , and let us assume that there exist positive real coefficients $\lambda_1, \dots, \lambda_k$ verifying $\sum_{r=1}^k \lambda_r = 1$, and vectors w_1, \dots, w_s such that *the effective potential*

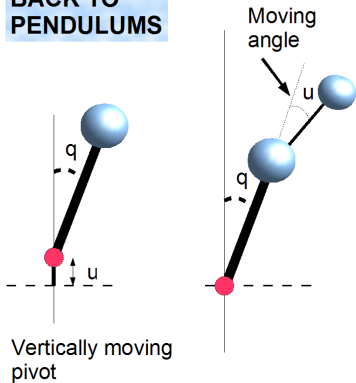
$$\hat{U}(q) \doteq U(\bar{u}, q) - \frac{1}{2} \sum_{r=1}^k \lambda_r \sum_{\alpha, \beta=1}^m e_{\alpha, \beta}(q, \bar{u}) w_r^\alpha w_r^\beta$$

has a strict minimum at $q = \bar{q}$.

Then, the control mechanical system can be stabilized to (\bar{q}, \bar{u}) .

- **Idea of the proof:** the choice of the λ_r, w_r selects from the corresponding differential inclusion $\dot{x} \in \mathcal{G}$ a conservative mechanical system with potential energy equal to \hat{U}
- If \bar{q} is unstable for the frozen control $u = \bar{u}$, a **necessary condition for stabilizability** is that **the matrix $e_{\alpha,\beta}$ be q -dependent.**

BACK TO PENDULUMS



**Is the state $q=0$
stabilizable
by a vibrational u ?**

Pendulum with moving pivot

$$\begin{cases} \dot{q} = p + (\sin q)\dot{u} \\ \dot{p} = -\frac{\partial U}{\partial q} - p \cos q \dot{u} - (\sin q \cos q)(\dot{u})^2, \end{cases}$$

where $U(q, c) \doteq \cos q$.

The kinetic matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -\sin q \\ -\sin q & 2 \end{pmatrix}$$

...so that the 1×1 matrix E is

$$E = e_{11} = 1 + \cos^2 q$$

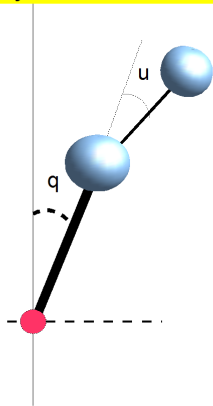
Notice: The necessary condition (i.e. " E depends on q ") is verified; Moreover: the **effective potential**

$$U_{\{1\}\{w\}} = \cos q - \frac{1}{2}(1 + (\cos q)^2)w^2.$$

has a minimum at $q = 0$ as soon as $w^2 \geq 1$. Therefore: The system is vibrationally stabilizable at 0.

Coda

Why NO stabilizability?



In order to give the answer one
need not even know the equation of motion...

Indeed the kinetic matrix is

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in other words the control equations are affine in \dot{u} .

SECOND MOVEMENT

—*Adagio*—

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- (iii) **Such a definition allows for jumps of u .**

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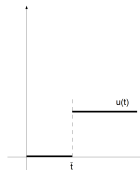
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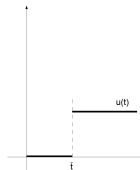
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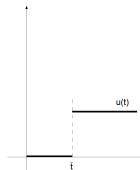
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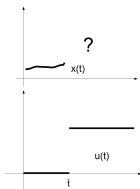
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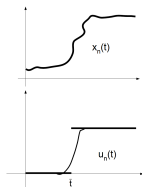
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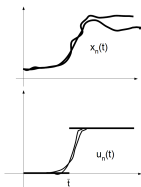
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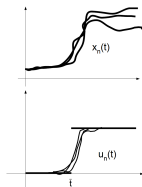
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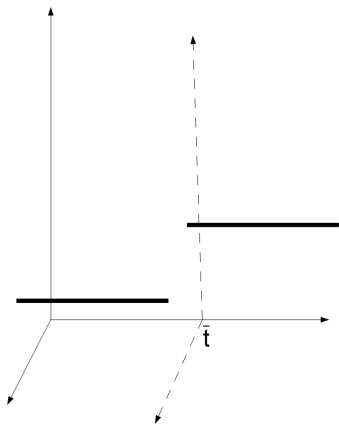
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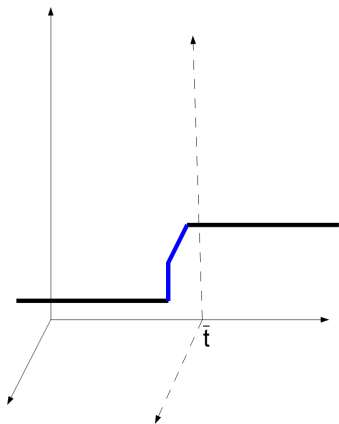
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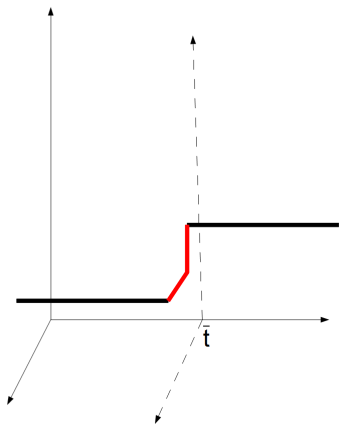
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Roughly speaking: at a jump of u one has to assign the *trajectory of the jump*







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H. Sussmann, A. Bressan, A.Bressan-F.R, M.Motta- F.R., B. Miller, Dal Maso-F.R., C. Sartori-F.R., J. Dorroh-G.Ferreyra, A. Sarychev, R. Vinter-G. Silva, F.Lobo Pereira, P. Mason, P. Wolenski-S.Zabic, H. Zidani...

THIRD MOVEMENT (*Scherzo*)

—*Con moto*—

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*Is the fact that the control system is **affine in \dot{u}** —generally it is **quadratic**— related to some differential geometric property?*

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The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

The Kinetic Energy

$$\mathcal{T} = \mathcal{T}[(q, u)(\dot{q}, \dot{u})]$$

is a **quadratic form** in (\dot{q}, \dot{u}) , i.e. , it is a **Riemannian metric** on the configuration manifold:

$$\mathcal{T} = (\dot{q}, \dot{u}) \cdot \mathbf{A} \cdot (\dot{q}, \dot{u})^t \quad \mathbf{A} = (a_{r,s})_{r,s=1,\dots,N+M}$$

(\mathbf{A} is the so-called *kinetic matrix*)

The Hamiltonian is nothing but its Legendre transform:

$$\mathcal{H}[(q, u)(p, \pi)] \doteq \mathcal{T}^*$$

In particular \mathcal{H} is quadratic in the momenta (p, π)

$$\mathcal{H} = (p, \pi) \mathbf{A}^{-1} (p, \pi)^t \quad \mathbf{A}^{-1} = (a^{r,s})_{r,s=1,\dots,N+M}$$

Remember: The Hamiltonian equations of motion before the imposition of the holonomic constraint (i.e.. the control) $u = u(t)$

$$\left\{ \begin{array}{l} \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{u}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi_\alpha} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{F}_i \\ \dot{\pi}_\alpha = -\frac{\partial \mathcal{H}}{\partial u_\alpha} + \mathcal{F}_{u_\alpha} \end{array} \right.$$

Remember: the equations of motion after the imposition of the constraint (=control) $u = u(t)$:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 \\ F^{u(\cdot)} \end{pmatrix}}^f + \phi + \sum_{\alpha=1}^M g_{\alpha} \dot{u}_{\alpha} + \sum_{\alpha,\beta=1}^M h_{\alpha,\beta} \dot{u}_{\alpha} \dot{u}_{\beta}$$

with suitable vector fields $f(q, p, u)$, $g_{\alpha}(q, p, u)$, $h_{\alpha,\beta}(q, p, u)$ determined by the Kinetic Energy and the applied forces.

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$$h_{\alpha,\beta} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \frac{\partial e_{\alpha,\beta}}{\partial q^1} \\ \cdot \\ \cdot \\ \frac{\partial e_{\alpha,\beta}}{\partial q^N} \end{pmatrix}$$

where

$$\begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

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(=inverse of the lower right $M \times M$ minor of \mathbf{A}^{-1})

Remember:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

IF AND ONLY IF

THE MATRIX

$$E = \begin{pmatrix} e_{11}, \dots, e_{1M} \\ \dots \\ e_{M1}, \dots, e_{MM} \end{pmatrix} = \begin{pmatrix} a^{N+1,N+1}, \dots, a^{N+1,N+M} \\ \dots \\ a^{N+M,1}, \dots, a^{N+M,N+M} \end{pmatrix}^{-1}$$

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The question now is:

Is this property a "geometric property"?

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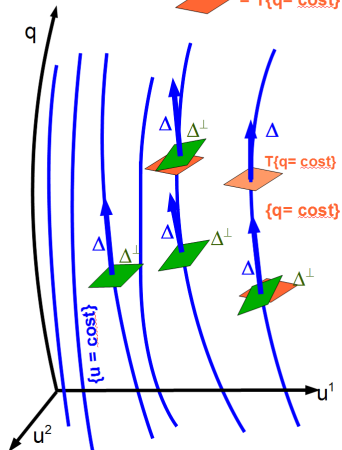
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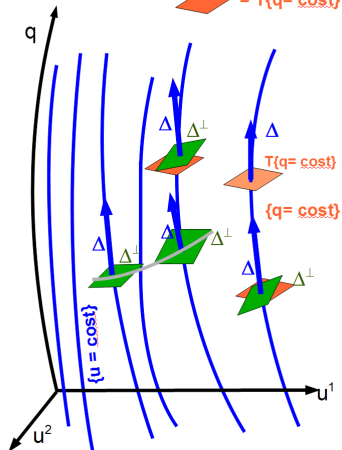
IF AND ONLY IF

(3) Any pair of leaves $\{u = \text{constant}\}$ have keep the same distance at every point

$\uparrow = \Delta = T\{u = \text{cost}\}$ $\text{green plane} = \Delta^\perp = \text{orth. bundle}$
 $\text{orange plane} = T\{q = \text{cost}\}$



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$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

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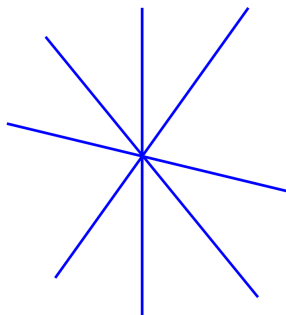
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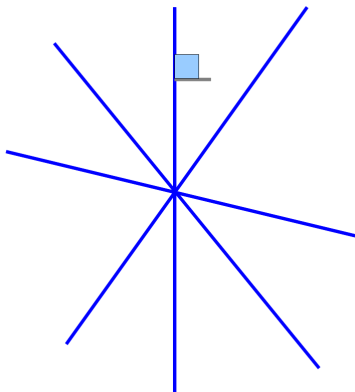
The orthogonal curvature $\frac{\partial e_{\alpha,\beta}}{\partial q^i}$ is a *tensor* with respect to the coordinate transformations that respect the foliation structure.

EXAMPLES



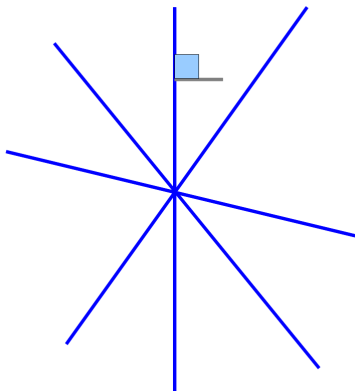
Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



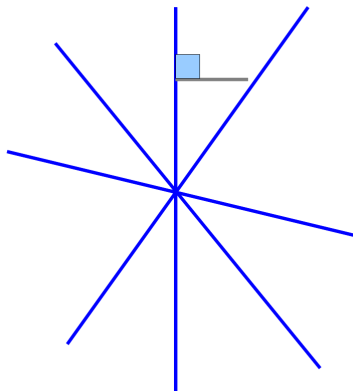
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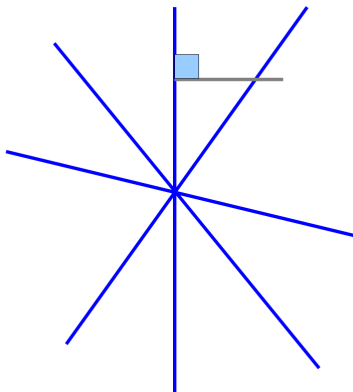
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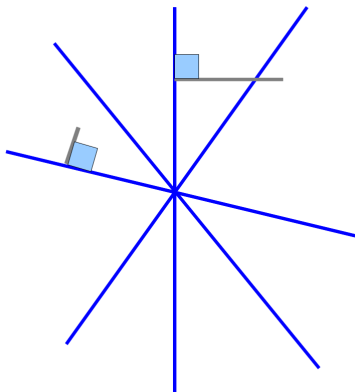
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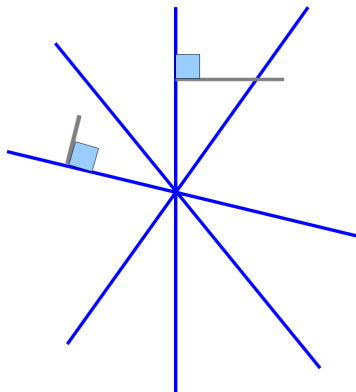
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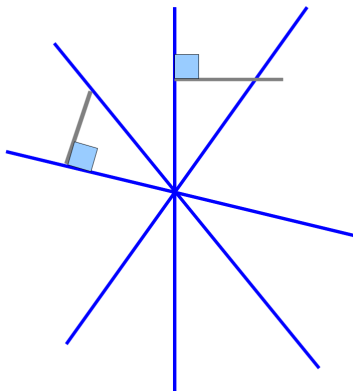
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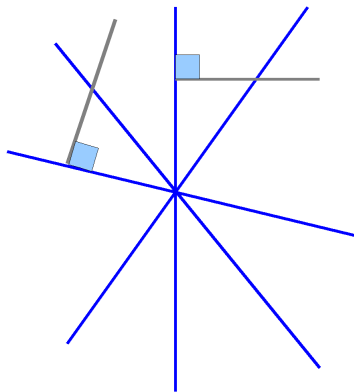
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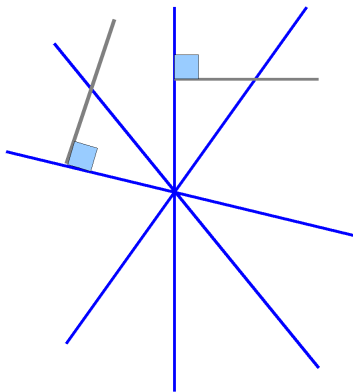
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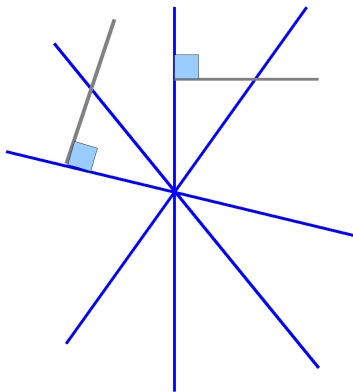
The angle as control:



Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:
quadratic part!

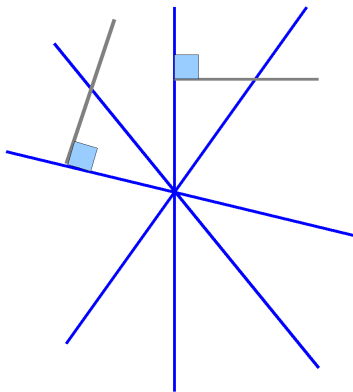
Non-zero



Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:
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Hence,

Non-zero

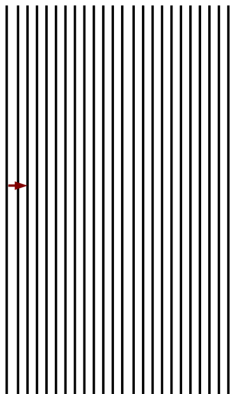


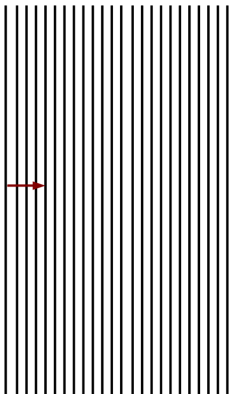
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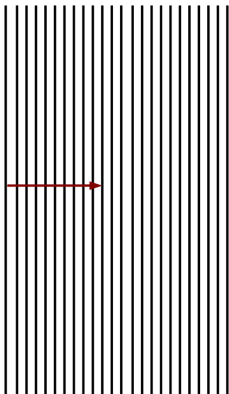
The angle as control:
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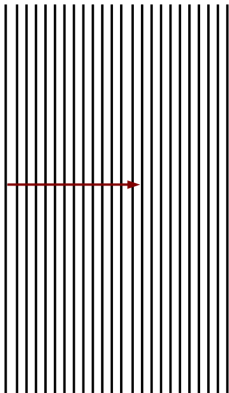
Hence, chance of vibrational stabilization!

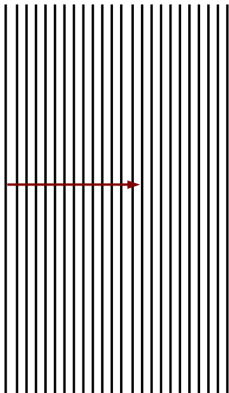
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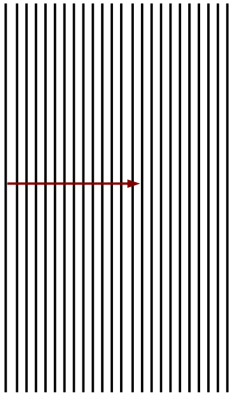






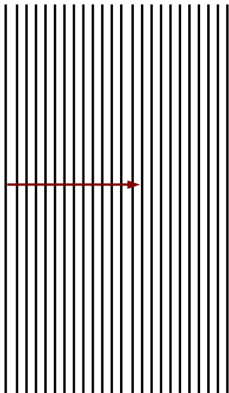


No quadratic part!



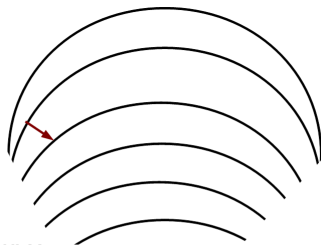
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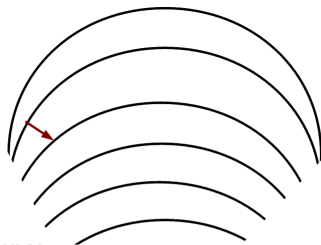


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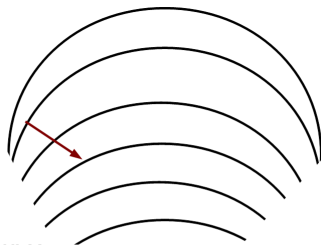
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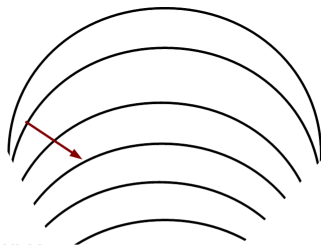
Pendulum with oscillating pivot



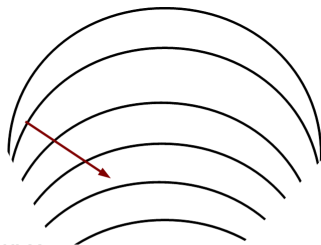
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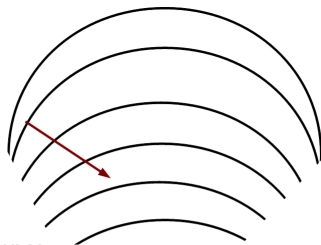
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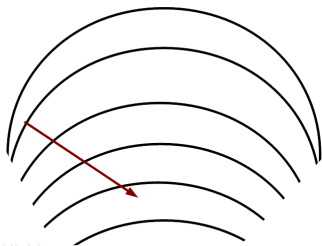


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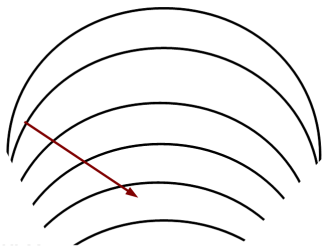


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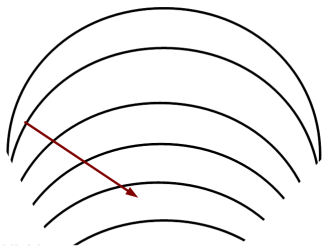


Pendulum with oscillating pivot



Non-zero quadratic part!

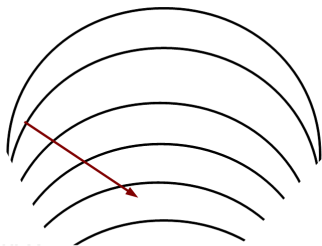
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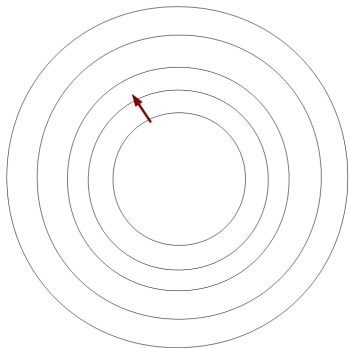
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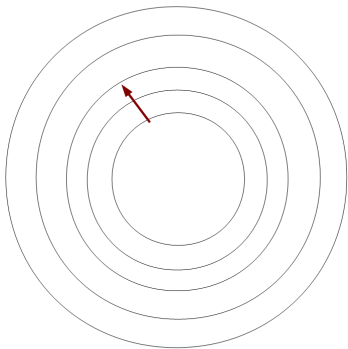
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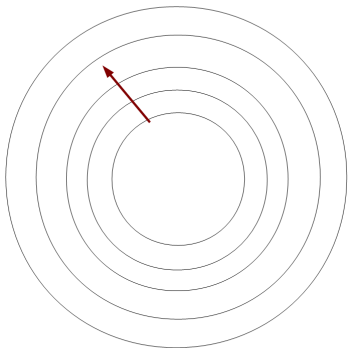
The length of the pendulum as control



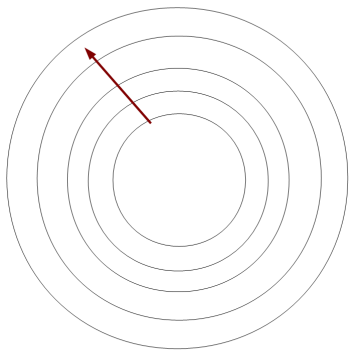
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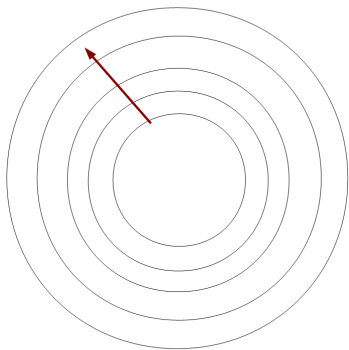
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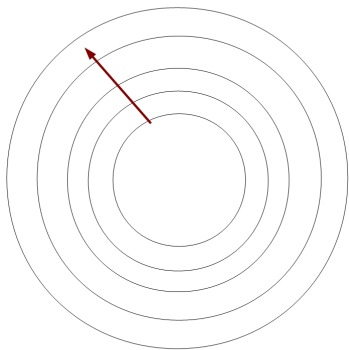


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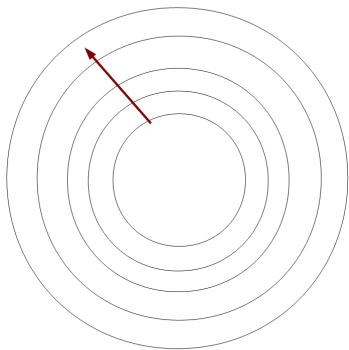
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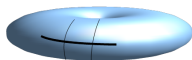


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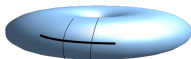
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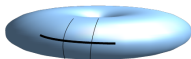


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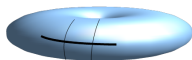
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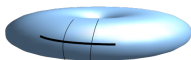
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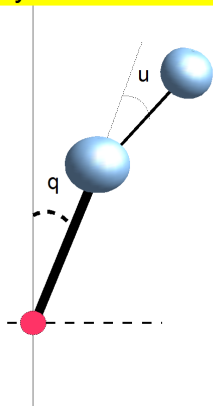
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Geodesics keep orthogonality to leaves. Hence no quadratic part! Hence, no vibrational stabilization!

Why NO stabilizability?



FOURTH MOVEMENT (*Finale*)

—*Allegro molto*—

The title of this tutorial contain the expression
Non holonomic

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Non holonomic

BUT

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Non holonomic

BUT

UP TO NOW EVERYTHING HAS BEEN HOLONOMIC...

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So, this Final is devoted to "Non holonomic constraints"

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A non holonomic constraint is a (linear) constraint on the velocity \dot{q}

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

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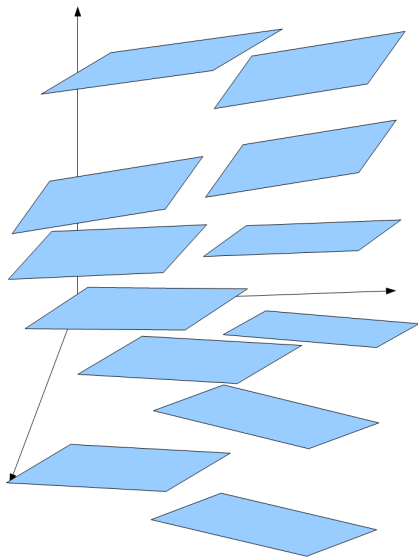
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i.e., there is no *foliation* of \mathcal{Q} whose leaves have $\Delta(q)$ as tangent space at any q .

EXAMPLE in \mathcal{R}^3



ANOTHER EXAMPLE in \mathcal{R}^3

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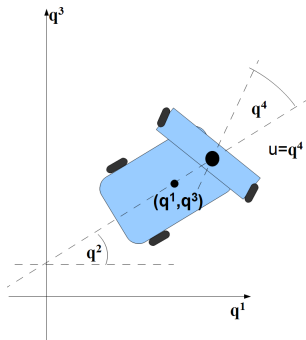
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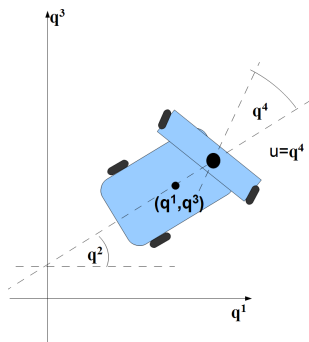
does not belong to Δ .

A mechanical example: The Roller Racer

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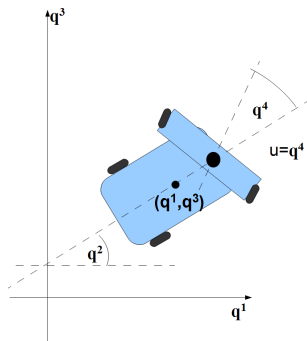


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The fact that the velocity of the first body must be directed as the angle q^2 and the analog fact for the velocity of the second body, is **non holonomic constraint** on the system.



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In particular,

no forces would be produced by rapid small oscillations of u .

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Physically one can say the non holonomic constraint produces a *centrifugal effect*

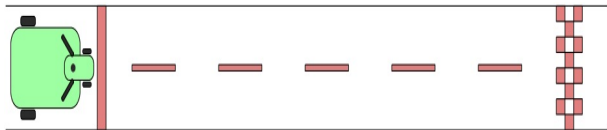
The control equations for the Roller Racer (on a surface with friction):

$$\left\{ \begin{array}{l} \dot{q}^1 = 2\rho \cos u \sin q^2 \cdot \xi - \frac{J\rho \sin q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{q}^2 = 2 \sin u \cdot \xi - \frac{J \sin^2 u}{\Delta_0} \cdot \dot{u} \\ \dot{q}^3 = 2\rho \cos q^2 \cos u \cdot \xi - \frac{J\rho \cos q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{\xi} = -\sin 2u \left(\frac{(I+J-\rho^2)}{\Delta_1} + \frac{1}{2(\rho^2/\Delta_4 + \sin^2 u)} \right) \cdot \xi \dot{u} - \frac{2J\rho^2 \cos u}{\Delta_1^2} \cdot \dot{u}^2. \end{array} \right.$$

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GRAZIE

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PER L'ATTENZIONE