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▶ **To cite this version:**

Olivier Bokanowski, Hasnaa Zidani. Minimal Time Problems with Moving Targets and Obstacles. 18th IFAC World Congress, 2011, Milano, Italy. pp.2589-2593, 10.3182/20110828-6-IT-1002.02261 . inria-00629166

**HAL Id: inria-00629166**

**https://inria.hal.science/inria-00629166**

Submitted on 5 Oct 2011

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# MINIMAL TIME PROBLEMS WITH MOVING TARGETS AND OBSTACLES

OLIVIER BOKANOWSKI[1,2,3] AND HASNAA ZIDANI[3]

Abstract. We consider minimal time problems governed by nonlinear systems under general time dependent state constraints and in the two-player games setting. In general, it is known that the characterization of the minimal time function, as well as the study of its regularity properties, is a difficult task in particular when no controllability assumption is made. In addition to these difficulties, we are interested here to the case when the target, the state constraints and the dynamics are allowed to be time-dependent.

We introduce a particular "reachability" control problem, which has a supremum cost function but is free of state constraints. This auxiliary control problem allows to characterize easily the backward reachable sets, and then, the minimal time function, without assuming any controllability assumption. These techniques are linked to the well known level-set approaches. Partial results of the study have been published recently by the authors in SICON. Here, we generalize the method to more complex problems of moving target and obstacle problems.

Our results can be used to deal with motion planning problems with obstacle avoidance.

**Keywords:** Minimal time problem, moving targets, time-dependent state constraints, motion planning, obstacle avoidance, Hamilton-Jacobi-Bellman equations, level set method, reachable set (attainable set).

## 1. Introduction

The regularity and characterization of the minimal time function is a widely studied topic (see [6], [10], [1], [8, 9] [5] and references therein).

When there is no state constraint, and under some local metric properties around the target, the minimal time function $\mathcal{T}$ is the unique continuous viscosity solution of an Hamilton-Jacobi (HJ) equation. See [1] in the one-player game setting and for corresponding Hamilton-Jacobi-Bellman (HJB) equation satisfied by $\mathcal{T}$.

In the presence of state constraints $\mathcal{K}$ (where $\mathcal{K}$ is a non-empty closed subset of $\mathbb{R}^d$), the minimum time function (and more generally the value function of an optimal control problem) is not necessarily continuous, unless some controllability assumption is made on the boundary of $\mathcal{K}$.

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In the work of [9], no controllability assumption is assumed and a characterization is obtained involving non-smooth analysis.

In the present paper, we follow some ideas developed in [5] in order to treat minimum time function and capture basin for two-player games. We are mainly interested by the case of a moving target and time-dependent state constraints, and the dynamics will be time-dependent as well.

We refer to [21, 11, 1, 19] and references therein for an introduction and some results for deterministic two-player games with infinite horizon.

**Acknowlegments.** *This work was supported by the EU under the 7th Framework Programme FP7-PEOPLE-2010-ITN, GA number 264735-SADCO.*

## 2. Setting of the problem

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty compact subset of $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. For $t \geq 0$, let $\mathbb{A}_t := \{\alpha : (0,t) \to \mathcal{A},\ \text{measurable}\}$ and $\mathbb{B}_t := \{\alpha : (0,t) \to \mathcal{B},\ \text{measurable}\}$. We consider a dynamics $f : \mathbb{R}^d \times [0,+\infty[ \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^d$ such that

**(H1)** $f$ is Lipschitz continuous

and, for every $x \in \mathbb{R}^d$ and $(\alpha,\beta) \in \mathbb{A}_t \times \mathbb{B}_t$, its associated trajectory $y = y_x^{\alpha,\beta}$ defined as the (absolutely continuous) solution of

$$\dot{y}(s) = f(y(s), s, \alpha(s), \beta(s)),\ \text{for a.e. } s \in [0,t], \tag{1a}$$

$$y(0) = x. \tag{1b}$$

To simplify, we shall assume that $f$ is globally Lipschitz in all its variables (although weaker assumptions can be made for part of the results).

We also assume that $(\mathcal{K}_t)_{t\geq 0}$ is a family of closed sets of $\mathbb{R}^d$ (set of "constraints"), and that $(\mathcal{C}_t)_{t\geq 0}$, an other family of closed sets (set of "targets").

We consider a game involving two players, starting at time $t = 0$. The first player wants to steer the system (initially at point $x$) to the target $\mathcal{C}_t$ in some minimal time $t \geq 0$, and by staying in $\mathcal{K}_t$ (and using her input $\alpha$), while the second player tries to steer the system away from $\mathcal{C}_t$ or from $\mathcal{K}_t$ (with her input $\beta$). More precisely we will say that the trajectory is "admissible on $[0,t]$" if it satisfies the constraints on the time interval $[0,t]$:

$$y_x^{\alpha,\beta}(\theta) \in \mathcal{K}_\theta, \quad \forall \theta \in [0,t].$$

We define the set of non-anticipative strategies for the first player, as follows:

$$\begin{aligned}\Gamma_t := \Big\{ & a : \mathbb{B}_t \to \mathbb{A}_t,\ \forall (\beta,\tilde{\beta}) \in \mathbb{B}_t \text{ and } \forall s \in [0,t], \\ & \Big(\beta(\theta) = \tilde{\beta}(\theta)\ a.e.\ \theta \in [0,s]\Big) \Rightarrow \\ & \qquad \Big(a[\beta](\theta) = a[\tilde{\beta}](\theta)\ a.e.\ \theta \in [0,s]\Big) \quad \Big\}.\end{aligned}$$

Then we are interested to characterize the following capture basin for the first player:

$$\begin{aligned}\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t) := \Big\{ & x \in \mathbb{R}^d,\ \exists a \in \Gamma_t,\ \forall \beta \in \mathbb{B}_t, \\ & \Big(y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t,\ \text{ and } y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}_\theta,\ \forall \theta \in [0,t]\Big)\Big\}\end{aligned} \tag{2}$$

Thus $x \in \mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$ means that there exists a non anticipative strategy $a \in \Gamma_t$ such that for any adverse strategy $\beta \in \mathbb{B}_t$, we have $y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t$ (we reach the target $\mathcal{C}_t$ at time $t$).

This setting includes the case of a fixed target $\mathcal{C}_t \equiv \mathcal{C}$ or of a fixed constraint $\mathcal{K}_t \equiv \mathcal{K}$. It also contains the particular case of a one-player game (it suffices to take $B = \{b_0\}$ a fixed value, $\mathbb{B}_t = \{\beta\}$ a fixed constant function and then any $\alpha \in \mathbb{A}_t$ represents an admissible non-anticipative strategy).

We are also interested in computing the minimal time function $\mathcal{T}(x)$ defined by

$$\begin{aligned}\mathcal{T}(x) := \inf\Big\{ & t \geq 0,\ \exists a \in \Gamma_t, \forall b \in \mathcal{B}_t, \\ & y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t \text{ and} \\ & y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}_\theta,\ \forall \theta \in [0,t]\Big\}\end{aligned}$$

Since we have $\mathcal{T}(x) = \inf\{t \geq 0,\ x \in \mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)\}$, it is sufficient to characterize the sets $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$.

It is well known that the set $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$ is linked to a control problem. Indeed, consider some Lipschitz continuous function $\vartheta_0 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that

$$\vartheta_0(x,t) \leq 0 \Longleftrightarrow x \in \mathcal{C}_t, \tag{3}$$

and consider the control problem:

$$\begin{aligned} u(x,t) \quad := \quad & \inf_{a\in\Gamma} \max_{\beta\in\mathbb{B}_t} \Big\{ \vartheta_0(y_x^{a[\beta],\beta}(t),t), & (4)\\ & \beta \text{ s.t. } y_x^{a[\beta],\beta} \text{ admissible on } [0,t]\Big\} & (5)\end{aligned}$$

(with value $u(x,t) = +\infty$ if for any strategy $a$ there is no admissible trajectory).

Let us assume that $f$ is affine in the variable $a$ :

$$(\mathbf{H2}) \quad f(x,t,a,b) = f_1(x,t,b) + f_2(x,t,b) \cdot a$$

(a similar assumption, such as $f(x,t,A,b)$ convex for all $b$, could also be assumed). As in [7] we have that $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$ is a closed set.

Under (H2), the capture basin is related to the negative region of $u(.,t)$ :

$$\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t) = \{x \in \mathbb{R}^d, u(x,t) \leq 0\}. \tag{6}$$

However the characterization of $u$ by means of an HJB equation is not easy because of the state constraints, unless some strong assumptions are satisfied. Our aim in this paper is to give a simple way to characterize the capture basin.

In section 3, we first discuss some existing and related results. In section 4 we present our results. An appendix contains the proofs.

## 3. Discussion

For the unconstrained case, several works have been devoted to the characterization of the value function $u$ as a continuous viscosity solution of a Hamilton-Jacobi equation, see [12, 1]. In presence of state constraints (and when $\mathcal{K}_t \equiv \mathcal{K}$ and is different from $\mathbb{R}^d$), the continuity of this value function is no longer satisfied, unless the dynamics satisfy a special controllability assumption on the boundary of the state constraints. This assumption called "inward pointing qualification (IPQ)" condition was first introduced by [18]. It asks that at each point of $\mathcal{K}$ there exists a field of the system pointing inward $\mathcal{K}$. Clearly this condition ensures the viability of $\mathcal{K}$ (from any initial condition in $\mathcal{K}$, there exists an admissible trajectory which could stay for ever in $\mathcal{K}$). Under the (IPQ) condition, the value function $u$ is the unique continuous *constrained viscosity solution* of a HJB equation with a suitable new boundary condition.

Unfortunately, in many control problems, the (IPQ) condition is not satisfied and the value function $u$ could be discontinuous. In this framework, Frankowska introduced in [13] another controllability assumption, called "outward pointing condition." Under this assumption it is still possible to characterize the value function as the unique lower semi-continuous (l.s.c. for short) solution of an HJB equation.

In absence of any controllability assumption, the function $u$ is discontinuous and its characterization becomes more complicate, see for instance [20, 4] and the references therein. We refer also to [8, 9] for a characterization based on viability theory.

Several papers in the literature deal with the link between reachability and HJB equations. In the case when $\mathcal{K} = \mathbb{R}^d$, we refer to [16] and the references therein. The case when $\mathcal{K}$ is an *open set* in $\mathbb{R}^d$ is investigated in [15]. We also refer to [14] for a short discussion linking the reachable sets under state constraints to HJB equations. The treatment in this reference assumed a $C^1$ value function.

In a recent work [5], the case $\mathcal{K}_t \equiv \mathcal{K}$, $\mathcal{C}_t \equiv \mathcal{C}$ and with no time dependency in the dynamics, has been investigated. It was shown that the capture basin $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}$ can be characterized by means of a control problem whose value function is continuous (even Lipschitz continuous). Let us recall here the main idea. For simplicity we consider also the one-player game ($f(x,t,a,b) \equiv f(x,a)$). We first consider continuous functions $g : \mathbb{R}^d \to \mathbb{R}$ and $\vartheta_0 : \mathbb{R}^d \to \mathbb{R}$ such that

$$g(x) \leq 0 \Longleftrightarrow x \in \mathcal{K} \quad \text{and} \quad \vartheta_0(x) \leq 0 \Longleftrightarrow x \in \mathcal{C}.$$

Then we introduce the new control problem:

$$\vartheta(x,t) \;:=\; \inf_{\alpha \in \mathbb{A}_t} \left\{ \max\left(\vartheta_0(y_x^\alpha(t)), \max_{\theta \in [0,t]} g(y_x^\alpha(\theta))\right)\right\}.$$

It is proved in [5] that the value function $\vartheta$ is the unique continuous viscosity solution of the equation:

$$\min\Big(\partial_t\vartheta(x,t) + H(x, D_x\vartheta(x,t)), \vartheta(x,t) - g(x))\Big) = 0, \quad \text{for } t \in [0,+\infty[,\ x \in \mathbb{R}^d, \tag{7a}$$

$$\vartheta(x,0) = \max(\vartheta_0(x), g(x)). \tag{7b}$$

where $H(x,p) := \max_{a\in\mathcal{A}}\big(-f(x,a)\cdot p\big)$, and furthermore we have

$$\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t) = \{x,\ \vartheta(x,t) \le 0\}.$$

The main feature of (7) is to use a modelization with a supremum cost, in order to handle easily the state constraints and to determine the corresponding capture basins. This idea generalizes the known level-set approach usually used for unconstrained problems. Moreover, the continuous setting opens a large class of numerical schemes to be used for such problems (such as Semi-Lagrangian or finite differences schemes). We refer to [5] for numerical results and comparison of various approaches for state-constrained problems.

We shall now consider the general problem of moving (or time-dependent) targets as well as moving obstacles.

*N*otations. Throughout the paper $|\cdot|$ is a given norm on $\mathbb{R}^d$ (for $d \ge 1$). For any closed set $K \subset \mathbb{R}^d$ and any $x \in K$, we denote by $d(x,K)$ the distance from $x$ to K: $d(x,K) := \inf\{|x-y|,\ y \in K\}$. We shall also denote by $d_K(x)$ the signed distance function to $K$, i.e., with $d_K(x) := d(x,K)$ for $x \notin K$, and $d_K(x) := -d(x, \mathbb{R}^d\backslash K)$ for $x \in K$.

## 4. Main results

We assume that

**(H3)** the set-valued applications $\theta \rightsquigarrow \mathcal{K}_\theta$ and $\theta \rightsquigarrow \mathcal{C}_\theta$ are upper semi-continuous.

We recall that if $(Q_t)_{t\ge0}$ denotes a family of subsets of $\mathbb{R}^{d+1}$, then the set valued map $t \rightsquigarrow Q_t$ is said to be "upper semi-continuous" if

$$\forall\epsilon > 0,\ \exists\alpha > 0,\ \forall\theta \in [t-\alpha, t+\alpha],\quad Q_\theta \subset Q_t + \epsilon B(0,1).$$

**Remark 1.** *For every $t \ge 0$, the set $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$ contains the initial positions which can be steered to the target (exactly) at time $t$. Of course, we can also define the "backward reachable set", which is the set of points from which one can reach the target $\mathcal{C}_t$ before time $t$. This set is also a capture basin for the dynamics $\tilde{f}$ where*

$$\tilde{f}(x,t,(a,\lambda),b) := \lambda f(x,t,a,b), \quad \lambda \in [0,1]$$

*(see* [16, 5]*).*

We start by embedding the position $y(t)$ and time $t$ into a "space time" space. To do so, we set for every $z = (y,t) \in \mathbb{R}^d \times \mathbb{R}$, the set-valued map $F : \mathbb{R}^{d+1} \times [0,+\infty) \times \mathcal{A} \times \mathcal{B} \rightsquigarrow \mathbb{R}^{d+1}$ such that :

$$F(z,a,b) := \big\{f(y,t,a,b)\big\} \times \big\{1\big\},$$

and we remark that $F$ satisfies similar Lipschitz continuity and linearity assumptions as in (H1) and (H2). For a given $\xi \in \mathbb{R}^d \times \mathbb{R}$ and $(\alpha,\beta) \in \mathbb{A}_t \times \mathbb{B}_t$, we can then consider $z = z_\xi^{\alpha,\beta}$, the absolutely continuous solution of

$$\dot{z}(s) = F(z(s),\alpha(s),\beta(s)), \text{a.e. } s \in [0,t], \quad z(0) = \xi.$$

(we shall simply denote $z = z_\xi$ if there is no ambiguity). Any solution $z_\xi(s) = (y(s),\eta(s))$ of the previous system satisfies equivalently, if $\xi = (x,t_0)$,

$$\begin{cases} \dot{y}(s) = f(y(s),t_0+s,\alpha(s),\beta(s)), \ s \geq 0, \quad y(0) = x, \\ \eta(s) = t_0 + s, \quad s \geq 0 \end{cases}$$

Moreover, let also introduce two subsets of $\mathbb{R}^{d+1}$:

$$\mathbb{C} := \bigcup_{t \geq 0} \mathcal{C}_t \times \{t\} \quad \text{and} \quad \mathbb{K} := \bigcup_{t \geq 0} \mathcal{K}_t \times \{t\}.$$

We have the following elementary result :

**Lemma 1.** *Under (H3), the sets $\mathbb{C}$ and $\mathbb{K}$ are closed subsets of $\mathbb{R}^{d+1}$.*

Hence there exists Lipschitz continuous functions $\vartheta_0 : \mathbb{R}^{d+1} \to \mathbb{R}$ and $g : \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\vartheta_0(\xi) \leq 0 \Leftrightarrow \xi \in \mathbb{C}. \tag{8}$$

and

$$g(\xi) \leq 0 \Leftrightarrow \xi \in \mathbb{K} \tag{9}$$

(for instance we may choose $\vartheta_0(\xi) := d_{\mathbb{C}}(\xi)$ and $g(\xi) := d_{\mathbb{K}}(\xi)$). In particular, for any $t < 0$ we have $\vartheta_0((x,t)) > 0$ and $g((x,t)) > 0$.

We can then define a capture basin associated to the new dynamics $F$:

$$\begin{aligned} \text{Cap}^F_{\mathbb{C},\mathbb{K}}(\tau) \ := \ \Big\{ & x \in \mathbb{R}^{d+1}, \ \exists a \in \Gamma_\tau, \ \forall \beta \in \mathbb{B}_\tau, \\ & \Big( z_\xi^{a[\beta],\beta}(\tau) \in \mathbb{C}, \ \text{ and } z_\xi^{a[\beta],\beta}(\theta) \in \mathbb{K}, \ \forall \theta \in [0,\tau] \Big) \Big\} \end{aligned} \tag{10}$$

Notice that in the case $\xi = (x,0)$, we have $z_\xi(t) = (y_x(t),t)$ where $y_x$ is a trajectory for the dynamics $f$. Hence

$$y_x(t) \in \mathcal{C}_t \quad \Leftrightarrow \quad z_\xi(t) \in \mathbb{C},$$

and in the same way,

$$y_x(t) \in \mathcal{K}_t \quad \Leftrightarrow \quad z_\xi(t) \in \mathbb{K}.$$

Therefore we can easily deduce the following result:

**Proposition 1.** *For all $t \geq 0$, we have*

$$x \in \text{Cap}^f_{\mathcal{C},\mathcal{K}}(t) \quad \Leftrightarrow \quad (x,0) \in \text{Cap}^F_{\mathbb{C},\mathbb{K}}(t). \tag{11}$$

Since $\mathrm{Cap}^F_{\mathbb{C},\mathbb{K}}(s)$ has a fixed state constraint $\mathbb{C}$ and fixed target $\mathbb{K}$ and an autonomous dynamics $F$, we can use the results of [5].

We consider the control problem, for $\xi \in \mathbb{R}^{d+1}$ and $\tau \geq 0$:

$$\vartheta(\xi,\tau) := \min_{a\in\Gamma_\tau}\max_{\beta\in\mathbb{B}_\tau}\left\{\max\left(\vartheta_0(z_\xi^{a[\beta],\beta}(\tau)),\ \max_{\theta\in[0,\tau]} g(z_\xi^{a[\beta],\beta}(\theta))\right)\right\} \tag{12}$$

where we recall that the Lipschitz function $g$ is related to the obstacle $\mathbb{K}$ by (9) (we note that for $t<0$ and $\tau\geq 0$, we have $\vartheta(x,t,\tau)>0$).

It is the use of the supremum norm that will enable us to deal with the controllability problem, because now (12) has no "explicit" state constraint. In fact, in this new setting, the term $\max_{\theta\in[0,\tau]} g(z_\xi(\theta))$ plays a role of a penalization that a trajectory $z_\xi$ would pay if it violates the state-constraints. Theorem 2 will show the advantage of considering (12), because $\vartheta$ will be characterized as the unique continuous solution of an HJB equation.

**Theorem 1** (Characterization of the capture basin)**.**
*Assume (H1)-(H3). Let $\vartheta_0$ (resp. $g$) be Lipschitz continuous functions satisfying* (8) *(resp.* (9)*). Let $\vartheta$ be the value function defined by* (12)*. For every $\tau\geq 0$, we have:*

$$\mathrm{Cap}^F_{\mathbb{C},\mathbb{K}}(\tau) = \left\{\xi\in\mathbb{R}^{d+1},\ \vartheta(\xi,\tau)\leq 0\right\}.$$

*In particular, we have,*

$$\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(\tau) = \left\{x\in\mathbb{R}^{d},\ \vartheta(x,0,\tau)\leq 0\right\}.$$

Now, the function $\vartheta$ can be characterized as the unique solution of a Hamilton-Jacobi equation. More precisely, considering the Hamiltonian

$$H^F(\xi,P) := \max_{a\in\mathcal{A}}\min_{b\in\mathcal{B}}\big(-F(\xi,a,b)\cdot P\big), \tag{13}$$

by using [3] (see also [17, 5]), we have

**Theorem 2.** *Assume (H1), and that $\vartheta_0$ and $g$ are Lipschitz continuous. Then $\vartheta$ is the unique continuous viscosity solution of the variational inequation (or "obstacle" problem)*

$$\min(\partial_\tau\vartheta + H^F(\xi,\, D_\xi\vartheta),\ \vartheta - g(\xi)) = 0, \qquad \tau>0,\ \xi\in\mathbb{R}^{d+1}, \tag{14a}$$

$$\vartheta(\xi,0) = \max(\vartheta_0(\xi), g(\xi)),\quad \xi\in\mathbb{R}^{d+1}. \tag{14b}$$

For sake of completeness the notion of viscosity solution is recalled in the appendix (see Definition 1).

Application. We are thus able to compute capture basin using regular functions. There are numerous schemes that can approximate the value function $\vartheta$ of the previous HJB or HJI equations. This gives a way to compute the set $\mathrm{Cap}^F_{\mathbb{C},\mathbb{K}}(t)$. Then in view of Theorem 1 we can find the points $x$ that belong to $\mathrm{Cap}^f_{\mathcal{C},\mathcal{K}}(t)$.

**Remark 2.** *From a theoretical point of view, the choice of $g$ is not important, and $g$ can be any Lipschitz function satisfying (9). Of course, the value function $\vartheta$ is dependent on $g$, while the set $\{\xi \in \mathbb{R}^{d+1}, \vartheta(\xi,t) \leq 0\}$ does not depend on $g$.*

There are other informations in the function $\vartheta$. For instance, let $y_{x,t}^{\alpha,\beta}$ denotes the solution of the differential equation (1a) and such that $y(t) = x$ (instead of $y(0) = x$). For $s \geq t$, we define

$$\text{Cap}^f_{\mathcal{C},\mathcal{K}}(t;s) := \Big\{ x \in \mathbb{R}^d,\ \exists a \in \Gamma,\ \forall \beta \in \mathbb{B}, \Big( y_x^{a[\beta],\beta}(s) \in \mathcal{C}_s, \quad \text{and } y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}_\theta,\ \forall \theta \in [t,s] \Big) \Big\} \tag{15}$$

Then we have

**Proposition 2.** *For any $\tau \geq 0$,*

$$\text{Cap}^f_{\mathcal{C},\mathcal{K}}(t;t+\tau) = \{x \in \mathbb{R}^d,\ \vartheta(x,t,\tau) \leq 0\}.$$

Indeed this comes from the fact that $y_x^{\alpha,\beta}(s) = y_{x,t}^{\bar\alpha,\bar\beta}(s+t)$ (where $\bar\alpha(s) := \alpha(s-t)$ and $\bar\beta(s) := \beta(s-t)$), and thus for $\xi = (x,t)$ we can deduce that

$$z_\xi(\tau) \in \mathbb{C} \quad \Leftrightarrow \quad y_{x,t}(\tau) \in \mathcal{C}_\tau$$

and

$$z_\xi(\tau \in \mathbb{K} \quad \Leftrightarrow \quad y_{x,t}(\tau) \in \mathcal{K}_\tau.$$

In other words, $\vartheta(x,t,\tau) \leq 0$ (for some $\tau \geq 0$) is equivalent to say that there exists some non-anticipative strategy $a[\cdot]$ such that (for any adverse strategy $\beta$) we can reach the target $\mathcal{C}_{t+\tau}$ at time $t+\tau$ starting from $x$ at time $t$.

## Appendix A. Proofs of the main results

For sake of simplicity these proofs are given in the one-player game setting.

*Proof of Theorem 1.* We reproduce here the idea of the proof that can be found in [5]. Assume that $\xi \in \text{Cap}^F_{\mathbb{C},\mathbb{K}}(\tau)$. Then there exists an admissible trajectory $z_\xi$ such that

$$\vartheta_0(z_\xi(t)) \leq 0, \text{ and } z_\xi(\theta) \in \mathbb{K} \text{ for every } \theta \in [0,\tau].$$

Hence, $\max_{\theta\in[0,\tau]} g(z_\xi(\theta)) \leq 0$, and we have:

$$\vartheta(\xi,\tau) \leq \max(\vartheta_0(z_\xi(\tau)), \max_{\theta\in[0,\tau]} g(z_\xi(\theta))) \leq 0.$$

Conversely, assume that $\vartheta(\xi,\tau) \leq 0$. Then there exists a trajectory $z_\xi$ for the dynamics $F$, such that

$$0 \geq \vartheta(\xi,\tau) = \max(\vartheta_0(z_\xi(\tau)), \max_{\theta\in[0,\tau]} g(z_\xi(\theta))).$$

Thus, for all $\theta \in [0,\tau]$, $g(z_\xi(\theta)) \leq 0$, i.e. $z_\xi(\theta) \in \mathbb{K}$, and $z_\xi$ is an admissible trajectory. Moreover, we have $\vartheta_0(z_\xi(\tau)) \leq 0$, hence $z_\xi(\tau) \in \mathbb{C}$ and we can conclude that $\xi \in \text{Cap}^F_{\mathbb{C},\mathbb{K}}(\tau)$. □

*Proof of Theorem 2.* It is based on a dynamic programming principle (DPP) for $\vartheta$ that we shall not reproduce here (see for instance [3, Proposition 3.1]).

We recall here the definition of viscosity solution for (7) (the definition in the case of (14) is similar).

**Definition 1** (Viscosity solution)**.** *An upper semi-continuous (resp. lower semi-continuous) function $\vartheta : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of* (7) *if $\vartheta(x,0) \leq \vartheta_0(x)$ in $\mathbb{R}^d$ (resp. $\vartheta(x,0) \geq \vartheta_0(x)$) and for any $(x,t) \in \mathbb{R}^d \times (0,\infty)$ and any test function $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\vartheta - \phi$ attains a maximum (resp. a minimum) at the point $(x,t) \in \mathbb{R}^d \times (0,\infty)$, then we have*

$$\min(\partial_t \phi + H(x, \nabla\phi),\ \vartheta - g(x)) \leq 0$$

$$(\text{resp.} \quad \min(\partial_t \phi + H(x, \nabla\phi),\ \vartheta - g(x)) \geq 0)\,.$$

*A continuous function $\vartheta$ is a viscosity solution of* (7) *if $\vartheta$ is a viscosity subsolution and a viscosity supersolution of* (7)*.*

The fact that $\vartheta$ is the unique solution of (7) follows from the comparison principle for (7) (which is classical, see for instance [2]), and the fact that the Hamiltonian function $H$ satisfies

$$|H(x_2,p) - H(x_1,p)| \leq C(1+|p|)\ |x_2 - x_1|,$$
$$|H(x,p_2) - H(x,p_1)| \leq C|p_2 - p_1|,$$

for some constant $C \geq 0$ and for all $x_i$, $p_i$, $x$ and $p$ in $\mathbb{R}^d$. □

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