



**HAL**  
open science

# Petri Net Reachability Graphs: Decidability Status of FO Properties

Philippe Darondeau, Stephane Demri, Roland Meyer, Christophe Morvan

► **To cite this version:**

Philippe Darondeau, Stephane Demri, Roland Meyer, Christophe Morvan. Petri Net Reachability Graphs: Decidability Status of FO Properties. IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, IARCS, Dec 2011, Bombay, India. inria-00627657

**HAL Id: inria-00627657**

**<https://inria.hal.science/inria-00627657v1>**

Submitted on 25 May 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Petri Net Reachability Graphs: Decidability Status of FO Properties

Philippe Darondeau<sup>1</sup>, Stéphane Demri<sup>2</sup>, Roland Meyer<sup>3</sup>, and Christophe Morvan<sup>4</sup>

1 IRISA/INRIA, Campus de Beaulieu, Rennes, France

philippe.darondeau@inria.fr

2 LSV, ENS Cachan, CNRS, INRIA, France [Stephane.Demri@lsv.ens-cachan.fr](mailto:Stephane.Demri@lsv.ens-cachan.fr)

3 University of Kaiserslautern, Germany [meyer@cs.uni-kl.de](mailto:meyer@cs.uni-kl.de)

4 Université Paris-Est, Marne-La-Vallée, France

[christophe.morvan@univ-paris-est.fr](mailto:christophe.morvan@univ-paris-est.fr)

---

## Abstract

We investigate the decidability and complexity status of model-checking problems on unlabelled reachability graphs of Petri nets by considering first-order, modal and pattern-based languages without labels on transitions or atomic propositions on markings. We consider several parameters to separate decidable problems from undecidable ones. Not only are we able to provide precise borders and a systematic analysis, but we also demonstrate the robustness of our proof techniques.

**1998 ACM Subject Classification** F.3.1 Specifying and Verifying and Reasoning about Programs, F.4.1 Mathematical Logic, D.2.4 Software/Program Verification

**Keywords and phrases** Petri nets, First order logic, Reachability graph

**Digital Object Identifier** 10.4230/LIPIcs.FSTTCS.2011.140

## 1 Introduction

**Decision problems for Petri nets.** Much effort has been dedicated to decision problems about Petri nets such as reachability or equivalence, or model checking logical fragments. Reachability is decidable [20] but this is a hard problem. Language equality is, by contrast, undecidable for labelled Petri nets [11, 1]. Many important problems have received decision procedures, e.g., boundedness [16], deadlock-freeness and liveness [10] (by reduction to reachability), semilinearity [12], etc. Hack's thesis [10] provides a comprehensive overview of problems equivalent to reachability. Hack showed that equality of reachability sets of two Petri nets with identical places is undecidable [11]. As our main contribution, we link this result to first-order logic expressing properties of general Petri net reachability graphs.

**Our motivations.** For Petri nets, model checking CTL formulae with atomic propositions expressing that a place contains at least one token is known to be undecidable [7]. This result carries over to all fragments of CTL containing the modalities EF or AF. Model checking CTL without atomic propositions but with next-time modalities indexed by action labels is undecidable too [7]. In contrast, LTL model-checking over VASS is EXPSPACE-complete [9] (atomic propositions are control states). These negative results do not compromise the search for decidable fragments of first-order logic that describe only purely graph-theoretically the reachability graphs. Our intention is to deliberately discard edge labels and atomic propositions on markings. As an example, we consider the structure  $(\mathbb{N}^n, \rightarrow)$  derived from a Petri net  $N$  with  $n$  places such that  $M \rightarrow M'$  iff  $M$  evolves to  $M'$  by firing a transition of  $N$ . Since  $(\mathbb{N}^n, \rightarrow, =)$  is an automatic structure, its first-order theory is decidable, see e.g. [5].



© P. Darondeau, S. Demri, R. Meyer, and C. Morvan;

licensed under Creative Commons License NC-ND

31<sup>st</sup> Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2011).

Editors: Supratik Chakraborty, Amit Kumar; pp. 140–151

Leibniz International Proceedings in Informatics



LIPIcs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

However, it is unclear what happens if we consider the first-order theory of  $\rightarrow$  over the more interesting structure  $(\text{Reach}(N), \rightarrow)$ . Here,  $\text{Reach}(N)$  denotes the set of all markings *reachable* from the initial marking of  $N$ . This paper investigates this question and therefore investigates the decidability status of first-order logic with a bit of MSO (via quantification over reachable markings) on  $\mathbb{N}^n$ , sharing with [21] a common motivation. We study properties of the Petri net reachability graph that are *purely graph-theoretical*; they do not refer to tokens or transition labels and they are mostly *local* in that they can often be expressed in terms of  $\rightarrow$  instead of its transitive closure. For instance, this contrasts with logics in [3] that state quantitative properties on markings and transitions, and evaluate formulae on runs.

**Our contributions.** We investigate the model-checking problem over structures of the form  $(\text{Reach}(N), \rightarrow, \overset{*}{\rightarrow})$  generated from Petri nets  $N$  with first-order languages including predicate symbols for  $\rightarrow$  and/or  $\overset{*}{\rightarrow}$ . As it is a classical fragment of first-order logic, we also consider the modal language  $\text{ML}(\Box, \Box^{-1})$  with forward and backward modalities. To conclude the study, we consider an alternative framework where the structures are *reachability sets*, subsets of  $\mathbb{N}^n$  when the underlying net has  $n$  places. For these structures, we study satisfiability of properties defined by *patterns*. Patterns are bounded  $n$ -dimensional sets of points that are colored black, white, or grey to mean “reachable”, “non-reachable”, or “don’t care”, respectively. Let us mention prominent features of our investigation. (1) Undecidability proofs are obtained by reduction from the equality problem (or the inclusion problem) between reachability sets defined by Petri nets, shown undecidable in [11]. We demonstrate that our proof schema is robust and can be adapted to numerous formalisms specifying local properties as in first-order logic. (2) To determine the cause of undecidability, we investigate logical fragments. At the same time, we strive for maximally expressive decidable fragments. (3) For decidable problems, we assess the computational complexity — either relative to standard complexity classes or by establishing a reduction from the reachability problem for Petri nets. Our main findings are as follows: Model-checking  $(\text{Reach}(N), \rightarrow)$  [resp.  $(\text{Reach}(N), \overset{*}{\rightarrow})$ ,  $(\text{Reach}(N), \overset{\pm}{\rightarrow})$ ] is undecidable for the appropriate first-order language with one binary predicate symbol. Undecidability is also shown for the positive fragment of  $\text{FO}(\rightarrow)$ , the forward fragment of  $\text{FO}(\rightarrow)$  and  $\text{FO}(\rightarrow)$  augmented with  $\overset{*}{\rightarrow}$  even if the reachability sets are effectively semilinear. We prove that model-checking the existential fragment of  $\text{FO}(\rightarrow)$  is decidable, but as hard as the reachability problem for Petri nets. As far as  $\text{ML}(\Box, \Box^{-1})$  is concerned, the global model-checking on  $(\text{Reach}(N), \rightarrow)$  is undecidable but it becomes decidable when restricted to  $\text{ML}(\Box)$  (even if extended with Presburger-definable predicates on markings); the latter problem is also as hard as the reachability problem for Petri nets. The satisfiability of properties defined by bounded patterns is undecidable.

## 2 Preliminaries

We recall basics on Petri nets and semilinear sets; we introduce Petri net reachability graphs as first-order structures. We define first-order logic and modal logic interpreted on these graphs. Finally, we present decidability results about model-checking problems.

### 2.1 Petri nets

A *Petri net* is a bi-partite graph  $N = (P, T, F, M_0)$ , where  $P$  and  $T$  are *finite* disjoint sets of *places* and *transitions*, and  $F : (P \times T) \cup (T \times P) \rightarrow \mathbb{N}$ . A *marking* of  $N$  is a function  $M : P \rightarrow \mathbb{N}$ .  $M_0$  is the *initial marking* of  $N$ . A transition  $t \in T$  is *enabled* at a marking

$M$ , written  $M[t]$ , if  $M(p) \geq F(p, t)$  for all places  $p \in P$ . If  $t$  is enabled at  $M$  then it can be fired. This leads to the marking  $M'$  defined by  $M'(p) = M(p) + F(t, p) - F(p, t)$  for all  $p \in P$ , in notation:  $M[t]M'$ . The definitions are extended to transition sequences  $s \in T^*$  in the expected way. A marking  $M'$  is *reachable* from a marking  $M$  if  $M[s]M'$  for some  $s \in T^*$ . A transition  $t$  is *in self-loop* with a place  $p$  iff  $F(p, t) = F(t, p) > 0$ . A transition is *neutral* if it has null effect on all places. The *reachability set*  $\text{Reach}(N)$  of  $N$  is the set of all markings that are reachable from the initial marking.

► **Theorem 2.1.** (I) [20] Given a Petri net  $N$  and two markings  $M$  and  $M'$ , it is decidable whether  $M'$  is reachable from  $M$ . (II) [11] Given two Petri nets  $N$  and  $N'$ , it is not decidable whether  $\text{Reach}(N) = \text{Reach}(N')$  [resp.  $\text{Reach}(N) \subseteq \text{Reach}(N')$ ].

A Petri net  $N$  induces several standard structures. The *unlabelled reachability graph* of  $N$  is the structure  $\text{URG}(N) = (D, \text{init}, \rightarrow, \xrightarrow{*}, \xrightarrow{+}, =)$  where  $D = \text{Reach}(N)$ ,  $\text{init} = \{M_0\}$ , and  $\rightarrow$  is the binary relation on  $D$  defined by  $M \rightarrow M'$  if  $M[t]M'$  for some  $t \in T$ . The relations  $\xrightarrow{*}$  and  $\xrightarrow{+}$  are the iterative and strictly iterative closures of  $\rightarrow$ , respectively. The *reachability graph*  $\text{RG}(N)$  of  $N$  is  $(\text{Reach}(N), \rightarrow)$ . The *unlabelled transition graph* of  $N$  is the structure  $\text{UG}(N) = (D, \text{init}, \rightarrow, \xrightarrow{*}, \xrightarrow{+}, =)$  with  $D = \mathbb{N}^P$ . Note that reachability of markings is not taken into account in  $\text{UG}(N)$ . In the sequel, by default  $\text{card}(P) = n$  and we identify  $\mathbb{N}^P$  and  $\mathbb{N}^n$ . We also call *1-loop* an edge  $M \rightarrow M'$  with  $M = M'$ .

*Semilinear subsets* of  $\mathbb{N}^n$  form an effective Boolean algebra and they coincide with sets definable in Presburger arithmetic (decidable first-order theory of natural numbers with addition). Hence, herein we use equally semilinearity or definability in Presburger arithmetic. Note that in [8], Ginsburg and Spanier gave an effective correspondence between semilinear subsets and subsets of  $\mathbb{N}^n$  definable in Presburger arithmetic. We know that given a Petri net  $N$  and a semilinear set  $E \subseteq \mathbb{N}^{|P|}$  one can decide whether  $\text{Reach}(N) \cap E \neq \emptyset$  [11, L. 4.3].

## 2.2 First-order languages

We introduce a first-order logic FO with atomic predicates  $x \rightarrow y$ ,  $x \xrightarrow{*} y$ ,  $x \xrightarrow{+} y$  and  $\text{init}(x)$ . Formulae in FO are defined by  $x \rightarrow y \mid x \xrightarrow{*} y \mid x \xrightarrow{+} y \mid \text{init}(x) \mid x = y \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x \varphi \mid \forall x \varphi$ . Given a set  $\mathbf{P}$  of predicate symbols from the above signature, we denote the *restriction of FO to the predicates in  $\mathbf{P}$*  by  $\text{FO}(\mathbf{P})$ . Formulae are interpreted either on  $\text{URG}(N)$  or on  $\text{UG}(N)$ . Observe that FO on  $\text{UG}(N)$  enables, using *init* and reachability predicates, to relativize formulae to  $\text{URG}(N)$ . We omit the standard definition of the satisfaction relation  $\mathcal{U}, \mathbf{v} \models \varphi$  with  $\mathcal{U}$  a structure ( $\text{URG}(N)$ ,  $\text{RG}(N)$  or  $\text{UG}(N)$ ) and  $\mathbf{v}$  a valuation of the free variables in  $\varphi$ . Typically,  $\forall x \varphi$  holds true whenever the formula  $\varphi$  holds true for all elements (markings) of the considered structure. *Sentences* are closed formulae, i.e. without free variables. If  $\mathcal{U} \models \varphi$  then  $\mathcal{U}$  is called a model of  $\varphi$ .

In the sequel, we consider several model-checking problems. The model-checking problem  $\text{MC}^{\text{URG}}(\text{FO})$  [resp.  $\text{MC}^{\text{UG}}(\text{FO})$ ] is stated as follows: given a Petri net  $N$  and a sentence  $\varphi \in \text{FO}$ , does  $\text{URG}(N) \models \varphi$  [resp.  $\text{UG}(N) \models \varphi$ ]? The logics  $\text{FO}(\mathbf{P})$  induce restricted model checking problems  $\text{MC}^{\text{URG}}(\text{FO}(\mathbf{P}))$  and  $\text{MC}^{\text{UG}}(\text{FO}(\mathbf{P}))$ , respectively. Formulae in FO can express standard structural properties, like deadlock-freeness ( $\forall x \exists y x \rightarrow y$ ) or cyclicity ( $\forall x \forall y x \xrightarrow{*} y \Rightarrow y \xrightarrow{*} x$ ). Semilinear sets and relations are known to be automatic (may be generated by finite synchronous automata [5]). In particular, it means that  $(\mathbb{N}^n, \rightarrow, =)$  is automatic. By [5],  $(\star) \text{MC}^{\mathcal{S}}(\text{FO})$  is decidable for each automatic structure  $\mathcal{S}$ . Proposition 2.2 below, consequence of  $(\star)$ , is our current state of knowledge.

► **Proposition 2.2.** (I)  $\text{MC}^{\text{UG}}(\text{FO}(\rightarrow, =))$  is decidable. (II) Let  $\mathcal{C}$  be a class of Petri nets  $N$  for which the restriction on  $\text{Reach}(N)$  of the reachability relation  $x \xrightarrow{*} y$  is effectively semilinear. Then,  $\text{MC}^{\text{URG}}(\text{FO})$  restricted to  $\mathcal{C}$  is decidable. (III) Let  $\mathcal{C}$  be a class of Petri nets  $N$  for which  $\text{Reach}(N)$  is effectively semilinear. Then,  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow, =))$  restricted to  $\mathcal{C}$  is decidable.

Here are some classes of Petri nets for which the reachability relation  $\xrightarrow{*}$  is effectively semilinear: cyclic Petri nets [2], communication-free Petri nets [6], vector addition systems with states of dimension 2 [18], single-path Petri nets [14], etc.

Note that given  $\varphi$  in  $\text{FO}(\rightarrow, =)$ , one can effectively build a Presburger formula that characterizes exactly the valuations satisfying  $\varphi$  in  $\text{UG}(N)$ . However, having  $\mathbb{N}^n$  as a domain does not always guarantee decidability, see the undecidability result in [21, Theorem 2] about a structure with domain  $\mathbb{N}^n$  but equipped with successor relations for each dimension and with regularity constraints on them.

### 2.3 Standard first-order fragments: modal languages

By moving along edges, modal languages provide a local view for graph structures. Note the contrast to first-order logic in which one quantifies over any element of the structure. Applications of modal languages include modelling temporal and epistemic reasoning, and they are central for designing logical specification languages. The modal language  $\text{ML}(\square, \square^{-1})$  (or simply  $\text{ML}$ ) defined below has no propositional variable (like Hennessy-Milner modal logic) and no label on modal operators. This allows us to interpret modal formulae on directed graphs of the form  $(\text{Reach}(N), \rightarrow)$ . The modal formulae in  $\text{ML}$  are defined by the grammar  $\perp \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \square\varphi \mid \square^{-1}\varphi$ . We write  $\text{ML}(\square)$  to denote the restriction of  $\text{ML}$  to  $\square$  and we use the standard abbreviations  $\diamond\varphi \stackrel{\text{def}}{=} \neg\square\neg\varphi$  and  $\diamond^{-1}\varphi \stackrel{\text{def}}{=} \neg\square^{-1}\neg\varphi$ . We interpret modal formulae on directed graphs  $(\text{Reach}(N), \rightarrow)$ . We provide the definition of the satisfaction relation  $\models$  relatively to an arbitrary directed graph  $\mathcal{M} = (W, R)$  and  $w \in W$  (clauses for Boolean connectives and logical constants are omitted):

- ★  $\mathcal{M}, w \models \square\varphi \stackrel{\text{def}}{=} \text{for every } w' \in W \text{ such that } (w, w') \in R, \text{ we have } \mathcal{M}, w' \models \varphi.$
- ★  $\mathcal{M}, w \models \square^{-1}\varphi \stackrel{\text{def}}{=} \text{for every } w' \in W \text{ such that } (w', w) \in R, \text{ we have } \mathcal{M}, w' \models \varphi.$

Model-checking problem  $\text{MC}^{\text{URG}}(\text{ML})$  is the following: given a Petri net  $N$  and  $\varphi \in \text{ML}$ , does  $(\text{Reach}(N), \rightarrow), M_0 \models \varphi$ ? Let  $\text{MC}^{\text{URG}}(\text{ML}(\square))$  denote  $\text{MC}^{\text{URG}}(\text{ML})$  restricted to  $\text{ML}(\square)$ .

► **Proposition 2.3.**  $\text{MC}^{\text{URG}}(\text{ML}(\square))$  is decidable and PSPACE-complete.

Adding  $\square^{-1}$  to  $\text{ML}(\square)$ , often does not change the computational complexity of model checking, see e.g. [4]. When it comes to Petri net reachability graphs  $\text{RG}(N)$ , adding  $\square^{-1}$  preserves decidability but at the cost of performing reachability checks. With a hardness result in Section 3.4, we argue that such checks cannot be avoided.

► **Proposition 2.4.**  $\text{MC}^{\text{URG}}(\text{ML}(\square, \square^{-1}))$  is decidable.

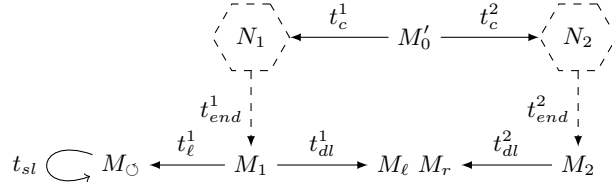
We introduce another decision problem about  $\text{ML}$  that is related to  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$ . The *validity problem*  $\text{VAL}^{\text{URG}}(\text{ML})$ , is stated as follows: given a Petri net  $N$  and  $\varphi \in \text{ML}$ , does  $(\text{Reach}(N), \rightarrow), M \models \varphi$  for every marking  $M \in \text{Reach}(N)$ ? As observed earlier, formulae from  $\text{ML}(\square, \square^{-1})$  can be viewed as first-order formulae in  $\text{FO}(\rightarrow)$ . Therefore, using modal languages in specifications is a way to consider fragments of  $\text{FO}(\rightarrow)$ . Indeed, given  $\varphi$  in  $\text{ML}(\square, \square^{-1})$ , one can compute in linear time a first-order formula  $\varphi'$  with only two individual variables (see e.g. [4]) that satisfies: for every Petri net  $N$  we have  $\text{RG}(N) \models \varphi'$  iff  $\text{RG}(N), M \models \varphi$  for every  $M$  in  $\text{Reach}(N)$ . Hence, the validity problem  $\text{VAL}^{\text{URG}}(\text{ML})$  appears as a natural counterpart to  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$ .

### 3 Structural Properties of Unlabelled Net Reachability Graphs

We study the decidability status of model checking unlabelled reachability graphs of Petri nets against the first-order and modal logics defined in the previous section.

#### 3.1 A proof schema for the undecidability of $\text{FO}(\rightarrow)$

To establish undecidability of  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$ , we provide a reduction of the equality problem for reachability sets, see Theorem 2.1(II). Given two Petri nets  $N_1$  and  $N_2$  with the same places, we build  $\overline{N}$  and  $\varphi$  in  $\text{FO}(\rightarrow)$  such that  $\text{Reach}(N_1) = \text{Reach}(N_2)$  iff  $\text{RG}(\overline{N}) \models \varphi$ . Interestingly,  $\varphi$  shall be independent of  $N_1$  and  $N_2$ .



■ **Figure 3.1** Reachability graph of  $\overline{N}$

In  $\overline{N}$ , the nets  $N_1$  and  $N_2$  to be compared for equality of reachability sets share all places except two added control places  $p_1$  and  $p_2$  (set in self-loop with the respective transitions of  $N_1$  and  $N_2$ ). The Petri net  $\overline{N}$  has one more extra place  $p$  initially marked. Two concurrent transitions  $t_c^1$  and  $t_c^2$  compete to consume the initial token and mark either  $p_1$  and all places marked in the initial configuration of  $N_1$  or  $p_2$  and all places marked in the initial configuration of  $N_2$  (see Figure 3.1). The first step in the execution of  $\overline{N}$  implements an arbitrary choice between simulating  $N_1$  or  $N_2$ .

Once the simulation of  $N_1$  or  $N_2$  has started, it may be stopped at any time. This is done by two transitions  $t_{end}^1$  and  $t_{end}^2$  that move the control token from  $p_1$  or  $p_2$  to a new control place  $p'_1$  or  $p'_2$ , thus leading to the marking  $M_1$  or  $M_2$  shown in Figure 3.1. After this, the token count on the places of  $N_1$  and  $N_2$  is not changed any more. Moving the token to  $p'_1$  or  $p'_2$  switches control to reporting subnets  $N'_1$  or  $N'_2$  that behave as indicated in Figure 3.1 starting from markings  $M_1$  and  $M_2$ .

By just emptying the control place  $p'_1$  or  $p'_2$ ,  $N'_1$  and  $N'_2$  may forget the index 1 or 2 of the net  $N_1$  or  $N_2$  that was simulated and enter a deadlock marking  $M$ , that reflects the last marking of  $N_1$  or  $N_2$  in the simulation. For this purpose,  $\overline{N}$  is provided with two transitions  $t_{dl}^1$  and  $t_{dl}^2$  (in Figure 3.1,  $M$  is denoted  $M_\ell$  and  $M_r$  indicating whether it emerged from the simulation of  $N_1$  (*left*) or  $N_2$  (*right*)).  $\text{Reach}(N_1) = \text{Reach}(N_2)$  iff every simulation result or deadlock marking  $M$  can be obtained from  $N_1$  and  $N_2$ . But inspecting  $M$  in isolation does not reveal whether it stemmed from  $N_1$  or  $N_2$ .

Deadlock markings ( $M$ ) and their immediate predecessor markings ( $M_1$  and/or  $M_2$ ) are easily characterized by first-order formulae. In order to express in  $\text{FO}(\rightarrow)$  that every simulation result  $M$  has exactly two direct ancestor markings  $M_1$  and  $M_2$  (such that  $M_1[t_{dl}^1]M$  and  $M_2[t_{dl}^2]M$ ), it is necessary that the behaviours of  $N'_1$  and  $N'_2$  from  $M_1$  or  $M_2$  can be distinguished by  $\text{FO}(\rightarrow)$  formulae. For this purpose, one gives to  $N'_1$  but not to  $N'_2$  the possibility to avoid the deadlock state  $M_\ell = M$  by firing from  $M_1$  a special transition  $t_\ell^1$  that leads to a marking ( $M_\circ$ ) with a 1-loop  $t_\circ$  (no new deadlock is introduced thus). In  $\overline{N}$ ,  $t_\ell^1$  competes with  $t_{dl}^1$  to move the token from the control place  $p'_1$  to another control place

$p_{\circlearrowleft}$ , controlling the 1-loop  $t_{\circlearrowleft}$ . In this way, the formula  $\varphi_{\ell}(x) \stackrel{\text{def}}{=} \exists y (x \rightarrow y \wedge y \rightarrow x)$  holds in markings  $M_1$  and does not hold in markings  $M_2$ .

A formula  $\varphi$  expressing that  $N_1$  and  $N_2$  have equal reachability sets is then:  $\forall z (\neg \exists z' z \rightarrow z') \Rightarrow (\exists z_1 z_1 \rightarrow z \wedge \varphi_{\ell}(z_1)) \wedge (\exists z_2 z_2 \rightarrow z \wedge \neg \varphi_{\ell}(z_2))$ . The formula  $\varphi$  requires that for any simulation result  $M$ , both logical experiments witnessing for  $N_1$  and  $N_2$  succeed. It is important to observe that the only deadlock markings of  $\overline{N}$  are the markings reached by the transitions  $t_{dl}^1$  and  $t_{dl}^2$ . Lemma 3.1 below, based on this remark, shows that the formula  $\varphi$  expresses in fact the equality of the reachability sets of  $N_1$  and  $N_2$ .

The strength of the construction stems from the combination of two ideas. A Petri net can (i) store choices over arbitrary long histories and (ii) reveal this propagated information by finite back and forth experiments determining local structures characterised by first-order formulae. The experiments consist here of one backward transition, reconstructing the initial choice, and some forward transitions checking the presence of a 1-loop.

► **Lemma 3.1.** *Reach( $N_1$ ) = Reach( $N_2$ ) if and only if  $\text{RG}(\overline{N}) \models \varphi$ .*

For the implication from left to right, consider a deadlock marking  $M$ .  $M$  is only reachable via  $t_{dl}^1$  or  $t_{dl}^2$ , say  $M'_1[t_{dl}^1]M$ . Then marking  $M'_1$  satisfies  $\varphi_{\ell}$  and stems from a marking  $M_1[t_{end}^1]M'_1$  of  $N_1$ . The hypothesis on equal reachability sets then yields a marking  $M_2$  of  $N_2$  that leads by transition  $t_{end}^2$  to a marking  $M'_2$  satisfying  $\neg \varphi_{\ell}$  as required.

In turn, if  $\varphi$  holds, then we prove two inclusions. To show  $\text{Reach}(N_1) \subseteq \text{Reach}(N_2)$ , consider marking  $M_1$  reachable via sequence  $s_1$  in  $N_1$ . In  $\overline{N}$ , the marking can be prolonged to a deadlock  $M$  with  $M'_0[t_c^1]M_0[s_1]M_1[t_{end}^1]M'_1[t_{dl}^1]M$ . Here,  $M'_1$  satisfies  $\varphi_{\ell}$ . But  $\varphi$  yields another predecessor  $M'_2$  of  $M$  with  $M'_2 \neq M'_1$ . To avoid the 1-loop, it has to result from a sequence  $M'_0[t_c^2]M_0[s_2]M_2[t_{end}^2]M'_2[t_{dl}^2]M$ . It is readily checked that  $M_1$  and  $M_2$  coincide up to the token on the control place. This means  $M_1 \in \text{Reach}(N_2)$  as required.

By recycling variables in  $\varphi$  above, we get a sharp result that marks the undecidability border of model checking against  $\text{FO}(\rightarrow)$  by two variables. Model checking  $\text{FO}(\rightarrow)$  restricted to a one variable is decidable.

► **Theorem 3.2.** *There exists a formula  $\varphi$  in  $\text{FO}(\rightarrow)$  with two individual variables such that  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$  restricted to  $\varphi$  is undecidable.*

The above undecidability result can be further sharpened since it is shown in [15] that the undecidability of the equality problem holds already for Petri nets with 5 unbounded places.

## 3.2 Robustness of the proof schema

Based on the previous proof schema, we present undecidability results for subproblems of  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$ . We consider the positive fragment, the forward fragment, the restriction when the direction of edges is omitted, and  $\text{ML}(\square, \square^{-1})$ . Let  $\lambda(x, x') \stackrel{\text{def}}{=} (x \rightarrow x') \vee (x' \rightarrow x)$ . Expressing properties about  $\text{RG}(N)$  in  $\text{FO}(\lambda)$  amounts to getting rid of the direction of edges of this graph. Despite this weakening, undecidability is still present. To instantiate the above argumentation, we have to identify deadlock markings and analyse their environment. In  $\text{FO}(\lambda)$ , we augment markings encountered during the simulation by 3-cycles. Then, the absence of 3-cycles and an environment without such cycles characterises deadlock markings.

► **Proposition 3.3.**  $\text{MC}^{\text{URG}}(\text{FO}(\lambda))$  is undecidable.

Proposition 3.4 below is proved by adapting the construction depicted in Figure 3.1.

► **Proposition 3.4.**  $\text{VAL}^{\text{URG}}(\text{ML}(\square, \square^{-1}))$  is undecidable.

This undecidability result is tight (see Section 3.3). Translating formulae in  $\text{ML}(\Box, \Box^{-1})$  to  $\text{FO}(\rightarrow)$  with two individual variables gives another evidence that  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$  with two variables is undecidable. Although  $\text{VAL}^{\text{URG}}(\text{ML}(\Box, \Box^{-1}))$  and  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$  are undecidable, we have identified decidable fragments of modal logic in Section 2.3. By analogy, one may expect to find decidable fragments of first-order logic. We prove that this is not the case. We consider here positive  $\text{FO}(\rightarrow)$  and forward  $\text{FO}(\rightarrow)$ . In a *positive formula*, atomic propositions occur only under the scope of an even number of negations. Let  $\text{FO}^+(\text{P})$  denote the positive fragment of  $\text{FO}(\text{P})$ . A *forward formula* is a formula in which every occurrence  $x \rightarrow y$  is in the scope of a quantifier sequence of the form  $Q_1 x \dots Q_2 y$  where  $x$  is bound before  $y$ . Let  $\text{FO}_f(\text{P})$  denote the forward fragment of  $\text{FO}(\text{P})$ .

► **Proposition 3.5.**  $\text{MC}^{\text{URG}}(\text{FO}^+(\rightarrow))$  is undecidable.

► **Proposition 3.6.**  $\text{MC}^{\text{URG}}(\text{FO}_f(\rightarrow))$  is undecidable.

While forward formulae can well identify the deadlock markings used in the proof schema, the difficulty is in the description of the local environment witnessing the simulation results.

### 3.3 Taming undecidability with fragments

In this section, we present the restrictions of  $\text{FO}(\rightarrow)$  that we found to have decidable model checking or validity problems. We write  $\exists\text{FO}$  for the fragment of  $\text{FO}$  whose formulae use only existential quantification when written in prenex normal form.

► **Proposition 3.7.**  $\text{MC}^{\text{URG}}(\exists\text{FO}(\rightarrow, =))$  is decidable.

Decidability of  $\text{MC}^{\text{URG}}(\exists\text{FO}(\rightarrow, =))$  is obtained by checking the existence of reachable markings satisfying Presburger constraints. As a corollary,  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow, =))$  restricted to Boolean combinations of existential formulae is decidable, and so is the subgraph isomorphism problem as follows: given a finite directed graph  $\mathcal{G}$  and a Petri net  $N$ , is there a subgraph of  $(\text{Reach}(N), \rightarrow)$  isomorphic to  $\mathcal{G}$ ? Section 3.2 proves that  $\text{VAL}^{\text{URG}}(\text{ML}(\Box, \Box^{-1}))$  is undecidable. To our surprise, and in contrast to the negative result on model checking the forward fragment of  $\text{FO}$ , this undecidability depends on the backward modality, see Proposition 3.8 below (it can be extended to allow labels on edges). We write  $\text{PAML}(\Box)$  to denote the extension of  $\text{ML}(\Box)$  by allowing as atomic formulae quantifier-free Presburger formulae about the number of tokens in places.

► **Proposition 3.8.** The validity problem  $\text{VAL}^{\text{URG}}(\text{PAML}(\Box))$  is decidable.

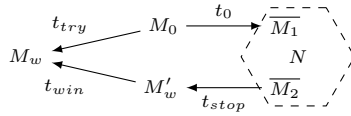
Decidability mainly holds because (non-)satisfaction of formulae in  $\text{PAML}(\Box)$  requires the existence of finite tree-like patterns and if the root is in  $\text{Reach}(N)$ , so are all its nodes (unlike with  $\text{ML}(\Box, \Box^{-1})$ ).

### 3.4 On the hardness of the decidable problems

Some of our decision procedures call subroutines for solving reachability in Petri nets. As this problem is not known to be primitive recursive, we provide here some complexity-theoretic justification for these costly invocations: we reduce the reachability problem for Petri nets to the decidable problems  $\text{MC}^{\text{URG}}(\text{ML}(\Box, \Box^{-1}))$  and to  $\text{MC}^{\text{URG}}(\exists\text{FO}(\rightarrow))$ . Besides reachability, we gave decision procedures that exploit the semilinearity of reachability sets or relations (see e.g. Proposition 2.2), but already for bounded Petri nets,  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$  is of high complexity.

► **Proposition 3.9.**  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$  restricted to bounded Petri nets is decidable but this problem has nonprimitive recursive complexity.





■ **Figure 3.2** Reachability graph in the hardness proof of  $ML(\square, \square^{-1})$ -model checking

► **Proposition 3.10.** There is a logarithmic-space reduction from the reachability problem for Petri nets to  $MC^{URG}(ML(\square, \square^{-1}))$ .

We reduce reachability of marking  $M_2$  from marking  $M_1$  in a Petri net  $N$  to an instance of  $MC^{URG}(ML(\square, \square^{-1}))$  for a larger net  $\bar{N}$ . The idea is to introduce a marking  $M_w$  (see Figure 3.2) such that the existence of a path to  $M_w$  of length greater than 1 witnesses for the existence of some path from  $M_1$  and  $M_2$  in  $RG(N)$ . To reach  $M_w$  by an ML formula, we place it close to the new initial marking. We sketch the argumentation. The initial marking  $M_0$  of  $\bar{N}$  contains a single marked place  $p_i$  on which compete two transitions  $t_{try}$  and  $t_0$ . Transition  $t_{try}$  moves the unique token from  $p_i$  to another place  $p_w$  and thus produces the marking  $M_w$  where no other place is marked. Transition  $t_0$  loads  $M_1$  in the places of  $N$  and moves the control token from  $p_i$  to another control place  $p_c$  set in self-loop with all transitions of  $N$ . This starts the simulation of  $N$  from  $M_1$ . The simulation may be interrupted whenever it reaches a marking of  $N$  greater than or equal to  $M_2$ . Then, transition  $t_{stop}$  consumes  $M_2$  from the places of  $N$  and moves the control token from  $p_c$  to a place  $p_{w'}$ . The control token is finally moved from  $p_{w'}$  to  $p_w$  by firing  $t_{win}$ .  $M_w$  is reached, after firing  $t_{stop} t_{win}$ , iff  $\bar{M}_2$  is reached. Therefore  $M_2$  is reachable from  $M_1$  iff  $M_w$  is reachable from  $\bar{M}_1$  (its restriction to places of  $N$  equals  $M_1$ ). This is equivalent to stating that  $M_w$  has a predecessor different from  $M_0$ . The shape of the reachability graph enables to formulate the latter as a local property in  $ML(\square, \square^{-1})$ :  $\varphi := \diamond(\square \perp \wedge \diamond^{-1} \diamond^{-1} \top)$ . Without loss of generality, we can assume that  $M_1$  is no deadlock and  $M_2 \neq M_1$ . Formula  $\varphi$  requires that  $M_0$  has a deadlock successor and has an incoming path of length two. That the successor is a deadlock means it is not  $\bar{M}_1$  but  $M_w$  obtained by firing  $t_{try}$ . The path from  $M_0$  to  $M_w$  is of length one and  $M_0$  has no predecessor. So the path of length two to  $M_w$  is not via  $t_{try}$  but stems from  $t_{win}$ . This means  $M_w$  is reachable from  $\bar{M}_1$ , which means  $M_2$  is reachable from  $M_1$  in  $N$ .

The proof of Proposition 3.10 can be adapted to  $\exists FO(\rightarrow)$  for which we also have shown decidability of the model-checking by reduction to the reachability problem for Petri nets.

► **Proposition 3.11.** There is a logarithmic-space reduction from the reachability problem for Petri nets to  $MC^{URG}(\exists FO(\rightarrow))$ .

## 4 FO with Reachability Predicates

We consider several first-order languages with reachability relations  $\overset{*}{\rightarrow}$  or  $\overset{\pm}{\rightarrow}$ , mainly without the one-step relation  $\rightarrow$ . Undecidability does not follow from Theorem 3.2 since we may exclude  $\rightarrow$ . Nonetheless we follow the same proof schema. Besides, we distinguish the case when reachability sets are semilinear leading to a surprising undecidability result (Proposition 4.4). Finally, we show that  $MC^{UG}(FO(\rightarrow, \overset{*}{\rightarrow}))$  is undecidable too.

### 4.1 FO with reachability relations

The decidability status of  $MC^{URG}(FO(\overset{\pm}{\rightarrow}))$  is not directly dependent upon the decidability status of  $MC^{URG}(FO(\rightarrow))$ . Still we are able to adapt the construction of Section 3.1 but

using now a formula  $\varphi$  in  $\text{FO}(\overset{\pm}{\rightarrow})$ . The Petri net  $\overline{N}$  is the one depicted on Figure 3.1. The formula  $\varphi$  is defined as follows:  $\varphi \stackrel{\text{def}}{=} \forall z \, dl(z) \Rightarrow (\exists z_1 \, (z_1 \overset{\pm}{\rightarrow} z) \wedge \varphi_2(z_1)) \wedge (\exists z_2 \, (z_2 \overset{\pm}{\rightarrow} z) \wedge \psi_2(z_2))$  where  $dl(z) \stackrel{\text{def}}{=} \neg \exists z' \, z \overset{\pm}{\rightarrow} z'$ ,  $sl(y) \stackrel{\text{def}}{=} y \overset{\pm}{\rightarrow} y \wedge \forall w \, [y \overset{\pm}{\rightarrow} w \Rightarrow w \overset{\pm}{\rightarrow} y]$ ,  $\varphi_2(z) \stackrel{\text{def}}{=} [\exists y \, z \overset{\pm}{\rightarrow} y \wedge sl(y)] \wedge [\forall y \, z \overset{\pm}{\rightarrow} y \Rightarrow (sl(y) \vee dl(y))]$ , and  $\psi_2(z) \stackrel{\text{def}}{=} [\exists y \, z \overset{\pm}{\rightarrow} y \wedge \forall y \, z \overset{\pm}{\rightarrow} y \Rightarrow dl(y)]$ . One can show that  $\text{Reach}(N_1) = \text{Reach}(N_2)$  iff  $\text{RG}(\overline{N}) \models \varphi$ .

► **Proposition 4.1.**  $\text{MC}^{\text{URG}}(\text{FO}(\overset{\pm}{\rightarrow}))$  is undecidable. Furthermore this results holds for the fixed formula  $\varphi$  defined earlier.

In order to prove undecidability of  $\text{MC}^{\text{URG}}(\text{FO}(\overset{*}{\rightarrow}))$  we have to adapt our usual proof schema, since, in contrast with  $\text{FO}(\overset{\pm}{\rightarrow})$ , we are no longer able to identify 1-loops.

► **Proposition 4.2.**  $\text{MC}^{\text{URG}}(\text{FO}(\overset{*}{\rightarrow}))$  is undecidable.

Even though  $\text{MC}^{\text{UG}}(\text{FO}(\rightarrow, =))$  is decidable (see Proposition 2.2), replacing  $\rightarrow$  by  $\overset{*}{\rightarrow}$  and adding *init* leads to undecidability.

► **Corollary 4.3.**  $\text{MC}^{\text{UG}}(\text{FO}(\textit{init}, \overset{*}{\rightarrow}))$  is undecidable.

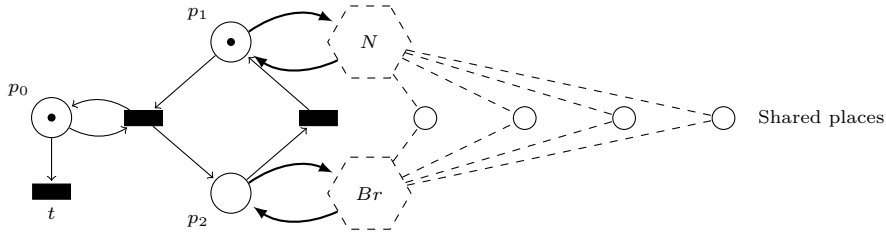
Indeed,  $\text{MC}^{\text{URG}}(\text{FO}(\overset{*}{\rightarrow}))$  reduces to  $\text{MC}^{\text{UG}}(\text{FO}(\textit{init}, \overset{*}{\rightarrow}))$  by relativization:  $\text{URG}(N) \models \varphi$  iff  $\text{UG}(N) \models \exists x_0 \, \textit{init}(x_0) \wedge f(\varphi)$  where  $\varphi$  and  $f(\varphi)$  are in  $\text{FO}(\overset{*}{\rightarrow})$ ,  $f$  is homomorphic for Boolean connectives and  $f(\forall x \, \psi) \stackrel{\text{def}}{=} \forall x \, (x_0 \overset{*}{\rightarrow} x) \Rightarrow f(\psi)$ .

## 4.2 When semilinearity enters into the play

We saw that  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow, =))$  restricted to Petri nets with effectively semilinear reachability sets is decidable (see Proposition 2.2), but it is unclear what happens if the relation  $\overset{*}{\rightarrow}$  is added. We establish that  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow, \overset{*}{\rightarrow}))$  restricted to Petri nets with semilinear reachability sets is undecidable, by a reduction from  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow))$ . Given a Petri net  $N$  and a sentence  $\varphi \in \text{FO}(\rightarrow)$ , we reduce the truth of  $\varphi$  in  $\text{RG}(N)$  to the truth of a formula  $\overline{\varphi}$  in  $\text{RG}(\overline{N})$  with a semilinear reachability set. The Petri net  $\overline{N}$  is defined from  $N$  by adding the new places  $p_0$ ,  $p_1$  and  $p_2$ ; each transition from  $N$  is in self-loop with  $p_1$ . Moreover, we add a new set of transitions that are in self-loop with  $p_2$  and that consist in adding or removing tokens from the original places of  $N$  (thus modifying its content arbitrarily). These transitions form a subnet denoted by  $Br$ . Three other transitions are added; see Figure 4.1 for a schematic representation of  $\overline{N}$  (initial marking  $M'_0$  of  $\overline{N}$  restricted to places in  $N$  is  $M_0$  with  $M'_0(p_0) = M'_0(p_1) = 1$  and  $M'_0(p_2) = 0$ ). Our intention is to enforce  $\text{Reach}(\overline{N})$  to be semilinear while being able to identify a subset from  $\text{Reach}(\overline{N})$  that is in bijection with  $\text{Reach}(N)$ ; this is a way to drown  $\text{Reach}(N)$  into  $\text{Reach}(\overline{N})$ . Indeed,  $\text{Reach}(\overline{N})$  contains all the markings such that the sum of  $p_1$  and  $p_2$  is 1 and  $p_0$  is at most 1. Moreover, if the transition  $t$  is fired first, then the subsequently reachable markings are precisely those of  $N$ ;  $\text{RG}(N)$  embeds isomorphically into  $\text{RG}(\overline{N})$ . Until  $t$  is fired, one may always come back to  $M'_0$ , using the brownian subnet  $Br$ , but this is impossible afterwards.

► **Proposition 4.4.**  $\text{MC}^{\text{URG}}(\text{FO}(\rightarrow, \overset{*}{\rightarrow}))$  restricted to Petri nets with semilinear reachability sets is undecidable.

**Proof.** In a first stage, we use *init* although this predicate cannot be expressed in  $\text{FO}(\rightarrow, \overset{*}{\rightarrow})$ . Let  $\overline{\varphi}$  be the formula  $\exists x_0 \, x_1 \, \textit{init}(x_0) \wedge x_0 \rightarrow x_1 \wedge \neg(x_1 \overset{*}{\rightarrow} x_0) \wedge f(\varphi)$  where  $f(\cdot)$  is homomorphic for Boolean connectives and  $f(\forall x \, \psi) \stackrel{\text{def}}{=} \forall x \, (x_1 \overset{*}{\rightarrow} x) \Rightarrow f(\psi)$  (relativization). In  $\overline{\varphi}$ ,  $x_0$  is interpreted as the initial marking, and  $x_1$  is interpreted as a successor of  $x_0$  from which  $x_0$  cannot be reached again. This may only happen by firing  $t$  from  $M'_0$ . Now the relativization of every other variable to  $x_1$  in  $\overline{\varphi}$  ensures that  $\text{RG}(N) \models \varphi$  iff  $\text{RG}(\overline{N}) \models \overline{\varphi}$ . To remove



■ **Figure 4.1** Petri net  $\overline{N}$

*init*, we construct a Petri net  $\overline{N'}$  similar to  $\overline{N}$ .  $\overline{N'}$  has an extra place  $p'_0$ , initially marked with one token, and a new transition that consumes this token and produces two tokens in  $p_0$  and  $p_1$ , which were initially empty. By construction, the initial marking of  $\overline{N'}$  is the sole marking in  $\text{RG}(\overline{N'})$  with no incoming edge and one outgoing edge. We use the formula  $\overline{\varphi}' = \exists x'_0 x_0 x_1 (\neg \exists y y \rightarrow x'_0) \wedge x'_0 \rightarrow x_0 \wedge x_0 \rightarrow x_1 \wedge (\neg x_1 \xrightarrow{*} x_0) \wedge f(\varphi)$ . For the same reasons as above,  $\text{RG}(N) \models \varphi$  iff  $\text{RG}(\overline{N'}) \models \overline{\varphi}'$ . ◀

### 4.3 The reachability relation and the structure $UG$

Corollary 4.3 states a first undecidable result for  $UG$ . In this section we examine two other cases where the model checking of formulas in  $\text{FO}(\rightarrow, \xrightarrow{*})$  are undecidable for this structure.

► **Proposition 4.5.**  $\text{MC}^{\text{UG}}(\text{FO}(\rightarrow, \xrightarrow{*}))$  is undecidable.

Proposition 4.5 holds even when the reachable set of the net is effectively semilinear.

► **Proposition 4.6.**  $\text{MC}^{\text{UG}}(\text{FO}(\rightarrow, \xrightarrow{*}))$  is undecidable for classes of Petri nets having an effective semilinear reachability set.

In this section we have examined several first-order sublanguages involving the reachability predicate. We obtained undecidability results, even when the reachable markings form a semilinear set, and even when  $UG(N)$  is considered instead of  $\text{URG}(N)$

## 5 Pattern Matching Problem

In this section, we do not consider the reachability graphs of Petri nets but their reachability sets ( $\text{Reach}(N)$ ), plain subsets of  $\mathbb{N}^n$  where  $n$  is the number of places of the net. In [17] the author characterizes such sets as *almost-semilinear* sets, a global property. On the opposite, we focus here on the shape of local neighborhoods by determining the existence of markings in  $\mathbb{N}^n$  whose surrounding satisfies a specific pattern of reachable and non-reachable positions.

Using such patterns, one may check for instance whether there exist two reachable markings that differ only on a fixed place and by exactly one token.

A *pattern*  $\mathcal{P}$  is defined as a map  $[0, N_1] \times \dots \times [0, N_n] \rightarrow \{\{\circ\}, \{\bullet\}, \{\circ, \bullet\}\}$  (values 'unreachable', 'reachable', 'dontcare'). A *constrained* position for  $\mathcal{P}$  is an element of  $[0, N_1] \times \dots \times [0, N_n]$  with  $\mathcal{P}$ -image different from  $\{\circ, \bullet\}$ . Observe that patterns have the full dimension of the state space of the net. Each Petri net  $N$  with  $n$  (ordered) places induces a map  $f_N : \mathbb{N}^n \rightarrow \{\{\circ\}, \{\bullet\}\}$  such that  $f_N(M) = \{\bullet\}$  iff  $M \in \text{Reach}(N)$ . Given a Petri net  $N$ , a pattern  $\mathcal{P}$  is *matched* by the net  $N$  at a point  $\vec{v} \in \mathbb{N}^n$  if, for all  $\vec{a} \in [0, N_1] \times \dots \times [0, N_n]$ ,  $f_N(\vec{v} + \vec{a}) \subseteq \mathcal{P}(\vec{a})$ . A pattern  $\mathcal{P}$  is *matched* by a Petri net  $N$  if it is matched by  $N$  at some point  $\vec{v} \in \mathbb{N}^n$  (that may not be a reachable marking). The *Pattern Matching Problem* (PMP) is defined as follows: given a Petri net  $N$  and a pattern  $\mathcal{P}$ , is  $\mathcal{P}$  matched by  $N$ ?

■ **Table 1** Summary

Problem	#	Arbitrary	Effectively semilinear $\text{Reach}(N)$
$\text{MC}^\sharp(\text{FO}(\rightarrow))$	<i>URG</i> <i>UG</i>	<b>UNDEC</b> (Theo. 3.2) DEC	<b>DEC</b> DEC
$\text{MC}^\sharp(\text{FO}(\overset{\rightarrow}{\rightarrow}))$	<i>URG</i>	<b>UNDEC</b> (Prop. 4.1)	open
$\text{MC}^\sharp(\text{FO}(\overset{*}{\rightarrow}))$	<i>URG</i>	<b>UNDEC</b> (Prop. 4.2)	open
$\text{MC}^\sharp(\text{FO}(\rightarrow, \overset{*}{\rightarrow}))$	<i>URG</i> <i>UG</i>	UNDEC <b>UNDEC</b> (Prop. 4.5)	<b>UNDEC</b> (Prop. 4.4) <b>UNDEC</b> (Prop. 4.6)
$\text{MC}^\sharp(\text{FO}^+(\rightarrow))$	<i>URG</i>	<b>UNDEC</b> (Prop. 3.5)	DEC
$\text{MC}^\sharp(\text{FO}_f(\rightarrow))$	<i>URG</i>	<b>UNDEC</b> (Prop. 3.6)	DEC
$\text{MC}^\sharp(\exists\text{FO}(\rightarrow, =))$	<i>URG</i>	<b>DEC</b> (Prop. 3.7)	DEC
$\text{MC}^\sharp(\text{FO}(\rightarrow, =))$	<i>UG</i>	<b>DEC</b> (Prop. 2.2)	DEC
$\text{MC}^\sharp(\text{ML}(\square))$	<i>URG</i>	<b>PSPACE-complete</b>	PSPACE-complete
$\text{MC}^\sharp(\text{ML}(\square, \square^{-1}))$	<i>URG</i>	<b>DEC</b> (Prop. 2.4)	DEC
$\text{VAL}^\sharp(\text{ML}(\square, \square^{-1}))$	<i>URG</i>	<b>UNDEC</b> (Prop. 3.4)	DEC
$\text{VAL}^\sharp(\text{PAML}(\square))$	<i>URG</i>	<b>DEC</b> (Prop. 3.8)	DEC
PMP	$\mathbb{N}^n$	<b>UNDEC</b> (Proposition 5.1)	<b>DEC</b> (Proposition 5.1)

► **Proposition 5.1.** (1) Let  $\mathcal{C}$  be a class of Petri nets with effectively semilinear reachability sets. Then, PMP restricted to Petri nets in  $\mathcal{C}$  is decidable. (2) PMP restricted to patterns with at most two constrained positions is undecidable.

Proposition 5.1(1) follows from the semilinearity of the set of marking satisfying patterns. To prove (2) we embed the reachable sets of two nets into two hyperplanes. Then these sets do not match iff there are two markings one reachable, the other not which may be encoded into a pattern. We use, here, a pattern with 2 adjacent, reachable and non-reachable, positions. It seems uneasy to prove this result using patterns having a single kind of constraints.

## 6 Concluding Remarks

We investigated mainly the model-checking problem over unlabelled reachability graphs of Petri nets with  $\text{FO}(\rightarrow)$ . The robustness of our main undecidability proof has been tested against standard fragments of  $\text{FO}(\rightarrow)$ , modal fragments, patterns and against the additional assumption that reachability sets are effectively semilinear. Table 1 provides a summary of the main results (observe that whenever the reachability relation is effectively semilinear, each problem is decidable). Results in bold are proved in the paper, whereas unbold ones are their consequences. Despite the quantity of results, a few rules of thumb can be synthesized: (1) undecidability of  $\text{MC}(\text{FO}(\rightarrow))$  is robust for several fragments of  $\text{FO}(\rightarrow)$ ; (2) decidability results with simple restrictions such as considering bounded Petri nets or  $\exists\text{FO}(\rightarrow)$  lead to computationally difficult problems (see Section 3.4); (3) the above points are still relevant for modal languages and patterns. Let us conclude by mentioning possible continuations of this work. Firstly, our taxonomy of results is partially incomplete.

New directions can also be followed. First, one could check geometrical properties of the reachability set  $\text{Reach}(N)$ , e.g., the existence of an homogeneous ball around some reachable marking. Second, one could ask decidability questions about infinite unfoldings of nets in place of net reachability graphs. Such unfoldings may be shaped as trees if they may be local event structures [13]. With tree-unfoldings, labelling arcs (or nodes) is required if one wants to be able to express non-trivial properties, but then markings can be encoded to trees in which each arc represents one token being removed from a place identifies by the label of the arc. With event structure unfoldings, labelling an event  $e$  by a (sufficiently large)

number  $k$  may always be simulated by adding  $k$  events triggered by  $e$  and in direct conflict with one another. In both cases, for obtaining decidable fragments of FO, one must avoid introducing any relation that would allow comparing for isomorphism two subtrees of two substructures triggered by two different events (like  $t_{dl}^1$  and  $t_{dl}^2$  in Fig. 3.1). The situation is different with *regular* trace event structures, although the substructure triggered by an event is characterized here by the label of this event. The decidability of FO over regular trace event structures has indeed been shown in [19]. However, *regular* trace event structures model safe Petri nets, whereas the model checking questions studied in this paper bear upon general and thus unbounded Petri nets.

**Acknowledgments:** We would like to thank the anonymous referees for helpful remarks and suggestions.

---

### References

- 1 T. Araki and T. Kasami. Some decision problems related to the reachability problem for Petri nets. *TCS*, 3:85–104, 1976.
- 2 T. Araki and T. Kasami. Decidability problems on the strong connectivity of Petri net reachability sets. *TCS*, 4:99–119, 1977.
- 3 M. F. Atig and P. Habermehl. On Yen’s path logic for Petri nets. *International Journal of Foundations of Computer Science*, 22(4):783–799, 2011.
- 4 P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. CUP, 2001.
- 5 A. Blumensath and E. Grädel. Automatic structures. In *LICS’00*, pages 51–62, 2000.
- 6 J. Esparza. Petri nets, commutative context-free grammars, and basic parallel processes. *Fundamenta Informaticae*, 31(13):13–26, 1997.
- 7 J. Esparza. Decidability and complexity of Petri net problems — an introduction. In *Advances in Petri Nets 1998*, volume 1491 of *LNCS*, pages 374–428. Springer, 1998.
- 8 S. Ginsburg and E. Spanier. Semigroups, Presburger formulas, and languages. *Pacific Journal of Mathematics*, 16:285–296, 1966.
- 9 P. Habermehl. On the complexity of the linear-time mu-calculus for Petri nets. In *IC-ATPN’97*, volume 1248 of *LNCS*, pages 102–116. Springer, 1997.
- 10 M. Hack. *Decidability Questions for Petri nets*. PhD thesis, MIT, 1975.
- 11 M. Hack. The equality problem for vector addition systems is undecidable. *TCS*, 2:77–96, 1976.
- 12 D. Hauschildt. Semilinearity of the reachability set is decidable for Petri nets. Technical Report FBI-HH-B-146/90, University of Hamburg, 1990.
- 13 P. W. Hoogers, H. C. M. Kleijn, and P. S. Thiagarajan. An event structure semantics for general Petri nets. *TCS*, 153:129–170, 1993.
- 14 R. Howell, P. Jančar, and L. Rosier. Completeness results for single-path Petri nets. *I & C*, 106(2):253–265, 1993.
- 15 P. Jančar. Undecidability of bisimilarity for Petri nets and some related problems. *TCS*, 148:281–301, 1995.
- 16 R. M. Karp and R. E. Miller. Parallel program schemata. *JCSS*, 3:147–195, 1969.
- 17 J. Leroux. Vector Addition System Reachability Problem (A Short Self-Contained Proof). In *POPL’11*, pages 307–316, 2011.
- 18 J. Leroux and G. Sutre. On Flatness for 2-Dimensional Vector Addition Systems with States. In *CONCUR’04*, volume 3170 of *LNCS*, pages 402–416. Springer, 2004.
- 19 P. Madhusudan. Model-checking trace event structures. In *LICS’03*, pages 371–380, 2003.
- 20 E. Mayr. An algorithm for the general Petri net reachability problem. *SIAM Journal of Computing*, 13(3):441–460, 1984.
- 21 S. Schulz. First-order logic with reachability predicates on infinite systems. In *FST&TCS’10*, pages 493–504. LIPICS, 2010.