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# Characterization of the value function of final state constrained control problems with BV trajectories\*

Ariela Briani<sup>†</sup>      Hasnaa Zidani<sup>‡</sup>

## Abstract

This paper aims to investigate a control problem governed by differential equations with Radon measure as data and with final state constraints. By using a known reparametrization method (by Dal Maso and Rampazzo [18]), we obtain that the value function can be characterized by means of an auxiliary control problem of absolutely continuous trajectories, involving time-measurable Hamiltonian. We study the characterization of the value function of this auxiliary problem and discuss its numerical approximations.

**Keywords:** Optimal control problem, differential systems with measures as data, measurable functions, Hamilton-Jacobi equations.

**AMS Classification:** 49J15, 35F21, 34A37

## 1 Introduction

In this paper we investigate, via a Hamilton-Jacobi-Bellman approach, a final state constrained optimal control problem with a Radon measure term in the dynamics.

Several real applications can be described by optimal control problems involving discontinuous trajectories. For instance, in space navigation area, when steering a multi-stage launcher, the separation of the boosters (once they are empty) lead to discontinuities in the mass variable [9]. In resource management, discontinuous trajectories are also used to modelize the problem of sequential batch reactors (see [23]). Many other applications can be found in the Refs. [8, 16, 17, 19].

Consider the controlled system:

$$dY(t) = \sum_{i=1}^M g_i(t, Y(t)) d\mu_i + g_0(t, Y(t), \alpha(t)) dt \quad \text{for } t \in (\tau, T] \quad (1a)$$

$$Y(\tau^-) = X. \quad (1b)$$

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<sup>†</sup>Laboratoire de Mathématiques et Physique Théorique, Faculté de sciences et Techniques, Université François Rabelais, Parc de Grandmont, 37200 Tours. **E-mail:** ariela.briani@lmpt.univ-tours.fr

<sup>‡</sup>Equipe Commands, INRIA Saclay - CMAP- ENSTA ParisTech, 32 Boulevard Victor, 75739 Paris cedex 15, France. **E-mail:** Hasnaa.Zidani@ensta.fr

where  $x \in \mathbb{R}^N$ , the measurable control  $\alpha : (0, +\infty) \rightarrow \mathbb{R}^m$  takes values in a compact set  $A \subset \mathbb{R}^m$ , and  $\mu = (\mu_1, \dots, \mu_M)$  is a given Radon measure. Let  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a given lower semicontinuous (lsc) function and consider the control problem:

$$v(X, \tau) := \inf\{\varphi(Y_{X,\tau}^\alpha(T)) : \alpha(\cdot) \in L^\infty(0, T; A) \text{ and } Y_{X,\tau}^\alpha \text{ satisfies (1)}\}. \quad (2)$$

Due to the presence of the measure  $\mu$ , the definition of solution for the state equation (1) is not classical. We will refer to the definition introduced by Dal Maso and Rampazzo in [18] using the technique of graph completion (Definition 2.1 in Section 2 below). Roughly speaking, by a suitable change of variable in both time and the primitive of  $\mu$ , we can reduce (1) to usual controlled ordinary differential equation with a measurable time-dependent dynamics (see Theorem 2.2 below).

Since the 80's impulsive control problems, i.e. when the measures appear as controls, have been extensively studied by many authors. Let us mention the pioneering works [26], [31] and [10], [11] and [32]. We refer to [30] for the study of the existence results of optimal trajectories, and to [1] for first and second necessary optimality conditions.

In problem (2), the measure is fixed and the state equation is controlled by means of a measurable function  $\alpha$ . Our main goal is to use the HJB approach in order to characterize the value function  $v$  and then to study a numerical method for the approximation of this function.

It is easy to check that the the value function  $v$  fulfills a Dynamic Programming Principle (DPP). Formally, we can derive then, the following HJB equation:

$$\begin{cases} -v_t(X, t) + \sup_{a \in A} \left\{ -Dv(X, t) \cdot \left( g_0(t, X, a) + \sum_{i=1}^M g_i(t, X) \mu_i \right) \right\} = 0; \\ v(X, T) = \varphi(X). \end{cases} \quad (3)$$

However, the sense of solutions for this equations is not clear. The main difficulty is to give a meaning to the term " $Dv \cdot \mu$ " knowing that one can not expect to have a differentiable value function. In order to overcome this difficulty, following the ideas in [14], we define a new value function  $\bar{v}$  such that:

$$v(X, \tau) = \bar{v}(X, \mathcal{W}(\tau)),$$

where  $\mathcal{W}$  is the known change of variable coming from the graph completion technique (See Theorem 2.3). The advantage of this change of variable comes from the fact that the HJB equation for  $\bar{v}$  has a  $t$ -measurable Hamiltonian and not a measure term. More precisely, we can prove that  $\bar{v}$  is a solution, in a suitable sense, of the following equation:

$$\begin{cases} -v_s(X, s) + H(s, X, D\bar{v}(X, s)) = 0; \\ \bar{v}(X, 1) = \varphi(X); \end{cases} \quad (4)$$

where  $H(t, x, p) = \sup_{a \in A} \{-p \cdot \mathcal{F}(t, x, a)\}$  and  $\mathcal{F}(t, x, a)$  is a  $t$ -measurable dynamics (see Section 2.2 for the definition of  $\mathcal{F}$ ). The main contribution of this paper is to study a precise sense of solution for (4) with  $t$ -measurable Hamiltonian and a lsc final function  $\varphi$ .

We recall that, in the case when  $\varphi$  is continuous the definition of viscosity solution for  $t$ -measurable Hamiltonians has been introduced by Ishii in 1985 (see

[24]) and extended to second order by Nunziante in [28]-[29](see also the work of Lions-Perthame [25] and Briani-Rampazzo [15]). Moreover, a very general stability result has been proved more recently by Barles in [4]. On the other hand, to deal with the case when the Hamiltonian  $H$  is continuous with respect to the time variable and the final data  $\varphi$  is lsc, the definition of bilateral viscosity solution has been introduced by Barron and Jensen in 1990 ([7]) and by Frankowska [20] in 1993.

In this paper, since we are interested in a target problem, the function  $\varphi$  is lsc. Moreover, the Hamiltonian in (4) is only  $t$ -measurable. Similar problems has been studied, via non smooth analysis, by H. Frankowska and co-authors in [21, 22]. Here we introduce a new definition of viscosity solution of (4), namely the definition of  $L^1$ -bilateral viscosity solution (Definition 3.1 below). This definition allows to characterize  $\bar{v}$  as the unique  $L^1$ -bilateral viscosity solution of equation (4) (Theorem 3.2). It gives also a suitable framework to deal with the numerical approximation of  $\bar{v}$  (and then of  $v$  by the change of variable  $\mathcal{W}$ ). More precisely, we prove in Theorem 3.3 a convergence result for monotone, stable and consistent numerical schemes, and give an example of a scheme satisfying these properties. Some numerical tests are presented in Subsection 3.2.

On the other hand, we study the properties of  $L^1$ - bilateral viscosity solution for a general HJB equation. In particular, in Section 4, we derive the consistency of the definition (Theorem 4.4), a general stability result w.r. to the Hamiltonian (Theorem 4.6) and uniqueness result (Theorem 4.8).

This paper is organized as follows. In Section 2 we set the optimal control problem we are considering. Subsection 2.1 is devoted to the definition of solution for the state equation and we construct the reparametrized optimal control problem in Subsection 2.2. In Section 3 we state the definition of  $L^1$ -bilateral viscosity solution and we study numerical approximation results for  $t$ -measurable HJB equations. Finally, in Section 4 we will prove the consistency (Theorem 4.4), stability (Theorem 4.6 and 4.7) and uniqueness (Theorem 4.8) result for  $L^1$ -bilateral viscosity solution.

**Notations.** For each  $r > 0$ ,  $x \in \mathbb{R}^N$  we will denote by  $B_r(x)$  the closed ball of radius  $r$  centered in  $x$ . Given a Radon measure  $\mu$  we will denote by  $L^1_\mu(\mathbb{R})$  the space of integrable functions with respect to the measure  $\mu$ .

For a function  $f : [a, b] \rightarrow \mathbb{R}^N$  we will denote by  $V_a^b(f)$  the classical variation on  $[a, b]$  and by  $BV([0, T]; \mathbb{R}^N)$  the set of functions  $f : [0, T] \rightarrow \mathbb{R}^N$  with bounded variation on  $[0, T]$ . Moreover, we will denote by  $BV^-([0, T]; \mathbb{R}^N)$  the set of left continuous functions of  $BV([0, T]; \mathbb{R}^N)$  which are continuous at 0.

In all the sequel, we will use the classical notations:  $f(t^+) := \lim_{s \rightarrow t^+} f(s)$  and  $f(t^-) := \lim_{s \rightarrow t^-} f(s)$ . And finally, we will denote by  $AC([0, 1]; \mathbb{R}^N)$  the set of absolutely continuous functions from  $[0, 1]$  to  $\mathbb{R}^N$ .

## 2 The optimal control problem with BV trajectories

In this section we will first recall the definition of solution for the state equation as introduced by Dal Maso and Rampazzo in [18] and we recall the graph completion construction. Then, we define the reparametrized optimal control problem.

## 2.1 The state equation

Let us fix  $0 \leq \tau < T$ , an initial datum  $X \in \mathbb{R}^N$ , a given Radon measure  $\mu = (\mu_1, \dots, \mu_M)$ , a control variable  $\alpha \in \mathcal{A}$ , and consider the controlled trajectory  $Y_{X,\tau}^\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^N$  solution of:

$$\begin{cases} dY(t) = \sum_{i=1}^M g_i(t, Y(t)) d\mu_i + g_0(t, Y(t), \alpha(t)) dt & \text{for } t \in (\tau, T] \\ Y(\tau^-) = X. \end{cases} \quad (5)$$

We assume the following:

- (Hco)** The set of admissible controls is  $\mathcal{A} := \{\alpha : (0, T) \rightarrow A \text{ measurable}\}$ , where  $A$  is a compact subset of  $\mathbb{R}^m$ ,  $m \geq 1$ .
- (Hg1)** The functions  $g_0(t, Y, a) : \mathbb{R}^+ \times \mathbb{R}^N \times A \rightarrow \mathbb{R}^N$ ,  $g_i(t, Y) : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ , ( $i = 1, \dots, M$ ), are measurable functions in  $t$  and continuous in  $Y, a$ . Moreover, for each  $Y \in \mathbb{R}^N$  we have  $g_0(\cdot, Y, a) \in L^1(\mathbb{R}^+)$  and  $g_i(\cdot, Y) \in L^1_{\mu_i}(\mathbb{R}^+)$ , ( $i = 1, \dots, M$ ).
- (Hg2)** There exists a function  $k_0 \in L^\infty(\mathbb{R}^+; \mathbb{R}^+)$  such that

$$|g_i(t, Y) - g_i(t, Z)| \leq k_0(t) |Y - Z| \quad \forall Y, Z \in \mathbb{R}^N, \quad \text{a.e. } t \in \mathbb{R}^+, \quad i = 1, \dots, M.$$

$$|g_0(t, Y, a) - g_0(t, Z, a)| \leq k_0(t) |Y - Z| \quad \forall Y, Z \in \mathbb{R}^N, \forall a \in A, \quad \text{a.e. } t \in \mathbb{R}^+.$$

- (Hg3)** There exists  $K > 0$  such that

$$|g_i(t, Y)| \leq K \quad \forall Y \in \mathbb{R}^N, \quad \text{a.e. } t \in \mathbb{R}^+, \quad i = 1, \dots, M.$$

$$|g_0(t, Y, a)| \leq K \quad \forall Y \in \mathbb{R}^N, \forall a \in A, \quad \text{a.e. } t \in \mathbb{R}^+.$$

Following [18], we introduce the left continuous primitive  $B$  of the Radon measure  $\mu$ , i.e.  $B \in BV^-([0, T]; \mathbb{R}^M)$  and its distributional derivative  $\dot{B}$  coincides with  $\mu$  on  $[0, T[$ . In all the sequel, we will denote by  $\mathcal{T} := \{t_i, i \in \mathbb{N}\}$  the countable subset of  $[0, T)$  which contains 0 and all the discontinuity points of  $B$ , and by  $E_c$  the set of all continuity points of  $B$ . Furthermore, let  $(\psi_t)_{t \in \mathcal{T}} := (\psi_t^1, \dots, \psi_t^M)$  be a family of Lipschitz continuous maps from  $[0, 1]$  into  $\mathbb{R}^M$  such that

$$\sum_{t \in \mathcal{T}} V_0^1(\psi_t) < \infty, \quad \psi_t(0) = B(t^-) \quad \text{and} \quad \psi_t(1) = B(t^+) \quad \forall t \in \mathcal{T}; \quad (6)$$

(if  $t = 0$  we require only  $\psi_t(1) = B(0^+)$ ). We will denote by  $\xi$  the solution of:

$$\frac{d\xi}{d\sigma} = \sum_{i=1}^M g_i(\sigma, \xi(\sigma)) \frac{d\psi_t^i}{d\sigma} \quad \text{for } \sigma \in (0, 1] \quad \xi(0) = \bar{\xi},$$

and we set  $\xi(\bar{\xi}, \psi_t) := \xi(1) - \bar{\xi}$ .

The definition of solution introduced by Dal Maso and Rampazzo in [18, Definition 5.1] is then the following:

**Definition 2.1.** Given an initial datum and time  $(X, \tau)$ , a control  $\alpha \in \mathcal{A}$  and a family of Lipschitz continuous maps  $(\psi_t)_{t \in \mathcal{T}}$  fulfilling (6), the function  $Y_{X, \tau}^\alpha \in BV^-([\tau, T]; \mathbb{R}^N)$  is a solution of (5) if for each Borel subset  $\mathcal{B}$  of  $]\tau, T[$  we have

$$\int_{\mathcal{B}} dY(t) = \int_{\mathcal{B}} g_0(t, Y(t), \alpha(t)) dt + \sum_{i=1}^M \int_{\mathcal{B} \cap E_c} g_i(t, Y(t)) d\mu_i + \sum_{t \in \mathcal{T} \cap \mathcal{B}} \xi(Y(t^-), \psi_t) \quad (7)$$

and  $Y(\tau^-) = X$ . Moreover, if  $\tau \in \mathcal{T}$  we have  $Y(\tau^+) = \xi(X, \psi_\tau)$ .

In order to prove the uniqueness of this solution we set

$$a_i := V_0^1(\psi_{t_i}), \quad a := \sum_{i=1}^{+\infty} a_i, \quad w(t) := \frac{t + V_0^t(B)}{T + V_0^T(B)},$$

and we define  $\mathcal{W} : [0, T] \rightarrow [0, 1]$  as follows:

$$\mathcal{W}(t) := \frac{1}{1+a} \left( w(t) + \sum_{t_i < t} a_i \right). \quad (8)$$

The graph completion of  $B$  corresponding to the family  $(\psi_t)_{t \in \mathcal{T}}$  is then defined by:

$$\Phi(s) := (\phi^0; \phi^1, \dots, \phi^M)(s) \quad (9a)$$

$$= \begin{cases} (t; B(t)) & \text{if } s = \mathcal{W}(t) & t \in [0, T] \setminus \mathcal{T} \\ (t_i; \psi_{t_i} \left( \frac{s - \mathcal{W}(t_i)}{\mathcal{W}(t_i^+) - \mathcal{W}(t_i)} \right)) & \text{if } s \in [\mathcal{W}(t_i), \mathcal{W}(t_i^+)] & t_i \in \mathcal{T}. \end{cases} \quad (9b)$$

We are now ready to construct the reparametrization of system (5). Let  $\sigma := \mathcal{W}(\tau)$ , for each control  $\alpha \in \mathcal{A}$  and initial datum  $X$  we denote by  $Z_{X, \sigma}^\alpha : [\sigma, 1] \rightarrow \mathbb{R}^N$  the solution of

$$\begin{cases} \frac{dZ}{ds}(s) = \sum_{i=1}^M g_i(\phi^0(s), Z(s)) \left( \mu_i^\alpha(\phi^0(s)) \frac{d\phi^0}{ds}(s) + \frac{d\phi^i}{ds}(s) \right) + \\ \quad g_0(\phi^0(s), Z(s), \alpha(\phi^0(s))) \frac{d\phi^0}{ds}(s) & \text{for } s \in (\sigma, 1] \\ Z(\sigma) = X \end{cases} \quad (10)$$

where  $\mu^\alpha$  is the absolutely continuous part of the measure  $\mu$  with respect to the Lebesgue measure, i.e.  $\mu(t) = \mu^\alpha(t)dt + \mu^s$ . Note that the derivatives of  $\phi^0, \phi^i$  are measurable functions, therefore assumptions **(Hg1)**-**(Hg2)** ensure the applicability of Caratheodory's Theorem to obtain the existence of a unique solution of (10) in  $AC([\sigma, 1]; \mathbb{R}^N)$ .

**Theorem 2.2.** Assume **(Hco)** and **(Hg1)**-**(Hg3)**. Let  $\mu$  be a Radon measure and  $(\psi_t)_{t \in \mathcal{T}}$  be a family fulfilling (6). Then  $Y_{X, \tau}^\alpha \in BV^-([\tau, T]; \mathbb{R}^N)$  is a solution of (5) if and only if there exists a solution  $Z_{X, \sigma}^\alpha \in AC([\sigma, 1]; \mathbb{R}^N)$  of (10) corresponding to the graph completion  $\Phi$  defined in (9) such that

$$Z_{X, \sigma}^\alpha(\mathcal{W}(t)) = y(t) \quad \forall t \in [\tau, T] \quad (11)$$

where  $\mathcal{W}$  is given by (8).

Moreover, for each Radon measure  $\mu$  and each family  $(\psi_t)_{t \in \mathcal{T}}$  equation (5) has a unique solution (up to a set of zero Lebesgue measure).

**Proof.** The equivalence (11) can be obtained by adapting the proof given for  $M = N = 1$  in [14, Theorem 2.8]. On the other hand, the uniqueness of the solution is a consequence of Caratheodory's Theorem applied to equation (10).  $\square$

**Remark 1.** We point out that this definition depends on the family  $(\psi_t)_{t \in \mathcal{T}}$  we choose. It is now a classical result that under commutativity conditions on the vector fields  $g_i$  ( $i = 1, \dots, M$ ) the solution does not depend on this choice as studied in the pioneering works of Bressan and Rampazzo [12, 13]. However, in this paper, the dependence on the choice of  $\psi_t$  does not imply any specific difficulty in the sequel. Of course, from an application point of view, one have to be aware of this dependence and made an accurate choice of the reparametrization.

## 2.2 The control problem

Let us now describe our optimal control problem. Given a lower semicontinuous function  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$  and a final time  $T$ , our aim is to calculate the following value function

$$v(X, \tau) := \inf_{\alpha \in \mathcal{A}} \varphi(Y_{X, \tau}^\alpha(T)) \quad (12)$$

where  $Y_{X, \tau}^\alpha$  is the solution of equation (5).

It is easy to prove that the following Dynamic Programming Principle (DPP) holds: for each  $\tau \leq h \leq T$  we have

$$v(X, \tau) = \inf_{\alpha \in \mathcal{A}} v(Y_{X, \tau}^\alpha(h), h).$$

Therefore we can formally derive a HJB equation:

$$\begin{cases} -v_t(X, t) + H(t, X, Dv(X, t)) = 0 & \text{for } (X, t) \in \mathbb{R}^N \times (0, T), \\ v(X, T) = \varphi(X) & \text{for } X \in \mathbb{R}^N \end{cases}$$

where the Hamiltonian is

$$H(t, X, P) = \sup_{a \in \mathcal{A}} \left\{ -P \cdot \left( g_0(t, X, a) + \sum_{i=1}^M g_i(t, X) \mu_i \right) \right\}.$$

As we pointed out in the Introduction, the problem is to give a meaning to the term  $Dv \cdot \mu$  knowing that one can not expect to have a differentiable value function.

In view of Theorem 2.2, it is then natural to consider the trajectories  $Z_{X, \sigma}^\alpha$  solution of the the reparametrized system (10). We define then the corresponding value function as follows:

$$\bar{v}(X, \sigma) = \inf_{\alpha \in \mathcal{A}} \varphi(Z_{X, \sigma}^\alpha(1)). \quad (13)$$

The link between the two problems is given by the following result.

**Theorem 2.3.** *Let  $v$  and  $\bar{v}$  be respectively defined in (12) and (13). For each  $X \in \mathbb{R}^N$  and  $\tau \in [0, T]$  we have*

$$v(X, \tau) = \bar{v}(X, \mathcal{W}(\tau)) \quad (14)$$

where  $\mathcal{W}$  is given by (8). Moreover

$$v_{\sharp}(X, \tau) = \bar{v}_{\sharp}(X, \mathcal{W}(\tau)) \quad \forall X \in \mathbb{R}^N, \quad \forall \tau \in [0, T] \setminus \mathcal{T} \quad (15)$$

and

$$v_{\sharp}(X, \tau) \geq \bar{v}_{\sharp}(X, \mathcal{W}(\tau)) \quad \forall X \in \mathbb{R}^N, \quad \forall \tau \in \mathcal{T}, \quad (16)$$

where we respectively denote by  $v_{\sharp}$  and  $\bar{v}_{\sharp}$  the lower semicontinuous envelope of  $v$  and  $\bar{v}$  w.r. to both variable  $(X, \tau)$  and  $(X, s)$ .

**Proof.** By Theorem 2.2 above we have  $Y_{X, \tau}^{\alpha}(T) = Z_{X, \mathcal{W}(\tau)}^{\alpha}(\mathcal{W}(T)) = Z_{X, \sigma}^{\alpha}(1)$  then (14) follows by the definitions of  $v$  and  $\bar{v}$ .

Since, by construction,  $\mathcal{W}(\tau)$  is monotone increasing in  $[0, T]$  and continuous in any  $\tau \in \mathcal{T}$ , (15) and (16) easily follow.  $\square$

**Remark 2.** In (15), (16) we stressed the link between the lsc envelopes of  $v$  and  $\bar{v}$  because is indeed the function  $\bar{v}_{\sharp}(X, s)$  that will be characterized as solution of an HJB equation.

Thanks to Theorem 2.3, it is clear that we turn now our attention to the HJB equation for the function  $\bar{v}$ . The advantage is that we do not have any more measure in the dynamics.

The new value function  $\bar{v}$  satisfies also a DPP:

$$\bar{v}(X, \sigma) = \inf_{\alpha \in \mathcal{A}} \bar{v}(Z_{X, \sigma}^{\alpha}(h), h) \quad \forall \sigma \leq h \leq 1, \quad \forall X \in \mathbb{R}^N. \quad (17)$$

From this DPP, one could expect to characterize  $\bar{v}$  through the following HJB equation:

$$\begin{cases} -\bar{v}_s(X, s) + H(s, X, D\bar{v}(X, s)) = 0 & \text{for } (X, s) \in \mathbb{R}^N \times (0, 1), \\ \bar{v}(X, 1) = \varphi(X) & \text{for } X \in \mathbb{R}^N \end{cases} \quad (18)$$

where the Hamiltonian is

$$H(s, X, P) = \sup_{a \in A} \left\{ -P \cdot \left( g_0(\phi^0(s), X, a) \frac{d\phi^0}{ds}(s) + \sum_{i=1}^M g_i(\phi^0(s), X) \left( \mu_i^a(\phi^0(s)) \frac{d\phi^0}{ds}(s) + \frac{d\phi^i}{ds}(s) \right) \right) \right\}. \quad (19)$$

Note that, by definition (9), the graph completion  $(\phi^0, \phi^i)$  is a Lipschitz function, therefore we can not expect to have a time continuous Hamiltonian. Moreover, our final condition  $\varphi$  is only lower semicontinuous. Thus, we should first give a precise meaning to the definition of the viscosity solution of the equation (18). This will be the aim of the following Section.

### 3 Optimal control problems with measurable time-dependent dynamics

In this section we characterize the value function  $\bar{v}$  as the unique  $L^1$ -bilateral viscosity solution of (18). Moreover, we investigate convergence results for numerical



schemes of equation (18). We will prove our results in the following more general framework.

Fix a final time  $T$ , given  $x \in \mathbb{R}^N$ ,  $\tau \geq 0$  and a control  $\alpha \in \mathcal{A}$ , we consider the trajectory  $y_{x,\tau}^\alpha$ , solution of the following system:

$$\begin{cases} \dot{y}(t) = \mathcal{F}(t, y(t), \alpha(t)), & \text{for } t \in (\tau, T) \\ y(\tau) = x. \end{cases} \quad (20)$$

For each initial point and time  $(x, \tau) \in \mathbb{R}^N \times \mathbb{R}^+$  we set:

$$\vartheta(x, \tau) := \inf_{\alpha \in \mathcal{A}} \varphi(y_{x,\tau}^\alpha(T)). \quad (21)$$

We assume the following :

**(HF1)**  $\mathcal{F}(t, x, a) : \mathbb{R}^+ \times \mathbb{R}^N \times A \rightarrow \mathbb{R}^N$  is measurable in  $t$  and continuous in  $x$  and  $a$ .  
Moreover, for each  $(x, a) \in \mathbb{R}^N \times A$  we have  $\mathcal{F}(\cdot, x, a) \in L^1(\mathbb{R}^+)$ .

**(HF2)** There exists  $k_0 \in L^\infty(\mathbb{R}^+; \mathbb{R}^+)$  such that

$$|\mathcal{F}(t, x, a) - \mathcal{F}(t, z, a)| \leq k_0(t)|x - z| \quad \forall x, z \in \mathbb{R}^N, a \in A, t \in \mathbb{R}^+.$$

**(HF3)** There exists a  $K > 0$  such that

$$|\mathcal{F}(t, x, a)| \leq K \quad \forall x \in \mathbb{R}^N, a \in A, t \in \mathbb{R}^+.$$

**(Hid)** The function  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$  is lower semi continuous and bounded.

**Remark 3.** Let us point out that if we assume **(Hg1)**-**(Hg3)**, then the function

$$\mathcal{F}(s, x, a) := \sum_{i=1}^M g_i(\phi^0(s), x) \left( \mu^a(\phi^0(s)) \frac{d\phi^0}{ds}(s) + \frac{d\phi^i}{ds}(s) \right) + g_0(\phi^0(s), x, a) \frac{d\phi^0}{ds}(s)$$

fulfills **(HF1)**-**(HF3)**. Therefore, all the results in this section will apply, in particular, to the value function  $\bar{v}$  defined in (13).

In all the sequel, we denote  $\mathcal{V}$  the lower semicontinuous envelope of  $\vartheta$  defined by:

$$\mathcal{V}(x, t) := \liminf_{y \rightarrow x, s \rightarrow t} \vartheta(y, s). \quad (22)$$

Our first aim is then to prove that we can characterize the function  $\mathcal{V}$  in (22) as the unique  $L^1$ -bilateral viscosity solution (see the definition below) of the following HJB equation:

$$\begin{cases} -\mathcal{V}_t(x, t) + H(t, x, D\mathcal{V}(x, t)) & = 0 & \text{for } (x, t) \in \mathbb{R}^N \times (0, T), \\ \mathcal{V}(x, T) & = \varphi(x) & x \in \mathbb{R}^N \end{cases} \quad (23)$$

where the Hamiltonian is

$$H(t, x, p) = \sup_{a \in A} \{-p \cdot \mathcal{F}(t, x, a)\}. \quad (24)$$

**Definition 3.1.**  $L^1$ -bilateral viscosity solution (L1Bvs)

Let  $u : \mathbb{R}^N \times (0, T) \rightarrow \mathbb{R}$  be a bounded lsc function. We say that  $u$  is a  $L^1$ -bilateral viscosity solution (L1Bvs) of (23) if:

for any  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x)$  we have

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,sup}_{|t-t_0| \leq \delta} \sup_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \geq 0$$

and

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq 0.$$

Moreover, the final condition is satisfied in the following sense:

$$\varphi(x) = \inf \left\{ \liminf_{n \rightarrow \infty} u(x_n, t_n) : x_n \rightarrow x, t_n \uparrow T \right\}.$$

**Remark 4.** For the sake of clarity, we will state and prove below (Section 4), the consistency, stability and uniqueness result for the viscosity sense (L1Bvs) defined in Definition 3.1.

Let us now prove the characterization of the value function.

**Theorem 3.2.** *Assume (HF1)-(HF3) and (Hid). The function  $\mathcal{V}$ , defined in (22), is the unique  $L^1$ -bilateral viscosity solution of (23), when the Hamiltonian is given in (24).*

**Proof.** The proof of  $\mathcal{V}$  being a  $L^1$ -bilateral viscosity solution of (23) can be obtained with the same kind of argument given in [2, Proposition 5.3, Chapter V]. The uniqueness follows by Theorem 4.8. □

### 3.1 Numerical approximations of (23).

In the case when the Hamiltonian is continuous (both in time and in space), numerical discretization of Hamilton-Jacobi equations has been studied by many authors. The general framework of Barles-Souganidis [5] ensures that the numerical scheme is convergent (to the viscosity solution) whenever this scheme is consistent, monotone and stable and the HJB equation satisfies a strong comparison principle. The class of schemes satisfying these properties is very large and includes upwind finite differences, Semi-Lagrangian methods, Markov-Chain approximations.

In this section, we extend the result of [5] to the case of equation (23), where the Hamiltonian is only  $t$ -measurable, and show that the  $t$ -measurable viscosity notion is still a good framework to analyze the convergence of numerical approximations. We give also an example of a monotone, stable and consistent scheme of (23) based on finite differences approximations. Finally, a numerical example is given in Subsection 3.2.

Let  $G$  be a space grid on  $\mathbb{R}^N$  with a uniform mesh size  $\Delta x > 0$  (of course a nonuniform grid could also be considered), and let  $\Delta t > 0$  be a time step (we assume that  $T/\Delta t$  belongs to  $\mathbb{N}$ ). In the sequel, we will use the following notations:

$$\Delta := (\Delta x, \Delta t), \quad t_n := n\Delta t, \quad x_j \text{ is a node in } G, \quad N_T := \frac{T}{\Delta t}. \quad (25)$$

Consider an approximation scheme of the following form:

$$\mathcal{S}^\Delta(t_n, x_j, v_j^n, v^{n+1}) = 0 \quad \forall x_j \in G, n = 0, \dots, N_T - 1; \quad v_j^{N_T} = \varphi(x_j) \quad \forall x_j \in G. \quad (26)$$

Thus, if  $v$  is a continuous function defined on  $[0, T] \times \mathbb{R}^N$ , the approximation scheme reads

$$\mathcal{S}^\Delta(t, x, v(x, t), v(\cdot, t + \Delta t)) = 0 \text{ in } (0, T) \times \mathbb{R}^N. \quad (27)$$

On  $\mathcal{S}^\Delta : (0, T) \times \mathbb{R}^N \times \mathbb{R} \times L^\infty(\mathbb{R}^N)$  we assume the following:

**(M) Monotonicity.** For each  $u \geq v$  we have

$$\mathcal{S}^\Delta(t, x, r, u) \leq \mathcal{S}^\Delta(t, x, r, v) \quad \forall t \in (0, T), x \in \mathbb{R}^N, r \in \mathbb{R}.$$

**(S) Stability.** There exists  $K > 0$  such that, if  $v^\Delta$  is solution of (27) then

$$\|v^\Delta\|_{L^\infty} \leq K,$$

$K$  being independent of  $\Delta x, \Delta t$ .

**(C) Consistency.** For every point  $(x_0, t_0)$ , for any  $b \in L^1(0, T)$  and any function  $\phi(x)$  such that:  $\phi \in C^1(\mathbb{R}^N)$ , by setting  $\psi(x, t) := \int_0^t b(s) ds + \phi(x)$ , we have:

$$\begin{aligned} & \operatorname{ess\,sup}_{|t-t_0| \leq \Delta t} \sup_{x \in B_{\Delta x}(x_0), p \in B_{\Delta x}(D\phi(x_0))} \{-b(t) + H(t, x, p)\} \geq \\ & \geq \mathcal{S}^\Delta(t_0, x_0, \psi(x_0, t_0), \psi(\cdot, t_0 + \Delta t)) + o_{\Delta x}(1) \geq \\ & \geq \operatorname{ess\,inf}_{|t-t_0| \leq \Delta t} \inf_{x \in B_{\Delta x}(x_0), p \in B_{\Delta x}(D\phi(x_0))} \{-b(t) + H(t, x, p)\}. \end{aligned} \quad (28)$$

An example of scheme fulfilling the above assumptions, when the Hamiltonian is given by (24), is the following

$$\begin{aligned} \mathcal{S}^\Delta(t, x, u(x, t), u(\cdot, t + \Delta t)) & := \frac{u(x, t) - u(x, t + \Delta t)}{\Delta t} + \\ & + \frac{1}{\Delta t} \int_t^{t+\Delta t} \sup_{a \in A} \left( (-\mathcal{F})^+(s, x, a) \cdot \frac{u(x, t + \Delta t) - u(x - \Delta x, t + \Delta t)}{\Delta x} + \right. \\ & \quad \left. + (-\mathcal{F})^-(s, x, a) \cdot \frac{u(x + \Delta x, t + \Delta t) - u(x, t + \Delta t)}{\Delta x} \right) ds, \end{aligned} \quad (29)$$

where we classically denoted  $g^+ := \max(g, 0)$  and  $g^- := \min(g, 0)$ .

**Proposition 1.** Assume that  $\mathcal{F}$  fulfills assumptions **(HF1)**-**(HF3)**, and consider the Hamiltonian in (24). Let  $\Delta = (\Delta x, \Delta t)$  be mesh sizes satisfying:

$$\frac{\Delta t}{\Delta x} |\mathcal{F}(s, x, a)| \leq 1 \quad \text{for a.e. } s \in (0, T), \forall x \in \mathbb{R}^N, a \in A. \quad (30)$$

Then, the scheme  $\mathcal{S}^\Delta$  given in (29) satisfies conditions **(M)**, **(S)** and **(C)**.

**Proof.** First remark that the Stability condition **(S)** easily follows from the boundedness of  $\mathcal{F}$  and **(HF3)**. Moreover, the monotonicity **(M)** follows from condition (30) by standard arguments.

To prove consistency, we fix  $(x_0, t_0)$  and consider a function  $\psi(x, t) = \int_0^t b(s) ds + \phi(x)$  for  $b \in L^1(0, T)$  and  $\phi \in C^1(\mathbb{R}^N)$ . By using the regularity of  $\psi$  and assumption **(HF3)** on  $\mathcal{F}$ , we get:

$$\mathcal{S}^\Delta(t_0, x_0, \psi(x_0, t_0), \psi(\cdot, t_0 + \Delta t)) = \frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} \{-b(s) + H(s, x_0, D\phi(x_0))\} ds + o_{\Delta x}(1).$$

Condition **(C)** follows.  $\square$

The general convergence result is the following.

**Theorem 3.3.** *Assume **(HF1)**-**(HF3)**. Let  $\mathcal{V}$  be defined as in (22) with  $\varphi$  fulfilling assumption **(Hid)**. Consider a sequence of continuous and bounded functions  $\varphi_m : \mathbb{R}^N \rightarrow \mathbb{R}$  (for  $m \geq 1$ ) such that  $(\varphi_m)_{m \in \mathbb{N}}$  is monotone increasing and*

$$\lim_{m \rightarrow \infty} \varphi_m(x) = \varphi(x) \quad \forall x \in \mathbb{R}^N.$$

Let  $\Delta = (\Delta x, \Delta t)$  be a mesh size such that the scheme  $\mathcal{S}^\Delta$  fulfills conditions **(M)**, **(S)** and **(C)**, and let  $v^{\Delta, m} := (v_j^n)_{n, j}$  be the solution of :

$$\mathcal{S}^\Delta(t_n, x_j, v_j^n, v_j^{n+1}) = 0 \quad \forall x_j \in G, n = 0, \dots, N_T - 1; \quad v_j^{N_T} = \varphi_m(x_j) \quad \forall x_j \in G. \quad (31)$$

Then, as  $\Delta t \rightarrow 0$ ,  $\Delta x \rightarrow 0$  and  $m \rightarrow +\infty$ ,  $v^{\Delta, m}$  converges pointwise to the function  $\mathcal{V}$ .

**Proof.** The proof will be given in two steps.

*Step 1.* We first suppose that the final data is continuous ( $\varphi_m \equiv \varphi$ ). We consider a  $\Delta_k = (\Delta x_k, \Delta t_k)$  and denote by  $v^{\Delta_k}$  the solution of (31) corresponding to  $\Delta_k$  and  $\varphi_m \equiv \varphi$ . We will prove that, as  $k \rightarrow 0$ , the sequence  $v^{\Delta_k}$  converges locally uniformly to the unique  $L^1$ -viscosity solution of (23).

For each  $k$ , we set  $(x_k, t_k) := (x_{j_k}, t_{n_k})$  where  $(x_{j_k}, t_{n_k})$  are the points defined in (25) when  $\Delta$  is  $\Delta_k$ . Let us first observe that by the stability assumption **(S)** the sequence  $v^{\Delta_k}$  is bounded, therefore the following weak semi-limits are well defined:

$$v_*(x, t) := \liminf_{k \rightarrow 0} \lim_{x_k \rightarrow x, t_k \rightarrow t} v^{\Delta_k}(x_k, t_k) \quad v^*(x, t) := \limsup_{k \rightarrow 0} \lim_{x_k \rightarrow x, t_k \rightarrow t} v^{\Delta_k}(x_k, t_k).$$

Note that both  $v_*$  and  $v^*$  trivially satisfy the final condition in (23). Therefore, the convergence result will follow once we prove that  $v_*$  and  $v^*$  are respectively a  $L^1$ -viscosity supersolution and a  $L^1$ -viscosity subsolution of (23). Indeed, if this is true, by the comparison result [24, Theorem 8.1], we have  $v^* \leq v_*$ . Since the reverse is true by definition, the two weak semi limits coincide and the thesis follows.

Let us now prove that  $v^*$  is a  $L^1$ -viscosity subsolution of (23). (The proof of  $v_*$  being a  $L^1$ -viscosity supersolution of (23) is completely similar and will not be detailed.)

Following Definition 4.3 below, for any  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local maximum point of  $v^*(x, t) - \int_0^t b(s) ds - \phi(x)$  we have to prove that

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq 0. \quad (32)$$

Note that, without loss of generality, we can assume that  $(x_0, t_0)$  is a *strict* local zero maximum of  $v^*(x, t) - \int_0^t b(s) ds - \phi(x)$ . There exists then a sequences of points  $(x_k, t_k)$  such that

- (a)  $(x_k, t_k) \rightarrow (x_0, t_0)$  as  $k \rightarrow 0$ .
- (b)  $(x_k, t_k)$  is a local maximum point of  $v^{\Delta_k}(x, t) - \int_0^t b(s)ds - \phi(x)$ .
- (c)  $\xi_k := v^{\Delta_k}(x_k, t_k) - \int_0^{t_k} b(s)ds - \phi(x_k) \rightarrow 0 = v^*(x_0, t_0) - \int_0^{t_0} b(s)ds - \phi(x_0)$  as  $k \rightarrow 0$ .

Thanks to (b), we can apply the monotonicity assumption **(M)** with  $v = v^{\Delta_k}$ ,  $u = \phi(x) + \int_0^t b(s)ds + \xi_k$  and  $r = v^{\Delta_k}(x_k, t_k) = \xi_k + \phi(x_k) + \int_0^{t_k} b(s)ds$  and obtain

$$\begin{aligned} \mathcal{S}^{\Delta_k} \left( t_k, x_k, \xi_k + \phi(x_k) + \int_0^{t_k} b(s)ds, \xi_k + \phi(\cdot) + \int_0^{t_k + \Delta t_k} b(s)ds \right) &\leq \\ &\leq \mathcal{S}^{\Delta_k}(t_k, x_k, v^{\Delta_k}(x_k, t_k), v^{\Delta_k}(\cdot, t_k + \Delta t_k)) = 0, \end{aligned} \quad (33)$$

where we also used that  $v^{\Delta_k}$  is a solution of (26).

Fix now a  $\delta > 0$ , by (a) and the regularity of  $\phi$  we can always find a  $\delta_k \leq \delta$  such that  $\min(\Delta x_k, \Delta t_k) \leq \delta_k$ ,  $B_{\delta_k}(t_k, x_k) \subseteq B_\delta(x_0, t_0)$ , and  $B_{\delta_k}(D\phi(x_k)) \subseteq B_\delta(D\phi(x_0))$ . Therefore, also by the consistency assumption **(C)** and (33) we get:

$$\begin{aligned} &\text{ess inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq \\ &\leq \text{ess inf}_{|t-t_k| \leq \delta_k} \inf_{x \in B_{\delta_k}(x_k), p \in B_{\delta_k}(D\phi(x_k))} \{H(t, x, p) - b(t)\} \leq \\ &\leq \mathcal{S}^{\Delta_k} \left( t_k, x_k, \xi_k + \phi(x_k) + \int_0^{t_k} b(s)ds, \xi_k + \phi(\cdot) + \int_0^{t_k + \Delta t_k} b(s)ds \right) + o_{\delta_k}(1) \leq o_{\delta_k}(1). \end{aligned} \quad (34)$$

Inequality (32) follows then by letting  $\delta \rightarrow 0^+$  (which implies  $\delta_k \rightarrow 0^+$ ).

*Step 2.* For every  $m \geq 1$ , by Step 1, as  $k \rightarrow 0$ , the sequence  $(v^{\Delta_k, m})_k$  converges to  $v_m$  the unique  $L^1$ -viscosity solution of

$$\begin{cases} -v_t(x, t) + H(t, x, Dv) &= 0 & \text{in } \mathbb{R}^N \times (0, T) \\ v(x, T) &= \varphi_m(x) & \text{in } \mathbb{R}^N. \end{cases} \quad (35)$$

The pointwise convergence of  $v_m$  to  $\mathcal{V}$  then follows by the stability result Theorem 4.7 below.  $\square$

**Remark 5.** In the case of Eikonal equation with  $t$ -measurable velocity function, a similar convergence result is proved, in the recent work of A. Monteillet [27], for a particular numerical scheme.

### 3.2 A numerical test.

In this section, we use the scheme given in (29) to solve Hamilton-Jacobi equations coming from a simple control problem with BV trajectories.

Consider the target  $\mathcal{C} := B(0, r)$ , which is the ball centered at the origin and of radius  $r = 0.25$ . Consider also a trajectory  $Y_{\tau, X}^{(\alpha, c)}$ , depending on the control

variables  $\alpha : (0, T) \rightarrow A := [0, 2\pi]$  and  $c : (0, T) \rightarrow U$ , and governed by the following dynamics

$$\begin{aligned}\dot{Y}(t) &= c(t) \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta_1 + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_2, \\ Y(\tau) &= X\end{aligned}$$

where  $C_1 := 0.5$ ,  $C_2 := 0.2$ , and  $\delta_u$  (for  $u = 1, 2$ ) denotes the Dirac measure at time  $t = u$ . The control variable  $c$  takes its values in a compact set  $U$ . Here we will consider two cases:

- **Case 1:**  $U \equiv \{0.5\}$  which amounts saying that we are allowed to move in any direction in the sphere centered at the origin and with radius 0.5.
- **Case 2:**  $U = [0, 0.5]$ , which means that we can move in any direction in the Ball centered at the origin and with radius 0.5.

In both cases, at time  $t = 1$  and  $t = 2$  the trajectories jump. We consider the value function corresponding to the *Rendez-Vous* problem:

$$v(t, X) := \inf\{\varphi(Y_{\tau, X}^{\alpha, c}(T)); \alpha \in L^\infty(0, T; A), c \in L^\infty(0, T; U)\},$$

where  $T = 3$ , and  $\varphi(x) = 0$  when  $x \in \mathcal{C}$  and 1 otherwise.

It is not difficult to compute the reparametrized function:

$$\Phi(s) = \begin{cases} (15s, 0, 0) & 0 \leq s \leq \frac{1}{15}; \\ (1, \frac{15}{6}s - \frac{1}{6}, 0) & \frac{1}{15} < s < \frac{7}{15}; \\ (15s - 6, 1, 0) & \frac{7}{15} < s < \frac{8}{15}; \\ (2, 1, \frac{15}{6}s - \frac{8}{6}) & \frac{8}{15} < s < \frac{14}{15}; \\ (15s - 12, 2, 1) & \frac{14}{15} < s < 1. \end{cases}$$

Let us notice that in Case 2, the value function  $\bar{v}$  corresponding to the parametrized problem is lsc.

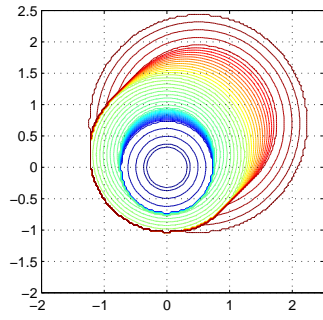
Fig. 1 shows the numerical solution in the Case 1, while Fig.2 shows the results corresponding to Case 2. These numerical experiments are performed by using the finite differences scheme with  $150^2$  grid points. Computations are done on the domain  $[-1.5, 3]^2$ . The final cost function is approximated by a function (with  $n = 10$ ):

$$\varphi_n(X) := 1/n \min(1, \|x\| - 0.5).$$

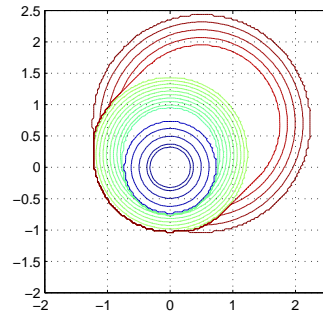
In the two cases, we compute first the value function  $\bar{v}$  corresponding to the parametrized control problem, and then we deduce the original value function by using a change of variable. The latter step is very easy to perform numerically, since  $v$  turns to be just the restriction of  $\bar{v}$  on  $[0, \frac{1}{15}] \cap [\frac{7}{15}, \frac{8}{15}] \cap [\frac{14}{15}, 1]$ . In Figs. 1 & 2, we plot only the 0-level sets.

## 4 Properties of the $L^1$ -bilateral viscosity solution of HJB equations.

This section is devoted to the main properties of the  $L^1$ -bilateral solutions defined in Definition 3.1. First, we give an equivalent formulation of this definition and we

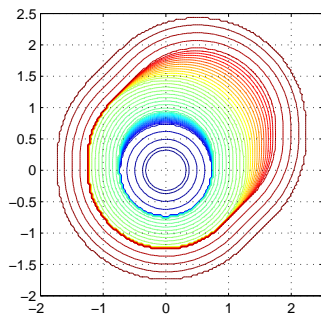


(a) 0-level sets of  $\bar{v}$

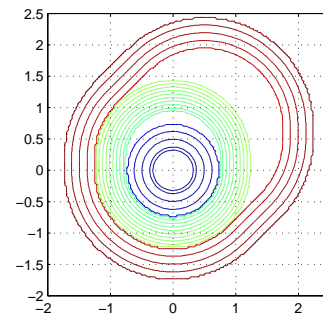


(b) 0-level sets of  $v$

Figure 1: Case 1: Numerical solutions with  $150^2$  grid nodes.



(a) 0-level sets of  $\bar{v}$



(b) 0-level sets of  $v$

Figure 2: Case 2: Numerical solutions with  $150^2$  grid nodes.

prove that it is consistent with the definitions of viscosity solutions given for a more regular HJB equation (Subsection 4.1). The stability result is given in Subsection 4.2.

Fix  $T > 0$ , and consider the general Hamilton-Jacobi-Bellman equation

$$\begin{cases} -u_t(x, t) + H(t, x, Du) &= 0 & \text{in } \mathbb{R}^N \times (0, T) \\ u(x, T) &= \varphi(x) & \text{in } \mathbb{R}^N. \end{cases} \quad (36)$$

On the Hamiltonian  $H : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  we assume the following:

- (H0)** The function  $H(t, x, p)$  is measurable in  $t$  and continuous in  $x$  and  $p$ . Moreover, for each  $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$  we have  $H(\cdot, x, p) \in L^1(\mathbb{R}^+)$ .
- (H1)** For each compact subset  $K$  of  $\mathbb{R}^N \times \mathbb{R}^N$  there exists a modulus  $m = m(K) : (0, T) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $t \rightarrow m(t, r) \in L^1(0, T)$  for all  $r \geq 0$ ,  $m(t, r)$  is increasing in  $r$ ,  $m(\cdot, r) \rightarrow 0$  in  $L^1(0, T)$  as  $r \rightarrow 0$ , and

$$|H(t, x, p) - H(t, y, q)| \leq m(t, |x - y| + |p - q|)$$

for almost every  $t$  and for any  $(x, p), (y, q) \in K$ .

Moreover, in the following, we may need some assumptions stronger than **(H1)**:

- (H2)** There exists a function  $k_0 \in L^\infty(\mathbb{R}^+; \mathbb{R}^+)$  such that

$$|H(t, x, p) - H(t, y, p)| \leq k_0(t)(1 + |p|)|x - y| \quad \text{for all } p \in \mathbb{R}^N, t \in \mathbb{R}^+, x, y \in \mathbb{R}^N.$$

- (H3)** For each  $(t, x)$  the function  $H(t, x, \cdot)$  is convex and there exists a constant  $L > 0$  such that

$$|H(t, x, p) - H(t, x, q)| \leq L|p - q| \quad \text{for all } p, q \in \mathbb{R}^N, t \in \mathbb{R}^+, x \in \mathbb{R}^N.$$

On the final data  $\varphi$  we suppose **(Hid)**.

**Remark 6.** It is easy to check that if the dynamics  $\mathcal{F}$  fulfills assumptions **(HF1)**-**(HF3)**, then the Hamiltonian defined in (24) satisfies assumptions **(H0)**-**(H3)**.

In order to give an equivalent formulation of the definition of  $L^1$ -bilateral viscosity solution we need to introduce the following sets of functions. Fix  $(x_0, t_0)$  and a function  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ , and set:

$$\begin{aligned} \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0)) &:= \\ &= \{G(t, x, p) \in C(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N), \text{ convex in } p, b(t) \in L^1(\mathbb{R}^+) \\ &\quad \text{such that } G(t, x, p) + b(t) \leq H(t, x, p) \\ &\text{for all } x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0, t_0)), \text{ a. e. } t \in B_\delta(t_0) \text{ and some } \delta > 0\} \end{aligned}$$

$$\begin{aligned} \mathcal{H}^+(t_0, x_0, D\phi(x_0, t_0)) &:= \\ &= \{G(t, x, p) \in C(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N), \text{ convex in } p, b(t) \in L^1(\mathbb{R}^+) \\ &\quad \text{such that } G(t, x, p) + b(t) \geq H(t, x, p) \\ &\text{for all } x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0, t_0)), \text{ a. e. } t \in B_\delta(t_0) \text{ and some } \delta > 0\}. \end{aligned}$$



**Definition 4.1.  $L^1$ -bilateral viscosity solution (L1Bvs) II**

Let  $u : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$  be a bounded lower semi-continuous function. We say that  $u$  is a  $L^1$ -bilateral viscosity solution (L1Bvs) of (36) if:

1. for any  $(x_0, t_0)$ ,  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ ,  $(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0))$  such that  $(x_0, t_0)$  is a local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$  we have

$$-\phi_t(x_0, t_0) + G(t_0, x_0, D\phi(x_0, t_0)) \leq 0,$$

2. for any  $(x_0, t_0)$ ,  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ ,  $(G, b) \in \mathcal{H}^+(t_0, x_0, D\phi(x_0, t_0))$  such that  $(x_0, t_0)$  is a local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$  we have

$$-\phi_t(x_0, t_0) + G(t_0, x_0, D\phi(x_0, t_0)) \geq 0.$$

3. The final condition is satisfied in the following sense:

$$\varphi(x) = \inf \left\{ \liminf_{n \rightarrow \infty} u(x_n, t_n) : x_n \rightarrow x, t_n \uparrow T \right\}.$$

**Remark 7.** Other formulations can be considered to define the  $L^1$ -bilateral viscosity notion. For instance, one can take the test function  $\phi(x, t) \in C^1(\mathbb{R}^N \times (0, T))$  in Definition 3.1 or  $\phi \in C^1(\mathbb{R}^N)$  in Definition 4.1. Of course, one can also replace  $\phi \in C^1(\mathbb{R}^N)$  by  $\phi \in C^2(\mathbb{R}^N), \dots, C^\infty(\mathbb{R}^N)$ . On the other hand, by classical arguments in the theory of viscosity solutions, we may replace the local minimum by global, or local strict or global strict.

**Proposition 2.** *Assume (H0) and (H1). Then, Definitions 3.1 and 4.1 are equivalent.*

**Proof.** The equivalence follows by remarking that for any  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x)$  we have

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \operatorname{ess\,sup}_{|t-t_0| \leq \delta} \sup_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} &= \\ &= \inf_{(G, b) \in \mathcal{H}^+(t_0, x_0, D\phi(x_0))} G(t_0, x_0, D\phi(x_0)) \end{aligned}$$

and

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} &= \\ &= \sup_{(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0))} G(t_0, x_0, D\phi(x_0)). \end{aligned}$$

□

## 4.1 Consistency

Let us now prove that Definition 4.1 is consistent with the definitions of viscosity solutions given for more regular HJB equations. In particular those considered for a time-continuous Hamiltonian and/or a continuous initial data. For the sake of completeness let us recall here the definition of viscosity solution in those cases.

**Definition 4.2** (bilateral viscosity solution (Bvs), [7]). Assume that  $H$  is continuous w.r. to the time variable. Let  $u \in LSC(\mathbb{R}^N \times (0, T))$  be a bounded function. We say that  $u$  is a bilateral viscosity solution (Bvs) of (36) if for any  $\phi \in C^1(\mathbb{R}^N \times (0, T))$  and  $(x_0, t_0)$  local minimum point of  $u(x, t) - \phi(x, t)$  we have

$$-\phi_t(x_0, t_0) + H(t_0, x_0, D\phi(x_0, t_0)) = 0,$$

and if the final condition is satisfied in the following sense:

$$\varphi(x) = \inf \left\{ \liminf_{n \rightarrow \infty} u(x_n, t_n) : x_n \rightarrow x, t_n \uparrow T \right\}.$$

**Definition 4.3** ( $L^1$ -viscosity solution (L1vs), [24, 25]). Assume that the final condition  $\varphi$  is a continuous function on  $\mathbb{R}^N$ .

We say that  $u \in LSC(\mathbb{R}^N \times (0, T))$  is a  $L^1$ -viscosity supersolution (L1vsp) of (36) if: for any  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local minimum point of  $u(x, t) - \int_0^t b(s)ds - \phi(x)$  we have

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,sup}_{|t-t_0| \leq \delta} \sup_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \geq 0.$$

We say that  $u \in USC(\mathbb{R}^N \times (0, T))$  is a  $L^1$ -viscosity subsolution (L1vsb) of (36) if: for any  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local maximum point of  $u(x, t) - \int_0^t b(s)ds - \phi(x)$  we have

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq 0.$$

We say that  $u \in C(\mathbb{R}^N \times (0, T))$  is a  $L^1$ -viscosity solution (L1vs) if it is both a  $L^1$ -viscosity subsolution and a  $L^1$ -viscosity supersolution and the final condition is satisfied pointwise:

$$u(x, T) = \varphi(x) \text{ in } \mathbb{R}^N.$$

The link between these two definitions and our Definition 4.1 is stated in the following Theorem.

**Theorem 4.4. (Consistency).** *Assume (H0)-(H3) and (Hid).*

(a) *If the final condition  $\varphi$  is a continuous function, then*

$$u \text{ is a } L^1\text{-bilateral viscosity solution} \iff u \text{ is a } L^1\text{-viscosity solution.}$$

(b) *If the Hamiltonian  $H$  is continuous also in the  $t$ -variable, then*

$$u \text{ is a } L^1\text{-bilateral viscosity solution} \iff u \text{ is a bilateral viscosity solution.}$$

**Proof.** The proof of statement (a) is based on some results introduced and developed in ([7, Theorem 1.1]). We recall here this result for the convenience of the reader.

**Lemma 4.5.** *Let  $W$  be a continuous function on  $[0, \infty) \times \mathbb{R}^n$  such that  $W$  has a zero maximum (minimum) at  $(\tau, \xi)$ . Let  $\varepsilon > 0$ . Then there is a smooth function  $\psi$ , a finite set of numbers  $\alpha_k \geq 0$  summing to one, and a finite collection of points  $(t_k, x_k)$  such that*

1.  $W - \psi$  has a zero minimum (maximum) at  $(t_k, x_k)$ ;
2.  $(t_k, x_k) \in B_{o_\varepsilon(1)\sqrt{\varepsilon}}(s, y)$  for some  $(s, y) \in B_{o_\varepsilon(1)}(\tau, \xi)$ ;
3.  $|D_{t,x}\psi(t_k, x_k)| = \frac{o_\varepsilon(1)}{\sqrt{\varepsilon}}$ ;
4.  $\sum_k \alpha_k D_{t,x}\psi(t_k, x_k) = 0$ .

Step 1. Assume that  $u$  is a L1vs and let us show that  $u$  is also a L1Bvs. For this let  $b$  be in  $L^1(0, T)$ ,  $\phi$  be in  $C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  be a local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x)$ , we have to show that

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,sup}_{|t-t_0| \leq \delta} \sup_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \geq 0 \quad (37)$$

and

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq 0. \quad (38)$$

Since  $u$  is a L1vs, then we have (37). To prove (38), for each  $\delta > 0$  we apply Lemma 4.5 above choosing  $\varepsilon$  small enough to ensure the existence of an  $\eta > 0$  such that  $\frac{o_\varepsilon(1)}{\sqrt{\varepsilon}} + o_\varepsilon(1) + \eta \leq \delta$  and  $o_\varepsilon(1)\sqrt{\varepsilon} + o_\varepsilon(1) + \eta \leq \delta$  (and with  $W(t, x) = u(x, t) + \int_0^t b(s)ds - \phi(x, t)$ ).

Therefore, there exists a smooth function  $\psi$  and a finite set of points  $(x_k, t_k)$  such that  $u - \int_0^t b - (\phi + \psi)$  has a zero maximum at  $(x_k, t_k)$  and for each  $k$

$$B_\eta(x_k, t_k) \subset B_\delta(t_0, x_0), \quad B_\eta(D\phi(x_k) + D\psi(x_k, t_k)) \subset B_\delta(D\phi(x_0)). \quad (39)$$

Thus

$$\begin{aligned} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} &\leq \\ &\leq \operatorname{ess\,inf}_{|t-t_k| \leq \eta} \inf_{x \in B_\eta(x_k), p \in B_\eta(D\phi(x_k) + D\psi(x_k, t_k))} \{H(t, x, p) - b(t)\}. \end{aligned} \quad (40)$$

Since  $u$  is a L1vs, in particular is a  $L^1$ -viscosity subsolution therefore in each point  $(t_k, x_k)$  we have

$$\lim_{\eta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_k| \leq \eta} \inf_{x \in B_\eta(x_k), p \in B_\eta(D\phi(x_k) + D\psi(x_k, t_k))} \{H(t, x, p) - b(t)\} \leq 0.$$

Letting  $\delta$  going to  $0^+$  ( $\Rightarrow \eta \rightarrow 0^+$ ) in (40) we obtain (38) and conclude the proof.

Step 2. Assume that  $u$  is a L1Bvs and let us show that  $u$  is also a L1vs. We first remark that, by Definition 3.1 if  $u$  is a L1Bvs, is in particular a  $L^1$ -viscosity supersolution. Therefore, to prove that  $u$  is a  $L^1$ -viscosity subsolution fix  $b \in L^1(0, T)$ ,  $\phi \in C^1(\mathbb{R}^N)$  and  $(x_0, t_0)$  local maximum point of  $u(x, t) - \int_0^t b(s)ds - \phi(x)$  our thesis is

$$\lim_{\delta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq 0. \quad (41)$$

As above, for each  $\delta > 0$  we apply Lemma 4.5 choosing  $\varepsilon$  small enough to ensure the existence of an  $\eta > 0$  such that  $\frac{o_\varepsilon(1)}{\sqrt{\varepsilon}} + o_\varepsilon(1) + \eta \leq \delta$  and  $o_\varepsilon(1)\sqrt{\varepsilon} + o_\varepsilon(1) + \eta \leq \delta$ ,

and with  $W(t, x) = u(x, t) + \int_0^t b(s) ds - \phi(x, t)$ . Therefore, there exists a smooth function  $\psi$  and a finite set of points  $(x_k, t_k)$  such that  $u - \int_0^t b - (\phi + \psi)$  has a zero minimum at  $(x_k, t_k)$  and for each  $k$

$$B_\eta(x_k, t_k) \subset B_\delta(t_0, x_0), \quad B_\eta(D\phi(x_k) + D\psi(x_k, t_k)) \subset B_\delta(D\phi(x_0)). \quad (42)$$

Thus,

$$\begin{aligned} & \operatorname{ess\,inf}_{|t-t_0| \leq \delta} \inf_{x \in B_\delta(x_0), p \in B_\delta(D\phi(x_0))} \{H(t, x, p) - b(t)\} \leq \\ & \leq \operatorname{ess\,inf}_{|t-t_k| \leq \eta} \inf_{x \in B_\eta(x_k), p \in B_\eta(D\phi(x_k) + D\psi(x_k, t_k))} \{H(t, x, p) - b(t)\}. \end{aligned} \quad (43)$$

Since  $u$  is a L1Bvs we have (3.1) at each point  $(t_k, x_k)$ , i.e.

$$\lim_{\eta \rightarrow 0^+} \operatorname{ess\,inf}_{|t-t_k| \leq \eta} \inf_{x \in B_\eta(x_k), p \in B_\eta(D\phi(x_k) + D\psi(x_k, t_k))} \{H(t, x, p) - b(t)\} \leq 0.$$

Letting  $\delta$  going to  $0^+$  ( $\Rightarrow \eta \rightarrow 0^+$ ) in (43) we obtain (41) and conclude the proof.

The proof of (b) is straightforward.  $\square$

## 4.2 Stability

We prove here a stability result with respect to the Hamiltonian. It will be proved assuming a very weak convergence in time as done, for  $L^1$ -viscosity solution, by Barles in [4]. (Our proof is indeed an adaptation to L1Bvs of the proof of [4, Theorem 1.1]). Note that in this proof we only need assumptions **(H0)**-**(H1)** on the Hamiltonian.

**Theorem 4.6. Stability w.r.to  $H$ .** *For each  $n \in \mathbb{N}$  let  $u_n$  be a  $L^1$ -bilateral viscosity solution of*

$$-u_t(x, t) + H_n(t, x, Du) = 0 \text{ in } \mathbb{R}^N \times (0, T) \quad (44)$$

*We assume that:*

- I) *For each  $n \in \mathbb{N}$  the Hamiltonian  $H_n$  fulfills hypotheses **(H0)**-**(H1)** for some modulus  $m_n = m_n(K)$  such that  $\|m_n(\cdot, r)\|_{L^1(0, T)} \rightarrow 0$  as  $r \rightarrow 0$  uniformly with respect to  $n$ , for any compact subset  $K$ .*
- II) *There exists a function  $H$  fulfilling hypotheses **(H0)**-**(H1)** such that, for any  $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$ ,*

$$\lim_{n \rightarrow \infty} \int_0^t H_n(s, x, p) ds = \int_0^t H(s, x, p) ds \text{ locally uniformly in } (0, T).$$

*Then, the function*

$$u(x, t) := \inf_{(x_n, t_n) \rightarrow (x, t)} \liminf_{n \rightarrow \infty} u_n(x_n, t_n),$$

*is a  $L^1$ -bilateral viscosity solution of*

$$-u_t(x, t) + H(t, x, Du) = 0 \text{ in } \mathbb{R}^N \times (0, T) \quad (45)$$

**Proof.** Following Definition 4.1 we have to prove that

- (1) for any  $(x_0, t_0)$ ,  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ ,  $(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0))$  such that  $(x_0, t_0)$  is a local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$  we have

$$-\phi_t(x_0, t_0) + G(t_0, x_0, D\phi(x_0, t_0)) \leq 0, \quad (46)$$

- (2) for any  $(x_0, t_0)$ ,  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ ,  $(G, b) \in \mathcal{H}^+(t_0, x_0, D\phi(x_0, t_0))$  such that  $(x_0, t_0)$  is a local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$  we have

$$-\phi_t(x_0, t_0) + G(t_0, x_0, D\phi(x_0, t_0)) \geq 0. \quad (47)$$

In order to prove statement (1), let us fix a  $(x_0, t_0)$ ,  $\phi \in C^1(\mathbb{R}^N \times \mathbb{R}^+)$ ,  $(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0))$  such that  $(x_0, t_0)$  is a *strict* local minimum point for  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$ .

Fix now a small  $\delta > 0$ , we consider a large compact subset  $K$  of  $\mathbb{R}^N \times \mathbb{R}^N$  and the functions  $m, m_n$  given by assumptions I), II). We construct a new sequence  $(u_n^\delta)_n$  defined by

$$u_n^\delta(x, t) := u_n(x, t) + \int_0^t [m_n(s, \delta) + m(s, \delta)] ds.$$

Note that for each  $n, \delta$  the function  $u_n^\delta$  is a L1Bvs of

$$-w_t + H_n(t, x, Dw) - m_n(t, \delta) - m(t, \delta) = 0. \quad (48)$$

Moreover, if we set  $u^\delta(x, t) := \inf_{(x_n, t_n) \rightarrow (x, t)} \liminf_{n \rightarrow \infty} u_n^\delta(x_n, t_n)$ , by the properties of  $m, m_n$  we have  $u \leq u^\delta \leq u + O_\delta(1)$ . Therefore, by classical results, since  $(x_0, t_0)$  is a *strict* local minimum point of  $u(x, t) - \int_0^t b(s)ds - \phi(x, t)$ , for  $\delta$  small enough there exists a local minimum point of  $u^\delta(x, t) - \int_0^t b(s)ds - \phi(x, t)$ , that we will denote  $(x_\delta, t_\delta)$ . Note that  $(x_\delta, t_\delta) \rightarrow (x_0, t_0)$  as  $\delta \rightarrow 0$ .

We set now

$$\psi_n(s) := H_n(s, x_\delta, D\phi(x_\delta, t_\delta)) - H(s, x_\delta, D\phi(x_\delta, t_\delta)).$$

Our aim is to use the fact that the function  $u_n^\delta$  is a L1Bvs of (48) by testing with the function  $\phi(x, t) + \int_0^t b - \int_0^t \psi_n$ . To do this we first observe that the convergence assumption II) implies  $\int_0^t \psi_n \rightarrow 0$  locally uniformly in  $(0, T)$  therefore, for each  $(x_\delta, t_\delta)$  local minimum point of  $u^\delta(x, t) - \int_0^t b(s)ds - \phi(x, t)$ , there exists a sequence  $(x_\delta^n, t_\delta^n) \rightarrow (x_\delta, t_\delta)$  as  $n \rightarrow \infty$  of local minimum points of  $u_n^\delta(x, t) - \phi(x, t) - \int_0^t b(s)ds + \int_0^t \psi_n(s)ds$ . (Recall that  $u^\delta(x, t) := \inf_{(x_n, t_n) \rightarrow (x, t)} \liminf_{n \rightarrow \infty} u_n^\delta(x_n, t_n)$ ).

Let  $(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0))$  we state now that there exists a  $n$  big enough and a  $\delta$  small enough such that we can find a  $\eta > 0$  for which

$$\begin{aligned} \psi_n(t) + G(t, x, p) + b(t) &\geq H_n(t, x, p) - m_n(t, \delta) - m(t, \delta) \\ \forall t \in B_\eta(t_\delta^n), x \in B_\eta(x_\delta^n), p \in B_\eta(D\phi(x_\delta^n, t_\delta^n)). \end{aligned} \quad (49)$$

Indeed, since  $(G, b) \in \mathcal{H}^-(t_0, x_0, D\phi(x_0, t_0))$  there exists a  $\beta$  such that

$$\begin{aligned} \psi_n(t) + G(t, x, p) + b(t) &\geq H_n(t, x, p) - m_n(t, \delta) - m(t, \delta) \\ \forall t \in B_\beta(t_0), x \in B_\beta(x_0), p \in B_\beta(D\phi(x_0, t_0)) \end{aligned}$$

(where we used also the definition of  $m, m_n$ ). Thus (49) follows from  $(x_\delta^n, t_\delta^n) \rightarrow (x_\delta, t_\delta)$  as  $n \rightarrow \infty$ ,  $(x_\delta, t_\delta) \rightarrow (x_0, t_0)$  as  $\delta \rightarrow 0$  and the regularity of  $\phi$ .

By definition of L1Bvs, condition (49) and the fact that  $(x_\delta^n, t_\delta^n)$  is a local minimum point of  $u_n^\delta(x, t) - \phi(x, t) - \int_0^t b(s)ds + \int_0^t \psi_n(s)ds$  imply that

$$-\phi_t(x_\delta^n, t_\delta^n) + G(x_\delta^n, t_\delta^n, D\phi(x_\delta^n, t_\delta^n)) \leq 0.$$

Therefore letting  $n \rightarrow \infty$  and  $\delta \rightarrow 0$  by the continuity of  $G$  we obtain (46) and conclude the proof of (1).

Point (2) can be proved with the same argument by remarking that the functions

$$u_n^\delta(x, t) := u_n(x, t) - \int_0^t [m_n(s, \delta) + m(s, \delta)] ds$$

are L1Bvs of

$$-w_t + H_n(t, x, Dw) + m_n(t, \delta) + m(t, \delta) = 0. \quad \square$$

**Theorem 4.7. (Stability w.r.to  $\varphi$ .)** *For each  $n \in \mathbb{N}$ , let  $u_n$  be a  $L^1$ -bilateral viscosity solution of equation (36) with final condition  $u_n(x, T) = \varphi_n(x)$  in  $\mathbb{R}^N$ , where for each  $n \in \mathbb{N}$ , the function  $\varphi_n \in C(\mathbb{R}^N)$  and is bounded. Assume that the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is monotone increasing and for every  $x \in \mathbb{R}^N$ , we have:  $\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$ . Let  $u$  be the  $L^1$ -bilateral viscosity solution of equation (36) with final condition  $u(x, T) = \varphi(x)$  in  $\mathbb{R}^N$ .*

*Then,  $u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$ , for each  $(x, t) \in \mathbb{R}^N \times (0, T)$ .*

**Proof.** We first remark that since  $\varphi_n \in C(\mathbb{R}^N)$ ,  $u_n$  is continuous in  $\mathbb{R}^N \times [0, T]$  (see [24, Corollary 1.10]). Therefore, by uniqueness of L1vs solution, the sequence  $(u_n)_{n \in \mathbb{N}}$  is monotone increasing. We set  $w(x, t) := \sup_{n \in \mathbb{N}} u_n(x, t)$  and we remark that

$$w(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = \inf_{(x_n, t_n) \rightarrow (x, t)} \liminf_{n \rightarrow \infty} u_n(x_n, t_n),$$

(see, for instance [2, Lemma 2.18]). Therefore, by Theorem 4.6 above we have that  $w$  is a L1Bvs of (36) in  $\mathbb{R}^N \times (0, T)$ . Moreover, for each sequence  $(x_n, t_n) \rightarrow (x, T)$ , we have

$$\varphi(x) = \lim_{n \rightarrow \infty} \varphi_n(x) = \lim_{n \rightarrow \infty} u_n(x, T) = \lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} u_n(x_n, t_n) = \lim_{n \rightarrow \infty} w(x_n, t_n).$$

Thus  $w$  verify also the final condition  $w(x, T) = \varphi(x)$ . By uniqueness (Theorem 4.8 below) we can that conclude that  $w(x, t) = u(x, t)$  for all  $(x, t) \in \mathbb{R}^N \times (0, T)$  and the proof is completed.  $\square$

### 4.3 Uniqueness

We finally prove the uniqueness result.

**Theorem 4.8.** *Assume (H0)-(H3) and (Hid). Then there exists at most one  $L^1$ -bilateral viscosity solution of (36).*

**Proof.** This proof will follow the idea of G.Barles of using the inf-convolution in the proof of uniqueness for bilateral viscosity solution [3, Theorem 5.14].

Suppose that there exist  $v$  and  $u$  two  $L^1$ -bilateral viscosity solution of (36). Since  $v$  is in particular a  $L^1$ -viscosity supersolution the main point is to look for a

sequence of  $L^1$ -viscosity subsolutions of (36) approximating  $u$ . The thesis will then follow by comparison result for  $L^1$ -viscosity solution.

The construction of the approximating sequence can be summarised in the following Lemma. The proof being an adaptation of the proof given in [3, Lemme 5.5] will be not detailed (see also [6, Lemma 19]).

**Lemma 4.9.** *Under the assumption of Theorem 4.8, if  $u$  is  $L^1$ -bilateral viscosity solution of (36), let  $u_\varepsilon$  be defined by*

$$u_\varepsilon(x, t) := \inf_{y \in \mathbb{R}^N} \left\{ u(y, t) + e^{-Kt} \frac{|x - y|^2}{\varepsilon^2} \right\}, \quad \varepsilon > 0.$$

*Then, the upper semi continuous envelope  $(u_\varepsilon)^*$  is a  $L^1$ -viscosity subsolution of*

$$-(u_\varepsilon)_t + H(t, x, Du_\varepsilon) - \|k_0\|_\infty e^{\frac{1}{2}KT} M\varepsilon = 0 \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, T), \quad (50)$$

*for  $K$  big enough and where  $M = \sqrt{2} \|u\|_\infty$  and  $k_0$  is given in assumption **(H2)**. Moreover,*

$$(u_\varepsilon)^*(x, T) \leq \varphi(x) \quad \text{for } x \in \mathbb{R}^N. \quad (51)$$

Since  $(u_\varepsilon)^*$  is a  $L^1$ -viscosity subsolution of (50) the function  $(u_\varepsilon)^* - (\|k_0\|_\infty e^{\frac{1}{2}KT} M\varepsilon)t$  is a  $L^1$ -viscosity subsolution of (36), therefore, by the comparison result for  $L^1$ -viscosity solutions (see [24, Theorem 8.1] or [29]) we obtain

$$(u_\varepsilon)^* - \|k_0\|_\infty e^{\frac{1}{2}KT} M\varepsilon \leq v(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, T)$$

where we used also (51). Letting  $\varepsilon \rightarrow 0$  we have

$$u(x, t) \leq v(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, T),$$

thus, reversing the roles of  $u$  and  $v$ , the uniqueness follows.  $\square$

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