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Observability and Detectability Analysis of Singular Linear Systems with Unknown Inputs

Francisco Javier Bejarano, Thierry Floquet, Wilfrid Perruquetti, and Gang Zheng

***Abstract*— We study the observability problem of a general class of singular linear systems with unknown inputs. It is shown that, under some assumptions, the problem is equivalent to study the observability of a standard linear system with unknown inputs satisfying algebraic constraints. We obtain necessary and sufficient conditions for observability in terms of the zeros of the system matrix.**

***Index Terms*— Singular systems, strong observability, algebraic observability.**

I. Introduction

The problem of designing an observer for a multivariable linear system partially driven by unknown inputs has been widely studied [1], [2], [3]. Such observers can be of important use for systems subject to disturbances or with inaccessible inputs, or when dealing with the fault diagnosis problem.

Observability and the problem of observer design have been quite widely studied for singular systems with perfectly known model ([4], [5],[6], [7], [8], [9]). However there exist few results dealing with the problem of observer design for singular systems with unknown inputs [10], [11], [12]. Most investigation have been devoted to designing Luenberger observers. Such observers can be designed under necessary and sufficient conditions.

In this note, the observability problem of a general class of singular linear systems with unknown inputs is studied. It is shown that, under some assumptions, the problem is equivalent to study the observability of a standard linear system with unknown inputs satisfying algebraic constraints. We obtain necessary and sufficient conditions for observability in terms of the zeros of the system matrix. The observer design is based on exact differentiators to generate additional independent output signals from the available measurements.

Notation. The following notation will be used throughout the paper. For a matrix X, we denote by $X^{\perp}$ a full row rank matrix such that $X^{\perp}X = 0$, and by $X^{\perp\perp}$ a full row rank matrix such that $\operatorname{rank} X^{\perp\perp}X = \operatorname{rank} X$. The Moore-Penrose pseudoinverse matrix of X is denoted by X^{+}. By $\|\cdot\|$, we mean the Euclidean norm. $\mathbb{C}^{-}$ denotes de set of complex numbers with strictly negative real part. I_r is the identity matrix of dimension r by r. $0_{r\times s}$ is the zero matrix of dimension r. And as usual $x\left(0^{+}\right) = \lim_{t\to 0^{+}} x\left(t\right)$.

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II. Observation Problem Formulation

$x_{\mu_1}(\xi, t)$

Let us consider the class of linear singular systems governed by the following equations

$$\Sigma : \left\{ \begin{array}{rcl} E\dot{x}\left(t\right) & = & Ax\left(t\right) + D\mu\left(t\right) \\ y\left(t\right) & = & Cx + F\mu\left(t\right) \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the system output, and $\mu \in \mathbb{R}^m$ is the unknown input vector. Matrices $E, A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $F \in \mathbb{R}^{p\times m}$ are all constant. The matrix E is assumed to be **singular**. Given a state $x_0 \in \mathbb{R}^n$ and a function μ, we denote by $x_\mu\left(x_0, t\right)$ the state of Σ at time t which results from taking the initial condition equal to x_0 and the input vector is equal to μ. Therefrom, in a straightforward manner we define the output $y_\mu\left(x_0, t\right) = Cx_\mu\left(x_0, t\right) + F\mu\left(t\right)$.

We are interested in the reconstruction of the state vector $x\left(t\right)$ given the output information $y\left(\tau\right)_{\tau\in[0,t)}$. In general, system Σ must not have a regular pencil [13], i.e. it is allowed that $\det(\lambda E - A) = 0$ for all $\lambda \in \mathbb{C}$. Nevertheless, it is necessary to assume that for all μ, there exists $x\left(t\right)$ as a solution of Σ which is piecewise continuous for all $t > 0$; however, an impulse may occur at $t = 0$. In order to give algebraic conditions allowing the reconstruction of $x\left(t\right)$, we consider the following definitions, which are based on classical definitions regarding the strong observability (SO) and strong detectability (SD) properties (see, e.g. [14]).

Definition 1 (Strong observability): The system Σ is strongly observable if for all $x_0 \in \mathbb{R}^n$ and for every input function μ, the following implication is satisfied

$$y_\mu\left(x_0, t\right) = 0 \ \forall t > 0 \text{ implies } x\left(0^{+}\right) = 0. \tag{2}$$

Definition 2 (Strong detectability): The system Σ is strongly detectable if for all $x_0 \in \mathbb{R}^n$ and for every input function μ, the following implication holds

$$y_\mu\left(x_0, t\right) = 0 \ \forall t > 0 \text{ implies } \lim_{t\to\infty} x_\mu\left(x_0, t\right) = 0. \tag{3}$$

In the next section, we show that checking the SO (resp. SD) of system Σ amounts to verifying two conditions: the SO (resp. SD) of a regular linear systems with unknown inputs and a rank condition.

III. Observability Analysis

Since E is singular, there exist non-singular matrices $T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ such that E can be transformed as

follows[1],

$$TES = \begin{bmatrix} I_{\rho_E} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \rho_E := \operatorname{rank} E \tag{4}$$

Thus, let us define the vector $z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = S^{-1}x$, where $z_1 \in \mathbb{R}^{\rho_E}$ and $z_2 \in \mathbb{R}^{n-\rho_E}$. In these new coordinates, Σ can be rewritten as follows

$$\Psi : \begin{cases} \dfrac{d}{dt}TESz(t) &= TASz(t) + TD\mu(t) \\ y(t) &= CSz(t) + F\mu(t) \end{cases} \tag{5}$$

It is clear that Σ is SO (resp. SD) if, and only if, Ψ is SO (resp. SD). Thus, due to (4), Ψ takes the following form

$$\dot{z}_1(t) = T_1AS_1z_1(t) + T_1AS_2z_2(t) + T_1D\mu(t) \tag{6a}$$
$$0 = T_2AS_1z_1(t) + T_2AS_2z_2(t) + T_2D\mu(t) \tag{6b}$$
$$y = CS_1z_1(t) + CS_2z_2(t) + F\mu(t) \tag{6c}$$

We might consider system Ψ as a regular system with unknown inputs (including vector z_2) and algebraic constraints. As we will see below a simple manner to study the observability of Σ is considering (6b) as part of the system output of a new pseudo system and considering z_2 as part of the unknown input vector. Indeed, consider the following system

$$\Phi : \begin{cases} \dot{z}_1(t) &= \bar{A}z_1(t) + \bar{D}v(t) \\ \bar{y}(t) &= \bar{C}z_1(t) + \bar{F}v(t) \end{cases} \tag{7}$$

where $v(t) \in \mathbb{R}^{n-\rho_E+m}$, $\bar{y}(t) \in \mathbb{R}^{n-\rho_E+p}$ and the matrices $\bar{A}$, $\bar{D}$, $\bar{C}$, and $\bar{F}$ are defined as follows

$$\bar{A} = T_1AS_1, \quad \bar{D} = \begin{bmatrix} T_1AS_2 & T_1D \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} T_2AS_1 \\ CS_1 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} T_2AS_2 & T_2D \\ CS_2 & F \end{bmatrix}$$

It is clear by (6) that Φ looks like system Ψ. In general, they do not represent identical systems. However, both systems are identical if these both identities hold: $\bar{y} = \begin{bmatrix} 0 \\ y \end{bmatrix}$ and $v(t) = \begin{bmatrix} z_2(t) \\ \mu(t) \end{bmatrix}$. We will show in the next theorem, the fulfillment of the SO (resp. SD) of Σ is equivalent to the fulfillment of the SO (resp. SD) of Φ plus a rank condition (needed to reconstruct z_2).

Theorem 1: System Σ is SO (resp. SD) if, and only if, Φ is SO (resp. SD) and the following rank condition holds

$$\operatorname{rank}\bar{B} = n - \rho_E + \operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix}, \text{ where } \bar{B} := \begin{bmatrix} \bar{D} \\ \bar{F} \end{bmatrix}. \tag{8}$$

Furthermore, the equivalence claimed in this theorem is independent of the choice of the matrices T and S.

[1]We might select $S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$ to be nonsingular and so that $\operatorname{im} S_2 = \ker E$. Thus, $ES = \begin{bmatrix} ES_1 & 0 \end{bmatrix}$ and $\operatorname{rank} ES_1 = \operatorname{rank} E$. Then a nonsingular matrix T might be selected as $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ so that $T_1ES_1 = I$ and $T_2ES_1 = 0$, one possibility is to select $T_1 = (ES_1)^+ = \left[(ES_1)^T(ES_1)\right]^{-1}(ES_1)^T$.

Proof: Firstly, notice that, since $\operatorname{rank}\begin{bmatrix} D^T & F^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} (TD)^T & F^T \end{bmatrix}$, the fulfillment of (8) is equivalent to say that $\bar{B}\begin{bmatrix} z_2^T & \mu^T \end{bmatrix}^T = 0$ implies $z_2 = 0$.

Necessity. Assume that Σ is SO (resp. SD). Hence, implication (2) (resp. implication (3)) holds. Now, if, for an input v and state z_1, $\bar{y}_v(z_1, t) = 0$ for all $t > 0$. By selecting $\begin{bmatrix} z_2(t) \\ \mu(t) \end{bmatrix} = v(t)$, we make that Ψ and Φ represent the same system. Thus, with $x_1 = Sz(0)$, we obtain that $y_\mu(x_1, t) = 0$ for all $t > 0$. Since (2) (resp. implication (3)) holds, $x(0^+) = 0$ (resp. $x(t)$ converges to zero), which in turn implies that $z(0^+) = 0$ (resp. $z(t)$ converges to zero), in particular $z_1(0^+) = 0$ (resp. $\lim_{t\to\infty} z_1(t) = 0$), i.e. Φ is SO (resp. SD).

Now, assume that (8) does not hold. Then, there exists a vector v which can be divided as $v^T = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}$ ($v_1 \in \mathbb{R}^{n-\rho_E}$, $v_2 \in \mathbb{R}^m$) so that $\bar{B}v = 0$ and $v_1 \neq 0$. By selecting $z_2(t) = v_1$ and $\mu(t) = v_2$, and $z_1(0^+) = 0$, eqs. (6) are fulfilled, and $y(t) = 0$ for all $t > 0$. Therefore $x(t) = Sz(t) = S\begin{bmatrix} 0 & v_1^T \end{bmatrix}^T = \text{const} \neq 0$. That is, in such a case Σ is not SO (resp. Σ is not SD).

Sufficiency. First, assume that Φ is SO and (8) is fulfilled. Then $\bar{y}(t) = 0$, for all $t > 0$, implies that $z_1(t) = 0$. Furthermore, it implies, (from (8)) that $z_2(t) = 0$ also. Suppose that $y_\mu(x_0, t) = 0$ for a state $x_0 \in \mathbb{R}^n$ and an input function μ. By taking $z_\mu(z_0, t) = S^{-1}x_\mu(x_0, t)$, we have that the algebraic constraint (6b) is fulfilled. Thus, we have that, for Φ, $\bar{y}_v(z_1, t) = 0$, for all $t > 0$, with v defined such that Φ and Ψ represent the same system. Then, from the assumption over Φ, $z_1(t) = 0$ and $z_2 = 0$ for all $t > 0$. That is $z(t)$ is identical to zero for all $t > 0$. Therefore, $x(0^+) = Sz(0^+) = 0$. Thus we conclude saying that Σ is SO.

Now, let us assume that Φ is SD and the rank condition (8) is fulfilled. If $y_\mu(x_0, t) = 0$ for an $x_0 \in \mathbb{R}^n$ and some input function μ, then, again, $\bar{y}_v(z_1, t) = 0$, for all $t \geq 0$, with v properly selected. Due to the SD assumption, $z_1(t)$ converges to zero. Due to the SD and condition (8), $v(t)$ must have the form $v(t) = K^*z_1(t) + Lw$ ($t > 0$) for a particular matrix K^*, a matrix L such that $\bar{B}L = 0$ and a function w (the properties of K, L are not relevant for this proof, see, e.g. [14]). Then, due to the convergence of z_1, we have that $\bar{B}v(t)$ converges to zero also, which in turn due to (8), implies that $z_2(t)$ converges to zero. Then we can say the same for the entire state $z(t)$. We finish concluding that $x_\mu(x_0, t)$ converges to zero also and, therefore, Σ is SD.

The independence from T and S is trivial. Indeed, let (T^1, S^1) and (T^2, S^2) be two pair of matrices such that T^i and S^i ($i = 1, 2$) satisfy (4). Let us call $\Phi(T^i, S^i)$ to the system Φ when $T = T^i$ and $S = S^i$. If $\Phi(T^1, S^1)$ is SO (resp. SD) and (8) is satisfied, then by the first part of theorem 1 Σ is SO (resp. SD) also, which in turn implies, by the same theorem (first part), that $\Phi(T^2, S^2)$ is SO (resp. SD) and (8) is fulfilled. So neither the SO nor the SD of Φ depends on the choice of pair of matrices T and S satisfying (4). ■

As for system Φ, SO and SD can be completely determined by the four-tuple $(\bar{A}, \bar{C}, \bar{D}, \bar{F})$. Therefore, as for Σ, we expect that those properties can be completely characterized by the five-tuple (E, A, C, D, F). Let $R(\lambda)$ be the system matrix of Σ, i.e.,

$$R(\lambda) = \begin{bmatrix} \lambda E - A & -D \\ C & F \end{bmatrix}, \lambda \in \mathbb{C}$$

We say that $\lambda_0 \in \mathbb{C}$ is a zero of Σ if $\operatorname{rank} R(\lambda_0) < n + \operatorname{rank} \begin{bmatrix} D \\ F \end{bmatrix}$. Let $\sigma_z(\Sigma)$ be defined as the set of zeros of Σ. Let us characterize SO and SD in terms of the zeros of Σ.

Corollary 1: System Σ is SO (resp. SD) if, and only if, $\sigma_z(\Sigma) = \varnothing$ (resp. $\sigma_z(\Sigma) \subset \mathbb{C}^-$).

Proof: Let $Q(\lambda)$ be the system matrix of Φ. We have that

$$\operatorname{rank} \begin{bmatrix} E\lambda - A & -D \\ C & F \end{bmatrix} = \\ = \operatorname{rank} \begin{bmatrix} Is - T_1AS_1 & -T_1AS_2 & T_1D \\ -T_2AS_1 & -T_2AS_2 & -T_2D \\ CS_1 & CS_2 & F \end{bmatrix}$$

That is, $\operatorname{rank} R(\lambda) = \operatorname{rank} Q(\lambda)$. Thus, the corollary follows from Theorem 1 and the fact that Φ is SO (resp. SD)[2] if and only if $\sigma_z(\Phi) = \varnothing$ (resp. $\sigma_z(\Phi) \subset \mathbb{C}^-$). ■

Notice that, as we have used it implicitly in the proof of Corollary 1, for the case $E = I$ Corollary 1 is a known fact. Furthermore, if we consider that no unknown inputs are affecting the system, i.e., $\mu = 0$, then from Corollary 1 we have that Σ is observable in the sense of definition 1 if, and only if, $\operatorname{rank} \begin{bmatrix} E\lambda - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$, which coincides with the condition obtained in [4] for observability of singular systems without unknown inputs.

IV. Algebraic Observability

As we might expect SO coincides with algebraic observability (reconstructability): we say that Σ is algebraic observable if x can be expressed as an algebraic function of y and a finite number of its derivatives (see, e.g. [16]). To find an algebraic function, one can use the weakly unobservable subspace. Precisely a recursive algorithm that allows constructing such a subspace was proposed in [17]. Furthermore, using that algorithm when Σ is SD we can expressed x as a function of y, a finite number of its derivatives, and a variable not known in finite time, but converging asymptotically to zero.

We have to recall some concepts concerning the strong observability and detectability (see, e.g. [14]). For the linear system (7), we say that $z_{10} \in \mathcal{V}(\Phi)$ if there exists an input function $v(t)$ such that $\bar{y}_v(z_{10}, t) = 0$ for all $t \geq 0$. $\mathcal{V}(\Phi)$ is called the weakly unobservable subspace of Φ. It is clear that Φ is SO if, and only if, $\mathcal{V}(\Phi) = \{0\}$. A recursive algorithm to construct $\mathcal{V}(\Phi)$ has been given as follows

$$\mathcal{V}_0 = \mathbb{R}^n, \ \mathcal{V}_{k+1} = \begin{pmatrix} \bar{A} \\ \bar{C} \end{pmatrix}^{-1} \left[(\mathcal{V}_k \times 0) + \operatorname{im} \begin{pmatrix} \bar{D} \\ \bar{F} \end{pmatrix} \right]$$

[2]Such a statement was proven in [15].

Between some other interesting facts, we have that if $\mathcal{V}_{k+1} = \mathcal{V}_k$, then $\mathcal{V}_k = \mathcal{V}_j$ for all $j \geq k$, and there exists $k \leq \rho_E$ such that $\mathcal{V}(\Phi) = \mathcal{V}_k$. In matrix terms, we can obtain $\mathcal{V}_{k+1}$ by the following algorithm (see, [17]).

$$\begin{aligned} M_{k+1} &= N_{k+1}^{\perp\perp} N_{k+1}, \quad M_1 = \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp} \bar{F}^{\perp}\bar{C} \\ N_{k+1} &= T_k \begin{pmatrix} M_k\bar{A} \\ \bar{C} \end{pmatrix}, \ T_k = \begin{pmatrix} M_k\bar{D} \\ \bar{F} \end{pmatrix}^{\perp} \end{aligned} \tag{9}$$

Thus, we have that $\mathcal{V}_{k+1} = \ker M_{k+1}$. Most important for us it is the fact that there exists an integer $k \leq \rho_E$ such that $\mathcal{V}(\Phi) = \ker M_k$. Let us denote by l, **the smallest integer such that** $\operatorname{rank} M_l = \operatorname{rank} M_{l+1}$, which yields the identity

$$\mathcal{V}(\Phi) = \ker M_l \tag{10}$$

For our purposes, we point out that Φ **is SO if, and only if**, $\operatorname{rank} M_l = \rho_E$. For the case of SD we have to work a bit more with system Φ. Indeed, let us assume that $\operatorname{rank} M_l < \rho_E$. Let V be a full column rank matrix so that $M_l V = 0$, i.e. $\operatorname{im} V = \mathcal{V}(\Phi)$. There exists a pair of matrices Q and K^* such that

$$\bar{A}V + \bar{D}K^* = VQ \quad \text{and} \quad \bar{C}V + \bar{F}K^* = 0. \tag{11}$$

From (11), it is clear that $\left(\bar{A} + \bar{D}K^*V^+\right)\mathcal{V}(\Phi) \subset \mathcal{V}(\Phi)$ and $\left(\bar{C} + \bar{F}K^*V^+\right)\mathcal{V}(\Phi) = 0$. We can define a non-singular matrix P of dimension ρ_E as follows,

$$P = \begin{bmatrix} M_l \\ V^+ \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} M_l^+ & V \end{bmatrix}$$

where V^+ and M_l^+. Explicitly, $V^+ = \left(V^TV\right)^{-1}V^T$ and $M_l^+ = M_l^T\left(M_lM_l^T\right)^{-1}$. By defining the vectors $w_1 = M_lz_1$ and $w_2 = V^+z_1$, we have that $z_1 = M_l^+w_1 + Vw_2$. System Φ in these new coordinates can be rewritten as follows:

$$\dot{w}_1 = \bar{A}_1w_1 + \bar{D}_1(v - K^*w_2) \tag{12a}$$
$$\dot{w}_2 = \bar{A}_2w_1 + \bar{A}_3w_2 + \bar{D}_2(v - K^*w_2) \tag{12b}$$
$$\bar{y} = \bar{C}_1w_1 + \bar{F}(v - K^*w_2) \tag{12c}$$

where

$$\begin{aligned} \bar{A}_1 &= M_l\left(\bar{A} + \bar{D}K^*V^+\right)M_l^+, \quad \bar{D}_1 = M_l\bar{D} \\ \bar{A}_2 &= V^+\left(\bar{A} + \bar{D}K^*V^+\right)M_l^+, \quad \bar{D}_2 = V^+\bar{D} \\ \bar{A}_3 &= V^+\left(\bar{A} + \bar{D}K^*V^+\right)V, \quad \bar{C}_1 = \bar{C}M_l^+ \end{aligned}$$

It is known also that system Φ **is SD if, and only if**, $\operatorname{rank} \begin{bmatrix} \bar{D}_1 \\ \bar{F} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \bar{D} \\ \bar{F} \end{bmatrix}$ and $\bar{A}_3$ is a Hurwitz matrix (see, e.g. [18]).

Let us return to system Φ described by (7). Define $\xi_1 = \left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}\bar{F}^{\perp}\bar{y} = M_1z_1$, with M_1 defined as in (9). Let us derive once the vector ξ_1:

$$\dot{\xi}_1(t) = M_1\bar{A}z_1(t) + M_1\bar{D}v(t) \tag{13}$$

Let us define a new vector ξ_2 as follows

$$\xi_2 := N_2^{\perp\perp}T_1 \begin{bmatrix} \dot{\xi}_1 \\ \bar{y}(t) \end{bmatrix} \tag{14}$$

with $N_2^{\perp\perp}$ and T_1 defined by (9). Thus, taking into account (7), (13), and (9), we have that

$$\frac{d}{dt}J_2\left[\begin{array}{c}\bar{y}\\ \int_{t_0}^{t}\bar{y}(\tau)\,d\tau\end{array}\right]=\xi_2=M_2z_1(t),\quad t>t_0\geq 0 \quad (15)$$

where

$$J_2=N_2^{\perp\perp}T_1\left[\begin{array}{cc}J_1 & 0\\ 0 & I_{\bar{p}}\end{array}\right],\qquad J_1=\left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}\bar{F}^{\perp}$$

In the first identity of (15), we take outside the differential operator from (14) and use the definition of ξ_1. From (7) and the second identity of (15), we obtain that the derivative of ξ_2 is equal to

$$\dot{\xi}_2=M_2\bar{A}z_1(t)+M_2\bar{D}v(t)$$

Now, define ξ_3 in the following way,

$$\xi_3:=N_3^{\perp\perp}T_2\left[\begin{array}{c}\dot{\xi}_2\\ \bar{y}(t)\end{array}\right] \quad (16)$$

Thus, by the same reasoning used to obtain (15), we can obtain the following identities

$$\frac{d^2}{dt^2}J_3\left[\begin{array}{c}\bar{y}(t)\\ \int_{t_0}^{t}\bar{y}(\tau)\,d\tau\\ \int_{t_0}^{t}\int_{t_0}^{\tau_2}\bar{y}(\tau_1)\,d\tau_1d\tau_2\end{array}\right]=\xi_3=M_3z_1,\qquad t_0\geq 0$$

where

$$J_3=N_3^{\perp\perp}T_2\left[\begin{array}{cc}J_2 & 0\\ 0 & I_{\bar{p}}\end{array}\right]$$

Thus we can follow the same procedure in an iterative manner, to arrive to the following set of equations, for $k\geq 1$,

$$\frac{d^k}{dt^k}J_{k+1}\left[\begin{array}{c}\bar{y}(t)\\ \vdots\\ \int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}\bar{y}(\tau_1)\,d\tau_1\cdots d\tau_k\end{array}\right]=M_{k+1}z_1 \quad (17)$$

where M_{k+1} defined by (9), and J_{k+1} defined by the following recursive algorithm, for $k\geq 1$,

$$J_1=\left(\bar{F}^{\perp}\bar{C}\right)^{\perp\perp}\bar{F}^{\perp},\qquad J_{k+1}=N_{k+1}^{\perp\perp}T_k\left[\begin{array}{cc}J_k & 0\\ 0 & I_{\bar{p}}\end{array}\right]$$

Thus M_kz_1 is expressed by a high order derivative of a function of $y(t)$. As we will see below, it is also possible to express x as a high order derivative of a function depending on y. In such a way a real-time differentiator could be used, two of them frequently used due to their finite time convergence can be found in [19], [20].

V. State Reconstruction of Σ

In order to match system Σ with system Φ, from now on, we define $\bar{y}=\left[\begin{array}{c}0_{n-\rho_E}\\ y\end{array}\right]\in\mathbb{R}^{\bar{p}}$ ($\bar{p}:=n-\rho_E+p$), and $v(t)=\left[\begin{array}{c}z_2(t)\\ \mu(t)\end{array}\right]\in\mathbb{R}^{q}$ ($q=n-\rho_E+m$), then in view of (6), equations (5) and (7) are identical. Therefore, the reconstruction of $x(t)$ and $\mu(t)$ of Σ is equivalent to the reconstruction of $z_1(t)$ and $z_2(t)$ of Φ. Below, we consider two cases: when Σ is SO and when it is SD, but not SO. Of course, since Φ is a standard linear system, there might be other methods, besides the one proposed below, that might be used to carry out the algebraic reconstruction of the state.

A. Finite time reconstruction

Let us consider that system Σ is SO. Then, the reconstruction of entire state vector $x(t)$ in a finite time: by means of an algebraic formula. Let us proceed in the following way. Since Φ is SO, $\operatorname{rank}M_l=\rho_E$ (section IV). In that case, from (17), we have that

$$\frac{d^{l-1}}{dt^{l-1}}M_l^{-1}J_l\left[\begin{array}{c}\bar{y}(t)\\ \vdots\\ \int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}\bar{y}(\tau_1)\,d\tau_1\cdots d\tau_{l-1}\end{array}\right]=z_1 \quad (18)$$

where $M_l\in\mathbb{R}^{\rho_E\times\rho_E}$ and $J_l\in\mathbb{R}^{\rho_E\times\bar{p}l}$. Let $\bar{m}$ be equal to $\operatorname{rank}\left[\begin{array}{c}D\\ F\end{array}\right]$. Let $U\in\mathbb{R}^{q\times\bar{m}}$ be a matrix so that $\operatorname{rank}\left[\begin{array}{c}D\\ F\end{array}\right]U=\bar{m}$. Since (8) must be satisfied according to Theorem 1, we have that

$$z_2(t)=\left[\begin{array}{cc}I_{n-\rho_E} & 0_{\bar{q}}\end{array}\right]\left(\left[\begin{array}{c}\bar{D}\\ \bar{F}\end{array}\right]\left[\begin{array}{cc}I & 0\\ 0 & U\end{array}\right]\right)^{+}\times \\ \times\left(\left[\begin{array}{c}\dot{z}_1(t)\\ \bar{y}\end{array}\right]-\left[\begin{array}{c}\bar{A}\\ \bar{C}\end{array}\right]z_1(t)\right) \quad (19)$$

where $\bar{q}:=n-\rho_E+\bar{m}$. Now, we are ready to give an algebraic formula to reconstruct x in finite time.

Theorem 2: If system Σ is SO, state x can be expressed algebraically by the following formula:

$$x(t)=\frac{d^l}{dt^l}\left[\begin{array}{cc}S & 0_{n\times\bar{m}}\end{array}\right]\left[\begin{array}{c}H_1\\ H_2\end{array}\right]\times \\ \times\left[\begin{array}{c}\bar{y}(t)\\ \vdots\\ \int_{t_0}^{t}\cdots\int_{t_0}^{\tau_1}\bar{y}(\tau_1)\,d\tau_1\cdots d\tau_l\end{array}\right] \quad (20)$$

where $H_1\in\mathbb{R}^{\rho_E\times\bar{p}(l+1)}$ and $H_2\in\mathbb{R}^{\bar{q}\times\bar{p}(l+1)}$ are matrices defined as follows:

$$H_1:=\left[\begin{array}{cc}0_{\rho_E\times\bar{p}} & M_l^{-1}J_l\end{array}\right],\quad H_2:=\left[\begin{array}{c}\bar{D}U\\ \bar{F}U\end{array}\right]^{+}(G_1-G_2)$$

$$G_1:=\left[\begin{array}{cc}M_l^{-1}J_l & 0_{\rho_E\times\bar{p}}\\ 0_{\bar{p}\times\bar{p}l} & I_{\bar{p}\times\bar{p}}\end{array}\right],\quad G_2:=\left[\begin{array}{cc}0_{\rho_E\times\bar{p}} & \bar{A}M_l^{-1}J_l\\ 0_{\bar{p}\times\bar{p}} & \bar{C}M_l^{-1}J_l\end{array}\right]$$

$$G_1,G_2\in\mathbb{R}^{\rho_E+\bar{p}\times\bar{p}(l+1)}$$

Proof: Let us define the extended vector $Y_k\in\mathbb{R}^{\bar{p}(k+1)}$ as follows

$$Y_k=\left[\begin{array}{c}\bar{y}(t)\\ \int_{t_0}^{t}\bar{y}(\tau_1)\,d\tau_1\\ \vdots\\ \int_{t_0}^{t}\cdots\int_{t_0}^{\tau_1}\bar{y}(\tau_1)\,d\tau_1\cdots d\tau_k\end{array}\right],\ k=1,2,\ldots \quad (21)$$

Then for a matrix X of suitable dimension the following identity holds:

$$\frac{d^k}{dt^k}XY_k=\frac{d^{k+1}}{dt^{k+1}}\left[\begin{array}{cc}0 & X\end{array}\right]Y_{k+1}$$

Thus, since $x = Sz$, by manipulating (19) and tanking into account (18), (20) is obtained. ■

B. Asymptotic Reconstruction

Now, let us assume that Σ is SD, but not SO. Once that we now how to reconstruct the state x for the case when Σ is SO, the reconstruction of x for the case considered in this section can be done following a quite simple procedure.

In this case, as Σ is not SO, by differentiation, we are able to reconstruct $w_1 = M_l z_1$ only, where $\operatorname{rank} M_l < \rho_E$. Since $\begin{bmatrix} M_l \bar{D} \\ \bar{F} \end{bmatrix} U$ has full column rank (see section IV), then, from (17) and from (12a) and (12c), we have the following expression for w_1 and w_2,

$$w_1(t) = \frac{d^{l-1}}{dt^{l-1}} J_l Y_{l-1}(t) \tag{22}$$

$$z_2(t) = \tfrac{d^l}{dt^l} \begin{bmatrix} I_{n-\rho_E} & 0_{n-\rho_E \times \bar{m}} \end{bmatrix} \bar{H}_2 Y_l(t) + \\ + \begin{bmatrix} I_{n-\rho_E} & 0_{n-\rho_E \times m} \end{bmatrix} K^* w_2 \tag{23}$$

where

$$\bar{H}_2 := \begin{bmatrix} \bar{D}_1 U \\ \bar{F} U \end{bmatrix}^+ \left(\bar{G}_1 - \bar{G}_2\right), \quad \bar{H}_2 \in \mathbb{R}^{\bar{q}+\bar{p} \times \bar{p}(l+1)}$$

$$\bar{G}_1 := \begin{bmatrix} J_l & 0_{\rho_M \times \bar{p}} \\ 0_{\bar{p} \times \bar{p} l} & I_{\bar{p}} \end{bmatrix}, \quad G_2 := \begin{bmatrix} 0_{\rho_M \times \bar{p}} & \bar{A}_1 J_l \\ 0_{\bar{p} \times \bar{p}} & \bar{C}_1 J_l \end{bmatrix}$$

$$\bar{G}_1, \bar{G}_2 \in \mathbb{R}^{\rho_M + \bar{p} \times \bar{p}(l+1)}, \quad \rho_M = \operatorname{rank} M_l$$

Thus, in view that $z_1 = M_l^+ w_1 + V w_2$, we obtain the identity

$$x(t) = \frac{d^l}{dt^l} \left(\begin{bmatrix} S & 0_{n \times \bar{m}} \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \end{bmatrix} Y_l(t) \right) + \\ + \begin{bmatrix} S & 0_{n \times m} \end{bmatrix} \begin{bmatrix} V \\ K^* \end{bmatrix} w_2 \tag{24}$$

where

$$\bar{H}_1 = \begin{bmatrix} 0_{\rho_E \times \bar{p}} & M_l^+ J_l \end{bmatrix} \in \mathbb{R}^{\rho_E \times \bar{p}(l+1)}$$

Let us define $\hat{w}_2$ as follows

$$\dot{\hat{w}}_2 = \bar{A}_3 \hat{w}_2 + \bar{A}_2 \frac{d^l}{dt^l} \left(\begin{bmatrix} 0_{\rho_E - \rho_M \times \bar{p}} & J_l \end{bmatrix} + \bar{D}_2 U \bar{H}_2 \right) Y_l \tag{25}$$

Eq. (25) follows from (23). Then, taking into account (12b), (22), and (23), we have that

$$\dot{w}_2 - \dot{\hat{w}}_2 = \bar{A}_3 \left(w_2 - \hat{w}_2\right)$$

Therefore, by the SD assumption, $\bar{A}_3$ is Hurwitz; hence $\hat{w}_2$ converges exponentially to w_2. Therefore, we have that x is equal to

$$x(t) = \hat{x}(t) + \begin{bmatrix} S & 0_{n \times \bar{m}} \end{bmatrix} \begin{bmatrix} V \\ K^* \end{bmatrix} \left(w(t) - \hat{w}_2(t)\right) \tag{26}$$

where

$$\hat{x}(t) = \frac{d^l}{dt^l} \left(\begin{bmatrix} S & 0_{n \times \bar{m}} \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \end{bmatrix} Y_l(t) \right) + \\ + \begin{bmatrix} S & 0_{n \times m} \end{bmatrix} \begin{bmatrix} V \\ K^* \end{bmatrix} \hat{w}_2 \tag{27}$$

and $\|x - \hat{x}\|$ converges to zero.

For the case when the output is sufficiently smooth, the differential operator in (25) and (27) might be moved to the right side of the constant matrices, and in such a case, by (26), x can be expressed as an algebraic function of y, $\dot{y}$, so till y^l, and a variable $(w(t) - \hat{w}_2(t))$ that converges to zero.

VI. Example

Example 1. Let us consider that Σ has the following matrices values

$$E = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$$

Choosing the matrices S and T as follows,

$$S = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain the new matrices $\bar{A}$, $\bar{C}$, $\bar{D}$, and $\bar{F}$, which take the following values:

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 2 & 3 & 1 \\ -2 & -6 & -1 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

For simulation purposes, we have chosen $\mu = 2\sin(x_1 - x_2 + x_3) + \cos(t)$ and $x_1 = x_2 - 2 - \cos(3t)$ Thus, by defining $\bar{y}^T = \begin{bmatrix} 0 & 0 & y^T \end{bmatrix}$ and following (20), we have that x can be reconstructed using the following formulas,

$$\begin{aligned} x_1 &= -\frac{1}{3} y_1 + \frac{7}{12} y_2 - \frac{1}{2} \dot{y}_1 + \frac{1}{6} \dot{y}_2 \\ x_2 &= \frac{2}{3} y_1 - \frac{1}{6} y_2 + \frac{1}{6} \dot{y}_2 \\ x_3 &= -\frac{1}{4} y_2 + \frac{1}{2} \dot{y}_1 \\ x_4 &= \frac{1}{3} y_1 - \frac{1}{6} y_2 + \frac{1}{6} \dot{y}_2 \end{aligned}$$

To obtain estimate the state x, we use two different differentiators, an algebraic numerical differentiator (ALND) [20] and a high order sliding mode differentiator (HOSMD) [19]. The original and estimated states are depicted in figures 1, using the ALND, and 2, using the HOSMD.

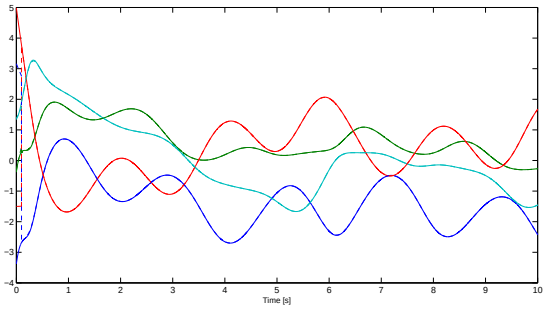

Fig. 1. States (solid line) and their (dashed line) estimation with an ALND.

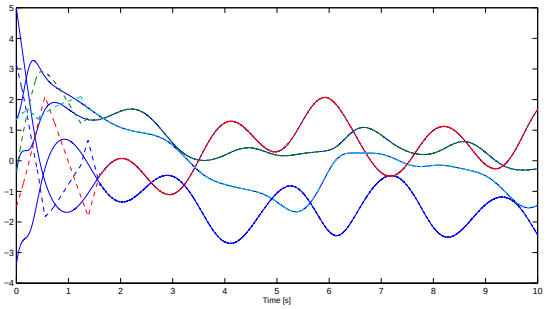

Fig. 2. States (solid line) and their (dashed line) estimation with a HOSMD.

CONCLUSIONS

We have given, under suitable assumptions, necessary and sufficient conditions to estimate the slow (non-impulsive) trajectories of the state vector. We have given explicit formulas to reconstruct in finite time and asymptotically the states. When an estimation of x is needed in practice, it might be better not to use an "excessive" number of derivatives. That is, if an asymptotic estimation is enough, we need to differentiate only the needed times allowing after to design a Luenberger-like observer. In that case, a simple but cumbersome modified procedure might be followed in order to reduce the number of derivatives required to estimate the state. (see, e.g., [2] and [21]).

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