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To cite this version:

Falk Hante, Mario Sigalotti, Marius Tucsnak. On conditions for asymptotic stability of dissipative infinite-dimensional systems with intermittent damping. Journal of Differential Equations, 2012, 252 (10), pp.5569-5593. 10.1016/j.jde.2012.01.037 . inria-00616474v2

HAL Id: inria-00616474

https://inria.hal.science/inria-00616474v2

Submitted on 10 Feb 2012

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ON CONDITIONS FOR ASYMPTOTIC STABILITY OF DISSIPATIVE INFINITE-DIMENSIONAL SYSTEMS WITH INTERMITTENT DAMPING

FALK M. HANTE, MARIO SIGALOTTI AND MARIUS TUCSNAK

Abstract. We study the asymptotic stability of a dissipative evolution in a Hilbert space subject to intermittent damping. We observe that, even if the intermittence satisfies a persistent excitation condition, if the Hilbert space is infinite-dimensional then the system needs not being asymptotically stable (not even in the weak sense). Exponential stability is recovered under a generalized observability inequality, allowing for time-domains that are not intervals. Weak asymptotic stability is obtained under a similarly generalized unique continuation principle. Finally, strong asymptotic stability is proved for intermittences that do not necessarily satisfy some persistent excitation condition, evaluating their total contribution to the decay of the trajectories of the damped system. Our results are discussed using the example of the wave equation, Schrödinger's equation and, for strong stability, also the special case of finite-dimensional systems.

Keywords: Intermittent damping; asymptotic behavior; persistent excitation; maximal dissipative operator.

1. Introduction

Consider a system of the form $\dot{z} = Az + Bu$ with z in some (finite- or infinite-dimensional) Hilbert space H, B bounded, and assume that there exists a stabilizing feedback law $u = u^* = Kz$. Now consider the system

$$\dot{z} = Az + \alpha(t)Bu, \tag{1}$$

where the *signal* α takes values in $[0, 1]$ and $\alpha(t) = 0$ for certain times t (i.e., the control may be switched off over possibly non-negligible subsets of time). Under which conditions imposed on α is the closed-loop system (1) with the same control u^* asymptotically stable? It must be stressed that a complete knowledge of α (and, in particular, the precise information on the set of times where it vanishes) would be a too restrictive condition to impose on α. We rather look for conditions valid for a whole class $\mathcal{G}$ of functions α and, therefore, we expect the closed-loop systems (1) with u^* to be asymptotically stable for every $\alpha \in \mathcal{G}$ (and, possibly, uniformly with respect to all such α). If α takes the values 0 and 1 only, then the system (1) actually switches between the uncontrolled system $\dot{z} = Az$ and the controlled one $\dot{z} = Az + Bu$.

If the uncontrolled dynamics are unstable then we should impose on α conditions guaranteeing a *sufficient amount of action* on the system. Actually, even if they

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are asymptotically stable, the stability of the overall system is not guaranteed in general (see [12]).

The question issued above may be motivated by some failure in the transmission from the controller to the plant, leading to instants of time at which the control is switched off, or to some time-varying phenomenon affecting the efficiency of the control action. It is also related to problems stemming from identification and adaptive control (see, e.g., [2]). In such type of problems, one is lead to consider the stability of linear systems of the kind $\dot{z} = -P(t)z$, $z \in \mathbb{R}^N$, where the matrix $P(\cdot)$ is symmetric non-negative definite. Under which conditions on P is the non-autonomous system stable? An answer for this particular case can be found in the seminal paper [21] which asserts that, if $P \geq 0$ is bounded and has bounded derivative, it is *necessary and sufficient*, for the global *exponential stability* of $\dot{z} = -P(t)z$, that P is also *persistently exciting*, i.e., that there exist $\mu, T > 0$ such that

$$\int_t^{t+T} \xi^T P(s)\xi \, ds \geq \mu,$$

for all unitary vectors $\xi \in \mathbb{R}^N$ and all $t \geq 0$.

The notion of persistent excitation, therefore, appears naturally as a reasonable additional assumption on α while studying the stabilization of (1). The papers [7, 8], whose results are detailed below, study the case of finite-dimensional systems of the form (1) under the assumption that there exist two positive constants μ, T such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha(s) \, ds \geq \mu. \tag{2}$$

Given two positive real numbers $\mu \leq T$, we say that α is a T-μ PE-signal (standing for *persistently exciting signal*) if it satisfies (2). Note that we do not consider here any extra assumption on the regularity of the PE-signal α (e.g., having a bounded derivative or being piecewise constant).

In [7] it is proved that if A is neutrally stable (and (A, B) is stabilizable) then $u^* = -B^T x$ stabilizes (1) exponentially, uniformly with respect to the class of T-μ PE-signals (see also [2]). The results in [7] cover also the first nontrivial case where A is not stable, namely the double integrator $\dot{z} = J_2 z + \alpha b_0 u$, where J_2 denotes the 2×2 Jordan block corresponding to the eigenvalue zero, the control is scalar and $b_0 = (0, 1)^T$. It is shown that, for every pair (T, μ), there exists a feedback $u^* = Kx$ such that the corresponding closed-loop system is exponentially stable, uniformly with respect to the class of T-μ PE-signals.

In [8] this last result is extended by proving that for the single-input case

$$\dot{z} = Az + \alpha(t)bu, \qquad u \in \mathbb{R}, \qquad z \in \mathbb{R}^N,$$

there exists a stabilizer uniform feedback $u^* = Kx$ for the class of T-μ PE-signals whenever (A, b) is controllable and the eigenvalues of A have non-positive real part. It is shown, moreover, that there exist controllable pairs (A, b) for which no such stabilizing feedback exists.

The scope of the present paper is to extend the analysis described above to infinite-dimensional systems. We focus on the case where A generates a strongly continuous contraction semigroup and $K = -B^*$, where B^* denotes the adjoint of B. This situation corresponds to the neutrally stable case studied, in the finite-dimensional setting, in [7]. Recall that the linear feedback control term $Bu = -BB^*z$ is a common choice to stabilize a dissipative linear system (see [29] and also [13]).

The motivating example, illustrating the new phenomena associated with the new setting, is the one of a string, fixed at both ends, and damped—when $\alpha(t) = 1$—on a proper subdomain. It is not hard to construct (see Example 2.1 for details) an example of periodic traveling wave on which the damping induced by a certain periodic nonzero signal α (hence, satisfying a persistent excitation condition) is ineffective. Therefore, the counterpart of the finite-dimensional stabilizability result does not hold and additional assumptions have to be made in order to guarantee the stability of the closed-loop system.

The first type of results in this direction (Section 3) concerns exponential stability. We prove that, if there exist $\vartheta, c > 0$ such that

$$\int_0^{\vartheta} \alpha(t)\|B^* e^{tA} z_0\|_H^2 \, dt \geq c\|z_0\|_H^2, \quad \text{for all } T\text{-}\mu \text{ PE-signal } \alpha(\cdot), \tag{3}$$

then there exist $M \geq 1$ and $\gamma > 0$ such that the solution $z(t)$ of

$$\dot{z} = Az - \alpha(t)BB^* z, \tag{4}$$

satisfies

$$\|z(t)\|_H \leq M e^{-\gamma t}\|z_0\|_H$$

uniformly with respect to z_0 and α. (See Theorem 3.2.) The counterpart of (3) in the unswitched case (i.e., when $\alpha \equiv 1$) is an observability inequality for the pair (A, B^*). Condition (3) can actually be seen as a generalized observability inequality. The proof of Theorem 3.2 is based on deducing from (3) a uniform decay for the solutions of (4) of the squared norm, chosen as Lyapunov function, on time-intervals of length T. The conclusion follows from standard considerations on the scalar-valued Lyapunov function (see, for instance, [1]). As an application of the general stability result we consider the example of the wave equation on a N-dimensional domain, damped everywhere. It should be stressed that generalized observability inequalities of the type discussed here have already been considered in the literature for the heat equation with boundary or locally distributed control ([9, 20, 24, 34]).

The second type of results presented in this paper (Section 4) deals with weak stability. We prove that, if there exists $\vartheta > 0$ such that

$$\int_0^{\vartheta} \alpha(s)\|B^* e^{sA} z_0\|_H^2 \, ds \neq 0 \qquad \text{for all } z_0 \neq 0 \text{ and all } T\text{-}\mu \text{ signal } \alpha,$$

then the solution $t \mapsto z(t)$ of system (4) converges weakly to 0 in H as $t \to \infty$ for any initial data $z_0 \in H$ and any T-μ PE-signal α. (See Theorem 4.2.) The counterpart of such condition in the unswitched case is the unique continuation property, ensuring approximate controllability (see, e.g., [33]). The proof of Theorem 4.2 is based on a compactness argument. The theorem is applied to the case of a Schrödinger equation with internal control localized on a subdomain. The generalized unique continuation property is then recovered by an analyticity argument (Privalov's theorem) and standard unique continuation (Holmgren's theorem).

Finally, a third type of results (Section 5) concerns strong (but not necessarily exponential) stability. In the spirit of [14], instead of imposing conditions on α which are satisfied on every time-window of prescribed length, we admit the "excitations" to be rarefied in time and of variable duration. Stability is guaranteed by asking that the total contribution of the excitations, suitably summed up, is "large enough". More precisely, it is proved that if there exist $\rho > 0$ and a continuous function $c\colon (0,\infty) \to (0,\infty)$ such that for all $T > 0$, (3) holds true with $\mu = \rho T$ and $c = c(T)$, and if there exists a sequence of disjoint intervals (a_n, b_n) in $[0,\infty)$ with $\int_{a_n}^{b_n} \alpha(t)\, dt \geq \rho(b_n - a_n)$ and $\sum_{n=1}^{\infty} c(b_n - a_n) = \infty$, then the solution $z(\cdot)$ of (4) satisfies $\|z(t)\|_H \to 0$ as $t \to \infty$. A function $c(\cdot)$ as above is explicitely found

in the case of the uniformly damped wave equation (it is of order T^3 for T small) and also in the finite-dimensional case, where it is of the same order as the one computed by Seidman in the unswitched case ([28]). A large literature is devoted to conditions ensuring stability of second order systems with time-varying parameters, mostly but not exclusively in the finite-dimensional setting. Let us mention, for instance, [15, 16, 25, 30] and the already cited paper [14]. Integral conditions in space, instead of in time, guaranteeing stabilizability of systems whose uncontrolled dynamics are given by a contraction semigroup have also been studied. Let us mention, for instance, [18, 22, 31] for the wave equation and [6] for the plate equation. In analogy with the function $c(\cdot)$ introduced above, in the mentioned papers the correct weight has to be used in order to sum up the contributions of the damping coefficients at different points. An interesting question, with possible applications to bang-bang control, would be to combine these two type of results, i.e., to consider controls supported in "sufficiently large" measurable subsets of the time-space domain.

2. Preliminaries

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_H$. In the following, we study systems modeled as

$$\begin{cases} \dot{z}(t) = Az(t) + \alpha(t)Bu(t) \\ u(t) = -B^* z(t) \\ z(0) = z_0 \end{cases} \tag{5}$$

with $A\colon H \supset D(A) \to H$ being a (possibly unbounded) linear operator generating a strongly continuous contraction semigroup $\{e^{tA}\}_{t\geq 0}$, $B\colon U \to H$ being a bounded linear operator on some Hilbert space U, $B^*\colon H \to U$ being its adjoint and $\alpha\colon [0,\infty) \to [0,1]$ being some signal possibly tuning the feedback control term $Bu(t) = -BB^*z(t)$. Note that under the above assumptions, the operator A is maximal dissipative (see [33, Proposition 3.1.13]).

We are interested in conditions on A, B and on a class of signals $\mathcal{G}$ ensuring the asymptotic decay of solutions $z \mapsto z(t)$ of (5) to the origin in a suitable sense as time t tends to infinity—independently of the initial data $z_0 \in H$ and of the specific $\alpha(\cdot)$ chosen in $\mathcal{G}$. The interesting case is when the uncontrolled evolution does not generate a strict contraction, i. e., when $\|e^{tA}\| = 1$ for $t \geq 0$, so that the energy of the system may stay constant in the absence of damping.

The classes of signals we mostly deal with are those defined by persistent excitation conditions. The latter are defined as follows: given two positive constants T and μ satisfying $\mu \leq T$, we say that a measurable signal $\alpha(\cdot)$ is a T-μ *PE-signal* if it satisfies

$$\int_t^{t+T} \alpha(s)\, ds \geq \mu, \text{ for all } t \in [0,\infty). \tag{6}$$

We note at this point that we will not make any further smoothness assumption on $\alpha(\cdot)$ for our stability results in this paper. Thus our analysis takes into account the modeling of abrupt actuator failures up to the extremal case when the system switches between an uncontrolled evolution when $\alpha(t) = 0$ and a fully controlled evolution when $\alpha(t) = 1$. Conditions of the type (6) however mean that to some extent the feedback control is active.

Solutions of (5) have to be interpreted in the mild sense, i.e., for any $t \geq 0$ and $z_0 \in H$, the solution $z(\cdot)$ of (5), evaluated at time t, is given by

$$z(t) = e^{tA}z_0 - \int_0^t e^{(t-s)A}\alpha(s)BB^*z(s)\, ds.$$

For any measurable signal α and for any finite time-horizon $\vartheta \geq 0$, there exists a unique mild solution $z(\cdot) \in C([0,\vartheta];H)$ (see, e. g., [4]). Occasionally, we write $z(t;z_0)$ to indicate the dependency of the mild solution on the initial data z_0.

As recalled in the introduction, it is shown in [7] that for $H = \mathbb{R}^N$ and (A,B) controllable (with A a dissipative $N \times N$-matrix) the solutions of (5) satisfy

$$\|z(t)\| \leq Me^{-\gamma t}\|z_0\|, \quad t \geq 0$$

uniformly in α satisfying (6), in the sense that the constants M and γ depend only on A, B, μ and T. Such result does not extend in full generality to infinite-dimensional spaces. We see this from the following example with (A,B) being a controllable pair, made of a skew-adjoint (and thus dissipative) operator A and a bounded operator B.

Example 2.1. (String equation) Let us consider a damped string of length normalized to one with fixed endpoints. Its dynamics can be described by

$$v_{tt}(t,x) = v_{xx}(t,x) - \alpha(t)d(x)^2 v_t(t,x), \qquad (t,x) \in (0,\infty) \times (0,1), \tag{7}$$

$$v(0,x) = y_0(x), \qquad x \in (0,1), \tag{8}$$

$$v_t(0,x) = y_1(x), \qquad x \in (0,1), \tag{9}$$

$$v(t,0) = v(t,1) = 0, \qquad t \in (0,\infty), \tag{10}$$

where $d \in L^\infty(0,1)$ and $\alpha \in L^\infty([0,\infty),[0,1])$.

We can express such dynamics as a system of type (5) with $H = U = H_0^1(0,1) \times L^2(0,1)$, $z(t) = (v(t,\cdot), v_t(t,\cdot))$, $A(z_1(t),z_2(t)) = (z_2(t), \partial_{xx}z_1(t))$, $B(z_1(t),z_2(t)) = (0, dz_2(t))$. The operator A is dissipative taking, as norm in H,

$$\|(z_1,z_2)\|^2 = \|\partial_x z_1\|^2_{L^2(0,1)} + \|z_2\|^2_{L^2(0,1)}.$$

Assume that

$$d = \chi_\omega \tag{11}$$

for some proper subinterval ω of $(0,1)$. Then there exist $T \geq \mu > 0$, a T-μ PE-signal α, and a corresponding nonzero periodic solution. This follows from the results in [19] (see also [14]) and can be illustrated by an explicit counterexample expressed in terms of d'Alembert solutions.

Let $\omega = (a,b)$ and assume, without loss of generality, that $b < 1$. Set $b' = \frac{1+b}{2}$. Take $T = 2$ and $\mu = 1 - b'$. Then

$$\alpha = \sum_{k=0}^{\infty} \chi_{[2k-\mu, 2k+\mu)}$$

is a T-μ signal and

$$v(t,x) = \sum_{k=0}^{\infty} (\chi_{[b'+2k,1+2k]}(x+t) - \chi_{[-1-2k,-b'-2k]}(x-t))$$

is a periodic, nonzero, mild solution of (7), (10) corresponding to α.

Notice, in particular, that even *weak* asymptotic stability fails to hold in this case. ◇

The scope of the reminder of the paper is to understand to which extent the finite-dimensional results obtained in [7] may be extended to the case where H is infinite-dimensional. A crucial remark in this perspective is the following energy decay estimate. Let

$$V(z) = \frac{1}{2}\|z\|_H^2 \tag{12}$$

denote the "energy" in H and observe that we have

$$V(z(t+s)) - V(z(t)) \leq -\int_t^{t+s} \alpha(s)\|B^* z(s)\|_U^2 \, ds \quad \text{for all } s \geq 0, \tag{13}$$

so that $V(\cdot)$ is non-increasing along trajectories for all signals $\alpha(\cdot)$. This can be shown by a standard approximation argument [23, Theorem 2.7] and using that A is maximal dissipative.

The estimate provided by the following lemma will be a key tool in the proof of some of the results in this paper.

Lemma 2.1. *Let $0 \leq a \leq b < \infty$. Then, for any measurable function $\alpha\colon [0,\infty) \to [0,1]$, the solution $z(\cdot)$ of system* (5) *satisfies*

$$V(z(b)) - V(z(a)) \leq -(2 + 2(b-a)^2\|B\|^4)^{-1} \int_0^{b-a} \alpha(t+a)\|B^* e^{tA} z(a)\|_U^2 \, dt.$$

Proof. Let

$$\phi_a(t) = e^{(t-a)A} z(a), \quad t \geq a,$$

and $\psi_a(\cdot)$ be the mild solution of

$$\begin{cases} \dot{\psi}_a(t) = A\psi_a(t) - \alpha(t) BB^* z(t), & t \geq a, \\ \psi_a(a) = 0. \end{cases}$$

Observe that

$$\phi_a(t) + \psi_a(t) = e^{(t-a)A} z(a) - \int_a^t e^{(t-\tau)A} \alpha(\tau) BB^* z(\tau) \, d\tau = z(t), \quad t \geq a, \tag{14}$$

and that

$$\sup_{\xi \in [a,b]} \|\psi_a(\xi)\|_H^2 \leq (b-a)\|B\|^2 \int_a^b \|\alpha(t) B^* z(t)\|_U^2 \, dt, \tag{15}$$

because, for $\xi \in [a,b]$,

$$\begin{aligned} \|\psi_a(\xi)\|_H^2 &\leq \left(\int_a^\xi \|e^{(\xi-t)A}\|_H \|B\| \|\alpha(t) B^* z(t)\|_U \, dt \right)^2 \\ &\leq \|B\|^2 \left(\int_a^\xi \|\alpha(t) B^* z(t)\|_U \, dt \right)^2 \\ &\leq (\xi - a)\|B\|^2 \int_a^\xi \|\alpha(t) B^* z(t)\|_U^2 \, dt, \end{aligned}$$

where we used that $\|e^{tA}\| \leq 1$.

From inequality (15) we get

$$\begin{aligned} \int_a^b \alpha(t)\|B^* \psi_a(t)\|_U^2 \, dt &\leq (b-a)\|B^*\|^2 \sup_{t \in [a,b]} \|\psi_a(t)\|_H^2 \\ &\leq (b-a)^2 \|B\|^2 \|B^*\|^2 \int_a^b \|\alpha(t) B^* z(t)\|_U^2 \, dt. \end{aligned} \tag{16}$$

Moreover, using (14), we obtain

$$\begin{aligned} \int_a^b \alpha(t)\|B^* \phi_a(t)\|_U^2 \, dt &= \int_a^b \alpha(t)\|B^*(z(t) - \psi_a(t))\|_U^2 \, dt \\ &\leq 2 \left(\int_a^b \alpha(t)\|B^* z(t)\|_U^2 \, dt + \int_a^b \alpha(t)\|B^* \psi_a(t)\|_U^2 \, dt \right). \end{aligned} \tag{17}$$

Plugging (16) in (17), we get

$$\int_a^b \alpha(t)\|B^*\phi_a(t)\|_U^2\,dt \le 2(1+(b-a)^2\|B\|^4)\int_a^b \|\alpha(t)B^*z(t)\|_U^2\,dt. \tag{18}$$

We get from the energy inequality (13) combined with (18) that

$$\begin{aligned} V(z(b))-V(z(a)) &\le -\int_a^b \alpha(t)\|B^*z(t)\|_U^2\,dt \\ &\le -\frac{1}{2(1+(b-a)^2\|B\|^4)}\int_a^b \alpha(t)\|B^*\phi_a(t)\|_U^2\,dt \\ &= -\frac{1}{2(1+(b-a)^2\|B\|^4)}\int_0^{b-a} \alpha(a+t)\|B^*e^{tA}z(a)\|_U^2\,dt, \end{aligned}$$

concluding the proof. □

Other useful facts which are used repeatedly below are the following remarks on the class of T-μ PE-signals. We note that if $\alpha(\cdot)$ is a T-μ PE-signal, then for every $t_0 \ge 0$, the same is true for $\alpha(t_0+\cdot)$. Moreover, the set of all T-μ PE-signals is weakly-$*$ compact, i. e., for any sequence $(\alpha_n(\cdot))_{n\in\mathbb{N}}$ in this set, there exists a subsequence $(\alpha_{n(\nu)}(\cdot))_{\nu\in\mathbb{N}}$ such that for some T-μ PE-signal $\alpha_\infty(\cdot)$

$$\int_0^\infty \alpha_\infty(s)g(s)\,ds = \lim_{\nu\to\infty}\int_0^\infty \alpha_{n(\nu)}(s)g(s)\,ds \quad \text{for all } g\in L^1([0,\infty)). \tag{19}$$

The existence of a function $\alpha_\infty \in L^\infty([0,\infty),[0,1])$ satisfying (19) follows from the weak-$*$ compactness of $L^\infty([0,\infty),[0,1])$ and one recovers (6) for α_∞ by choosing as g in (19) the indicator function of the interval $[t,t+T]$.

3. Exponential stability under persistent excitation

We next show that, under the following condition, asymptotic exponential stability holds.

Hypothesis 3.1. *There exist two constants $c,\vartheta>0$ such that*

$$\int_0^\vartheta \alpha(t)\|B^*e^{tA}z_0\|_U^2\,dt \ge c\|z_0\|_H^2, \quad \textit{for all } z_0\in H \textit{ and all } T\text{-}\mu \textit{ PE-signals } \alpha(\cdot). \tag{20}$$

Theorem 3.2. *Under Hypothesis 3.1, there exist two constants $M\ge 1$ and $\gamma>0$ such that the mild solution $z(\cdot)$ of system* (5) *satisfies*

$$\|z(t)\|_H \le Me^{-\gamma t}\|z_0\|_H, \quad t\ge 0, \tag{21}$$

for any initial data $z_0\in H$ and any T-μ PE-signal $\alpha(\cdot)$.

Proof. Fix some T-μ PE-signal $\alpha(\cdot)$ and some $s\ge 0$, and define V by (12). Lemma 2.1 with $a=s$ and $b=s+\vartheta$, where ϑ is as in Hypothesis 3.1, then yields

$$V(z(s+\vartheta))-V(z(s)) \le -\frac{1}{2(1+\vartheta^2\|B\|^4)}\int_0^\vartheta \alpha(t+s)\|B^*e^{tA}z(s)\|_U^2\,dt.$$

Again using that $\alpha(\cdot+s)$ is a T-μ PE-signal, Hypothesis 3.1 then implies

$$V(z(s+\vartheta))-V(z(s)) \le -\frac{c}{(1+\vartheta^2\|B\|^4)}V(z(s)).$$

The desired estimate (21) then follows from standard arguments. □

Example 3.1 below illustrates an application of Theorem 3.2. We consider again the model of a damped string introduced in Example 2.1, replacing the localized damping given in (11) (which, as we proved, gives rise to non-stabilizability) by a damping acting almost everywhere. The argument is presented for the general case of the damped wave equation (the string corresponding to the case $N = 1$).

Example 3.1. (Wave equation) Let $N \geq 1$ and consider a N-dimensional version of system (7)–(10) introduced in Example 2.1:

$$v_{tt}(t,x) = \Delta v(t,x) - \alpha(t)d(x)^2 v_t(t,x), \qquad (t,x) \in (0,\infty)\times\Omega, \tag{22}$$

$$v(0,x) = y_0(x), \qquad x \in \Omega, \tag{23}$$

$$v_t(0,x) = y_1(x), \qquad x \in \Omega, \tag{24}$$

$$v(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\partial\Omega, \tag{25}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $d \in L^\infty(\Omega)$ satisfies

$$|d(x)| \geq d_0 > 0 \quad \text{for almost all } x \in \Omega.$$

We claim that in this case Hypothesis 3.1 is satisfied with $\vartheta = T$, taking $H = H_0^1(\Omega) \times L^2(\Omega)$ with norm

$$\|(z_1, z_2)\|^2 = \|\nabla z_1\|^2_{L^2(\Omega)} + \|z_2\|^2_{L^2(\Omega)}.$$

Denote by $(\phi_n)_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions of the Laplace–Dirichlet operator on Ω. For each $n \in \mathbb{N}$, let $\lambda_n > 0$ be the eigenvalue corresponding to ϕ_n. Recall that λ_n goes to infinity as $n \to \infty$.

Let $t \mapsto z(t) = (v(t,\cdot), v_t(t,\cdot))$ be a solution of (22)–(25) with initial condition $(y_0(\cdot), y_1(\cdot)) = (\sum_{n\in\mathbb{N}} a_n\phi_n(\cdot), \sum_{n=1}^\infty \sqrt{\lambda_n} b_n \phi_n(\cdot))$, where $(\sqrt{\lambda_n}a_n)_{n\in\mathbb{N}}$ and $(\sqrt{\lambda_n}b_n)_{n\in\mathbb{N}}$ belong to ℓ^2. By definition, $\|z(0)\|_H^2 = \sum_{n\in\mathbb{N}} \lambda_n(a_n^2 + b_n^2)$ and

$$v(t,x) = \sum_{n\in\mathbb{N}} a_n\phi_n(x)\cos(\sqrt{\lambda_n}t) + \sum_{n=1}^\infty b_n\phi_n(x)\sin(\sqrt{\lambda_n}t).$$

Then

$$\begin{aligned}
\int_0^T \alpha(t)\|B^*z(t)\|_U^2\,dt &\geq d_0^2 \int_0^T \int_\Omega \alpha(t)|v_t(x,t)|^2\,dx\,dt \\
&= d_0^2 \int_0^T \int_\Omega \alpha(t)\left(\sum_{n\in\mathbb{N}} \lambda_n(-a_n\sin(\sqrt{\lambda_n}t) + b_n\cos(\sqrt{\lambda_n}t))\phi_n(x)\right)^2 dx\,dt \\
&= d_0^2 \sum_{n\in\mathbb{N}} \lambda_n \int_0^T \alpha(t)(-a_n\sin(\sqrt{\lambda_n}t) + b_n\cos(\sqrt{\lambda_n}t))^2 dt,
\end{aligned}$$

where we used that, for all $n, m \in \mathbb{N}$,

$$\int_0^1 \phi_n(x)\phi_m(x)dx = \delta_{nm}.$$

We are left to prove that there exist $c_0 > 0$ independent of $n \in \mathbb{N}$, $a_n, b_n \in \mathbb{R}$ and of the T-μ signal α such that

$$\int_0^T \alpha(t)(-a_n\sin(\sqrt{\lambda_n}t) + b_n\cos(\sqrt{\lambda_n}t))^2 dt \geq c_0(a_n^2 + b_n^2). \tag{26}$$

For every $\epsilon \in (0,1)$, let

$$A_n^\epsilon = \{t \in [0,T] \mid |-a_n\sin(\sqrt{\lambda_n}t) + b_n\cos(\sqrt{\lambda_n}t)| > \epsilon\sqrt{a_n^2 + b_n^2}\}. \tag{27}$$

Notice that $-a_n \sin(\sqrt{\lambda_n} t) + b_n \cos(\sqrt{\lambda_n} t) = \sqrt{a_n^2 + b_n^2} \sin(\sqrt{\lambda_n} t + \theta_n)$ for some $\theta_n \in \mathbb{R}$. Hence,

$$| - a_n \sin(\sqrt{\lambda_n} t) + b_n \cos(\sqrt{\lambda_n} t)| \leq \lambda_n \sqrt{a_n^2 + b_n^2} |t - t_0|$$

for every t_0 belonging to $\{t_0 \mid \sin(\lambda_n t_0 + \theta_n) = 0\} = \frac{\pi}{\lambda_n}\mathbb{Z} - \frac{\theta_n}{\lambda_n}$. In particular, $[0, T] \setminus A_n^\epsilon$ is contained in the set of points with a distance from $\frac{\pi}{\lambda_n}\mathbb{Z} - \frac{\theta_n}{\lambda_n}$ smaller than ϵ/λ_n, i.e., in the union of intervals of length $2\epsilon/\lambda_n$ centered at elements of $\frac{\pi}{\lambda_n}\mathbb{Z} - \frac{\theta_n}{\lambda_n}$. Therefore,

$$\begin{aligned} \text{meas}(A_n^\epsilon) &\geq T - 2\frac{\epsilon}{\lambda_n} \# \left([0, T] \cap \left(\frac{\pi}{\lambda_n}\mathbb{Z} - \frac{\theta_n}{\lambda_n} \right) \right) \geq T - 2\frac{\epsilon}{\lambda_n} \left(\frac{T\lambda_n}{\pi} + 1 \right) \\ &\geq T \left(1 - \frac{2\epsilon}{\pi} \right) - 2\frac{\epsilon}{\min_{n \in \mathbb{N}} \lambda_n}. \end{aligned} \tag{28}$$

Thus, the measure of A_n^ϵ tends to T as ϵ goes to zero, uniformly with respect to the triple (n, a_n, b_n). In particular, there exists $\bar{\epsilon} > 0$ such that for every $n \in \mathbb{N}$, $a_n, b_n \in \mathbb{R}$ and every T-μ signal α,

$$\int_{A_n^{\bar{\epsilon}}} \alpha(t) dt \geq \frac{\mu}{2}.$$

Then

$$\begin{aligned} \int_0^T \alpha(t)(-a_n \sin(\sqrt{\lambda_n} t) + b_n \cos(\sqrt{\lambda_n} t))^2 dt &\geq \bar{\epsilon}^2 \left(a_n^2 + b_n^2\right) \int_{A_n^{\bar{\epsilon}}} \alpha(t) dt \\ &\geq \frac{\mu \bar{\epsilon}^2}{2} (a_n^2 + b_n^2), \end{aligned}$$

proving (26) with $c_0 = \mu \bar{\epsilon}^2/2$. ◇

Remark 3.1. The example presented above shows that the sufficient condition for asymptotic stability of abstract second order evolution equations with *on/off damping* considered in [14] is not necessary, as detailed here below. The question of its necessity had been raised in [10, p. 2522].

Extending a result of [30] for ordinary differential equations, it was shown in [14] that existence of a sequence of open disjoint intervals I_n of length T_n such that

$$\sum_{n=1}^{\infty} m_n T_n \min\left(T_n^2, (1 + m_n M_n)^{-1}\right) = \infty \tag{29}$$

and existence of constants $M_n \geq m_n > 0$ such that

$$m_n \leq \alpha(t) \leq M_n, \quad t \in I_n, \tag{30}$$

implies asymptotic stability of systems whose prototype is (22)–(25).

Taking, for example, $I_n = (s_n, s_n + \frac{1}{n})$ with

$$s_n = \sum_{k=1}^{n-1} \frac{2}{k}$$

and $\alpha(\cdot)$ piecewise constant such that (30) holds with $m_n = M_n = 1$, the sum in (29) converges, but for $T = 2$,

$$\int_t^{t+T} \alpha(s)\, ds \geq \mu, \qquad t \geq 0,$$

for some $\mu > 0$, as it easily follows by noticing that $\lim_{t \to +\infty} \int_t^{t+T} \alpha(s)\, ds = 1/2$. ◇

Another example that one could consider is the Schrödinger equation with internal damping. Because of the infinite speed of propagation of the Schrödinger equation, it is a natural question whether, differently form the case of the wave equation, stability can be achieved by a localized damping. We are not able to give an answer to this question (detailed below), which we leave as an open problem.

Example 3.2. (Schrödinger equation) Consider

$$iy_t(t,x) + y_{xx}(t,x) + i\alpha(t)d(x)^2 y(t,x) = 0, \qquad (t,x) \in (0,\infty)\times(0,1), \quad (31)$$

$$y(t,0) = y(t,1) = 0, \qquad t \in (0,\infty), \quad (32)$$

$$y(0,x) = y_0(x), \qquad t \in (0,1), \quad (33)$$

with $d(\cdot) \in L^\infty(0,1)$ and $\alpha(\cdot)$ being a T-μ PE-signal. Assume that $d = \chi_\omega$ with $\omega = (a,b)$ a nonempty subinterval of $(0,1)$.

In order to write system (31)–(33) in the form (5), let $H = U = L^2(0,1)$, define A as $Az = iz_{xx}$, acting on $D(A) = \mathcal{H}^2(0,1) \cap \mathcal{H}^1_0(0,1)$, and let $B\colon z \mapsto \chi_\omega z$ be the multiplication operator by the function $\chi_\omega = d$. Then, for $y_0 \in H$, the mild solution $z(\cdot)$ of (5) with this choice of A and B corresponds to the weak solution $y(\cdot)$ of (31)–(33) (see [3]).

Since A is skew-adjoint, in order to apply Theorem 3.2 we should prove that Hypothesis 3.1 holds true. More explicitly, we should prove that there exist $\vartheta, c > 0$ such that, for each $z_0 \in L^2(0,1)$ and each T-μ PE-signal α,

$$\int_0^\vartheta \int_a^b \alpha(t)|(e^{tA}z_0)(x)|^2\,dx\,dt \geq c\int_0^1 |z_0(x)|^2 dx.$$

In order to fix the ideas, let us take $\vartheta = T > \mu$. The question can be rephrased by asking whether there exists $c > 0$ such that for every $\Xi \subset [0,T]$ of measure equal to μ,

$$\int_\Xi \int_a^b |\sum_{n\in\mathbb{N}} \langle \phi_n, z_0\rangle_{L^2(0,1)} \phi_n(x) e^{in^2\pi^2 t}|^2\,dx\,dt \geq c\int_0^1 |z_0(x)|^2 dx, \qquad (34)$$

with $\phi_n(x) = \sqrt{2}\sin(n\pi x)$. This problem is, up to our knowledge, open.

The question is somehow related with a discussion presented by Seidman in [27], where it is conjectured that, among all such sets Ξ, the maximal constant in (34) (uniform with respect to z_0) is obtained for intervals.

Notice that in the case $\omega = (0,1)$ inequality (34) is satisfied because the L^2 norm of $z(t)$ is constant with respect to t. Because of the full damping in space, the techniques developed by Fattorini in [9] would also apply, yielding the required generalized observability inequality. ◇

4. Weak stability under persistent excitation

Our main result is that weak asymptotic stability holds when the pair (A,B) has the following T-μ PE unique continuation property, which weakens Hypothesis 3.1.

Hypothesis 4.1. *There exists $\vartheta > 0$ such that for all T-μ PE-signals $\alpha(\cdot)$*

$$\int_0^\vartheta \alpha(t)\|B^* e^{tA} z_0\|_U^2\,dt = 0 \quad \Rightarrow \quad z_0 = 0. \qquad (35)$$

We will prove the following.

Theorem 4.2. *Under Hypothesis 4.1, the mild solution $t \mapsto z(t)$ of system (5) converges weakly to 0 in H as $t \to \infty$ for any initial data $z_0 \in H$ and any T-μ PE-signal $\alpha(\cdot)$.*

Proof. It suffices to prove that, for each $z_0 \in H$ and for each T-μ PE-signal $\alpha(\cdot)$, the *weak ω-limit set*

$$\omega(z_0, \alpha(\cdot)) = \{z \in H \mid \text{there exists a sequence } \{s_n\}_{n\in\mathbb{N}},\ s_n \to \infty, \text{ so that } z(s_n; z_0) \rightharpoonup z \text{ as } n \to \infty\}$$

is non-empty and, taking $\vartheta > 0$ as in Hypothesis 4.1,

$$z_\infty \in \omega(z_0, \alpha(\cdot)) \Rightarrow \exists\, \alpha_\infty\ T\text{-}\mu \text{ PE-signal s. t. } \int_0^\vartheta \alpha_\infty(t)\|B^* e^{tA} z_\infty\|_U^2\, dt = 0. \quad (36)$$

The assertion of the theorem then follows from (35).

Let $z_0 \in H$ and a T-μ-persistent excitation signal $\alpha(\cdot)$ be given. Let $z(\cdot; z_0)$ be the unique mild solution of the system (5) and define V as in (12).

From the energy inequality (13), one obtains that the weak ω-limit set $\omega(z_0, \alpha(\cdot))$ is non-empty. So let $z_\infty \in \omega(z_0, \alpha(\cdot))$ be an element of the weak ω-limit set and let $\{s_n\}_{n\in\mathbb{N}}$, $s_n \to \infty$ be a sequence of times such that $z(s_n; z_0) \rightharpoonup z_\infty$ as $n \to \infty$.

We consider the translations

$$z_n(s) = z(s + s_n; z_0) \quad \alpha_n(s) = \alpha(s + s_n)$$

and we note that $z_n(\cdot)$ is the mild solution of system (5) for the T-μ PE-signal $\alpha_n(\cdot)$ and initial condition $z_n(0) = z(s_n; z_0)$, i. e., $z_n(\cdot)$ satisfies

$$z_n(s) = e^{sA} z(s_n; z_0) - \int_0^s e^{(s-t)A} \alpha_n(t) BB^* z_n(t)\, dt. \quad (37)$$

Therefore, we have the energy estimates

$$V(z_n(s)) - V(z(s_n; z_0)) \le -\int_0^s \alpha_n(t)\|B^* z_n(t)\|_U^2\, dt \quad \text{for all } s \ge 0. \quad (38)$$

From (38) and (13) we get

$$\|z_n(s)\|_H \le \|z(s_n; z_0)\|_H \le \|z_0\|_H, \quad s \ge 0, \quad (39)$$

and thus, for any $\vartheta \ge 0$, we have that $\{z_n(\cdot)\}_{n\in\mathbb{N}}$ is a bounded subset of $C([0, \vartheta]; H)$. Choose $\vartheta > 0$ as in Hypothesis 4.1.

We claim that

$$z_n(s) \rightharpoonup z_\infty(s) \text{ as } n \to \infty, \quad \text{for all } s \in [0, \vartheta], \quad (40)$$

where $z_\infty(\cdot)$ is the mild solution of the undamped equation

$$\begin{cases} \dot{z}(s) = Az(s) \\ z(0) = z_\infty. \end{cases} \quad (41)$$

Indeed, much as in [5], we can show that $\{z_n\}_{n\in\mathbb{N}}$ is equicontinuous in $C([0, \vartheta]; H_w)$, where H_w is H endowed with the weak topology. To verify this, let $s_r \searrow s$ in $[0, \vartheta]$ and select some $\psi \in H$. From (37), we have that

$$\begin{aligned} |\langle z_n(s_r) - z_n(s), \psi\rangle| \le\ & \left|\left\langle [e^{s_r A} - e^{sA}] z(s_n; z_0), \psi \right\rangle\right| \\ & + \int_0^s \left|\left\langle [e^{(s_r - t)A} - e^{(s-t)A}] \alpha_n(t) BB^* z_n(t), \psi \right\rangle\right| dt \\ & + \int_s^{s_r} \left|\langle e^{(s_r - t)A} \alpha_n(t) BB^* z_n(t), \psi\rangle\right| dt. \end{aligned} \quad (42)$$

Moreover, using (39), we have that

$$\|\alpha_n(t) BB^* z_n(t)\|_H \le \|\alpha_n(t)\|_{\mathbb{R}} \|BB^*\|_{\mathcal{L}(H)} \|z_n(t)\|_H \le \text{const.} \|z_0\|_H$$

and, as proved in [5, Theorem 2.3],

$$a_r = \sup_{\|\phi\|_H \le 1,\ 0 \le t \le s} |\langle [e^{(s-t)A} - e^{(s_r - t)A}]\phi, \psi\rangle| \to 0 \quad \text{as } r \to \infty.$$

Thus, from (42) we get

$$|\langle z_n(s_r) - z_n(s), \psi\rangle| \leq \text{const.} a_r \|z_0\|_H + \text{const.} |s_r - s|,$$

and hence

$$|\langle z_n(s_r) - z_n(s), \psi\rangle| \to 0 \quad \text{uniformly as } r \to \infty. \tag{43}$$

Similarly, one shows that (43) holds for $s_r \nearrow s$ in $[0, \vartheta]$. Thus, $\{z_n\}_{n\in\mathbb{N}}$ is equicontinuous in $C([0, \vartheta]; H_w)$. Again using that $\{z_n(s) \mid n \in \mathbb{N},\ s \in [0, \vartheta]\}$ is bounded in H by (39), we may view $\{z_n\}_{n\in\mathbb{N}}$ as an equibounded set of curves in H endowed with the metrized weak topology. Hence we can apply the Arzela–Ascoli theorem for metric spaces to conclude that there exists $z_\infty(\cdot) \in C([0, \vartheta]; H_w)$ and a subsequence that we re-label by $n \in \mathbb{N}$ so that $z_n(s) \rightharpoonup z_\infty(s)$ uniformly on $[0, \vartheta]$ as $\nu \to \infty$. Moreover, for any $\psi \in H$ we have from (37) by adding and subtracting $\langle e^{(s-t)A}\alpha_n(t)BB^* z_\infty(t), \psi\rangle$ under the integral that

$$\begin{aligned} \langle z_n(s), \psi\rangle = \langle e^{sA} z(s_n; z_0), \psi\rangle - \int_0^s \alpha_n(t)\langle e^{(s-t)A} BB^* z_\infty(t), \psi\rangle\, dt \\ - \int_0^s \alpha_n(t)\langle e^{(s-t)A} BB^*[z_n(t) - z_\infty(t)], \psi\rangle\, dt. \end{aligned} \tag{44}$$

Using that $\alpha_n(t)$ is a bounded sequence for $t \in [0, s]$ and that

$$\langle e^{(s-t)A} BB^*[z_n(t) - z_\infty(t)], \psi\rangle \to 0 \quad \text{as } \nu \to \infty$$

for all $t \in [0, s]$, we can conclude from the dominated convergence theorem that

$$\int_0^s \alpha_n(t)\langle e^{(s-t)A} BB^*[z_n(t) - z_\infty(t)], \psi\rangle\, dt \to 0, \quad \text{as } n \to \infty.$$

Hence, by sequential weak*-compactness of $L^\infty([0, \infty); [0, 1])$, we can extract another subsequence that we again re-label by $n \in \mathbb{N}$ and pass to the limit in (44), obtaining that, for every $s \geq 0$,

$$\langle z_\infty(s), \psi\rangle = \langle e^{sA} z_\infty, \psi\rangle - \int_0^s \langle e^{(s-t)A}\alpha_\infty(t) BB^* z_\infty(t), \psi\rangle\, dt, \tag{45}$$

where $\alpha_\infty(\cdot)$ again is a T-μ PE-signal. Since (45) holds for all $\psi \in H$, we have

$$z_\infty(s) = e^{sA} z_\infty - \int_0^s e^{(s-t)A}\alpha_\infty(t) BB^* z_\infty(t)\, dt.$$

Next we show that

$$\int_0^s e^{(s-t)A}\alpha_\infty(t) BB^* z_\infty(t)\, dt = 0. \tag{46}$$

Since $V(z_n(0))$ is bounded and monotone, it has a limit $V^* = \lim_{n\to\infty} V(z_n)$, so that

$$\int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2\, dt \leq V(z_n(0)) - V^* \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2\, dt \to 0 \quad \text{as } n \to \infty. \tag{47}$$

Morover, (40) and $\alpha_n \overset{*}{\rightharpoonup} \alpha_\infty$ imply

$$\liminf_{n\to\infty} \int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2\, dt \geq \int_0^\vartheta \alpha_\infty(t)\|B^* z_\infty(t)\|_U^2\, dt. \tag{48}$$

To see this, observe that (40) implies

$$\|B^* z_\infty(t)\|_U \leq \liminf_{n\to\infty} \|B^* z_n(t)\|_U, \text{ for all } t \in [0, \vartheta]. \tag{49}$$

Fix any $\epsilon > 0$ and define, for all $m \in \mathbb{N}$,

$$S_m^\epsilon = \{t \in [0, \vartheta] \mid \|B^* z_n(t)\|_U^2 \geq \|B^* z_\infty(t)\|_U^2 - \epsilon \text{ for all } n \geq m\}.$$

From (49) we have

$$[0, \vartheta] = \bigcup_m S_m^\epsilon \quad (S_m^\epsilon \supseteq S_{m-1}^\epsilon),$$

hence there exists $m(\epsilon)$ such that $|S_{m(\epsilon)}^\epsilon| > T - \epsilon$. Then we have, for $n \geq m(\epsilon)$,

$$\begin{aligned} \int_0^T \alpha_n(t)\|B^* z_n(t)\|_U^2 \, dt = & \int_{S_{m(\epsilon)}^\epsilon} \alpha_n(t) \left(\|B^* z_n(t)\|_U^2 - \|B^* z_\infty(t)\|_U^2 + \epsilon\right) dt \\ & + \int_{S_{m(\epsilon)}^\epsilon} \alpha_n(t) \left(\|B^* z_\infty(t)\|_U^2 - \epsilon\right) dt \\ & + \int_{[0,\vartheta] \setminus S_{m(\epsilon)}^\epsilon} \alpha_n(t)\|B^* z_n(t)\|_U^2 \, dt. \end{aligned} \tag{50}$$

The first integral in the right-hand side of (50) is non-negative because $\|B^* z_n(t)\|_U^2 - \|B^* z_\infty(t)\|_U^2 + \epsilon \geq 0$ for all $t \in S_{m(\epsilon)}^\epsilon$ and $\alpha_n(t) \geq 0$ for all $t \in [0, \vartheta]$. The second integral is bounded from below by

$$\int_{S_{m(\epsilon)}^\epsilon} \alpha_n(t)\|B^* z_\infty(t)\|_U^2 \, dt - \epsilon\vartheta \geq \int_0^\vartheta \alpha_n(t)\|B^* z_\infty(t)\|_U^2 \, dt - \epsilon(\text{const.} + \vartheta)$$

as it follows from (39). Finally, the third integral is non-negative, again because $\alpha_n(t) \geq 0$ for all $t \in [0, \vartheta]$. Thus, for all $n \geq m(\epsilon)$,

$$\int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2 \, dt \geq \int_0^\vartheta \alpha_n(t)\|B^* z_\infty(t)\|_U^2 \, dt - \epsilon(C + \vartheta).$$

Hence, by the convergence $\alpha_n(\cdot) \overset{*}{\rightharpoonup} \alpha_\infty(\cdot)$,

$$\liminf_{n\to\infty} \int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2 \, dt \geq \int_0^\vartheta \alpha_\infty(t)\|B^* z_\infty(t)\|_U^2 \, dt - \epsilon(C + T),$$

proving (48) from the fact that ϵ is arbitrary.

From (48) and (47), we have

$$0 = \lim_{n\to\infty} \int_0^\vartheta \alpha_n(t)\|B^* z_n(t)\|_U^2 \, dt = \int_0^\vartheta \alpha_\infty(t)\|B^* z_\infty(t)\|_U^2 \, dt, \tag{51}$$

so either $\alpha_\infty(t) = 0$ or $B^* z_\infty(t) = 0$ for almost every $t \in [0, \vartheta]$. This proves (46) and hence $z_\infty(\cdot)$ solves, as claimed, the undamped equation (41).

Finally, since z_∞ solves (41), we have $z_\infty(s) = e^{sA} z_\infty$ and thus (51) implies (36), as required. □

In the example below we go back to the internally damped Schrödinger equation considered in Example 3.2, where we were not able to conclude whether such equation is strongly stable, uniformly with respect to all T-μ signals ($T > \mu > 0$ given). We prove here, in the general N-dimensional case, that weak stability holds true.

Example 4.1. (Schrödinger equation) Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 1$, and consider the internally damped Schrödinger equation

$$iy_t(t,x) + \Delta y(t,x) + i\alpha(t)d(x)^2 y(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \Omega, \tag{52}$$

$$y(t,x) = 0, \qquad t \in (0,\infty) \times \partial\Omega, \tag{53}$$

$$y(0,x) = y_0(x), \qquad t \in \Omega, \tag{54}$$

where $d(\cdot)$ belongs to $L^\infty(\Omega)$ and $\alpha(\cdot)$ is a T-μ PE-signal. Assume that there exist $d_0 > 0$ and an open nonempty $\omega \subset \Omega$ such that

$$|d(x)| \geq d_0 \text{ for a. e. } x \text{ in } \omega. \tag{55}$$

As in Example 3.2, system (52)–(54) can be written in the form (5) with $H = U = L^2(\Omega)$, $Az = i\Delta z$, $D(A) = \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$, and $B\colon z \mapsto dz$. Since A is skew-adjoint, it generates a contraction semigroup.

As to apply Theorem 4.2, it remains to show that the pair (A, B) has the T-μ PE unique continuation property stated in Hypothesis 4.1. To this end, fix some $z_0 \in L^2(\Omega)$, some T-μ PE-signal $\alpha(\cdot)$, and choose any $\vartheta > T - \mu$. Observe that, since

$$\int_0^\vartheta \alpha(t)\|B^* e^{tA} z_0\|_U^2 \, dt = \int_0^\vartheta \alpha(t)\|d e^{tA} z_0\|_H^2 \, dt,$$

then either $\alpha(t) = 0$ or $de^{tA}z_0 = 0$ for almost every $t \in [0, \vartheta]$. But (6) implies that $\alpha(\cdot) > 0$ on a set $\Xi \subset (0, \vartheta)$ with $\text{meas}(\Xi) \geq \vartheta - T + \mu > 0$ and (55) yields $d(\cdot) \neq 0$ a. e. on the open set $\omega \subset \Omega$. Hence,

$$(t, x) \mapsto (e^{tA} z_0)(x) \equiv 0 \quad \text{on } \Xi \times \omega.$$

Let us now adapt the unique continuation argument proposed in [26] in order to prove that $(t, x) \mapsto (e^{tA} z_0)(x) = 0$ on the open set $(0, \vartheta) \times \omega$. Write the spectrum of A as $(\lambda_k)_{k\in\mathbb{N}}$ (with eigenvalues repeated according to their multiplicities). Then the sequence $(i\lambda_k)_{k\in\mathbb{N}}$ is contained in $\mathbb{R}$ and is bounded from below. Denote by $(\phi_k)_{k\in\mathbb{N}}$ an orthonormal basis of H such that $A\phi_k = \lambda_k \phi_k$. Fix any $\varphi \in L^2(\omega)$ and consider the function $F\colon t \mapsto \sum_{k\in\mathbb{N}} e^{\lambda_k t} \langle \phi_k, z_0\rangle \langle \phi_k, \varphi\rangle$. Notice that $F(t) = \langle e^{tA} z_0, \varphi\rangle$ and that F can be extended from $\mathbb{R}$ to $\mathbb{C}^- = \{w \in \mathbb{C} \mid \text{Im}(w) \leq 0\}$, thanks to the lower boundedness (in $\mathbb{R}$) of $(i\lambda_k)_{k\in\mathbb{N}}$. Moreover, F is complex analytic in the interior of $\mathbb{C}^-$ and continuous up to its boundary. Since F is zero on a subset of the boundary of $\mathbb{C}^-$ of positive (one-dimensional) measure, then it follows from Privalov's uniqueness theorem (see [37, Vol. II, Theorem 1.9, p. 203]) that F vanishes identically. By the arbitrariness of $\varphi \in L^2(\omega)$ it follows, as required, that $e^{tA} z_0$ vanishes on ω for $t \in (0, \theta)$.

Applying Holmgren's uniqueness theorem (see [17, Theorem 8.6.8] and also [36]), we deduce that z_0 vanishes on Ω, proving Hypothesis 4.1. ◇

5. Strong stability

Condition (6) means that the feedback control $Bu = -BB^*z$ is, to some extent, active on *every* interval of the length T. From an application point of view it is also interesting to study the case when there are intervals of arbitrary length where no feedback control is active, in the spirit of the results in, e. g., [14, 15, 25, 30]. A natural question is then to ask which conditions imposed on A, B, and on the distribution and length of these intervals suffice to ensure stability.

Below we give an abstract result ensuring the strong asymptotic stability of the closed-loop system (5) using observability estimates for the open-loop system. Stressing the importance, in order to apply such result, of having explicit estimates for control costs (i.e., the constants c appearing in inequalities of the type (20)), we then show on several examples how this can lead to stabilizing conditions.

Definition 5.1. *We say that $\alpha(\cdot) \in L^\infty([0, T], [0, 1])$ is of class $\mathcal{K}(A, B, T, c)$ if*

$$\int_0^T \alpha(t)\|B^* e^{sA} z_0\|_U^2 \, dt \geq c\|z_0\|_H^2, \quad \textit{for all } z_0 \in H. \tag{56}$$

With this definition, we can state the following abstract result.

Theorem 5.2. *Suppose that (a_n, b_n), $n \in \mathbb{N}$, is a sequence of disjoint intervals in $[0,\infty)$, that c_n, $n \in \mathbb{N}$, is a sequence of positive real numbers and that $\alpha(\cdot) \in L^\infty([0,\infty),[0,1])$ is such that its restriction $\alpha(a_n+\cdot)|_{[0,b_n-a_n]}$ to the interval (a_n,b_n) is of class $\mathcal{K}(A,B,b_n-a_n,c_n)$ for all $n \in \mathbb{N}$. Moreover, assume that $\sup_{n\in\mathbb{N}}(b_n - a_n) < \infty$ and $\sum_{n=1}^\infty c_n = \infty$.*

Then the mild solution of (5) satisfies $\|z(t)\|_H \to 0$ as $t \to \infty$.

Proof. First of all notice that (56) implies that $c \le T\|B^*\|^2$. Hence, the sum of the c_n corresponding to intervals (a_n, b_n) contained in a given bounded interval $[\tau_0, \tau_1]$ is finite and can be approximated arbitrarily well by the sum of finitely many of such c_n. Therefore, we can extract a locally finite subsequence of intervals, still denoted by (a_n, b_n), $n \in \mathbb{N}$, such that $\sum_{n=1}^\infty c_n = \infty$ and, up to a reordering, $b_n \le a_{n+1}$ for all $n \in \mathbb{N}$.

Using the energy inequality (13) we get $V(z(a_{n+1})) \le V(z(b_n))$ while Lemma 2.1, with $a = a_n$ and $b = b_n$, implies that

$$V(z(b_n)) - V(z(a_n)) \le -\frac{1}{2(1+(b_n-a_n)^2\|B\|^4)} \int_0^{b_n-a_n} \alpha(a_n+t)\|B^* e^{tA} z(a_n)\|_U^2 \, dt.$$

Thus, since $\alpha(a_n+\cdot)|_{[0,b_n-a_n]}$ is of class $\mathcal{K}(A,B,b_n-a_n,c_n)$, we have

$$V(z(a_{n+1})) - V(z(a_n)) \le -\frac{c_n}{1+(b_n-a_n)^2\|B\|^4} V(z(a_n)). \tag{57}$$

Using the estimate (57) recursively, we obtain

$$V(z(a_{n+1})) \le \prod_{j=1}^n \left(1 - \frac{c_j}{1+(b_j-a_j)^2\|B\|^4}\right) V(z_0).$$

Since

$$\begin{aligned}
\log \prod_{j=1}^\infty \left(1 - \frac{c_j}{1+(b_j-a_j)^2\|B\|^4}\right) &= \sum_{j=1}^\infty \log\left(1 - \frac{c_j}{1+(b_j-a_j)^2\|B\|^4}\right) \\
&\le -\sum_{j=1}^\infty \frac{c_j}{1+(b_j-a_j)^2\|B\|^4} \\
&\le -\frac{1}{1+\|B\|^4 \sup_{j=1}^\infty (b_j-a_j)^2} \sum_{j=1}^\infty c_j = -\infty,
\end{aligned}$$

then $V(z(a_{n+1}))$ tends to zero as n goes to infinity. □

As a direct application of this abstract result, we consider again the Schrödinger equation in one space dimension.

Example 5.1. (Schrödinger equation) Consider

$$iy_t(t,x) + y_{xx}(t,x) + i\alpha(t)d(x)^2 y(t,x) = 0, \qquad (t,x) \in (0,\infty)\times(0,1), \tag{58}$$

$$y(t,0) = y(t,1) = 0, \qquad t \in (0,\infty), \tag{59}$$

$$y(0,x) = y_0(x), \qquad t \in (0,1), \tag{60}$$

with $d(\cdot) \in L^\infty(0,1)$ and $\alpha(\cdot) \in L^\infty([0,\infty),[0,1])$. Assume that $d = \chi_\omega$ with ω a nonempty subinterval of $(0,1)$ and assume that (a_n,b_n), $n \in \mathbb{N}$, is a sequence of disjoint intervals in $[0,\infty)$ such that $\sup_{n\in\mathbb{N}}(b_n-a_n) < \infty$ and $\alpha(\cdot)|_{(a_n,b_n)} \equiv 1$.

As in Example 3.2, we write system (58)–(60) in the form (5) with $H = U = L^2(0,1)$, the skew-adjoint operator A given by $Az = iz_{xx}$ acting on $D(A) = \mathcal{H}^2(0,1)\cap\mathcal{H}_0^1(0,1)$, and the multiplication operator $B\colon z \mapsto \chi_\omega z$, so that for $y_0 \in H$, the mild solution $z(\cdot)$ of (5) with this choice of A, B corresponds to the weak solution $y(\cdot)$ of (58)–(60).

It is well known that for any interval (a_n, b_n), $n \in \mathbb{N}$, there exists a positive constant c_n such that

$$\int_{a_n}^{b_n} \int_\omega |e^{tA} z(x)|^2 \, dx \, dt \geq c_n \|z\|_H^2, \qquad z \in H, \tag{61}$$

that is, $\alpha(a_n + \cdot)|_{[0, b_n - a_n]}$ is of class $\mathcal{K}(A, B, b_n - a_n, c_n)$ (see, for instance, [33, Remark 6.5.4]). Moreover, rewriting (61) as

$$\int_{a_n}^{b_n} \int_\omega \left| \sum_{k \in \mathbb{N}} \langle \phi_k, z \rangle_{L^2(0,1)} \phi_k(x) e^{in^2\pi^2 t} \right|^2 dx \, dt \geq c_n \|z\|_H^2,$$

with $\phi_k(x) = \sqrt{2} \sin(n\pi x)$ we get from [32, Corollary 3.2] that c_n can be taken satisfying

$$c_n \geq C e^{-\frac{2}{b_n - a_n}}$$

for some positive constant C independent of n.

Hence, Theorem 5.2 guarantees that the mild solution of (5) converges strongly to the origin in H if

$$\sum_{n=1}^{\infty} e^{-\frac{2}{b_n - a_n}} = \infty.$$

◇

Remark 5.1. The results in the above example and, more generally, the methodology employed in this section, can be adapted to the case of some unbounded control operators and thus to boundary stabilization problems. As an example, consider the system

$$iy_t(t,x) + y_{xx}(t,x) = 0, \qquad (t,x) \in (0,\infty) \times (0,1), \tag{62}$$

$$y_x(t,0) = -i\alpha(t) y(t,0), \qquad t \in (0,\infty), \tag{63}$$

$$y(t,1) = 0, \qquad t \in (0,\infty), \tag{64}$$

$$y(0,x) = y_0(x), \qquad x \in (0,1). \tag{65}$$

with a piecewise constant $\alpha \colon [0,\infty) \to [0,1]$ satisfying conditions as in Example 5.1 for some sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$. Clearly, Theorem 5.2 does not apply in this case. However, we can retrieve similar conditions on the intervals (a_n, b_n) in order to have the strong stability property as in Example 5.1. Indeed, it suffices to check the exact observability for the undamped dynamics and to show that an energy estimate such in Lemma 2.1 holds for the constant damping case. The operator $A \colon D(A) \to L^2(0,1)$ corresponding to the undamped case (i.e., $\alpha = 0$ in (63)) is

$$D(A) = \{\varphi \in H^2(0,1) \mid \varphi_x(0) = 0, \ \varphi(1) = 0\},$$

$$A\varphi = i\varphi_{xx} \qquad (\varphi \in D(A)),$$

whereas the control operator is given by $B = \delta_0$, where δ_0 is the Dirac mass concentrated at the origin.

Using the results in [32], it is not difficult to check that for $\alpha = 0$ there exist $C_1, \ C_2 > 0$ such that

$$C_1 e^{\frac{C_2}{T}} \int_0^T |y(t,0)|^2 \, \mathrm{d}t \geqslant \|y_0\|_{L^2(0,1)}^2 \quad (T > 0, \ y_0 \in D(A)).$$

The last formula is, according to the above definitions of A and B, equivalent to the inequality

$$C_1 e^{\frac{C_2}{T}} \int_0^T |B^* e^{sA} y_0| \, dt \geq c \|y_0\|_{L^2(0,1)} \qquad (y_0 \in D(A)), \tag{66}$$

so that we have indeed the exact observability in any time $T > 0$ for the undamped dynamics.

To check an energy estimate similar to the one in Lemma 2.1, one can first prove (13) for y_0 in the domain of the generator (which is done via integration by parts). One can then check (using, for instance, a transfer function like in Guo and Shao [11]) that the system (A, B, B^*) is well-posed in the sense of Salamon and Weiss (see [35]). ◇

Sufficient conditions for strong stability as those obtained in Theorem 5.2 can be specified more precisely in the case of integral "excitations".

Theorem 5.3. *Suppose that there exist constants $\rho, T_0 > 0$ and a positive, continuous function $c\colon (0,\infty) \to \mathbb{R}$ such that for all $T \in (0, T_0]$, if for some $\tilde{\alpha} \in L^\infty([0,T],[0,1])$*

$$\int_0^T \tilde{\alpha}(t)\,dt \geq \rho T$$

then $\tilde{\alpha}(\cdot)$ is of class $\mathcal{K}(A, B, T, c(T))$. Let (a_n, b_n), $n \in \mathbb{N}$, be a sequence of disjoint intervals in $[0,\infty)$ and $\alpha \in L^\infty([0,\infty),[0,1])$. Assume that $\int_{a_n}^{b_n} \alpha(t)\,dt \geq \rho(b_n - a_n)$ and $\sum_{n=1}^\infty c(b_n - a_n) = \infty$. Then the mild solution of (5) satisfies $\|z(t)\|_H \to 0$ as $t \to \infty$.

Proof. In the case where $\sup_{n\in\mathbb{N}}(b_n - a_n) \leq T_0$ the conclusion follows directly from Theorem 5.2.

Now assume that for infinitely many $n \in \mathbb{N}$, $b_n - a_n > T_0$. Let n be such that $b_n - a_n > T_0$ and split $I_n = (a_n, b_n)$ into finitely many pairwise disjoint subintervals $I_n^1, \ldots, I_n^r$ of common length $l_n \in [T_0/2, T_0]$. Since $\sum_{j=1}^r \int_{I_n^j} \alpha(t)\,dt \geq \rho(b_n - a_n) = r\rho\, l_n$, then there exists $j \in \{1, \ldots, r\}$ such that $\int_{I_n^j} \alpha(t)\,dt \geq \rho l_n = \rho |I_n^j|$. Denote I_n^j by (a_n', b_n'). If n is such that $b_n - a_n \leq T_0$, set $a_n' = a_n$ and $b_n' = b_n$.

Again applying Theorem 5.2 to the sequence of intervals (a_n', b_n'), $n \in \mathbb{N}$, we can conclude by showing that $\sum_{n=1}^\infty c(b_n' - a_n') = \infty$, since $\sup_{n\in\mathbb{N}}(b_n' - a_n') < \infty$. The unboundedness of $\sum_{n=1}^\infty c(b_n' - a_n')$ follows from the remark that, for infinitely many $n \in \mathbb{N}$, $c(b_n' - a_n') \geq \min_{T\in[T_0/2,T_0]} c(T) > 0$. □

Example 5.2. (Wave equation) Let $t \mapsto z(t) = (v(t,\cdot), v_t(t,\cdot))$ be a solution of the wave equation (22)–(25) where, as in Example 3.1, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $d \in L^\infty(\Omega)$ satisfies

$$|d(x)| \geq d_0 > 0 \quad \text{for almost all } x \in \Omega. \tag{67}$$

Consider $T, \rho > 0$ and some $\alpha(\cdot) \in L^\infty([0,T],[0,1])$ satisfying

$$\int_0^T \alpha(t)\,dt \geq T\rho. \tag{68}$$

Using the same notation as in Example 3.1 and, in particular, fixing an initial condition and defining the set A_n^ϵ as in (27), we have, according to (28),

$$\operatorname{meas}(A_n^\epsilon) \geq T\left(1 - \frac{2\epsilon}{\pi}\right) - 2\frac{\epsilon}{\min_{n\in\mathbb{N}} \lambda_n}$$

for any $\epsilon \in (0,1)$. Without loss of generality, we can assume that $\min_{n\in\mathbb{N}} \lambda_n = \lambda_1$. For T small enough, choosing $\bar{\epsilon} = \frac{\rho\lambda_1}{6}T$, we get $\operatorname{meas}(A_n^{\bar{\epsilon}}) \geq T\left(1 - \frac{\rho}{2}\right)$, leading to

$$\int_{A_n^{\bar{\epsilon}}} \alpha(t)\,dt \geq \frac{T\rho}{2},$$

because of (68). The definition of $A_n^{\bar\epsilon}$ yields the observability estimate

$$\int_0^T \alpha(t)(-a_n \sin(\sqrt{\lambda_n} t) + b_n \cos(\sqrt{\lambda_n} t))^2 \, dt \geq \bar\epsilon^2 \, (a_n^2 + b_n^2) \int_{A_n^{\bar\epsilon}} \alpha(t) \, dt$$
$$\geq \frac{\rho^3 \lambda_1^2}{72} T^3 (a_n^2 + b_n^2).$$

Reasoning as in Example 3.1, we obtain that the function $c(T)$ appearing in the statement of Theorem 5.3 for the system (22)–(25) with uniform damping (67) can be chosen of order T^3 for T small.

In particular, Theorem 5.3 states that a sufficient condition for the strong asymptotic stability of the solutions of (22)–(25) with uniform damping (67) is that $\alpha(\cdot) \in L^\infty([0,\infty),[0,1])$ satisfies

$$\int_{a_n}^{b_n} \alpha(t) \, dt \geq \rho(b_n - a_n), \quad n \in \mathbb{N},$$

for some positive constant ρ and some sequence (a_n, b_n), $n \in \mathbb{N}$, of disjoint intervals in $[0,\infty)$ such that

$$\sum_{n=1}^{\infty} (b_n - a_n)^3 = \infty.$$

For the particular case of the wave equation, the sufficient condition obtained here weakens the one considered in [14] where $\alpha(\cdot)$ is bounded away from 0 by a constant m_n on each interval (a_n, b_n) in order to guarantee asymptotic stability (cf. also Remark 3.1).

◇

Example 5.3. (Finite-dimensional linear systems) Let us characterize the function $c(\cdot)$ appearing in the statement of Theorem 5.3 in the case of finite-dimensional systems, that is, when $\dim(H) < \infty$. We prove below that T behaves polynomially and that its degree for T small can be taken equal to the sharp estimate computed by Seidman in the case $\alpha \equiv 1$ (see [28]).

In the finite-dimensional case, the assumption that A generates a strongly continuous contraction semigroup is standardly weakened into the requirement that A is neutrally stable, that is, its eigenvalues are of non-positive real part and all Jordan blocks corresponding to pure imaginary eigenvalues are trivial.

Clearly, a necessary condition for ensuring the convergence to the origin of all trajectories of the system $\dot{x} = Ax + \alpha Bu$ for some $\alpha = \alpha(t) \in [0,1]$ is that the pair (A, B) is stabilizable. We will make this assumption from now on.

Up to a linear change of variables, A and B can be written as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_3 \end{pmatrix},$$

where A_1 is Hurwitz and all the eigenvalues of A_3 are purely imaginary. From the neutral stability assumption and up to a further linear change of coordinates, we may assume that A_3 is skew-symmetric. From the stabilizability assumption on (A, B), moreover, we deduce that (A_3, B_3) is controllable.

Setting $x = (x_1, x_3)$ according to the above decomposition, the system $\dot{x} = Ax + \alpha Bu$ can be written as

$$\dot{x}_1 = A_1 x_1 + A_2 x_3 + \alpha(t) B_1 u, \tag{69}$$
$$\dot{x}_3 = A_3 x_3 + \alpha(t) B_3 u. \tag{70}$$

Assume that, for a given $\alpha(\cdot)$, all solutions of (70) with $u = -B_3^\top x_3$ converge to the origin. Then all trajectories of system (69)-(70), with the choice of feedback $u = -B_3^\top x_3$, converge to the origin, since (69) becomes an autonomous linear

Hurwitz system subject to a perturbation whose norm converges to zero as time goes to infinity.

The previous discussion allows us to focus on the special case where A is skew-symmetric and (A, B) is controllable.

Denote by $K_{(A,B)}$ the minimal non-negative integer such that

$$\operatorname{rank}[B, AB, \dots, A^{K_{(A,B)}}B] = N, \tag{71}$$

where N is the dimension of H. We have the following result.

Proposition 5.4. *Let A be skew-symmetric and (A, B) controllable. Then for every $\rho > 0$ there exists $\kappa > 0$ such that, for every $T \in (0, 1]$ and every $\alpha \in L^\infty([0, T], [0, 1])$, if $\int_0^T \alpha(t)dt \geq \rho T$ then α is of class $\mathcal{K}(A, B, T, \kappa T^{2K_{(A,B)}+1})$.*

Proof. Let $K = K_{(A,B)}$ and fix $\rho > 0$. We should prove that, for some $\kappa > 0$, given any $z_0 \in \mathbb{R}^n$ and any $\alpha \in L^\infty([0, T], [0, 1])$ such that $T \in (0, 1]$ and $\int_0^T \alpha(t)dt \geq \rho T$, we have

$$\int_0^T \alpha(t)\|B^\top e^{tA} z_0\|^2 dt \geq \kappa T^{2K+1}\|z_0\|^2.$$

Denote by $b_1, \dots, b_r$ the columns of B and assume, by contradiction, that there exist $(T_n)_{n\in\mathbb{N}} \subset (0, 1]$, $(z_0^n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ with $\|z_0^n\| = 1$, and $(\alpha^n)_{n\in\mathbb{N}} \subset L^\infty([0, 1], [0, 1])$ with $\int_0^{T_n} \alpha^n(t)dt \geq \rho T_n$ such that $\lim_{n\to\infty} \kappa_n = 0$ where

$$\kappa_n = \frac{\int_0^{T_n} \alpha^n(t) \sum_{i=1}^r (b_i^\top e^{tA} z_0^n)^2 dt}{T_n^{2K+1}}, \quad n \in \mathbb{N}.$$

Let $\beta^n(t) = \alpha^n(T_n t)$ for $n \in \mathbb{N}$ and $t \in [0, 1]$. Then $\int_0^1 \beta^n(t)dt \geq \rho$ and

$$\kappa_n = \frac{\int_0^1 \beta^n(t) \sum_{i=1}^r (b_i^\top e^{tT_n A} z_0^n)^2 dt}{T_n^{2K}}, \quad n \in \mathbb{N}.$$

By compactness, up to extracting a subsequence, $T_n \to T_\infty$ in $[0, 1]$, $z_0^n \to z_0^\infty$ in $\mathbb{R}^N$ and $\beta^n \overset{*}{\rightharpoonup} \beta^\infty$ in $L^\infty([0, 1], [0, 1])$. In particular, $\|z_0^\infty\| = 1$, $\int_0^1 \beta^\infty(t)dt \geq \rho$ and

$$\lim_{n\to\infty} \kappa_n T_n^{2K} = \int_0^1 \beta^\infty(t) \sum_{i=1}^r \left(b_i^\top e^{tT_\infty A} z_0^\infty\right)^2 dt = 0.$$

Assume first that $T_\infty > 0$. Then the analytic function $t \mapsto \sum_{i=1}^r \left(b_i^\top e^{tT_\infty A} z_0^\infty\right)^2$ annihilates on the support of β^∞, which has positive measure, and is thus identically equal to zero, contradicting the controllability of the pair (A, B).

Let then $T_\infty = 0$. Rewrite κ_n as

$$\kappa_n = \int_0^1 \beta^n(t) \sum_{i=1}^r \left(c_0^{i,n} + tc_1^{i,n} + \cdots + t^K c_K^{i,n} + r^{i,n}(t)\right)^2 dt,$$

where

$$c_j^{i,n} = \frac{b_i^\top A^j z_0^n}{j! T_n^{K-j}}, \qquad \|r^{i,n}\|_{L^\infty(0,1)} \leq M T_n,$$

for some $M > 0$ only depending on A, B, and K. Define the vector

$$C^n = (c_0^{1,n}, \dots, c_K^{1,n}, c_0^{2,n}, \dots, c_K^{2,n}, \dots, c_0^{r,n}, \dots, c_K^{r,n})$$

belonging to $\mathbb{R}^{r(K+1)}$. Since $\|z_0^n\| = 1$, $T_n \leq 1$, and because of (71), there exists $\nu > 0$ only depending on A and B such that $\|C^n\| \geq \nu$. Thus,

$$\kappa_n \geq \nu^2 \frac{\int_0^1 \beta^n(t) \sum_{i=1}^r \left(c_0^{i,n} + tc_1^{i,n} + \cdots + t^K c_K^{i,n} + r^{i,n}(t)\right)^2 dt}{\|C^n\|^2}.$$

Up to extracting a subsequence, $C^n/\|C^n\|$ converges in the unit sphere of $\mathbb{R}^{r(K+1)}$. Denote its limit by $(\gamma_0^1, \dots, \gamma_K^r)$. Then

$$t \mapsto \sum_{i=1}^{r} \left(c_0^{i,n} + t c_1^{i,n} + \cdots + t^K c_K^{i,n} + r^{i,n}(t) \right)^2$$

converges uniformly on $[0,1]$ to $t \mapsto \sum_{i=1}^{r} \left(\gamma_0^i + t\gamma_1^i + \cdots + t^K \gamma_K^i \right)^2$. We can conclude that

$$\int_0^1 \beta^\infty(t) \sum_{i=1}^{r} \left(\gamma_0^i + t\gamma_1^i + \cdots + t^K \gamma_K^i \right)^2 dt = 0,$$

leading to a contradiction, since β^∞ is nonzero on a subset of $[0,1]$ of positive measure and $(\gamma_0^1, \dots, \gamma_K^r)$ is a nonzero vector. □

Proposition 5.4 and Theorem 5.3 imply the following.

Corollary 5.5. *Let A be skew-symmetric and (A, B) be controllable. Then for every $\rho > 0$, every $\alpha \in L^\infty([0,\infty),[0,1])$ such that there exist a sequence (a_n, b_n), $n \in \mathbb{N}$, of disjoint intervals in $[0,\infty)$ with $\int_{a_n}^{b_n} \alpha(t)\,dt \geq \rho(b_n - a_n)$ and $\sum_{n=1}^{\infty}(b_n - a_n)^{2K_{(A,B)}+1} = \infty$, and every solution $z(\cdot)$ of (5) corresponding to α, we have $\|z(t)\|_{\mathbb{R}^N} \to 0$ as $t \to \infty$.*

◇

Acknowledgments. This work was supported by the ANR grant ArHyCo, Program ARPEGE, contract number ANR-2008 SEGI 004 01-30011459. The research presented in this article was mostly carried out while F. M. Hante and M. Sigalotti were with Institut Élie Cartan (IECN) and CORIDA, INRIA Nancy–Grand Est.

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