



HAL
open science

Propriétés spectrales des opérateurs de Schrödinger magnétiques invariant par translation

Willy Dantin-Carrère

► **To cite this version:**

Willy Dantin-Carrère. Propriétés spectrales des opérateurs de Schrödinger magnétiques invariant par translation. Annales Henri Poincaré, 2008, 9, page 181-207. inria-00614636

HAL Id: inria-00614636

<https://inria.hal.science/inria-00614636>

Submitted on 13 Aug 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**Réponse à une question ouverte
d'analyse spectrale
(niveau thèse)**

Question ouverte
d'un article de recherche de D.YAFAEV
**« Propriétés spectrales
des opérateurs de Schrödinger
magnétiques invariant par translation. »**

**Suite d'une micro-thèse effectuée en
Master 2 Analyse Géométrie et Arithmétique**
M. Dantin-Carrère Willy

Déposé à l'I.N.P.I. sous enveloppe Soleau

Remerciements : Pour la correction des calculs par le Professeur Michel Vaugon de l'université de Paris VI.

L'article de Dimitri Yafaev se nomme :

« **On spectral properties of translationally invariant magnetic Schrödinger operators** ».

Il est paru dans la revue : **Annales Henri Poincaré 9 (2008).**

Article se situe : **p181-207.**

Revue de l'**Institut Henri Poincaré (I.H.P.).**

La question ouverte se trouve après le **théorème 4-9** page 21 de l'article.

transformations) by the equation $\text{curl } A(x) = B(x)$, can be chosen as

$$A(x) = (0, 0, -a(x_1, x_2)) \quad (1.1)$$

so that it has the constant direction. In contrast to fields of the first class, now the variable x_3 cannot be separated in the Schrödinger equation. Nevertheless due to the invariance with respect to translations along the x_3 -axis the operator (we always suppose that the charge of a particle is equal to 1)

$$\mathbf{H} = (i\nabla + A(x))^2 \quad (1.2)$$

can be realized, after the Fourier transform in the variable x_3 , in the space $L^2(\mathbb{R}; L^2(\mathbb{R}^2))$ as the operator of multiplication by the operator-valued function

$$H(p) = -\Delta + (a(x_1, x_2) + p)^2 : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2), \quad p \in \mathbb{R}. \quad (1.3)$$

Moreover, if $a(x_1, x_2) = a(r)$, then the subspaces with fixed magnetic quantum number $m \in \mathbb{Z}$ are invariant subspaces of $H(p)$ so that the operator $H(p)$ reduces to the orthogonal sum over $m \in \mathbb{Z}$ of the operators

OPÉRATEUR UTILISÉ
DANS LA RÉPONSE

$$\left(H_m(p) = -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \frac{m^2}{r^2} + (a(r) + p)^2 \right) \quad (1.4)$$

acting in the space $\mathcal{H} = L^2(\mathbb{R}_+; r dr)$. In this case the field is given by the equation

$$B(x) = b(r)(-\sin \theta, \cos \theta, 0) \quad (1.5)$$

where $b(r) = a'(r)$ and θ is the polar angle. Thus, vectors $B(x)$ are tangent to circles centered at the origin. An important example of such type is a field created by a current along an infinite straight wire (coinciding with the x_3 -axis). In this case $b(r) = b_0 r^{-1}$ so that $a(r) = b_0 \ln r$. The Schrödinger operator with such magnetic potential was studied in [15].

1.2. In this article we consider magnetic fields (1.5) with a sufficiently arbitrary function $b(r)$. Our goal is to study basic spectral properties of the corresponding Schrödinger operator \mathbf{H} such as the absolute continuity, location and multiplicity of the spectrum, as well as the long-time behaviour of the unitary group $\exp(-i\mathbf{H}t)$. We emphasize that for magnetic fields considered here, the problem is genuinely three-dimensional, and actually the motion of a particle in the x_3 -direction is of a particular interest.

Using the cylindrical invariance of field (1.5), we can start either from translational or from rotational (around the x_3 -axis) symmetries. The rotational invariance implies that the operator \mathbf{H} is the orthogonal sum of its restrictions \mathbf{H}_m on the subspaces of functions with magnetic quantum number $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. It can be identified with the operator (we keep the same notation for this operator)

$$\mathbf{H}_m = -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \frac{m^2}{r^2} + \left(i \frac{d}{dx_3} + a(r) \right)^2 \quad (1.6)$$

acting in the space $\mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R}; r dr dx_3)$. In view of the translation invariance, every operator \mathbf{H}_m can be realized (again after the Fourier transform in the variable x_3) in the space $L^2(\mathbb{R}; L^2(\mathbb{R}_+; r dr))$ as the operator of multiplication by the operator-valued function $H_m(p)$ defined by (1.4).

Suppose now that $b(r)$ does not tend to zero too fast so that $a(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then the spectrum of each operator $H_m(p)$ is discrete. Let $\lambda_{n,m}(p)$, $n \in \mathbb{N} = \{1, 2, \dots\}$, be the increasing sequence of its eigenvalues (they are simple and positive), and let $\psi_{n,m}(r, p)$ be the corresponding sequence of its eigenfunctions. The functions $\lambda_{n,m}(p)$ are known as dispersion curves of the problem. They determine the spectral properties of the operator \mathbf{H}_m .

Note that if $a(r)$ is replaced by $-a(r)$, then $\lambda_{n,m}(p)$ is replaced by $-\lambda_{n,m}(p)$, so that the case $a(r) \rightarrow -\infty$ as $r \rightarrow \infty$ is automatically included in our considerations. Recall that, for a magnetic field $B(x)$, the magnetic potential $A(x)$ such that $\text{curl } A(x) = B(x)$ is defined up to a gauge term $\text{grad } \varphi(x)$. In particular for magnetic fields (1.5) in the class of potentials $A(x) = (0, 0, -a(r))$ one can always add to $a(r)$ an arbitrary constant c . This leads to the transformations $\lambda_{n,m}(p) \mapsto \lambda_{n,m}(p - c)$ and $\psi_{n,m}(r, p) \mapsto \psi_{n,m}(r, p - c)$.

1.3. The precise definitions of the operators \mathbf{H}_m and \mathbf{H} and their decompositions into the direct integrals over the operators $H_m(p)$ and $H(p)$ are given in Section 2. To put it differently, we construct a complete set of eigenfunctions of the operator \mathbf{H} . They are parametrized by the magnetic quantum number m , the momentum p in the direction of the x_3 -axis and the number n of an eigenvalue $\lambda_{n,m}(p)$ of the operator $H_m(p)$. Thus, if we set

$$\psi_{n,m,p}(r, \theta, x_3) = e^{ipx_3} e^{im\theta} \psi_{n,m}(r, p), \quad (1.7)$$

then

$$\mathbf{H}\psi_{n,m,p} = \lambda_{n,m}(p)\psi_{n,m,p}. \quad (1.8)$$

In Section 3, we show that for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$:

- Under very general assumptions $\lambda_{n,m}(p) \rightarrow \infty$ as $p \rightarrow \infty$ (Proposition 3.3).
- If $b(r) \rightarrow 0$ as $r \rightarrow \infty$, then $\lambda_{n,m}(p) \rightarrow 0$ as $p \rightarrow -\infty$ (Proposition 3.5).
- If $b(r)$ admits a finite positive limit b_0 as $r \rightarrow \infty$, then $\lambda_{n,m}(p) \rightarrow (2n - 1)b_0$ for all m as $p \rightarrow -\infty$ (Proposition 3.6).
- If $b(r) \rightarrow \infty$ as $r \rightarrow \infty$, then $\lambda_{n,m}(p) \rightarrow \infty$ as $p \rightarrow -\infty$ (Proposition 3.6).

Related results concerning the dispersion curves for Schrödinger operator with constant magnetic fields defined on unbounded domains $\Omega \subset \mathbb{R}^2$ have been obtained in [5] (the case where Ω is a strip) and in [4], [7, Section 4.3] (the case where Ω is a half-plane).

In Theorem 3.8 we formulate the main spectral results which follow from the asymptotic properties of the dispersion curves $\lambda_{n,m}(p)$, $p \in \mathbb{R}$. First, the analyticity and the asymptotics as $p \rightarrow \infty$ of $\lambda_{n,m}(p)$ imply immediately that the spectra $\sigma(\mathbf{H}_m)$ and $\sigma(\mathbf{H})$ of the operators \mathbf{H}_m , $m \in \mathbb{Z}$, and \mathbf{H} are purely absolutely continuous. Moreover,

$$\sigma(\mathbf{H}_m) = [\mathcal{E}_m, \infty), \quad m \in \mathbb{Z}, \quad \sigma(\mathbf{H}) = [\mathcal{E}_0, \infty), \quad (1.9)$$

where

$$\mathcal{E}_m = \inf_{p \in \mathbb{R}} \lambda_{1,m}(p) \geq 0. \quad (1.10)$$

Next, in the case where the magnetic field tends to 0 as $r \rightarrow \infty$, the spectra of \mathbf{H}_m , $m \in \mathbb{Z}$, coincide with $[0, \infty)$ and have infinite multiplicity. On the other hand, in the case where the magnetic field tends as $r \rightarrow \infty$ to a positive finite limit, or to infinity,

we have that $\mathcal{E}_m > 0$ for all $m \in \mathbb{Z}$ and each of the spectra $\sigma(\mathbf{H}_m)$ contains infinitely many thresholds.

Further, in Section 4, we obtain a convenient formula for the derivatives $\lambda'_{n,m}(p)$ which play the role of asymptotic group velocities. Our formula for $\lambda'_{n,m}(p)$ yields sufficient conditions (see Theorem 4.3) for positivity of these functions. The leading example when these conditions are met, is

$$b(r) = b_0 r^{-\delta}, \quad b_0 > 0, \quad \delta \in [0, 1], \quad (1.11)$$

and $m \neq 0$. If $\delta = 1$, this result remains true for all m (cf. [15]). On the contrary, if $\delta = 0$ and $m = 0$, then $\lambda'_{n,0}(p) < 0$ for all n on some interval of p (lying on the negative half-axis). Similar results concerning for the dispersion curves for the Schrödinger operator with constant magnetic field, defined on the half-plane with Dirichlet (resp., Neumann) boundary conditions, can be found in [4] (resp., [3] and [7, Section 4.3]).

Finally, in Section 5 we discuss the long-time behaviour of a quantum particle. The time evolution of a quantum system is determined by the unitary groups $\exp(-i\mathbf{H}_m t)$, $m \in \mathbb{Z}$, so that an analysis of its asymptotics as $t \rightarrow \pm\infty$ relies on spectral properties of the operators \mathbf{H}_m . Since these operators have discrete spectra, a quantum particle remains localized in the (x_1, x_2) -plane. Its propagation in the x_3 -direction is governed by the group velocities $\lambda'_{n,m}(p)$. In particular, the condition $\lambda'_{n,m}(p) > 0$ for all $n \in \mathbb{N}$ and $p \in \mathbb{R}$ implies that a quantum particle with the magnetic quantum number m propagates as $t \rightarrow +\infty$ in the positive direction of the x_3 -axis.

Let us compare these results with the long-time behaviour of a classical particle in magnetic field (1.5). As shown in [15], the function $x'_3(t)$ is periodic with period T determined by initial conditions. Its drift $x_3(T) - x_3(0)$ over the period is nonnegative if $b(r) \geq 0$ and $b'(r) \geq 0$. Moreover, it is strictly positive if $b'(r) > 0$ for all r . In the case $b(r) = \text{const}$ it is still strictly positive if the angular momentum m of a particle is not zero. Thus, our results for functions (1.11) correspond completely to the classical picture if $\delta = 1$ or $\delta \in [0, 1)$ and $m \neq 0$. In the case $\delta = 0$ and $m = 0$ the behaviour of quantum and classical particles turn out to be qualitatively different.

2. HAMILTONIANS AND THEIR DIAGONALIZATIONS

Here we give precise definitions of the Hamiltonians and discuss their reductions due to the cylindrical symmetry.

2.1. For an arbitrary magnetic potential $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $A \in L^2_{\text{loc}}(\mathbb{R}^3)^3$, the self-adjoint Schrödinger operator (1.2) can be defined via its quadratic form

$$\mathbf{h}[\mathbf{u}] = \int_{\mathbb{R}^3} |i(\nabla \mathbf{u})(x) + A(x)\mathbf{u}(x)|^2 dx, \quad x = (x_1, x_2, x_3). \quad (2.1)$$

It is easy to see that this form is closed on the set of functions $u \in L^2(\mathbb{R}^3)$ such that $\nabla \mathbf{u} \in L^1_{\text{loc}}(\mathbb{R}^3)^3$ and $i\nabla \mathbf{u} + A\mathbf{u} \in L^2(\mathbb{R}^3)^3$. Similarly, if $a \in L^2_{\text{loc}}(\mathbb{R}^2)$, then the self-adjoint Schrödinger operator (1.3) can be defined via its quadratic form

$$h[u; p] = \int_{\mathbb{R}^2} (|(\nabla u)(x)|^2 + (a(x) + p)^2 |u(x)|^2) dx, \quad x = (x_1, x_2), \quad p \in \mathbb{R}. \quad (2.2)$$

finite limits $b_0(2n-1)$ as $p \rightarrow -\infty$. If $\delta < 0$, then these functions tend to $+\infty$ as $p \rightarrow -\infty$.

3.3. Let us return to the Hamiltonians \mathbf{H}_m and \mathbf{H} defined in Section 2.

Theorem 3.8. Assume (2.4) and (2.10).

(i) Then all operators \mathbf{H}_m , $m \in \mathbb{Z}$, and hence \mathbf{H} are absolutely continuous and their spectra coincide with the half-axes defined by equations (1.9) and (1.10).

(ii) If the hypotheses of Proposition 3.5 hold true, then $\mathcal{E}_m = 0$ for all $m \in \mathbb{Z}$. Moreover, the multiplicities of all spectra $\sigma(\mathbf{H}_m)$ and hence of $\sigma(\mathbf{H})$ are infinite.

(iii) Let the hypotheses of Proposition 3.6 hold true. If $b(r) \rightarrow \infty$, then the infimum in (1.10) is attained (at a finite point) so that for all $m \in \mathbb{Z}$

$$\mathcal{E}_m = \min_{p \in \mathbb{R}} \lambda_{1,m}(p) > 0.$$

(iv) Let the hypotheses of Proposition 3.6 hold true. If $b(r)$ admits a finite positive limit b_0 as $r \rightarrow \infty$, then $\mathcal{E}_m \in (0, b_0]$ for all $m \in \mathbb{Z}$.

Proof. It suffices to prove only the assertions concerning the operators \mathbf{H}_m . In view of decomposition (2.12) they reduce to corresponding statements about the operators

$\Lambda_{n,m}$. These operators are absolutely continuous because the eigenvalues $\lambda_{n,m}(p)$ are real analytic functions of $p \in \mathbb{R}$ which are non constants since according to Corollary 3.4 $\lambda_{n,m}(p) \rightarrow \infty$ as $p \rightarrow \infty$. Moreover, we have that

$$\sigma(\Lambda_{n,m}) = [\mathcal{E}_{n,m}, \infty) \quad \text{where} \quad \mathcal{E}_{n,m} = \inf_{p \in \mathbb{R}} \lambda_{n,m}(p) \geq 0 \quad (3.52)$$

because $\lambda_{n,m}(p) > 0$ for all $p \in \mathbb{R}$. This implies relations (1.9) with \mathcal{E}_m defined by (1.10).

In case (ii) it suffices to use that according to (3.9) $\mathcal{E}_{n,m} = 0$ and hence $\sigma(\Lambda_{n,m}) = [0, \infty)$ for all m and n .

In case (iii) Proposition 3.6 implies that $\lambda_{n,m}(p) \rightarrow \infty$ as $p \rightarrow -\infty$ for all n and m so that

$$\mathcal{E}_{n,m} = \min_{p \in \mathbb{R}} \lambda_{n,m}(p) > 0 \quad (3.53)$$

and hence infimum in (1.10) can be replaced by minimum.

In case (iv) we use that according to (3.25) $\mathcal{E}_{n,m} \leq (2n-1)b_0$. Moreover, $\mathcal{E}_{n,m} > 0$ because $\lambda_{n,m}(p) > 0$ for all $p \in \mathbb{R}$. For $n=1$, this gives the desired result. \square

Remark 3.9. According to (2.12) and (3.52) the spectrum of the operator \mathbf{H}_m consists of the "branches" $[\mathcal{E}_{n,m}, \infty)$ where the points $\mathcal{E}_{n,m}$ are called thresholds. In cases (iii) and (iv)

$$\mathcal{E}_{n,m} < \mathcal{E}_{n+1,m} \quad (3.54)$$

for all $n \in \mathbb{N}$. Indeed, in case (iii) (3.54) is a consequence of the estimate $\lambda_{n,m}(p) < \lambda_{n+1,m}(p)$ valid for all $p \in \mathbb{R}$ and of formula (3.53). In case (iv) one has to take additionally into account that the limit of $\lambda_{n,m}(p)$ as $p \rightarrow -\infty$ is strictly smaller than that of $\lambda_{n+1,m}(p)$. Inequality (3.54) means that there are infinitely many distinct thresholds in each of the spectra $\sigma(\mathbf{H}_m)$, $m \in \mathbb{Z}$, and hence in $\sigma(\mathbf{H})$.

HYPOTHÈSE sur $\lambda_{n,m}(p)$ fonction analytique réelle de p

Now we are in a position to obtain a formula for the derivative $\lambda'_{n,m}(p)$. In addition to our usual assumptions that $b(r)$ is not too singular at $r = 0$, an integration-by-parts machinery requires that $b(r)$ does not vanish too rapidly as $r \rightarrow 0$. The precise conditions are formulated rather differently in the cases $m \neq 0$ and $m = 0$. We start with the first case.

Theorem 4.3. *Let $m \neq 0$. Suppose that $b \in C^3(\mathbb{R}_+)$ and $b(r) > 0$, $r \in \mathbb{R}_+$. Assume (2.10) and that $b(r) = O(e^{cr})$ for some $c > 0$ as $r \rightarrow \infty$. At $r = 0$ we suppose that $b(r) = O(r^{-\gamma})$ where $\gamma < 3/2$. Moreover, we assume that for some $\beta < 2|m| - 1$*

$$|(b(r)^{-1})^{(k)}| \leq Cr^{-\beta-k}, \quad k = 0, 1, 2, 3, \quad r \rightarrow 0. \quad (4.9)$$

Put

$$v(r) = r(r^{-1}(rb(r)^{-1})')' \quad \left. \begin{array}{l} v(r) \text{ utilisé pour la} \\ \text{condition } \lambda_{n,m}(p) \geq 0 \end{array} \right\}$$

Then

$$\begin{aligned} \lambda'_{n,m}(p) = & -2 \int_0^\infty rb^{-2}(r)b'(r)\psi'_{n,m}(r;p)^2 dr \\ & -2^{-1} \int_0^\infty v'(r)\psi_{n,m}^2(r;p) dr + 2m^2 \int_0^\infty r^{-2}b^{-1}(r)\psi_{n,m}^2(r;p) dr, \end{aligned} \quad (4.10)$$

where the eigenfunctions $\psi_{n,m}(r;p)$ of the operator $H_m(p)$ are real and normalized, that is $\|\psi_{n,m}\| = 1$.

Proof. In view of the equation

$$(a(r) + p)^2 \psi_{n,m} = r^{-1}(r\psi'_{n,m})' - m^2 r^{-2} \psi_{n,m} + \lambda_{n,m} \psi_{n,m} \quad (4.11)$$

we can apply to the function $\psi_{n,m}$ the results of Lemmas 4.1 and 4.2 where $q(r) = (a(r) + p)^2 - \lambda_{n,m}$. Thus, Lemma 4.2 implies that $\psi_{n,m}(r;p) = O(r^{|m|})$ and $\psi'_{n,m}(r;p) = O(r^{|m|-1})$ as $r \rightarrow 0$ which ensures that non-integral terms disappear at $r = 0$.

HYPOTHÈSE (To prove the same for non-integral terms corresponding to $r \rightarrow \infty$, we use super-exponential decay of eigenfunctions $\psi_{n,m}(r;p)$ of the operators $H_m(p)$. This result is valid [14] (see also [6]) for all one-dimensional Schrödinger operators with discrete spectra. In view of the condition $a(r) = O(e^{cr})$, it follows from equation (4.11) that the derivatives $\psi'_{n,m}(r;p)$ also decay super-exponentially.

Let us proceed from the formula of the first order perturbation theory (known as the Feynman-Hellman formula)

FORMULE UTILISÉE POUR

$$\lambda'_{n,m}(p) = \int_0^\infty \frac{\partial(a(r) + p)^2}{\partial p} \psi_{n,m}^2(r;p) r dr = \int_0^\infty \frac{\partial(a(r) + p)^2}{\partial r} \psi_{n,m}^2(r;p) \tau(r) dr \quad (4.12)$$

where $\tau(r) = rb(r)^{-1}$. Using that $a(r) = O(r^{1-\gamma})$ and $\tau(r) = O(r^{1-\beta})$, we integrate by parts and get

$\lambda'_{n,m}(p) = 0$
 $\partial_p \lambda_{n,m}(p) = 0$

$$\lambda'_{n,m}(p) = - \int_0^\infty (a(r) + p)^2 \psi_{n,m}(r;p) (\tau'(r) \psi_{n,m}(r;p) + 2\tau(r) \psi'_{n,m}(r;p)) dr.$$

Now it follows from equation (4.11) that

$$\begin{aligned} \lambda'_{n,m}(p) = & -\lambda_{n,m}(p) \int_0^\infty (\tau(r)\psi_{n,m}^2(r;p))' dr + m^2 \int_0^\infty r^{-2}(\tau(r)\psi_{n,m}^2(r;p))' dr \\ & - \int_0^\infty r^{-1}(r\psi'_{n,m}(r;p))'(\tau'(r)\psi_{n,m}(r;p) + 2\tau(r)\psi'_{n,m}(r;p)) dr. \end{aligned} \quad (4.13)$$

By the condition $\tau(r)\psi_{n,m}^2(r;p) \rightarrow 0$ as $r \rightarrow 0$, the first term in the right-hand side equals zero. In the second term we integrate by parts which yields

$$\int_0^\infty r^{-2}(\tau(r)\psi_{n,m}^2(r;p))' dr = 2 \int_0^\infty r^{-3}\tau(r)\psi_{n,m}^2(r;p) dr$$

because $r^{-2}\tau(r)\psi_{n,m}^2(r;p) \rightarrow 0$.

In the last integral in the right-hand side of (4.13), we also integrate by parts using that $\psi'_{n,m}(r;p)\psi_{n,m}(r;p)\tau'(r) \rightarrow 0$ as $r \rightarrow 0$. Thus, we have that

$$\begin{aligned} - \int_0^\infty r^{-1}(r\psi'_{n,m}(r;p))' \tau'(r)\psi_{n,m}(r;p) dr &= \int_0^\infty \tau'(r)\psi'_{n,m}(r;p)^2 dr \\ &+ \int_0^\infty v(r)\psi'_{n,m}(r;p)\psi_{n,m}(r;p) dr. \end{aligned} \quad (4.14)$$

The last integral in the right-hand side equals

$$-2^{-1} \int_0^\infty v'(r)\psi_{n,m}^2(r;p) dr$$

because $v(r)\psi_{n,m}^2(r;p) \rightarrow 0$ as $r \rightarrow 0$. Similarly, we get that

$$\begin{aligned} -2 \int_0^\infty r^{-1}(r\psi'_{n,m}(r;p))' \tau(r)\psi'_{n,m}(r;p) dr &= - \int_0^\infty r^{-2}\tau(r) d(r\psi'_{n,m}(r;p)^2) \\ &= \int_0^\infty r^2(\tau^{-2}\tau(r))' \psi'_{n,m}(r;p)^2 dr \end{aligned}$$

since $\tau(r)\psi'_{n,m}(r;p)^2 \rightarrow 0$ as $r \rightarrow 0$. Putting the results obtained together, we arrive at representation (4.10). \square

CONDITION
Pour avoir
 $\lambda'_{n,m}(p) \geq 0$

Corollary 4.4. *If $v'(r) \leq 0$ and $r^2b(r)v'(r) \leq 4m^2$ for all $r \geq 0$, then $\lambda'_{n,m}(p) \geq 0$ for all $p \in \mathbb{R}$ and n . If, moreover, one of these inequalities is strict on some interval, then $\lambda'_{n,m}(p) > 0$.*

Corollary 4.5. *If $b(r) = b_0r^{-\delta}$, $\delta \in [0, 1]$, then $\tau(r) = b_0^{-1}r^{1+\delta}$, $v(r) = b_0^{-1}(\delta^2 - 1)r^{\delta-1}$ and*

$$\begin{aligned} \lambda'_{n,m}(p) = & 2b_0^{-1}\delta \int_0^\infty r^\delta \psi'_{n,m}(r;p)^2 dr \\ & + b_0^{-1}(2m^2 - 2^{-1}(1-\delta)^2(1+\delta)) \int_0^\infty r^{-2+\delta} \psi_{n,m}^2(r;p) dr. \end{aligned}$$

For $b_0 > 0$, this expression is strictly positive (so that the functions $\lambda_{n,m}(p)$ are strictly increasing for all $p \in \mathbb{R}$) for $m \neq 0$ since $(1 - \delta)^2(1 + \delta) \leq 1$. Moreover, for $\delta = 1$ this result is true for all $m \in \mathbb{Z}$.

In the case $m = 0$ we consider for simplicity only fields (1.11).

Proposition 4.6. *If $b(r) = b_0 r^{-\delta}$, $\delta \in [0, 1]$, then*

$$\lambda'_{n,0}(p) = 2b_0^{-1} \delta \int_0^\infty r^\delta \psi'_{n,0}(r; p)^2 dr - b_0^{-1} 2^{-1} (1 - \delta)^2 (1 + \delta) \int_0^\infty r^{-2+\delta} (\psi_{n,0}^2(r; p) - \psi_{n,0}^2(0; p)) dr.$$

If $b_0 > 0$ and $\delta = 1$, then $\lambda'_{n,0}(p) > 0$ for all $p \in \mathbb{R}$.

Proof. Let us proceed again from formula (4.12). We use now that the function $\psi_{n,0}(|x|; p)$ of $x \in \mathbb{R}^2$ belongs to the Sobolev class $H_{loc}^2(\mathbb{R}^2)$, and therefore $\psi_{n,0}(r; p)$ has a finite limit as $r \rightarrow 0$. Thus, by Lemma 4.1 $\psi'_{n,0}(r; p) = O(r^{1-\varepsilon})$ for any $\varepsilon > 0$ as $r \rightarrow 0$. These results allow us to integrate by parts as in the case $m \neq 0$. The only difference is with the second integral in the right-hand side of (4.14). Now $v(r) = b_0^{-1}(\delta^2 - 1)r^{\delta-1}$ and this integral equals

$$\begin{aligned} \int_0^\infty v(r) \psi'_{n,0}(r; p) \psi_{n,0}(r; p) dr &= 2^{-1} \int_0^\infty v(r) d(\psi_{n,0}^2(r; p) - \psi_{n,0}^2(0; p)) \\ &= -2^{-1} \int_0^\infty v'(r) (\psi_{n,0}^2(r; p) - \psi_{n,0}^2(0; p)) dr \end{aligned}$$

because $v(r)(\psi_{n,0}^2(r; p) - \psi_{n,0}^2(0; p)) \rightarrow 0$ as $r \rightarrow 0$. □

4.2. In this subsection we show that for linear potentials, that is for magnetic fields not depending on τ , all eigenvalues $\lambda_{n,0}(p)$, $n \in \mathbb{N}$, of the operator $H_0(p)$ are not monotonous functions of $p \in \mathbb{R}$. We follow closely the proof of the first part of Proposition 3.6. However we now use that eigenfunctions of the harmonic oscillator decay faster than any power of r^{-1} at infinity (actually, they decay super-exponentially).

HYPOTHÈSES

Proposition 4.7. *Assume that for sufficiently large τ*

$$b(\tau) = b_0 > 0. \tag{4.15}$$

HYPOTHÈSE

Then, for all $n \in \mathbb{N}$, some $\gamma_n > 0$ and sufficiently large $k > 0$, we have

$$\lambda_{n,0}(-k) \leq (2n - 1)b_0 - \gamma_n k^{-2}. \tag{4.16}$$

Proof. Let ζ be the same function as in the proof of the first part of Proposition 3.6. We set $\rho_k = b_0^{-1}k$, $\gamma_k = 2b_0^{1/2}k^{-1}$ and define the functions $\zeta(r; k)$ and $\varphi_j(r; k)$ by formulas (3.31) and (3.32), respectively. It suffices to check that

$$\langle L_0(-k)\varphi(k), \varphi(k) \rangle \leq 2n - 1 - \gamma_n k^{-2}. \tag{4.17}$$

for sufficiently large k and all normalized functions from subspace (3.16). Let us proceed from formula (3.35). Since the functions $\psi_j(x; k)$ decay faster than any power of $|x|^{-1}$

Comparing this result with (3.27), we see that, for potentials $a(r) = b_0 r$,

$$\lim_{p \rightarrow -\infty} \lambda_{n,m}(p) = b_0(2n-1) < 2b_0(2n-1 + |m|) = \lambda_{n,m}(0).$$

Together with (4.16), this implies that the functions $\lambda_{n,0}(p)$ have negative local minima. Thus, we get the following

Theorem 4.9. *Under the hypotheses of Proposition 4.7 the eigenvalues $\lambda_{n,0}(p)$, $n \in \mathbb{N}$, of the operator $H_0(p)$ are not monotonous functions of $p \in \mathbb{R}$. Moreover, if (4.15) is satisfied for all $r > 0$ and $a(r) = b_0 r$, then the functions $\lambda_{n,0}(p)$ lose their monotonicity for $p < 0$.*

QUESTION
OUVERTE

We do not know how many minima have the functions $\lambda_{n,0}(p)$. The problem of monotonicity of the eigenvalues $\lambda_{n,0}(p)$ for fields $b(r) = b_0 r^{-\delta}$ where $\delta \in (0, 1)$ remains also open.

4.3. In a somewhat similar situation the break down of monotonicity of group velocities was exhibited in [7]. In this paper one considers the Schrödinger operator $\mathbf{H}^{(N)} = -\frac{\partial^2}{\partial x^2} + \left(i\frac{\partial}{\partial y} - bx\right)^2$ with constant magnetic field $b > 0$, defined on the semi-plane $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ with the Neumann boundary condition at $x = 0$. Let $H^{(N)}(p) = -d^2/dx^2 + (bx + p)^2$, $p \in \mathbb{R}$, be the self-adjoint operator in the space $L^2(\mathbb{R}_+)$ corresponding to the boundary condition $u'(0) = 0$. Then the operator $\mathbf{H}^{(N)}$ is unitarily equivalent under the partial Fourier transform with respect to y , to the direct integral $\int_{\mathbb{R}}^{\oplus} H^{(N)}(p) dp$. It is shown in [7, Section 4.3] that the lowest eigenvalue $\mu_1(p)$ of $H^{(N)}(p)$ is not monotonous for $p < 0$. This follows from the inequality $\mu_1'(0) > 0$ proven¹ in [3] and the relations

$$\lim_{p \rightarrow -\infty} \mu_1(p) = \mu_1(0) = b. \quad (4.22)$$

Our proof of non-monotonicity of the functions $\lambda_{n,0}(p)$ is essentially different since in contrast with (4.22) we have $\lim_{p \rightarrow -\infty} \lambda_{n,0}(p) < \lambda_{n,0}(0)$.

5. ASYMPTOTIC TIME EVOLUTION

5.1. Combined with the stationary phase method, the spectral analysis of the operators $\mathbf{H} = \mathbf{H}(a)$ allows us to find the asymptotics for large t of solutions $u(t) = \exp(-i\mathbf{H}t)u_0$ of the time dependent Schrödinger equation. It follows from (1.2) that

$$\overline{\exp(-i\mathbf{H}(a)t)u_0} = \exp(i\mathbf{H}(-a)t)\overline{u_0}.$$

Therefore it suffices to consider the case $a(r) \rightarrow +\infty$. Moreover, on every subspace \mathfrak{H}_m with a fixed magnetic quantum number m , the problem reduces to the asymptotics of the function $u(t) = \exp(-i\mathbf{H}_m t)u_0$.

Let us proceed from decomposition (2.12). Suppose that $\mathcal{F}_m u_0 \in \text{Ran } \Psi_{n,m}$. Then (see (2.11))

$$(\mathcal{F}_m u_0)(r, p) = \psi_{n,m}(r, p) f(p) \quad (5.1)$$

¹Note that in [3] and [7] the parameter p is chosen with the opposite sign.

* CONDITION SUR $\lambda_{m,m}(p)$ pour avoir un point minimum $\lambda''_{m,m}(p) > 0$

* D'APRÈS LA FORMULE DE FEYNMAN-HELLMAN (Formule de la théorie de perturbation de premier ordre)

$$\lambda'_{m,m}(p) = \int_0^{+\infty} \frac{\partial(a(z)+p)}{\partial p} \Psi_{m,m}^2(z;p) \cdot z \, dz \quad (4-12) \text{ page 17}$$

$$\lambda''_{m,m}(p) = \int_0^{+\infty} \partial_p (2 \cdot (a(z)+p) \cdot \Psi_{m,m}^2(z;p)) \cdot z \, dz > 0$$

$$\lambda''_{m,m}(p) = \int_0^{+\infty} \left[2 \cdot \Psi_{m,m}^2(z;p) + 4 \cdot (a(z)+p) \cdot \Psi'_{m,m}(z;p) \cdot \Psi_{m,m}(z;p) \right] \cdot z \, dz$$

on pose $\Psi_{m,m}(z;p) = \Psi$

$$\lambda''_{m,m}(p) = \int_0^{+\infty} \left[2 \cdot \Psi^2 + 4(a(z)+p) \Psi' \Psi \right] \cdot z \, dz > 0$$

$$2 \int_0^{+\infty} \Psi^2 \cdot z \, dz > - \int_0^{+\infty} 4(a(z)+p) (\partial_p \Psi) \cdot \Psi \cdot z \, dz$$

$$2 \int_0^{+\infty} \Psi^2 \cdot z \, dz + \int_0^{+\infty} 4a(z) (\partial_p \Psi) \cdot \Psi \cdot z \, dz > -4p \int_0^{+\infty} (\partial_p \Psi) \cdot \Psi \cdot z \, dz$$

* CONDITION SUR P :

$$p > \frac{-\frac{1}{2} \int_0^{+\infty} \Psi_{m,m}^2(z;p) \cdot z \, dz - \int_0^{+\infty} a(z) (\partial_p \Psi_{m,m}(z;p)) \cdot \Psi_{m,m}(z;p) \cdot z \, dz}{\int_0^{+\infty} (\partial_p \Psi_{m,m}(z;p)) \cdot \Psi_{m,m}(z;p) \cdot z \, dz}$$

* CONDITION POUR AVOIR $\lambda'_{m,m}(p) \geq 0$ } D'après le corollaire 4-4
page 18 article YATAEV

* corollaire 4-4 : Si $b'(r) \leq 0$ et $r^2 b(r) v'(r) \leq 4 m^2$ pour tout $r \geq 0$
alors $\lambda'_{m,m}(p) \geq 0$, pour tout $p \in \mathbb{R}$ et m . Si de plus, une de ces
inégalités est stricte sur un quelconque intervalle alors $\lambda'_{m,m}(p) > 0$

* Dans notre cas il nous faut $\lambda'_{m,m}(p) \geq 0$ sachant $b_0 > 0$
On a $b(r) = a'(r)$ } $\delta \in [0, 1]$

Soient $\left\{ \begin{array}{l} a(r) = b_0 \ln r \\ a(r) = \frac{b_0}{-\delta+1} r^{-\delta+1} \\ a(r) = b_0 \cdot r \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b(r) = b_0 \cdot r^{-1} \\ b(r) = b_0 \cdot r^{-\delta} \\ b(r) = b_0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b'(r) = -b_0 \cdot r^{-2} \\ b'(r) = -\delta b_0 \cdot r^{-\delta-1} \\ b'(r) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b'(r) < 0 \text{ ①} \\ b'(r) \leq 0 \text{ ② si } \delta = 0 \\ b'(r) = 0 \text{ ③} \end{array} \right.$

pour le 3^{ème} cas $b'(r) = 0 \Rightarrow b'(r) \leq 0 \Rightarrow \lambda'_{m,m}(p) \geq 0$
pour le 2^{ème} cas $b'(r) < 0$ sauf si $\delta = 0 \Rightarrow b'(r) = 0$, on retombe sur le }
pour le 1^{er} cas $b'(r) < 0 \Rightarrow \lambda'_{m,m}(p) > 0$ } 3^{ème} cas

$\lambda'_{m,m}(p) \geq 0$ pour $a(r) = b_0 \cdot r$ et $b(r) = b_0$

* D'après corollaire 4-4 : $b'(r) \leq 0$ } \Rightarrow
 $r^2 b(r) v'(r) \leq 4 m^2 \Rightarrow r^2 b(r) (r (r^{-1} (r b(r)^{-1})'))' \leq 4 m^2$ } $\lambda'_{m,m}(p) \geq 0$

* Application du calcul pour $b(r) = b_0$

$$v(r) = r (r^{-1} (r b(r)^{-1})')' = r (r^{-1} (r b_0^{-1})')' = r (r^{-1} b_0^{-1})'$$

$$v(r) = r (-b_0^{-1} r^{-2}) = -b_0^{-1} r^{-1} \Rightarrow v'(r) = \frac{b_0^{-1}}{r^2}$$

$$r^2 b(r) \cdot r'(r) = r^2 b_0 \cdot \frac{b_0^{-1}}{r^2} = 1 \Rightarrow \text{on obtient donc } \boxed{1 \leq 4 m^2}$$

$$m^2 - \frac{1}{4} \geq 0$$

$$\boxed{-\frac{1}{2} \geq m \geq \frac{1}{2}} \text{ pour } b(r) = b_0 \text{ et } a(r) = b_0 \cdot r$$

* CONDITION POUR AVOIR $\lambda'_{m,m}(p) = 0$

D'après page 17 de l'article de YAFAEV (4-12)

$$\lambda'_{m,m}(p) = \int_0^{+\infty} \partial_p ((a(r)+p)^2) \Psi_{m,m}^2(r;p) r dr$$

$$\lambda'_{m,m}(p) = \int_0^{+\infty} 2(a(r)+p) \cdot \Psi_{m,m}^2(r;p) r dr$$

$$\lambda'_{m,m}(p) = 0 \quad \int_0^{+\infty} a(r) \cdot r \cdot \Psi_{m,m}^2(r;p) dr = \int_0^{+\infty} p \cdot r \cdot \Psi_{m,m}^2(r;p) dr$$

$$p = - \int_0^{+\infty} a(r) \cdot r \cdot \Psi_{m,m}^2(r;p) dr / \int_0^{+\infty} r \cdot \Psi_{m,m}^2(r;p) dr$$

$$p = \frac{- \int_0^{+\infty} a(r) \cdot r \cdot \Psi_{m,m}^2(r;p) dr}{\int_0^{+\infty} r \cdot \Psi_{m,m}^2(r;p) dr}$$