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# Planar products of linearized polynomials

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**Abstract.** Let  $L_1(x)$  and  $L_2(x)$  be linearized polynomials over  $\mathbb{F}_{q^n}$ . We determine conditions when the product  $L_1(x) \cdot L_2(x)$  is a planar mapping on  $\mathbb{F}_{q^n}$ . For a linearized polynomial  $L$  over  $\mathbb{F}_{q^n}$ , let  $\mathcal{M}(L) = \{\alpha \in \mathbb{F}_{q^n} : L(x) + \alpha \cdot x \text{ is bijective on } \mathbb{F}_{q^n}\}$ . We show that the planarity of the product  $L_1(x) \cdot L_2(x)$  is linked with the set  $\mathcal{M}(L)$  of a suitable linearized polynomial  $L$ . We use this relation to describe some families of such planar mappings and we give some nonexistence results.

**Keywords:** planar mapping, quadratic mapping, Dembowski-Ostrom polynomial, linearized polynomial, directions defined by linear functions

## 1 Introduction

Let  $p$  be an odd prime number and  $q$  a power of  $p$ . Given a mapping  $f : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  and a nonzero element  $a \in \mathbb{F}_{q^n}$ , we call

$$D_{f,a} : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}, \quad x \mapsto f(x+a) - f(x) - f(a)$$

the difference mapping of  $f$  defined by  $a$ . A mapping  $f : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  is called planar (or perfect nonlinear) if all its difference mappings are bijective. Planar mappings were introduced in [7] as a tool to construct projective spaces. In Cryptology planar mappings are called perfectly nonlinear, and they provide the optimal resistance to differential attacks [16]. In [8], [9], planar mappings are used to construct optimal constant-composition codes and signal sets. Planar mappings do not exist in finite fields of even characteristic. A mapping  $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$  is called almost perfect nonlinear (APN) if every difference mapping of it has an image set of the maximal possible cardinality  $2^{n-1}$ .

The  $q$ -weight of a nonnegative integer  $m$  is the sum of the digits in its  $q$ -adic representation, i.e. if  $m = \sum_i b_i q^i$  then the  $q$ -ary weight of  $m$  is  $\sum_i b_i \in \mathbb{Z}$ . Recall, that any mapping of  $\mathbb{F}_{q^n}$  can be represented by a polynomial over  $\mathbb{F}_{q^n}$  of degree less than  $q^n$ . Moreover, different such polynomials define different mappings. This allows us to identify the set of mappings of  $\mathbb{F}_{q^n}$  with the set of polynomials over

$\mathbb{F}_{q^n}$  with degree less than  $q^n$ . We use the notation  $F(X)$  to denote a polynomial, while  $F(x)$  is used for the mapping induced by  $F(X)$ . The algebraic  $q$ -degree of a polynomial over  $\mathbb{F}_{q^n}$  is the maximal  $q$ -weight of the exponents in its terms. We use briefly the term algebraic degree, since  $q$  is fixed all over the paper.

The  $\mathbb{F}_q$ -linear mappings  $L : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  are represented by the polynomials of algebraic degree 1 with the zero constant term, that is  $L(x) = \sum_{i=0}^{n-1} c_i x^{q^i}$ ,  $c_i \in \mathbb{F}_{q^n}$ . Such polynomials are called linearized or  $q$ -polynomials. The affine mappings are given by the polynomials of algebraic degree 1.

Two mappings  $F, G : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  are called extended affine equivalent (EA-equivalent), if  $G = A_1 \circ F \circ A_2 + A$  for some affine permutations  $A_1, A_2$  and an affine mapping  $A$ . EA-equivalent non-constant mappings have the same algebraic degree. It is easy to see that EA-equivalence preserves the planarity of mappings.

A mapping is called quadratic if it is represented by a polynomial of algebraic degree 2. The polynomials of algebraic degree 2 with no terms of algebraic degree 1,

$$\sum_{i,j=0}^{n-1} a_{i,j} x^{q^i+q^j}, \quad a_{i,j} \in \mathbb{F}_{q^n},$$

are called Dembowski-Ostrom polynomials in [6]. Observe that any quadratic mapping is EA-equivalent to a one represented by a Dembowski-Ostrom polynomial. Dembowski-Ostrom planar polynomials describe finite commutative semifields and vice-versa [7, 5].

In this paper we continue the study of products of linearized polynomials. To our knowledge such polynomials were first considered in [2]. In [2, 12] bijectiveness of these polynomials is studied. APN products of linearized polynomials are considered in [3], where it is shown that if  $X \cdot L(X)$  is APN with a linearized polynomial  $L(X)$  then  $L(X)$  must be a monomial. In this paper we investigate the planar products of linearized polynomials. In particular we observe that to contrary to the APN case there are planar polynomials  $X \cdot L(X)$  with a non-monomial linearized polynomial  $L(X)$ .

## 2 Planar products of linearized polynomials

**Proposition 1.** *Let  $L_1, L_2 \in \mathbb{F}_{q^n}[X]$  be linearized polynomials. If the mapping  $L_1 \cdot L_2 : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  is planar, then necessarily the mappings  $L_1$  and  $L_2$  are bijective on  $\mathbb{F}_{q^n}$ .*

*Proof.* Observe that  $L_1(X) \cdot L_2(X)$  is a Dembowski-Ostrom polynomial. In [14, 18] it is shown that the only zero of a planar Dembowski-Ostrom polynomial is  $X = 0$ , which yields the statement.  $\square$

The next proposition shows that the study of planar polynomials of shape  $L_1(X) \cdot L_2(X)$  can be reduced to the one of shape  $X \cdot L(X)$ .

**Proposition 2.** *Let  $L_1, L_2 : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  be  $\mathbb{F}_q$ -linear. Then  $L_1(x) \cdot L_2(x)$  is planar on  $\mathbb{F}_{q^n}$  if and only if  $x \cdot L_1(L_2^{-1}(x))$  is planar on  $\mathbb{F}_{q^n}$ , where  $L_2^{-1}(x)$  is the inverse mapping of  $L_2$ .*

*Proof.* The mappings  $L_1(x) \cdot L_2(x)$  is EA-equivalent to  $x \cdot L_1(L_2^{-1}(x))$ , since

$$(L_1(x) \cdot L_2(x)) \circ L_2^{-1}(x) = L_1(L_2^{-1}(x)) \cdot L_2(L_2^{-1}(x)) = L_1(L_2^{-1}(x)) \cdot x. \quad \square$$

For any linearized polynomial  $L(X)$  over  $\mathbb{F}_{q^n}$  we define the polynomial  $Q_L$  by

$$Q_L(X) := \frac{L(X)}{X}.$$

**Lemma 1.** *Let  $L(X) \in \mathbb{F}_{q^n}[X]$  be a linearized permutation polynomial. The mapping  $x \cdot L(x)$  is planar on  $\mathbb{F}_{q^n}$  if and only if, for any non-zero  $\beta \in \mathbb{F}_{q^n}$ , at most one of  $\beta$  or  $-\beta$  belongs to the set  $\{Q_L(x) : x \in \mathbb{F}_{q^n}^*\}$ .*

*Proof.* Consider

$$D_a(x) = (x+a)L(x+a) - xL(x) - aL(a) = aL(x) + xL(a).$$

Since  $D_a(x)$  is a linear mapping, it is bijective if and only if it has a trivial kernel. It remains to note that the condition  $aL(x) + xL(a) = 0$  if and only if  $x = 0$  is equivalent to

$$\frac{L(x)}{x} \neq -\frac{L(a)}{a}$$

for any non-zero  $a, x \in \mathbb{F}_{q^n}$ .  $\square$

Further we show that the planarity of  $X \cdot L(X)$  is linked with the set

$$\mathcal{M}(L) = \{\alpha \in \mathbb{F}_{q^n} : L(x) + \alpha \cdot x \text{ is bijective on } \mathbb{F}_{q^n}\}.$$

**Theorem 1.** *Let  $L(X) \in \mathbb{F}_{q^n}[X]$  be linearized. The mapping  $x \cdot L(x)$  is planar on  $\mathbb{F}_{q^n}$  if and only if  $0 \in \mathcal{M}(L)$  and at least one of  $\beta$  or  $-\beta$  belongs to the set  $\mathcal{M}(L)$  for any non-zero  $\beta \in \mathbb{F}_{q^n}$ .*

*Proof.* The condition  $0 \in \mathcal{M}(L)$  states that  $L(x)$  is bijective on  $\mathbb{F}_{q^n}$ . Using Lemma 1, for any non-zero  $\beta \in \mathbb{F}_{q^n}$  it must hold that either  $\beta$  or  $-\beta$  is not contained in  $\{Q_L(x) : x \in \mathbb{F}_{q^n}^*\}$ . The proof now follows from the observation that  $\beta \notin \{Q_L(x) : x \in \mathbb{F}_{q^n}^*\}$  if and only if  $L(x) - \beta \cdot x$  is bijective on  $\mathbb{F}_{q^n}$ .  $\square$

Note that the monomial planar mappings  $x^2$  and  $x^{q^i+1}$  on  $\mathbb{F}_{q^n}$  with  $n/\gcd(n, i)$  odd are covered by Theorem 1. The conditions of Theorem 1 are trivially fulfilled for  $x^2$ . In the case  $x \cdot x^{q^i}$  it is easy to see, that there is no  $\beta$  such that both  $x^{q^i} + \beta x$  and  $x^{q^i} - \beta x$  are not bijective on  $\mathbb{F}_{q^n}$  if and only if  $-1$  is not a  $(q^i - 1)$ th power in  $\mathbb{F}_{q^n}$ , or equivalently  $n/\gcd(n, i)$  odd.

In [3] it is shown that if  $X \cdot L(X)$  is APN with a linearized polynomial  $L(X)$  then  $L(X)$  must be a monomial. Does a similar result hold for planar products  $X \cdot L(X)$ ? The answer is negative as the following observation shows.

**Proposition 3.** *Let  $1 \leq k \leq n - 1$  with  $n/\gcd(n, k)$  odd and let  $u \in \mathbb{F}_{q^n}$  be not a  $(q^k - 1)$ st power in  $\mathbb{F}_{q^n}$ . Then the mapping  $F(x) = x \cdot (x^{q^{n-k}} - ux^{q^k})$  is planar on  $\mathbb{F}_{q^n}$ .*

*Proof.* We show that  $F(x)$  is EA-equivalent to the planar monomial  $x^{q^k+1}$ . Indeed,

$$F(x) = x^{q^{n-k}+1} - ux^{q^k+1} = (x^{q^{n-k}} - ux) \circ x^{q^k+1},$$

and the assumptions on  $k$  and  $u$  ensure that  $x^{q^{n-k}} - ux$  is bijective and  $x^{q^k+1}$  is planar.

Our next goal is to describe further families of planar products  $X \cdot L(X)$  using Theorem 1. To our knowledge the only linearized polynomials  $L$  for which the sets  $\mathcal{M}(L)$  are known are  $X$ ,  $X^{q^k} - uX$ ,  $Tr(X) = X + X^q + \dots + X^{q^{n-1}}$  or some natural transformations of these polynomials.

**Open Problem.** Find new families of linearized permutation polynomials  $L$  with  $|\{\beta, -\beta\} \cap \mathcal{M}(L)| \geq 1$  for any  $\beta \in \mathbb{F}_{q^n}$ .

The next result was proved in [10, 11] by different approaches. In [10] it is shown that the mappings of Theorem 2 can be obtained from the mapping  $x \mapsto x^2$  via EA-equivalence. We show that this result can be proved by using Theorem 1 as well.

**Theorem 2.** *Let  $u \in \mathbb{F}_{q^2}$ . The mapping  $F(x) = x^2 + ux^{q+1}$  is planar on  $\mathbb{F}_{q^2}$  if and only if  $1 - u^{q+1}$  is a square in the subfield  $\mathbb{F}_q$ .*

*Proof.* The statement is true for  $u = 0$ , so let  $u \neq 0$ . By Proposition 1 if  $F(x)$  is planar then  $x + ux^q$  is bijective on  $\mathbb{F}_{q^2}$ , which is equivalent to  $u^{q+1} \neq 1$ . Further we apply Theorem 1 with  $L(x) = x + ux^q$ . Let  $\beta \in \mathbb{F}_{q^2}$  be such that both  $L(x) + \beta x = ux^q + (1 + \beta)x$  and  $L(x) - \beta x = ux^q + (1 - \beta)x$  are not bijective on  $\mathbb{F}_{q^2}$ . This is the case if and only if the elements  $-(1 + \beta)/u$  and  $-(1 - \beta)/u$  are  $(q - 1)$ st powers in  $\mathbb{F}_{q^2}$ , or equivalently if it holds:

$$\left(\frac{\beta + 1}{u}\right)^{q+1} = \left(\frac{\beta - 1}{u}\right)^{q+1} = 1.$$

In particular, it must hold  $(\beta + 1)^{q+1} = (\beta - 1)^{q+1}$ , which implies  $\beta^q = -\beta$ . Hence,

$$\left(\frac{\beta + 1}{u}\right)^{q+1} = \frac{-\beta^2 - \beta + \beta + 1}{u^{q+1}} = 1,$$

which yields

$$\beta^2 = -u^{q+1} + 1.$$

Note that  $1 - u^{q+1}$  is an element in the subfield  $\mathbb{F}_q$ . Moreover  $\beta \notin \mathbb{F}_q$ , since  $\beta = -\beta^q$ . Hence,  $F(x)$  is planar on  $\mathbb{F}_{q^2}$  only if  $1 - u^{q+1}$  is a square in  $\mathbb{F}_q$ . Finally, we show that this condition is necessary as well. Indeed, let  $1 - u^{q+1} \neq 0$

be a non-square in  $\mathbb{F}_q$ . Suppose,  $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  satisfies  $\beta^2 = -u^{q+1} + 1$ . Then necessarily  $\beta^q = -\beta$ , since  $\beta$  and  $-\beta$  are the zeros of the irreducible polynomial  $X^2 - (1 - u^{q+1})$  in  $\mathbb{F}_{q^2}$  and therefor  $-\beta$  is the conjugate of  $\beta$  over  $\mathbb{F}_q$ . To complete the proof it remains to note that for this  $\beta$  it holds

$$\left(\frac{\beta + 1}{u}\right)^{q+1} = \left(\frac{\beta - 1}{u}\right)^{q+1} = 1.$$

□

In the remaining part of the paper we consider the case  $L(X) = Tr(X)$ , where  $Tr(X)$  is the polynomial describing the trace mapping from  $\mathbb{F}_{q^n}$  onto  $\mathbb{F}_q$ .

**Proposition 4.** [1, 13] For the trace mapping  $Tr$  from  $\mathbb{F}_{q^n}$  onto  $\mathbb{F}_q$  it holds  $\mathcal{M}(Tr) = \{\gamma \in \mathbb{F}_{q^n}^* : Tr(\gamma^{-1}) \neq -1\}$ .

**Theorem 3.** Let  $a \in \mathbb{F}_{q^n}^*$  with  $Tr((2a)^{-1}) \neq -1$ . Then  $F(X) = X \cdot (Tr(X) + aX)$  is planar on  $\mathbb{F}_{q^n}$  if and only if there is no  $z \in \mathbb{F}_{q^n}$  such that  $Tr(z^{-1}) = -1$  and  $Tr((z - 2a)^{-1}) = 1$ .

*Proof.* By Proposition 4 it holds

$$\mathcal{M} := \mathcal{M}(Tr(x) + ax) = \{\alpha \in \mathbb{F}_{q^n} : \alpha \neq -a \text{ and } Tr((a + \alpha)^{-1}) \neq -1\}.$$

By Theorem 1,  $F(x)$  is planar if and only if at least one of  $\beta$  or  $-\beta$  belongs to  $\mathcal{M}$  for any non-zero  $\beta \in \mathbb{F}_{q^n}$ . First we consider the case that  $\beta \in \{-a, a\}$ . Note that  $a \in \mathcal{M}$  as  $a \neq -a$  and  $Tr((2a)^{-1}) \neq -1$ . Hence if  $\beta \in \{-a, a\}$ , then either  $\beta$  or  $-\beta$  is contained in  $\mathcal{M}$ . It remains to show that if  $\beta \in \mathbb{F}_{q^n} \setminus \{0, -a, a\}$ , then either  $\beta$  or  $-\beta$  is contained in  $\mathcal{M}$ . Assume the contrary and hence we have  $\beta \in \mathbb{F}_{q^n} \setminus \{0, -a, a\}$  with  $Tr((a + \beta)^{-1}) = -1$  and  $Tr((a - \beta)^{-1}) = -1$ . Putting  $z = a + \beta$  we obtain  $z \in \mathbb{F}_{q^n}$  with  $a - \beta = -(z - 2a)$ ,  $Tr(z^{-1}) = -1$  and  $Tr((z - 2a)^{-1}) = -Tr((a - \beta)^{-1}) = 1$ . Using the condition in the statement we complete the proof. □

*Remark 1.* If  $a = 0$  in Theorem 3, then  $F(x) = x \cdot (Tr(x))$  is not planar by Proposition 1, since  $Tr(x)$  is not bijective. Further let  $a \in \mathbb{F}_{q^n}^*$  with  $Tr((2a)^{-1}) = -1$ . Then  $a \notin \mathcal{M}(Tr(x) + ax)$  and  $-a \notin \mathcal{M}(Tr(x) + ax)$ . Hence also for such an  $a$  the mapping  $F(x) = x \cdot (Tr(x) + ax)$  is not planar (see Theorem 1).

Corollaries 1 and 2 show, that the conditions of Theorem 3 are satisfied for  $n = 3$  and  $a = -1, -2$ .

**Corollary 1.** The mapping  $F(x) = x^{q^2+1} + x^{q+1}$  is planar on  $\mathbb{F}_{q^3}$ . Equivalently, there is no  $z \in \mathbb{F}_{q^3}$  satisfying  $Tr\left(\frac{1}{z}\right) = -1$  and  $Tr\left(\frac{1}{z+2}\right) = 1$ .

*Proof.* The statement follows from Proposition 3 and Theorem 3 □

**Corollary 2.** The mapping  $G(x) = x^{q^2+1} + x^{q+1} - x^2$  is planar on  $\mathbb{F}_{q^3}$ .

*Proof.* Note that  $G(x) = x(\text{Tr}(x) - 2x)$  and hence we can use Theorem 3 with  $a = -2$ . Suppose there is  $z \in \mathbb{F}_{q^3}$  with

$$\text{Tr}\left(\frac{1}{z}\right) = -1 \quad \text{and} \quad \text{Tr}\left(\frac{1}{z+4}\right) = 1. \quad (1)$$

Observe that then  $z \notin \mathbb{F}_q$ . Indeed, for  $z \in \mathbb{F}_q$  condition (1)

$$\text{Tr}\left(\frac{1}{z}\right) = \frac{3}{z} = -1 \quad \text{and} \quad \text{Tr}\left(\frac{1}{z+4}\right) = \frac{3}{z+4} = 1.$$

reduces to  $z = -3$  and  $z = -1$ , which cannot be satisfied. Let  $z \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$  which satisfies (1). As in the proof of Corollary 1 already shown the minimal polynomial of  $z$  over  $\mathbb{F}_q$  is of the following shape

$$M_z(X) = X^3 + a_2X^2 + bX + b.$$

The minimal polynomial of  $z+4$  is  $M_{z+4}(X) = M_z(X-4)$ , and hence

$$\begin{aligned} M_{z+4}(X) &= (X-4)^3 + a_2(X-4)^2 + b(X-4) + b \\ &= X^3 + (a_2 - 12)X^2 + (48 - 8a_2 + b)X - 64 + 16a_2 - 3b. \end{aligned}$$

To ensure  $\text{Tr}((z+4)^{-1}) = 1$  it must hold

$$48 - 8a_2 + b = 64 - 16a_2 + 3b,$$

and hence  $b = 4a_2 - 8$ . Substituting this value of  $b$  in  $M_z(X)$ , we get

$$M_z(X) = X^3 + a_2X^2 + (4a_2 - 8)X + 4a_2 - 8,$$

a contradiction, since  $X = -2$  is a zero of the latter polynomial.  $\square$

*Remark 2.* Observe that the planar mappings described in Corollaries 1 and 2 are EA-equivalent to the monomial planar mapping  $x^{q+1}$ . Indeed, for  $F(x)$  this follows from the proof of Proposition 3, and in the case of  $G(x)$  from the following equality

$$\left(x^{q^2} + x^q - x\right)^{q+1} = \left(-x^{q^2} + x^q + x\right) \circ G(x).$$

In the following theorem we show that if  $n$  is sufficiently large, then there is no  $a \in \mathbb{F}_{q^n}$  such that  $F(X) = X \cdot (\text{Tr}(X) + aX)$  is planar on  $\mathbb{F}_{q^n}$ . This is a result obtained using Theorem 3 and Remark 1, which is in the opposite direction of the existence results given above.

**Theorem 4.** *Let  $a \in \mathbb{F}_{q^n}$  and  $F(X) = X \cdot (\text{Tr}(X) + aX)$ . If one the following conditions hold, then  $F(X)$  is not planar on  $\mathbb{F}_{q^n}$ :*

- $q \geq 23$  and  $n \geq 12$ ,
- $q \in \{11, 13, 17, 19\}$  and  $n \geq 13$ ,

- $q = 9$  and  $n \geq 14$ ,
- $q = 7$  and  $n \geq 15$ .
- $q = 5$  and  $n \geq 16$ .
- $q = 3$  and  $n \geq 19$ .

*Proof.* Using Remark 1, we can assume that  $a \neq 0$ . Let  $f_1(z)$  and  $f_2(z)$  be the rational functions in  $\mathbb{F}_{q^n}(z)$  given by

$$f_1(z) = \frac{1}{z} \text{ and } f_2(z) = \frac{1}{z - 2a}.$$

We use Theorem 1.1 from [4]. Let  $p$  be the characteristic of  $\mathbb{F}_{q^n}$ . Recall that the set  $\{f_1(z), f_2(z)\}$  is *strongly linearly independent over  $\mathbb{F}_q$*  (cf. [4]) if there is no  $a_1, a_2, \epsilon \in \mathbb{F}_{q^n}$  and  $h(z) \in \mathbb{F}_{q^n}(z)$  such that  $(a_1, a_2) \neq (0, 0)$  and

$$a_1 f_1(z) + a_2 f_2(z) = h(z)^p - h(z) + \epsilon.$$

We use some simple facts from the theory of algebraic function fields (cf. [17] or [15]). Let  $P_1$  be the place of  $\mathbb{F}_{q^n}(z)$  corresponding to the zero of  $z$  and let  $\nu_{P_1}$  denote the normalized discrete valuation corresponding to  $P_1$ .

Assume that  $a_1 \neq 0$ . Then

$$\nu_{P_1}(h(z)^p - h(z) + \epsilon) = \nu_{P_1}(a_1 f_1(z) + a_2 f_2(z)) = \nu_{P_1}(f_1(z)) = -1. \quad (2)$$

Hence  $\nu_{P_1}(h(z)) < 0$ , indeed otherwise, using the triangle inequality of valuations (cf. [17, Definition 1.1.9]) we get

$$\nu_{P_1}(h(z)^p - h(z) + \epsilon) \geq \min\{\nu_{P_1}(h(z)^p), \nu_{P_1}(h(z)), \nu_{P_1}(\epsilon)\} \geq 0,$$

which is a contradiction to (2). As  $\nu_{P_1}(h(z)) < 0$ , we have  $\nu_{P_1}(\epsilon) > \nu_{P_1}(h(z)) > \nu_{P_1}(h(z)^p)$ . Then using the strict triangle inequality of valuations (cf. [17, Lemma 1.1.11]) we obtain that  $\nu_{P_1}(h(z)^p - h(z) + \epsilon) = p\nu_{P_1}(h(z)) = p\ell$ , where  $\ell = \nu_{P_1}(h(z))$  is a negative integer. This is again a contradiction to (2). Therefore we get that  $a_1 = 0$ . Similarly we show that  $a_2 = 0$ . This proves that the set  $\{f_1(z), f_2(z)\}$  is strongly linearly independent over  $\mathbb{F}_q$ .

Under the notation of [4, Theorem 1.1] we have  $r = 2$ ,  $\deg f_1(z) = \deg f_2(z) = 1 = m$ ,  $t_1 = -1$ ,  $t_2 = 1$ ,  $\ell = 1$ . Then there exists a primitive element  $\gamma \in \mathbb{F}_{q^n}$  such that  $Tr(\gamma^{-1}) = -1$  and  $Tr((\gamma - 2a)^{-1}) = 1$  if

$$n > 4(2 + \log_q(9.8 \cdot 1 \cdot 2 \cdot 1)), \quad (3)$$

where  $\log_q$  is the logarithm with respect to base  $q$ . It is easy to check the (3) is satisfied if one of the conditions in the statement of the theorem holds. For example if  $q \geq 23$ , then  $4(2 + \log_q(9.8 \cdot 2)) \leq 4(2 + \log_{23}(9.8 \cdot 2)) = 11.7959 \dots$ . This completes the proof.  $\square$

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