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UNIQUE DECOMPOSITION FOR A POLYNOMIAL OF LOW RANK

EDOARDO BALLICO, ALESSANDRA BERNARDI

ABSTRACT

Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the s -th secant variety of the d -uple Veronese embedding of \mathbb{P}^m into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d -th powers of linear forms requires more than s addenda. We show that if $s \leq d$ then F can be uniquely written as $F = M_1^d + \dots + M_t^d + Q$, where M_1, \dots, M_t are linear forms with $t \leq (d-1)/2$, and Q a binary form such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i 's linear forms and m_i 's forms of degree d_i such that $\sum (d_i + 1) = s - t$.

INTRODUCTION

In this paper we will always work with an algebraically closed field K of characteristic 0. Let $X_{m,d} \subset \mathbb{P}^N$, with $m \geq 1$, $d \geq 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d -uple Veronese embedding $\nu_d : \mathbb{P}^m \rightarrow \mathbb{P}^N$. The s -th secant variety $\sigma_s(X_{m,d})$ of the Veronese variety $X_{m,d}$ is the Zariski closure in \mathbb{P}^N of the union of all linear spans $\langle P_1, \dots, P_s \rangle$ with $P_1, \dots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\text{sbr}(P) = s$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P . Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V^*)$, with V an $(m+1)$ -dimensional vector space over K , the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (a symmetric tensor) of type:

$$(1) \quad F = L_1^d + \dots + L_r^d, \quad (T = v_1^{\otimes d} + \dots + v_r^{\otimes d}).$$

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem that is having a lot of attentions not only from classical Algebraic Geometry ([1], [7], [5], [6], [9]), but also from applications like Computational Complexity ([8]) and Signal Processing ([10]).

At the Workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented the following result:

([2], Corollary 1) Let $F \in K[x_0, \dots, x_m]_d$ be such that $\text{sbr}(F) + \text{sr}(F) \leq 2d+1$ and $\text{sbr}(F) < \text{sr}(F)$. Then there are an integer $t \geq 0$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \dots + M_t^d$, $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$, and $\text{sr}(F) = \text{sr}(Q) + t$. Moreover t , M_1, \dots, M_t and the linear span of L_1, L_2 are uniquely determined by F .

In terms of tensors it can be translated as follows:

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([2], Corollary 2) Let $T \in S^d V^*$ be such that $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$ and $\text{sbr}(T) < \text{sr}(T)$. Then there are an integer $t \geq 0$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$, $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$, and $\text{sr}(T) = \text{sr}(v) + t$. Moreover t, w_1, \dots, w_t and $\langle v_1, v_2 \rangle$ are uniquely determined by T .

The natural questions that arose at that workshop from applied people, were about the possible uniqueness of the binary form Q in [2], Corollary 1 (ie. the vector v in [2], Corollary 2) and a bound on the number t of linear forms (ie. rank 1 symmetric tensors). We are finally able to give the most possible complete answer to this question. The main result of this paper is the following.

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d} - 1$. Suppose that:*

$$\begin{aligned} \text{sbr}(P) &< \text{sr}(P) \text{ and} \\ \text{sbr}(P) + \text{sr}(P) &\leq 2d + 1. \end{aligned}$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P , and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$ and $\deg \mathcal{Z} \leq \text{sbr}(P)$. There is a unique rational normal curve $C_d \subset X_{m,d}$ such that $C_d \cap (\mathcal{S} \cup \mathcal{Z}) \geq d + 2$. Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \quad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.

Moreover C_d, \mathcal{S}_2 and \mathcal{Z} are unique, $\deg(\mathcal{Z}) = \text{sbr}(P)$, $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$, $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ and \mathcal{Z} is the unique zero-dimensional subscheme N of $X_{m,d}$ such that $\deg(N) \leq \text{sbr}(P)$ and $P \in \langle N \rangle$.

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let $F \in K[x_0, \dots, x_m]_d$ be such that $\text{sbr}(F) + \text{sr}(F) \leq 2d + 1$ and $\text{sbr}(F) < \text{sr}(F)$. Then there are an integer $0 \leq t \leq (d - 1)/2$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \dots + M_t^d$, $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$, and $\text{sr}(F) = \text{sr}(Q) + t$.*

Moreover the line $\langle L_1, L_2 \rangle$, the forms M_1, \dots, M_t and Q such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i 's linear forms and m_i 's forms of degree d_i such that $\sum (d_i + 1) = s - t$, are uniquely determined by F .

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let $T \in S^d V^*$ be such that $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$ and $\text{sbr}(T) < \text{sr}(T)$. Then there are an integer $0 \leq t \leq (d - 1)/2$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$, $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$, and $\text{sr}(T) = \text{sr}(v) + t$. Moreover the line $\langle v_1, v_2 \rangle$, the vectors v_1, \dots, v_t and the tensor v such that $v = \sum_{i=1}^q u_i^{\otimes (d-d_i)} \otimes z_i$ with $u_i \in \langle v_1, v_2 \rangle$ and $z_i \in S^{d_i}(\langle v_1, v_2 \rangle)$ such that $\sum (d_i + 1) = s - t$, are uniquely determined by T .*

Moreover in Theorem 2 and in Corollary 4, by introducing the notion of linearly general position of a scheme (Definition 1), we can also extend to a geometric description the condition for the uniqueness of the scheme \mathcal{Z} of Theorem 1. We can rephrase their contents in terms of homogeneous polynomials and symmetric tensors in the following Corollary.

Corollary 3. *Fix integers $m \geq 2$ and $d \geq 4$. Fix F an homogeneous polynomial in $m + 1$ variables of degree d ($T \in S^d V$ respectively) such that $\text{sbr}(F) \leq d$ ($\text{sbr}(T) \leq d$). Let $Z \subset \mathbb{P}^m$ be*

any smoothable zero-dimensional scheme such that $\nu_d(Z)$ computes $\text{sbr}(F)$ ($\text{sbr}(T)$). Assume that Z is in linearly general position. Then Z is the unique scheme computing $\text{sbr}(P)$ ($\text{sbr}(F)$).

1. PROOFS

The existence of such a scheme \mathcal{Z} was known from [3] and [4] (see Remark 1 of [2]).

Lemma 1. *Fix integers $m \geq 2$ and $d \geq 2$, a line $\ell \subset \mathbb{P}^m$ and any finite set $E \subset \mathbb{P}^m \setminus \ell$ such that $\sharp(E) \leq d$. Then $\dim(\langle \nu_d(E) \rangle) = \sharp(E) - 1$ and $\langle \nu_d(\ell) \rangle \cap \langle \nu_d(E) \rangle = \emptyset$.*

Proof. Since $h^0(\ell \cup E, \mathcal{O}_{\ell \cup E}(d)) = d + 1 + \sharp(E)$, to get both statements it is sufficient to prove $h^1(\mathcal{I}_{\ell \cup E}(d)) = 0$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing ℓ . Since E is finite and H is general, we have $H \cap E = \emptyset$. Hence the residual exact sequence of the scheme $\ell \cup E$ with respect to the hyperplane H is the following exact sequence on \mathbb{P}^m :

$$(2) \quad 0 \rightarrow \mathcal{I}_E(d-1) \rightarrow \mathcal{I}_{\ell \cup E}(d) \rightarrow \mathcal{I}_{\ell, H}(d) \rightarrow 0$$

Since $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell, H}(d)) = 0$, we get the lemma. \square

Proof of Theorem 1. All the statements are contained in [2], Theorem 1, except the uniqueness of \mathcal{Z} , that $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ and that $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$. Let $\ell \subset \mathbb{P}^m$ be the line such that $\nu_d(\ell) = C_d$. Take $Z, S, Z_i, S_i \subset \mathbb{P}^m$, $i = 1, 2$, such that $\nu_d(Z) = \mathcal{Z}$, $\nu_d(S) = \mathcal{S}$, $\nu_d(Z_i) = \mathcal{Z}_i$, and $\nu_d(S_i) = \mathcal{S}_i$. Assume the existence of another subscheme $\mathcal{Z}' \subset X_{m,d}$ such that $P \in \langle \nu_d(\mathcal{Z}') \rangle$ and $\deg(\mathcal{Z}') \leq \text{sbr}(P)$. Set $\mathcal{Z}'_1 := \mathcal{Z}' \cap C_d$. The proof of [2], Theorem 1, gives $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup \mathcal{S}_2$. Since C_d is a smooth curve, $\mathcal{Z}_1 \cup \mathcal{Z}'_1 \subset C_d$, $\mathcal{S}_2 \cap \ell = \emptyset$, and $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}'_1) \sqcup \mathcal{S}_2$, the schemes \mathcal{Z} and \mathcal{Z}' are curvilinear. Hence all subschemes of \mathcal{Z} and \mathcal{Z}' are smoothable. Hence any subscheme of either \mathcal{Z} or \mathcal{Z}' may be used to compute the border rank of some point of \mathbb{P}^N . Since $\deg(\ell \cap (Z \cup S)) \geq d + 2$, $\nu_d((Z \cup S) \cap \ell)$ spans $\langle C_d \rangle$. Lemma 1 implies $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$. Since $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$ and $\sharp(\mathcal{S}) = \text{sr}(P)$, we have $P \notin \langle \mathcal{A} \rangle$ for any $\mathcal{A} \subsetneq \mathcal{S}$. Therefore we get that $\langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle \mathcal{S}_1 \rangle$ is a unique point. Call P_1 this point. Similarly, $\langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point and we call it P_2 . Since $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$, the set $\langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ is at most one point. Since $P_i \in \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$, $i = 1, 2$, we have $P_1 = P_2$ and $\{P_1\} = \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$. Since $P_1 = P_2$ and $P_1 \in \langle \mathcal{S}_1 \rangle$ and $P_2 \in \langle \mathcal{Z}'_1 \rangle$, we have $P_1 \in \langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_1 \rangle$. Take any $E \subseteq \mathcal{Z}_1$ such that $P_1 \in \langle E \rangle$. Since $P \in \langle \{P_1\} \cup \mathcal{S}_2 \rangle \subseteq \langle E \cup \mathcal{S}_2 \rangle$ and $P \notin \langle \mathcal{U} \rangle$ for any $\mathcal{U} \subsetneq \mathcal{Z}$, we get $E \cup \mathcal{S}_2 = \mathcal{Z}$. Hence $E = \mathcal{Z}_1$. Therefore \mathcal{Z}_1 computes $\text{sbr}(P_1)$ with respect to C_d . Similarly, \mathcal{Z}'_1 computes $\text{sr}(P_2)$ with respect to the same rational normal curve C_d . Since $P_1 = P_2$, we have $\mathcal{Z}'_1 = \mathcal{Z}_1$ (as for all curves we get the uniqueness of \mathcal{Z}_1 for $\text{sbr}(P_1) \leq \lfloor (d+2)/2 \rfloor$). Since $\text{sbr}(P_1) \neq \text{sr}(P_1)$, a theorem of Sylvester gives $\text{sbr}(P_1) + \text{sr}(P_1) = d + 2$, i.e. $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$.

Definition 1. A scheme $Z \subset \mathbb{P}^m$ is said to be in *linearly general position* if for every linear subspace $R \subsetneq \mathbb{P}^m$ we have $\deg(R \cap Z) \leq \dim(R) + 1$.

Notice that the next theorem is false if either $d = 2$ or $m = 1$.

Theorem 2. *Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $P \in \langle \nu_d(Z) \rangle$ and $P \notin \langle \nu_d(Z') \rangle$. Assume $\deg(Z) \leq d$ and that Z is in linearly general position. Then Z is the unique scheme $Z' \subset \mathbb{P}^m$ such that $\deg(Z') \leq d$ and $P \in \langle \nu_d(Z') \rangle$. Moreover $\nu_d(Z)$ computes $\text{sbr}(P)$.*

Proof. The existence of a scheme computing $\text{sbr}(P)$ follows from [2], Remark ++= and the assumption “ $\text{sbr}(P) \leq d$ ”. Fix any scheme $Z' \subset \mathbb{P}^m$ such that $Z' \neq Z$, $\deg(Z') \leq d$, $P \in \langle \nu_d(Z') \rangle$, and $P \notin \langle \nu_d(Z'') \rangle$ for any $Z'' \subsetneq Z'$. Assume $Z' \neq Z$. Since $\deg(Z \cup Z') \leq 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z \cup Z'}(d)) > 0$ ([2], Lemma 1), there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (Z \cup Z')) \geq d + 2$. Since Z is in linearly general position and $m \geq 2$, we have $\deg(Z \cap D) \leq 2$. Hence $\deg(Z' \cap D) \geq d$.

Hence $\deg(Z') = d$. Since $\deg(Z') = d$, we get $Z' \subset D$. Hence $P \in \langle \nu_d(D) \rangle$. Hence $\text{sr}(P) = d$. As for all curves we get $\text{sbr}(P) \leq \lfloor (d+2)/2 \rfloor$. Since $\deg(Z') = d$, we assumed $\deg(Z') \leq \text{sbr}(P)$, contradicting the assumption $d \geq 4$. \square

Corollary 4. *Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$ such that $\text{sbr}(P) \leq d$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ computes $\text{sbr}(P)$. Assume that Z is in linearly general position. Then Z is the unique scheme computing $\text{sbr}(P)$.*

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