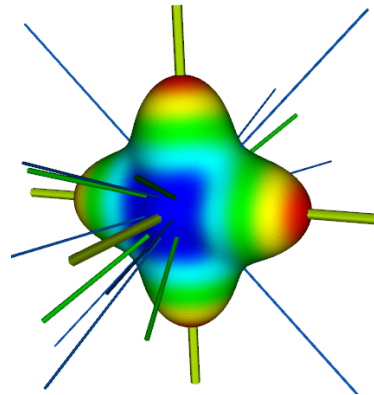
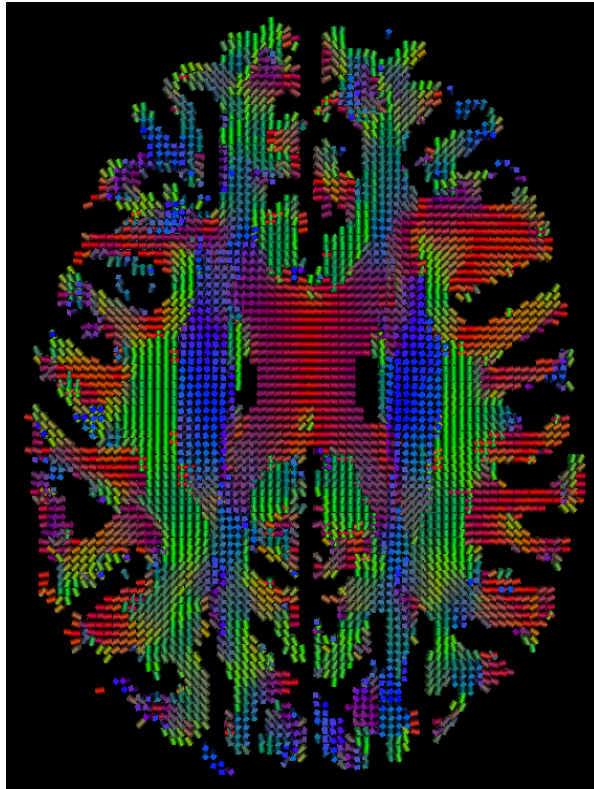


A Polynomial Approach for Maxima Extraction and its Application to Tractography in HARDI

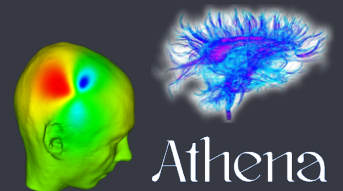


Aurobrata Ghosh⁽¹⁾
Demian Wassermann⁽²⁾
Rachid Deriche⁽¹⁾

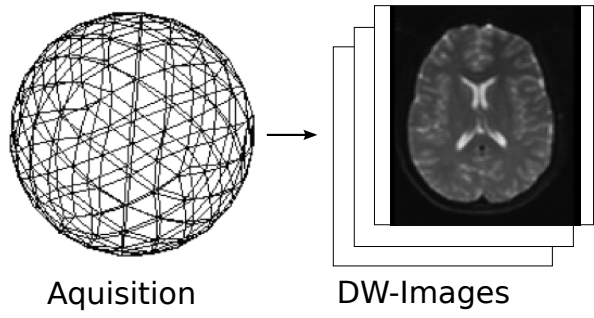
07-07-2011



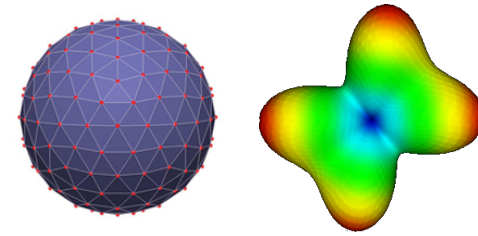
- (1) ATHENA Project Team, INRIA Sophia Antipolis-Méditerranée, France
- (2) Department of Radiology, Harvard Medical School & Brigham and Women's Hospital, Boston, USA



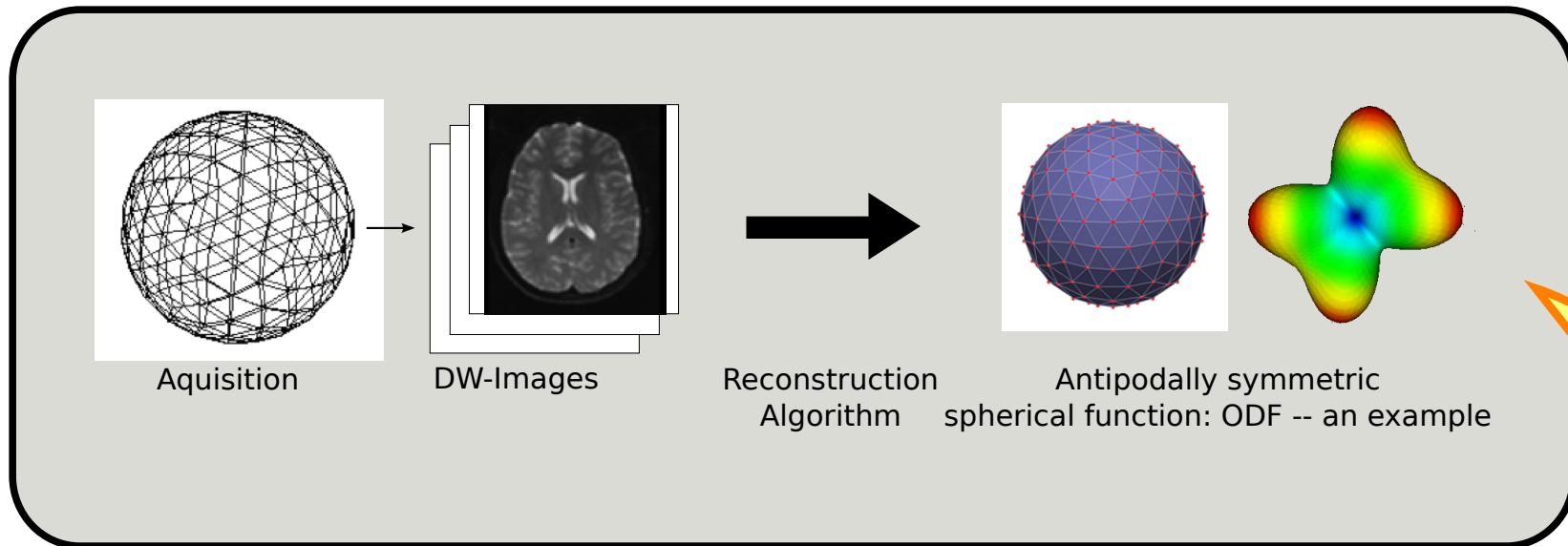
1. Spherical Functions in Diffusion MRI
 - DTI: the Ellipsoid
 - Crossing fibers: Spherical Functions
2. Maxima Extraction
 - A Polynomial Approach
3. An Application: Tractography



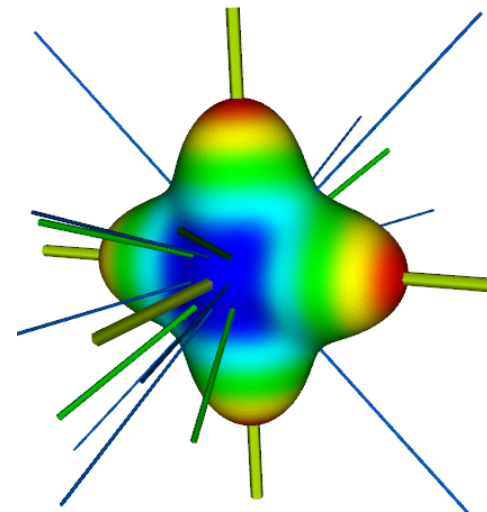
Reconstruction
Algorithm



Antipodally symmetric
spherical function: ODF -- an example



- ★ Maxima extraction from **ANY** spherical function
- ★ **ALL** maxima are found together with *high precision*
- ★ Application: Tractography

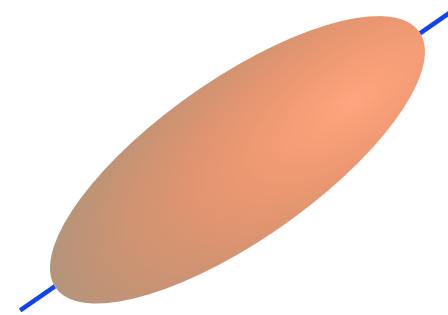


- ★ Spherical Functions play a central role in Diffusion MRI

★ Spherical Functions play a central role in Diffusion MRI

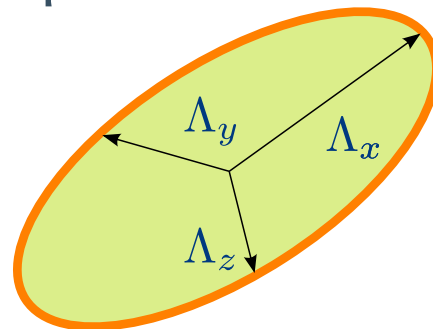
★ DTI Spherical Function: Diffusion Ellipsoid

$$\frac{\mathbf{r}^T \mathbf{D}^{-1} \mathbf{r}}{2t} = 1.$$



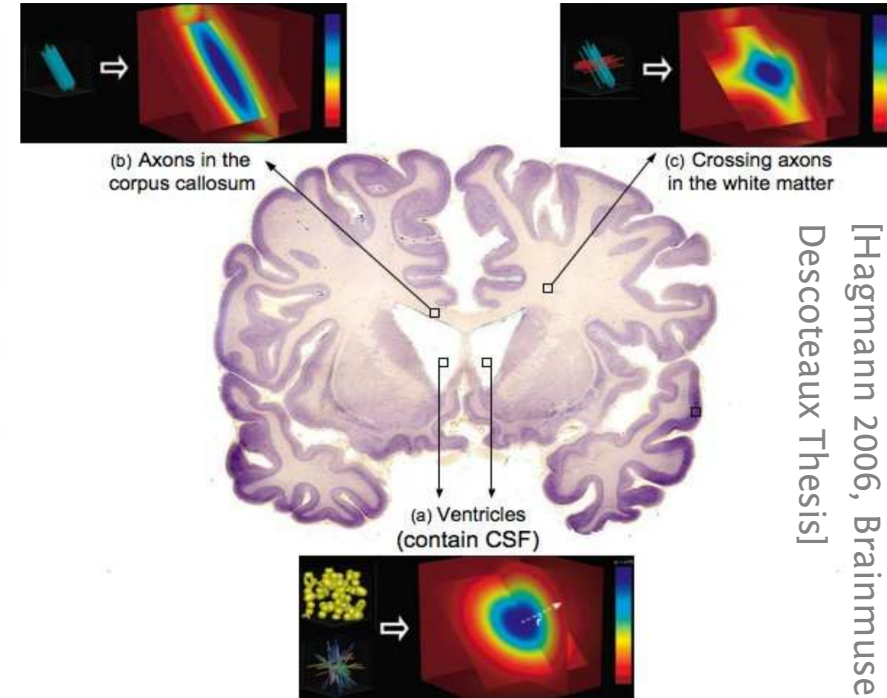
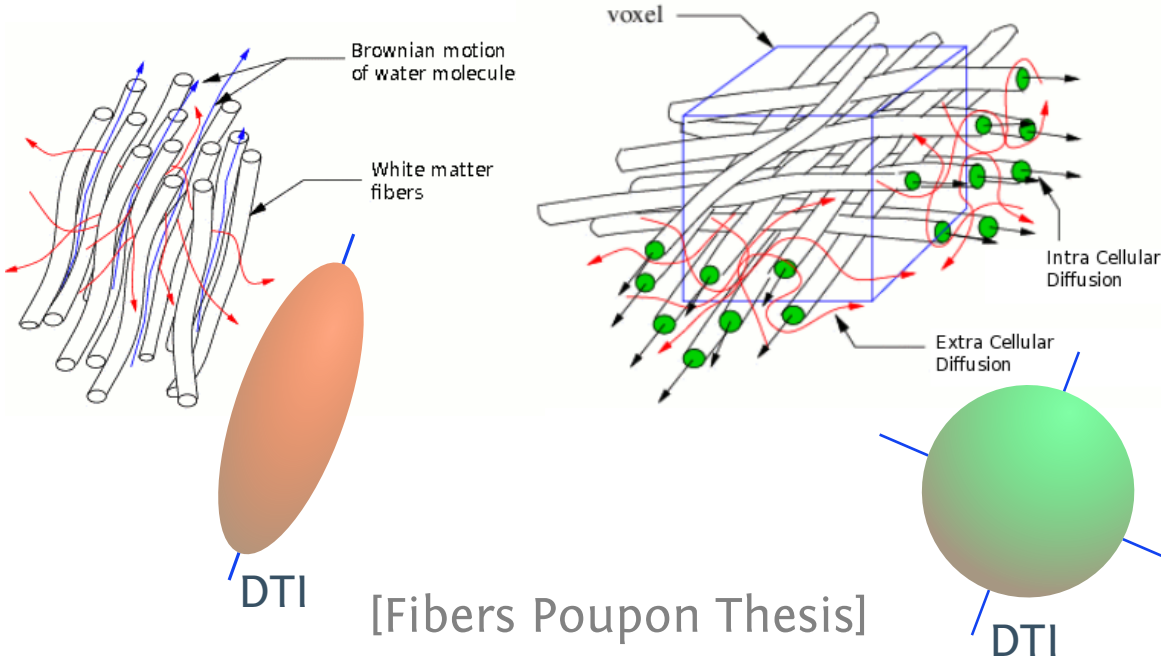
★ Geometric interpretation:

🌀 Spectral Decomposition
(Fast & Easy)



{ *Tractography*
Fractional Anisotropy
Other scalars (biomarkers)

DTI has limitations in regions with fiber-crossings

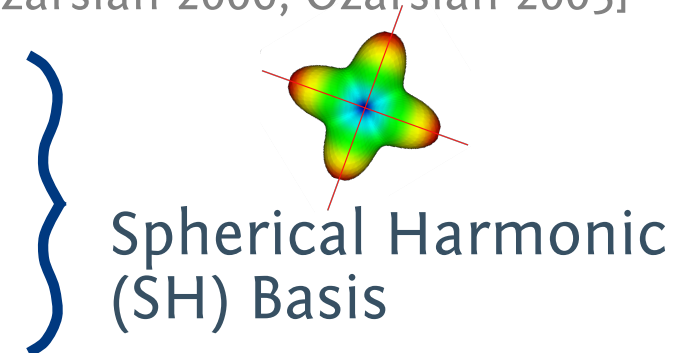


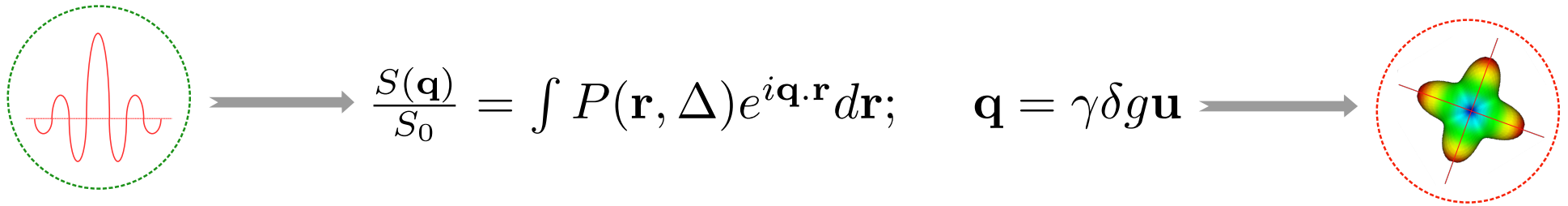
[Hagmann 2006, Brainmuseum Descoteaux Thesis]

★ Other Spherical Functions (SF) in dMRI:

[Tuch Thesis, Tournier 2004, Jansons 2003, Alexander 2005, Ozarslan 2006, Ozarslan 2003]

- ④ Orientation Distribution Function (ODF)
- ④ Fiber Orientation Density (FOD)
- ④ PASMRI/MESD
- ④ DOT, ADC ... etc...





★ **ODF-Tuch:** [Tuch Thesis]

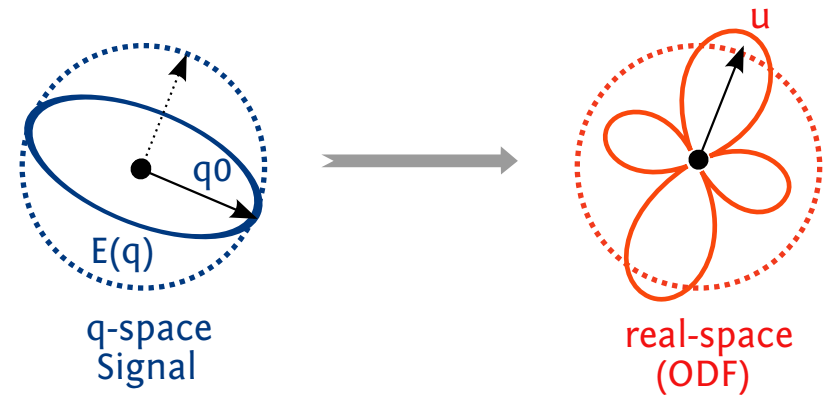
$$\Psi_T(\hat{u}) = \int_0^\infty P(r\hat{u})dr$$

★ **ODF-Solid Angle:**

$$\Psi_{SA}(\hat{u}) = \int_0^\infty P(r\hat{u})r^2dr$$

(True marginal distribution)

[Wedeen 2000, Aganj 2010, Vega 2010]



★ **Spherical Harmonics:**

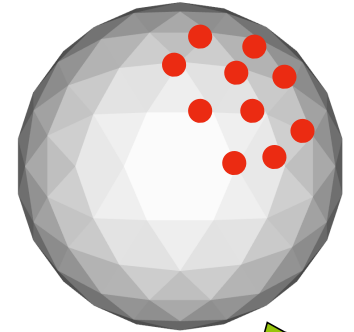
[Anderson 2005, Hess 2006, Descoteaux 2007]

$$\Psi(\theta, \phi) = \sum_{j=1}^N \underbrace{2\pi P_{d(j)}(0)c_j}_{f_j} Y_j(\theta, \phi)$$

⊖ Geometric Characteristics? Maxima, minima ...?

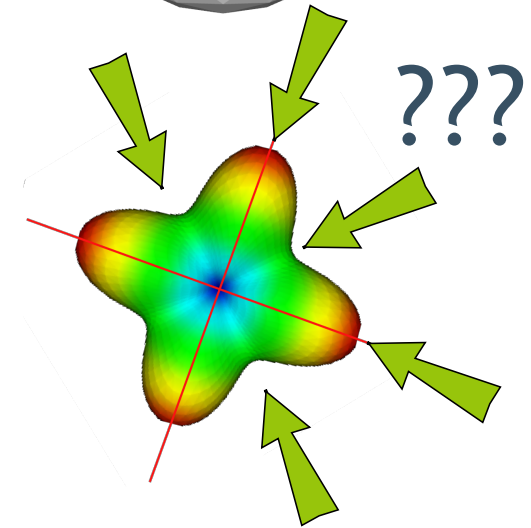
★ Discrete Mesh Search [Descoteaux 2008]

⊖ heuristic, precision ~ mesh



★ Optimization Approaches [Jansons 2003, Tournier 2004]

⊖ local search



★ Polynomial Approaches

- Cartesian coordinates

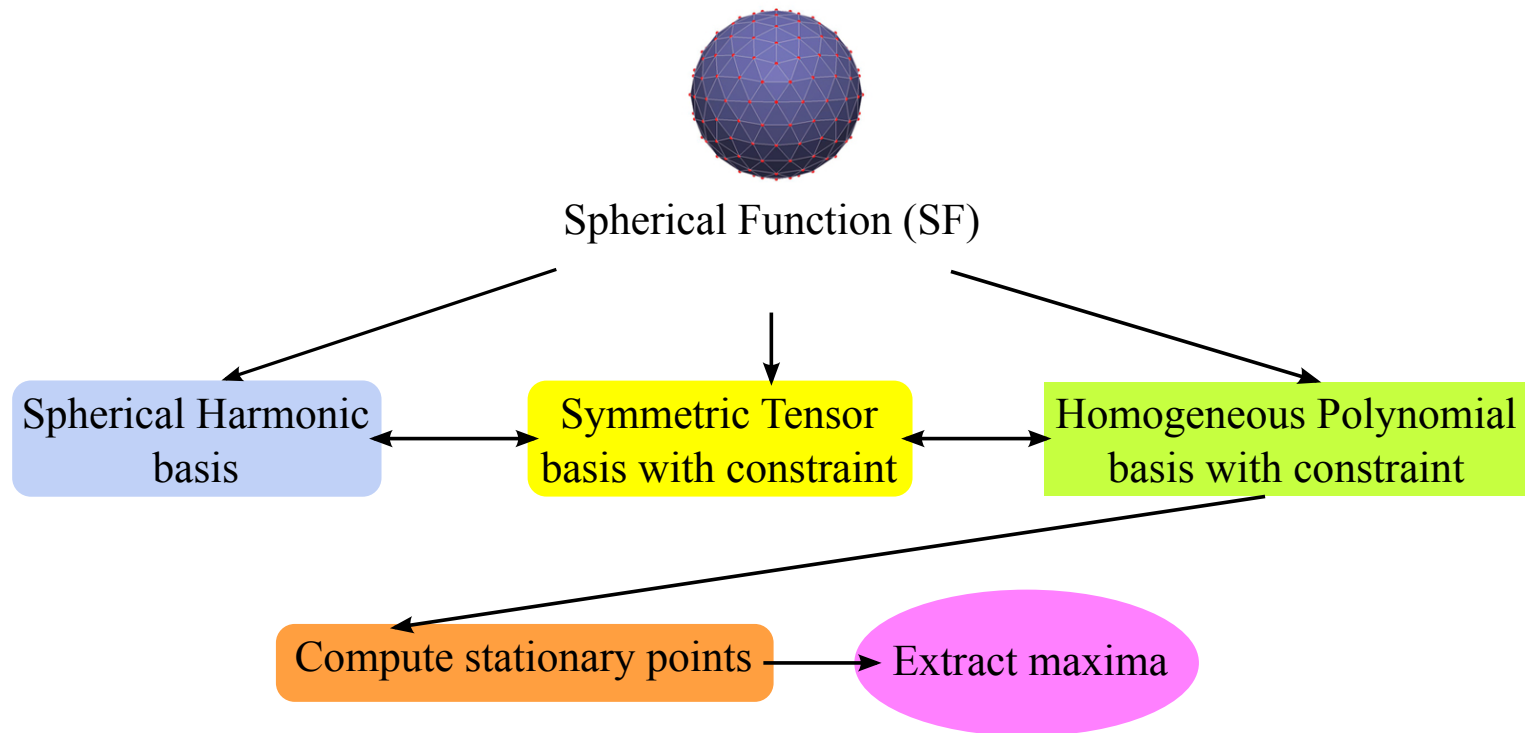
[Bloy 2008, Ghosh 2008]

- Spherical coordinates

[Aganj 2010]

- ★ ALL Stationary Points (*extrema*) of a Spherical Function (SF)
 1. Homogeneous Polynomial (HP) representation
 2. Constrained Optimization formulation
 - ⦿ System of Hom. Polys.
 3. Categorize extrema: maxima, minima, saddle-points

- ★ 3 equivalent bases to represent a SF:



★ SH basis to Tensor basis: $c_i = \int_{\Omega} \left(\underbrace{A_{j_1 j_2 \dots j_d} g_{j_1} g_{j_2} \dots g_{j_d}}_{\text{Spherical Function}} \right) Y_i(\theta, \phi) d\Omega$

$\implies \mathbf{C} = \mathbf{M}\mathbf{A}$

(SH Transform)

★ Tensor basis to Hom. Poly. basis:

$$A_{j_1 j_2 \dots j_d} g_{j_1} g_{j_2} \dots g_{j_d} = \sum_{i_1 + i_2 + i_3 = d} A_{i_1 i_2 i_3} g_1^{i_1} g_2^{i_2} g_3^{i_3}$$

(Re-ordering of terms)

★ ODF as a Hom. Poly.:

$$P_{\Psi}(\mathbf{x} = [x_1, x_2, x_3]^T) = \sum_{i_1 + i_2 + i_3 = d} A_{i_1 i_2 i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

($\|\mathbf{x}\|_2 = 1$)

1. Homogeneous Polynomial (HP) representation
2. Constrained Optimization formulation
 - ④ System of Hom. Polys.
3. Categorize extrema: maxima, minima, saddle-points

★ Non-linear Polynomial optimization:

$$F(\mathbf{x}, \lambda) = P(\mathbf{x}) - \lambda (\|\mathbf{x}\|_2^2 - 1).$$

(Lagrange Multipliers)

★ System of HPs: $\nabla F : \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = \|\mathbf{x}\|_2^2 - 1 = 0$

★ Polynomial system solver: *Finds all extrema*

(Find all real roots)

[Mourrain 2009]

[Mourrain 2009]

- ★ Convert HP to **Bernstein Basis**: $p(x) = \sum_{i=0}^d c_i B_i^d(x; a, b)$
(exact arithmetic) (in the interval $[a, b]$)

Theorem (Descartes Rule 1D)

The number of real roots of $p(x)$ in $]a, b[$ is bounded by the number $V(\mathbf{c})$ of sign changes of $\mathbf{c} = (c_i)_{i=0..d}$, and is equal modulo 2. (*exclusion test* $V(\mathbf{c})=0$ no roots)

- ★ Analytically bracket / Numerically refine roots
- ★ ALL real roots (extrema) found by exclusion

1. Homogeneous Polynomial (HP) representation
2. Constrained Optimization formulation
 - © System of Hom. Polys.
3. Categorize extrema: maxima, minima, saddle-points

★ Bordered Hessian

$$\overline{H}_{(m+n) \times (m+n)}[F(\mathbf{x}, \lambda)] = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_n} & \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} & 0 & \cdots & 0 \end{pmatrix}$$

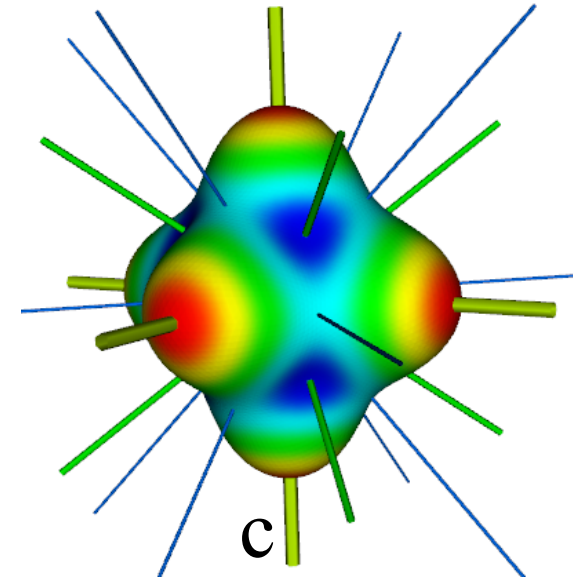
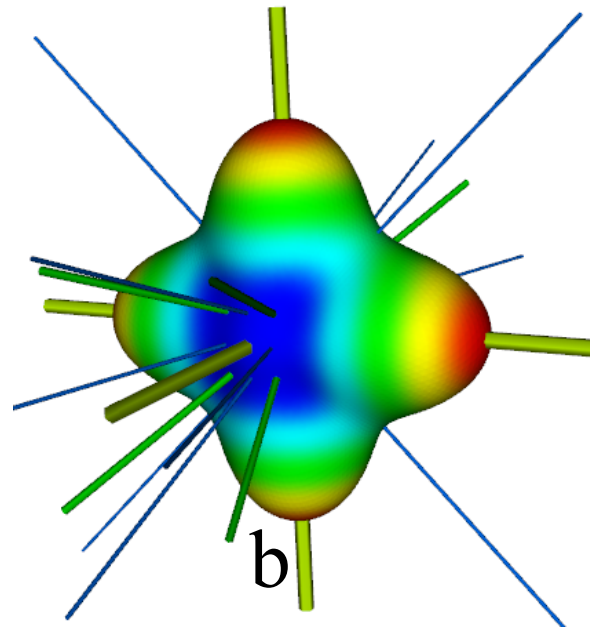
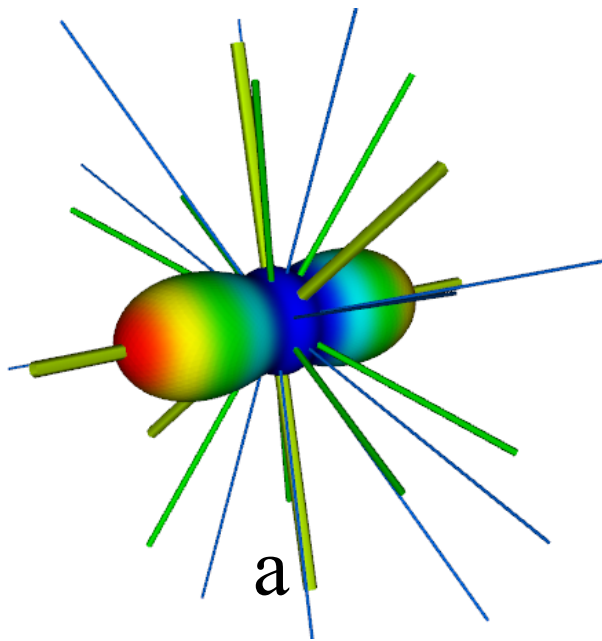
★ Bordered Hessian

$$\bar{H}_{(m+n) \times (m+n)} [F(\mathbf{x}, \lambda)] = \begin{pmatrix} \boxed{\begin{matrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_n} \end{matrix}} & \boxed{\begin{matrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{matrix}} \\ \boxed{\begin{matrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{matrix}} & \boxed{\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}} \end{pmatrix}$$

$r \times r$ $r \times m$
 $H[F(\mathbf{x}, \lambda)]_{n \times n}$ $\frac{\partial}{\partial \mathbf{x}} \text{constraint}_{m \times n}$
 $m \times r$ $\frac{\partial}{\partial \mathbf{x}} \text{constraint}_{m \times n}$ $0\text{-padding}_{m \times m}$ $m \times m$

★ Maximum: $(-1)^m |\bar{H}_r [F(\mathbf{x}, \lambda)]| > 0; \quad r = m + 1, \dots, n,$

Minimum: $(-1)^r |\bar{H}_r [F(\mathbf{x}, \lambda)]| > 0 \quad (m = 1 \quad n = 3)$

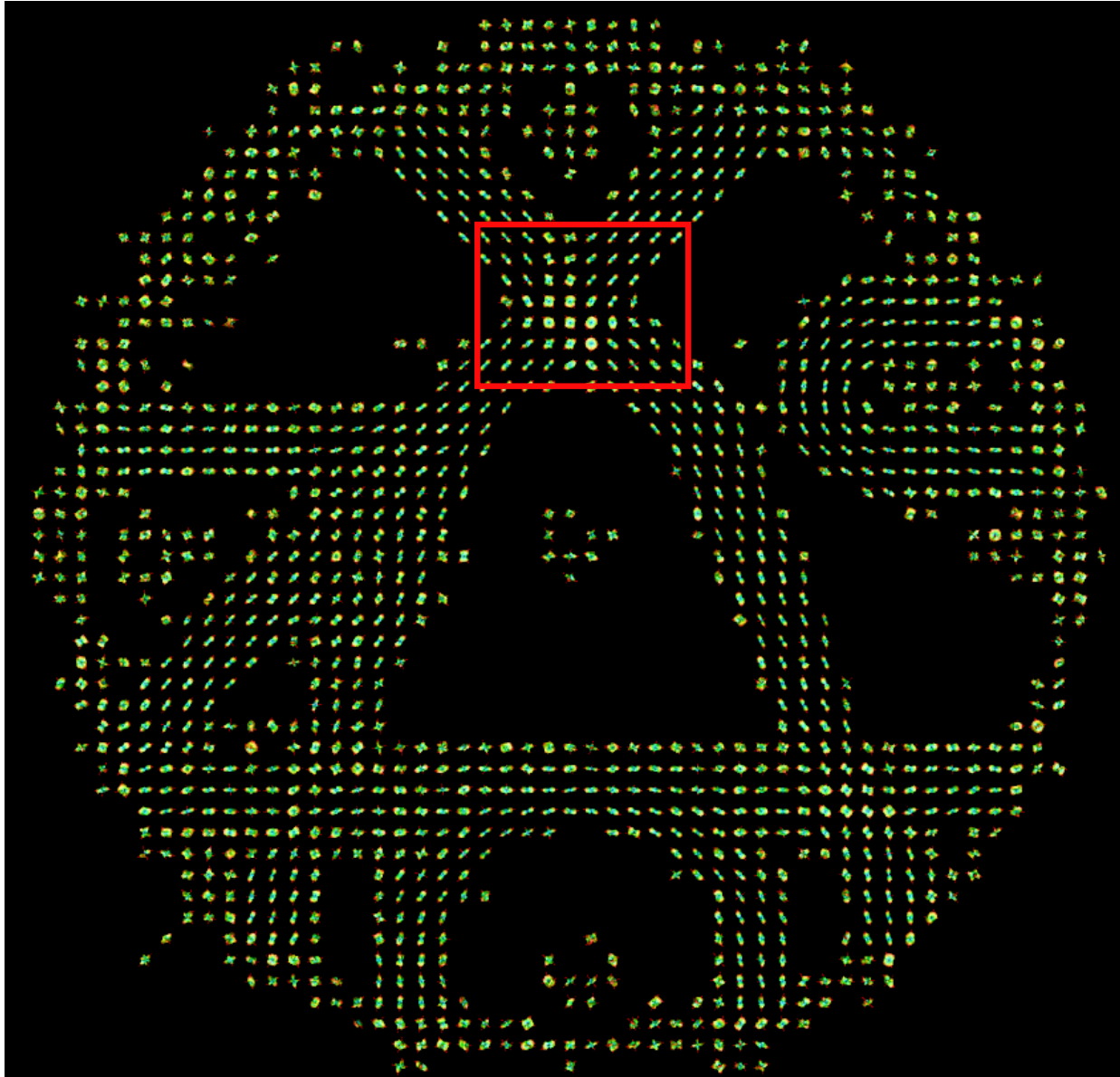


Glyph-colour: **red: high anisotropy**, blue: low anisotropy

Lines: **thick-green: Maxima**, thin-blue: Saddle points, **green: Minima**.

a) 1-fiber b) 2-fibers c) 3-fibers

[C. Poupon et al. 2008]



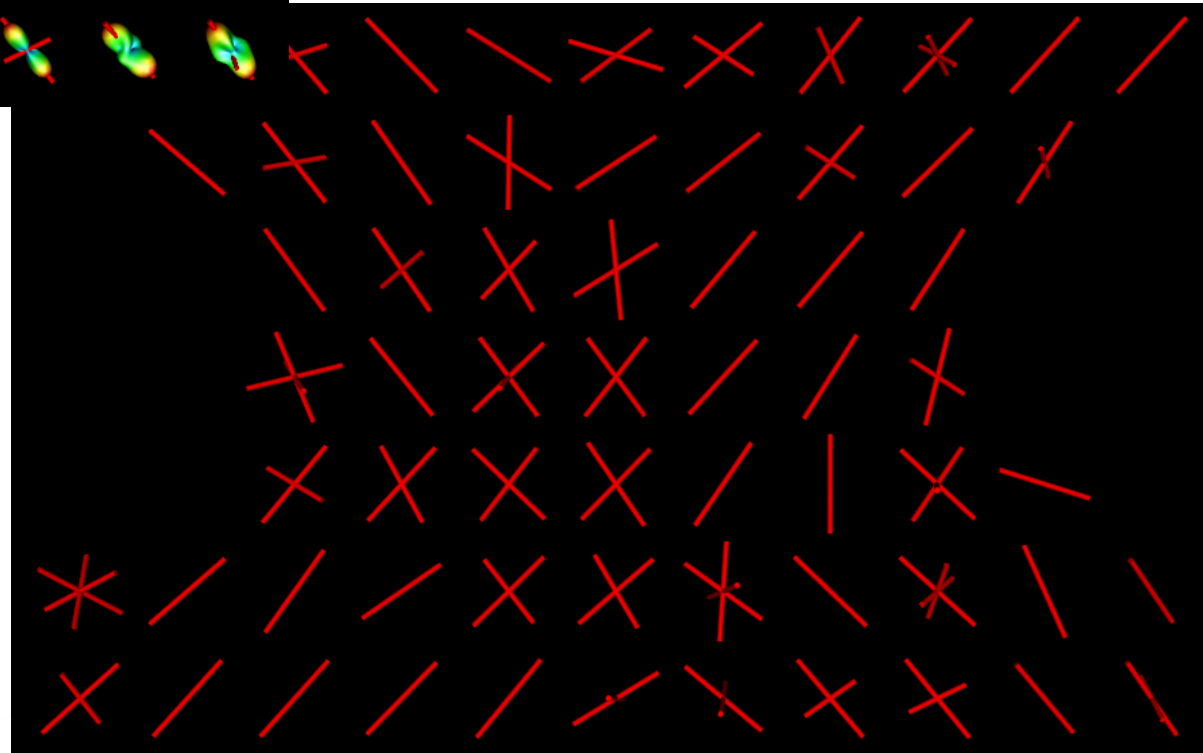
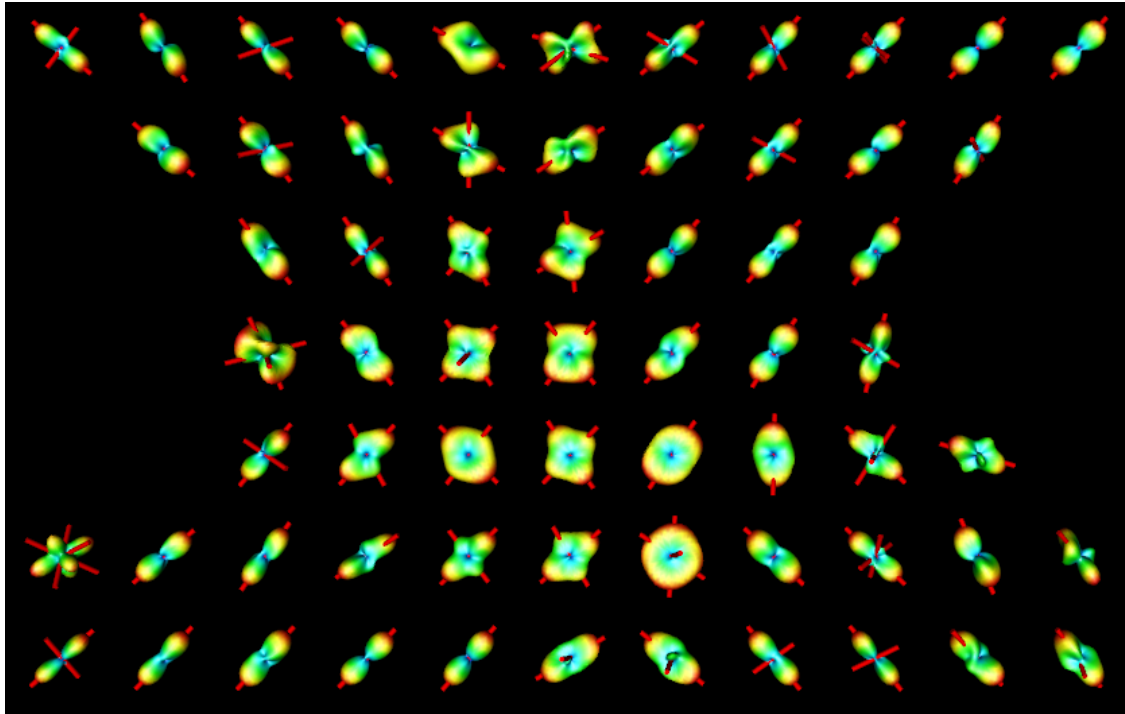
Spacing: 3x3x3

$b = 1500\text{s/mm}^2$

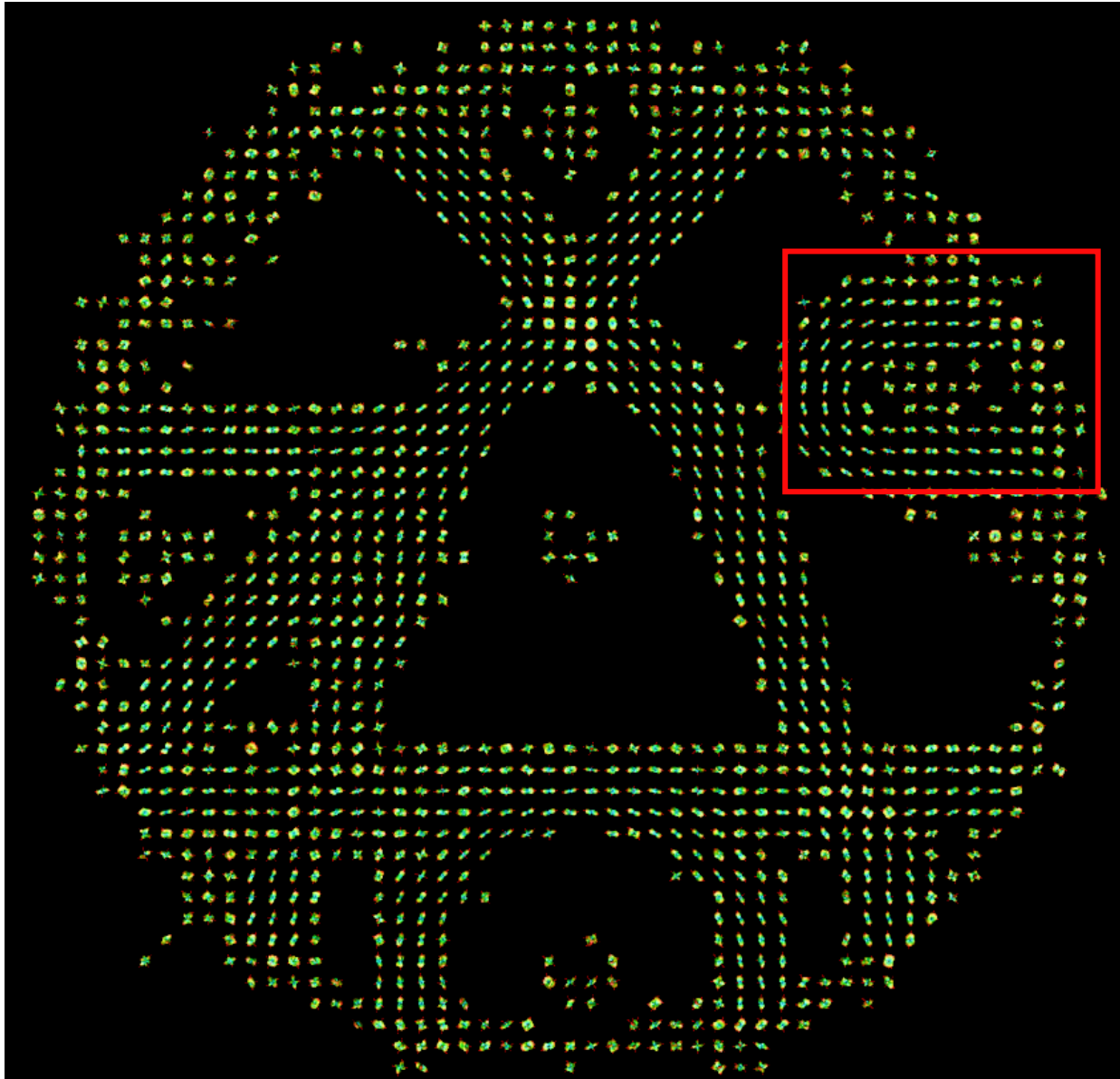
64 DWIs + T0

ODF-Tuch

[C. Poupon et al. 2008]



[C. Poupon et al. 2008]



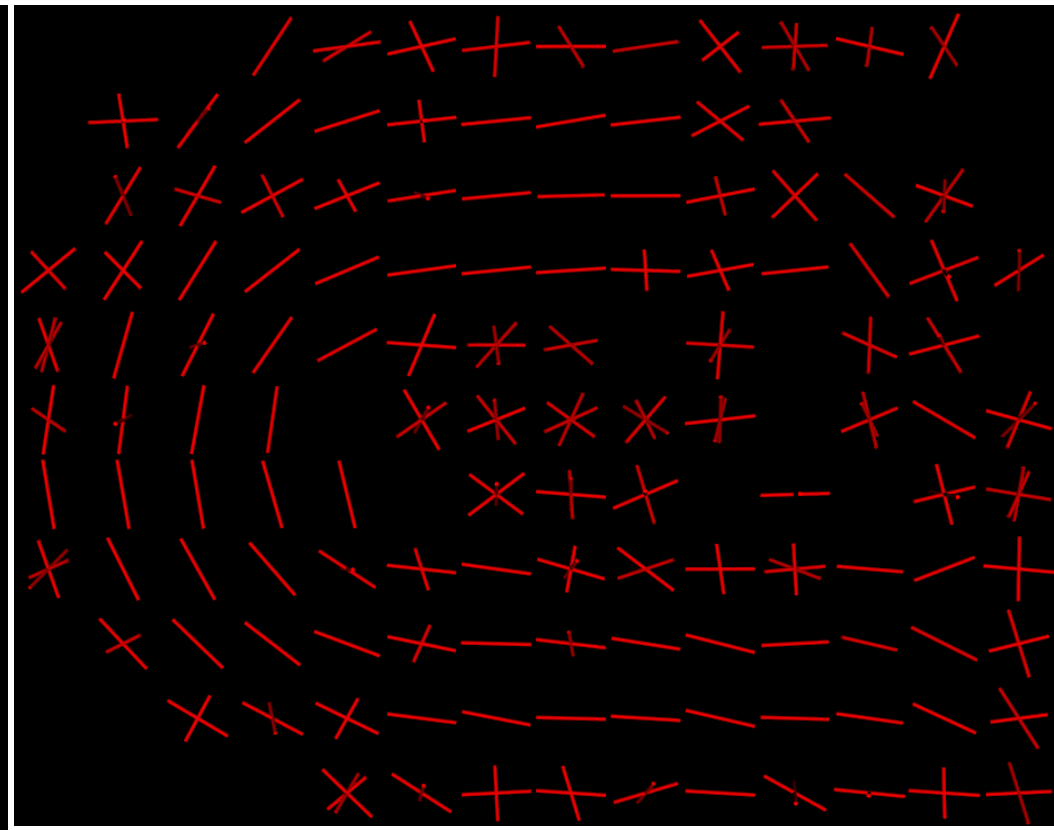
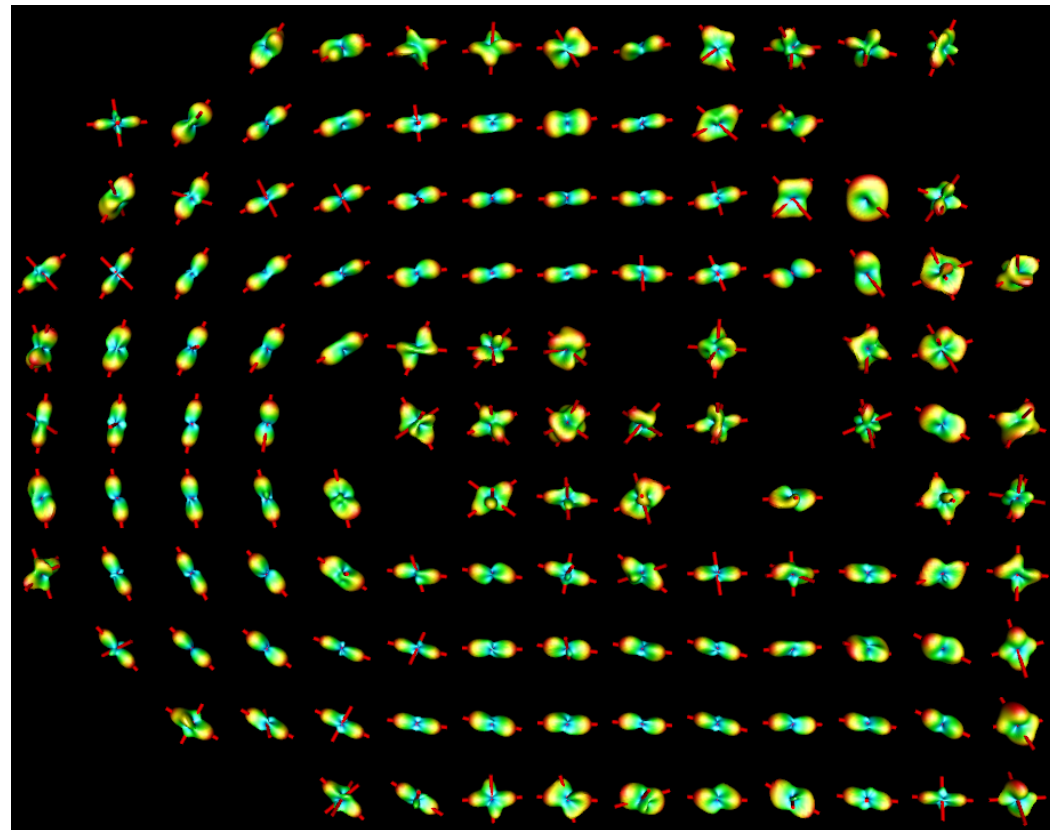
Spacing: 3x3x3

$b = 1500\text{s/mm}^2$

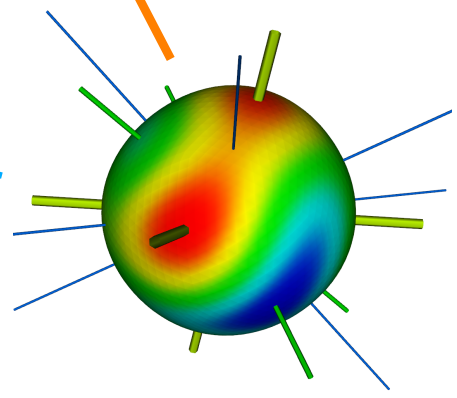
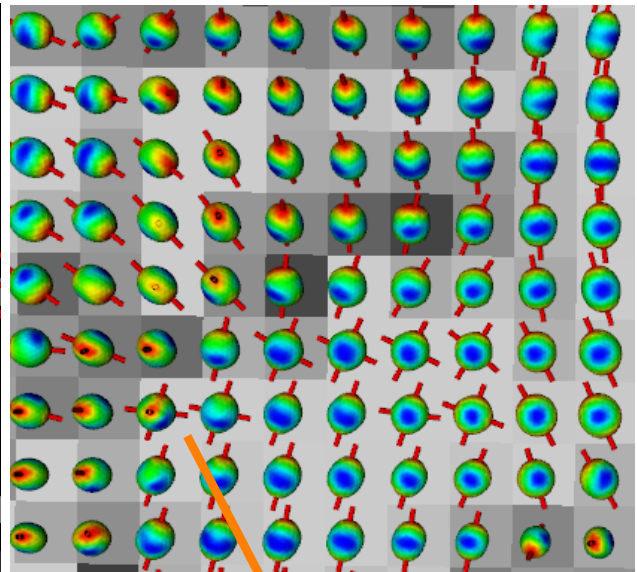
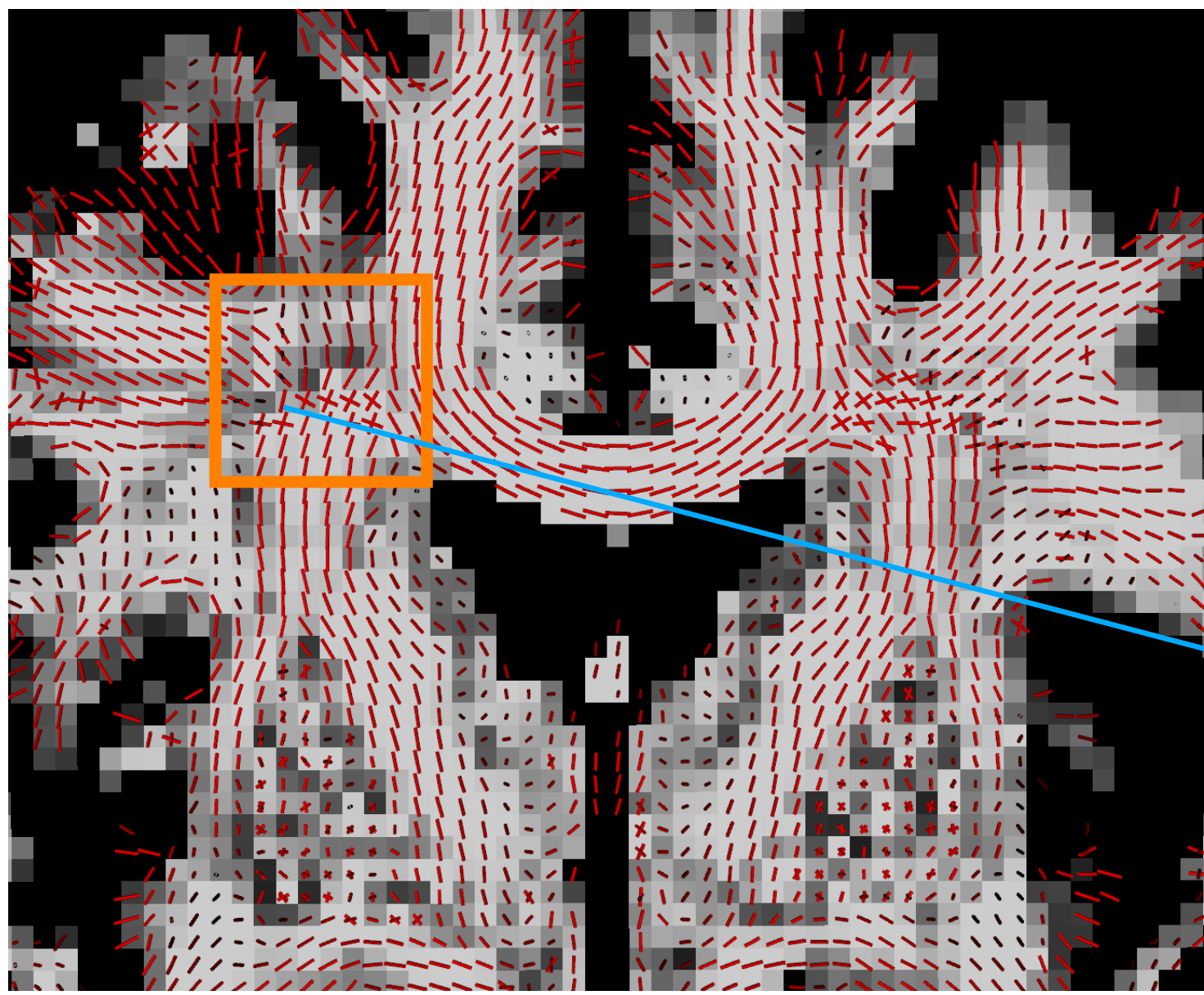
64 DWIs + T0

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[C. Poupon et al. 2008]

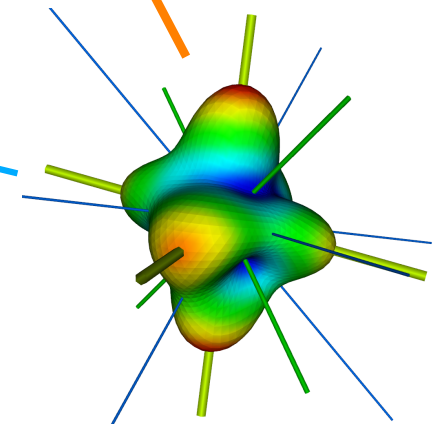
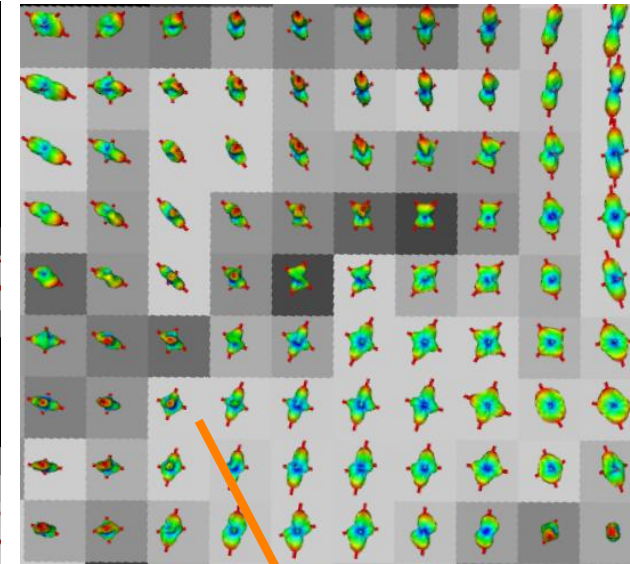
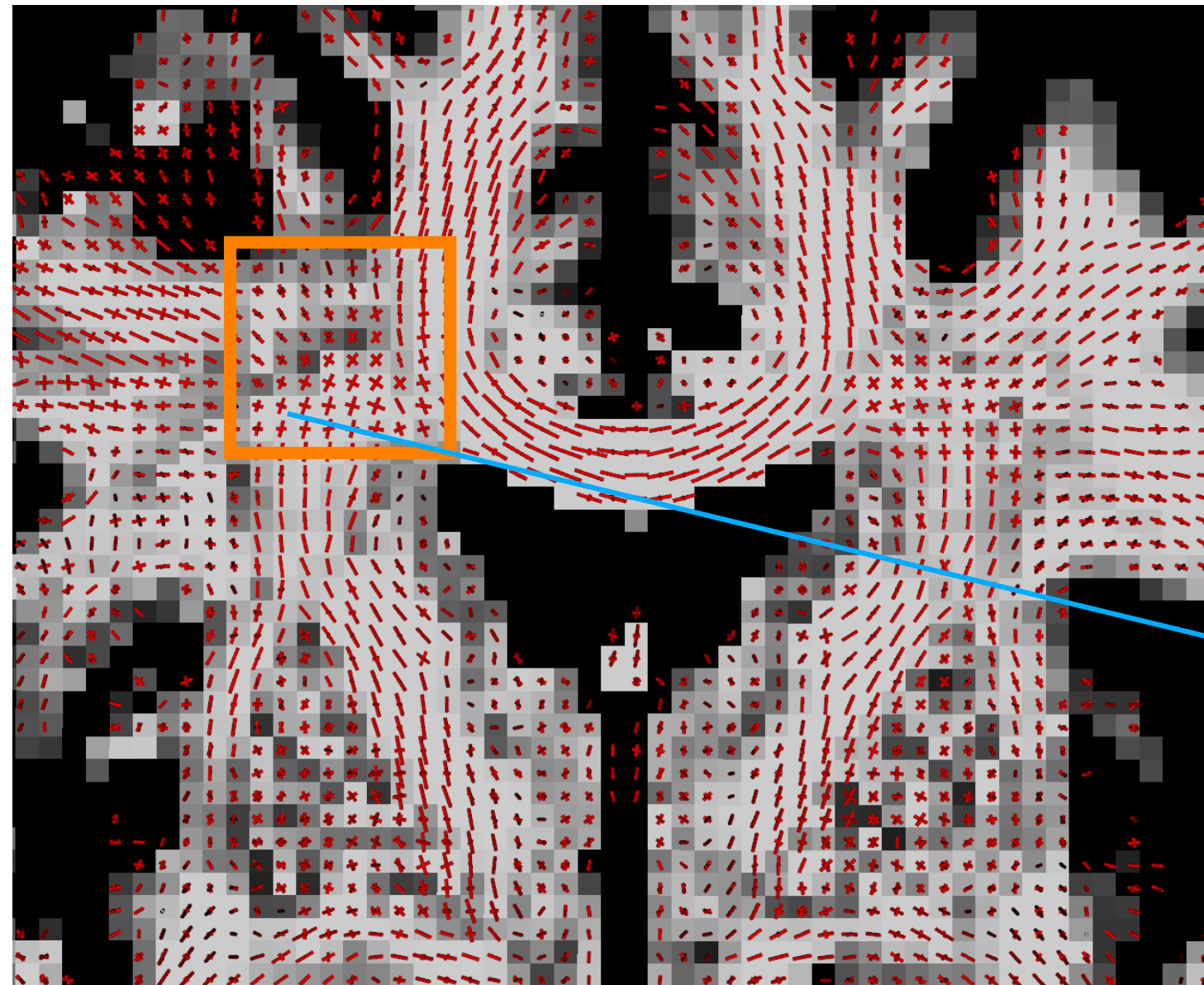


[Anwander 2007]



ODF-Tuch

[Anwander 2007]



ODF-SA

(Synthetic data)

★ How good was the estimation?

- ① Take ground truth directions $\{d_i\}$ and generate *signal*
- ② Estimate ODF-Ts from signal
- ③ Compute maxima of ODF-Ts: $\{d'_i\}$
- ④ Compute (F= ODF as HP): $\|\nabla F(d)\|$ & $\|\nabla F(d')\|$

$\|\nabla F(d)\|$: estimation error

$\|\nabla F(d')\|$: maxima extraction error

	$ang(d, d')^\circ$		$\ \nabla F(d')\ $		$\ \nabla F(d)\ $	
	m	v	m	v	m	v
1-Fib	0.01	4.22e-05	2.07e-11	1.5e-21	4.9e-04	9.17e-08
2-Fib _{90°}	0.025	3.85e-05	5.58e-11	1.76e-21	3.6e-03	2.48e-06

m = mean, v = variance

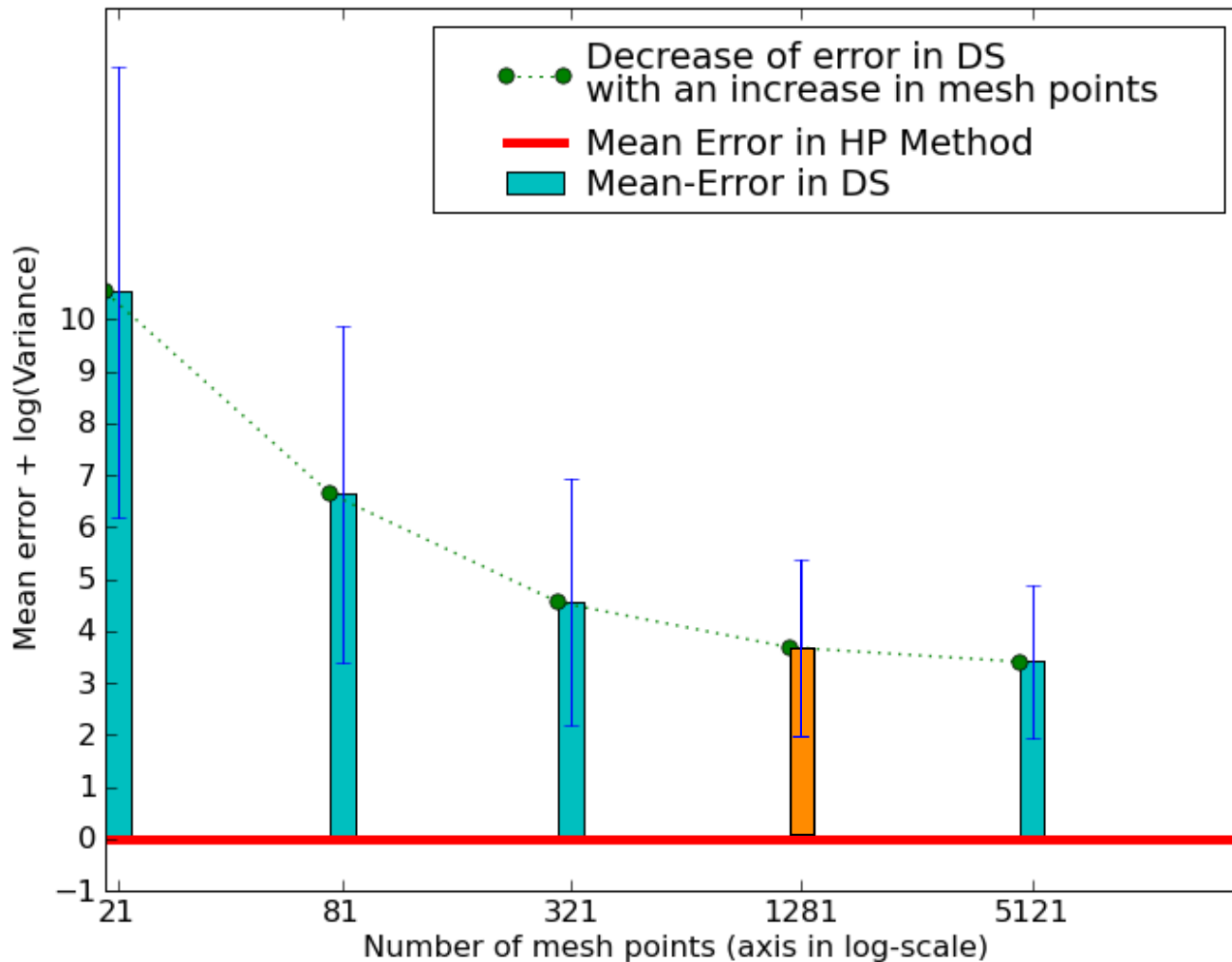
★ Noise-free 100 simulations (each)

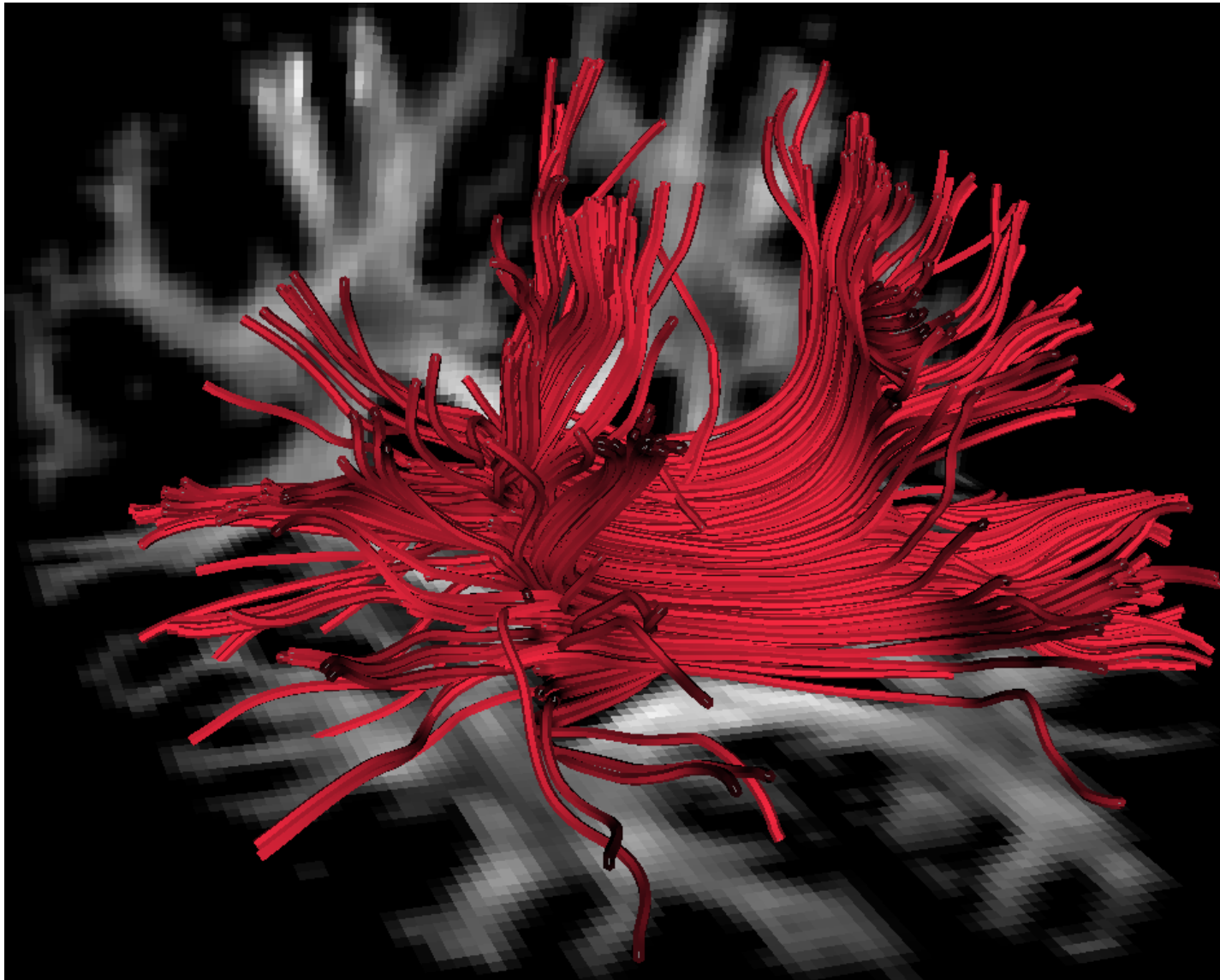
★ Given F (ODF), $\|\nabla F\| = 0 \Rightarrow$ Maxima (Human brain data)

$\|\nabla F(d)\|$: HP maxima error

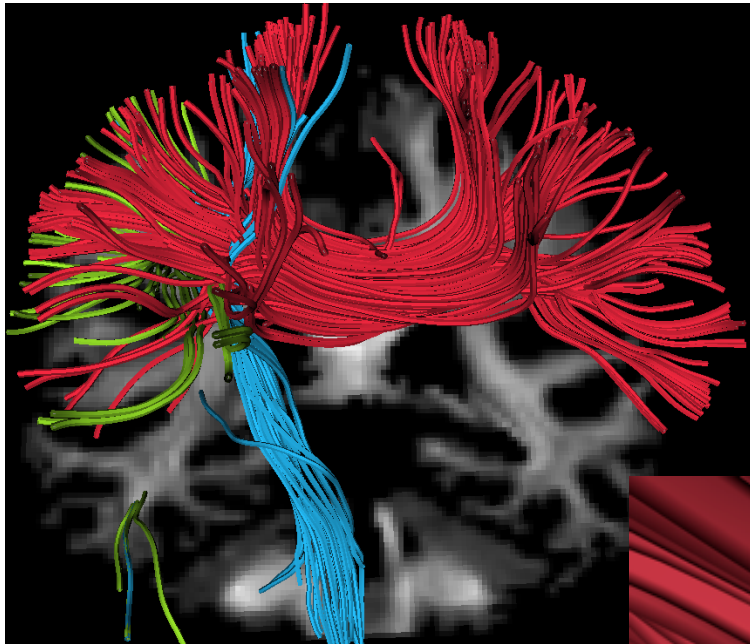
$\|\nabla F(d')\|$: Discrete mesh search (DS) maxima error

Comparison of error in Discrete Search method (DS) vs the HP Method

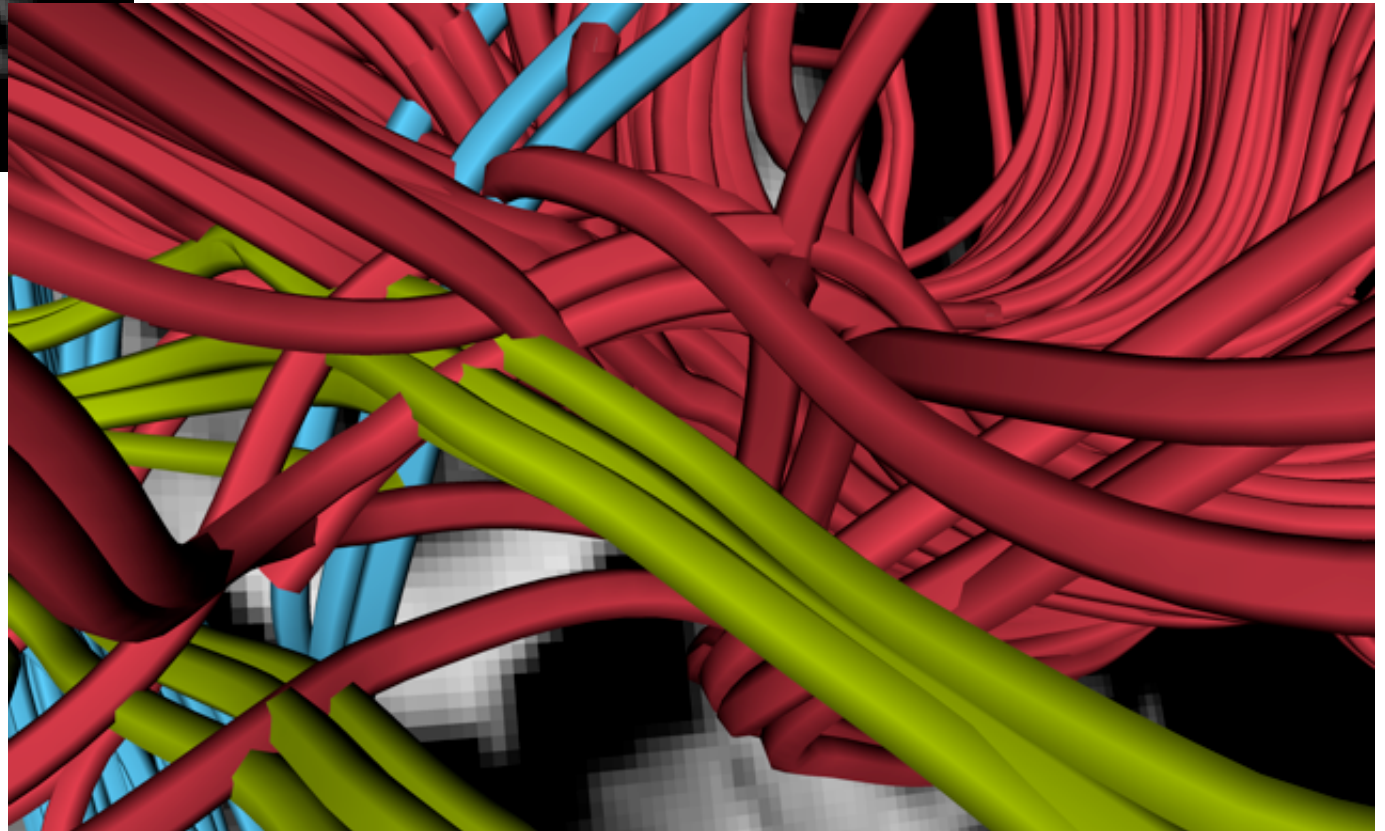




seeds: Corpus Callosum



blue: Cortico-Spinal Tract
green: Superior Longitudinal Fasciculus
red: Corpus Callosum



[Ghosh 2008b, Ghosh2011]

- ★ Maxima extraction from **ANY** kind of spherical function
- ★ **ALL** maxima are found together
- ★ Maxima found with *high precision*
- ★ Application: Tractography with maxima-extraction

- ④ No spatial regularization
- ④ Multiple roots (high orders) implies computational burden



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Dr. Demian Wassermann

