

A Polynomial Approach for Maxima Extraction and its Application to Tractography in HARDI

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07-07-2011



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1. Spherical Functions in Diffusion MRI

- DTI: the Ellipsoid
- Crossing fibers: Spherical Functions

- 2. Maxima Extraction
 - A Polynomial Approach

3. An Application: Tractography









Context and Contribution





Spherical Functions

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Spherical Functions play a central role in Diffusion MRI



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Spherical Functions play a central role in Diffusion MRI

DTI Spherical Function: Diffusion Ellipsoid

$$\frac{\mathbf{r}^T \mathbf{D}^{-1} \mathbf{r}}{2t} = 1$$



Geometric interpretation:

© Spectral Decomposition (Fast & Easy)



Tractography Fractional Anisotropy Other scalars (biomarkers)



Crossing Fibers

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DTI has limitations in regions with fiber-crossings



✓ Other Spherical Functions (SF) in dMRI:

[Tuch Thesis, Tournier 2004, Jansons 2003, Alexander 2005, Ozarslan 2006, Ozarslan 2003]

- Orientation Distribution Function (ODF)
- Siber Orientation Density (FOD)
- PASMRI/MESD
- DOT, ADC ... etc...

Spherical Harmonic (SH) Basis



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$$\longrightarrow \frac{S(\mathbf{q})}{S_0} = \int P(\mathbf{r}, \Delta) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}; \quad \mathbf{q} = \gamma \delta g \mathbf{u} \longrightarrow \mathbf{q}$$

V ODF-Tuch: [Tuch Thesis] $\Psi_T(\hat{u}) = \int_0^\infty P(r\hat{u})dr$ **ODF-Solid Angle:** $\Psi_{SA}(\hat{u}) = \int_0^\infty P(r\hat{u})r^2dr$ (True marginal distribution) [Wedeen 2000, Aganj 2010, Vega 2010]



[Anderson 2005, Hess 2006, Descoteaux 2007]







Maxima Extraction

- Geometric Characteristics? Maxima, minima ...?
- Discrete Mesh Search [Descoteaux 2008]
 heuristic, precision ~ mesh

- Optimization Approaches [Jansons 2003,
 Iocal search
- Polynomial Approaches
 Cartesian coordinates
 [Bloy 2008, Ghosh 2008]
 Spherical coordinates
 - Spherical coordinates [Aganj 2010]





ALL Stationary Points (extrema) of a Spherical Function (SF)

- 1. Homogeneous Polynomial (HP) representation
- 2. Constrained Optimization formulation
 - System of Hom. Polys.
- 3. Categorize extrema: maxima, minima, saddle-points



⅔ 3 equivalent bases to represent a SF:





1. Hom. Poly. Representation

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SH basis to Tensor basis:
$$c_i = \int_{\Omega} \left(\underbrace{A_{j_1 j_2 \dots j_d} g_{j_1} g_{j_2} \dots g_{j_d}}_{Spherical \ Function} \right) Y_i(\theta, \phi) d\Omega$$

 $\Rightarrow \mathbf{C} = \mathbf{MA}$
(SH Transform)

Tensor basis to Hom. Poly. basis:

$$A_{j_1 j_2 \dots j_d} g_{j_1} g_{j_2} \dots g_{j_d} = \sum_{i_1 + i_2 + i_3 = d} A_{i_1 i_2 i_3} g_1^{i_1} g_2^{i_2} g_3^{i_3}$$

(Re-ordering of terms)





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Non-linear Polynomial optimization:

$$F(\mathbf{x}, \lambda) = P(\mathbf{x}) - \lambda \left(\|\mathbf{x}\|_2^2 - 1 \right).$$

(Lagrange Multipliers)

System of HPs:
$$\nabla F$$
: $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = \|\mathbf{x}\|_2^2 - 1 = 0$

Polynomial system solver: Finds all extrema
 (Find all real roots)
 [Mourrain 2009]

Athena 2. Subdivision Poly. System Solver 15

[Mourrain 2009]

Convert HP to Bernstein Basis: $p(x) = \sum_{i=0}^{a} c_i B_i^d(x; a, b)$

(exact arithmetic)

(in the interval [a,b])

Theorem (Descartes Rule 1D)

The number of real roots of p(x) in]a,b[is bounded by the number V(c) of sign changes of $\mathbf{c} = (c_i)_{i=0..d}$, and is equal modulo 2.(exclusion test V(c)=0 no roots)

Analytically bracket / Numerically refine roots

ALL real roots (extrema) found by exclusion



- 1. Homogeneous Polynomial (HP) representation
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3. Categorize Extrema 17

Bordered Hessian

$$\overline{H}_{(m+n)\times(m+n)}[F(\mathbf{x},\lambda)] = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_n} & \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} & 0 & \cdots & 0 \end{pmatrix}$$



3. Categorize Extrema 18

Bordered Hessian

$$\overline{H}_{(m+n)\times(m+n)}[F(\mathbf{x},\lambda)] = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_n} \\ H[F(\mathbf{x},\lambda)]_{n\times n} & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_m}{\partial x_n} & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_m}{\partial x_n} & 0 \\ \frac{\partial g_m}{\partial x_n} & 0 \\ \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_m}{\partial x_n} & 0 \\ \frac{\partial g_m}{\partial x_n} \\$$



3. Categorize Extrema 19

🔀 Bordered Hessian



X Maximum: $(-1)^m |\overline{H}_r[F(\mathbf{x}, \lambda)]| > 0; r = m + 1, ..., n,$

Minimum: $(-1)^r |\overline{H}_r[F(\mathbf{x}, \lambda)]| > 0 \quad (m = 1 \ n = 3)$



Results: Voxel 20



Glyph-colour: red: high anisotropy, blue: low anisotropy Lines: thick-green: Maxima, thin-blue: Saddle points, green: Minima. a) 1-fiber b) 2-fibers c) 3-fibers



[C. Poupon et al. 2008]

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Spacing: 3x3x3 b = 1500s/mm^2 64 DWIs + T0 ODF-Tuch



[C. Poupon et al. 2008]







[C. Poupon et al. 2008]

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Spacing: 3x3x3 b = 1500s/mm^2 64 DWIs + T0 ODF-Tuch



[C. Poupon et al. 2008]

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Human Brain 25

[Anwander 2007]





Human Brain 26

[Anwander 2007]





(Synthetic data)

How good was the estimation?

- Take ground truth directions {*d*_{*i*}} and generate *signal*
- Estimate ODF-Ts from signal \bigcirc
- Compute maxima of ODF-Ts: {*d'i*} \bigcirc
- Compute (F= ODF as HP): $||\nabla F(d)|| \& ||\nabla F(d')||$

 $||\nabla F(d)||$: estimation $||\nabla F(d')||$: maxima error

extraction error

	$\ ang(d,d^{'})^{o} \ $		$ abla { m F}(d^{'}) $		$ abla { m F}(d) $	
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1-Fib	0.01	4.22e-05	2.07e-11	1.5e-21	4.9e-04	9.17e-08
2-Fib _{90°}	0.025	3.85e-05	5.58e-11	1.76e-21	3.6e-03	2.48e-06

m = mean, v = variance

Noise-free 100 simulations (each)

Compare Discrete Mesh Search

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Number of mesh points (axis in log-scale)



Results: ODF-SA Tractography



seeds: Corpus Callosum



Results: ODF-SA Tractography



blue: Cortico-Spinal Tract green: Superior Longitudinal Fasciculus red: Corpus Callosum

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Take Home Messages

[Ghosh 2008b, Ghosh2011]



ALL maxima are found together

Axima found with *high precision*

Application: Tractography with maxima-extraction



• No spatial regularization

Multiple roots (high orders) implies computational burden



Thank you

ANR project NucleiPark France-Parkinson Association

Acknowledgements:



Dr. Rachid Deriche Dr. Maxime Descoteaux Dr. Pierre Comon Dr. Bernard Mourrain Dr. Elias Tsigaridas Dr. Demian Wassermann

