Characteristics of Invariant Weights Related to Code Equivalence over Rings

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Abstract. The Extension Theorem states that, for a given weight on the alphabet, every linear isometry between linear codes extends to a monomial transformation of the entire space. This theorem has been proved for several weights and alphabets, including the original MacWilliams' Equivalence Theorem for Hamming weight on codes over finite fields. Now we ask: What conditions must a weight satisfy so that the Extension Theorem will hold? In this paper we provide an algebraic framework for determining such conditions, generalising the approach taken in [5].

Keywords: MacWilliams' Equivalence Theorem, Extension Theorem, Weights, Ring-Linear Codes.

Introduction

Two linear codes of the same length over a given alphabet are said to be equivalent if there exists a (weight preserving) monomial transformation mapping one to the other. MacWilliams in her Equivalence Theorem [11] proved that when the alphabet is a finite field any linear Hamming isometry between linear codes will extend to a monomial transformation. Thus the equivalence question can be seen as an extension problem. A character theoretic proof of this Extension Theorem in [16] led to a generalisation of this theorem for codes over finite Frobenius rings in [17]. Indeed in [19] it was shown that linear Hamming isometries extend precisely when the ring is Frobenius.

In the seminal paper on ring linear coding [8] it was already noticed that weights other than the Hamming weight would play a significant role, such as the Lee weight over $\mathbb{Z}_4$. The concept of a homogenous weight was first introduced in [3] where a combinatorial proof of the Extension Theorem for this weight and codes over $\mathbb{Z}_m$ is provided. In [7] we see that every homogeneous isometry is a Hamming isometry yielding the Extension Theorem for homogeneous weight and codes over finite Frobenius rings. This paper followed the combinatorial tack

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of [3] for the $\mathbb{Z}_m$ case. For the more general case of codes as quasi-Frobenius modules the Extension Theorem holds for Hamming weight as seen in [6].

Following from the chain ring result of [4], obtained by examining the generation of invariant weights, in [5] a complete characterisation of those weights for which the Equivalence Theorem holds for codes over $\mathbb{Z}_m$ is supplied. Here we extend the ideas of that paper to more general rings, outlining a strategy for attaining necessary and sufficient conditions for a weight to satisfy the Extension Theorem.

We begin in Section 1 with defining codes, weights and the equivalence condition for the ring case. In Section 2 we revise some key properties of chain rings and provide a thorough introduction to the Möbius Function. In Section 3 we describe the structural context so crucial to the elegance and seeming simplicity of our results. Finally in Section 4 we provide a concise condition for an invariant weight to satisfy the generalised MacWilliams' Equivalence Theorem.

1 Weight Functions and the Equivalence Theorem

In all of our discussion let R be a finite associative ring with identity 1. Denote by $R^\times$ the group of invertible elements of R. By a weight on R we mean any function $w : R \to \mathbb{C}$ satisfying $w(0) = 0$. The left symmetry group of w is $\mathrm{Sym}_L(w) := \{u \in R^\times \mid w(x) = w(ux) \ \ \forall x \in R\}$ and the right symmetry group is $\mathrm{Sym}_R(w) := \{u \in R^\times \mid w(xu) = w(x) \ \ \forall x \in R\}$. The weight w is called *invariant* if both of these symmetry groups are maximal, i.e. if they coincide with $R^\times$. Note if $\mathrm{Sym}_L(w) = R^\times$ we have $Rx = Ry$ implies $w(x) = w(y)$.

Definition 1. An invariant weight w on R is called *homogeneous*, if there exists a real number $c \geq 0$ such that for all $x \in R$ there holds:

$$\sum_{y \in Rx} w(y) \ = \ c\,|Rx| \quad \text{if } x \neq 0\,.$$

The concept of a homogeneous weight was originally introduced in [2] and further generalised in [7] in such a way that homogeneous weights exist for every finite ring. There are several definitions of homogeneous and prehomogeneous weights, particularly for more general codes as modules as given in [14]. We now define the normalised homogeneous weight w_{hom}.

Definition 2. The normalised homogeneous weight $w_{\mathrm{hom}} : R \to \mathbb{R}$ is given by

$$w_{\mathrm{hom}}(x) = 1 - \frac{\mu(0, Rx)}{|R^\times x|}\,,$$

where μ is the Möbius function on the lattice of principal ideals of R, defined in the following section, and $|R^\times x|$ counts the number of generators of the ideal Rx.

Given a positive integer n, any weight $w : R \to \mathbb{C}$ shall be extended to a function on R^n by defining $w(x) := w(x_1) + w(x_2) + \cdots + w(x_n)$ for $x \in R^n$. Suppose that C is a linear code of length n over R , i. e. an R-submodule of R^n. A linear map $\phi : C \to R^n$ is called a w-isometry if $w(\phi(x)) = w(x)$ for all $x \in C$.

A bijective module homomorphism $\phi : R^n \to R^n$ is called a *monomial transformation* if there exists a permutation π of $\{1 \dots n\}$ and units $u_1, \dots, u_n \in R^\times$ such that $\phi(x) = (x_{\pi(1)}u_1, \dots, x_{\pi(n)}u_n)$ for every $x = (x_1, \dots, x_n) \in R^n$. If all the units u_i are contained in a subgroup G of $R^\times$ we call it a *G-monomial transformation*.

Clearly any $\mathrm{Sym}_R(w)$-monomial transformation will be a w-isometry for any weight w and hence restricts to a w-isometry on every linear code $C \subseteq R^n$. Conversely we may ask if a given linear w-isometry $\phi : C \to R^n$, defined on a linear subcode C of R^n is a restriction of an appropriate monomial transformation of R^n. This is the essence of MacWilliams' Equivalence Theorem:

Theorem 3 (MacWilliams [11]). *Every linear Hamming isometry between linear codes of the same length over a finite field can be extended to a monomial transformation of the ambient vector space.*

Definition 4. Suppose w is an arbitrary weight. We say that *MacWilliams' Equivalence Theorem* (or the *Extension Theorem*) holds for w if for any positive integer n, any linear code C in R^n and any linear w-isometry $\phi : C \to R^n$ there exists a $\mathrm{Sym}_R(w)$-monomial transformation of R^n which extends ϕ.

An obvious necessary condition for MacWilliams' Equivalence Theorem to hold for a weight w on R is that all w-isometries are injective.

2 Chain Rings and Möbius Inversion

In the following sections we will harness the power of Möbius Inversion to prove our most vital results. For this reason we begin with a short primer, for more details see [15]. First we include a brief summary of the key properties of chain rings (c.f. [10], [12], [9]).

Definition 5. A ring R is called a *left chain ring* if the lattice of left ideals of R form a chain under the partial ordering of inclusion. Similarly for *right chain ring*. If R is both a left and right chain ring then it is called a *chain ring*.

The following theorem, combining Theorem 1.1 of [13] and Lemma 1 of [1], demonstrates the numerous equivalent definitions of a finite chain ring, so giving us a variety of different approaches to studying chain rings. Recall a *principal left ideal ring* is a ring with identity in which each left ideal is left principal, and a *principal ideal ring* is a ring which is both a principal left ideal ring and a principal right ideal ring.

Theorem 6. *The following are equivalent:*

(i) *R is a local principal ideal ring.*
(ii) *R is a left chain ring.*
(iii) *R is a chain ring.*
(iv) *R is a local ring and* $\mathrm{rad}(R)$ *is a left principal ideal.*
(v) *Every one-sided ideal of R is two-sided and belongs to the chain* $R \rhd \mathrm{rad}(R) \rhd ... \rhd \mathrm{rad}(R)^{n-1} \rhd \mathrm{rad}(R)^n = \{0\}$*, for some* $n \in \mathbb{N}$.

Remark 7. Note that in the above if $n > 1$, then we have $\mathrm{rad}(R)^i = R\pi^i = \pi^i R$ for any $\pi \in \mathrm{rad}(R) \backslash \mathrm{rad}(R)^2$, $i \in \{1 \dots n\}$. Also any element $a \in R$ can be decomposed uniquely into a representation $a = a_0 + a_1\pi + \cdots + a_{n-1}\pi^{n-1}$, where the a_i are from a co-ordinate set Γ. This is called the π-adic representation of a. Wood noted in [18] that

$$\mathrm{rad}(R)^i \backslash \mathrm{rad}(R)^{i+1} = R^\times \pi^i = \pi^i R^\times \ .$$

This property extends in a natural way to finite direct products of chain rings and, combined with our structural approach, facilitates the proof of the main theorems herein.

It is fitting to begin our discussion of the Möbius function in the context of the incidence algebra. We describe the incidence algebra of a finite partially ordered set. Note that locally finite is sufficient for the following definitions.

Definition 8. Let P be a finite partially ordered set and $\mathbb{F}$ a field. Set

$$\mathcal{A}(P) = \{f : P \times P \longrightarrow \mathbb{F} \mid f(x,y) = 0 \text{ if } x \not\leq y\} \ ,$$

with addition and scalar multiplication defined by

$$\begin{aligned} (f+g)(x,y) &= f(x,y) + g(x,y) \\ (kf)(x,y) &= kf(x,y) \ . \end{aligned}$$

Also define multiplication by:

$$(f \odot g)(x,y) \ = \sum_{x \leq z \leq y} f(x,z) g(z,y) \ .$$

Then $\mathcal{A}(P)$ is an algebra, called the *incidence algebra* of P, with identity

$$\delta(x,y) := \begin{cases} 1 & : \quad x = y \\ 0 & : \quad \text{otherwise} \ . \end{cases}$$

Theorem 9. *An element $f \in \mathcal{A}(P)$ has a multiplicative inverse if and only if $f(x,x) \neq 0 \ \ \forall x \in P$. If g is the inverse of f then $g(x,x) = \frac{1}{f(x,x)}$ and*

$$g(x,y) = -g(x,x) \sum_{x < z \leq y} f(x,z) g(z,y) \ .$$

Definition 10. We define the zeta function $\zeta \in \mathcal{A}(P)$ to be

$$\zeta(x,y) := \begin{cases} 1 & : \quad x \leq y \\ 0 & : \quad \text{otherwise}\,. \end{cases}$$

Its inverse is called the Möbius function which we now define.

Definition 11. Consider a field $\mathbb{F}$ and a locally finite partially ordered set P with partial ordering $\leq$. The *Möbius function*, $\mu : P \times P \longrightarrow \mathbb{F}$, is defined by any of the four equivalent statements:

$$\begin{array}{lllll}
\text{(i)} & \mu(x,x) = 1 & \text{and} & \displaystyle\sum_{x \leq z \leq y} \mu(z,y) = 0 & \text{for } x < y \\
\text{(ii)} & \mu(x,x) = 1 & \text{and} & \displaystyle\sum_{x \leq z \leq y} \mu(x,z) = 0 & \text{for } x < y \\
\text{(iii)} & \mu(x,x) = 1 & \text{and} & \mu(x,y) = -\displaystyle\sum_{x < z \leq y} \mu(z,y) & \text{for } x < y \\
\text{(iv)} & \mu(x,x) = 1 & \text{and} & \mu(x,y) = -\displaystyle\sum_{x \leq z < y} \mu(x,z) & \text{for } x < y
\end{array}$$

Theorem 12. *Let P, $\mathbb{F}$, and μ be as above and let f, g be functions from P to $\mathbb{F}$. If P has least element 0 then:*

$$g(x) = \sum_{y \leq x} f(y) \;\; \textit{for all } x \in P \quad \Leftrightarrow \quad f(x) = \sum_{y \leq x} g(y)\mu(y,x) \;\; \textit{for all } x \in P.$$

If additionally the partially ordered set P has a greatest element 1 then

$$g(x) = \sum_{x \leq y} f(y) \;\; \textit{for all } x \in P \quad \Leftrightarrow \quad f(x) = \sum_{x \leq y} g(y)\mu(x,y) \;\; \textit{for all } x \in P.$$

3 Convolution and Correlation

We now describe a structural context for proving the Extension Theorem. Two key operations, convolution and correlation, allow us to define a module of weights over an algebra of complex functions. Consider the set $\mathbb{C}^R$ of all functions $\{f \mid f : R \to \mathbb{C}\}$. If for f, g elements of $\mathbb{C}^R$ and for $c \in \mathbb{C}$ we define addition and scalar multiplication by

$$\begin{aligned} (f+g)(x) &:= f(x) + g(x) \\ (cf)(x) &:= cf(x)\,. \end{aligned}$$

then $V = [\mathbb{C}^R, +, 0; \mathbb{C}]$ is a $\mathbb{C}$-vector space.

Definition 13. Let f and g be elements of $\mathbb{C}^R$. We define the *multiplicative convolution* of f and g, $f * g : \mathbb{C}^R \times \mathbb{C}^R \longrightarrow \mathbb{C}^R$ by

$$(f * g)(x) := \sum_{\substack{a,b \in R, \\ ab = x}} f(a)g(b)\,.$$

Consider the function δ_A, where A is a subset of R, defined by:

$$\delta_A(x) := \begin{cases} 1 & : \quad x \in A \\ 0 & : \quad \text{otherwise}\,. \end{cases}$$

For a singleton $A = \{r\}$ we denote this simply by δ_r. The multiplicative identity of the $*$ operation is δ_1.

Lemma 14. *$\mathbb{C}^R$, with addition and scalar multiplication as above and the operation $*$, is an algebra over $\mathbb{C}$, which we call $\mathbb{C}[R, *]$.*

Note that $\delta_r * \delta_s = \delta_{rs}$ and that $\{\delta_r \mid r \in R\}$ form a basis of $\mathbb{C}[R, *]$.

Definition 15. Let f, g and w be elements of $\mathbb{C}^R$. The *multiplicative correlation* of f and w on the left, $f \circledast' w$, and of w and g on the right,$w \circledast g$, are defined by

$$(f \circledast' w)(x) := \sum_{r \in R} f(r)w(xr)$$
$$(w \circledast g)(x) := \sum_{r \in R} w(rx)g(r).$$

Lemma 16. *Let $f, g, w \in \mathbb{C}^R$, then convolution and correlation have the following relationships:*

$$(f * g) \circledast' w = f \circledast' (g \circledast' w)$$
$$w \circledast (f * g) = (w \circledast f) \circledast g$$
$$g \circledast' (w \circledast f) = (g \circledast' w) \circledast f.$$

Lemma 17. *The complex vector space V is a $\mathbb{C}[R, *]$-bimodule under the left and right $\mathbb{C}[R, *]$-actions*

$$(f, w) \longrightarrow f \circledast' w$$
$$(w, g) \longrightarrow w \circledast g.$$

Lemma 18. *The set $\mathbb{C}\delta_0$ is a two sided ideal in the algebra $\mathbb{C}[R, *]$ where*

$$\mathbb{C}\delta_0 = \{c\delta_0 \mid c \in \mathbb{C}\}.$$

With this 2-sided ideal we can immediately form the factor ring $\mathbb{C}[R, *]/\mathbb{C}\delta_0$ which we call $\mathbb{C}_0[R]$.

Definition 19. We define the set V_0 to be those functions w in V which satisfy $w(0) = 0$.

$$V_0 := \{w \in V \mid w(0) = 0\}.$$

As $w \circledast \delta_0 = 0$ for all $w \in V_0$ this induces a natural right action of $\mathbb{C}_0[R]$ on V_0 by

$$w \circledast (f + \mathbb{C}\delta_0) := w \circledast f,$$

where $g = f + \mathbb{C}\delta_0$ is any element of $\mathbb{C}_0[R]$ and $w \in V_0$. Similarly there exists a left action via $\circledast'$.

For any function $f \in \mathbb{C}[R]$ we can define left and right symmetry groups as we did for weights: $\mathrm{Sym}_R(f) := \{u \in R^\times \mid f(xu) = f(x)\ \forall x \in R\}$. It follows $\mathrm{Sym}_L(f * g) \supseteq \mathrm{Sym}_L(f)$ and $\mathrm{Sym}_R(f * g) \supseteq \mathrm{Sym}_R(g)$.

Lemma 20. *Symmetry groups are inherited as follows for correlation*

$$\mathrm{Sym}_L(w \circledast g) \supseteq \mathrm{Sym}_R(g)$$
$$\mathrm{Sym}_R(f \circledast' w) \supseteq \mathrm{Sym}_L(f).$$

Lemma 21. *Define* $S = \{f \in \mathbb{C}_0[R] \mid f(xu) = f(x) \quad \forall x \in R, u \in R^\times\}$ *and let the invariant weights from Section 1 be denoted by* W*. Then* W *is a right* S*-module under correlation* $\circledast$ *in a naturally inherited way.*

4 MacWilliams' Extension Theorem by Module Generation

We re-examine the Extension Theorem with this new perspective. We aim to classify all weights that generate W as a right S-module. This will then yield MacWilliams' Equivalence Theorem for these weights due to the following results, equivalent to those in [4].

Lemma 22. *If* ϕ *is a* w*-isometry then* ϕ *is a* $(w \circledast s)$*-isometry for all* $s \in S$*.*

Remark 23. Let R be a Frobenius ring. If $w \circledast S = W$ then $w \circledast h = w_H$ for some $h \in S$ where w_H denotes the Hamming weight. Since every w-isometry is a $w \circledast h$ isometry, by Lemma 22, we have that MacWilliams' Extension Theorem holds for w.

Let the ring R be a finite product of finite chain rings R_i, say $R = R_1 \times R_2 \times \cdots \times R_r$, with Jacobson radicals generated by (distinct) $p_1, p_2, \ldots, p_r$ of nilpotency $d_1, d_2, \ldots, d_r$ respectively. We view elements of R as r-tuples of chain ring elements i.e. $a \in R$ represented as $a = (a_1, a_2, \ldots, a_r)$ where each $a_i \in R_i$. Operations, including multiplication, are performed component-wise. The set of generators of the ideals of R is given by $\{R^\times e \mid e \in E\}$ where E are the representatives

$$E = \{{p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r} = e \mid 0 \le e_i \le d_i\}.$$

The lattice of principal left ideals of R may be described by $E({}_RR) = \{Re \mid e \in E\}$.

We have for $x, y \in E$ the relations $x \le y \quad \Leftrightarrow \quad x_i \le y_i \quad \forall i$ and hence $Rx \ge Ry \quad \Leftrightarrow \quad x_i \le y_i \quad \forall i$. The *socle* of any R-module M is the sum of the minimal submodules of M. When M is the ring itself this is the sum of the minimal ideals. Here the representative of the socle is $s = {p_1}^{d_1-1}{p_2}^{d_2-1} \ldots {p_r}^{d_r-1}$ since $\mathrm{Soc}(R \times S) = \mathrm{Soc}(R) \times \mathrm{Soc}(S)$.

Definition 24. For each $e \in E$ define the basis element

$$\delta_e := \frac{1}{|R^\times e|} \sum_{a \in R^\times e} \delta_a \, .$$

Denote by $z/x \in E$ the representative generator $a \in E$ with $a_i = z_i - x_i \ \ \forall i$. We define a basis $\{\eta_x \mid x \in E \backslash \{0\}\}$ by

$$\eta_x := \sum_{Rz \leq Rx} \mu(0, Rz) \delta_{\frac{z}{x}} \, ,$$

where μ is the Möbius function induced by the lattice of left principal ideals under the partial order of inclusion.

Lemma 25. *The Möbius function elements contained in η_x are, for all $e \in E$,*

$$\mu(0, Re) = \begin{cases} (-1)^{\Sigma(d_i - e_i)} & : \ e \leq s \\ 0 & : \ e \not\leqslant s \, . \end{cases}$$

Since $\mu(0, Rz) = 0$ for $Rz \not\leqslant Soc(R)$ we need only include those z with indices $z_i = d_i$ or $z_i = d_i - 1$ in the sum.

Proposition 26.

$$(w \circledast \eta_x)(y) = \begin{cases} \sum_{z \leq x} \mu(0, Rz) w(\frac{zy}{x}) & : \ y \leq x \\ 0 & : \ y \not\leqslant x \, . \end{cases}$$

Thus the matrix of coefficients of the weight w with respect to the basis $\{\eta_x \mid x \in E \backslash \{0\}\}$ is triangular. We require for w to generate W that the diagonal elements are nonzero, indeed this is sufficient. Combining all of these elements we arrive at our main theorem.

Theorem 27. *Let R be a finite direct product of finite chain rings with E the set of representatives of the ideals of R. If $w \in W$ with*

$$\sum_{z \leq x} \mu(0, Rz) w(z) \neq 0 \quad \textit{for all } x \in E \backslash \{0\} \, ,$$

then MacWilliams' Equivalence Theorem holds for w.

We remark that a finite commutative ring is a direct product of chain rings if and only if it is a principal ideal ring. Hence the theorem applies in particular to finite commutative principal ideal rings.

Conclusion

By considering the module of invariant weights in terms of an algebra of complex functions we have determined the conditions an invariant weight defined on a direct product of chain rings must satisfy for MacWilliams' equivalence theorem to hold. Thus provided these conditions are satisfied all isometries of that weight will extend to monomial transformations.

References

1. W. E. Clark and D. A. Drake, *Finite chain rings*, Abh. Math. Sem. Univ. Hamburg **39** (1973), 147–153.
2. I. Constantinescu and W. Heise, *On the concept of code-isomorphy*, J. Geom. **57** (1996), no. 1-2, 63–69.
3. I. Constantinescu, W. Heise, and T. Honold, *Monomial extensions of isometries between codes over Z_m*, Proceedings of the Fifth International Workshop in Algebraic and Combinatorial Coding Theory (ACCT-5) (Sozopol, Bulgaria), 1996, pp. 98–104.
4. M. Greferath and T. Honold, *On weights allowing for MacWilliams equivalence theorem*, Proceedings of the Fourth International Workshop in Optimal Codes and Related Topics (Pamporovo, Bulgaria), 2005, pp. 182–192.
5. ———, *Monomial extensions of isometries of linear codes II: Invariant weight functions on Z_m*, Proceedings of the Tenth International Workshop in Algebraic and Combinatorial Coding Theory (ACCT-10) (Zvenigorod, Russia), 2006, pp. 106–111.
6. M. Greferath, A. A. Nechaev, and R. Wisbauer, *Finite quasi-Frobenius modules and linear codes*, J. Algebra Appl. **3** (2004), no. 3, 247–272.
7. M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams' equivalence theorem*, J. Combin. Theory Ser. A **92** (2000), no. 1, 17–28.
8. A. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, *The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes*, IEEE Trans. Inform. Theory **40** (1994), no. 2, 301–319.
9. F. Kasch, *Modules and rings*, London Mathematical Society Monographs, vol. 17, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982.
10. S. Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.
11. J. MacWilliams, *A theorem on the distribution of weights in a systematic code*, Bell System Tech. J. **42** (1963), 79–94.
12. Bernard R. McDonald, *Finite rings with identity*, Marcel Dekker Inc., New York, 1974.
13. A. A. Nechaev, *Finite rings of principal ideals*, Mat. Sb. (N.S.) **91(133)** (1973), 350–366, 471.
14. A. A. Nechaev and T. Honold, *Fully weighted modules and representations of codes*, Problemy Peredachi Informatsii **35** (1999), no. 3, 18–39.
15. S. Roman, *Coding and information theory*, Graduate Texts in Mathematics, vol. 134, Springer-Verlag, New York, 1992.
16. H. N. Ward and J. A. Wood, *Characters and the equivalence of codes*, J. Combin. Theory Ser. A **73** (1996), no. 2, 348–352.
17. J. A. Wood, *Extension theorems for linear codes over finite rings*, Applied algebra, algebraic algorithms and error-correcting codes (Toulouse, 1997), Lecture Notes in Comput. Sci., vol. 1255, Springer, Berlin, 1997, pp. 329–340.
18. ———, *Factoring the semigroup determinant of a finite commutative chain ring*, Coding theory, cryptography and related areas (Guanajuato, 1998), Springer, Berlin, 2000, pp. 249–259.
19. ———, *Code equivalence characterizes finite Frobenius rings*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 699–706 (electronic).