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Generalized framework for adjoint error estimation of PG method in linear problems

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Abstract

The current work concerns the study and the implementation of a modern algorithm for error estimation in CFD computations. This estimate involves the dealing of the adjoint argument. By solving the adjoint problem, it is possible to obtain important information about the transport of the error towards the quantity of interest. The aim is to apply for the first time this procedure into Petrov-Galerkin (PG) method. Streamline Upwind Petrov-Galerkin, stabilised Residual Distribution and bubble method are involved for the implementation. Scalar hyperbolic problems are firstly used as test cases.

Keywords: Error Estimation, adjoint problem, Petrov-Galerkin method, Residual Distribution scheme, advection-reaction problem

1. Introduction

Over the last decade, much progress has been made in the area of Error Estimation. This theory provides a way to construct error indicators for CFD computations of PDE's. These error indicators can be used to drive automatic mesh adaptation algorithms. By adapting the mesh, we optimize the mesh spacings or reduce memory usage.

In this field, the *a posteriori* error analysis is one of the most used procedures to compute numerical error indicators. The relevance and generality of this estimation has been powerfully argued in the work of Johnson and his collaborators [1]. The *a posteriori* error bounds resulting from this analysis involve the numerical residual, obtained by inserting the computed solution into the current problem equations; this residual measures the extent to which the numerical approximation to the analytical solution fails to satisfy the current problem. From this study, the *Type II a posteriori* error bounds have been defined.

Becker and Rannacher worked also on this issue and they developed the so-called weighted-residual-based, or *Type I*, *a posteriori* error estimation ([2] and [3]). Here, the error representation formula defines the error in the target functional with the numerical residual, weighted by the solution of an adjoint problem. The key ingredient is this auxiliary problem, involving the formal adjoint of the current partial differential operator. For computing this product, also this adjoint problem will have to be implemented and solved numerically. The data for the adjoint problem is a quantity of interest depending on the application. In engineering applications, this is typically a functional of the analitical solution such as a mean, point value, boundary flux. In fluid dynamics, it may be the pression at the stagnation point, the pressure-drop between inflow and outflow or the drag or lift coefficients of a body immersed into the fluid.

For solving the adjoint problem, added cost rises. However, it is paid back by important information which helps us to identify where the real source of the error comes from. An example is given by Hartmann ([4]) who shows the relevance and the advent-

ages of *Type I* indicators over the *Type II* for the adaptive mesh design for a supersonic flow past an airfoil.

Definition in continuous

Primal problem model. Let Ω be a bounded open domain R^d with boundary Γ. Given the *primal problem*

$$Lu = f \quad \text{in} \quad \Omega \qquad Bu = g \quad \text{on} \quad \Gamma \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, L denotes a linear differential operator on Ω and B denotes a linear boundary operator on Γ.
To solve this problem a numerical discretization is given and an approximated solution, u_h, is sought in the place of u, by using a discrete numerical method. We consider the numerical solution as a linear combination of piecewise polynomial functions of degree p on a partition $\mathcal{T}_h$ of the domain Ω. For this reason we can write $u_h \in V_{h,p}$.

The functional $J(\cdot)$. In many problems of physical interest the quantity of interest for the current problem is an output or target functional of the solution rather the solution itself. This target functional is defined as $J(\cdot)$. Depending on the problem, it can be a different quantity, for example the drag or the lift coefficient or a point value of the solution.
This functional will be computed numerically and evaluated by the numerical solution, $J_h(u_h)$. The final purpose is to compute as good as possible this quantity, trying to estimate the order of error that the numerical approximations generate. Given a tolerance $\mathsf{TOL} > 0$, then the problem might be defined as finding $u_h \in V_{h,p}$ such that

$$|J(u) - J_h(u_h)| \leq \mathsf{TOL} \tag{2}$$

On this paper, we neglet the approximation of the functional $J_h(\cdot)$ and we simply assume to compute the exact functional by the numerical solution $J(u_h)$.

Associated adjoint problem. Mathematical theory tells us that, by the (continuous) *compatibility condition*

$$(Lu, v)_\Omega + (Bu, C^* v)_\Gamma = (u, L^* v)_\Omega + (Cu, B^* v)_\Gamma \tag{3}$$

for each operator appearing in the primal problem, L, B and C, there exists a corresponding adjoint one, L^*, B^* and C^*. By these so-called *adjoint operators*, we build the adjoint problem associated to (1)

$$L^* z = \hat{\jmath}_\Omega \quad \text{in} \quad \Omega, \qquad B^* z = \hat{\jmath}_\Gamma \quad \text{on} \quad \Gamma. \tag{4}$$

Terms $\hat{\jmath}_\Omega$ and $\hat{\jmath}_\Gamma$ on the right-hand side depend on the target quantity that we want to investigate and that, according to the theory, it is defined as follows

$$J(\omega) = (\omega, \hat{\jmath}_\Omega)_\Omega + (C\omega, \hat{\jmath}_\Gamma)_\Gamma \tag{5}$$

where $\hat{\jmath}_\Omega \in L^2(\Omega)$ and $\hat{\jmath}_\Gamma \in L^2(\Gamma)$ and C is a differential boundary operator on Γ.
The associated adjoint problem is hugely important. It provides how the information is transported towards the current quantity of interest. This property is extremely useful to derive where and how the source of the error of the target quantity is carried over the domain. The error indicator derived by this further information will be able to refine only the terms that really affect the error of the quantity of interest.

Numerical discretization

Let Ω be subdivided into shape-regular mesh $\mathcal{K} = \{\kappa\}$ consisting of elements κ. Let $\mathcal{V}_{h,p}$ be the standard finite element space of piecewise polynomials of complete degree p with C^0 continuity between elements

$$\mathcal{V}_{h,p} = \{v : v \in C^0(\Omega), v|_\kappa \in \mathcal{P}_p(\kappa), \forall \kappa \in \mathcal{K}\} \tag{6}$$

with $\mathcal{P}(\kappa)$ the space of polynomials of degree $\leq p$ defined on an element κ. Let then define a second space $\mathcal{V}^B_{h,q}$ which is the mesh dependent broken space of piecewise polynomials of complete degree q in each κ with no continuity between elements

$$\mathcal{V}^B_{h,q} = \{v : v|_\kappa \in \mathcal{P}_q(\kappa), \forall \kappa \in \mathcal{K}\} \tag{7}$$

Now, let discretize and numerically solve both problems (1) and (4) by using a *Petrov-Galerkin* (PG) method, where trial and test functions belong to $\mathcal{V}_{h,p}$ and $\mathcal{V}^B_{h,q}$, depending on the current problem.
According to (6) and (7), the boundary conditions are not included in the functional spaces and then, they are added and solved thorough the current problem. The approch with a strong formulation for the boundary conditions is hint in Appendix A.
Then following (3), the (discrete) compatibility condition holds as follows

$$\begin{aligned} &(Lu_h, v_B)_\Omega + (Bu_h, C^* v_B)_\Gamma \\ &= \\ &(u_h, L^* v_B)_\Omega + (Cu_h, B^* v_B)_\Gamma + \sum_k (H(u_h, \boldsymbol{n}), v_B^+)_{\partial\kappa\setminus\Gamma} \end{aligned} \tag{8}$$

where $H(w_h, \boldsymbol{n})$ is the numerical flux over $\partial\kappa$, which is continuous because the $w_h \in \mathcal{V}_{h,p}$, $\boldsymbol{n}$ is the outward normal along the element boundary and v_B^+ the outward traces of v_B over $\partial\kappa$.
We can so define a bilinear operator $\mathcal{B}(\cdot,\cdot)$ as

$$\mathcal{B}(u_h, v_B) = (u_h, L^* v_B)_\Omega + (C u_h, B^* v_B)_\Gamma + \sum_k (H(u_h, \boldsymbol{n}), v_B^+)_{\partial\kappa\setminus\Gamma} \quad (9)$$

By the identity (8), we state that this operator is *adjoint consistent*. The latter property is very important as it provides that using it into the numerical adjoint problem, this will be a consistent discretization of the continuous problem.
The (discrete) primal problem is then defined as

PRIMAL PROBLEM Find $u_h \in \mathcal{V}_{h,p}$ such that

$$\mathcal{B}(u_h, v_B) = F(v_B) \qquad \forall v \in \mathcal{V}_{h,q}^B \quad (10)$$

while the corresponding adjoint problem is given by

ADJOINT PROBLEM Find $z_B \in \mathcal{V}_{h,q}^B$ such that

$$\mathcal{B}(w_h, z_B) = J(w_h) \qquad \forall w_h \in \mathcal{V}_{h,p} \quad (11)$$

As we see, both problems use the same operator $\mathcal{B}(\cdot,\cdot)$ but in the primal one, the solution $u_h \in \mathcal{V}_{h,p}$ and the test function belongs to $\mathcal{V}_{h,q}^B$, while for the adjoint problem, the solution $z_B \in \mathcal{V}_{h,q}^B$ and the test function is taken from $\mathcal{V}_{h,p}$.
Let now remember that

$$u_h = \sum_{\kappa\in\mathcal{K}} \sum_{j\in\kappa} \varphi_j u_j \quad \text{and} \quad z_B = \sum_{\kappa\in\mathcal{K}} \sum_{i\in\kappa} \Psi_i z_i$$

Gathering all the coefficients u_j, into a vector, $\boldsymbol{u} = \{u_1, u_2, \ldots, u_N\}$, let write the primal problem in the following matrix form

$$\boldsymbol{B}\,\boldsymbol{u} = \boldsymbol{F} \quad (12)$$

Then, we can prove that since the bilinear operator $\mathcal{B}(\cdot,\cdot)$ is symmetrical and adjoint consistent, the adjoint problem can be written in a matrix form as

$$\boldsymbol{B}^T\,\boldsymbol{z} = \boldsymbol{J} \quad (13)$$

where the vector $\boldsymbol{z} = \{z_1, z_2, \ldots, z_N\}$ and $\boldsymbol{B}^T$ is simply the transpose of the matrix used in the primal problem. Besides the matrix approach, we also deal with an alternative solver by using the iterative explicit method in Appendix B.

Broken space $\mathcal{V}_{h,q}^B$
A pletora of possible Broken spaces $\mathcal{V}_{h,q}^B$ are available. Herebelow let list the three Petrov-Galerkin spaces that will be taken into account for the next computations. Therefore, let remind that for a PG method, all of the broken spaces $\mathcal{V}_{h,q}^B$ can be always defined by a sum of two contributions

$$\mathcal{V}_{h,q}^B = \text{span}\{\Psi^0 + \Psi^1\}$$

where Ψ^0 is the "main" shape function of the space and Ψ^1 is a element stabilizer term.

Streamline Upwind Petrov-Galerkin method. In case of a Stremline Upwind Petrov-Galerkin (SUPG) method, the broken space $\mathcal{V}_{h,q}^B$ is the same than $\mathcal{V}_{h,p}$ with still the addition of a stabilizer term

$$\mathcal{V}_{h,q}^B = \text{span}\{\varphi_i^q + \tau_\kappa \boldsymbol{b} \cdot \nabla\varphi_i^q\} \quad (14)$$

where τ_κ is usually the size of the element κ and φ_i^q the Lagrange interpolant polynomial of the degree of freedom, i, and order q; while, fanally, $\boldsymbol{b}$ is the local advection of the current problem.
So for this functional space $\Psi_j^0 = \varphi_j^q$ and $\Psi_j^1 = \tau_\kappa \boldsymbol{b} \cdot \nabla\varphi_j^q$.

Residual-Distribution Method. If the stabilised Residual Distribution (RD) space is chosen, the broken space, $\mathcal{V}_{h,q}^B$, is defined as

$$\begin{aligned} \mathcal{V}_{h,q}^B &= \text{span}\{\chi^\kappa \beta_i^\kappa\} && \text{for} \quad q = 1 \\ \mathcal{V}_{h,q}^B &= \text{span}\{\chi^\kappa \beta_i^\kappa + \tau_\kappa \boldsymbol{b} \cdot \nabla\varphi_i^q\} && \text{for} \quad q > 1 \end{aligned} \quad (15)$$

where $\beta_i^\kappa \in C^0$, χ^κ is 1 in κ and 0 everywhere else, while τ_κ is usually the size of the element κ and φ_i^q the Lagrange interpolant polynomial of the degree of freedom, i, and order q. So $\Psi_j^0 = \chi^\kappa \beta_j^\kappa$ and $\Psi_j^1 = 0$ or $\tau_\kappa \boldsymbol{b} \cdot \nabla\varphi_j^q$.

"Bubble" method. The "bubble" method comes out from the equivalence of the RD and SUPG method. Indeed, it takes the following broken space $\mathcal{V}_{h,q}^B$

$$\mathcal{V}_{h,q}^B = \text{span}\{\varphi_i^q + S^\kappa \alpha_i^\kappa\} \quad (16)$$

where φ_i^q the Lagrange interpolant polynomial of the degree of freedom, i, and order q, while S^κ is a bubble function null along $\partial\kappa$. In this case, it is either a linear bubble, unary in the baricentric point $\boldsymbol{x}_g$, or a cubic

one, coming out from the product of the three vertex linear functions. Finally α_i^κ is given by solving the equivalence

$$\int_\kappa \varphi_i^q + S^\kappa \alpha_i^\kappa d\boldsymbol{x} = \int_\kappa \beta_i^\kappa d\boldsymbol{x}$$

Then here $\Psi_j^0 = \varphi_j^q$ and $\Psi_j^1 = S^\kappa \alpha_i^\kappa$.

Superconvergence of Functionals

Let now examinate the convergence rates for functionals. Barth [5] proved that functionals show a superconvergence property when the Galerkin orthogonality of the error $u - u_h$ holds with respect to the all test functions of the bilinear operator. So even if the convergence rates for the global error measures might be not affected, the convergence rates for the functionals can be quite different with or without this orthogonality property. Barth showed in fact that for finite volumes with reconstruction, because of this orthogonality loss the superconvergence for the functionals is not attained.

In the setting of linear advection problems, this superconvergence theory for functionals has been treated as well by Süli and Houston [6] for streamline diffusion discretization. For problems with sufficiently smooth primal and adjoint solutions, the basic theoretical result states that if the primal method converges at the rate $O(h^{p+1/2})$ then functionals converge at the rate $O(h^{2p+1})$.

This theory comes from by using the classic definition of streamline diffusion method where the stabilizer term appears in the bilinear operator and not in the projected functional space, $\mathcal{V}_{h,q}^B$ like in the current approach. Both being the same method simply rewritten in two different ways, the same convergence ratio is expected. So now, to prove it, let use this alternative approach with the SUPG as broken space and numerically verify the convergence rate of functionals.

The simple channel case is then taken, where the advection is unary and parallel to the walls and with g=sin$^2(y)$ and $\hat{j}_\Gamma$=sin$^2(y)$. Table 1 tabulates values of the global solution error and the error in the funtional using a sequence of four nested meshes. The numerical results using p=1 and p=2 approximation confirm the theoretically order convergence in the global L^2 error norm and in the weighted outflow functional. The superconvergence of functionals, here verified for SUPG method, will later have to be done also for the RD and bubble broken space.

Method	dofs	p	$\|\|u - u_h\|\|_{L^2}$		$\|J(u) - J(u_h)\|$	
SUPG	524	1	.134e-1		.120e-2	
SUPG	2013	1	.256e-2	(2.39)	.161e-3	(2.90)
SUPG	7841	1	.490e-3	(2.38)	.203e-4	(2.99)
SUPG	31233	1	.104e-3	(2.24)	.253e-5	(3.00)
SUPG	2013	2	.829e-3		.190e-2	
SUPG	7889	2	.882e-4	(3.23)	.243e-3	(4.64)
SUPG	31041	2	.109e-4	(3.02)	.301e-4	(4.84)
SUPG	124289	2	.148e-5	(2.89)	.370e-5	(4.91)

Table 1: Convergence rates of SUPG method for channel problem. Tabulated data for the global $L^2(\Omega)$ error and error in the outflow flux functional.

Error Representation Formula

Let consider the primal numerical problem (10) and that holds the *Galerkin orthogonality condition*

$$\mathcal{B}(u - u_h, v_B) = 0 \qquad \forall v \in \mathcal{V}_{h,p} \tag{17}$$

Let then notice that by the *compatibility condition* (3), using infinite-dimentional trial and test space, the adjoint problem can be redefined as

$$\mathcal{B}(w, z) = J(w) \qquad \forall w \in \mathcal{V}$$

So, an exact error representation formula for a given functional $J(\cdot)$ results from the following steps, where π_B denotes any suitable projection operator (i.e. interpolation, L_2 projection) into $\mathcal{V}_{h,p}^B$,

$$\begin{aligned} J(u) - J(u_h) &= J(u - u_h) \\ &= \mathcal{B}(u - u_h, z) \\ &= \mathcal{B}(u - u_h, z - \pi_B z) \\ &= \mathcal{B}(u, z - \pi_B z) - \mathcal{B}(u_h, z - \pi_B z) \\ &= F(z - \pi_B z) - \mathcal{B}(u_h, z - \pi_B z) \end{aligned} \tag{18}$$

so in summary

$$J(u) - J(u_h) = F(z - \pi_B z) - \mathcal{B}(u_h, z - \pi_B z) \tag{19}$$

where no dependence on the exact solution u appears. Comptutationally, this error representation formula is not suitable for obtaing computable *a posteriori* error estimation unless the function $z - \pi_B z$ is unknown, since $z \in \mathcal{V}^B$ is a solution of the infinite-dimensional adjoint problem. So z has to be computed by the discrete adjoint problem (11). Since the (8), we can solve the adjoint problem by using the same bilinear operator, $\mathcal{B}(w_h, z)$, used for the primal problem. However, due to the Galerkin orthogonality, the adjoint numerical problem must be approximated in a larger space

of functions than that utilized in the primal numerical problem. Here, this is achieved by solving the adjoint problem using a polynomial space that is one polynomial degree higher than the primal numerical problem, i.e. if $v_B \in \mathcal{V}^B_{h,q}$ then $z \approx z_{B'} \in \mathcal{V}^B_{h,q+1}$.
Besides, the error representation formula written in the global abstract form of the (19) does not indicate which elements in the mesh should be refined to reduce the measured error in a functional. So, now the goal is to estimate the *local* contribution of each element in the mesh to the functional error. This local cell contribution will then be used as an error indicator for choosing which elements to refine or coarsen in the adaptive mesh procedure. By applying the *triangle inequality*, indeed, we have

$$\begin{aligned}\left|J(u) - J(u_h)\right| &= \left|F(z - \pi_B z) - \mathcal{B}(u_h, z - \pi_B z)\right| \\ &= \Big|\sum_{\kappa\in\mathcal{K}} F_\kappa(z - \pi_B z) - \mathcal{B}_\kappa(u_h, z - \pi_B z)\Big| \\ &\leq \sum_{\kappa\in\mathcal{K}} \left|F_\kappa(z - \pi_B z) - \mathcal{B}_\kappa(u_h, z - \pi_B z)\right|\end{aligned} \tag{20}$$

This direct estimate let us define for each partition element κ the *adaptation element indicator* η_κ

$$|\eta_\kappa| \equiv \left|\mathcal{R}_\kappa(u_h, z - \pi z)\right| \tag{21}$$

with

$$\mathcal{R}_\kappa(u, v) = (R(u), v)_\kappa + (r(u), C^* v)_{\partial\kappa\cap\Gamma}$$

where $R(u) = f - Lu$ and $r(u) = g - Bu$. Such that the simplest adaptation stopping criteria will be

$$|J(u) - J(u_h)| = \Big|\sum_\kappa \eta_\kappa\Big|$$

and according to a particular adaptive meshing strategy the element will be refine or coarse depening on the corresponding η_κ.

Projection Operation π_B. As seen, the unkonwn function $z - \pi_B z$ is computed by approximating $z \approx z_{B'} \in \mathcal{V}^B_{h,q+1}$, such that seeking the adjoint solution in a larger space than the trial one used for the primal problem and later it is projected onto $\mathcal{V}^B_{h,q}$. Let

$$z = \sum_i^N \Psi_i z_i \qquad \text{and} \qquad \pi z = \sum_i^{N^\pi} \Psi_i^\pi \pi z_i$$

Because $\Psi_i(\boldsymbol{x}_i) = 1$ but $\Psi_i(\boldsymbol{x}_j) \neq 0$, then both $z(\boldsymbol{x}_i) \neq z_i$ and $\pi z(\boldsymbol{x}_i) \neq \pi z_i$. So, to get the coefficients πz_i we need to solve the following identity

$$\int_\Omega \Psi^\pi_{i_B} z \, d\boldsymbol{x} = \int_\Omega \Psi^\pi_{i_B} \pi z \, d\boldsymbol{x} \tag{22}$$

According to the z and πz definitions and because of the local element support of the shape functions, we end up to a local element linear system as follows

$$\boldsymbol{P}_\kappa \, \boldsymbol{z}_\kappa = \boldsymbol{M}_\kappa \, \pi \boldsymbol{z}_\kappa \tag{23}$$

where the pedex κ means by the current κ-element and

$$\boldsymbol{M}_\kappa = \int_\kappa \Psi_i^\pi \Psi_j^\pi, \; \boldsymbol{P}_\kappa = \int_\kappa \Psi_i^\pi \Psi_j, \qquad \boldsymbol{z}_\kappa = z_i^\kappa, \; \pi \boldsymbol{z}_\kappa = \pi z_i^\kappa$$

so finally,

$$\pi \boldsymbol{z}_\kappa = \boldsymbol{M}_\kappa^{-1}(\boldsymbol{P}_\kappa \, \boldsymbol{z}_\kappa)$$

The projected solution πz is then computed and hence, we can obtain the local error indicator defined in (21).

Conclusions

Among different types of error representation formula, we interested in Type I error indicator. Unlike Type II error bounds where the only local residual is taken into account, Type I involves also the solution of the associated adjoint problem. By it, we are able to obtain further information concerning the trasport of the error of computing the quantity of interest.
For the first time, we want to develop this error analysis applied a Petrov-Galerkin numerical discretization by using methods like Streamline Upwind PG, stabilised Residual Distribution and bubble method.
In the current PG discretization, the asymmetry of both function spaces, $\mathcal{V}_{h,p}$ and $\mathcal{V}^B_{h,q}$, is kept into their own definitions, so without affecting the bilinear operator terms. This and the (discrete) compatibility condition let us define a unique adjoint consistent $\mathcal{B}(\cdot,\cdot)$ as in the Galerkin form.
By now, we are implementing this error estimation theory on a code. We start by simple scalar first-order hyperbolic problem (using for the primal solution both p=1 and p=2) to pass later to more complicated problems like the 2D compressible Euler equations.

Appendix A. Strong boundary conditions

Strong boundary conditions are realized by considering an appropriate function (sub)space $\mathcal{V}_{h,p,0} \subset \mathcal{V}_{h,p}$ (or $\mathcal{V}_{B,q,0} \subset \mathcal{V}_{B,q}$) which incorportates directly the conditions on the boundary Γ. Then, the function spaces to search the primal and adjoint solutions that

also include homogenous boundary conditions are, respectively,

$$\mathcal{V}_{h,p,0} = \{v : v \in C^0(\Omega), v|_\kappa \in \mathcal{P}_p(\kappa), \forall \kappa \in \mathcal{K},\ Bv = 0 \text{ on } \Gamma\}$$
$$\mathcal{V}^B_{h,q,0} = \{v : v|_\kappa \in \mathcal{P}_q(\kappa), \forall \kappa \in \mathcal{K},\ B^* v = 0 \text{ on } \Gamma\} \tag{A.1}$$

and so the corresponding bilinear operator is given by

$$\mathcal{B}_0(u_{h,0}, v_B) = (u_{h,0}, L^* v_B)_\Omega + \sum_k (H(u_{h,0}, \boldsymbol{n}), v_B^+)_{\partial\kappa\setminus\Gamma}$$

For the realization of the inhomogeneous inflow boundary conditions, assume that $u_h = u_{h,0} + u_g$ with $u_g \in \mathcal{V}_{h,p}$ and such that $Bu_h = g$ on Γ; since $Bu_{h,0}\big|_\Gamma = 0$, by definition, then $Bu_g = g$ on Γ. The same approch is taken for $z_B = z_{B,0} + z_{\hat{j}_\Gamma}$, with $z_{\hat{j}_\Gamma} \in \mathcal{V}_{h,q}$ and where $B^* z_B = \hat{j}_\Gamma$ and $B^* z_{B,0} = 0$ on Γ such that $B^* z_{\hat{j}_\Gamma} = \hat{j}_\Gamma$ on Γ.

Appendix B. Iterative explicit Method

Let remember the bilinear operator definition

$$\mathcal{B}(u_h, v_B) = (u_h, L^* v_B)_\Omega + (Cu_h, B^* v_B)_\Gamma + \sum_k (H(u_h, \boldsymbol{n}), v_B^+)_{\partial\kappa\setminus\Gamma}$$

and the definitions of both solutions

$$u_h = \sum_{\kappa\in\mathcal{K}} \sum_{j\in\kappa} \varphi_j\, u_j \quad \text{and} \quad z_B = \sum_{\kappa\in\mathcal{K}} \sum_{i\in\kappa} \Psi_i\, z_i$$

Hence, we could solve both problems by computing the coefficients u_i and z_j througth the explicit iterative method

$$u_i^{n+1} = u_i^n - \omega \Phi_i^u \qquad \text{with} \qquad \Phi_i^u = \mathcal{B}(u_h, \Psi_i) - F(\Psi_i)$$
$$z_j^{n+1} = z_j^n - \omega \Phi_j^z \qquad \text{with} \qquad \Phi_j^z = \mathcal{B}(w_j, z_B) - J(w_j) \tag{B.1}$$

where $\Psi_i \in \mathcal{V}_{h,p}$ and $w_j \in \mathcal{V}_{h,q}$ and u_i^0 and z_j^0 are given initial solutions.

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