Adaptive Coordinate Descent
Ilya Loshchilov, Marc Schoenauer, Michèle Sebag

To cite this version:

HAL Id: inria-00587534
https://inria.hal.science/inria-00587534
Submitted on 20 Apr 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ABSTRACT

Independence from the coordinate system is one source of efficiency and robustness for the Covariance Matrix Adaptation Evolution Strategy (CMA-ES). The recently proposed Adaptive Encoding (AE) procedure generalizes CMA-ES adaptive mechanism, and can be used together with any optimization algorithm. Adaptive Encoding gradually builds a transformation of the coordinate system such that the new coordinates are as decorrelated as possible with respect to the objective function. But any optimization algorithm can then be used together with Adaptive Encoding, and this paper proposes to use one of the simplest of all, that uses a dichotomy procedure on each coordinate in turn. The resulting algorithm, termed Adaptive Coordinate Descent (ACiD), is analyzed on the Sphere function, and experimentally validated on BBOB testbench where it is shown to outperform the standard (1 + 1)-CMA-ES, and is found comparable to other state-of-the-art CMA-ES variants.

Categories and Subject Descriptors
I.2.8 [Computing Methodologies]: Artificial Intelligence Problem Solving, Control Methods, and Search

General Terms
Algorithms

Keywords
Continuous Optimization, Adaptive Encoding, Line Search, Adaptive Coordinate Descent, Covariance Matrix Adaptation

1. INTRODUCTION

Separable continuous optimization problems are problems in which the objective function can be optimized coordinate-wise. Finding the global optimum of a separable function in $\mathbb{R}^d$ amounts to perform $d$ simple line searches along each of the $d$ coordinates. Unfortunately, interesting problems are usually not separable. Nevertheless, many optimization methods implicitly assume some form of separability of the objective function, or at least are much more efficient on separable functions as they explicitly use the coordinate system in their search. A well-known exception is the Covariance Matrix Adaptation Evolution Strategy (CMA-ES) [7], that performs a rotation-invariant search, and is thus independent of any coordinate system. The basic idea of CMA-ES is to evolve, besides a population of solutions to the optimization problem at hand, a “Covariance Matrix” that can be viewed as a coordinate transform: in case of a quadratic objective function, CMA-ES Covariance Matrix has been empirically demonstrated to gradually converge to the inverse Hessian matrix of the objective function. In the coordinate system defined by this inverse Hessian, the quadratic objective function has become separable, and the optimization problem, trivial. Of course, CMA-ES covariance matrix is only, in the quadratic case, an approximation of the inverse Hessian. And interesting problems are not quadratic indeed (and if they were, they would be easy to solve directly). Nevertheless, twice continuously differentiable objective functions can be viewed as close-to-quadratic around their optima (local or global), and adapting the coordinate system with respect to the “cumulative path” of the search makes it easier and faster to find the optimum.

The basic principles of this adaptive coordinate transformation have been generalized to general search strategies, under the name of Adaptive Encoding in [4], and experimented with Cauchy mutations in a stochastic search framework. The resulting optimization method is hence heavily coordinate-dependent, and its results deteriorate when the degree of non-separability of the objective function increases. However, this limitation of Cauchy mutation almost vanishes with Adaptive Encoding, demonstrating the usefulness of a well-designed adaptive coordinate system.

Putting things together, a natural idea is then to couple some simple optimization methods, i.e., some successive coordinate-wise line searches, with Adaptive Encoding: coordinate line-searches only work well for separable functions, but Adaptive Encoding should gradually lead the search toward a transformed coordinate system where the objective function resembles more a separable function than in the original system, paving the road for the coordinate line-search. Though the resulting algorithm has little to do with Evolutionary Computation, it heavily relies on Adaptive Encoding, the backbone of CMA-ES algorithms.
The paper is organized the following way: Section 2 first introduces the algorithmic background, namely Adaptive Encoding and some Coordinate Descent Method, before detailing their coupling into the Adaptive Coordinate Descent algorithm. Section 3 presents the experiments that validate Adaptive Coordinate Descent first on the Sphere function, the well-known separable test function, establishing performance bounds for the proposed approach. Extensive experiments on the BBOB testbench [5] are then presented and discussed. Finally, Section 4 concludes the paper and sketches directions for further researches.

2. ALGORITHMS

2.1 Adaptive Encoding

Though historically introduced as a derandomization of self-adaptive Evolution Strategies (ES) [7], CMA-ES was only recently revisited as a hybrid between some ES with adaptive step-size and some Adaptive Encoding (AE) procedure [4]. AE can be applied to any continuous domain search algorithm, in order to make it independent from any given coordinate system. As a result, some search algorithm that performed rather poorly on non-separable functions can be tremendously boosted (e.g., by a factor up to 3 orders of magnitude for Evolution Strategy with Cauchy distribution [4]).

An iteration of CMA-ES, decomposed into Adaptive Encoding and Evolution Strategy with step-size adaptation, is described in Algorithm 1. In standard Evolution Strategy, \( \lambda \) offspring are sampled (line 1) from a normal distribution with step-size \( \sigma \) and mean \( m \), where \( m \) is the centroid of best \( \mu \) individuals of the previous iteration. The \( \lambda \) offspring are evaluated (line 2 with \( B = I \)) the identity matrix). Depending on the choice of the step-size adaptation rule, the step-size is then adapted, either by some rule similar to the one-fifth rule [13, 11] (line 4), or using the Cumulative Step-size Adaptation [6] (line 6).

Algorithm 1 CMA-ES = Adaptive Encoding + ES

\[
\begin{align*}
1: & \quad x_i \leftarrow m + \sigma N(0, I), \text{ for } i = 1 \ldots \lambda \\
2: & \quad f_i \leftarrow f(Bx_i), \text{ for } i = 1 \ldots \lambda \\
3: & \quad \text{if Evolution Strategy (ES) with 1/5th success rule then}
4: & \quad \sigma \leftarrow \sigma \exp \frac{n}{\lambda (\text{expected success rate})} \\
5: & \quad \text{if Cumulative Step-Size Adaptation ES (CSA-ES) then}
6: & \quad \sigma \leftarrow \sigma \exp \frac{n}{\lambda (\text{expected success rate})} \\
7: & \quad B \leftarrow \text{AdaptiveEncoding}(Bx_1, \ldots, Bx_\lambda)
\end{align*}
\]

CMA-ES differs from standard lines 2 and 7, that describe the use of the Adaptive Encoding procedure. CMA-ES maintains a coordinate system transformation matrix \( B \), and though it evaluates the individuals in the original coordinate system of \( \mathbb{R}^d \) (line 2), it generates the offspring, using some isotropic normal distribution, in some transformed coordinate system (line 1). The \( d \times d \) matrix \( B \) is the matrix of the transformation. In Algorithm 1, offspring \( x_i \) are represented in this transformed coordinate system, and \( Bx_i \) are their images in the original coordinate system. Matrix \( B \) is iteratively adapted by the AE procedure using information from the most successful \( \mu \) offspring (line 7).

The CMA update rule for \( B \), denoted as AE:CMA, derived from the original Covariance Adaptation rule of the \((\mu, \lambda)\)-CMA-ES [7], is detailed in Algorithm 2. The covariance matrix update is similar to some Principal Component Analysis (PCA) of the successful search steps. The goal of PCA is to find an orthogonal transformation to convert the set of possibly correlated variables into a set of uncorrelated variables, called principal components, that are the eigenvectors of the covariance matrix of the data. However, while PCA is usually used to reduce the dimensionality of the data by taking into account only the main principal components (corresponding to the largest eigenvalues), CMA retains all principal components. These components are determined at each iteration by the eigen-decomposition of the current covariance matrix \( C \) (line 15 of Algorithm 2). The transformation matrix \( B \) is the square-root of the covariance matrix \( C \) (line 16). An illustration of Principal Components Analysis is shown in Fig. 1.(a), where the principal components are depicted as the dotted lines, such that the largest variance by any projection of the data comes to lie on the first principal component, the second largest variance on the second, and so on. The idea of such a transformation is to make the objective function in the transformed space as similar as possible to the Sphere function, which is known to be simple for analysis and optimization.

Algorithm 2 Adaptive Encoding

\[
\begin{align*}
1: & \quad \text{Input: } x_1, \ldots, x_\mu \\
2: & \quad \text{if Initialize then}
3: & \quad w_i \leftarrow \frac{1}{\lambda}; \ c_i \leftarrow \frac{c_i}{c_0}; \ \sigma \leftarrow \frac{\sigma}{\sigma_0} \\
4: & \quad \rho \leftarrow 0 \\
5: & \quad C \leftarrow I; \ B \leftarrow I \\
6: & \quad m \leftarrow \sum_{i=1}^{\lambda} x_i w_i \\
7: & \quad \text{return.} \\
8: & \quad \mu \leftarrow m \\
9: & \quad m \leftarrow \sum_{i=1}^{\lambda} x_i w_i \\
10: & \quad z_0 \leftarrow \left[ \left[ B^{-1}(m - m^\ast) \right] - (m - m^\ast) \right] \\
11: & \quad z_i \leftarrow \left[ B^{-1}(x_i - m^\ast) \right] - (x_i - m^\ast) \\
12: & \quad p \leftarrow (1 - c_0)p + \sqrt{c_0(2 - c_0)}z_0 \\
13: & \quad C_\mu \leftarrow \sum_{i=1}^{\lambda} w_i z_i z_i^T \\
14: & \quad C \leftarrow (1 - c_1 - c_0)C + c_1 p p^T + c_0 C_\mu \\
15: & \quad B' \leftarrow \text{eigendecomposition}(C) \\
16: & \quad B \leftarrow B' D \\
17: & \quad \text{Output: } B
\end{align*}
\]

Fig. 1.(b) illustrates an Adaptive Encoding Update iteration, where only the \( \mu \) (green/bold) best among \( \lambda \) generated offspring are used to compute a partial covariance matrix \( C_\mu \) (line 13), which replaces a fraction \( c_\mu \) of the current covariance matrix \( C \) (line 14). Additionally, the path of the mean of distribution (evolution path \( p \)) is recorded in order to increase the variance of favorable directions (line 12, and bold arrow in Fig. 1.(b)). A fraction \( c_1 \) of the current covariance matrix \( C \) is also replaced by the rank-one matrix of eigen direction \( p \) (line 14). However, the update in line 14 might become instable if \( c_\mu + c_1 > 1 \). Hence the parameter setting (line 3) needs to be chosen specifically for the algorithm at hand.

It has been shown in [4] that the AE:CMA-update applied to Evolution Strategy with Cauchy distribution improves the performance on non-separable functions by a factor of roughly one thousand. These results and the fact that the Sphere-like transformed space is usually simpler to analyse
than the original one, make it reasonable to explicitly exploit
this property in search.

2.2 Coordinate Descent by Dichotomy

Coordinate Descent (CD) is probably one of the oldest
multidimensional optimization method. It became espe-
cially popular in numerical linear algebra under the name
of Gauss-Seidel method for solving systems of linear equa-
tions. In Evolutionary Computation community, when used
for optimization, this method is called Coordinate Strategy
[14]. CD is based on the idea that an n-dimensional opti-
mization problem can be decomposed into n one-dimensional
sub-problems. Each variable is updated in turn, while all
other variables remain fixed, by solving a one-dimension
imization sub-problem using any suitable one-dimension
imization optimization algorithm. Note that CD can be viewed as
a special case of Block Coordinate Descent, that partitions the
coordinates into N blocks: f is iteratively optimized with
respect to one of the coordinate block while other coordi-
nates are fixed [15]. Obviously, it is reasonable to use CD
when dealing with unimodal separable problems.

2.2.1 Adaptive Dichotomy

One of the simplest one-dimension optimization algorithm
to use is a dichotomy method (inspired by the bisection
method to find a zero of a given function). Let us consider
an interval [a, b] where the optimum is known to lie, and
assume that the value of the objective function f is known at
the center \( m = \frac{a + b}{2} \) of the interval. Evaluate the two points
\( X_1 = m - \frac{(b - a)}{4} \) and \( X_2 = m + \frac{(b - a)}{4} \), centers of the left
and right parts of [a, b]. If f is unimodal, only three cases are
possible: \( X_1 \) is better than \( m \) and \( X_2 \), \( X_2 \) is better than \( m \)
and \( X_1 \), or both \( X_1 \) and \( X_2 \) are worse than \( m \) (if \( X_1 \) and \( X_2 \)
are both better than \( m \), then the problem is multimodal).

If \( X_1 \) is better than \( m \) and \( X_2 \), then the optimum lies in
the interval \([a, m)\): replace \( b \) with \( m \) and \( m \) with \( X_1 \).
Similarly, if \( X_2 \) is better than \( m \) and \( X_1 \), replace \( a \) with
\( m \) and \( m \) with \( X_2 \). Finally, if \( X_1 \) and \( X_2 \) are worse than
\( m \), then the optimum lies in the interval \( (X_1, X_2) \): replace \( a \)
with \( X_1 \) and \( b \) with \( X_2 \). In all 3 cases, we end up with a new
interval \([a, b]\) which contains the optimum, whose length is
half that of the original \([a, b]\), and for which we know the
value of \( f \) at its center.

When dealing with multi-dimensional problems, dichotomy
steps can be achieved on each coordinate successively: Fig-
ure 1.(c) illustrates the 2D-case and displays an example of
one dichotomy step in each direction.

Another point of view on the dichotomy method is to con-
sider it as a derandomized \((1 + 2) - ES\) algorithm with
step-size adaptation: Assuming the current step-size is \( \sigma \)
and current solution is \( m \), the basic step of the dichotomy
method described above generates 2 offspring \( X_1 \) and \( X_2 \) in
deterministic way. The best of \( m, X_1, \text{and } X_2 \) becomes the
next parent, and \( \sigma \) is divided by 2. In the case of one-
dimensional unimodal problems, if the initial interval con-
tains the global optimum, this algorithm will find it. Simi-
larly, in the case of multi-dimensional unimodal problems, if
the initial rectangle contains the global optimum, the algo-
rithm will find it, either by running the dichotomy method
on each coordinate up to a given precision, or by alternating
one step of the dichotomy method in each direction in turn.

Fig. 2.(a) shows the result of such an optimization of the
Sphere function \( f(x) = \|x\|^2 \) starting from the initial point
\( X_0 = (-3.1, -4.1) \). Dichotomy proceeds for 20 iterations
for the first coordinate and then for 20 iterations for the
second coordinate, reaching the target function value \( 10^{-10} \)
after 80 evaluations. Exactly the same result is obtained by
cyclically repeating this procedure over each coordinate in
turn, as shown on Fig. 2.(b). The second variant, however,
seems to be more robust if the problem is not perfectly
separable, exploring a larger region of the search space rather
than rapidly reducing one dimension to a single value.

However, if the optimum lies outside the initial interval,
or if the interval is somehow transformed after a rotation of
the coordinate system (e.g., due to Adaptive Encoding, see
Section 2.3), it might be necessary to allow more exploration
in case of successful sampling (one offspring was better than
the parent \( m \)). Such dichotomy method with step-size (in-
terval) adaptation will be called Adaptive Dichotomy (AD),
and works as follows: Generate two offspring \( X_1 \) and \( X_2 \) as

Figure 1: AE-CMA-like Adaptive Encoding Update (b) mostly based on Principal Component Analysis (a) is used to extend
some Coordinate Descent method (c) to the optimization of non-separable problems (d). See text for details.
there is a constant $c$ well studied in this context. It that of standard Evolution Strategies, whose behavior is
Adaptive Dichotomy on the Sphere function, and compare
functions, let us analyze the convergence rate of CD with
will be given in Section 2.3 (Algorithm 3).

A more formal description of CD with Adaptive Dichotomy
at the price of additional functions evaluations.

In the case of standard dichotomy,
\[ k_{\text{succ}} = 0 \]
with the dichotomy \((a-e)\) and Adaptive Coordinate Descent (ACiD) \((f)\) and Rosenbrock 
\((e,f)\) functions. The initial point \(X_0 = (-3.1, -4, 1)\) and the target point \(X_{\text{target}}\) with \(f(X_{\text{target}}) < 10^{-10}\). While the CD
with the dichotomy \((a,b)\) performs best on the Sphere function cyclically dividing by two the step size for the corresponding coordinate (depicted as the line), the increasing of the step size by factor \(k_{\text{succ}}\) in the case of successful sampling leads to a better but still slow convergence on non-separable Rosenbrock function \((e)\). The adaptation of the coordinate system allows significantly speed up the search \((f)\).

2.2.2 Convergence Rates

Before turning to Adaptive Encoding and non-separable functions, let us analyze the convergence rate of CD with
Adaptive Dichotomy on the Sphere function, and compare it that of standard Evolution Strategies, whose behavior is
well studied in this context.

Linear convergence to the optimal point \(X^*\) takes place if there is a constant \(c \neq 0\), such that

\[
\frac{1}{T_k} \ln \frac{\|X_k - X^*\|}{\|X_0 - X^*\|} \to c,
\]

where \(X_0\) is the initial point and \(X_k\) the best point found after \(k\) iterations for a cost of \(T_k\) fitness function evaluations.

The empirical convergence rate on the Sphere function of the proposed CD with \(k_{\text{succ}} = 0.5, 1.0\) and \(2.0\), as well as that of two variants of \((1+1)\)-Evolution Strategy, as estimated from the median of 101 independent runs, is shown in Fig. 3. The algorithm denoted as \((1+1)\)-ES corresponds to the \((1+1)\)-Evolution Strategy with the initial step-size \(\sigma_0 = 1.8\), while the search interval is \([-3; 3]^d\). The \((1+1)\)-ES opt algorithm is the Evolution Strategy with the scale-invariant step-size: the optimal step-size for ES on Sphere function is proved to be \(\sigma = \frac{1}{d} \ln \|X - X^*\|\), i.e., proportional to the distance to the optimum.

It is clear that the convergence rate of the CD with \((1+1)\) dichotomy \((k_{\text{succ}} = 0.5)\) is linear with dimension \(d\), and is equal to \(-\ln(2)/2d = -0.3465/d\). The rates for CD with Adaptive Dichotomy, with \(k_{\text{succ}} = 1.0\) and \(k_{\text{succ}} = 2.0\) are 1.5 and 2.0 times slower, respectively, than with \(k_{\text{succ}} = 0.5\).

The recently proposed technique of mirrored sampling and sequential selection for Evolution Strategy [2], can also be
used with the CD method proposed here, because the sampled points are symmetric by definition. We hence propose
a modified version of the CD with Adaptive Dichotomy algorithm, called CD \(k_{\text{succ}}=2.0\) ms, in which the second offspring is not evaluated if the first one is better than the parent. This does reduce the number of fitness function evaluations, though marginally, as can be seen on Fig. 3 (line with label “ms”).

It is important to note that the optimal convergence rate of (1+1)-ES opt is not achievable in practice, because the optimal step-size is unknown for a given black-box function. In the case of CD, parameter \(k_{\text{succ}}\), which controls the exploration rate, can be used to implicitly tune the target convergence rate.

A final remark on one-dimensional algorithms: Obviously, any other one-dimensional optimization method could be used instead of the Dichotomy method. The Golden Section method (also called Fibonacci method) is known to have explored new points with respect to two evaluated points on a line. Therefore, after a change of coordinate (due to Adaptive Encoding, see next Section) will require to recompute the fitness of these rotated point whereas dichotomy only requires the fitness value of the center of the current domain, that is preserved by the change of coordinate.

### 2.3 Adaptive Coordinate Descent

The Adaptive Encoding procedure (Section 2.1) iteratively learns the coordinate systems in which the objective function is “as close to separable as possible”. The Coordinate Descent method (Section 2.2) takes advantage of the separability of the problem at hand, iteratively optimizing \(d\) independent one-dimensional problems. Combining both approaches leads to propose Adaptive Coordinate Descent (ACiD), which benefits from these two ideas, interleaving CD and AE, learning the same coordinate transform than the original CMA-ES inspired AE, and performing CD steps in the transformed space.

Fig. 1(d) illustrates how AE acts on the iterations of one-dimensional search steps of Fig. 1(c). The advantages of ACiD over CD become obvious on non-separable functions. Fig. 2(e) and (f) show sample runs of CD and ACiD respectively, on Rosenbrock function in 2-D: ACiD (with \(k_{\text{succ}} = 2.0\)) quickly adapts an appropriate coordinate system and finds the optimum 70 times faster than CD.

#### Algorithm 3 ACiD

1: \(m \leftarrow x_{i,d}^{\text{min}} + \mathbb{I}_{i,d}(x_{i,d}^{\text{max}} - x_{i,d}^{\text{min}})\)
2: \(f_{\text{best}} \leftarrow \text{evaluate}(m)\)
3: \(\sigma_{i,d} \leftarrow (x_{i,d}^{\text{max}} - x_{i,d}^{\text{min}})/4\)
4: \(\mathbf{B} \leftarrow \mathbf{I}\)
5: \(i_x \leftarrow 0\)
6: \(\text{while NOT Stopping Criterion do}\)
7: \(i_x \leftarrow i_x + 1 \mod d\) \(/\!\!/\text{ Cycling over [1, }d]\)
8: \(x_{i,d}^{\prime} \leftarrow 0\)
9: \(x_{i,d}^{\prime} \leftarrow -\sigma_{i,d} ; x_1 \leftarrow m + \mathbf{B}x' ; f_1 \leftarrow \text{evaluate}(x_1)\)
10: \(x_{i,d}^{\prime} \leftarrow +\sigma_{i,d} ; x_2 \leftarrow m + \mathbf{B}x' ; f_2 \leftarrow \text{evaluate}(x_2)\)
11: \(\text{succ} \leftarrow 0\)
12: \(\text{if } f_1 < f_{\text{best}} \text{ then}\)
13: \(f_{\text{best}} \leftarrow f_1 ; m \leftarrow x_1 ; \text{ succ} \leftarrow 1\)
14: \(\text{if } f_2 < f_{\text{best}} \text{ then}\)
15: \(f_{\text{best}} \leftarrow f_2 ; m \leftarrow x_2 ; \text{ succ} \leftarrow 1\)
16: \(\text{if succ} = 1 \text{ then}\)
17: \(\sigma_{i,d} \leftarrow k_{\text{succ}} \cdot \sigma_{i,d}\)
18: \(\text{else}\)
19: \(\sigma_{i,d} \leftarrow k_{\text{nsucc}} \cdot \sigma_{i,d}\)
20: \(x_{i,(2i-1)}^{\prime} \leftarrow x_i ; f_{i,(2i-1)}^{\prime} \leftarrow f_1\)
21: \(x_{i,2i}^{\prime} \leftarrow x_2 ; f_{2i}^{\prime} \leftarrow f_2\)
22: \(\text{if } i_x = d \text{ then}\)
23: \(x^{\prime} \leftarrow \{x_{i,a}^{\prime} \mid 1 \leq i \leq 2d\}\)
24: \(\mathbf{B} \leftarrow \text{AdaptiveEncoding}(x_1^{\prime}, \ldots, x_d^{\prime})\)

Algorithm 3 describes the proposed ACiD algorithm. Note that it also describes the non-adaptive CD method by taking \(\mathbf{B} = \mathbf{I}\) and removing the call to AdaptiveEncoding (line 24). Algorithm starts by randomly initializing the current parent \(m\) uniformly in the given rectangle domain \(\Pi_i(x_{i,\text{min}}, x_{i,\text{max}})\), and evaluating it. Initial step-sizes \(\sigma_i\) are set to \(\frac{1}{d}\) of the corresponding interval length. \(i_x\) is the index of the cur-
1.4
1
2.8
4
2.6
2.4
1.8
2.2
3
1.6
cmu
0.8

Comparison procedures provided by this framework, ACiD
starting point – and the permutation of variables if any.
The result solution for the noiseless functions only depends on the
µ best offspring encountered during the k = +1, −1, −4, −8.

The vertical black line indicates the maximum number of function evaluations. Light yellow (or gray in b&w) lines in the background show similar ECDFs for target value $10^{-8}$ of all algorithms benchmarked during BBOB 2009.

rent coordinate, going from 1 to d cyclically (line 7). The
two offspring $x_1$ and $x_2$ are generated from $m$ with offset
±σ on coordinate $i$ (lines 8-10). If one of them has bet-
tter fitness value than $m$ (minimization assumed here), $m$ and $f_{best}$ are updated accordingly (line 13 or 15). The $i$ step-size $σ_i$ is then updated multiplicatively depending on the success indicator (line 17 or 19). The coordinates and fitness of both offspring are then stored (lines 20 and 21). At the end of the coordinate loop (i.e., when $i = d$, line 23), the Adaptive Encoding procedure is called to update the transformation matrix $B$, using information from the µ best offspring encountered during the $d$ coordinate steps (line 24).

The proposed algorithm is deterministic, therefore the result solution for the noiseless functions only depends on the starting point – and the permutation of variables if any.

3. EXPERIMENTAL VALIDATION

Adaptive Coordinate Descent has been benchmarked on the noiseless Black-Box Optimization Benchmarking (BBOB) testbed [5]. Thanks to the publicly available results of many algorithms on the same testbench, and to automatic comparison procedures provided by this framework, ACiD results will be compared to those of the state-of-the-art algorithms: Bipop-CMA-ES, IPOC-CMA-ES, IPOC-aCMA-ES, (1+1)-CMA-ES and (1+2n)-CMA-ES [1].

3.1 Experimental Settings

In order to validate ACiD with a robust version, a value of $k_{succ} = 2$ will be used as a baseline (while $k_{success} = 0.5$ as usual). Indeed, while other values for $k_{success}$ ∈ [0.5, 2.0] may lead to faster convergence, they also sometimes result in premature convergence on some problems, even on the Sphere function, as demonstrated by preliminary experiments. Also

note that the basic rates for the covariance matrix update are set to the simple values $c_1 = \frac{1}{r}$ and $c_2 = \frac{1}{2\sqrt{r}}$ rather than the default ones [4]. However, several other experiments not presented here due to space limitations, demonstrated that slightly different variants of ACiD could improve over the baseline. Hence, a total of 4 variants will be tested in the following: the baseline described above is denoted ACiD; ACiD-$k_{succ}1.5$ is the same variant, but with $k_{succ} = 1.5$; ACiD-prom uses the value $k_{succ} = 2$, but implements mirrored sampling and sequential selection (Section 2.2.2); finally, ACiD-$a$-cma0.8 uses $c_2 = \frac{0.5}{\sqrt{r}}$ rather than the default value, as some extensive experiments varying all parameters have shown that $c_2$ is a sensitive parameter indeed, and the value of $\frac{0.5}{\sqrt{r}}$ can bring up to 20% improvement on some problems.

Because ACiD can be considered as $(1+2) - ES$ algorithm, a restart procedure is necessary to improve the performance on multi-modal functions. Similarly to the other $(1+1) - ES$ that ACiD will be compared to within BBOB testbench, the algorithm is restarted if the improvements of the best solution is smaller than $10^{-25}$ during the last 10 + $[20d^{1.5}]$ function evaluations. The maximum number of function evaluations is $10^d$, and the initial interval is $[-3,3]^d$. All results are statistics over 15 independent runs.

The performances of all algorithms will be measured using the expected running time (ERT), i.e., the expected number of function evaluation to reach a target precision for the first time [5]. The ERT computes to $ERT(f_{target}) = RT_{s} + \frac{1-p_s}{p_s}RT_{us}$, where the running times $RT_{s}$ and $RT_{us}$ denote the average number of function evaluations for successful and unsuccessful trials respectively, and $p_s$ denotes the fraction of successful trials – a run being successful if it does reach the target precision.

The MatLab source code of ACiD is available online at http://sites.google.com/site/acdgecco/.
for all functions and subgroups in 10-D (Left) and 40-D (Right). The best ever line corresponds to the best result obtained by at least one algorithm from BBOB 2009 for each of the targets.

### 3.2 Results and Discussion

The first experiment is concerned with the sensitivity of parameter $k_{succ}$. Fig.4.(Left) shows the performance of ACiD (in terms of BBOB-SP1) depending on $k_{succ}$, for several problems in 10-D. $k_{succ}$ determines how fast the step-size will increase for a given coordinate if the last step along that coordinate was successful. There is a strong link between $k_{succ}$ and the covariance matrix learning coefficients $c_1$ and $c_p$, since they both determine the comparative impact of the new steps.

The experiments show that ACiD does not converges for $k_{succ} \leq 1$ on non-separable problems, whereas on the Sphere function, ACiD obtains nearly the same results than CD with $0.5 \leq k_{succ} < 1.0$. The reason for this is easy to understand on a small example: if $k_{succ} = 0.5$ and $k_{succ} = 1.1$, then in the case of 2 consecutive unsuccessful steps, the step-size $\sigma$ is divided by 4; so in order to come back to initial step size, 13 consecutive successful steps are needed! The optimal pair $k_{succ}, k_{unsucc}$ of course depends on the problem, and on the other parameters of the ACiD, too. But $k_{succ} = \frac{1}{k_{unsucc}} = 2.0$ seems to be both robust and simple, at least for the given problems.

The BBOB noiseless testbed [5] contains separable, ill-conditioned and multi-modal functions with adequate and weak global structures, with in total 24 functions. Furthermore, standard BBOB outputs include aggregated Empirical Cumulative Distribution Functions (ECDF) that give the proportion of runs that reached a given precision for a given computational effort.

Fig.4.(Right) thus presents the ECDFs of both ACiD and (1+1)-CMA-ES aggregated over the 24 noiseless benchmark problems in 20-D. A closer look at the results (details not shown here) reveals that for 50% of the solved functions, ACiD is about by 40% faster than (1+1)-CMA-ES. ACiD converges on Attractive Sector function $f_6$, while (1+1)-CMA-ES does not. On Rosenbrock $f_5$ and $f_9$, Ellipsoid $f_2$ and $f_{50}$ and Discus $f_{11}$, some ACiD algorithms are up to 2 times faster than (1+1)-CMA. Furthermore, with a budget of 1000$d$ function evaluations, ACiD performs best in 20-D among all algorithms that entered BBOB-2009 competition.

Fig. 5 gives a more general view of similar experiments in 10-D and 40-D – but here experiments with 50 different target values are aggregated. Overall, the functions are not easy to solve. Even the best CMA-ES algorithm, BIPOP-CMA-ES, can solve less than 90% of the problems using maximum number of function evaluations (the large crosses). Furthermore, the elitist algorithms (ACiD, (1+1)-CMA-ES, (1+2m)-CMA-ES) are more likely to get stuck in a local optimum than generational algorithms (BIPOP-CMA-ES, IPOP-CMA-ES and IPOP-aCMA-ES), especially in the case where the step-size decreases after each unsuccessful step. The increase of the population size after restart in the case of the premature convergence is the main tool, which leads to a superior performance of the generational CMA-ES algorithms over the single-population algorithms on the multi-modal problems. Indeed, only 10 out of 24 problems are unimodal and this 42% threshold can clearly be seen on the figure. However, the good news is that all ACiD algorithms have at least comparable performance with (1+1)-CMA-ES and (1+2m)-CMA-ES: they outperform one another depending on the problem, but the differences are not significant.

The superiority of the ACiD as an absolutely deterministic algorithm was not obvious a priori. These experiments confirm the hypothesis that the efficiency of CMA-ES is mostly due to the Adaptive Encoding procedure, and that the second component of the algorithm (Evolution Strategies, and Gaussian mutations) can be replaced without significant (or even any) loss of performance, at least in the case of the single-individual algorithms. The IPOP-aCMA-ES is only algorithm among the ones presented in this paper that uses all $\lambda$ offspring in its covariance matrix update. While the best $\mu$ points are used to increase the variance along the successful directions, the worst $\lambda - \mu = \mu$ points are used with negative sign to exclude irrelevant directions of search. It is clear that such kind of negative update can be applied to ACiD too. This will be the subject of further work.
4. CONCLUSION AND PERSPECTIVES
The very powerful Covariance Matrix Adaptation part of the state-of-the-art Covariance Matrix Adaptation Evolution Strategy (CMA-ES) [7] has been generalized into the so-called Adaptive Encoding (AE) [4] that can be used in conjunction with any optimization algorithm, gradually learning an optimal coordinate system where the objective function at hand is (in the best case, and at least locally) separable. Simple yet powerful algorithms can be used to optimize separable functions, as for instance the Coordinate Descent (CD), that performs up to 2 times faster than the (1 + 1)-ES on the Sphere function. The Adaptive Coordinate Descent algorithm (ACiD), proposed in this paper, uses AE coupled with adaptive CD. ACiD has been shown to be competitive with the CMA-ES algorithms and even up to 2 times faster than (1 + 1)-CMA-ES on several functions of the BBOB test suite. There are, however, a large number of issues that remain open.

Of course, the generational versions of CMA-ES outperform the single-individual ones like (1 + 1)-CMA-ES and ACiD, on most multi-modal problems. But this is essentially due to the restarts with increasing population size. Some further work will be concerned with designing a generational extension of ACiD.

Partial experiments indicate that the off-line tuning of the covariance matrix learning rates $c_i$ and $c_o$ can lead to 30-50% speed-up depending on the problem and dimension: more detailed experiments must be made in this direction.

Borrowing ideas from [10], an extension of the Adaptive Encoding procedure to the non-linear case using Kernel Principal Component Analysis (KPCA) [12] is envisioned. Such extension should for instance make it feasible to sample the non-linear distribution along the parabolic shaped optimal valley of the Rosenbrock function.

In ACiD, the evolution path somehow approximates the gradient of the fitness function, and this information is used in the Adaptive Encoding update. However, the line search along the gradient could also be performed explicitly, as in quasi-Newton methods (e.g., BFGS method [3]) and Pattern Search methods (e.g., Hooke and Jeeves method [8]).

We anticipate successful applications of ACiD algorithm to constrained problems. For CMA-ES in large dimensions, resampling the infeasible points does not work, and leads to a rapid decrease of the step-size that further limits the exploration. Within ACiD, the resampling on a line is easy, both in the transformed and in the original spaces.

Another possible extension of ACiD is concerned with surrogate models: the computationally cheap meta-model assisted one-dimensional search becomes favorable even with some budget of 3 to 5 function evaluations, at least for unimodal problems. Furthermore, in order to preserve the invariance properties of the ACiD, comparison-based surrogate models can be used, as advocated in [9].

Finally, the one-dimensional search procedure used in ACiD could be replaced by some $k$-dimensional search ($k \leq d$). For $k = 2$, the budget is $2k = 4$ function evaluations to find the best of 8 possible states (see Fig.1.(c)). By taking into account all available information, such as the projection of the evolution path on 2-D, we could increase the chances to directly find new best points, for example in the corner. In this case, by simply increasing both step-sizes, the resulting speed-up would increase to 4. We suppose that even such simple strategies, together with sequential selection, can make ACiD significantly faster.

5. ACKNOWLEDGEMENTS
This work is partially funded by UFI of System@tic Paris-Region ICT cluster through contract DGT 117 407 Complex Systems Design Lab (CSDL).

6. REFERENCES