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# Computability of the Radon-Nikodym derivative

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**Abstract.** We show that computability of the Radon-Nikodym derivative of a measure  $\mu$  absolutely continuous w.r.t. some other measure  $\lambda$  can be reduced to a single application of the non-computable operator  $EC$ , which transforms enumeration of sets (in  $\mathbb{N}$ ) to their characteristic functions. We also give a condition on the two measures (in terms of the computability of the norm of a certain linear operator involving the two measures) which is sufficient to compute the derivative.

## 1 Introduction

**Theorem 1 (Radon-Nikodym).** *Let  $(\Omega, \mathcal{A}, \lambda)$  be a measured space where  $\lambda$  is  $\sigma$ -finite. Let  $\mu$  be a finite measure that is absolutely continuous w.r.t.  $\lambda$ . There exists a unique function  $h \in L^1(\lambda)$  such that for all  $f \in L^1(\mu)$ ,*

$$\int f \, d\mu = \int fh \, d\lambda.$$

*$h$  is called the **Radon-Nikodym derivative**, or **density**, of  $\mu$  w.r.t.  $\nu$ .*

Is this theorem computable? Can  $h$  be computed from  $\mu$  and  $\lambda$ ? In [3] a negative answer was given.

In this paper we investigate to what extent this theorem is non-computable. We first give an upper bound for its non-computability, showing that it can be computed using a single application of the operator  $EC$  (which transforms enumerations of sets of natural numbers into their characteristic functions). In proving this result we use two classical theorems: Levy's zero-one law and Radon-Nikodym Theorem itself. We then give a sufficient condition on the measures to entail the computability of the RN derivative: this condition is the computability of the norm of a certain integral operator associated to the measures.

## 2 Preliminaries

### 2.1 Little bit of Computability via Representations

To carry out computations on infinite objects we encode those objects into infinite symbolic sequences, using representations (see [6] for a complete development). Let  $\Sigma = \{0, 1\}$ . A **represented space** is a pair  $(X, \delta)$  where  $X$  is a set

and  $\delta \subset \Sigma^{\mathbb{N}} \rightarrow X$  is an onto partial map. Every  $p \in \text{dom}(\delta)$  such that  $\delta(p) = x$  is called a  $\delta$ -**name** of  $x$  (or **name** of  $x$  when  $\delta$  is clear from the context).

Let  $(X, \delta_X)$  be a represented space. An element  $x \in X$  is **computable** if it has a computable name. Let  $(Y, \delta_Y)$  be another represented space. A realizer for a function  $f : X \rightarrow Y$  is a (partial) function  $F : \Sigma^{\mathbb{N}} \rightarrow \Sigma^{\mathbb{N}}$  such that  $f \circ \delta_X = \delta_Y \circ F$  (with the expected compatibilities between domains).  $f$  is **computable** if it has a computable realizer. Of course the image of a computable element by a computable function is computable.

*Example 1.* Let  $2^{\mathbb{N}}$  be the powerset of  $\mathbb{N}$ . The classical notions of recursive and recursively enumerable sets can be grasped as the computable elements of the space  $2^{\mathbb{N}}$  endowed with two representations, Cf and En respectively, defined by:

$$\begin{aligned} \text{En}(p) &= \{n \in \mathbb{N} : 100^n 1 \text{ is a subword of } p\} \\ \text{Cf}(p) &= \{n \in \mathbb{N} : p_n = 1\} \end{aligned}$$

where  $p \in \Sigma^{\mathbb{N}}$ ,  $p_n$  is the  $n$ th symbol in  $p$ .  $\text{En}, \text{Cf} : \Sigma^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  are total functions.

We define the operator  $EC$  as the identity from represented space  $(2^{\mathbb{N}}, \text{En})$  to represented space  $(2^{\mathbb{N}}, \text{Cf})$ . It transforms an enumeration of a set into its characteristic function.  $EC$  is not computable.

The non-computability of functions  $f, g$  between represented spaces can be compared using a notion of reducibility introduced in [5]:  $f \leq_W g$  if there are computable (partial) functions  $K, H$  such that for every realizer  $G$  of  $g$ ,  $p \mapsto K(p, G \circ H(p))$  is a realizer of  $f$ . In other words,  $f \leq_W g$  if  $f$  can be computed using one single application of  $g$  (provided by an oracle) in the computation.<sup>4</sup>

## 2.2 Computable Measurable Spaces

We start by briefly recalling some basic definitions from measure theory. See for example [4, 2, 1] for a complete treatment. A **ring**  $\mathcal{R}$  over a set  $\Omega$  is a collection of subsets of  $\Omega$  which contains the empty set and is closed under finite unions and relative complementation ( $B \setminus A \in \mathcal{R}$ , for  $A, B \in \mathcal{R}$ ). A  **$\sigma$ -algebra**  $\mathcal{A}$  (over the set  $\Omega$ ) is a collection of subsets of  $\Omega$  which contains  $\Omega$  and is closed under complementation and countable unions (and therefore also closed under countable intersections). A ring  $\mathcal{R}$  generates a unique  $\sigma$ -algebra, denoted by  $\sigma(\mathcal{R})$ , and defined as the smallest  $\sigma$ -algebra containing  $\mathcal{R}$ .

In this paper we will work with the **measurable space**  $(\Omega, \mathcal{A})$ , where  $\mathcal{A}$  is a  $\sigma$ -algebra generated by a countable ring  $\mathcal{R}$ . Members of  $\mathcal{A} = \sigma(\mathcal{R})$  will be referred to as **measurable sets**.

A **measure** over a collection  $\mathcal{C}$  (which is at least closed by finite unions) of subsets of  $\Omega$  is a function  $\mu : \mathcal{C} \rightarrow \mathbb{R}^{\infty} (= \mathbb{R} \cup \{\infty\})$  such that i)  $\mu(\emptyset) = 0$ ,  $\mu(E) \geq 0$  for all  $E \in \mathcal{C}$ , and ii)  $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$  for pairwise disjoint sets  $E_0, E_1, \dots \in \mathcal{C}$  such that  $\bigcup_i E_i \in \mathcal{C}$ . A measure  $\mu$  over a collection  $\mathcal{C}$  is said to

<sup>4</sup> the relation is denoted  $\leq_W$  as it is a generalization of Wadge reducibility.

be  **$\sigma$ -finite**, if there are sets  $E_0, E_1, \dots \in \mathcal{C}$  such that  $\mu(E_i) < \infty$  for all  $i$  and  $\Omega = \bigcup_i E_i$ .

It is well known that a  $\sigma$ -finite measure over ring  $\mathcal{R}$  has a unique extension to a measure over the  $\sigma$ -algebra  $\sigma(\mathcal{R})$ . For measures  $\mu$  and  $\lambda$ , we say that  $\mu$  is **absolutely continuous** w.r.t.  $\lambda$ , and write  $\mu \ll \lambda$ , if  $(\lambda(A) = 0 \implies \mu(E) = 0)$  for all measurable sets  $E$ .

We now introduce the effective counterparts.

**Definition 1.** A **computable measurable space** is a tuple  $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$  where

1.  $(\Omega, \mathcal{A})$  is a measurable space,  $\mathcal{R}$  is a countable ring such that  $\bigcup \mathcal{R} = \Omega$  and  $\mathcal{A} = \sigma(\mathcal{R})$ ,
2.  $\alpha : \mathbb{N} \rightarrow \mathcal{R}$  is a computable enumeration such that the operations  $(A, B) \rightarrow A \cup B$  and  $(A, B) \rightarrow A \setminus B$  are computable w.r.t.  $\alpha$ .

In the following  $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$  will be a computable measurable space. We will consider only measures  $\mu : \mathcal{A} \rightarrow [0; \infty]$  such that  $\mu(E) < \infty$  for every  $E \in \mathcal{R}$ . Observe that such a measure  $\mu$  is  $\sigma$ -finite, and therefore well-defined by its values on the ring  $\mathcal{R}$ . Conversely if a measure  $\mu$  over  $\mathcal{A}$  is  $\sigma$ -finite, one can choose a countable generating  $\mathcal{R}$  such that  $\mu(E) < \infty$  for all  $E \in \mathcal{R}$ .

Computability on the space of measures over  $(\Omega, \mathcal{A}, \mathcal{R})$  will be expressed via representations.

**Definition 2.** Let  $\mathcal{M}$  be the set of measures  $\mu$  such that  $\mu(E) < \infty$  for all  $E \in \mathcal{R}$  and let  $\mathcal{M}_{<\infty}$  be the set of all finite measures. Define representations  $\delta_{\mathcal{M}} : \Sigma^{\mathbb{N}} \rightarrow \mathcal{M}$  and  $\delta_{\mathcal{M}_{<\infty}} : \Sigma^{\mathbb{N}} \rightarrow \mathcal{M}_{<\infty}$  as follows:

1.  $\delta_{\mathcal{M}}(p) = \mu$ , iff  $p$  is (more precisely, encodes) a list of all  $(l, n, u) \in \mathbb{Q}^3$  such that  $l < \mu(E_n) < u$ , for every  $E_n \in \mathcal{R}$ .
2.  $\delta_{\mathcal{M}_{<\infty}}(p) = \mu$ , iff  $p = \langle p_1, p_2 \rangle$  such that  $\delta_{\mathcal{M}}(p_1) = \mu$  and  $p_2$  is (more precisely, encodes) a list of all  $(l, u) \in \mathbb{Q}^2$  such that  $l < \mu(\Omega) < u$ .

Thus, a  $\delta_{\mathcal{M}}$ -name  $p$  allows to compute  $\mu(A)$  for every ring element  $A$  with arbitrary precision. A  $\delta_{\mathcal{M}_{<\infty}}$ -name allows additionally to compute  $\mu(\Omega)$ . Obviously,  $\delta_{\mathcal{M}_{<\infty}} \leq \delta_{\mathcal{M}}$  and  $\delta_{\mathcal{M}_{<\infty}} \equiv \delta_{\mathcal{M}}$  if  $\Omega \in \mathcal{R}$ . But in general, not even the restriction of  $\delta_{\mathcal{M}}$  to the finite measures is reducible to  $\delta_{\mathcal{M}_{<\infty}}$ .

We will also work with the spaces  $L^1$  of integrable functions and  $L^2$  of square-integrable functions (w.r.t. some measure). A **rational step function** is a finite sum

$$s = \sum_{k=1}^p \mathbf{1}(E_{i_k}) q_{j_k},$$

where  $E_{i_k} \in \mathcal{R}$  and  $q_{j_k} \in \mathbb{Q}$ .

The computable numberings of the ring  $\mathcal{R} = (E_0, E_1, \dots)$  and of the rational numbers  $\mathbb{Q} = (q_0, q_1, \dots)$  induce a canonical numbering of the collection  $\mathcal{RSF} = (s_0, s_1, \dots)$  of rational step functions. Since the collection  $\mathcal{RSF}$  is dense in the spaces  $L^1$  and  $L^2$ , we can use it to handle computability over these spaces via the following representations:

**Definition 3.** For every  $\mu \in \mathcal{M}$  define Cauchy representations  $\delta_\mu^k : \Sigma^\mathbb{N} \rightarrow L^k(\mu)$  of  $L^k(\mu)$  ( $k = 1, 2$ ) by:  $\delta_\mu^k(p) = f$  iff  $p$  is (encodes) a sequence  $(s_{n_0}, s_{n_1}, \dots)$  of rational step functions such that  $\|s_{n_i} - f\|_\mu^k \leq 2^{-i}$  for all  $i \in \mathbb{N}$ .

### 3 Effective Radon-Nikodym Theorem

#### 3.1 An upper bound

In the following, we present our first main result which, in words, says that computability of the Radon-Nikodym derivative is reducible to a single application of the (non-computable) operator EC.

**Theorem 2.** The function mapping every  $\sigma$ -finite measure  $\lambda \in \mathcal{M}$  and every finite measure  $\mu$  such that  $\mu \ll \lambda$  to the function  $h \in L^1(\lambda)$  such that  $\mu(E) = \int_E h d\lambda$  for all  $E \in \sigma(\mathcal{R})$  is computable via the representations  $\delta_\mathcal{M}$ ,  $\delta_{\mathcal{M}_{<\infty}}$  and  $\delta_\lambda^1$  with a single application of the operator EC.

For the proof, we will use the following classical result on convergence of conditional expectations, which is a consequence of the more general Doob's martingale convergence theorems (see for example [2], Section 10.5). We recall that a filtration  $(\mathcal{F}_n)_n$  is an increasing sequence of  $\sigma$ -algebras on a measurable space. In some sense,  $\mathcal{F}_n$  represents the information available at time  $n$ . In words, the following result says that if we are learning gradually the information that determines the outcome of an event, then we will become gradually certain what the outcome will be.

**Theorem 3 (Levy's zero-one law).** Consider a measured space  $(\Omega, \sigma(\mathcal{R}), \lambda)$  with  $\lambda$  finite. Let  $h \in L^1(\lambda)$ . Let  $(\mathcal{F}_n)_n$  be any filtration such that  $\sigma(\mathcal{R}) = \sigma(\bigcup_n \mathcal{F}_n)$ . Then

$$\mathbb{E}(h|\mathcal{F}_n) \xrightarrow{n \rightarrow \infty} h$$

both  $\lambda$ -almost everywhere and in  $L^1(\lambda)$ .

*Proof (of Theorem 2).* We start by proving the result for finite measures.

**Lemma 1.** From descriptions of finite measures  $\mu \ll \lambda$ , one can compute a sequence  $\{h_n\}_{n \in \mathbb{N}}$  of  $L^1(\lambda)$ -computable functions which converges in  $L^1(\lambda)$  to the Radon-Nikodym derivative  $h = \frac{d\mu}{d\lambda} \in L^1(\lambda)$ .

*Proof.* Consider the computable enumeration  $E_0, E_1, \dots$  of  $\mathcal{R}$ . For every  $n$ , consider the partition  $\mathcal{P}_n$  of the space given by the cells:

- all the possible  $E_0^* \cap E_1^* \cap \dots \cap E_n^*$  where  $E^*$  is either  $E$  or  $(E_0 \cup \dots \cup E_n) \setminus E$ ,
- all the  $E_{k+1} \setminus (E_0 \cup \dots \cup E_k)$  for every  $k \geq n$ .

Observe that all the cells are elements of the ring  $\mathcal{R}$ . The partitions  $\mathcal{P}_n$  induce a filtration which generates the  $\sigma$ -algebra  $\sigma(\mathcal{R})$ . Let  $h$  be the RN-derivative<sup>5</sup> of  $\mu$  w.r.t.  $\lambda$ .  $h \in L^1(\lambda)$  so by Levy's zero-one law,

$$\mathbb{E}(h|\mathcal{P}_n) \xrightarrow{n \rightarrow \infty} h$$

both  $\lambda$ -almost everywhere and in  $L^1(\lambda)$ .

Now,  $h_n := \mathbb{E}(h|\mathcal{P}_n)$  is constant on each cell  $C$  of  $\mathcal{P}_n$ , with value  $\frac{\mu(C)}{\lambda(C)}$  if  $\lambda(C) > 0$  and 0 otherwise. We now show that the functions  $h_n$  are (uniformly) computable elements of  $L^1(\lambda)$ . For a given  $\epsilon$ , one can find cells  $C_1, \dots, C_k$  in  $\mathcal{P}_n$  such that  $\lambda(C_i) > 0$  for all  $i = 1, \dots, k$  and  $\sum_{i=1}^k \mu(C_i) > \mu(\Omega) - \epsilon$ . Define  $h_n^\epsilon$  to be constant with value  $\frac{\mu(C_i)}{\lambda(C_i)}$  for  $i = 1, \dots, k$  and 0 on the other cells. As these values are uniformly computable, the functions  $h_n^\epsilon$  are uniformly  $L^1(\lambda)$ -computable. Moreover

$$\int |h_n - h_n^\epsilon| d\lambda = \int_{\Omega \setminus \bigcup_{i=1}^k C_i} h_n d\lambda < \epsilon$$

and thus, the functions  $h_n$  are (uniformly) computable elements of  $L^1(\lambda)$ , and converge to  $h$  in  $L^1(\lambda)$ . The lemma is proved.

Assume now that  $\mu \in \mathcal{M}_{<\infty}$  and  $\lambda \in \mathcal{M}$ . Let once again  $(E_0, E_1, \dots)$  be the computable numbering of the ring  $\mathcal{R}$ . Let  $F_0 := E_0$  and  $F_{n+1} := E_{n+1} \setminus \{F_0, \dots, F_n\}$ . Then  $(F_i)_i$  is a computable numbering of ring elements such that  $F_i \cap F_j = \emptyset$  for  $i \neq j$  and  $\Omega = \bigcup_i F_i$ .

Since  $i \mapsto \mu(F_i)$  is computable, there is a computable function  $d : \mathbb{N} \rightarrow \mathbb{N}$  such that  $d(i) > \mu(F_i) \cdot 2^i$ . Define a function  $w : \Omega \rightarrow \mathbb{R}$  by  $w(x) := 1/d(i)$  if  $x \in F_i$ . Then

$$\int w d\lambda = \sum_i \mu(F_i)/d(i) < 2.$$

Define a new measure  $\nu$  by

$$\nu(E) := \int_E w d\lambda. \tag{1}$$

for all  $E \in \mathcal{R}$ . Then  $\nu$  is a finite measure, which is equivalent to  $\lambda$  (i.e.  $\nu \ll \lambda \ll \nu$ ) and such that a  $\delta_{\mathcal{M}}$ -name of  $\nu$  can be computed from a  $\delta_{\mathcal{M}}$ -name of  $\lambda$ . Apply Lemma 1 to  $\mu$  and  $\nu$ : one can compute a sequence of  $L^1(\nu)$ -computable functions  $\{h'_n\}$  whose limit (in  $L^1(\nu)$ ) is the density  $h' = \frac{d\nu}{d\nu}$ . The sequence  $\{h'_n w\}$  is computable and converges (in  $L^1(\lambda)$ ) to  $h = \frac{d\mu}{d\lambda} = wh'$ . At this point we use the operator EC to extract a fast convergent subsequence, and hence to compute (a  $\delta_{\lambda}^1$ -name of)  $h$ . The proof is complete.

The above theorem shows that the Radon-Nikodym theorem is reducible to the (non-computable) operator EC. On the other hand, it was shown in [3] that

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<sup>5</sup> at this point we use the classical RN theorem

there exists a computable measure on the unit interval, absolutely continuous w.r.t. Lebesgue measure, and such that the operator EC can be reduced to the computation of the density (which is therefore not  $L^1$ -computable). This give us the following corollary.

**Corollary 1.** *For nontrivial computable measurable spaces, the Radon-Nikodym operator and EC are equivalent:  $RN \equiv_W EC$ .*

This result characterize the extent to which the Radon-Nikodym theorem is non-computable. However, it doesn't give us much information on "where" is the non-effectivity. In what follows we present a result which gives an explicit condition (in terms of the computability of the norm of a certain linear operator involving the two measures) allowing to compute the Radon-Nikodym derivative. The proof of this result is somewhat more involved, and some preparation will be required.

### 3.2 Locating the non-computability

Let  $\mu \in \mathcal{M}_{<\infty}$  be a finite measure over  $(\Omega, \mathcal{A}, \mathbb{R})$ . Let  $u : L^2(\mu) \rightarrow \mathbb{R}$  be a linear functional. Classically we have that the following are equivalent:

1.  $u$  is continuous,
2.  $u$  is uniformly continuous,
3. there exists  $c$  (a **bound for  $u$** ) such that for every  $f \in L^2(\mu)$ ,  $|u(f)| \leq c\|f\|_2$ .

The smallest bound for  $u$  is called **the norm** of  $u$  and is denoted by  $\|u\|$ . That is,

$$\|u\| := \sup\{c \in \mathbb{R} : |u(f)| \leq c\|f\|_2\} = \sup_{\|f\|_2=1} |u(f)|.$$

Suppose that the operator  $u$  is computable in the sense that one can compute the real number  $u(f)$  from (a  $\delta_{\mathcal{M}}$ -name of)  $\mu$  and (a  $\delta_{\mu}^2$ -name of)  $f$ . Consider now the numbered collection  $\mathcal{RSF} = \{r_n\}_{n \in \mathbb{N}}$ . Since the sequence  $s_i := \frac{r_i}{\|r_i\|_2}$  is uniformly computable (from  $\mu$ ) and dense in  $\{f \in L^2(\mu) : \|f\|_2 = 1\}$ , it follows that the norm  $\|u\|$  of a computable operator  $u$  is always a lower-computable (from  $\mu$ ) number. It is not, in general, computable. In case it is computable, we will say that the operator  $u$  is **computably normable**.

The following result is an effective version of Riesz-Fréchet Representation Theorem. For simplicity, we state the result in the particular Hilbert space  $L^2(\mu)$ , but the same proof works for any Hilbert Space provided that the inner product is computable.

**Theorem 4.** *Let  $u$  be a non-zero computably normable (from  $\mu$ ) linear functional over  $L^2(\mu)$ . Then from  $\mu$  one can compute a name of (the unique)  $g \in L^2(\mu)$  such that*

$$u(f) = \int fg \, d\mu$$

for all  $f \in L^2(\mu)$ .

*Proof.* In the following, all what we will compute will be from a  $\delta_{\mathcal{M}}$ -name of  $\mu$ . Chose any computable  $x \in L^2(\mu)$  out of  $\ker(u)$ . As  $\|u\|$  is computable, from the classical formula  $d(x, \ker(u)) = \frac{|u(x)|}{\|u\|}$ , it follows that  $d(x, \ker(u))$  is computable too. We can then enumerate a sequence of points  $y_n \in E = \ker(u) + x$  which is dense in  $E$ . Note that  $d(0, E) = d(x, \ker(u))$ . Let  $z_0 \in E$  be such that  $\|z_0\| = d(0, E)$ . Since

$$\|y_n - z_0\|^2 = \|y_n\|^2 - \|z_0\|^2$$

we can compute a subsequence  $y_{n_i}$  converging effectively to  $z_0$  which is therefore computable. Put  $z = \frac{z_0}{\|z_0\|}$  so that  $\|z\| = 1$ . All what remains is to show that  $g = u(z)z$  (which is computable) satisfies the required property. This is done as in the classical proof, namely:  $z$  has the property of being orthogonal to  $\ker(u)$ . That is,  $\int z f d\mu = 0$  for any  $f \in \ker(u)$ . Put

$$r := u(f)z - u(z)f.$$

We have  $u(r) = u(f)u(z) - u(z)u(f) = 0$  so that  $r \in \ker(u)$  and then  $\int rz d\mu = 0$ . This gives,

$$u(f) = u(f) \int z^2 d\mu = \int u(f)z^2 d\mu = \int (r + u(z)f)z d\mu = \int f u(z)z d\mu$$

and hence  $g = u(z)z$ , as was to be shown.

*Remark 1.* Let  $\mu$  be a finite measure. For  $f, g$  in  $L^2(\mu)$ , Hölder's inequality implies that  $fg \in L^1(\mu)$  and:

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad (2)$$

so that, in particular, since  $\mu$  is finite, if  $f \in L^2$  then  $f \in L^1$  (taking  $g \equiv 1$ ). Moreover, from a  $\delta_{\mu}^2$ -name of  $f$  one can compute a  $\delta_{\mu}^1$ -name.

Now, let  $\mu$  and  $\lambda$  be finite measures and consider a new measure  $\varphi := \mu + \nu$ . Let  $L_{\mu} : L^2(\varphi) \rightarrow \mathbb{R}$  be defined by  $L_{\mu}(f) := \int f d\mu$ . This is a bounded operator and it is easy to see that from  $\delta_{\mathcal{M}}$ -names of  $\mu$  and  $\lambda$  and from a  $\delta_{\varphi}^2$ -name of  $f$ , one can compute the value  $L_{\mu}(f)$ .

**Definition 4.** A finite measure  $\mu$  is said to be **computably normable relative to** some other finite measure  $\lambda$ , if the norm of the operator  $L_{\mu}$  (as defined above) is computable from  $\mu$  and  $\lambda$ .

At this point, we are ready to state our second main result.

**Theorem 5.** Let  $\mu, \lambda \in \mathcal{M}_{<\infty}$  be such that:

- (i)  $\mu \ll \lambda$ ,
- (ii)  $\mu$  is computably normable relative to  $\lambda$ .

Then the Radon-Nikodym derivative  $\frac{d\mu}{d\lambda}$  can be computed as an element of  $L^1(\lambda)$ , from  $\mu$  and  $\lambda$ .



*Proof.* We follow Von Neumann's proof. The measure  $\varphi := \mu + \lambda$  is computable and by hypothesis the operator  $L_\mu : L^2(\varphi) \rightarrow \mathbb{R}$  defined by  $L_\mu(f) := \int f d\mu$  is computably normable. Hence, by Theor. 4 one can compute a name of  $g \in L^2(\varphi)$  (and hence a name of  $g$  as a point in  $L^1(\lambda)$ ) such that for all  $f \in L^2(\varphi)$  the equality:

$$\int f d\mu = \int fg d\varphi = \int fg d\lambda + \int fg d\mu \quad (3)$$

holds. This relation can be rewritten as:

$$\int fg d\lambda = \int f(1-g) d\mu. \quad (4)$$

Note that (4) holds for any  $f \geq 0$  (take  $f_n = f \mathbf{1}_{\{f < n\}}$  and apply monotone convergence theorem). Taking  $f = \mathbf{1}_{\{g=1\}}$  in (4) we see that  $\lambda(\{g=1\}) = 0$ . Hence the following function is defined  $\lambda$ -almost everywhere:

$$h := \frac{g}{1-g}.$$

Taking  $f = \mathbf{1}_{\{g < 0\}}$  and  $\mathbf{1}_{\{g > 1\}}$  in (4) we see that  $0 \leq g \leq 1$   $\lambda$ -a.e., so  $h \geq 0$   $\lambda$ -a.e. Therefore,

$$\begin{aligned} \int fh d\lambda &= \int \left( \frac{f}{1-g} \right) g d\lambda \\ &= \int \left( \frac{f}{1-g} \right) (1-g) d\mu && \text{by (4)} \\ &= \int f d\mu. \end{aligned}$$

Now, taking  $f$  to be the constant function equal to 1, we conclude that  $\int h d\lambda = 1$  and then it is in  $L^1(\lambda)$ . This shows that  $h$  is the Radon-Nikodym derivative.

It remains to show that the function  $h = \frac{g}{1-g}$  is  $L^1(\lambda)$ -computable.

As  $g$  is  $L^1(\lambda)$ -computable, we can effectively produce a sequence  $u_i$  of rational step functions such that  $\|u_i - g\|_\lambda < 2^{-i}$ . As  $g \geq 0$   $\lambda$ -a.e. we can assume w.l.o.g. that  $u_i \geq 0$  (otherwise replace  $u_i$  with  $\max(u_i, 0)$ ).

For  $n \in \mathbb{N}$  let

$$\begin{aligned} g_n &:= \min(g, 1 - 2^{-n}) & h_n &:= g_n / (1 - g_n) \\ u_{in} &:= \min(u_i, 1 - 2^{-n}) & v_{in} &:= u_{in} / (1 - u_{in}) \end{aligned}$$

Since the function  $x \mapsto x/(1-x)$  is nondecreasing over  $(0, +\infty)$ ,

$$\begin{aligned} g_n &\leq g_{n+1} \text{ and } \sup_n g_n = g, \\ h_n &\leq h_{n+1} \text{ and } \sup_n h_n = h. \end{aligned}$$

Given a rational number  $\epsilon > 0$  we show how to compute  $n$  and  $i$  such that

$$\|h - v_{in}\|_\lambda < \epsilon.$$

$v_{in}$  will then be a rational step function approximating  $h$  up to  $\epsilon$ , in  $L^1(\lambda)$ . As a result, it will enable to compute a  $\delta_\mu^1$ -name of  $h$ .

To find  $n$  and  $i$ , we use the following inequality

$$\|h - v_{in}\|_\lambda \leq \|h - h_n\|_\lambda + \|h_n - v_{in}\|_\lambda \quad (5)$$

We first make the first term small. As for all  $n$ ,  $0 \leq h_n \leq h$ , one has  $\|h - h_n\|_\lambda = \|h\|_\lambda - \|h_n\|_\lambda$ . As  $\|h\|_\lambda = \mu(\Omega)$  is given as input and  $\|h_n\|_\lambda$  can be computed from  $n$ , one can effectively find  $n$  such that  $\|h\|_\lambda - \|h_n\|_\lambda < \epsilon/2$ .

We then make the second term in (5) small:

$$\begin{aligned} \|h_n - v_{in}\|_\lambda &= \left\| \frac{g_n}{1 - g_n} - \frac{u_{in}}{1 - u_{in}} \right\|_\lambda \\ &= \left\| \frac{g_n - u_{in}}{(1 - g_n)(1 - u_{in})} \right\|_\lambda \\ &\leq \|g_n - u_{in}\|_\lambda \cdot 2^{2n} \\ &\leq \|g_n - u_{in}\|_\varphi^1 \cdot 2^{2n} \\ &\leq \|g - u_i\|_\varphi^1 \cdot 2^{2n} \\ &\leq 2^{-i} \cdot 2^{2n}. \end{aligned}$$

We then compute  $i$  such that  $2^{-i} \cdot 2^{2n} < \epsilon/2$ . One finally gets the expected inequality

$$\|h - v_{in}\|_\lambda \leq \|h - h_n\|_\lambda + \|h_n - v_{in}\|_\lambda < \epsilon$$

and the result follows.

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