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# Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

CNRS AND UNIVERSITÉ PIERRE ET MARIE CURIE

**Kick Off Meeting SADCO**

Paris, March 3-4, 2011



# Control System under State Constraints

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U \quad \text{a.e. in } [0, 1] \\ x(0) = x_0 \\ x(t) \in K \quad \text{for all } t \in [0, 1] \end{cases}$$

$U$  is a complete separable metric space,  $K \subset \mathbb{R}^n$  is closed

$$f : [0, 1] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n$$

**Controls** are Lebesgue measurable functions  $u(\cdot) : [0, 1] \rightarrow U$

A **viable trajectory**  $x(\cdot)$  of control system is an absolutely continuous function satisfying  $x(0) = x_0$ ,

$$x'(t) = f(t, x(t), u(t)) \text{ almost everywhere in } [0, 1]$$

for some control  $u(\cdot)$  and  $x(t) \in K$  for all  $t \in [0, 1]$



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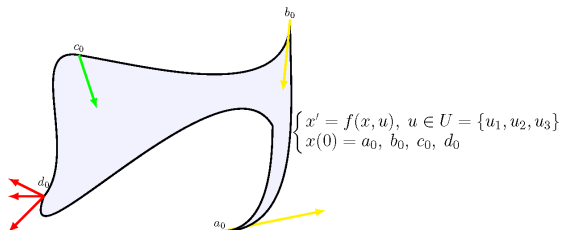
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# Tangent Vectors



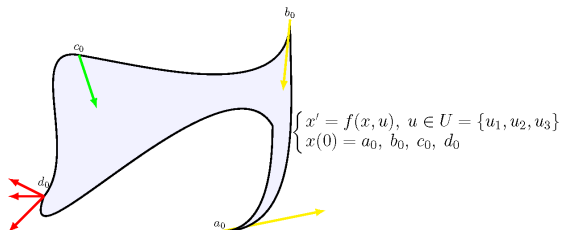
$f(a_0, u_1)$ ,  $f(b_0, u_2)$ ,  $f(c_0, u_2)$  are tangent to  $K$  at  $a_0$ ,  $b_0$  and  $c_0$ .  
At  $d_0$  there is no  $u \in U$  such that  $f(d_0, u)$  is tangent to  $K$ .

The **contingent** cone to  $K$  at  $x \in K$  is defined by

$$T_K(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x + hv; K)}{h} = 0 \right\}$$



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# Viability Theorem

$\mathcal{S}_K(x_0)$  – set of all viable trajectories starting at  $x_0$ ,

$$W(t, x) := \{u \in U \mid f(t, x, u) \in T_K(x)\}$$

Then  $x(\cdot) \in \mathcal{S}_K(x_0)$  **if and only if**  $x(0) = x_0$  and  $x(\cdot)$  is a trajectory of the following control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in W(t, x(t)) \quad \text{a.e. in } [0, 1]$$

for some control  $u(\cdot)$

If  $x \rightsquigarrow f(x, U) := \bigcup_{u \in U} \{f(x, u)\}$  is "Marchaud",  $f$  is continuous and  $K$  is closed, then the following statements are equivalent

- (i)  $\mathcal{S}_K(x_0) \neq \emptyset \quad \forall x_0 \in K$
- (ii)  $f(x, U) \cap T_K(x) \neq \emptyset \quad \forall x \in \partial K$



# Viability Theorem

J. W. Bebernes & J. D. Schuur (1970)

For functional differential inclusions G. Haddad (1981)

For measurable time dependence of dynamics

HF & S. Plaskacz (1994); R. Vinter & A. Rapaport (1996)

There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems

J.-P. Aubin & G. Da Prato; S. Shuzhong; M. Bardi & R. Jensen;  
R. Buckhdahn, R. Peng, M. Quincampoix & C. Rainer, ...

When  $K$  is not viable one considers its **Viability Kernel** - the largest viable subset of  $K$  J.-P. Aubin

In general however there is no Lipschitz dependence of viable trajectories on the initial conditions.





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In general however there is **no Lipschitz dependence** of viable trajectories on the initial conditions.



## Example : Milyutin, 2000 (also for higher order)

In the presence of state constraints **optimal trajectories** may behave quite badly.

$$\begin{cases} \min \int_0^T \left( x(t) + \frac{|u(t)|^p}{p} \right) dt, \quad p > 1 \\ x'''(t) = u(t), \quad x(0) = \xi_0, \quad x'(0) = \xi_1, \quad x''(0) = \xi_2 \\ u(t) \in \mathbb{R}, \quad x(t) \geq 0 \end{cases}$$

If an initial condition  $(\xi_0, \xi_1, \xi_2)$  is admissible, then the optimal solution exists and is unique.

For “most” of the admissible initial conditions  $(\xi_0, \xi_1, \xi_2)$  there exist  $T = T(\xi_0, \xi_1, \xi_2) > 0$  such that the optimal trajectory reaches the boundary of constraints an **infinite** (countable) number of times with  $T$  being their **only** accumulation point.



# Lipschitz Dependence on Initial Conditions

$$C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{h \rightarrow 0+, K \ni y \rightarrow x} \frac{\text{dist}(y + hv; K)}{h} = 0\}$$

**Inward Pointing Condition**  $f(t, x, U) \cap \text{Int } C_K(x) \neq \emptyset \quad \forall x \in \partial K$

Assume  $f(t, x, U)$  are closed,  $f(t, x, \cdot)$  is continuous,  $f(\cdot, \cdot, u)$  is locally Lipschitz (uniformly in  $u$ ),  $\sup_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$

**Theorem (HF & Rampazzo, 2000; FP4 1994-1998)**

If the *inward pointing condition* holds true, then the set-valued map  $S_K(\cdot)$  is locally  $\mathcal{C}$ -Lipschitz on  $K$ .

$S_K(\cdot)$  is  $\mathcal{C}$ -Lipschitz at  $x_0 \in K$  if it is Lipschitz as a set-valued map from a neighborhood of  $x_0$  in  $K$  into  $\mathcal{C} = C([0, 1]; \mathbb{R}^n)$ :

$\exists L \geq 0, \forall x_1, x_2 \in K$  close to  $x_0, \forall \bar{x}_1(\cdot) \in S_K(x_1),$

$\exists \bar{x}_2(\cdot) \in S_K(x_2)$  satisfying  $\|\bar{x}_1(\cdot) - \bar{x}_2(\cdot)\|_{\mathcal{C}} \leq L \|x_1 - x_2\|$



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# State Constraints with Smooth Boundaries

When the boundary of  $K$  is **smooth** and  $f$  depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If  $K$  is an intersection of closed sets  $K_j \subset \mathbb{R}^n$  with smooth boundaries

$$K = \bigcap_{j=1}^m K_j$$

a counterexample to (NFT) was recently given by  
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Below  $d_j$  denotes the oriented distance to  $K_j$  and  $J(x)$  the set of all active indices at  $x$ .



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# Assumptions

Assume  $K_j \subset \mathbb{R}^n$  is closed and has  $C_{loc}^{1,1}$ -boundary for every  $j$ .

- (i)  $f(\cdot, x, u)$  is measurable  $\forall x, u$ ;
- (ii)  $f(t, x, \cdot)$  is continuous  $\forall t, x$ ;
- (iii)  $\forall r > 0, \exists \lambda_r > 0$  such that  $\forall t, u$   
 $f(t, \cdot, u)$  is  $\lambda_r$ -Lipschitz on  $rB$ ;
- (iv)  $\exists \gamma > 0$  such that  $\forall t, x, \max_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$ ;
- (v)  $f(t, x, U)$  is closed  $\forall t, x$ ;
- (vi)  $0 \notin \text{conv} \{ \nabla d_j(x) \mid j \in J(x) \}$  **transversality condition**.

Then

$$N_K(x) := C_K(x)^- = \mathbb{R}_+ \text{conv} \{ \nabla d_j(x) \mid j \in J(x) \}$$



## Multiple State Constraints $K = \bigcap_{j=1}^m K_j$

**Relaxed Inward Pointing Condition** :  $\exists \rho > 0 \forall x \in \partial K, \forall t$

$$\exists v_{t,x} \in \text{co} f(t, x, U), \quad \langle n_x, v_{t,x} \rangle \leq -\rho \quad \forall n_x \in N_K(x), \quad |n_x| = 1$$

**Theorem (HF & Bettiol, to appear, DCDS-A, FP6 2002-06)**

Assume the *relaxed inward pointing condition* and that either  $f$  is differentiable with respect to  $x$  or the sets  $f(t, x, U)$  are convex. Then the set-valued map  $\mathcal{S}_K(\cdot)$  is locally  $\mathcal{C}$ -Lipschitz on  $\text{Int } K$ .

**Corollary (Approximation from the interior)**

Assume that the *relaxed inward pointing condition* holds true, that  $f$  is differentiable with respect to  $x$  and let  $x_0 \in \text{Int } K$ . Then for every  $\bar{x} \in \mathcal{S}_K(x_0)$  and every  $\varepsilon > 0$  there exists  $x_\varepsilon \in \mathcal{S}_K(x_0)$  such that  $\|x_\varepsilon - \bar{x}\|_{\mathcal{C}} < \varepsilon$  and  $x_\varepsilon(t) \in \text{Int } K$  for all  $t \in [0, 1]$ .



# Value Function of the Mayer Problem

$$\text{minimize } \{\varphi(x(1)) \mid x(\cdot) \in \mathcal{S}_K(x_0)\}$$

$\varphi : K \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous,  $\text{Dom } \varphi \neq \emptyset$ .

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e. in } [t_0, 1], \quad x(t_0) = y_0 \quad (1)$$

$$V(t_0, y_0) = \inf\{\varphi(x(1)) \mid x(\cdot) \text{ satisfies (1), } x(t) \in K \quad \forall t \in [t_0, 1]\}$$

Even for **smooth** data and in the absence of state constraints the value function is, in general, **nonsmooth**. For this reason derivatives of  $V$  are replaced by “generalized” derivatives.



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# Value Function and Optimal Synthesis

If  $K = \mathbb{R}^n$  and  $V \in C^1$ , then optimal trajectories are those satisfying

$$x'(t) \in f(t, x(t), \Lambda(t, x(t))) \quad \text{a.e.}, \quad x(0) = x_0,$$

$$\Lambda(t, y) := \{u \in U \mid H(t, y, -V'_x(t, y)) = \langle -V'_x(t, y), f(t, y, u) \rangle\}$$

and a trajectory-control pair  $(\bar{x}(\cdot), \bar{u}(\cdot))$  is optimal **if and only if**

$$\bar{u}(t) \in \Lambda(t, \bar{x}(t)) \quad \text{a.e.}$$

Generalizations to more natural **nonsmooth** value functions lead to **highly irregular**  $\Lambda(\cdot, \cdot)$  and  $f(\cdot, \cdot, \Lambda(\cdot, \cdot))$ .

There is **no** analogous characterization for the state constrained case, but some sufficient conditions are available.



# Local Lipschitz Continuity of Value Function

Assume that  $K = \bigcap_{j=1}^m K_j$ ,  $K_j \subset \mathbb{R}^n$  is closed with  $C_{loc}^{1,1}$ -boundary and satisfy the transversality condition.

## Theorem ( $m=1$ )

If the **(relaxed) inward pointing condition** holds true and  $\varphi$  is locally Lipschitz, then  $V$  is locally Lipschitz on  $[0, 1] \times K$ .

## Theorem ( $m > 1$ )

Assume that the **relaxed inward pointing condition** holds true, that  $\varphi$  is locally Lipschitz and that either  $f$  is differentiable with respect to  $x$ , or  $f(t, x, U)$  is convex for every  $(t, x) \in [0, T] \times \mathbb{R}^n$ . Then  $V$  is locally Lipschitz on  $[0, 1] \times \text{Int } K$ .



## Optimal Synthesis - necessity

$\forall t \in [0, T)$ ,  $x \in K$  such that  $V(t, x) \neq +\infty$  and all  $\bar{v} \in \mathbb{R}^n$  define the directional derivative in the direction  $(1, \bar{v}) \in \mathbb{R}^{n+1}$  by

$$D_{\uparrow} V(t, x)(1, \bar{v}) := \liminf_{h \rightarrow 0+, v \rightarrow \bar{v}} \frac{V(t+h, x+hv) - V(t, x)}{h}$$

and consider the sets

$$F(t, x) = \{u \in U \mid D_{\uparrow} V(t, x)(1, f(t, x, u)) \leq 0\}.$$

Then the dynamic programming principle implies that if  $x(\cdot) \in \mathcal{S}_K(x_0)$  is optimal for the Mayer problem and  $u(\cdot)$  is a corresponding optimal control, then

$$u(t) \in F(t, x(t)) \text{ a.e. in } [0, T].$$

In other words,  $x(\cdot)$  satisfies

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.$$





## Optimal Synthesis - sufficiency

The above condition is also sufficient when  $V$  is locally Lipschitz on  $K$ .

Assume that  $K = \bigcap_{j=1}^m K_j$ , that for every  $j$ ,  $K_j \subset \mathbb{R}^n$  is closed with  $\mathcal{C}_{loc}^{1,1}$ -boundary and that the transversality condition holds true.

### Theorem (Sufficient optimality condition)

Assume that  $\varphi$  is locally Lipschitz, that the **relaxed inward pointing condition** holds true and that  $V$  is **continuous** on  $[0, 1] \times K$ . Let  $\bar{x} \in W^{1,1}$  satisfy

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.$$

If the set  $\{t \in [0, T] \mid \bar{x}(t) \in \partial K\}$  is finite, then  $\bar{x}(\cdot)$  is an optimal solution to the Mayer problem.



# Optimal Trajectories

$$\text{epi}(V) = \{(t, x, r) \in [0, 1] \times K \times \mathbb{R} \mid r \geq V(t, x)\}$$

Assume that  $V(0, x_0) < +\infty$  and consider the viability problem

$$\begin{cases} s'(t) = 1, & s(0) = 0 \\ x'(t) = f(t, x(t), u(t)), & u(t) \in U \text{ a.e., } x(0) = x_0 \\ z'(t) = 0, & z(0) = V(0, x_0) \\ (s(t), x(t), z(t)) \in \text{epi}(V) & \text{for all } t \in [0, 1]. \end{cases} \quad (2)$$

Then a trajectory  $\bar{x}(\cdot)$  is optimal **if and only if** the mapping  $[0, 1] \ni t \mapsto (t, \bar{x}(t), V(0, x_0))$  satisfies (2) on  $[0, 1]$ .

Some **algorithms** were developed to get  $\text{epi}(V)$  as the **viability kernel** of an auxiliary control system under state constraints.



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# Maximum Principle + Sensitivity Relation

Let  $(\bar{x}(\cdot), \bar{u}(\cdot))$  be optimal and  $V(0, \cdot)$  be locally Lipschitz at  $x_0$ . Assume that transversality condition on  $K_j$  holds true, that  $\varphi$  is differentiable and  $f$  is differentiable with respect to  $x$ . Then  $\exists \lambda \in \{0, 1\}$ ,  $\psi(\cdot) \in NBV([0, 1]; \mathbb{R}^n)$  and  $p(\cdot) \in W^{1,1}([0, 1]; \mathbb{R}^n)$ ,  $(\lambda, p, \psi) \neq 0$  satisfying the **sensitivity relation**  $-p(0) \in \lambda \partial_x V(0, x_0)$  (**generalized gradient** of  $V(0, \cdot)$  at  $x_0$ ) and the **maximum principle**

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) \quad \text{a.e. in } [0, 1],$$

$$\langle p(t) + \psi(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U} \langle p(t) + \psi(t), f(t, \bar{x}(t), u) \rangle \quad \text{a.e.}$$

$$-p(1) - \psi(1) = \lambda \nabla \varphi(\bar{x}(1)), \quad \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall t \in (0, 1].$$

for a positive Radon measure  $\mu$  on  $[0, 1]$  and a Borel measurable  $\nu(s) \in N_K(\bar{x}(s)) \cap B$   $\mu$ -a.e.



## Normality of the Maximum principle

Assume that  $K = \bigcap_{j=1}^m K_j$ , that for every  $j$ ,  $K_j \subset \mathbb{R}^n$  is closed with  $\mathcal{C}_{loc}^{1,1}$ -boundary and that the transversality condition holds true.

### Theorem

Assume that  $x_0 \in \text{Int } K$  and the **relaxed inward pointing condition** holds true. Then  $V(0, \cdot)$  is locally Lipschitz at  $x_0$  and in the maximum principle  $\lambda = 1$ .

Sensitivity relations link the value function and the maximum principle and imply some sufficient optimality conditions.



# Thank you for your attention!

