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Viable Trajectories and Optimal Synthesis in Optimal Control under State Constraints

Hélène Frankowska

CNRS AND UNIVERSITÉ PIERRE ET MARIE CURIE

Kick Off Meeting SADCO

Paris, March 3-4, 2011
Control System under State Constraints

\[
\begin{align*}
  x'(t) &= f(t, x(t), u(t)), \quad u(t) \in U \quad \text{a.e. in } [0, 1] \\
  x(0) &= x_0 \\
  x(t) &\in K \quad \text{for all } t \in [0, 1]
\end{align*}
\]

\(U\) is a complete separable metric space, \(K \subset \mathbb{R}^n\) is closed

\[f : [0, 1] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n\]

**Controls** are Lebesgue measurable functions \(u(\cdot) : [0, 1] \to U\)

A viable trajectory \(x(\cdot)\) of control system is an absolutely continuous function satisfying \(x(0) = x_0\),

\[x'(t) = f(t, x(t), u(t)) \text{ almost everywhere in } [0, 1]\]

for some control \(u(\cdot)\) and \(x(t) \in K\) for all \(t \in [0, 1]\).
Control System under State Constraints

\[
\begin{cases}
  x'(t) = f(t, x(t), u(t)), & u(t) \in U \text{ a.e. in } [0, 1] \\
  x(0) = x_0 \\
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\end{cases}
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for some control \(u(\cdot)\) and \(x(t) \in K\) for all \(t \in [0, 1]\).
Tangent Vectors

$f(a_0, u_1), f(b_0, u_2), f(c_0, u_2)$ are tangent to $K$ at $a_0$, $b_0$ and $c_0$. At $d_0$ there is no $u \in U$ such that $f(d_0, u)$ is tangent to $K$.

The contingent cone to $K$ at $x \in K$ is defined by

$$T_K(x) := \{ v \in \mathbb{R}^n \mid \liminf_{h \to 0^+} \frac{\text{dist}(x + hv; K)}{h} = 0 \}$$
Tangent Vectors

\[ f(a_0, u_1), f(b_0, u_2), f(c_0, u_2) \] are tangent to \( K \) at \( a_0, b_0 \) and \( c_0 \). At \( d_0 \) there is no \( u \in U \) such that \( f(d_0, u) \) is tangent to \( K \).

The **contingent** cone to \( K \) at \( x \in K \) is defined by

\[
T_K(x) := \{ v \in \mathbb{R}^n \mid \liminf_{h \to 0^+} \frac{\text{dist}(x + hv; K)}{h} = 0 \}
\]
Viability Theorem

\[ S_K(x_0) - \text{set of all viable trajectories starting at } x_0, \]

\[ W(t, x) := \{u \in U \mid f(t, x, u) \in T_K(x)\} \]

Then \( x(\cdot) \in S_K(x_0) \) if and only if \( x(0) = x_0 \) and \( x(\cdot) \) is a trajectory of the following control system

\[ x'(t) = f(t, x(t), u(t)), \quad u(t) \in W(t, x(t)) \quad \text{a.e. in } [0, 1] \]

for some control \( u(\cdot) \).

If \( x \leadsto f(x, U) := \bigcup_{u \in U} \{f(x, u)\} \) is "Marchaud", \( f \) is continuous and \( K \) is closed, then the following statements are equivalent

\[(i) \quad S_K(x_0) \neq \emptyset \quad \forall \ x_0 \in K \]
\[(ii) \quad f(x, U) \cap T_K(x) \neq \emptyset \quad \forall \ x \in \partial K \]
Viability Theorem

J. W. Bebernes & J. D. Schuur (1970)
For measurable time dependence of dynamics

There exist also extensions of the above theorem to stochastic control systems and to infinite dimensional control systems
J.-P. Aubin & G. Da Prato; S. Shuzhong; M. Bardi & R. Jensen;
R. Buckdhahn, R. Peng, M. Quincampoix & C. Rainer, ...

When $K$ is not viable one considers its **Viability Kernel** - the largest viable subset of $K$

J.-P. Aubin

In general however there is no Lipschitz dependence of viable trajectories on the initial conditions.
Viability Theorem

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When $K$ is not viable one considers its Viability Kernel - the largest viable subset of $K$ J.-P. Aubin

In general however there is no Lipschitz dependence of viable trajectories on the initial conditions.
Example: Milyutin, 2000 (also for higher order)

In the presence of state constraints optimal trajectories may behave quite badly.

\[
\begin{array}{ll}
\min \int_0^T \left( x(t) + \frac{|u(t)|^p}{p} \right) dt, & p > 1 \\
x'''(t) = u(t), & x(0) = \xi_0, \ x'(0) = \xi_1, \ x''(0) = \xi_2 \\
u(t) \in \mathbb{R}, & x(t) \geq 0
\end{array}
\]

If an initial condition \((\xi_0, \xi_1, \xi_2)\) is admissible, then the optimal solution exists and is unique.

For “most” of the admissible initial conditions \((\xi_0, \xi_1, \xi_2)\) there exist \(T = T(\xi_0, \xi_1, \xi_2) > 0\) such that the optimal trajectory reaches the boundary of constraints an infinite (countable) number of times with \(T\) being their only accumulation point.
Lipschitz Dependence on Initial Conditions

\[ C_K(x) := \{ v \in \mathbb{R}^n \mid \lim_{h \to 0^+, K \ni y \to x} \frac{\text{dist}(y + hv; K)}{h} = 0 \} \]

**Inward Pointing Condition** \( f(t, x, U) \cap \text{Int} \ C_K(x) \neq \emptyset \ \forall \ x \in \partial K \)

Assume \( f(t, x, U) \) are closed, \( f(t, x, \cdot) \) is continuous, \( f(\cdot, \cdot, u) \) is locally Lipschitz (uniformly in \( u \)), \( \sup_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|) \)

**Theorem (HF & Rampazzo, 2000; FP4 1994-1998)**

If the inward pointing condition holds true, then the set-valued map \( S_K(\cdot) \) is locally \( C- \)Lipschitz on \( K \).

\( S_K(\cdot) \) is \( C- \)Lipschitz at \( x_0 \in K \) if it is Lipschitz as a set-valued map from a neighborhood of \( x_0 \) in \( K \) into \( C = C([0, 1]; \mathbb{R}^n) \) :

\( \exists \ L \geq 0, \ \forall \ x_1, x_2 \in K \) close to \( x_0, \ \forall \ \bar{x}_1(\cdot) \in S_K(x_1), \ \exists \ \bar{x}_2(\cdot) \in S_K(x_2) \) satisfying \( \| \bar{x}_1(\cdot) - \bar{x}_2(\cdot) \|_C \leq L |x_1 - x_2| \)
Lipschitz Dependence on Initial Conditions

\[ C_K(x) := \{ v \in \mathbb{R}^n \mid \lim_{h \to 0^+} \frac{\text{dist}(y + hv; K)}{h} = 0 \} \]

**Inward Pointing Condition** \( f(t, x, U) \cap \text{Int } C_K(x) \neq \emptyset \quad \forall x \in \partial K \)

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**Theorem (HF & Rampazzo, 2000; FP4 1994-1998)**

*If the inward pointing condition holds true, then the set-valued map \( S_K(\cdot) \) is locally \( C-\)Lipschitz on \( K \).*
Lipschitz Dependence on Initial Conditions

\[ C_K(x) := \{ v \in \mathbb{R}^n \mid \lim_{h \to 0^+, \, K \ni y \to x} \frac{\text{dist}(y + hv; K)}{h} = 0 \} \]

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**Theorem (HF & Rampazzo, 2000; FP4 1994-1998)**

*If the inward pointing condition holds true, then the set-valued map \( S_K(\cdot) \) is locally \( C^- \)Lipschitz on \( K \).*

\( S_K(\cdot) \) is \( C^- \)Lipschitz at \( x_0 \in K \) if it is Lipschitz as a set-valued map from a neighborhood of \( x_0 \) in \( K \) into \( C = C([0, 1]; \mathbb{R}^n) \):

\[ \exists \, L \geq 0, \, \forall x_1, x_2 \in K \text{ close to } x_0, \, \forall \bar{x}_1(\cdot) \in S_K(x_1), \]

\[ \exists \, \bar{x}_2(\cdot) \in S_K(x_2) \text{ satisfying } \| \bar{x}_1(\cdot) - \bar{x}_2(\cdot) \|_C \leq L |x_1 - x_2| \]
State Constraints with Smooth Boundaries

When the boundary of $K$ is smooth and $f$ depends on time in a measurable way a Neighboring Feasible Trajectory Theorem (NFT) proved via a constructive argument implies Lipschitz dependence of viable trajectories on initial conditions.

If $K$ is an intersection of closed sets $K_j \subset \mathbb{R}^n$ with smooth boundaries

$$K = \bigcap_{j=1}^{m} K_j$$

a counterexample to (NFT) was recently given by P. Bettiol, A. Bressan & R. Vinter (2010)

Below $d_j$ denotes the oriented distance to $K_j$ and $J(x)$ the set of all active indices at $x$. 
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Below $d_j$ denotes the oriented distance to $K_j$ and $J(x)$ the set of all active indices at $x$. 
Assumptions

Assume $K_j \subset \mathbb{R}^n$ is closed and has $C^{1,1}_{loc}$-boundary for every $j$.

(i) $f(\cdot, x, u)$ is measurable $\forall x, u$;

(ii) $f(t, x, \cdot)$ is continuous $\forall t, x$;

(iii) $\forall r > 0, \exists \lambda_r > 0$ such that $\forall t, u$

$$f(t, \cdot, u) \text{ is } \lambda_r\text{-Lipschitz on } rB;$$

(iv) $\exists \gamma > 0$ such that $\forall t, x, \max_{u \in U} |f(t, x, u)| \leq \gamma(1 + |x|)$;

(v) $f(t, x, U)$ is closed $\forall t, x$;

(vi) $0 \notin \text{conv} \{\nabla d_j(x) \mid j \in J(x)\}$ transversality condition.

Then

$$N_K(x) := C_K(x)^- = \mathbb{R}_+ \text{conv} \{\nabla d_j(x) \mid j \in J(x)\}$$
Multiple State Constraints $K = \bigcap_{j=1}^{m} K_j$

Relaxed Inward Pointing Condition: $\exists \rho > 0 \ \forall x \in \partial K, \ \forall t$

$\exists \nu_{t,x} \in \text{co } f(t, x, U), \langle n_x, \nu_{t,x} \rangle \leq -\rho \ \forall n_x \in N_K(x), \ |n_x| = 1$

Theorem (HF & Bettiol, to appear, DCDS-A, FP6 2002-06)

Assume the relaxed inward pointing condition and that either $f$ is differentiable with respect to $x$ or the sets $f(t, x, U)$ are convex. Then the set-valued map $S_K(\cdot)$ is locally $C-$Lipschitz on $\text{Int } K$.

Corollary (Approximation from the interior)

Assume that the relaxed inward pointing condition holds true, that $f$ is differentiable with respect to $x$ and let $x_0 \in \text{Int } K$. Then for every $\bar{x} \in S_K(x_0)$ and every $\varepsilon > 0$ there exists $x_{\varepsilon} \in S_K(x_0)$ such that $\|x_{\varepsilon} - \bar{x}\|_C < \varepsilon$ and $x_{\varepsilon}(t) \in \text{Int } K$ for all $t \in [0, 1]$. 
Value Function of the Mayer Problem

\[
\text{minimize } \{ \varphi(x(1)) \mid x(\cdot) \in S_K(x_0) \}
\]

\[\varphi : K \rightarrow \mathbb{R} \cup \{+\infty\} \text{ is lower semicontinuous, Dom } \varphi \neq \emptyset.\]

\[
x'(t) = f(t, x(t), u(t)), \ u(t) \in U \text{ a.e. in } [t_0, 1], \ x(t_0) = y_0 \quad (1)
\]

\[
V(t_0, y_0) = \inf \{ \varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1), \ x(t) \in K \ \forall \ t \in [t_0, 1] \}
\]

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of \( V \) are replaced by “generalized” derivatives.
Value Function of the Mayer Problem

\[
\minimize \{ \varphi(x(1)) \mid x(\cdot) \in S_K(x_0) \}
\]

\(\varphi : K \to \mathbb{R} \cup \{+\infty\}\) is lower semicontinuous, \(\text{Dom } \varphi \neq \emptyset\).

\[
x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e. in } [t_0, 1], \quad x(t_0) = y_0 \quad (1)
\]

\(V(t_0, y_0) = \inf \{ \varphi(x(1)) \mid x(\cdot) \text{ satisfies (1), } x(t) \in K \forall t \in [t_0, 1] \}\)

Even for smooth data and in the absence of state constraints the value function is, in general, nonsmooth. For this reason derivatives of \(V\) are replaced by “generalized” derivatives.
minimize \{ \varphi(x(1)) \mid x(\cdot) \in S_K(x_0) \} \\

\varphi : K \to \mathbb{R} \cup \{+\infty\} is lower semicontinuous, \text{Dom} \varphi \neq \emptyset. \\

x'(t) = f(t, x(t), u(t)), u(t) \in U \text{ a.e. in } [t_0, 1], \ x(t_0) = y_0 \quad (1) \\

V(t_0, y_0) = \inf\{ \varphi(x(1)) \mid x(\cdot) \text{ satisfies } (1), x(t) \in K \ \forall \ t \in [t_0, 1] \} \\

Even for \textit{smooth} data and in the absence of state constraints the value function is, in general, \textit{nonsmooth}. For this reason derivatives of \( V \) are replaced by “generalized” derivatives.
Value Function and Optimal Synthesis

If $K = \mathbb{R}^n$ and $V \in C^1$, then optimal trajectories are those satisfying

$$x'(t) \in f(t, x(t), \Lambda(t, x(t))) \quad \text{a.e.}, \quad x(0) = x_0,$$

$$\Lambda(t, y) := \{ u \in U \mid H(t, y, -V_x'(t, y)) = \langle -V_x'(t, y), f(t, y, u) \rangle \}$$

and a trajectory-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal if and only if

$$\bar{u}(t) \in \Lambda(t, \bar{x}(t)) \quad \text{a.e.}$$

Generalizations to more natural nonsmooth value functions lead to highly irregular $\Lambda(\cdot, \cdot)$ and $f(\cdot, \cdot, \Lambda(\cdot, \cdot))$.

There is no analogous characterization for the state constrained case, but some sufficient conditions are available.
Assume that $K = \bigcap_{j=1}^{m} K_j, \ K_j \subset \mathbb{R}^n$ is closed with $C^{1,1}_{loc}$-boundary and satisfy the transversality condition.

**Theorem (m=1)**

If the (relaxed) inward pointing condition holds true and $\varphi$ is locally Lipschitz, then $V$ is locally Lipschitz on $[0, 1] \times K$.

**Theorem ($m > 1$)**

Assume that the relaxed inward pointing condition holds true, that $\varphi$ is locally Lipschitz and that either $f$ is differentiable with respect to $x$, or $f(t, x, U)$ is convex for every $(t, x) \in [0, T] \times \mathbb{R}^n$. Then $V$ is locally Lipschitz on $[0, 1] \times \text{Int } K$. 
**Optimal Synthesis - necessity**

∀ \( t \in [0, T) \), \( x \in K \) such that \( V(t, x) \neq +\infty \) and all \( \bar{v} \in \mathbb{R}^n \) define the directional derivative in the direction \((1, \bar{v})\in\mathbb{R}^{n+1}\) by

\[
D^\uparrow V(t, x)(1, \bar{v}) := \liminf_{h \to 0+, v \to \bar{v}} \frac{V(t + h, x + hv) - V(t, x)}{h}
\]

and consider the sets

\[
F(t, x) = \{ u \in U \mid D^\uparrow V(t, x)(1, f(t, x, u)) \leq 0 \}.
\]

Then the dynamic programming principle implies that if \( x(\cdot) \in S_K(x_0) \) is optimal for the Mayer problem and \( u(\cdot) \) is a corresponding optimal control, then

\[
u(t) \in F(t, x(t)) \text{ a.e. in } [0, T].
\]

In other words, \( x(\cdot) \) satisfies

\[
x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.
\]
The above condition is also sufficient when $V$ is locally Lipschitz on $K$.

Assume that $K = \bigcap_{j=1}^{m} K_j$, that for every $j$, $K_j \subset \mathbb{R}^n$ is closed with $C^{1,1}_{loc}$-boundary and that the transversality condition holds true.

**Theorem (Sufficient optimality condition)**

Assume that $\varphi$ is locally Lipschitz, that the relaxed inward pointing condition holds true and that $V$ is continuous on $[0, 1] \times K$. Let $\bar{x} \in W^{1,1}$ satisfy

$$x'(t) \in f(t, x(t), F(t, x(t))) \text{ a.e., } x(0) = x_0.$$ 

If the set $\{t \in [0, T] \mid \bar{x}(t) \in \partial K\}$ is finite, then $\bar{x}(\cdot)$ is an optimal solution to the Mayer problem.
Optimal Trajectories

\[ epi(V) = \{(t, x, r) \in [0, 1] \times K \times \mathbb{R} \mid r \geq V(t, x)\} \]

Assume that \( V(0, x_0) < +\infty \) and consider the viability problem

\[
\begin{aligned}
&s'(t) = 1, \quad s(0) = 0 \\
&x'(t) = f(t, x(t), u(t)), \quad u(t) \in U \text{ a.e., } \quad x(0) = x_0 \\
z'(t) = 0, \quad z(0) = V(0, x_0) \\
(s(t), x(t), z(t)) \in epi(V) \quad \text{for all } \quad t \in [0, 1].
\end{aligned}
\] (2)

Then a trajectory \( \bar{x}(\cdot) \) is optimal if and only if the mapping \( [0, 1] \ni t \mapsto (t, \bar{x}(t), V(0, x_0)) \) satisfies (2) on \([0, 1]\).

Some algorithms were developed to get \( epi(V) \) as the viability kernel of an auxiliary control system under state constraints.
\[ epi(V) = \{(t, x, r) \in [0, 1] \times K \times \mathbb{R} | r \geq V(t, x)\} \]

Assume that \( V(0, x_0) < +\infty \) and consider the viability problem

\[
\begin{cases}
  s'(t) = 1, & s(0) = 0 \\
  x'(t) = f(t, x(t), u(t)), & u(t) \in U \text{ a.e.}, \ x(0) = x_0 \\
  z'(t) = 0, & z(0) = V(0, x_0) \\
  (s(t), x(t), z(t)) \in epi(V) \text{ for all } t \in [0, 1].
\end{cases}
\]

Then a trajectory \( \bar{x}(\cdot) \) is optimal if and only if the mapping \([0, 1] \ni t \mapsto (t, \bar{x}(t), V(0, x_0))\) satisfies (2) on \([0, 1]\).

Some algorithms were developed to get \( epi(V) \) as the viability kernel of an auxiliary control system under state constraints.
Let \((\bar{x}(\cdot), \bar{u}(\cdot))\) be optimal and \(V(0, \cdot)\) be locally Lipschitz at \(x_0\). Assume that transversality condition on \(K_j\) holds true, that \(\varphi\) is differentiable and \(f\) is differentiable with respect to \(x\). Then \(\exists \lambda \in \{0, 1\}, \psi(\cdot) \in NBV([0, 1]; \mathbb{R}^n)\) and \(p(\cdot) \in W^{1,1}([0, 1]; \mathbb{R}^n)\), 

\((\lambda, p, \psi) \neq 0\) satisfying the sensitivity relation \(-p(0) \in \lambda \partial_x V(0, x_0)\) (generalized gradient of \(V(0, \cdot)\) at \(x_0\)) and the maximum principle

\[
-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) \text{ a.e. in } [0, 1],
\]

\[
\langle p(t) + \psi(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U} \langle p(t) + \psi(t), f(t, \bar{x}(t), u) \rangle \text{ a.e.}
\]

\[-p(1) - \psi(1) = \lambda \nabla \varphi(\bar{x}(1)), \quad \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \quad \forall \ t \in (0, 1].\]

for a positive Radon measure \(\mu\) on \([0, 1]\) and a Borel measurable \(\nu(s) \in N_K(\bar{x}(s)) \cap B \quad \mu\text{-a.e.}\)
Normality of the Maximum principle

Assume that $K = \bigcap_{j=1}^{m} K_j$, that for every $j$, $K_j \subset \mathbb{R}^n$ is closed with $C^{1,1}_{loc}$-boundary and that the transversality condition holds true.

**Theorem**

Assume that $x_0 \in \text{Int} K$ and the relaxed inward pointing condition holds true. Then $V(0, \cdot)$ is locally Lipschitz at $x_0$ and in the maximum principle $\lambda = 1$.

Sensitivity relations link the value function and the maximum principle and imply some sufficient optimality conditions.
Thank you for your attention!